The Poincaré-Hopf Theorem

Connecting topology and analysis

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July 12^{th}, 2021

Overview & Preliminaries

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Definition

A **diffeomorphism** is a map $f : X \to Y$ that carries X homeomorphically onto Y if both f and f^{-1} are smooth.

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So, we say a manifold is a topological space that locally looks like $\mathbb{R}^m$.

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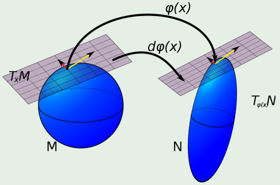
Answer

For each $x \in M$, we associate a linear subspace $TM_x \subset \mathbb{R}^k$ of dimension m called the **tangent space** of M at x, elements of which are tangent vectors to M at x. $d\phi$ is then a linear map from TM_x to $TN_{\phi(x)}$. So, we have an analogue between

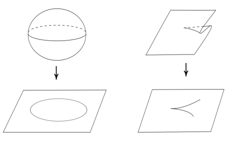
$$\phi : M \to N, \; d\phi_x : TM_x \to TN_{\phi(x)}$$



Overview & Preliminaries

Definition

We say x in M is a **regular point** (and f(x) a **regular value**) if df is non-singular (in other words, df_x is invertible).



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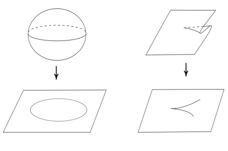
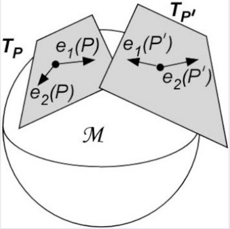


Definition

An **orientation** for a finite dimensional real vector space is an equivalence class of ordered bases as follows: the ordered basis $(b_1, ..., b_n)$ preserves the orientation of the basis $(b'_1, ..., b'_n)$ (where $b'_i = \sum a_{ij} b_j$) if $det(a_{ij}) > 0$, and reverses orientation if the determinant is negative.



Overview & Preliminaries

Definition

Let $x \in M$ be a regular point of f. The sign of df is $+1$ or -1 depending if df preserves or reverses orientation. Then, for any regular value $y \in N$, we define the **degree** of f at y as

$$deg(f; y) = \sum_{x \in f^{-1}(y)} \text{sign } df_x.$$

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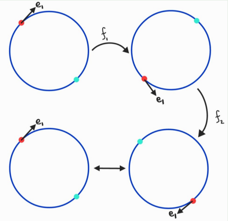
Example

The antipodal map on the 1-sphere is composed of two reflections, f_1 and f_2. The orientation of the basis vector e_1 is reversed twice (in other words, preserved), so the degree of the mapping is 1.



Poincaré-Hopf Overview

Let M be a compact manifold and w a smooth vector field on M with isolated zeros. If M has a boundary, then w is required to point outward at all boundary points.

Theorem

The sum $\sum$ i of the indices at the zeros of w is a topological invariant of M, and does not depend on the particular choice of vector field w.

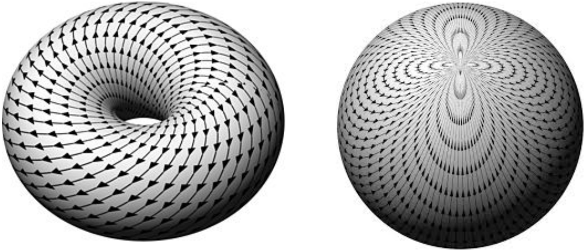


Figure: Vector fields on the 2-torus and 2-sphere.

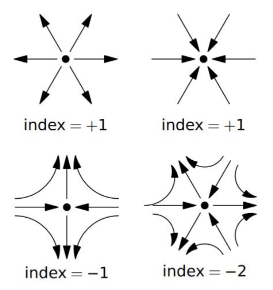
Index of a Vector Field

Definition

Consider first an open set $U \subset \mathbb{R}^m$ and a smooth vector field $v : U \to \mathbb{R}^m$ with an isolated zero at the point $z \in U$. Then, the function $\bar{v}(x) = v(x)/||v(x)||$ maps a small sphere centered at z into the unit sphere. The degree of this mapping is called the **index** i of v at the zero z.

Example

In two dimensions, vector fields which are sources or sinks have a positive index, while vector fields which are saddles have a negative index.



Primary Result

Theorem

Consider a compact, boundaryless manifold $M \subset \mathbb{R}^k$. Let N_ϵ denote the closed ϵ-neighborhood of M. For any vector field v on M with only nondegenerate zeros, the index sum is equal to the degree of the Gauss mapping

$$g : \partial N_\epsilon \to S^{k-1}.$$

In particular, this sum does not depend on the choice of the vector field.

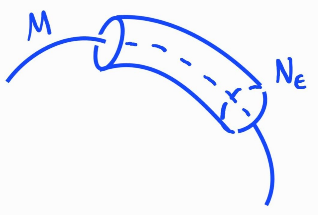


Figure: The ϵ-neighborhood of M.

