SHO Green's Function

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We consider a classical damped forced harmonic oscillator whose equation of motion is given by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \tag{1}$$

with driving force $f(t)$ and where $\gamma < \omega_0$. Note that we have deviated from the notation in the homework with the coefficient of 2γ above because it makes the final solution look nicer.

We wish to find the Green's function of the differential operator above. That is, we wish to find a function, $g(t, t')$, such that

$$\ddot{g} + 2\gamma\dot{g} + \omega_0^2 g = \delta(t - t') \tag{2}$$

subject to the initial condition, $g(t', t') = 0$.

First, let $\hat{f}(\omega)$ be the frequency-time Fourier transform of $f(t)$, defined by $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int dt\, e^{-i\omega t} f(t)$. Then the function $\hat{g}(\omega, t')$ must satisfy

$$\begin{aligned} -\omega^2\hat{g} + 2i\gamma\omega\hat{g} + \omega_0^2\hat{g} &= \frac{1}{\sqrt{2\pi}} e^{-i\omega t'} \\ \hat{g}(\omega, t') &= \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega t'}}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \\ \implies g(t, t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \end{aligned} \tag{3}$$

Which can be solved with a contour integral. Define $\omega^{\mp} = i\gamma \mp \sqrt{\omega_0^2 - \gamma^2}$. Then we can rewrite $g(t, t')$ as

$$\implies g(t, t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-t')}}{(\omega - \omega^+)(\omega^- - \omega)} \tag{4}$$

Let Γ_1 be a counterclockwise semicircular contour of radius R in the upper half plane containing $\omega^{\mp}$ and let Γ_2 be a clockwise semicircular contour of radius R in the lower half plane. In order for the integral along the semicircular portion to decay exponentially, we must have $\mathrm{Re}\left(i\omega(t - t')\right) < 0$ which occurs in the upper half plane when $t > t'$ and in the lower half plane when $t < t'$. So we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-t')}}{(\omega-\omega^+)(\omega^- -\omega)} &= \frac{1}{2\pi}\int_{\Gamma_1} dz \frac{e^{iz(t-t')}}{(z-\omega^+)(\omega^- - z)} \text{ , if } \ t>t' \\
\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-t')}}{(\omega-\omega^+)(\omega^- -\omega)} &= \frac{1}{2\pi}\int_{\Gamma_2} dz \frac{e^{iz(t-t')}}{(z-\omega^+)(\omega^- - z)} \text{ , if } \ t<t'
\end{aligned}
\tag{5}
$$

For convenience, define $\omega^* = \sqrt{\omega_0^2-\gamma^2}$. Then, by the Residue Theorem we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\Gamma_1} dz \frac{e^{iz(t-t')}}{(z-\omega^+)(\omega^- - z)} &= i\left(\frac{e^{i\omega^+(t-t')}}{\omega^- - \omega^+} - \frac{e^{i\omega^-(t-t')}}{\omega^- - \omega^+}\right) \\
&= \frac{-ie^{-\gamma(t-t')}}{2\omega^*}\left(e^{i\omega^*(t-t')} - e^{-i\omega^*(t-t')}\right) \\
&= e^{-\gamma(t-t')}\frac{\sin(\omega^*(t-t'))}{\omega^*} \\
\frac{1}{2\pi}\int_{\Gamma_2} dz \frac{e^{iz(t-t')}}{(z-\omega^+)(z-\omega^-)} &= 0
\end{aligned}
\tag{6}
$$

Since the integrand is analytic on the entire lower half place. Thus, the Green's function of the linear differential operator that yields the equation of motion for the damped harmonic oscillator is

$$
\begin{aligned}
g(t,t') &= \begin{cases} e^{-\gamma(t-t')}\frac{\sin\left(\omega^*(t-t')\right)}{\sqrt{\omega_0^2-\gamma^2}} \ , \ \ t>t' \\ 0 \ , \ \ t<t' \end{cases} \\
g(t,t') &= \Theta(t-t')e^{-\gamma(t-t')}\frac{\sin(\omega^*(t-t'))}{\omega^*}
\end{aligned}
\tag{7}
$$

where $\omega^* = \sqrt{\omega_0^2-\gamma^2}$ and where $\Theta(t)$ is the Heaviside step function. Note again, the different convention we have used for the damping coefficient, γ. In particular, this now agrees with the Green's function for the damped oscillator given on this Wikipedia page.

We may also write

$$
g(t) = \Theta(t)e^{-\gamma t}\frac{\sin(\omega^* t)}{\omega^*}
\tag{8}
$$

Define $L = \partial_t^2 + 2\gamma\partial_t + \omega_0^2$. Suppose $x(t)$ solves

$$
\begin{aligned}
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x &= f(t) \\
L\left[x(t)\right] &= f(t)
\end{aligned}
\tag{9}
$$

for some arbitrary "forcing" function $f(t)$. Recall that $L\left[g(t-t')\right] = \delta(t-t')$. So we can rewrite the above equation as follows

$$
\begin{aligned}
L\left[x(t)\right] &= f(t) \\
L\left[x(t)\right] &= \int_{-\infty}^{\infty} dt'\ \delta(t-t')f(t') \\
L\left[x(t)\right] &= \int_{-\infty}^{\infty} dt'\ L\left[g(t-t')\right] f(t') \\
L\left[x(t)\right] &= L\left[\int_{-\infty}^{\infty} dt'\ g(t-t')f(t')\right]
\end{aligned}
\tag{10}
$$

where the last step is justified because L only acts on functions of t. We can assume that solutions to $L\left[x(t)\right] = f(t)$ are unique provided initial conditions are specified. Therefore, we have

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} dt'\ g(t-t')f(t') \\
x(t) &= \int_{-\infty}^{\infty} dt'\ g(t')f(t-t')
\end{aligned}
\tag{11}
$$

for arbitrary forcing function $f(t)$ and where $g(t)$ is as defined in part a). This is clearly a convolution of f with g.

To find the frequency space Green's function, we solve

$$
\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = e^{-i\omega t}
\tag{12}
$$

for $x(t)$. The frequency space Green's function will be the amplitude, $\tilde{g}(\omega)$, of the solution, $x(t) = \tilde{g}(\omega)e^{-i\omega t}$. We can solve for $x(t)$ by forming the convolution with $g(t-t')$ shown above. $x(t)$ is then given by

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} dt'\ g(t')f(t-t') \\
x(t) &= \int_{-\infty}^{\infty} dt'\ e^{-i\omega(t-t')}\Theta(t')e^{-\gamma t'}\frac{\sin(\omega^* t')}{\omega^*} \\
x(t) &= \frac{1}{\omega^*}\int_{0}^{\infty} dt'\ e^{-i\omega(t-t')}e^{-\gamma t'}\sin(\omega^* t') \\
x(t) &= \frac{e^{-it\omega}}{(\gamma - i\omega)^2 + \omega^{*2}}
\end{aligned}
\tag{13}
$$

So we have that

$$\tilde{g}(\omega) = \frac{1}{(\gamma - i\omega)^2 + \omega^{*2}} \tag{14}$$

The convolution theorem states that if

$$x(t) = \int_{-\infty}^{\infty} dt' \; g(t - t') f(t') \tag{15}$$

then the Fourier transform of each function satisfies

$$\tilde{x}(\omega) = \tilde{g}(\omega) \tilde{f}(\omega) \tag{16}$$

Taking the inverse Fourier transform of both sides gives

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\omega) \tilde{f}(\omega) e^{-i\omega t} \tag{17}$$

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