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On the use of inexact subdomain solvers for BDDC algorithms

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Abstract

The standard BDDC (balancing domain decomposition by constraints) preconditioner is shown to be equivalent to a preconditioner built from a partially subassembled finite element model. This results in a system of linear algebraic equations which is much easier to solve in parallel than the fully assembled model; the cost is then often dominated by that of the problems on the subdomains. An important role is also played, both in theory and practice, by an averaging operator and in addition exact Dirichlet solvers are used on the subdomains in order to eliminate the residual in the interior of the subdomains. The use of inexact solvers for these problems and even the replacement of the Dirichlet solvers by a trivial extension are considered. It is established that one of the resulting algorithms has the same eigenvalues as the standard BDDC algorithm, and the connection of another with the FETI-DP algorithm with a lumped preconditioner is also considered. Multigrid methods are used in the experimental work and under certain assumptions, it is established that the iteration count essentially remains the same as when exact solvers are used, while considerable gains in the speed of the algorithm can be realized since the cost of the exact solvers grows superlinearly with the size of the subdomain problems while the multigrid methods are linear.

Keywords: Domain decomposition; Non-overlapping; FETI-DP; BDDC; Multigrid; Multi-level; Inexact solvers

1. Introduction

Domain decomposition methods based on nonoverlapping subdomains have been widely used and studied for solving large, symmetric, positive definite linear systems arising from the discretization of elliptic partial differential equations; see [31, Chapters 4–6] and the references therein. Three of the main families of these iterative substructuring methods are the balancing Neumann–Neumann methods [24,26,8], the FETI methods [12,10], and the BDDC methods [6,27,28]. The BDDC algorithms represent an interesting redesign of the Neumann–Neumann algorithms with the coarse, global component expressed in terms of a set of primal constraints. All these methods are closely related. Thus, common tools were developed in [21] for the study of the one-level FETI and the classical balancing Neumann–Neumann algorithms; cf. also [31, Chapter 6]. Fragakis and Papadrakakis [13] found experimentally that pairs of such methods have essentially identical spectra; they also discussed primal iterative substructuring methods which are close counterparts to various FETI algorithms. In an important contribution to the theory, Mandel et al. [28] established that the preconditioned operators of a pair of BDDC and FETI-DP algorithms, with the same primal constraints, have the same eigenvalues except possibly those equal to 0 or 1. In a recent paper [25], the authors rederived the BDDC and FETI-DP algorithms and also gave a short proof of the main result in [28]. A key to these simplifications is a change of variables so that a primal constraint on the average over an interface edge or face is represented by a single primal variable in the new coordinate

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system. Brenner and Sung [5] have also recently established that any such common eigenvalue, not equal to 0 or 1, has the same multiplicity. In addition, they give an example for which the eigenvalues of the FETI-DP operator all exceed 1 while the BDDC operator has an eigenvalue equal to 1.

In the current standard BDDC, Neumann–Neumann, and FETI algorithms, the subdomain problems are always solved by direct solvers. We note that the computational work of a direct sparse solver grows faster than linearly with the number of unknowns, and that it can become quite expensive for problems with subdomains with many degrees of freedom. Storage considerations can also, in practice, severely limit the size of the subdomain problems when direct solvers are used.

The computational cost of some inexact solvers, e.g., a multigrid V-cycle iteration, grows only linearly with the problem size. If the subdomain (and coarse level) problems can be solved approximately, and the resulting algorithm retains a good condition number, substantial gains in efficiency can be realized. We note that inexact solvers have been considered in domain decomposition preconditioners in [1,16,17,4,30]. In these methods, subdomain Dirichlet problems are solved by multigrid V-cycles and a well-chosen preconditioner for the subdomain interface Schur operator is also required. It has been shown that there is very little deterioration of convergence rates when replacing the exact solutions of the subdomain problems by multigrid approximations, and that the convergence rates of the algorithms are primarily determined by the performance of the preconditioner chosen for the interface Schur operator.

Inexact solvers have also recently been considered for FETI-DP and BDDC algorithms. Thus, Klawonn and Rheinbach [18] studied the effect of using inexact solutions of subdomain problems as well as inexact coarse solvers in FETI-DP algorithms, further developing ideas from [20]. For an approach to solving the coarse level problem approximately in the BDDC algorithms, see Tu [32–34]. Approximate solutions in the BDDC preconditioners, which satisfy certain conditions related to the substructure null spaces, are also discussed by Dohrmann [7]. We believe that much of this work illustrates the fact that it is easier to modify BDDC than FETI-DP algorithms. In this paper, we consider a different approach of using inexact subdomain solvers in BDDC algorithms and we also consider a related question concerning FETI-DP algorithms.

In this paper, we consider a different approach of using inexact subdomain solvers in BDDC algorithms and we also consider a related question concerning FETI-DP algorithms. The standard BDDC preconditioner is shown to be equivalent to a preconditioner defined in terms of a partially subassembled problem, which is obtained from subdomain problems by assembling only with respect to a few select primal interface degrees of freedom for each subdomain. Compared with the original discrete problem, the partially subassembled one has much fewer connections between neighboring subdomains and therefore its solution is more suitable for parallel implementation. An averaging operator, which involves discrete harmonic extensions of the interface jump, is used to connect the solutions of the partially subassembled problem, which in general are discontinuous across the subdomain interface, with the original problem defined on a space of continuous finite element functions. We also establish that one of the proposed algorithms, which uses a trivial extension of the jump, has the same nonzero eigenvalues as a FETI-DP algorithm with a lumped preconditioner with the possible exception of 1.

This paper is organized as follows. The preconditioners using a partially subassembled problem are described in Sections 2 and 3. Connections with the standard BDDC operator and a FETP-DP algorithm with a lumped preconditioner are shown in Section 4. Condition number bounds of several preconditioned operators are established in Section 5. Different choices of the coarse level primal set of degrees of freedom are discussed in Section 6 for both two- and three-dimensional problems. The use of multigrid V-cycles for solving the subdomain problems and the partially subassembled problem is discussed in Section 7. In Section 8, numerical experiments solving a two-dimensional Poisson equation further demonstrate the connections between the related algorithms as well as the effects of using multigrid in the algorithms.

2. Discretization and decomposition

Let us consider Poisson's equation on a bounded, polyhedral domain Ω, in two or three dimensions, with homogeneous Dirichlet boundary conditions. The equivalent variational problem is: find $u \in H_0^1(\Omega) = \{w \in H^1(\Omega) | w = 0 \text{ on } \partial\Omega\}$, such that,

$$a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$.

We denote by $\widehat{W}$ a conforming finite element space of functions which are continuous across the element boundaries. The finite element solution $u \in \widehat{W}$, of the problem (1), satisfies

$$Au = f, \tag{2}$$

where the stiffness matrix A is symmetric positive definite.

The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i, $i = 1,2,\ldots,N$, of a characteristic diameter H. Each subdomain is a union of shape regular elements and the nodes on the boundaries of neighboring subdomains match across the interface $\Gamma = (\cup\partial\Omega_i)\backslash\partial\Omega$. The interface Γ is composed of the subdomain faces and/or edges, which are regarded as open sets, and of the subdomain vertices, which are end points of edges. In three dimensions, the subdomain faces are shared by two subregions, and the edges typically shared by more than two; in two dimensions, each edge is shared by two subregions. The interface of the subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. The characteristic diameter of the elements of the underlying triangulation is denoted by h. We also

denote the set of nodes on Ω_i and Γ_i by $\Omega_{i,h}$ and $\Gamma_{i,h}$, respectively. We note that all the algorithms considered are well defined for the less regular subdomains that are obtained by mesh partitioners. When developing theory, we will assume, as is customary in domain decomposition theory, that the triangulation of each subdomain is quasi uniform and that each subdomain is the union of a bounded number of shape regular elements with diameters on the order of H; cf. [31, Section 4.2].

The discrete solution space $\widehat{W}$ is decomposed into subspaces of subdomain interior type and of subdomain interface type, i.e.,

$$\widehat{W} = W_{\mathrm{I}} \oplus \widehat{W}_{\Gamma} = \left(\prod_{i=1}^{N} W_{\mathrm{I}}^{(i)} \right) \oplus \widehat{W}_{\Gamma},$$

where W_{I} is the product of subdomain interior variable spaces $W_{\mathrm{I}}^{(i)}$. The elements of $W_{\mathrm{I}}^{(i)}$ are supported in the subdomain Ω_i and vanish on the subdomain interface Γ_i. $\widehat{W}_{\Gamma}$ is the space of traces on Γ of functions in $\widehat{W}$. For any function in the space $\widehat{W}$, the neighboring subdomains share the same degrees of freedom on the common subdomain interface Γ. We also denote the space of interface variables on the subdomain Ω_i by $W_{\Gamma}^{(i)}$, and the associated product space by $W_{\Gamma} = \prod_{i=1}^{N} W_{\Gamma}^{(i)}$. Functions in the space W_{Γ} are in general discontinuous across the subdomain interface.

