Representation Theory of SU(2)

Jeremy Krill

University of Texas at Austin
Directed Reading Program

jeremy.krill@utexas.edu

December 7th, 2020

Description of SU(2)

- SU(2) is the Lie group of special unitary transformations on complex 2D vectors
  - It consists of 2x2 complex, self-adjoint matrices with determinant one
- Since it is a real Lie group, SU(2) has a compatible structure of a real manifold
- Being isomorphic to the universal covering group of SO(3)- the 3D rotation group- representations of SU(2) describe non-relativistic spin in quantum mechanics

- A representation (π,V) of a group G on a complex vector space V (where V $\simeq \mathbf{C}^n$) is a homomorphism

$$\pi : G \to GL(n, \mathbf{C})$$

  - As a space, GL(n,$\mathbf{C}$) is the space of all n by n invertible complex matrices

- A representation π is irreducible if it has no subrepresentations
  - A subrepresentation is a nonzero proper subspace W $\subset$ V such that ($\pi_{|W}$,W) is a representation

Representations of SU(2)

- Consider an arbitrary 2 by 2 complex matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

- Unitarity implies that the rows are orthonormal, which results from the condition that the matrix times its conjugate-transpose is the identity

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Orthogonality of the two rows gives the relation

$$\gamma\overline{\alpha} + \delta\overline{\beta} = 0 \implies \delta = -\frac{\gamma\overline{\alpha}}{\overline{\beta}}$$

Representations of SU(2)

- The condition that the first row has length one gives

$$\alpha\overline{\alpha} + \beta\overline{\beta} = 1$$

- The determinant is computed to be 1 and the following is found

$$\alpha\delta - \beta\gamma = -\frac{\alpha\overline{\alpha}\gamma}{\overline{\beta}} - \beta\gamma = -\frac{\gamma}{\overline{\beta}}(\alpha\overline{\alpha} + \beta\overline{\beta}) = 1$$

$$\gamma = -\overline{\beta}, \delta = \overline{\alpha}$$

- So, an SU(2) matrix will have the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where $(\alpha,\beta) \in \mathbf{C}^2$ and $|\alpha|^2 + |\beta|^2 = 1$

Irreducibles of U(1)

- U(1) is the group of unitary 1x1 complex matrices (also known as the circle group) that acts on the complex plane by rotation about the origin
  - Elements of U(1) are points on the unit circle and can be labeled by a unit complex number $e^{i\theta}$
- Choosing U(1) $\subset$ SU(2) implies that given any representation (π,V) of SU(2) of dimension m, there is a representation ($\pi_{|U(1)}$,V) of U(1)
- Knowing the classification of U(1) irreducibles as $\pi_k(\theta) = e^{ik\theta} \in U(1)$ and applying weight decomposition yields

$$(\pi_{|U(1)}, V) = \mathbf{C}_{q_1} \oplus \mathbf{C}_{q_2} \oplus ... \oplus \mathbf{C}_{q_m}$$

for some $q_1, q_2, ..., q_m \in$ $\mathbf{Z}$ where $\mathbf{C}_q$ denotes the representation of U(1) associated with the integer q

Homomorphisms from SU(2) to U(m)

- A unitary representation (π,V) of SU(2) of dimension m is given by a homomorphism
$$\pi : SU(2) \to U(m)$$
- Taking the derivative of this to get a map between the tangent spaces of SU(2) and of U(m) at the identity of both groups gives a Lie algebra representation which takes skew-adjoint 2 by 2 matrices to skew-adjoint m by m matrices
$$\pi' : \mathbf{su}(2) \to \mathbf{u}(m)$$

- The Lie algebra **su**(2) can be thought of as the tangent space $\mathbf{R}^3$ to SU(2) at the identity element with a basis given by three skew-adjoint versions of the Pauli matrices

$$X_1 = -i\frac{1}{2}\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$X_2 = -i\frac{1}{2}\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$X_3 = -i\frac{1}{2}\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Take the self-adjoint versions $S_j = iX_j$ that satisfy

$$[S_1, S_2] = iS_3, [S_2, S_3] = iS_1, [S_3, S_1] = iS_2$$

Lie Algebra Representations of S_3

- With the choice of U(1) as matrices of the form

$$e^{i2\theta S_3} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

and $e^{i\theta}$ going around U(1) once as θ goes from 0 to 2π, a basis of V can be chosen so that

$$\pi(e^{i2\theta S_3}) = \begin{pmatrix} e^{i\theta q_1} & 0 & \dots & 0 \\ 0 & e^{i\theta q_2} & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & e^{i\theta q_m} \end{pmatrix}$$

