COMP3821/COMP9801 Workshop Week ~~4~~ 5

Reductions and NP-completeness

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Announcements

- Problem Set 3 Marks and Feedback on Thursday.
- Problem Set 4 Submission due on Friday Week 4. 5
  - Submissions on Gradescope.
- Workshop 5 has no submissions. No workshop in Week 6.
- Midterm Exam in Friday Week 7.
- Project progress report due Sunday Week 7.
- Help sessions on Friday 11am - 1pm in K17 G05.

Recap: NP

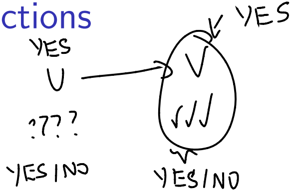
Definition: NP Decision Problem

A decision problem A is in the class NP (nondeterministic polynomial time, denoted $A \in \mathrm{NP}$) if there exists a decision problem $B \in \mathrm{P}$ such that, for every input x,

$$A(x) \text{ is YES} \iff \exists\, y \text{ with } |y| \leq |x|^{O(1)} \text{ such that } B(x, y) \text{ is YES.}$$

Here, y is the *certificate* and B is the *certifier*.

Recap: Reductions



Definition: Karp Reduction

Let U and V be two decision problems. We say that U is *polynomially reducible* to V if there exists a function f such that:

(a) f maps each instance x of U to an instance $f(x)$ of V;

(b) f preserves answers: $U(x) = \text{YES} \iff V(f(x)) = \text{YES}$;

(c) $f(x)$ can be computed in polynomial time.

$\Rightarrow$

$\cancel{X} \to \checkmark$

~~YES~~ NO $\to$ YES ✗

$\checkmark$

Recap: NP-hard and NP-complete

A NP-Hard
$\downarrow$
B NP-Hard

sorting $O(n\log n)$
$\downarrow$
convex hull $\Omega(n\log n)$

Definition: NP-hard

A decision problem V is NP-hard if every problem in NP is polynomially reducible to V.

Definition: NP-complete

A decision problem is NP-complete if it is both in NP and NP-hard.

Guided Problem 1

Guided Problem

Consider the following pair of problems.

- IndSet: Given a graph $G = (V, E)$ and an integer $1 \leq k \leq n$, does there exist a subset $S \subseteq V$ of at least k vertices such that any pair of vertices in S are not connected by a direct edge?
- HighDegreeIndSet: Given a graph $G = (V, E)$ and an integer $1 \leq k \leq n$, does there exist a subset $S \subseteq V$ of at least k vertices such that S forms an independent set and each vertex in S has degree at least k in G?

HDIS

(a) Show that HighDegreeIndSet is in NP. NP + NP-Hard

(b) It is known that IndSet is NP-complete. Show that HighDegreeIndSet is NP-hard and hence conclude that HighDegreeIndSet is NP-complete.

Guided Problem 1 Working

HDIS (G, k) → YES, $\boxed{I \subseteq V}$

a) Certificate : $I \subseteq V$

1. independent set of size at least k → $O(n^2)$
2. degree at least k → $O(|E| + k)$

$n = |V|$

HDIS $\in$ NP

Guided Problem 1 Working

b) ~~Inde~~ Indset $\longrightarrow$ HDIS

polynomial time

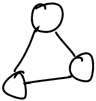
NP-Hard NP-Hard

Indset (G, k) $\longleftarrow$

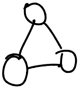
ADIS (G', k') $\longrightarrow$ $(G, 0)$

Indset

$k=3$



HDIS $k' = j$



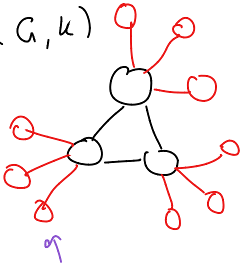
$k = 1$

NO $\rightarrow$ YES

Guided Problem 1 Working

YES $\Leftrightarrow$ YES
in ID   in HDIS

(G, k)



(H, k)

HDIS

$G = (V, E)$

Claim G has an independent set of size k iff H has a HDIS of degree k

$H = (V', E')$

$$V' = V \cup \{u_{i,v} : 1 \le i \le k, v \in V\}$$

$$E' = E \cup \{\{v, u_{i,v}\} : 1 \le i \le k, v \in V\}$$

Guided Problem 1 Working

($\Rightarrow$) Suppose G has an independent set I

I ind. in H.

