# Lie Groups and Lie Algebras

Jeremy Krill

University of Texas at Austin
Directed Reading Program

jeremy.krill@utexas.edu

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# What Are Representations?

- In representation theory, we care about studying groups that act as linear transformations on vector spaces
- A *representation* of a group $G$ is a homomorphism that maps $G$ to $GL(V)$, the group of linear invertible matrices
  - We require invertibility because groups must contain inverse elements
- Symbolically speaking, we say a representation is a map

$$\rho : G \to GL(V) \text{ s.t. } \rho(gh) = \rho(g)\rho(h),\ \forall g, h \in G$$

- An example is the group $S_3$, which has a representation $S_3 \to GL(\mathbb{R}^3)$

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};\ (23) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

# Some Light Topology

- $GL(V)$ can be seen as a topological group defined by viewing it as a subspace of $F^{n^2}$
  - Thus, by viewing $GL(V)$ as a manifold, both the multiplication and inversion operations are compatible (continuous, smooth, or regular) maps

$$m : G \times G \to G, (g, h) \mapsto gh, \; i : G \to G, g \mapsto g^{-1}$$

- When $G$ is a smooth manifold with $m$ and $i$ as smooth maps, $G$ is a *Lie group*
- A *morphism* (map) between two Lie groups $G$ and $H$ is a map $\rho : G \to H$ that is both differentiable and a group homomorphism

# An Example: SO(2)

- The group SO(2) is the set of all rotations on $\mathbb{R}^2$
  - If we take a point $P$ at $(a, b)$ in $\mathbb{R}^2$ and rotate it by $\theta$, we get
    $$(a', b') = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$$
- Therefore, the linear transformation can be defined as

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

- So, we can define an element of $SO(2)$ as

$$SO(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : 0 \leq \theta \leq 2\pi \right\}.$$

- Since each element of $SO(2)$ can be viewed as the matrix representation of $\cos\theta + i\sin\theta$, if we were to plot all elements of $SO(2)$, we'd get the unit circle

# Some More Topological Properties of Lie Groups

- Lie groups have the property, that for a Lie group $G$, all points in $G$ are generated any neighborhood $U \subset G$ of the identity
  - This is because the set $S$ generated by $U$ is both open and closed, and since $G$ is connected, $S = G$
- With this property, we are able to use "infinitesimal" neighborhoods of $G$- in this case, the tangent plane of $G$ at the identity
- Recall that a homomorphism $\rho(gh) = \rho(g)\rho(h)$ respects the action of a group on itself by left or right multiplication
  - For any $g \in G$, the differentiable map given by multiplication by $g$ is denoted by $m_g : G \to G$. Any $C^\infty$ map $\rho : G \to H$ of Lie groups will be a homomorphism if the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle m_g}\downarrow & & \downarrow{\scriptstyle m_{\rho(g)}} \\ G & \xrightarrow{\rho} & H \end{array}$$

# Deriving the Lie Bracket

- Although $m_g$ is a diffeomorphism, it isn't useful to describe operations on the tangent space at one point since maps $m_g$ have no fixed points. Instead, if we define the map $\Psi_g : G \to G$ by *conjugation*

  $$\Psi_g(h) = ghg^{-1}$$

  we get an automorphism $\rho$ that respects the action of $G$ on itself by conjugation, as seen by the diagram commuting

  $$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle \Psi_g}\downarrow & & \downarrow{\scriptstyle \Psi_{\rho(g)}} \\ G & \xrightarrow{\rho} & H \end{array}$$

- We therefore have a natural map $\Psi : G \to \mathrm{Aut}(G)$ that allows $\Psi_g$ to fix the identity element $e \in G$

# Deriving the Lie Bracket

- By looking at the differential of $\Psi_g$ at $e$, we get a linear map $d(\Psi_g)_e$ that maps the tangent space of $G$ at the identity, $T_eG$, to $T_{\Psi_g}G$
  - By setting $\mathrm{Ad}(g) = (d\Psi_g)_e$, we get

$$\mathrm{Ad}(gh) = d\Psi_{gh} = d(\Psi_g\Psi_h) = d(\Psi_g)d(\Psi_h) = \mathrm{Ad}(g)\mathrm{Ad}(h)$$

- Taking this differential of the adjoint representation at the identity, we get

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$$

- We can view the image $\mathrm{ad}(X)(Y)$ of a tangent vector $Y$ under the map $\mathrm{ad}(X)$ as a function of $X$ and $Y$ that gives us a bilinear map

$$T_eG \times T_eG \to T_eG$$

- In this bilinear map, for a pair of tangent vectors $X$ and $Y$ to $G$ at $e$, we say

$$[X, Y] := \mathrm{ad}(X)(Y)$$

# Properties of Lie Brackets

- A *Lie algebra* $\mathfrak{g}$ is a vector space where the *Lie bracket* $[X, Y]$ maps $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
- To make the bracket operation reasonably explicit, with $X, Y \in \mathfrak{g}$ and $\gamma : I \to G$ being an arc with $\gamma(0) = e, \gamma'(0) = X$, we have

$$[X, Y] = \mathrm{ad}(X)(Y) = \frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}(\gamma(t))(Y))$$

This then becomes

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}(\gamma(t))(Y)) &= \gamma'(0)Y\gamma'(0) + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}) \\ &= XY - YX\end{aligned}$$

# Properties of Lie Brackets

- The previous explains why we use the bracket notation- in any case where a Lie group is given as a subgroup of $GL_n(\mathbb{R})$, the Lie bracket is simply the matrix commutator
  - Note that the Lie bracket is not necessarily the matrix commutator, as this only holds for subgroups of $GL_n(\mathbb{R})$, which is why we define it using the adjoint representation
- The Lie bracket will have the following two properties:
  1. Bilinear: [cX,Y] = [X,cY] = c[X,Y], where it maps the vector space to itself
  2. Skew-Symmetric: [X,Y] = -[X,Y]
  3. Jacobi Identity: [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 (for any three tangent vectors $X$, $Y$, $Z$)
- So, a Lie algebra is a vector space equipped with a skew-symmetric bilinear form that satisfies the Jacobi Identity

# The Exponential Map

- Up to this point, we've defined Lie groups and Lie algebras and shown how to construct them. Our next task is to connect the two
- To study the relationship between a Lie group $G$ and its Lie algebra $\mathfrak{g}$, we use the exponential map

$$\exp : \mathfrak{g} \to G$$

- For a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, we have

$$\exp X = I + \sum_{n=1}^{\infty} \frac{X^n}{n!}$$

- This implies that the exponential map is natural in that for the map $\psi : G \to H$, the diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\psi"] \arrow[d, "exp"'] & \mathfrak{h} \arrow[d, "exp"] \\ G \arrow[r, "\psi"] & H \end{tikzcd}$$

# The Exponential Map

- For any subgroup of $GL_n(\mathbb{R})$, using the power series for $e^x$, and for $X \in \text{End}(V)$,

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \dots$$

- This converges and is invertible, and its inverse is $\exp(-X)$. The differential of this map at the origin is the identity
    - The restriction of the exponential map to some line through the origin in $\mathfrak{g}$ is a *one-parameter subgroup* of $G$, which is the image of a continuous group homomorphism $\psi : \mathbb{R} \to G$
- So, we can find the Lie algebras of certain Lie groups by taking the derivative of the exponential map

$$\frac{d}{dt} \exp(tX) = X \exp(tX)$$

# References I

William Fulton & Joe Harris.
*Representation Theory: A First Course*.
Springer, 2004.