Lie Algebra Representations of S_3

- Taking the derivative of this representation to get a Lie algebra representation using

$$\pi' = \frac{d}{d\theta}\pi(e^{\theta X})|_{\theta=0}$$

for X = i2S$_3$ yields

$$\pi'(i2S_3) = \frac{d}{d\theta}\begin{pmatrix} e^{i\theta q_1} & 0 & \dots & 0 \\ 0 & e^{i\theta q_2} & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & e^{i\theta q_m} \end{pmatrix}_{|\theta=0} = \begin{pmatrix} iq_1 & 0 & \dots & 0 \\ 0 & iq_2 & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & iq_m \end{pmatrix}$$

so $\pi'(S_3)$ will have half-integer eigenvalues

$$\pi'(S_3) = \begin{pmatrix} \frac{q_1}{2} & 0 & \dots & 0 \\ 0 & \frac{q_2}{2} & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & \frac{q_m}{2} \end{pmatrix}$$

Constructing SU(2) Raising and Lowering Operators

- If $\pi'(S_3)$ has an eigenvalue $\frac{n}{2}$, n is a weight of the representation (π,V), and the subspace $V_n \in$ V of the representation V satisfying

$$v \in V_n \implies \pi'(S_3)v = \frac{n}{2}v$$

  is called the n'th weight space of the representation- all vectors in it are eigenvectors of $\pi'(S_3)$ with eigenvalue $\frac{n}{2}$
- To construct raising and lowering operators for SU(2), let

$$S_+ = S_1 + iS_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_- = S_1 - iS_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $\pi'(S_+)$ is a raising operator for the representation (π,V), and $\pi'(S_-)$ is a lowering operator

Constructing SU(2) Raising and Lowering Operators

- The reason for this definition comes from

$$[S_3, S_+] = [S_3, S_1 + iS_2] = iS_2 + i(-iS_1) = S_1 + iS_2 = S_+$$

and, since π' is a Lie algebra homomorphism, implies

$$\pi'(S_3)\pi'(S_+) - \pi'(S_+)\pi'(S_3) = \pi'([S_3, S_+]) = \pi'(S_+)$$

- For v $\in V_n$,

$$\pi'(S_3)\pi'(S_+)v = \pi'(S_+)\pi'(S_3)v + \pi'(S_+)v = (\frac{n}{2} + 1)\pi'(S_+)v$$

$$v \in V_n \implies \pi'(S_+)v \in V_{n+2}$$

Irreducible Representations and Weights

- From this, it's clear that $\pi'(S_+)$ is a linear operator called a "raising operator" that takes vectors with a specific weight (V_n) to vectors with a weight 2 dimensions higher (V_{n+2})
  - Similarly, $\pi'(S_-)$ takes (V_n) to (V_{n-2}) and is called a "lowering operator"
- Finite dimensional irreducible representations of SU(2) have weights of the form

$$-n, -n+2, ..., n-2, n$$

for n a non-negative integer, each with multiplicity 1, with n a highest weight

Irreducible Representations and Weights

- Since representations can be studied by looking at the set of their weights under the action of $U(1) \subset SU(2)$, irreducible representations of SU(2) can be labeled by n, the highest weight
- The representation will be of dimension n+1 with weights

$$-n, -n+2, ..., n-2, n$$

- Each weight occurs with multiplicity 1, and

$$(\pi_{|U(1)}, V) = \mathbf{C}_{-n} \oplus \mathbf{C}_{-n+2} \oplus ... \oplus \mathbf{C}_{n-2} \oplus \mathbf{C}_{n}$$

Raising and Lowering Operators & Weights

- Starting with a highest weight or lowest weight vector, a basis for the representation can be generated by repeatedly applying raising or lowering operators where all vector spaces are copies of **C**, and all the maps are isomorphisms

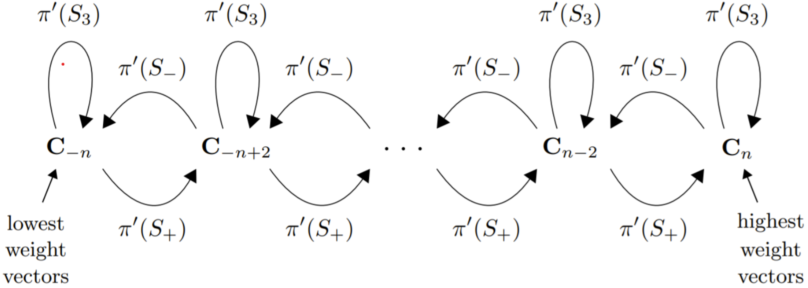


Figure: Basis for a representation of SU(2)

Irreducible Representations of SU(2)

- To construct irreducible representations of SU(2), take V_n for n $\in Z_{\geq 0}$ to be the n+1 dimensional vector space of homogeneous polynomials of two variables, z_1 and z_2, over **C**. Each f $\in V_n$ is of the form

$$f(z_1, z_2) = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \ldots + a_{n-1} z_1 z_2^{n-1} + a_n z_2^n$$

  where $a_j \in \mathbf{C}$

- The representation (π_n, V_n) of SU(2) on A $= \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ is defined by

$$(\pi_n(A)f)(z_1, z_2) = f(A^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix})^T = f(\overline{\alpha} z_1 - \beta z_2, \overline{\beta} z_1 + \alpha z_2)$$