The subdomain problems with Neumann boundary conditions on the subdomain interface can be written as

$$A^{(i)} u^{(i)} = \begin{bmatrix} A_{\mathrm{II}}^{(i)} & A_{\Gamma\mathrm{I}}^{(i)^T} \\ A_{\Gamma\mathrm{I}}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_{\mathrm{I}}^{(i)} \\ u_{\Gamma}^{(i)} \end{bmatrix} = \begin{bmatrix} f_{\mathrm{I}}^{(i)} \\ f_{\Gamma}^{(i)} \end{bmatrix} = f^{(i)}, \quad i = 1, 2, \ldots, N, \tag{3}$$

where $u^{(i)} = (u_{\mathrm{I}}^{(i)}, u_{\Gamma}^{(i)}) \in W^{(i)} := (W_{\mathrm{I}}^{(i)}, W_{\Gamma}^{(i)})$. The global problem (2) can then be assembled from the subdomain problems (3), i.e., it can be represented as

$$A = \sum_{i=1}^{N} R^{(i)^T} A^{(i)} R^{(i)}, \quad \text{and} \quad f = \sum_{i=1}^{N} R^{(i)^T} f^{(i)},$$

where $R^{(i)}$ is the restriction operator from the global vector space $\widehat{W}$ to the subdomain vector space $W^{(i)}$, i.e., it maps vectors in the space $\widehat{W}$ to their components on the subdomain Ω_i.

3. Preconditioners using a partially subassembled problem

We introduce a space of partially subassembled variables as

$$\widetilde{W} = W_r \oplus \widehat{W}_{\Pi} = \left(\prod_{i=1}^{N} W_r^{(i)} \right) \oplus \widehat{W}_{\Pi}.$$

The space $\widehat{W}_{\Pi}$ corresponds to a few select subdomain interface degrees of freedom, for each subdomain, and is typically spanned by subdomain vertex nodal basis functions, and/or by interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions correspond to the primal interface continuity constraints enforced in the BDDC algorithm; they often take the form of common averages over edges or faces of Γ. These interface degrees of freedom are shared by neighboring subdomains, and they are called the coarse level, primal degrees of freedom. The remaining interface degrees of freedom are the dual interface variables. We will always assume that the basis has been changed so that we have explicit primal unknowns corresponding to the primal continuity constraints on edges or faces; see Remark 1 in Section 6. The complementary space W_r is the product of the subdomain spaces $W_r^{(i)}$, which corresponds to the subdomain interior and dual interface degrees of freedom and it is spanned by all the basis functions which vanish at the primal degrees of freedom. Thus, functions in the space $\widetilde{W}$ have a continuous coarse level, primal part and typically a discontinuous dual part across the subdomain interface.

The partially subassembled problem matrix, corresponding to the variables in the space $\widetilde{W}$, is obtained by assembling the subdomain matrices (3) only with respect to the coarse level primal variables, and we have

$$\widetilde{A} = \sum_{i=1}^{N} \overline{R}^{(i)^T} A^{(i)} \overline{R}^{(i)},$$

where $\overline{R}^{(i)}$ is the restriction operator from the space $\widetilde{W}$ to $W^{(i)}$. Denoting the injection of $\widehat{W}$ into $\widetilde{W}$ by $\widetilde{R}$, we have $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$, i.e., the matrix A can be formed from $\widetilde{A}$ by assembling with respect to the dual interface variables on the subdomain interface. Thus, we can obtain the fully assembled stiffness matrix by two stages of subassembly.

In order to define certain scaling operators, we need to introduce a positive scaling factor $\delta_i^{\dagger}(x)$ for each node x on the interface Γ_i of the subdomain Ω_i. In applications, these scaling factors will depend on the heat conduction coefficient and the first of the Lamé parameters for scalar elliptic problems and the equations of linear elasticity, respectively; see [23,22]. Here, with $\mathcal{N}_x$ the set of indices of the subdomains which have x on their boundaries, we will only need to use inverse counting functions defined by $\delta_i^{\dagger}(x) = 1/\mathrm{card}(\mathcal{N}_x)$, where $\mathrm{card}(\mathcal{N}_x)$ is the number of the subdomains in the set $\mathcal{N}_x$. It is easy to see that $\sum_{j \in \mathcal{N}_x} \delta_j^{\dagger}(x) = 1$. Given the scaling factors at the subdomain interface nodes, we can define the scaled injection operator $\widetilde{R}_D$; each row of $\widetilde{R}$ corresponds to a degree of freedom of the space $\widetilde{W}$, and multiplying each such row, which corresponds to a dual interface degree of freedom, with the scaling factor $\delta_i^{\dagger}(x)$, where $x \in \Gamma_{i,h}$ is the corresponding interface node, gives us $\widetilde{R}_D$.

The first preconditioner introduced in this paper for solving problem (2) is

$$M_1^{-1} = \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D.$$

To multiply $\widetilde{A}^{-1}$ with a vector $\widetilde{g}$, which belongs to the space of right hand sides corresponding to $\widetilde{W}$, we need to solve the following partially subassembled problem, with a leading block diagonal submatrix,

$$\widetilde{A}\widetilde{u} = \begin{bmatrix} A_{rr}^{(1)} & & & \widetilde{A}_{\Pi r}^{(1)^T} \\ & \ddots & & \vdots \\ & & A_{rr}^{(N)} & \widetilde{A}_{\Pi r}^{(N)^T} \\ \widetilde{A}_{\Pi r}^{(1)} & \dots & \widetilde{A}_{\Pi r}^{(N)} & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_r^{(1)} \\ \vdots \\ u_r^{(N)} \\ u_\Pi \end{bmatrix} = \begin{bmatrix} g_r^{(1)} \\ \vdots \\ g_r^{(N)} \\ g_\Pi \end{bmatrix} = \tilde{g}, \tag{4}$$

where $\widetilde{A}_{\Pi\Pi} = \sum_{i=1}^{N} R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)}$ and $\widetilde{A}_{\Pi r}^{(i)} = R_\Pi^{(i)^T} A_{\Pi r}^{(i)}$. Here $R_\Pi^{(i)}$ is the restriction operator which maps functions in the space $\widehat{W}_\Pi$ onto their components of the subdomain Ω_i. By using a block Cholesky factorization, cf. [25], the inverse of $\widetilde{A}$ can be written as

$$\widetilde{A}^{-1} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{\Pi r}^T \\ I \end{bmatrix} \widetilde{S}_\Pi^{-1} \begin{bmatrix} -\widetilde{A}_{\Pi r} A_{rr}^{-1} & I \end{bmatrix}, \tag{5}$$

where A_{rr} and $\widetilde{A}_{\Pi r}$ consist of the corresponding blocks of $\widetilde{A}$ in Eq. (4), and $\widetilde{S}_\Pi = \sum_{i=1}^{N} R_\Pi^{(i)^T} (A_{\Pi\Pi}^{(i)} - A_{\Pi r}^{(i)} A_{rr}^{(i)^{-1}} A_{\Pi r}^{(i)^T}) R_\Pi^{(i)}$. $\widetilde{S}_\Pi$ is formed by solving subdomain Neumann problems with given values of the primal degrees of freedom. We see from Eq. (5) that to multiply $\widetilde{A}^{-1}$ by a vector, subdomain Neumann problems with given primal values and a coarse problem need to be solved.

Since $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$, the preconditioned operator $M_1^{-1}A$ can be written as $\widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D \widetilde{R}^T \widetilde{A} \widetilde{R}$, which has the same non-zero eigenvalues as the operator $\widetilde{R}\widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D \widetilde{R}^T \widetilde{A}$. We define an averaging operator

$$E_{D,1} = \widetilde{R}\widetilde{R}_D^T : \ \widetilde{W} \to \widetilde{W},$$

which preserves the values of the coarse level primal component and the subdomain interior component of the vectors, and provides an average of the dual subdomain interface values. Given any $w \in \widetilde{W}$, we denote its subdomain interior components by $w_{\mathrm{I}}^{(i)}$, its coarse level component on subdomain Ω_i by $w_\Pi^{(i)}$, and its dual interface components by $w_\Delta^{(i)}$. For two-dimensional problems, for which the coarse level primal set of variables contain all the subdomain corner variables, the component of $E_{D,1}w$ on subdomain Ω_i can be written as,

$$(E_{D,1}w)^{(i)}(x) = w_{\mathrm{I}}^{(i)}(x) + w_\Pi^{(i)}(x) + \sum_{\mathscr{E}^{ij} \subset \partial\Omega_i} (\delta_i^\dagger(x) w_{\Delta,\mathscr{E}^{ij}}^{(i)}(x) + \delta_j^\dagger(x) w_{\Delta,\mathscr{E}^{ij}}^{(j)}(x))$$

