Least Squares Pseudoinverse

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Here we calculate a least squares solution for the Moore-Penrose psuedoinverse of an $M \times N$ matrix, A.
The Moore-Penrose psuedoinverse of an $M \times N$ matrix, A, has the property that

$$AA^+A = A \tag{1}$$

Thus, we wish to solve for the matrix A^+. Restated in terms of the coefficients, A_{ij}, of A, this is equivalent to

$$\sum_{l=1}^{N} \sum_{k=1}^{M} A_{il} A^+_{lk} A_{kj} = A_{ij} \tag{2}$$

The least squares solution to this equation is the solution which minimizes the Total Sum of Squares (TSS), defined by

$$\begin{aligned} \text{TSS} &= \sum_{i=1}^{M} \sum_{j=1}^{N} \left(\sum_{l=1}^{N} \sum_{k=1}^{M} A_{il} A^+_{lk} A_{kj} - A_{ij} \right)^2 \\ \text{TSS}_{ij} &= \left(\sum_{l=1}^{N} \sum_{k=1}^{M} A_{il} A^+_{lk} A_{kj} - A_{ij} \right)^2 \end{aligned} \tag{3}$$

The TSS is minimized by differentiating with respect to A^+_{nm}, setting the result equal to zero, and solving for A^+_{nm}. This yields $M \times N$ equations in $M \times N$ variables:

$$\begin{aligned} 0 = \frac{\partial}{\partial A^+_{nm}} \text{TSS}_{ij} &= 2 \left(\sum_{l=1}^{N} \sum_{k=1}^{M} A_{il} A^+_{lk} A_{kj} - A_{ij} \right) \left(A_{in} A^+_{nm} A_{mj} \right) \\ A_{ij} &= \sum_{l=1}^{N} \sum_{k=1}^{M} A_{il} A^+_{lk} A_{kj} \\ A_{ij} &= \sum_{l \neq n}^{N} \sum_{k \neq m}^{M} A_{il} A^+_{lk} A_{kj} + A_{in} A^+_{nm} A_{mj} \\ A_{in} A^+_{nm} A_{mj} &= A_{ij} - \sum_{l \neq n}^{N} \sum_{k \neq m}^{M} A_{il} A^+_{lk} A_{kj} \\ A^+_{nm} &= \frac{A_{ij} - \sum_{l \neq n}^{N} \sum_{k \neq m}^{M} A_{il} A^+_{lk} A_{kj}}{A_{in} A_{mj}} \end{aligned} \tag{4}$$