Irreducible Representations of SU(2)

- Although irreducible representations of SU(2) have been constructed at this point, to confirm unitarity of the representation, Hermitian inner products on the spaces V^n need to be defined such that they are preserved by the action of SU(2)
- Hermitian inner products on functions on a space M are defined as

$$\langle f, g \rangle = \int_M \overline{f} g$$

so an inner product on polynomial functions on $\mathbf{C}^2$ can be defined by

$$\langle f, g \rangle = \frac{1}{\pi^2} \int_{\mathbf{C}^2} \overline{f(z_1, z_2)} g(z_1, z_2) e^{-(|z_1|^2 + |z_2|^2)} dx_1 dy_1 dx_2 dy_2$$

Orthonormal Bases for SU(2) Irreducibles

- The integral yields that the polynomials

$$\frac{z_1^j z_2^k}{\sqrt{j!k!}}$$

  will be an orthonormal basis of the space of polynomial functions, and the operators $\pi'(X), X \in \mathbf{su}(2)$ will be skew-adjoint
- Orthonormal bases for the representation space V^n of homogeneous polynomials will be as follows
  - For n = s = 0

$$1$$

  - For n = 1, s = $\frac{1}{2}$

$$z_1, z_2$$

  - For n = 2, s = 1

$$\frac{1}{\sqrt{2}} z_1^2, z_1 z_2, \frac{1}{\sqrt{2}} z_2^2$$

SU(2) in Quantum Mechanics

- A key to developments in modern physics is Noether's Theorem, which states that a Lie group action on a Lagrangian system that leaves the action invariant provides conserved quantities for each element of the Lie algebra
  - Essentially, physical symmetries give rise to conservation laws
- There are many clear examples of physical systems that are symmetric under the action of SO(3), such as with angular momentum. In order to see how SU(2) comes into play, note that SU(2) is isomorphic to Spin(3), which is a double cover of SO(3)
- So, the double cover homomorphism Φ takes

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in SU(2) \to \begin{pmatrix} cos(2\theta) & sin(2\theta) & 0 \\ -sin(2\theta) & cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

SU(2) in Quantum Mechanics

- Since there is a representation of Spin(3) $\simeq$ SU(2), and there is a covering map from Spin(3) to SO(3), the representations (π_n, V_n) for even values of n are also irreducible representations for SO(3)
- The representations (π_n, V_n) for odd n turn out to be found in fundamental particles. The spin of the representation, or particle, is defined as $s = \frac{n}{2}$
- The existence of spin $\frac{1}{2}$ particles shows that Spin(3) corresponds to rotations of fundamental quantum systems as opposed to SO(3)
    - Such spin $\frac{1}{2}$ particles are called "spinors," as they are elements of the representation of Spin(3) = SU(2) on $\mathbf{C}^2$ given by

$$g \in SU(2) \to \pi_{spinor}(g) = g$$

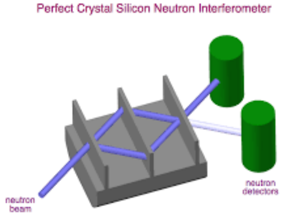
Particles as Spinors

particle	spin
electron	$\hbar/2$
proton	$\hbar/2$
neutron	$\hbar/2$
photon	$\hbar$
graviton	$2\hbar$
gluon	$\hbar$

- The vector space of angular momentum states for a particle with spin s is isomorphic to (π_{2s}, V_{2s})
  - In the case of an electron with spin $\frac{1}{2}$, $dim V_{2s} = 2s + 1 = 2$, which leads to the law that electron spins can either be "up" or "down"

Particles as Spinors

- Another consequence of $SU(2) \simeq Spin(3)$ and angular momentum is the existence of spinors
  - An electron must undergo two full rotations to return to its original state
- This can be thought of heuristically (rather than literally) as the electron's axis rotating halfway as the electron completes a full rotation
- Neutron interferometer experiments have demonstrated that single full rotations physically change the behavior of neutrons as opposed to two full rotations leaves the behavior unchanged



References I

Peter Woit.
Quantum Theory, Groups, and Representations: An Introduction.
Springer, 2016.

Andrew M. Steane.
An Introduction to Spinors.
University of Oxford, 2000.
https://arxiv.org/pdf/1312.3824.pdf

David Urbanik.
Quantum Physics and the Representation Theory of SU(2).
https://csclub.uwaterloo.ca/ dburbani/work/quantumreptheory.pdf

Qiaochu Yuan.
The Representation Theory of SU(2).
https://qchu.wordpress.com/2011/06/26/the-representation-theory-of-su2/