SHO Green's function

jiwk

Consider the equation for the (undamped) simple harmonic oscillator (SHO)

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \tilde{\omega}^2 x(t) = f(t) \tag{1}$$

where $f(t)$ is a driving force, and $\tilde{\omega} \in \mathbb{R}$. We would like to find the Green's function for the SHO equation, $G(t,t')$. Which requires solving the following equation

$$\frac{\partial^2 G(t,t')}{\partial t^2} + \tilde{\omega}^2 G(t,t') = \delta(t-t') \tag{2}$$

where $\delta(t)$ is the Dirac delta "function". To solve this, we shall perform a Fourier Transform in t.

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left(\frac{\partial^2}{\partial t^2} + \tilde{\omega}^2 \right) G(t,t') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \delta(t-t') \\ \left(\tilde{\omega}^2 - \omega^2 \right) \hat{G}(\omega,t') &= \frac{1}{\sqrt{2\pi}} e^{-i\omega t'} \end{aligned} \tag{3}$$

Where $\hat{G}(\omega,t')$ is the FT in t of $G(t,t')$. Note that we have performed integration-by-parts twice to eliminate the partial derivative with respect to t. The boundary terms from the integration-by-parts go to zero under the appropriate regularization of the integral. So we are left with

$$\hat{G}(\omega,t') = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega t'}}{\tilde{\omega}^2 - \omega^2} \tag{4}$$

Now to find $G(t,t')$ we perform an inverse FT on $\hat{G}(\omega,t')$. This yields

$$
\begin{aligned}
G(t,t') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \hat{G}(\omega,t') \\
G(t,t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{i\omega(t-t')}}{\tilde{\omega}^2 - \omega^2} \\
G(t,t') &= \frac{1}{2\tilde{\omega}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{i\omega(t-t')}}{\tilde{\omega} - \omega} + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{i\omega(t-t')}}{\tilde{\omega} + \omega} \right] \\
G(t,t') &= \frac{1}{2\tilde{\omega}} \left[\frac{e^{i\tilde{\omega}(t-t')}}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{-i\omega(t-t')}}{\omega} + \frac{e^{-i\tilde{\omega}(t-t')}}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{i\omega(t-t')}}{\omega} \right] \\
G(t,t') &= \frac{1}{2\tilde{\omega}} \left[\frac{e^{i\tilde{\omega}(t-t')}}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{-i\omega(t-t')}}{\omega} + \frac{e^{-i\tilde{\omega}(t-t')}}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{i\omega(t-t')}}{\omega} \right] \\
G(t,t') &= \frac{1}{\tilde{\omega}} \, \mathrm{Re} \left[\frac{e^{i\tilde{\omega}(t-t')}}{2\pi} \int_{-\infty}^{\infty} d\omega \; \frac{e^{-i\omega(t-t')}}{\omega} \right]
\end{aligned}
\tag{5}
$$

♣