at any node $x \in \overline{\Omega}_{i,h}$. Here $\mathscr{E}^{ij}$ represents an edge of the subdomain Ω_i, which is shared with the subdomain Ω_j, and $w_{\Delta,\mathscr{E}^{ij}}^{(i)}$ represents the restriction of $w_\Delta^{(i)}$ to the edge $\mathscr{E}^{ij}$. From the fact that $\sum_{j \in \mathscr{N}_x} \delta_j^\dagger(x) = 1, \ \forall x \in \Gamma_{i,h}$, i.e., $\delta_i^\dagger(x) = 1 - \delta_j^\dagger(x)$, in this case, we have

$$(E_{D,1}w)^{(i)} = w^{(i)} - \sum_{\mathscr{E}^{ij} \subset \partial\Omega_i} \delta_j^\dagger(\mathscr{E}^{ij}) (w_{\Delta,\mathscr{E}^{ij}}^{(i)} - w_{\Delta,\mathscr{E}^{ij}}^{(j)}), \tag{6}$$

where we have also used that fact that $\delta_j^\dagger(x)$ have the same value, $\delta_j^\dagger(\mathscr{E}^{ij})$, at all the nodes of $\mathscr{E}^{ij}$. We can see from Eq. (6) that the interface jump $w^{(i)} - w^{(j)}$ is extended by zero to the interior of subdomain Ω_i in $E_{D,1}w$.

In Section 6, we will establish a result on the stability of this averaging operator, i.e., we will bound the jump of the interface variables in Eq. (6) by the norms of $w^{(i)}$ and $w^{(j)}$. Since the discrete harmonic extension of the interface values to the subdomain interiors gives the minimum in the energy norm, cf. [31, Section 4.4], a better stability bound can be obtained if, in the above example, the component of the averaged vector on the subdomain Ω_i would be

$$(E_{D,2}w)^{(i)} = w^{(i)} - (\mathscr{H}^{(i)} \oplus I) \sum_{\mathscr{E}^{ij} \subset \partial\Omega_i} \delta_j^\dagger(\mathscr{E}^{ij}) (w_{\Delta,\mathscr{E}^{ij}}^{(i)} - w_{\Delta,\mathscr{E}^{ij}}^{(j)}). \tag{7}$$

Here $\mathscr{H}^{(i)} = -A_{\mathrm{II}}^{(i)^{-1}} A_{\Gamma\mathrm{I}}^{(i)^T}$ maps the subdomain interface jump to the interior of the subdomain by solving a Dirichlet problem; $\mathscr{H}^{(i)} \oplus I$ corresponds to the subdomain discrete harmonic extension. Correspondingly, we define our second averaging operator by

$$E_{D,2} = \widetilde{R}(\widetilde{R}_D^T - \mathscr{H} J_D),$$

where $\mathscr{H}$ is the direct sum of the $\mathscr{H}^{(i)}$. $J_D : \widetilde{W} \to \widetilde{W}$, is the jump operator for the dual interface variables across the subdomain interface. For any $w \in \widetilde{W}$, the component on the subdomain Ω_i of $J_D w$ is defined by

$$(J_D w(x))^{(i)} = \sum_{j \in \mathscr{N}_x} \delta_j^\dagger(x) (w^{(i)}(x) - w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h}.$$

We note that $J_D w$ always vanishes in the interior of the subdomain and for the coarse level primal component. For a matrix form of the jump operator, see [31, Section 6.3.3].

The preconditioner corresponding to the use of the averaging operator $E_{D,2}$ in the algorithm is defined by

$$M_2^{-1} = (\widetilde{R}_D^T - \mathscr{H} J_D) \widetilde{A}^{-1} (\widetilde{R}_D - J_D^T \mathscr{H}^T). \tag{8}$$

The component on subdomain Ω_i of $J_D^T w$ is given by

$$(J_D^T w(x))^{(i)} = \sum_{j \in \mathscr{N}_x} (\delta_j^\dagger(x) w^{(i)}(x) - \delta_i^\dagger(x) w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h}.$$

The subdomain interior and the coarse level primal components of $J_D^T w$ always vanish.

4. Connections between BDDC and FETI-DP algorithms

We have shown in [25] that the original BDDC operators, introduced by Dohrmann, cf. [6,27], can be written as $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$, after a change of variables corresponding to the coarse level, primal continuity constraints. Here S_Γ and $\widetilde{S}_\Gamma$ are the subdomain interface Schur complements of the matrices of problems (2) and (4), respectively. $\widetilde{R}_{D,\Gamma}$ is formed in the same way as $\widetilde{R}_D$ and maps the space of continuous interface variables to the space of partially subassembled interface variables with the appropriate scaling. We will also denote the restriction of the operators $\widetilde{R}$ and J_D to the space of the partially subassembled interface variables by $\widetilde{R}_\Gamma$ and $J_{D,\Gamma}$, respectively. In this section we will show that the preconditioned operator $M_2^{-1}A$ has essentially the same eigenvalues as the BDDC operator

$\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$. We will also establish a connection between the preconditioned operator $M_1^{-1}A$ and the FETI-DP algorithms using lumped preconditioners, cf. [10,12].

We first prove the following lemma:

Lemma 1. $A_{\Gamma\mathrm{I}}^T \widetilde{R}_{D,\Gamma}^T + \widetilde{A}_{\Gamma\mathrm{I}}^T J_{D,\Gamma} - \widetilde{A}_{\Gamma\mathrm{I}}^T = 0.$

Proof. Since $A_{\Gamma\mathrm{I}}$ can be obtained from $\widetilde{A}_{\Gamma\mathrm{I}}$ by assembling with respect to the dual interface variables, we have $A_{\Gamma\mathrm{I}} = \widetilde{R}_\Gamma^T \widetilde{A}_{\Gamma\mathrm{I}}$. Then, from the fact that $\widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T + J_{D,\Gamma} = I$, cf. [25, Lemma 1], we have

$$A_{\Gamma\mathrm{I}}^T \widetilde{R}_{D,\Gamma}^T + \widetilde{A}_{\Gamma\mathrm{I}}^T J_{D,\Gamma} - \widetilde{A}_{\Gamma\mathrm{I}}^T = \widetilde{A}_{\Gamma\mathrm{I}}^T \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T + \widetilde{A}_{\Gamma\mathrm{I}}^T J_D - \widetilde{A}_{\Gamma\mathrm{I}}^T = 0. \quad \square$$

Theorem 1. *The preconditioned operator $M_2^{-1}A$ has the same eigenvalues as the* BDDC *operator $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$, except for some eigenvalues equal to* 1.

Proof. We know, by using block Cholesky factorizations, that A and $\widetilde{A}^{-1}$ can be written as

$$A = \begin{bmatrix} A_{\mathrm{II}} & A_{\Gamma\mathrm{I}}^T \\ A_{\Gamma\mathrm{I}} & A_{\Gamma\Gamma} \end{bmatrix} = \begin{bmatrix} I & \\ A_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}} & \\ & S_\Gamma \end{bmatrix} \begin{bmatrix} I & A_{\mathrm{II}}^{-1} A_{\Gamma\mathrm{I}}^T \\ & I \end{bmatrix}, \tag{9}$$

$$\widetilde{A}^{-1} = \begin{bmatrix} A_{\mathrm{II}} & \widetilde{A}_{\Gamma\mathrm{I}}^T \\ \widetilde{A}_{\Gamma\mathrm{I}} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{\mathrm{II}}^{-1} \widetilde{A}_{\Gamma\mathrm{I}}^T \\ & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \times \begin{bmatrix} I & \\ -\widetilde{A}_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & I \end{bmatrix}. \tag{10}$$

Replacing $\widetilde{A}^{-1}$ and A in the preconditioned operator $M_2^{-1}A$ by the products in Eqs. (9) and (10), we find that $M_2^{-1}A$ has the same eigenvalues as the operator

$$\begin{bmatrix} I & A_{\mathrm{II}}^{-1} A_{\Gamma\mathrm{I}}^T \\ & I \end{bmatrix} \begin{bmatrix} I & A_{\mathrm{II}}^{-1} \widetilde{A}_{\Gamma\mathrm{I}}^T J_{D,\Gamma} \\ & \widetilde{R}_{D,\Gamma}^T \end{bmatrix} \begin{bmatrix} I & -A_{\mathrm{II}}^{-1} \widetilde{A}_{\Gamma\mathrm{I}}^T \\ & I \end{bmatrix} \times \begin{bmatrix} A_{\mathrm{II}}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & \\ -\widetilde{A}_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & I \end{bmatrix} \begin{bmatrix} I & \\ J_{D,\Gamma}^T \widetilde{A}_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & \widetilde{R}_{D,\Gamma} \end{bmatrix} \times \begin{bmatrix} I & \\ A_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}} & \\ & S_\Gamma \end{bmatrix}. \tag{11}$$