$I \subseteq V$ has degree at least k

I forms HDIS in H of degree at least k.

($\Leftarrow$) Suppose in H has a HDIS I' of degree at least k.

Verticies added only have degree, cannot be part of I'.

$v \in I'$ is in G, I' is independent so I' is an ind. set in G.

.. This polynomial time. Because indset is NP-Hard

HDIS is NP-Hard. NP $\leq$ NP-Hard

Guided Problem 1 Working

Adding $k|V|$ vertices and $k|V|$ edges

$O(k|V|)$ time - polynomial

This shows HDIS is NP-Hard because we know IndSet is NP-Hard. Since HDIS is in NP and in NP-Hard then ~~HH~~ HDIS is in NP-C.

Guided Problem 1 Part (a) Solution

HDIS is NP-C.

Given a set I of vertices, we check two conditions:

- I forms an independent set of size at least k. For every pair of vertices, $u, v \in I$ check that $\{u, v\} \notin E$. This takes $O(|V|^2)$ time, we can also keep track of the number of vertices in I.
- Every vertex in I has degree at least k. There are many ways to do this in polynomial time.

Guided Problem 1 Part (b) Solution

- Let $G = (V, E)$ be an arbitrary graph. Construct $H = (V', E')$ by attaching a *fan* of k vertices to each vertex of G.
- Formally, construct $H = (V', E')$ where

$$V' = V \cup \{u_{i,v} : 1 \leq i \leq k, v \in V\}$$
$$E' = E \cup \{\{v, u_{i,v}\} : 1 \leq i \leq k, v \in V\}$$

Guided Problem 1 Part (b) Solution

Claim. G has an independent set of size at least k if and only if H has a high-degree independent set of degree at least k.

- ($\implies$) Suppose G has an independent set I of size at least k. Then I forms an independent set in H of size at least k. Moreover, each vertex in I has degree at least k, by construction.
- ($\impliedby$) Suppose H has high-degree independent set I' of degree at least k. The vertices added all have degree 1, so cannot be part of I'. So I' only contains vertices from G, so I' forms an independent set of size at least k in G.
    - If $k = 1$, then there is always an independent set for $|V| \geq 1$. Similarly there will always be a high-degree independent set of degree at least k in our construction.

Guided Problem 1 Part (b) Solution

- The construction can be done in $O(k|V|)$ which is polynomial.
- Hence, since $\textsc{IndSet}$ is NP-Hard, so is $\textsc{HighDegreeIndSet}$.
- Together with part (a), this shows that $\textsc{HighDegreeIndSet}$ is NP-Complete.

NP-completeness Guided Problem II

Guided Problem

A *balloon of size k* is an undirected graph that consists of a simple cycle of length k and a simple path of size k, where the path has one endpoint lying on the cycle. Every other vertex on the cycle is disjoint from every other vertex on the path.

A balloon of size k therefore has $2k$ vertices: k vertices in the cycle and $k+1$ vertices in the path (including the shared endpoint on the cycle).

Given an undirected graph G and an integer $1 \leq k \leq n$, prove that it is NP-complete to decide whether G contains a balloon of size at least k.

NP-completeness Guided Problem II Example

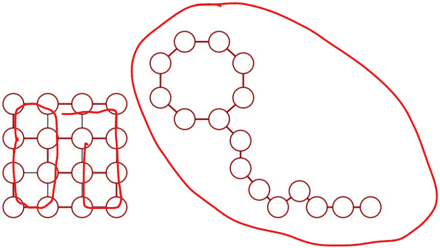


Figure: A balloon subgraph of size 8 on the 4×4 grid graph.

