COMP3821/COMP9801 Workshop Week 3

Greedy Algorithms on Graphs

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Announcements

- Problem Set 1 Marks and Feedback on Thursday (hopefully).
- Problem Set 2 Submission due on Friday Week 3.
    - Submissions on Gradescope.
- Project
    - Proposal due Sunday Week 3.
    - Mentor allocations out on Moodle.
- Help sessions on Friday 11am - 1pm for Weeks 1 - 10 in K17 G05.

Guided Problem 1

Guided Problem

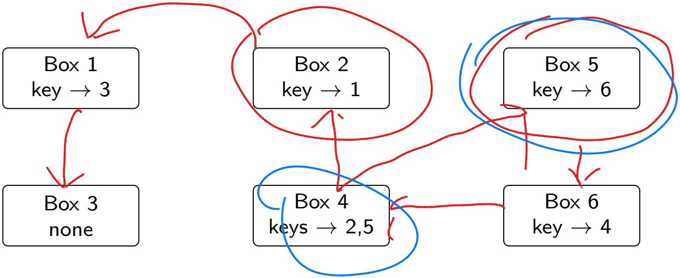
You have a collection of n lockboxes and m gold keys. Each key unlocks at most one box; however, each box might be unlocked by one key, by multiple keys, or by no keys at all. There are only two ways to open each box once it is locked: unlock it properly which requires having a matching key in your hand, or smash it to bits with a hammer.

Your baby brother, who loves playing with shiny objects, has somehow managed to lock all your keys inside the boxes! Luckily, your home security system recorded everything, so you know exactly which keys (if any) are inside each box and you can retrieve this information in constant-time; that is, you can assume that you know if box u contains a key to box v.

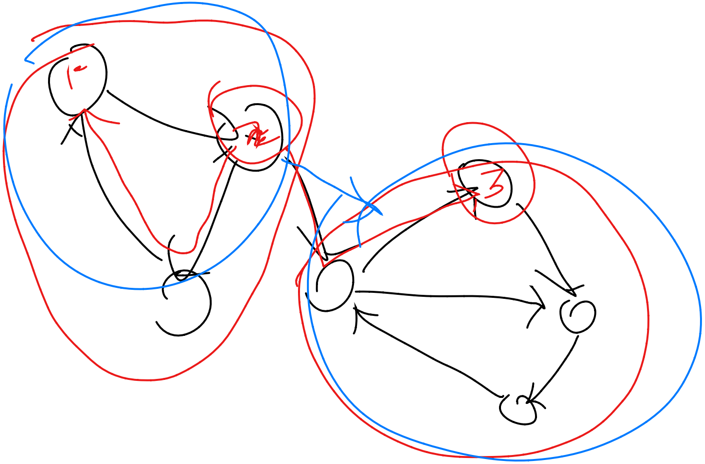
You need to get all the keys back out of the boxes, because they are made of gold. Clearly, you have to smash at least one box.

Describe and analyse a polynomial-time algorithm that computes the smallest number of boxes that need to be smashed in order to retrieve all of the keys.

Greedy Guided Problem 1 Example



Guided Problem 1 Working



Greedy Guided Problem 1 Solution

1) k is sufficient $ans \leq k$

2) k is minimal $k \leq ans$

Algorithm

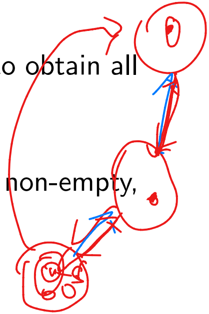
- Create a graph G by
  - creating a vertex for each non-empty box,
  - draw an edge $u \to v$ if box u contains a key to box v.
- Use Kosaraju's or Tarjan's algorithm to obtain condensation graph G'.
- Report number of source strongly connected components $= k$ (SCC) as the smallest number of boxes we need to smash.

Greedy Guided Problem 1 Proof A

Claim 1: Breaking 1 box per source SCC will allow us to obtain all keys.

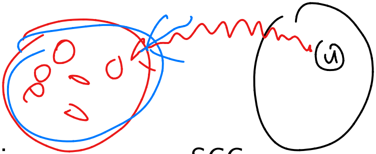
$ans \le k$

Proof:



- Consider any box u that contains a key. Since it is non-empty, it will be represented as a node in our graph G.
- Consider the SCC S that contains u, either
  - S is a source SCC, or
  - we can start from S and keep traversing to other SCC's along incoming edges in G'. Since G' is a directed acyclic graph (DAG), we must eventually reach a SCC with no incoming edges.
- In both cases, we have shown that there is a path from a source SCC to S.
- Therefore, there is a path from one of the smashed boxes, to all boxes u in G.