Here we have written the operator, $\widetilde{R}_D^T - \mathscr{H} J_D$, and its transpose in matrix form, and moved the last term of the operator to the front, which does not change the spectrum of the operator. From Lemma 1 and a simple computation, we see that the product of the matrices in (11) equals

$$\begin{bmatrix} I & \\ & \widetilde{R}_{D,\Gamma}^T \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & \\ & \widetilde{R}_{D,\Gamma} \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}} & \\ & S_\Gamma \end{bmatrix} = \begin{bmatrix} I & \\ & \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma \end{bmatrix},$$

which has the same eigenvalues as the BDDC operator, $\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma$, except for some eigenvalues equal to 1. $\square$

Lumped preconditioners were first used in one-level FETI algorithms, cf. [12]. Their application in FETI-DP algorithms is discussed in [10]. They are cheaper computationally than Dirichlet preconditioners since the latter require the solution of subdomain Dirichlet problems in each iteration. Results on scalable convergence are well established for FETI-DP algorithms using Dirichlet preconditioners, cf. [29,23,22,11] but theoretical results for lumped preconditioners are mostly missing. Here we will establish the spectral equivalence of the preconditioned operator $M_1^{-1}A$ and the FETI-DP operator using a lumped preconditioner. In Section 5, we will give a condition number bound of $M_1^{-1}A$, which therefore also is valid for the FETI-DP algorithms using lumped preconditioners. We note that in a recent paper, Fragakis and Papadrakakis [14] show experimentally that the performances of FETI algorithms with lumped preconditioners and their primal counterparts are very similar.

The FETI-DP operators can be written in terms of subdomain interface Schur complements, cf. [29,23]. We have shown in [25, Section 5] that the preconditioned FETI-DP operator with the Dirichlet preconditioner has the same non-zero eigenvalues as the operator $J_{D,\Gamma}^T \widetilde{S}_\Gamma J_{D,\Gamma} \widetilde{S}_\Gamma^{-1}$. Using the same arguments, the preconditioned FETI-DP operator with a lumped preconditioner has the same non-zero eigenvalues as $J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \widetilde{S}_\Gamma^{-1}$, where the Dirichlet preconditioner $\widetilde{S}_\Gamma$ is simply replaced by $\widetilde{A}_{\Gamma\Gamma}$, known as the lumped operator.

Lemma 2. *The operator $J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \widetilde{S}_\Gamma^{-1}$ has same eigenvalues as $J_D^T \widetilde{A} J_D \widetilde{A}^{-1}$, except for some eigenvalues equal to* 0.

Proof. Since

$$\begin{aligned} J_D^T \widetilde{A} J_D \widetilde{A}^{-1} &= \begin{bmatrix} 0 & 0 \\ 0 & J_{D,\Gamma}^T \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}} & \widetilde{A}_{\Gamma\mathrm{I}}^T \\ \widetilde{A}_{\Gamma\mathrm{I}} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & J_{D,\Gamma} \end{bmatrix} \begin{bmatrix} A_{\mathrm{II}} & \widetilde{A}_{\Gamma\mathrm{I}}^T \\ \widetilde{A}_{\Gamma\mathrm{I}} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \end{bmatrix} \begin{bmatrix} I & -A_{\mathrm{II}}^{-1} \widetilde{A}_{\Gamma\mathrm{I}}^T \\ & I \end{bmatrix} \times \begin{bmatrix} A_{\mathrm{II}}^{-1} & \\ & \widetilde{S}_\Gamma^{-1} \end{bmatrix} \begin{bmatrix} I & \\ -\widetilde{A}_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \widetilde{S}_\Gamma^{-1} \widetilde{A}_{\Gamma\mathrm{I}} A_{\mathrm{II}}^{-1} & J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \widetilde{S}_\Gamma^{-1} \end{bmatrix}, \end{aligned}$$

we see that $J_D^T \widetilde{A} J_D \widetilde{A}^{-1}$ has the same nonzero eigenvalues as $J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \widetilde{S}_\Gamma^{-1}$, except for some eigenvalues equal to 0. $\square$

Theorem 2. *The preconditioned operator $M_1^{-1}A$ has the same eigenvalues as the preconditioned* FETI-DP *operator with a lumped preconditioner, except for some eigenvalues equal to* 0 *and* 1.

Proof. We know that the preconditioned operator $M_1^{-1}A$ has the same non-zero eigenvalues as the operator $E_{D,1} \widetilde{A}^{-1} E_{D,1}^T \widetilde{A}$. Using exactly the same arguments as in the proof of [25, Theorem 1], we can show that $E_{D,1} \widetilde{A}^{-1} E_{D,1}^T \widetilde{A}$ and $(I - E_{D,1}^T) \widetilde{A} (I - E_{D,1}) \widetilde{A}^{-1}$ have the same eigenvalues

with the possible exception of 0 and 1. From the fact that $J_D = I - E_{D,1}$, cf. [25, Lemma 1], we see that $M_1^{-1}A$ has the same eigenvalues as $J_D^T \widetilde{A} J_D \widetilde{A}^{-1}$, with the possible exception of 0 and 1. The proof is then completed by using Lemma 2 and the fact that the preconditioned FETI-DP operator with a lumped preconditioner has the same non-zero eigenvalues as $J_{D,\Gamma}^T \widetilde{A}_{\Gamma\Gamma} J_{D,\Gamma} \widetilde{S}_\Gamma^{-1}$. □

5. Condition number bounds

Condition number bounds of the BDDC algorithms have been established in [27,28,25,5], and they also apply to the preconditioned operator $M_2^{-1}A$ as just established in Section 4. In this section, we will follow the analysis in [28] and [21], and reduce the condition number bounds of $M_1^{-1}A$ and $M_2^{-1}A$ to estimates of the norm of the averaging operators $E_{D,1}$ and $E_{D,2}$.

We define the $\widetilde{A}-$seminorm on the space $\widetilde{W}$ by

$$|w|_{\widetilde{A}}^2 = w^T \widetilde{A} w = \sum_{i=1}^N w^T \overline{R}^{(i)^T} A^{(i)} \overline{R}^{(i)} w = \sum_{i=1}^N |\overline{R}^{(i)} w|_{A^{(i)}}^2.$$

We know that the seminorms $|\cdot|_{A^{(i)}}$ and $|\cdot|_{H^1(\Omega_i)}$ are the same on the space $W^{(i)}$. The following assumption on the stability of $E_{D,1}$ and $E_{D,2}$, will be further considered in Section 6 for several different cases.

Assumption 1. For the averaging operators $E_{D,1}$ and $E_{D,2}$,

$$\sum_{i=1}^N |\overline{R}^{(i)}(E_{D,k} w)|_{H^1(\Omega_i)}^2 \leqslant \Phi_k(H,h) \sum_{i=1}^N |\overline{R}^{(i)} w|_{H^1(\Omega_i)}^2, \quad \forall w \in \widetilde{W},\ k = 1,2$$

or equivalently

$$|E_{D,k} w|_{\widetilde{A}}^2 \leqslant \Phi_k(H,h) |w|_{\widetilde{A}}^2, \quad \forall w \in \widetilde{W},\ k = 1,2,$$

where the $\Phi_k(H,h)$ are functions of the mesh sizes H and h.

With Assumption 1, we are ready to prove the following theorem.

Theorem 3. *Let Assumption 1 hold. Then, the preconditioned operators $M_k^{-1}A$, $k = 1,2$, are symmetric, positive definite with respect to the bilinear from $\langle\cdot,\cdot\rangle_A$ and*

$$\langle u,u\rangle_A \leqslant \langle M_k^{-1}Au, u\rangle_A \leqslant \Phi_k(H,h)\langle u,u\rangle_A, \quad \forall u \in \widehat{W}.$$

Proof. Here we only give a proof for the preconditioned operator $M_2^{-1}A$. The same arguments are also valid for $M_1^{-1}A$.

