Euler Reflection Formula

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In this document, we shall compute the Euler Reflection formula of the Gamma function. The Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{1}$$

And satisfies the following two relations

$$\begin{aligned} \Gamma(1) &= 1 \\ \Gamma(z+1) &= z\Gamma(z) \end{aligned} \tag{2}$$

The Euler reflection formula states that the Gamma function satisfies

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \tag{3}$$

We first write $\Gamma(1-z)\Gamma(z)$ by definition and attempt to reduce the expression to a single integral.

$$\begin{aligned} \Gamma(1-z)\Gamma(z) &= \int_0^\infty x^{1-z-1} e^{-x} dx \int_0^\infty y^{z-1} e^{-y} dy \\ &= \int_0^\infty \int_0^\infty y^{z-1} x^{-z} e^{-(x+y)}\ dx\ dy \\ &= \int_0^\infty \int_0^\infty \frac{1}{y} \left(\frac{y}{x}\right)^z e^{-(x+y)}\ dx\ dy \\ x &\to u^2 \\ y &\to v^2 \\ dx\ dy &\to 4uv\ du\ dv \\ \to \Gamma(1-z)\Gamma(z) &= 4 \int_0^\infty \int_0^\infty uv \frac{1}{v^2} \left(\frac{v}{u}\right)^{2z} e^{-(u^2+v^2)}\ du\ dv \\ &= 4 \int_0^\infty \int_0^\infty \frac{u}{v} \left(\frac{v}{u}\right)^{2z} e^{-(u^2+v^2)}\ du\ dv \\ &= 4 \int_0^\infty \int_0^\infty \left(\frac{v}{u}\right)^{2z-1} e^{-(u^2+v^2)}\ du\ dv \end{aligned} \tag{4}$$

We can now convert to polar coordinates, in which we will have a Gaussian integral in r.

$$
\begin{aligned}
u &\to r\cos\theta \\
v &\to r\sin\theta \\
du\ dv\ &\to r\ dr\ d\theta \\
\to \Gamma(1-z)\Gamma(z) &= 4\int_0^\infty \int_0^{\pi/2} (\tan\theta)^{2z-1}\, e^{-r^2} r\ d\theta\ dr \\
&= 2\int_0^{\pi/2} (\tan\theta)^{2z-1}\ d\theta
\end{aligned} \tag{5}
$$

We would now like to make a substitution that allows us to compute the integral using familiar tools from complex analysis.

$$
\begin{aligned}
\Gamma(1-z)\Gamma(z) &= 2\int_0^{\pi/2} (\tan\theta)^{2z-1}\ d\theta \\
\theta &\to \arctan t \\
d\theta &\to \frac{1}{1+t^2} dt \\
\to \Gamma(1-z)\Gamma(z) &= 2\int_0^\infty \frac{t^{2z-1}}{1+t^2}\ dt
\end{aligned} \tag{6}
$$

Another substitution...

$$
\begin{aligned}
\Gamma(1-z)\Gamma(z) &= 2\int_0^\infty \frac{t^{2z}}{t\,(1+t^2)}\ dt \\
t^2 &\to s \\
2t\ dt &\to ds \\
2\frac{1}{t}\ dt &\to \frac{1}{t^2} ds \\
2\frac{1}{t}\ dt &\to \frac{1}{s} ds \\
\to \Gamma(1-z)\Gamma(z) &= \int_0^\infty \frac{s^{z-1}}{1+s}\ ds
\end{aligned} \tag{7}
$$

Ok, one more substitution before we actually compute this thing.

$$
\begin{aligned}
\Gamma(1-z)\Gamma(z) &= \int_0^\infty \frac{s^{z-1}}{1+s}\,ds \\
s &\to e^t \\
ds &\to e^t dt \\
\to \Gamma(1-z)\Gamma(z) &= \int_{-\infty}^\infty \frac{e^{t(z-1)}}{1+e^t} e^t\,dt \\
&= \int_{-\infty}^\infty \frac{e^{zt}}{1+e^t}\,dt
\end{aligned} \tag{8}
$$