Greedy Guided Problem 1 Proof B



$k \leq ans$

Claim 2: We must break at least 1 box in every source SCC.

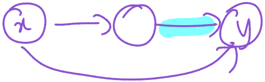
Proof:

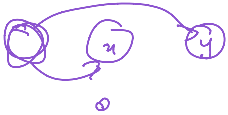
- Suppose that we did not break any box from some source SCC S but still obtained all keys.
- This means there must be a path from some broken box to all boxes in S.
- However, since there are no broken boxes in S, this implies that there is some path from some other SCC S' to S in G'.
- This implies that S has at least 1 incoming edge.
- Contradiction, since S is a source SCC.

Greedy Guided Problem 1 Time Complexity

- Make graph: $\mathcal{O}(n+m)$
- Tarjan's/Kosaraju's graph condensation algorithm: $\mathcal{O}(n+m)$
- Counting number of source SCC's: $\mathcal{O}(m)$
- Overall: $\mathcal{O}(2n+3m) = \mathcal{O}(n+m)$

Guided Problem 2





VC

Guided Problem

Let $G = (V, E)$ be a directed graph. We say that G is *unilaterally connected* if, for every pair of distinct vertices $x, y \in V$, there exists a directed path from x to y or there exists a directed path from y to x, or there exist directed paths in both directions. Describe and analyse a polynomial-time algorithm that decides whether G is unilaterally connected.

A B

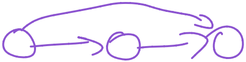
$x \to y$ in A

or $x \to y$ in B

$O(|V|^2)$

Guided Problem 2 Working

$V' \subseteq V$



$\Sigma_G = (\Sigma_V, \Sigma_E)$

condensation of G

G is UC $\iff$ Σ_G is UC (1)

$\iff$ Σ_G has unique topological ordering (2)

$\iff$ edge between every pair of adjacent vertices in the top order (3)

$O(|V| + |E|)$

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Compute condensation graph, compute top order

scan thru verticies ..

$O(|V| + |E|)$

Guided Problem 2 Working

1. G is VC $\iff$ Σ_G is VC

($\Rightarrow$) Suppose G is VC. G has connected components $\{C_1, \ldots, C_k\}$

Aside: x, y $\quad x \to y$ or $y \to x$ $\quad$ WLOG

Let $v_i, v_j \in \Sigma_V$ $\quad v_i \neq v_j$ $\quad v_i \in C_i$

$v_j \in C_j$

$\therefore$ let $x, y \in V$ with $x \in C_i$ $\quad y \in C_j$

$x \rightsquigarrow y$ $\quad x \rightsquigarrow v_i$

$y \rightsquigarrow v_j$ $\quad v_i$ to v_j in Σ_G

Guided Problem 2 Working

$\boxed{x \to y}$

$(\Leftarrow)$ Let $x, y \in V$ $x \neq y$

Suppose $x \in C_i$ and $y \in C_j$

if $C_i = C_j$ have path from x to y

else

$$\begin{array}{ccc} v_i & \rightsquigarrow & v_j \\ \downarrow & & \downarrow \\ C_i & & C_j \end{array}$$

$$\overbrace{x \rightsquigarrow v_i}^{cc} \underbrace{\rightsquigarrow}_{\Sigma_a \text{ is } UC} \overbrace{v_j \rightsquigarrow y}^{cc}$$

Guided Problem 2 Working

(2) G is UC $\iff$ G has unique top order
G is DAG

Σ_G is DAG - directed acyclic graph

($\Rightarrow$) Suppose G has 2 topological orderings σ, σ'

in σ x before y

σ' y before x

if $x \to y$ in top order x comes before y in any top ordering

if $y \to x$ y must before x σ' not top order

σ not top order unique top order

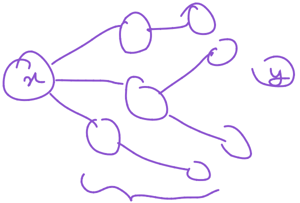
Guided Problem 2 Working

($\Leftarrow$) Let σ be the unique top ordering for G

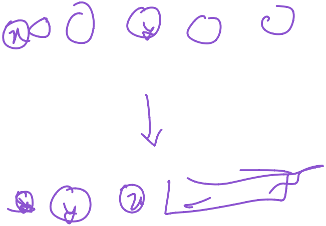
Let $x, y \in V$, $x \neq y$

WLOG x comes before y

Suppose no path from x to y



reachable by x as set S



no edge from anything in S to y

$x \rightsquigarrow y$ so G is VC.