Lower bound: Given $u \in \widehat{W}$, let

$$w = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T \mathscr{H}^T) A u \in \widetilde{W}. \tag{12}$$

We have $\widetilde{A} w = (\widetilde{R}_D - J_D^T \mathscr{H}^T) A u$. From the fact that $\widetilde{R}^T \widetilde{R}_D = I$ and $\mathrm{range}(J_D^T) \subset \mathrm{null}(\widetilde{R}^T)$, we have

$$\begin{aligned}\langle u,u\rangle_A &= u^T A u = u^T \widetilde{R}^T (\widetilde{R}_D - J_D^T \mathscr{H}^T) A u = u^T \widetilde{R}^T \widetilde{A} w \\ &= \langle w, \widetilde{R} u\rangle_{\widetilde{A}}. \end{aligned} \tag{13}$$

From the Cauchy–Schwarz inequality and the fact that $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$, we have

$$\langle w, \widetilde{R} u\rangle_{\widetilde{A}} \leqslant \langle w,w\rangle_{\widetilde{A}}^{1/2} \langle \widetilde{R} u, \widetilde{R} u\rangle_{\widetilde{A}}^{1/2} = \langle w,w\rangle_{\widetilde{A}}^{1/2} \langle u,u\rangle_A^{1/2}. \tag{14}$$

Therefore, from (13) and (14), we have, $\langle u,u\rangle_A \leqslant \langle w,w\rangle_{\widetilde{A}}$. Since

$$\begin{aligned}\langle w,w\rangle_{\widetilde{A}} &= u^T A (\widetilde{R}_D - J_D^T \mathscr{H}^T)^T \widetilde{A}^{-1} \widetilde{A} \widetilde{A}^{-1} (\widetilde{R}_D - J_D^T \mathscr{H}^T) A u \\ &= u^T A (\widetilde{R}_D^T - \mathscr{H} J_D) \widetilde{A}^{-1} (\widetilde{R}_D - J_D^T \mathscr{H}^T) A u \\ &= \langle u, M_2^{-1} A u\rangle_A, \end{aligned} \tag{15}$$

we have, $\langle u,u\rangle_A \leqslant \langle u, M_2^{-1}Au\rangle_A$, which gives the lower bound of the theorem.

Upper bound: Given $u \in \widehat{W}$, take $w \in \widetilde{W}$ as in Eq. (12). We have, $(\widetilde{R}_D^T - \mathscr{H} J_D) w = M_2^{-1} A u$. Using that $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$ and Assumption 1, we have

$$\begin{aligned}\langle M_2^{-1}Au, M_2^{-1}Au\rangle_A &= \langle (\widetilde{R}_D^T - \mathscr{H} J_D) w, (\widetilde{R}_D^T - \mathscr{H} J_D) w\rangle_A \\ &= \langle \widetilde{R}(\widetilde{R}_D^T - \mathscr{H} J_D) w, \widetilde{R}(\widetilde{R}_D^T - \mathscr{H} J_D) w\rangle_{\widetilde{A}} \\ &= |E_{D,2} w|_{\widetilde{A}}^2 \leqslant \Phi_2(H,h) |w|_{\widetilde{A}}^2. \end{aligned}$$

Therefore, from Eq. (15), we have

$$\langle M_2^{-1}Au, M_2^{-1}Au\rangle_A \leqslant \Phi_2(H,h) \langle u, M_2^{-1}Au\rangle_A. \tag{16}$$

Using the Cauchy–Schwarz inequality and Eq. (16), we have

$$\begin{aligned}\langle u, M_2^{-1}Au\rangle_A &\leqslant \langle u,u\rangle_A^{1/2} \langle M_2^{-1}Au, M_2^{-1}Au\rangle_A^{1/2} \\ &\leqslant \sqrt{\Phi_2(H,h)} \langle u,u\rangle_A^{1/2} \langle u, M_2^{-1}Au\rangle_A^{1/2}. \end{aligned}$$

This gives

$$\langle u, M_2^{-1}Au\rangle_A \leqslant \Phi_2(H,h) \langle u,u\rangle_A$$

and the upper bound of the theorem. □

6. Examples related to Assumption 1

In this section, we specify the functions $\Phi_k(H,h)$ in Assumption 1 for both two and three dimensional problems, and for different choices of the coarse level primal set degrees of freedom. Here, we always denote by C a positive constant, which is independent of H, h, and the number of subdomains.

6.1. Two-dimensional problems

For a two-dimensional subdomain Ω_i, we denote its edge shared with subdomain Ω_j by $\mathscr{E}^{ij}$. For any finite element function $u^{(i)} \in W^{(i)}$, let $I^{H,\mathscr{E}^{ij}} u^{(i)}$ be the linear interpolant of $u^{(i)}$ on $\mathscr{E}^{ij}$ between its two end point, and let $\overline{u}_{\mathscr{E}^{ij}}^{(i)}$ be its average value on the edge $\mathscr{E}^{ij}$, which is defined by

$$\overline{u}^{(i)}_{\mathscr{E}^{ij}} = \frac{\int_{\mathscr{E}^{ij}} u^{(i)}\,\mathrm{d}s}{\int_{\mathscr{E}^{ij}} 1\,\mathrm{d}s}. \tag{17}$$

The following lemma can be found in [31, Lemma 4.15].

Lemma 3. *Let Ω_i be a two-dimensional subdomain. For any $u^{(i)} \in W^{(i)}$,*

$$\|u^{(i)}\|^2_{L^\infty(\Omega_i)} \leqslant C(1+\log(H/h))\|u^{(i)}\|^2_{H^1(\Omega_i)}.$$

By using Lemma 3, we can prove the following lemma.

Lemma 4. *Let Ω_i be a two-dimensional subdomain. For any $u^{(i)} \in W^{(i)}$,*

$$\|u^{(i)} - I^{H,\mathscr{E}^{ij}}u^{(i)}\|^2_{L^2(\mathscr{E}^{ij})} \leqslant CH(1+\log(H/h))|u^{(i)}|^2_{H^1(\Omega_i)},$$
$$\|u^{(i)} - \bar{u}^{(i)}_{\mathscr{E}^{ij}}\|^2_{L^2(\mathscr{E}^{ij})} \leqslant CH|u^{(i)}|^2_{H^1(\Omega_i)}.$$

Proof. We first use a trace theorem, cf. [31, Lemma A.6],

$$\|u^{(i)}\|^2_{L^2(\mathscr{E}^{ij})} \leqslant H\|u^{(i)}\|^2_{H^1(\Omega_i)}, \tag{18}$$

where the factor H results from a scaling argument; more precisely, it can be obtained by mapping the domain Ω_i, of diameter H, into a reference domain of diameter 1, by isotropic dilation, cf. [31, Section 3.4].

We denote the two finite element nodal basis functions on the coarse mesh associated with the two end points of the edge $\mathscr{E}^{ij}$ by ϕ^H_1 and ϕ^H_2, and denote the function values of $u^{(i)}$ at the two end points by u_1 and u_2. We have

$$\|I^{H,\mathscr{E}^{ij}}u^{(i)}\|^2_{L^2(\mathscr{E}^{ij})} \leqslant 2|u_1|^2\|\phi^H_2\|^2_{L^2(\mathscr{E}^{ij})} + 2|u_2|^2\|\phi^H_2\|^2_{L^2(\mathscr{E}^{ij})}.$$

We know that $\|\phi^H_1\|^2_{L^2(\mathscr{E}^{ij})}$ and $\|\phi^H_2\|^2_{L^2(\mathscr{E}^{ij})}$ can be bounded by CH, cf. [31, Lemma B.5]. Then, from Lemma 3, we have

$$\|I^{H,\mathscr{E}^{ij}}u^{(i)}\|^2_{L^2(\mathscr{E}^{ij})} \leqslant CH(1+\log(H/h))\|u^{(i)}\|^2_{H^1(\Omega_i)}.$$

From (18) and the above inequality, we obtain the first inequality of the lemma, by using a Poincaré inequality, cf. [31, Theorem A.18], and the fact that $u^{(i)} - I^{H,\mathscr{E}^{ij}}u^{(i)}$ does not change with the addition of a constant to $u^{(i)}$.

For the second inequality, from the definition of the average value in Eq. (17), the Cauchy–Schwarz inequality, and (18), we have

$$|\overline{u}^{(i)}_{\mathscr{E}^{ij}}|^2 = \left|\int_{\mathscr{E}^{ij}} u^{(i)}\,\mathrm{d}s\right|^2 \Big/ \left|\int_{\mathscr{E}^{ij}} 1\,\mathrm{d}s\right|^2 \leqslant C\frac{1}{H}\|u^{(i)}\|^2_{L^2(\mathscr{E}^{ij})}$$
$$\leqslant C\|u^{(i)}\|^2_{H^1(\Omega_i)}.$$

Therefore, $\|\overline{u}^{(i)}_{\mathscr{E}^{ij}}\|^2_{L^2(\mathscr{E}^{ij})} \leqslant CH\|u^{(i)}\|^2_{H^1(\Omega_i)}$. Combining with (18) and noting that $u^{(i)} - \overline{u}^{(i)}_{\mathscr{E}^{ij}}$ is unchanged with respect to a constant shift of the function $u^{(i)}$, we have the second inequality of the lemma, by using a Poincaré inequality. □

From Eq. (6), we know that, for any $w \in \widetilde{W}$, we can write the component of $E_{D,1}w$ on the subdomain Ω_i as

$$(E_{D,1}w)^{(i)} = w^{(i)} - \sum_{\mathscr{E}^{ij}\subset\partial\Omega_i} \delta^\dagger_j(\mathscr{E}^{ij})I^h(\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - w^{(j)})), \tag{19}$$

where we assume that all the subdomain corner variables are coarse level primal variables. $\vartheta_{\mathscr{E}^{ij}}$ is the standard linear finite element cut-off functions of the edge $\mathscr{E}^{ij}$; it equals 1 at the interior nodes of the edge $\mathscr{E}^{ij}$ and vanishes at the other nodes of $\overline{\Omega}_{i,h}$. The standard interpolation operator I^h brings us back to the finite element space; it acts on a piecewise quadratic, continuous function. To bound the norm of $(E_{D,1}w)^{(i)}$, as required in Assumption 1, we only need to bound the terms with the interface jump in Eq. (19). It is known, see [31, Lemma 4.31], that we can ignore the I^h operator when we derive our estimates.

For the case where only the subdomain corner variables are the primal variables, $I^{H,\mathscr{E}^{ij}}w^{(i)}$ and $I^{H,\mathscr{E}^{ij}}w^{(j)}$ are the same, and we have,

$$|\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - w^{(j)})|_{H^1(\Omega_i)} \leqslant |\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)})|_{H^1(\Omega_i)}$$
$$+ |\vartheta_{\mathscr{E}^{ij}}(w^{(j)} - I^{H,\mathscr{E}^{ij}}w^{(j)})|_{H^1(\Omega_i)}.$$

A bound of the first term on the right hand side will be sufficient; the same argument can be used for the second term. The support of the function $\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)})$ in the subdomain Ω_i is only in the strip of elements, of width h, next to the edge $\mathscr{E}^{ij}$. We denote such an element by K and denote its nodes on the edge $\mathscr{E}^{ij}$ by x_s, $s = 1,\ldots,n_K$, where n_K is the number of such nodes. Then, the slope of the function $\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)})$ in the element K can be bounded in terms of its values at the nodes x_s. We have

$$\int_K |\nabla(\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)}))|^2$$
$$\leqslant Ch^2\sum_{s=1}^{n_K}\frac{(w^{(i)}(x_s) - I^{H,\mathscr{E}^{ij}}w(x_s))^2}{h^2}.$$

Summing over the elements K next to the edge $\mathscr{E}^{ij}$, we have

$$|\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)})|^2_{H^1(\Omega_i)} \leqslant C\frac{1}{h}\|w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)}\|^2_{L^2(\mathscr{E}^{ij})}.$$

Then, from the first inequality in Lemma 4,

$$|\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - I^{H,\mathscr{E}^{ij}}w^{(i)})|^2_{H^1(\Omega_i)}$$
$$\leqslant C(H/h)(1+\log(H/h))|w^{(i)}|^2_{H^1(\Omega_i)},$$

i.e., $\Phi_1(H,h) = C(H/h)(1+\log(H/h))$, in Assumption 1, for the case where only subdomain corner degrees of freedom are primal.

For the case where the coarse level primal variables also include averages over the subdomain edges, we can bound the terms with the interface jump in Eq. (19) by

$$|\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - w^{(j)})|_{H^1(\Omega_i)} \leqslant |\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - \bar{w}_{\mathscr{E}^{ij}})|_{H^1(\Omega_i)}$$
$$+ |\vartheta_{\mathscr{E}^{ij}}(w^{(j)} - \bar{w}_{\mathscr{E}^{ij}})|_{H^1(\Omega_i)},$$

where $\bar{w}_{\mathscr{E}^{ij}}$ represents the common edge average value of $w^{(i)}$ and $w^{(j)}$ on $\mathscr{E}^{ij}$. Following the same argument as above

and using the second inequality in Lemma 4, we can bound the first term in the right hand side by

$$|\vartheta_{\mathcal{E}^{ij}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ij}})|^2_{H^1(\Omega_i)} \leqslant C\frac{1}{h}\|w^{(i)}-\bar{w}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{E}^{ij})} \leqslant C(H/h)|w^{(i)}|^2_{H^1(\Omega_i)},$$

i.e., $\Phi_1(H,h)=CH/h$, in Assumption 1, for the case where the set of coarse level primal variables contain both the subdomain corner and the edge average degrees of freedom.

Remark 1. To enforce the continuity of the edge integrals (17) across the subdomain interface, a change of basis is implemented such that there is an explicit degree of freedom for each edge corresponding to a finite element basis function with weights at the nodes of the edge determined by the integral (17). Since subdomain corner variables, i.e., the variables at the end points of the edges, are also primal variables and therefore continuous, the weights at the corner points can be taken as zero. Then only the interior points of each edge are involved in the edge average basis function, which makes the change of basis completely independent on different edges. For an discussion of how to implement the change of variables, see [25, 22, Section 4.2, 19]. We also note that a change of basis does not change the subdomain Dirichlet problem matrices, but will have an effect on the matrices of Neumann problems. In Section 7, we will implement multigrid V-cycles to solve the partially subassembled problem. We will see from the numerical experiments in Section 8 that the effects of a change of basis on the performance of multigrid V-cycles are very limited.

6.2. Three-dimensional problems

For a three-dimensional subdomain Ω_i, we denote its face shared with the subdomain Ω_j by $\mathcal{F}^{ij}$. $\vartheta_{\mathcal{F}^{ij}}$ is the face finite element cut-off function, which equals 1 at the interior nodes of $\mathcal{F}^{ij}$ and vanishes at the other nodes of $\overline{\Omega}_{i,h}$. For each edge $\mathcal{E}^{ik}$ of the subdomain Ω_i, we denote by $\mathcal{M}_{\mathcal{E}^{ik}}$ the index set of all neighboring subdomains which have $\mathcal{E}^{ik}$ as an edge. The edge cut-off function $\vartheta_{\mathcal{E}^{ik}}$ equals 1 at the interior nodes of $\mathcal{E}^{ik}$ and vanishes at the other nodes of $\overline{\Omega}_{i,h}$.

The following results can be found in [31, Lemmas 4.16 and 4.21].

Lemma 5. *Let Ω_i be a three-dimensional subdomain. For any $u^{(i)}\in W^{(i)}$,*

$$\|u^{(i)}-\bar{u}^{(i)}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{E}^{ik})} \leqslant C(1+\log(H/h))|u^{(i)}|^2_{H^1(\Omega_i)},$$
$$\|u^{(i)}-\bar{u}^{(i)}_{\mathcal{E}^{ij}}\|^2_{L^2(\mathcal{F}^{ij})} \leqslant CH(1+\log(H/h))|u^{(i)}|^2_{H^1(\Omega_i)}.$$

In this section, we only specify the bound for $\Phi_1(H,h)$ in Assumption 1 for the case where the coarse level primal variables consist of all the subdomain vertex and edge average degrees of freedom; for other options, see [23] or [31, Section 6.4]. Very much as for the two-dimensional problems, we can write the component of the subdomain Ω_i of $E_{D,1}w$ as

$$(E_{D,1}w)^{(i)} = w^{(i)} - \sum_{\mathcal{F}^{ij}\subset\partial\Omega_i}\delta^\dagger_j(\mathcal{F}^{ij})I^h(\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-w^{(j)})) - \sum_{\mathcal{E}^{ik}\subset\partial\Omega_i}\sum_{l\in\mathcal{M}_{\mathcal{E}^{ik}}}\delta^\dagger_l(\mathcal{E}^{ik})I^h(\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-w^{(l)})). \quad (20)$$

In the following, we give bounds for the edge and face terms.

Edge terms: Denote the common edge average value of $w^{(i)}$ and $w^{(l)}$ on $\mathcal{E}^{ik}$ by $\bar{w}_{\mathcal{E}^{ik}}$, and we have,

$$|\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-w^{(l)})|^2_{H^1(\Omega_i)} \leqslant 2|\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} + 2|\vartheta_{\mathcal{E}^{ik}}(w^{(l)}-\bar{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)}.$$

To bound the first term on the right hand side, we only need to look at the elements next to the edge $\mathcal{E}^{ik}$ in the subdomain Ω_i. For each such element K, we denote its nodes on $\mathcal{E}^{ik}$ by x_s, $s=1,\ldots,n_K$, and we have

$$\int_K|\nabla(\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}}))|^2 \leqslant Ch^3\sum_{s=1}^{n_K}\frac{(w^{(i)}(x_s)-\bar{w}_{\mathcal{E}^{ik}})^2}{h^2}.$$

Summing over the elements K, we have

$$|\vartheta_{\mathcal{E}^{ik}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} \leqslant C\|w^{(i)}-\bar{w}_{\mathcal{E}^{ik}}\|^2_{L^2(\mathcal{E}^{ik})} \leqslant C(1+\log(H/h))|w^{(i)}|^2_{H^1(\Omega_i)},$$

where the last inequality is a result of Lemma 5.

Face terms: Denote one edge of the face $\mathcal{F}^{ij}$ by $\mathcal{E}^{ik}$ and the common edge average value of $w^{(i)}$ and $w^{(j)}$ on $\mathcal{E}^{ik}$ by $\bar{w}_{\mathcal{E}^{ik}}$. We have

$$|\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-w^{(j)})|^2_{H^1(\Omega_i)} \leqslant 2|\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} + 2|\vartheta_{\mathcal{F}^{ij}}(w^{(j)}-\bar{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)}.$$

For the first term, we note that $\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}})$ is supported only in the elements next to the face $\mathcal{F}^{ij}$ in the subdomain Ω_i. Let us denote such an element by K and denote its nodes on the face $\mathcal{F}^{ij}$ by x_s, $s=1,\ldots,n_K$, where n_K is the number of such nodes. We then have

$$\int_K|\nabla(\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}}))|^2 \leqslant Ch^3\sum_{s=1}^{n_K}\frac{(w^{(i)}(x_s)-\bar{w}_{\mathcal{E}^{ik}})^2}{h^2}.$$

Summing over the elements which are next to $\mathcal{F}^{ij}$, we have

$$|\vartheta_{\mathcal{F}^{ij}}(w^{(i)}-\bar{w}_{\mathcal{E}^{ik}})|^2_{H^1(\Omega_i)} \leqslant \frac{1}{h}\|w^{(i)}-\bar{w}_{\mathcal{E}^{ik}}\|^2_{L^2(\mathcal{F}^{ij})} \leqslant C(H/h)(1+\log H/h)|w^{(i)}|^2_{H^1(\Omega_i)},$$

where the last step is a result of Lemma 5.

Combining the bounds on the edge and face terms in the right hand side of Eq. (20), we have $\Phi_1(H,h)=C(H/h)(1+\log(H/h))$, in Assumption 1, for the case where the set of coarse level primal degrees of freedom consist of

all the subdomain vertex and edge average degrees of freedom.

6.3. Using discrete harmonic extensions

Better stability results can be obtained for the averaging operator $E_{D,2}$ which employs a discrete harmonic extension of the interface jump to the interior of subdomains. For two-dimensional problems, the components of $E_{D,2}w$ in the subdomain Ω_i can be written, as in Eq. (7),

$$(E_{D,2}w)^{(i)} = w^{(i)} - (\mathscr{H}^{(i)} \oplus I) \sum_{\mathscr{E}^{ij} \subset \partial\Omega_i} \delta_j^{\dagger}(\mathscr{E}^{ij}) I^h(\vartheta_{\mathscr{E}^{ij}}(w^{(i)} - w^{(j)})), \tag{21}$$

for any given $w \in \widetilde{W}$. A bound on the edge terms can be found in [35, Lemma 3.3]. For three dimensional problems, where both subdomain vertex and edge average degrees of freedom are chosen as primal variables, the results can be found in [31, Lemma 6.34]. In both cases, $\Phi_2(H,h) = C(1+\log(H/h))^2$, in Assumption 1.

To summarize, when only subdomain corner variables are chosen as coarse level primal variables, we have $\Phi_1(H,h) = C(H/h)(1+\log(H/h))$, for the averaging operator $E_{D,1}$, and $\Phi_2(H,h) = C(1+\log(H/h))^2$, for $E_{D,2}$, in Assumption 1, for two-dimensional problems. When the edge average degrees of freedom are also chosen as primal degrees of freedom, we have $\Phi_1(H,h) = CH/h$, for $E_{D,1}$, for two-dimensional problems and $\Phi_1(H,h) = C(H/h)(1+\log(H/h))$ for $E_{D,1}$, and $\Phi_2(H,h) = C(1+\log(H/h))^2$ for $E_{D,2}$, for three-dimensional problems.

7. Inexact solvers

Our BDDC preconditioner with inexact solvers is defined by

$$M_3^{-1} = (\widetilde{R}_D^T - \mathscr{H}_V J_D)\widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T \mathscr{H}_V^T).$$

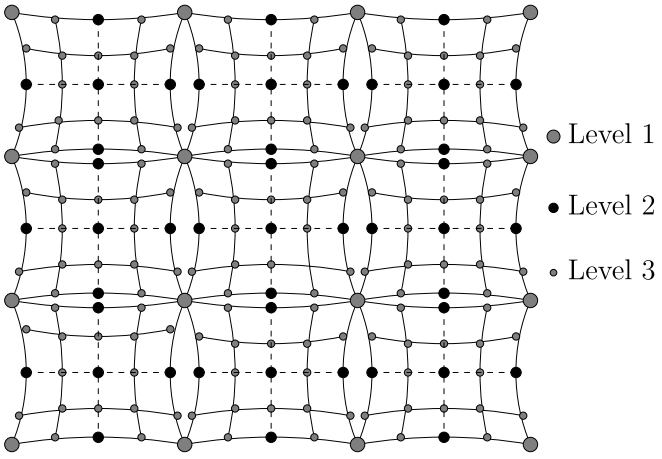

Fig. 1. Three nested meshes for the partially subassembled problem.

Here $\mathscr{H}_V$ represents an approximation of the operator $\mathscr{H}$ obtained by replacing the exact subdomain Dirichlet problem solutions by one multigrid V-cycle on each subdomain. $\widetilde{A}_V^{-1}$ represents the action of one or several multigrid V-cycles or W-cycles for solving the global, partially subassembled problem (4).

A few nested levels of meshes for the multigrid implementation, for solving the partially assembled problem (4), are shown in Fig. 1, with nine subdomains in two dimensions and for the case where the coarse level, primal variables are those at the subdomain corners. Problem matrices at different levels can be obtained by Galerkin projections, i.e., by multiplying prolongation and restriction operators with the matrices at the finer levels. From the structure of the matrix in (4), we can see that at each level of the V-cycle, a Gauss–Seidel smoothing step can be implemented subdomain by subdomain independently at first, and then followed by a Gauss–Seidel step on the coarse level primal variables, which has to be done sequentially. In our implementation, the coarsest level mesh in the multigrid methods is always given in terms of the nodes related to the coarse level primal variables, e.g., as in Fig. 1. At the coarsest level, the problem is solved exactly. When the size of the coarse level problem is relatively small, the main effort of a multigrid V-cycle for solving the partially subassembled problem (4) will be related to subdomain computations that can be carried out in parallel.

Uniform convergence rates of multigrid V-cycles for solving the subdomain Dirichlet problems are well established; cf. [2,3]. If we denote the subdomain problem matrix by $A^{(i)}$ and denote the bilinear form corresponding to a multigrid V-cycle for solving the subdomain Dirichlet problem by $A_V^{(i)}$, it was established that, cf. [3, Theorem 3.2],

$$c u^{(i)^T} A^{(i)} u^{(i)} \leqslant u^{(i)^T} A_V^{(i)} u^{(i)} \leqslant u^{(i)^T} A^{(i)} u^{(i)}, \quad \forall u^{(i)} \in W^{(i)}, \tag{22}$$

where c is a positive constant independent of the mesh size. These bounds hold uniformly for all subdomains, under the assumption of a regular decomposition of subdomains. We see in the next lemma that a bound similar to that of Assumption 1 can be proved for the averaging operator $E_{D,3} = \widetilde{R}(\widetilde{R}_D^T - \mathscr{H}_V J_D)$, using the same function $\Phi_2(H,h)$ as for $E_{D,2}$ in Assumption 1.

Lemma 6. *Let Assumption 1 hold for the averaging operator $E_{D,2}$. Then,*

$$\sum_{i=1}^{N} |\overline{R}^{(i)}(E_{D,3}w)|^2_{H^1(\Omega_i)} \leqslant C(1+\Phi_2(H,h)) \sum_{i=1}^{N} |\overline{R}^{(i)}w|^2_{H^1(\Omega_i)}, \quad \forall w \in \widetilde{W},$$

where C is a positive constant independent of H, h, and the number of subdomains.

Proof. We only consider the two-dimensional case; as mentioned in Section 6.3 the same arguments also work

for three-dimensional problems. We denote the second term of the right side in Eq. (21), by $(P_{D,2}w)^{(i)}$; it represents the discrete harmonic extension of the interface jump to the subdomain Ω_i. Similarly, we denote by $(P_{D,3}w)^{(i)}$ the same term, with $\mathscr{H}^{(i)}$ replaced by $\mathscr{H}_V^{(i)}$. This corresponds to an approximate discrete harmonic extension using a multigrid V-cycle to solve the subdomain Dirichlet problems. From Assumption 1 and the fact that $P_{D,2}w = w - E_{D,2}w$, we have

$$\sum_{i=1}^{N} |(P_{D,2}w)^{(i)}|^2_{H^1(\Omega_i)} \leqslant (2 + 2\Phi_2(H,h)) \sum_{i=1}^{N} |\overline{R}^{(i)}w|^2_{H^1(\Omega_i)}, \quad \forall w \in \widetilde{W}. \tag{23}$$

From (22), we know that $|(P_{D,3}w)^{(i)}|^2_{H^1(\Omega_i)} \leqslant C|(P_{D,2}w)^{(i)}|^2_{H^1(\Omega_i)}$, $\forall i = 1,2,\ldots,N$. Therefore, we have

$$\sum_{i=1}^{N} |(P_{D,3}w)^{(i)}|^2_{H^1(\Omega_i)} \leqslant C(2 + 2\Phi_2(H,h)) \sum_{i=1}^{N} |\overline{R}^{(i)}w|^2_{H^1(\Omega_i)}, \quad \forall w \in \widetilde{W}. \tag{24}$$

The proof is completed by using that $E_{D,3}w = w - P_{D,3}w$. □

To prove a condition number bound for the preconditioned operator $M_3^{-1}A$ in Theorem 4, we need an assumption on the convergence rate of multigrid V-cycles for solving the global partially subassembled problem (4).