This converges provided $0 < \mathrm{Re}(z) < 1$, but using the recursion relation, we can compute every other value of $\Gamma(1-z)\Gamma(z)$ for which it is defined. Observe that this integral has a simple pole at $t = i\pi$. Therefore we can use the residue theorem to compute the integral. Let $\gamma = \{t : -R < t < R\} \cup \{t + 2i\pi : -R < t < R\} \cup \{R + it : 0 < t < 2\pi\} \cup \{-R + it : 0 < t < 2\pi\}$. Then we have

$$
\begin{aligned}
\int_\gamma \frac{e^{zt}}{1+e^t}\,dt &= 2\pi i \lim_{t\to i\pi} (t-i\pi)\,\frac{e^{zt}}{1+e^t} \\
&= 2\pi i \lim_{t\to i\pi} (t-i\pi)\,\frac{e^{zt}}{1+e^{i\pi}(t-i\pi)+O(t-i\pi)^2} \\
&= 2\pi i \lim_{t\to i\pi} (t-i\pi)\,\frac{e^{zt}}{1+e^{i\pi}+e^{i\pi}(t-i\pi)+O(t-i\pi)^2} \\
&= 2\pi i e^{i\pi} \lim_{t\to i\pi} (t-i\pi)\,\frac{e^{zt}}{(t-i\pi)+O(t-i\pi)^2} \\
&= 2\pi i e^{i\pi(z-1)}
\end{aligned} \tag{9}
$$

Applying the parameterizations of γ, we have

$$
\begin{aligned}
2\pi i e^{i\pi(z-1)} &= \int_\gamma \frac{e^{zt}}{1+e^t}\,dt \\
&= \int_{-R}^R \frac{e^{zt}}{1+e^t}\,dt + \int_R^{-R} \frac{e^{z(t+2\pi i)}}{1+e^{t+2\pi i}}\,dt + 2\pi i \int_0^{2\pi} \frac{e^{z(R+it)}}{1+e^{R+it}}\,dt + 2\pi i \int_{2\pi}^0 \frac{e^{z(-R+it)}}{1+e^{-R+it}}\,dt \\
&= \int_{-R}^R \frac{e^{zt}}{1+e^t}\,dt - e^{2\pi i z} \int_{-R}^R \frac{e^{zt}}{1+e^t}\,dt + 2\pi i \left[\int_0^{2\pi} \frac{e^{z(R+it)}}{1+e^{R+it}} - \frac{e^{z(-R+it)}}{1+e^{-R+it}}\,dt\right] \\
&= \left(1-e^{2\pi i z}\right) \int_{-R}^R \frac{e^{zt}}{1+e^t}\,dt + 2\pi i \left[\int_0^{2\pi} \frac{e^{z(R+t)}}{1+e^{R+it}} - \frac{e^{z(-R+it)}}{1+e^{-R+it}}\,dt\right]
\end{aligned} \tag{10}
$$

We will attempt to bound the second two integrals above.

$$
\left[\int_0^1 \frac{e^{z(R+2\pi it)}}{1+e^{R+2\pi it}} - \frac{e^{z(-R+2\pi it)}}{1+e^{-R+2\pi it}}\,dt\right] = e^{zR}\int_0^{2\pi} \frac{e^{it}}{1+e^R e^{it}}\,dt - e^{-zR}\int_0^{2\pi} \frac{e^{it}}{1+e^{-R}e^{it}}\,dt \tag{11}
$$

We can write each of these as a contour integral around the unit circle, C.

$$e^{zR}\int_0^{2\pi}\frac{e^{it}}{1+e^Re^{it}}\,dt-e^{-zR}\int_0^{2\pi}\frac{e^{it}}{1+e^{-R}e^{it}}\,dt=-ie^{zR}\int_C\frac{1}{1+e^Rw}\,dw+ie^{-zR}\int_C\frac{1}{1+e^{-R}w}\,dw \tag{12}$$

But for R large and $0<\mathrm{Re}(z)<1$, the integral on the left has no poles within the contour and the integral on the left goes to zero since $e^{-zR}\to 0$. Therefore, we have

$$\begin{aligned}\lim_{R\to\infty}\left(1-e^{2\pi iz}\right)\int_{-R}^{R}\frac{e^{zt}}{1+e^t}\,dt&=2\pi ie^{i\pi(z-1)}\\ \int_{-\infty}^{\infty}\frac{e^{zt}}{1+e^t}\,dt&=-\frac{2\pi ie^{i\pi z}}{(1-e^{2\pi iz})}\\ \Gamma(1-z)\Gamma(z)\,dt&=\frac{2\pi ie^{i\pi z}}{(e^{2\pi iz-1})}\\ \Gamma(1-z)\Gamma(z)\,dt&=\frac{2\pi i}{(e^{\pi iz}-e^{-\pi iz})}\\ \Gamma(1-z)\Gamma(z)\,dt&=\frac{\pi}{\sin\pi z}\end{aligned} \tag{13}$$

Which is the desired result.

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