Hopf Lemma

Lemma

If $v : X \to \mathbb{R}^m$ is a smooth vector field with isolated zeros, and if v points out of X along the boundary, then the index sum $\sum i$ is equal to the degree of the Gauss mapping from ∂X to S^{m-1}. In particular, $\sum i$ does not depend on the choice of v.

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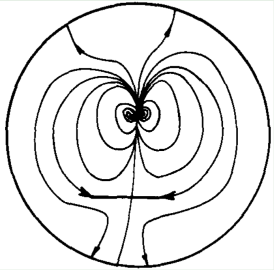
Example

If a vector field on the disk D^m points outward along the boundary, $\sum i = +1$. Likewise, a map from ∂D^m to S^{m-1} will have a degree of 1.



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Now, by removing an ϵ-ball around each zero, the function $\bar{v}(x) = v(x)/||v(x)||$ maps this manifold into S^{m-1}. Therefore, the sum of the degrees of $\bar{v}|\partial X$ is 0, and is homotopic to g. The other boundary components sum to $-\sum i$ since the ϵ-spheres get the wrong orientation. So, we have

$$\deg(g) - \sum i = 0$$
$$\implies \deg(g) = \sum i.$$

Main Theorem

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For any vector field v on M with only nondegenerate zeros, $\sum i$ is equal to the degree of the Gauss mapping

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Let $r(x) \in M$ be the closest point of M to $x \in N_\epsilon$. The vector $x - r(x)$ will therefore be perpendicular to $TM_{r(x)}$. Now, let

$$\phi(x) = ||x - r(x)||^2$$

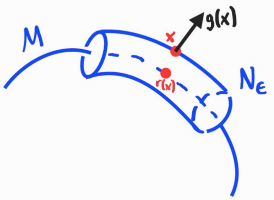
which implies

$$\text{grad}(\phi) = 2(x - r(x)).$$

Main Theorem

We can therefore see that the unit normal vector is given by

$$g(x) = \mathrm{grad}\phi / ||\mathrm{grad}\phi|| = (x - r(x))/\epsilon$$



Now, define a vector field w on N_ϵ as

$$w(x) = (x - r(x)) + v(r(x)).$$

w will therefore point outward along the boundary, and can only vanish at the zeros of v in M.

Main Theorem

Recall the Hopf Lemma:

Lemma

If $v : X \to \mathbb{R}^m$ is a smooth vector field with isolated zeros, and if v points out of X along the boundary, then the index sum $\sum i$ is equal to the degree of the Gauss mapping

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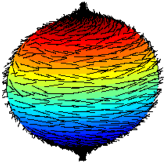
The orientations of w and v will be the same, so the index of w at a zero $z \in M$ will be equal to the index of v at z. Therefore, the index sum $\sum i$ is equal to the degree of g, and we've proven that this lemma can be extended to any vector field v on M with nondegenerate zeros!

Hairy Ball Theorem

Theorem

As a bonus, we can prove that every vector field on an even sphere has a zero.

Consider a vector field on the sphere S^n where every v points north. The south pole will be a source, with index $+1$. At the north pole, since the vectors converge inwards, the index will be $(-1)^n$.



So, for even-dimensional spheres, the index sum is $\sum i = 2$, meaning they must admit a vector field with at least one zero.

Poincaré-Hopf and the Euler Characteristic

The full version of the Poincaré-Hopf theorem states that the vector field index sum $\sum i$ is equal to the Euler characteristic

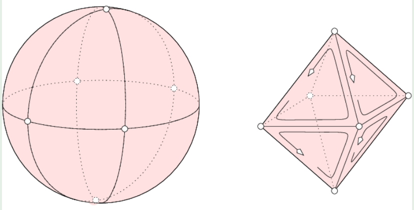
$$\chi(M) = \sum_{i=0}^{m} (-1)^i \text{ rank } H_i(M).$$

The classical definition of the characteristic was for polyhedra, with the formula

$$\chi = V - E + F$$

Example

For convex polyhedra, such as a sphere or an octahedron, the Euler characteristic is equal to 2 (6 vertices, 12 edges, 8 faces).

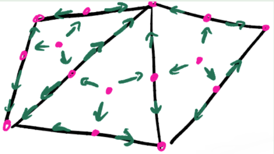


Intuitive Example of Poincaré-Hopf

To illustrate the Poincaré-Hopf theorem, consider an example of a polyhedron. We define a vector field with zeros at points on each face, edge, and vertex, and a corresponding vector between each zero.

Example

We can see that the vertices and faces all have positive indices, while the edges have negative indices.



Therefore, we get

$$\chi = \sum_{i=0}^{m} (-1)^i \text{ rank } H_i(M) = \sum_{x \in \text{zeros}} \text{index}(x)$$
$$= (+1) \text{ faces } (-1) \text{ edges } (+1) \text{ vertices}$$

References I

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Thanks

A huge thank you to my mentor, Charlie Reid, for guiding me through this topic! Thanks for listening!

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