COMP3821/COMP9801 Workshop Week 7

Linear Programming and Approximation Algorithms

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Guided Problem 2

Guided Problem

Let $G = (V, E)$ be an undirected graph. The *neighbourhood* of a vertex v is the vertex v and every vertex adjacent to v. A *double-dominating set* in G is a set $S \subseteq V$ of vertices such that, for each vertex $v \in V$, the neighbourhood of v contains at least two vertices in S.

Let $d > 2$, and let G be a graph such that every vertex has degree $d - 1$ (i.e. the neighbourhood of every vertex contains exactly d vertices). Furthermore, suppose that each vertex v has non-negative weight $w(v) \geq 0$. The goal is to find a double-dominating set $S \subseteq V$ whose total weight $\sum_{v \in S} w(v)$ is minimised. It is known that this problem is NP-hard.

Guided Problem 2

Guided Problem

(a) Write the integer linear program that captures this problem exactly; in particular, prove that each solution to the integer linear program describes a double-dominating set and that each double-dominating set corresponds to a solution to the integer linear program.

(b) Describe and analyse a polynomial-time $(d-1)$-approximation algorithm for the problem. Prove that your algorithm returns a feasible double-dominating set and that it achieves an approximation ratio of $d-1$.

Guided Problem 2 Solution Outline - Part (a)

Integer Linear Program

- For each $v \in V$, let $x_v \in \{0, 1\}$ denote whether we include v.
- Let $N(v)$ denote the neighbourhood of v.
- We obtain the following integer linear program:

$$\begin{aligned} \text{minimise} \quad & \sum_{v \in V} w(v) \cdot x_v \\ \text{subject to} \quad & \sum_{v \in N(u)} x_v \geq 2 \quad \forall u \in V, \\ & x_v \in \{0, 1\} \quad \forall u \in V. \end{aligned}$$

Guided Problem 2 Solution Outline - Part (a)

Proof of Correspondence

- Let $S \subseteq V$ be a feasible double-dominating set.
- Define integral vector $x^* = (x_1^*, \dots x_{|V|}^*)$ where

$$x_u^* = \begin{cases} 1 & \text{if } u \in S, \\ 0 & \text{otherwise.} \end{cases}$$

- ($\implies$) S is feasible double-dominating set, so each vertex $u \in V$ has at least two vertices in the neighbourhood of u belong to S. Hence $\sum_{v \in N(u)} x_v \geq 2$. Thus x^* is feasible with cost,

$$w(x^*) = \sum_{v \in V} w(v) \cdot x_v^* = \sum_{v \in S} w(v) = w(S).$$

- ($\impliedby$) x^* denote feasible integral solution. Define $S = \{u \in V : x_u^* = 1\}$. For every $u \in V$, we have $\sum_{v \in N(u)} x_v^* \geq 2$, so at least two vertices in the neighbourhood of u belong to S. Thus S is forms a double-dominating set with cost

$$w(S) = \sum_{v \in S} w(v) = \sum_{v \in V} w(v) \cdot x_v^* = w(x^*).$$

Guided Problem 2 Solution Outline - Part (b)

Algorithm - Relaxation

- Relax integrality constraint to obtain linear program by replacing the last constraints with $0 \leq x_v \leq 1$.
- Since relaxation is a linear program, we can solve it in polynomial time.
- Let $x^* = (x_1^*, \ldots, x_{|V|}^*)$ denote the optimal factional solution to the linear program.
- Obtain integral solution x': for each vertex $u \in V$

$$x'_u = \begin{cases} 1 & \text{if } x_u^* \geq 1/(d-1), \\ 0 & \text{otherwise.} \end{cases}$$

- Return $T = \{u \in V : x'_u = 1\}$.

Guided Problem 2 Solution Outline - Part (b)

Feasibility Proof

- We prove by contradiction with two cases. Let $u \in V$ be an arbitrary vertex.
- *Case 1.* No vertex in $N(u)$ belongs to T. Then $x'_v = 0$ for all $v \in N(u)$ so $x^*_v < 1/(d-1)$ for all v. Implying

$$\sum_{v \in N(u)} x^*_v < |N(u)|/(d-1) = d/(d-1) < 2,$$

  since G is a graph where every vertex has degree $d - 1 > 1$.
- *Case 2.* Only one vertex $v \in T$ is from $N(u)$. For each $v' \in N(u) \setminus \{v\}$, we have $x'_{v'} = 0$ implying $x^*_{v'} < 1/(d-1)$. So

$$\sum_{y \in N(u)} x^*_y = x^*_v + \sum_{y \in N(u) \setminus \{v\}} x^*_y < 1 + \frac{|N(u)| - 1}{d - 1} = 2.$$

- Both cases give contradiction, so T is feasible.

Guided Problem 2 Solution Outline - Part (b)

Approximation Proof

- Let OPT denote cost of optimal integral solution. Let $OPT^* = \sum_{v \in V} w(v) \cdot x_v^*$.
- Since each integral solution is a feasible solution to the relaxation, $OPT^* \leq OPT$.
- For each $u \in V$, we have $x_u' \leq (d-1) \cdot x_u^*$. Then

$$\begin{aligned} w(T) &= \sum_{v \in T} w(v) = \sum_{v \in V} w(v) \cdot x_v' \\ &\leq \sum_{v \in V} w(v) \cdot ((d-1) \cdot x_v^*) = (d-1) \sum_{v \in V} w(v) \cdot x_v^* \\ &= (d-1) \cdot OPT^* \leq (d-1) \cdot OPT. \end{aligned}$$

- Thus, our rounding algorithm returns feasible double-dominating set such that

$$OPT \leq w(T) \leq (d-1) \cdot OPT$$

which implies that it is a $(d-1)$-approximation.