Hamiltonian Cycle Problem

Definition: Hamiltonian Cycle

A Hamiltonian cycle in a graph G is a cycle that visits every vertex in G exactly once. It is known that deciding whether G has a Hamiltonian cycle is NP-complete.

NP-completeness Guided Problem II Working

Certificate = G' k

1) path of size k

2) Single cycle k

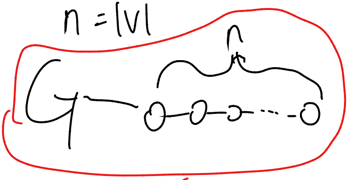
HamCycle

Balloon problem

G

G'

$n = |V|$



Yes

G'

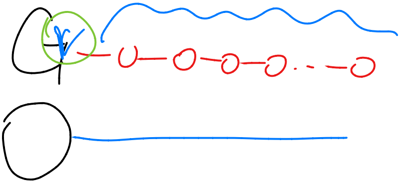
yes

NP-completeness Guided Problem II Working

G has hamCycle $\Leftrightarrow$ G' has balloon of size n

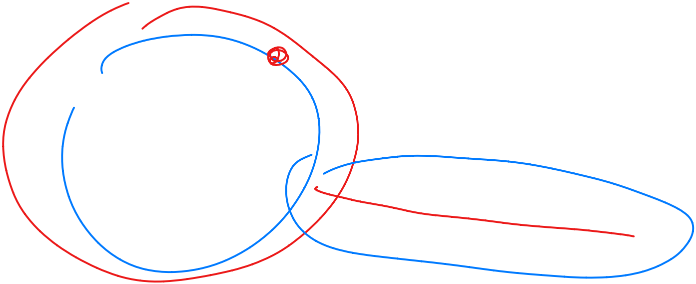
$\Rightarrow$ G has hamCycle C

$v \in C$

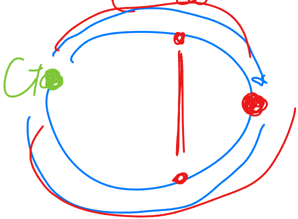


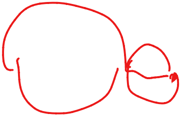
NP-completeness Guided Problem II Working

G' has a balloon of size n ⇒ G has hamCycle



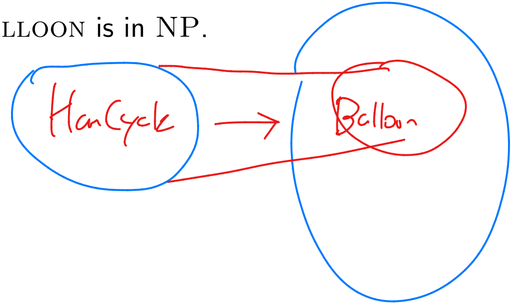
Claim C does not contain any of the added nodes





NP-completeness Guided Problem II Solution

Claim: BALLOON is in NP.



Balloon is in Np-Hard

NP-completeness Guided Problem II Solution

Claim: BALLOON is in NP-hard
Reduce HAMCYCLE to BALLOON

- Let G be an undirected graph with n nodes.
- Construct G' by adding a new path P of n nodes to some arbitrary node v in G.

We claim that

$$G' \text{ has a balloon of size } n \Leftrightarrow G \text{ has a Hamiltonian Cycle}$$

NP-completeness Guided Problem II Solution

G' has a balloon of size $n \Leftarrow G$ has a Hamiltonian Cycle

- Suppose that G has a Hamiltonian Cycle C
- C contains node v
- $C \cup P$ forms a balloon of size n in G'.

NP-completeness Guided Problem II Solution

G' has a balloon of size $n \Rightarrow G$ has a Hamiltonian Cycle

- Suppose that G' has a balloon of size n
- This balloon is the union of a path and a cycle C of length n
- C cannot contain any nodes in P, so it must lie entirely in the original graph
- Hence C is a Hamiltonian cycle.

NP-completeness Guided Problem II Solution

Reduction Time Complexity: