Sum of Cubes Identity

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This is an example of an easy induction proof of an identity that I find pleasing.

Proposition: For $n \in \mathbb{N}$, $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Proof. (*by induction*)

Let $n = 2$. Then $1^3 + 2^3 = 9 = 3^3 = (1 + 2)^3$.

Assume that, for all $k \leq n$, $1^3 + \cdots + k^3 = (1 + \cdots + k)^2$. Then we have

$$
\begin{aligned}
(1 + \cdots + n + n + 1)^2 &= (1 + \cdots + n)^2 + 2(n+1)(1 + \cdots + n) + (n+1)^2 \\
&= 1^3 + \cdots + n^3 + 2(n+1)\frac{n(n+1)}{2} + (n+1)^2 \\
&= 1^3 + \cdots + n^3 + n(n+1)^2 + (n+1)^2 \\
&= 1^3 + \cdots + n^3 + (n+1)(n+1)^2 \\
&= 1^3 + \cdots + n^3 + (n+1)^3
\end{aligned}
\tag{1}
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