Cauchy Distribution Characteristic Function

jiwk

In this document, we shall compute the characteristic function of the Cauchy distribution, the probability density function of which is given by

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \tag{1}$$

Define the characteristic function, $\varphi(t)$, of the Cauchy distribution by

$$\begin{aligned} \varphi(t) &= \mathbb{E}\left[e^{itx}\right] \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx \end{aligned} \tag{2}$$

To compute this integral, we will consider the following integral in the complex plane

$$I(t) = \int_{\Gamma} \frac{e^{itz}}{1+z^2} dz \tag{3}$$

where Γ is a closed semicircle of radius R in the complex plane containing $[-R, R] \subset \mathbb{R}$.

For $t > 0$, we compute

$$I(t) = \int_{\gamma_1} \frac{e^{itz}}{1+z^2} dz \tag{4}$$

where $\gamma_1 = [-R, R] \cup \{Re^{i\theta} : \theta \in [0, \pi]\}$ oriented counter clockwise. Since the integrand has a simple pole at $z = i$, we may use the residue theorem to write

$$\begin{aligned} I(t) &= 2\pi i \lim_{z \to i} (z-i) \frac{e^{itz}}{1+z^2} \\ &= 2\pi i \lim_{z \to i} (z-i) \frac{e^{itz}}{(z-i)(z+i)} \\ &= 2\pi i \lim_{z \to i} \frac{e^{itz}}{z+i} \\ &= 2\pi i \frac{e^{-t}}{2i} \\ &= \pi e^{-t} \end{aligned} \tag{5}$$

In terms of our parameterization of γ_1, we have

$$\pi e^{-t} = \int_{-R}^{R} \frac{e^{itx}}{1+x^2}dx + \int_0^{\pi} \frac{iRe^{i\theta}e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}d\theta \tag{6}$$

We now attempt to bound the integral over the arc in the upper half plane for $t > 0$,

$$\begin{aligned}
\left|\int_0^{\pi} \frac{iRe^{i\theta}e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}d\theta\right| &\leq \int_0^{\pi} \left|\frac{iRe^{i\theta}e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}\right| d\theta \\
&= \int_0^{\pi} \left|\frac{e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}\right| d\theta \\
&= \int_0^{\pi} \sqrt{\frac{e^{itRe^{i\theta}}e^{-itRe^{-i\theta}}}{1+2R^2\cos(2\theta)+R^4}}d\theta \\
&= \int_0^{\pi} \sqrt{\frac{e^{itR(e^{i\theta}-e^{-i\theta})}}{1+2R^2\cos(2\theta)+R^4}}d\theta \\
&= \int_0^{\pi} \frac{e^{-tR\sin(\theta)}}{\sqrt{1+2R^2\cos(2\theta)+R^4}}d\theta \\
&\leq \frac{\pi}{R^2-1}
\end{aligned} \tag{7}$$

Therefore, as $R \to \infty$, we have

$$\left|\int_0^{\pi} \frac{iRe^{i\theta}e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}d\theta\right| \leq \lim_{R\to\infty} \frac{\pi}{R^2-1} = 0 \tag{8}$$

And finally,

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{itx}}{1+x^2}dx + \lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}e^{itRe^{i\theta}}}{1+R^2e^{2i\theta}}d\theta &= \pi e^{-t} \\
\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}dx &= \pi e^{-t} \\
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}dx &= e^{-t} \\
\varphi(t) &= e^{-t}
\end{aligned} \tag{9}$$

for $t > 0$.

Now, for $t < 0$, we compute

$$I(t) = \int_{\gamma_2} \frac{e^{itz}}{1+z^2}dz \tag{10}$$

with $\gamma_2 = [-R, R] \cup \{Re^{-i\theta} : \theta \in [0, \pi]\}$ oriented clockwise. By the residue theorem we have

$$\begin{aligned} I(t) &= -2\pi i \lim_{z \to -i} (z+i) \frac{e^{itz}}{1+z^2} \\ &= \pi e^t \end{aligned} \tag{11}$$

Note the sign difference in the residue due to the orientation of γ_2. Proceeding as above,

$$\pi e^t = \int_{-R}^{R} \frac{e^{itx}}{1+x^2} dx + \int_0^{\pi} \frac{e^{itRe^{-i\theta}}}{1+Re^{-2i\theta}} d\theta \tag{12}$$

Observe that, for $t < 0$, we have

$$\begin{aligned} \left| \int_0^{\pi} \frac{iRe^{-i\theta} e^{itRe^{-i\theta}}}{1+R^2 e^{-2i\theta}} d\theta \right| &\le \int_0^{\pi} \left| \frac{iRe^{-i\theta} e^{itRe^{-i\theta}}}{1+R^2 e^{-2i\theta}} \right| d\theta \\ &\le \frac{\pi}{(1-R)^2} \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{itx}}{1+x^2} dx + \lim_{R \to \infty} \int_0^{\pi} \frac{e^{itRe^{-i\theta}}}{1+Re^{-2i\theta}} d\theta &= \pi e^t \\ \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx &= \pi e^t \\ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx &= e^t \\ \varphi(t) &= e^t \end{aligned} \tag{14}$$

for $t < 0$. Thus we find

$$\begin{aligned} \varphi(t) &= \begin{cases} 1\ , & t = 0 \\ e^{-t}\ , & t > 0 \\ e^{t}\ , & t < 0 \end{cases} \\ \varphi(t) &= e^{-|t|} \end{aligned} \tag{15}$$

which is a pretty weird result if you ask me.

Just for fun, here's a hacky calculation of the same thing.

$$
\begin{aligned}
\varphi(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx \\
\varphi''(t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2 e^{itx}}{1+x^2} dx \\
\varphi''(t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{itx}\left[1 - \frac{1}{1+x^2}\right] dx \\
\varphi''(t) &= \varphi(t) - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{itx} dx
\end{aligned} \tag{16}
$$

Recalling that the right hand side is just a delta function, we have

$$
\varphi''(t) = \varphi(t) - 2\delta(t) \tag{17}
$$

Now it seems more or less reasonable to postulate that $\varphi(t) = e^{f(t)}$ for some function $f(t)$.

$$
\begin{aligned}
\varphi(t) &= e^{f(t)} \\
\varphi'(t) &= f'(t)\varphi(t) \\
\varphi''(t) &= f''(t)\varphi(t) + f'(t)^2\varphi(t)
\end{aligned} \tag{18}
$$

Matching the sign of the coefficients of the two differential equations gives

$$
\begin{aligned}
f'(t)^2 &= 1 \\
f''(t) &= -2\delta(t)
\end{aligned} \tag{19}
$$

The second of these can be solved simply

$$
f'(t)^2 = 1 \implies
\begin{aligned}
f'(t) &= -2\int \delta(t) dt \\
f'(t) &= -2\Theta(t) + c \\
f'(t) &= -\left[2\Theta(t) - 1\right] \\
f'(t) &= -\mathrm{sgn}(t) \\
f(t) &= -|t|
\end{aligned} \tag{20}
$$

And we may omit the constant of integration because we know that $\varphi(0) = 1$. Thus, we have

$$
\varphi(t) = e^{-|t|} \tag{21}
$$

♣