Guided Problem 2 Working

x, S, y

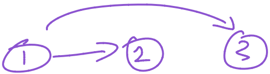
Since there's no path from x to y.

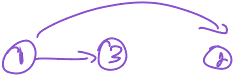
No path from anything in S to y.

Form a new top ordering where y goes before x and any vertex in S.

3 Unique top order $\iff$ "chain"

$\iff$ Hamiltonian path





Guided Problem 2 Working

[2, 3, 1, 2, - - -]

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Guided Problem 2 Solution Overview

- Let $\Sigma_G = (\Sigma_V, \Sigma_E)$ denote the condensation graph of G, computable in $O(|V| + |E|)$ using Kosaraju's algorithm.
- *Claim 1.* G is unilaterally connected $\iff$ Σ_G is unilaterally connected.
- *Claim 2.* A DAG G is unilaterally connected $\iff$ G has a unique topological ordering.
- *Claim 3.* G has a unique topological ordering $\iff$ there is an edge between every pair of consecutive vertices in the topological order.
- If above claims are true, we can compute topological ordering then check pairs of consecutive vertices in linear time.
- Thus, we can decide if G is unilaterally connected in $O(|V| + |E|)$ time.

Guided Problem 2 Solution Overview

- For this solution, when considering two distinct vertices $x, y \in V$, in a unilaterally connected graph it suffices to consider the existence of a path from x to y without loss of generality.
- If the path instead existed from y to x we could simply swap the labels of the vertices.
- If the path existed in both direction, then a path from x to y still exists.

Guided Problem 2 Solution Overview

Proof of Claim 1. G is unilaterally connected $\iff$ Σ_G is unilaterally connected.

- Let G have connected components $\{C_1, \ldots, C_k\}$. Additionally let $x, y \in V$ and $v_i, v_j \in \Sigma_V$ with $x \neq y, v_i \neq v_j$, $x \in C_i$.
- $(\implies)$ Suppose G is unilaterally connected. Further, let $x \in C_i$ and $y \in C_j$. Since G is unilaterally connected, WLOG suppose we have a path from x to y. This implies a path from v_i to v_j in Σ_G.
- $(\impliedby)$ Suppose Σ_G is unilaterally connected. If $C_i = C_j$ then we have a path from x to y. Else, since Σ_G is unilaterally connected, WLOG suppose we have path from v_i to v_j. Then x to v_i to v_j to y is a feasible path since $x, v_i \in C_i$ and $v_j, y \in C_j$. Thus there is a path from x to y.

Guided Problem 2 Solution Overview

Proof of Claim 2. A DAG G is unilaterally connected $\iff$ G has a unique topological ordering.

- ($\implies$) Suppose G is unilaterally connected. Let σ, σ' be two topological ordering of G. Let the difference be between $x, y \in V$ where x is before y in σ but after y in σ'. Then since G is unilaterally connected, WLOG suppose we have a path from x to y. Then any topological ordering must have x before y. Then this contradicts that σ' is a topological ordering. Thus $\sigma = \sigma'$.
- ($\impliedby$) Suppose G has a unique topological ordering, σ. Let $x, y \in V$ with $x \neq y$. WLOG suppose x before y in σ. Suppose no path from x to y. Denote S as set of vertices reachable from x. Then no edge exists from any vertex in S to y, so we can order y before x and S. Contradiction, so path from x to y.

Guided Problem 2 Solution Overview

Claim 3. G has a unique topological ordering $\iff$ there is an edge between every pair of consecutive vertices in the topological order.

- The proof of Claim 3 is left as an exercise to the reader. However we provide a few interesting observations.
- A DAG has a unique topological ordering if and only if it has a Hamiltonian Path.
- A Hamiltonian Path implies a chain/linked list structure within the topological ordering of the graph.