Lie Groups and Lie Algebras

Jeremy Krill

University of Texas at Austin Directed Reading Program jeremy.krill@utexas.edu

May 6th, 2021

What Are Representations?

- In representation theory, we care about studying groups that act as linear transformations on vector spaces
- A representation of a group G is a homomorphism that maps G to GL(V), the group of linear invertible matrices
 - We require invertibility because groups must contain inverse elements
- Symbolically speaking, we say a representation is a map

$$\rho: G \to GL(V) \text{ s.t. } \rho(gh) = \rho(g)\rho(h), \forall g, h \in G$$

ullet An example is the group S_3 , which has a representation $S_3 o GL(\mathbb{R}^3)$

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \ (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ (23) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

May 6th, 2021

Some Light Topology

- GL(V) can be seen as a topological group defined by viewing it as a subspace of F^{n^2}
 - Thus, by viewing GL(V) as a manifold, both the multiplication and inversion operations are compatible (continuous, smooth, or regular) maps

$$m: G \times G \rightarrow G, (g,h) \mapsto gh, i: G \rightarrow G, g \mapsto g^{-1}$$

- When G is a smooth manifold with m and i as smooth maps, G is a Lie group
- A morphism (map) between two Lie groups G and H is a map $\rho: G \to H$ that is both differentiable and a group homomorphism

An Example: SO(2)

- The group SO(2) is the set of all rotations on \mathbb{R}^2
 - If we take a point P at (a,b) in \mathbb{R}^2 and rotate it by θ , we get $(a',b')=(a\cos\theta-b\sin\theta,a\sin\theta+b\cos\theta)$
- Therefore, the linear transformation can be defined as

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

• So, we can define an element of SO(2) as

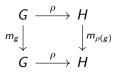
$$SO(2) = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} : 0 \le \theta \le 2\pi \right\}.$$

• Since each element of SO(2) can be viewed as the matrix representation of $\cos \theta + i \sin \theta$, if we were to plot all elements of SO(2), we'd get the unit circle

May 6th, 2021

Some More Topological Properties of Lie Groups

- Lie groups have the property, that for a Lie group G, all points in G are generated any neighborhood $U \subset G$ of the identity
 - This is because the set S generated by U is both open and closed, and since G is connected, S = G
- With this property, we are able to use "infinitesimal" neighborhoods of *G* in this case, the tangent plane of *G* at the identity
- Recall that a homomorphism $\rho(gh) = \rho(g)\rho(h)$ respects the action of a group on itself by left or right multiplication
 - For any $g \in G$, the differentiable map given by multiplication by g is denoted by $m_g : G \to G$. Any C^{∞} map $\rho : G \to H$ of Lie groups will be a homomorphism if the following diagram commutes



Deriving the Lie Bracket

• Although m_g is a diffeomorphism, it isn't useful to describe operations on the tangent space at one point since maps m_g have no fixed points. Instead, if we define the map $\Psi_g: G \to G$ by conjugation

$$\Psi_g(h) = ghg^{-1}$$

we get an automorphism ρ that respects the action of G on itself by conjugation, as seen by the diagram commuting

$$\begin{array}{ccc} G & \stackrel{\rho}{\longrightarrow} & H \\ \psi_g & & & \psi_{\rho(g)} \\ G & \stackrel{\rho}{\longrightarrow} & H \end{array}$$

• We therefore have a natural map $\Psi: G \to \operatorname{Aut}(G)$ that allows Ψ_g to fix the identity element $e \in G$

Deriving the Lie Bracket

- By looking at the differential of Ψ_g at e, we get a linear map $d(\Psi_g)_e$ that maps the tangent space of G at the identity, T_eG , to $T_{\Psi_g}G$
 - ullet By setting $\operatorname{Ad}(g)=(d\Psi_g)_e$, we get

$$Ad(gh) = d\Psi_{gh} = d(\Psi_g\Psi_h) = d(\Psi_g)d(\Psi_h) = Ad(g)Ad(h)$$

 Taking this differential of the adjoint representation at the identity, we get

$$\operatorname{\mathsf{ad}}: T_e G o \operatorname{\mathsf{End}}(T_e G)$$

• We can view the image ad(X)(Y) of a tangent vector Y under the map ad(X) as a function of X and Y that gives us a bilinear map

$$T_eG \times T_eG \rightarrow T_eG$$

 In this bilinear map, for a pair of tangent vectors X and Y to G at e, we say

$$[X,Y] := \operatorname{ad}(X)(Y)$$



Properties of Lie Brackets

- A Lie algebra $\mathfrak g$ is a vector space where the Lie bracket [X,Y] maps $\mathfrak g imes \mathfrak g o \mathfrak g$
- To make the bracket operation reasonably explicit, with $X,Y\in\mathfrak{g}$ and $\gamma:I\to G$ being an arc with $\gamma(0)=e,\gamma'(0)=X$, we have

$$[X,Y] = \operatorname{ad}(X)(Y) = \frac{d}{dt}\Big|_{t=0} (\operatorname{Ad}(\gamma(t))(Y))$$

This then becomes

$$\frac{d}{dt}\Big|_{t=0} (Ad(\gamma(t))(Y)) = \gamma'(0)Y\gamma'(0) + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1})$$

= XY - YX

Properties of Lie Brackets

- The previous explains why we use the bracket notation- in any case where a Lie group is given as a subgroup of $GL_n(\mathbb{R})$, the Lie bracket is simply the matrix commutator
 - Note that the Lie bracket is not necessarily the matrix commutator, as this only holds for subgroups of $GL_n(\mathbb{R})$, which is why we define it using the adjoint representation
- The Lie bracket will have the following two properties:
 - ① Bilinear: [cX,Y] = [X,cY] = c[X,Y], where it maps the vector space to itself
 - 2 Skew-Symmetric: [X,Y] = -[X,Y]
 - $\mbox{\bf 3}$ Jacobi Identity: [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 (for any three tangent vectors X,Y,Z)
- So, a Lie algebra is a vector space equipped with a skew-symmetric bilinear form that satisfies the Jacobi Identity

The Exponential Map

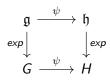
- Up to this point, we've defined Lie groups and Lie algebras and shown how to construct them. Our next task is to connect the two
- To study the relationship between a Lie group G and its Lie algebra \mathfrak{g} , we use the exponential map

$$\exp:\mathfrak{g} o G$$

• For a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, we have

$$\exp X = I + \sum_{n=1}^{\infty} \frac{X^n}{n!}$$

• This implies that the exponential map is natural in that for the map $\psi: G \to H$, the diagram commutes



The Exponential Map

• For any subgroup of $GL_n(\mathbb{R})$, using the power series for e^x , and for $X \in \operatorname{End}(V)$,

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \dots$$

- This converges and is invertible, and its inverse is $\exp(-X)$. The differential of this map at the origin is the identity
 - The restriction of the exponential map to some line through the origin in $\mathfrak g$ is a *one-parameter subgroup* of G, which is the image of a continuous group homomorphism $\psi:\mathbb R\to G$
- So, we can find the Lie algebras of certain Lie groups by taking the derivative of the exponential map

$$\frac{d}{dt}\exp(tX) = X\exp(tX)$$



References I



William Fulton & Joe Harris.

Representation Theory: A First Course.

Springer, 2004.