Assumption 2. There are positive constants $\tilde{c}$ and $\widetilde{C}$, such that,

$$\tilde{c}u^T\widetilde{A}u \leqslant u^T\widetilde{A}_V u \leqslant \widetilde{C}u^T\widetilde{A}u, \quad \forall u \in \widetilde{W}. \tag{25}$$

Theorem 4. *Let Assumptions 1 and 2 hold. The preconditioned operator* $M_3^{-1}A = (\widetilde{R}_D^T - \mathscr{H}_V J_D)\widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T\mathscr{H}_V^T)A$, *is symmetric, positive definite with respect to the bilinear form* $\langle\cdot,\cdot\rangle_A$ *and*

$$\frac{1}{\widetilde{C}}\langle u,u\rangle_A \leqslant \langle M_3^{-1}Au,u\rangle_A \leqslant \frac{C(1+\Phi_2(H,h))}{\tilde{c}}\langle u,u\rangle_A, \quad \forall u \in \widehat{W},$$

where C is a positive constant independent of H, h, and the number of subdomains.

Proof. The proof is very similar to that of Theorem 3.

Lower bound: Given $u \in \widehat{W}$, let

$$w = \widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T\mathscr{H}_V^T)Au \in \widetilde{W}. \tag{26}$$

We have $\widetilde{A}_V w = (\widetilde{R}_D - J_D^T\mathscr{H}_V^T)Au$. As in the proof of Theorem 3, we have

$$\langle u,u\rangle_A = u^T\widetilde{R}^T(\widetilde{R}_D - J_D^T\mathscr{H}_V^T)Au = u^T\widetilde{R}^T\widetilde{A}_V w = \langle w,\widetilde{R}u\rangle_{\widetilde{A}_V}. \tag{27}$$

We have, by using Assumption 2,

$$\begin{aligned}\langle w,\widetilde{R}u\rangle_{\widetilde{A}_V} &\leqslant \langle w,w\rangle^{1/2}_{\widetilde{A}_V}\langle\widetilde{R}u,\widetilde{R}u\rangle^{1/2}_{\widetilde{A}_V} \\ &\leqslant \sqrt{\widetilde{C}}\langle w,w\rangle^{1/2}_{\widetilde{A}_V}\langle\widetilde{R}u,\widetilde{R}u\rangle^{1/2}_{\widetilde{A}} \\ &= \sqrt{\widetilde{C}}\langle w,w\rangle^{1/2}_{\widetilde{A}_V}\langle u,u\rangle^{1/2}_{A}.\end{aligned} \tag{28}$$

Therefore, from (27) and (28), we have, $\langle u,u\rangle_A \leqslant \widetilde{C}\langle w,w\rangle_{\widetilde{A}_V}$. Since

$$\begin{aligned}\langle w,w\rangle_{\widetilde{A}_V} &= u^TA(\widetilde{R}_D - J_D^T\mathscr{H}_V^T)^T\widetilde{A}_V^{-1}\widetilde{A}_V\widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T\mathscr{H}_V^T)Au \\ &= u^TA(\widetilde{R}_D^T - \mathscr{H}_V J_D)\widetilde{A}_V^{-1}(\widetilde{R}_D - J_D^T\mathscr{H}_V^T)Au \\ &= \langle u, M_3^{-1}Au\rangle_A,\end{aligned} \tag{29}$$

we have, $\langle u,u\rangle_A \leqslant \widetilde{C}\langle u, M_3^{-1}Au\rangle_A$, which gives the lower bound.

Upper bound: As in the proof of Theorem 3, take $w \in \widetilde{W}$ as in Eq. (26), and we obtain

$$\begin{aligned}\langle M_3^{-1}Au, M_3^{-1}Au\rangle_A &= \langle(\widetilde{R}_D^T - \mathscr{H}_V J_D)w, (\widetilde{R}_D^T - \mathscr{H}_V J_D)w\rangle_A \\ &= \langle\widetilde{R}(\widetilde{R}_D^T - \mathscr{H}_V J_D)w, \widetilde{R}(\widetilde{R}_D^T - \mathscr{H}_V J_D)w\rangle_{\widetilde{A}} \\ &= |E_{D,3}w|^2_{\widetilde{A}} \leqslant C(1+\Phi_2(H,h))|w|^2_{\widetilde{A}},\end{aligned}$$

where we use Lemma 6 in the last step. Therefore, using Assumption 2 and Eq. (29), we find

$$\begin{aligned}\langle M_3^{-1}Au, M_3^{-1}Au\rangle_A &\leqslant \frac{C(1+\Phi_2(H,h))}{\tilde{c}}|w|^2_{\widetilde{A}_V} \\ &= \frac{C(1+\Phi_2(H,h))}{\tilde{c}}\langle u, M_3^{-1}Au\rangle_A.\end{aligned}$$

Then, using the Cauchy–Schwarz inequality, as in the proof of Theorem 3, we have

$$\langle u, M_3^{-1}Au\rangle_A \leqslant \frac{C(1+\Phi_2(H,h))}{\tilde{c}}\langle u,u\rangle_A,$$

and the upper bound of the theorem. □

Our multigrid problem on our partially assembled region is not standard and it is therefore not certain that $\tilde{c}$ and $\widetilde{C}$ in Assumption 2 can be made completely independent of the mesh sizes. Additional technical difficulties arise because of the use of basis functions representing edge or face averages. In the two-dimensional case represented in Fig. 1, we have been able to show that $\widetilde{C}$ is uniformly bounded while $1/\tilde{c} \leqslant C(1+\log(H/h))^2$.

Instead of multigrid V-cycles, we can also use other approaches for solving the partially subassembled problem (4) inexactly. One would be to construct submatrices $\widehat{A}^{(i)}$ which are spectrally equivalent to $A^{(i)}$. Such ideas have proven very successful in the design of many primal iterative substructuring algorithms; see Chapter 5 and in particular Section 5.2 of [31]. We note that all the matrices $A^{(i)}$ corresponding to interior subdomains will be singular and that therefore any such pair of $A^{(i)}$ and $\widehat{A}^{(i)}$ must have a common null space. Another requirement is that there exists a fast algorithm for solving the linear systems with the submatrices $\widehat{A}^{(i)}_{rr}$. If the pairs of subdomain matrices are spec-

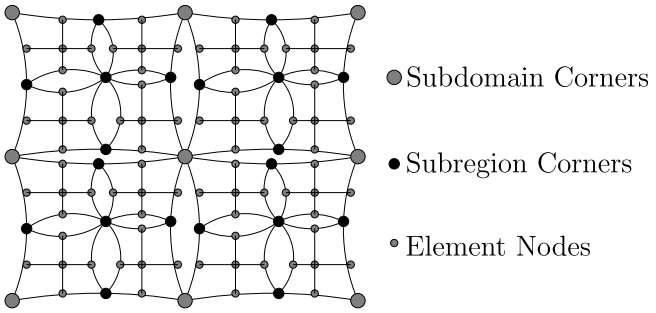

Fig. 2. Partially assembled mesh for a three-level BDDC preconditioner.

trally equivalent, it can be easily shown that the matrix obtained by partially subassembling the $\widehat{A}^{(i)}$ will be spectrally equivalent to $\widetilde{A}$. This then straightforwardly leads to a strong overall result if the inexact Dirichlet solvers also are of good quality. Algorithmically, an algorithm based on such a preconditioner would closely resemble that with exact subdomain solvers; an approximation of $\widetilde{S}_{\Pi}$ in (5) would be computed and then factored replacing the $A^{(i)}$ by the $\widehat{A}^{(i)}$ in the algorithm.

Another approach would be to implement a BDDC preconditioner as an inexact solver of the partially subassembled problem (4). We can further decompose each subdomain mesh to a partially assembled one; see Fig. 2 for an example of four subdomains, where each subdomain contains four smaller subregions of characteristic diameter H_S. Let $\widetilde{\widetilde{W}}$ be the space of functions which are continuous at the subregion corners and are in general discontinuous elsewhere on the subregion interface. Denote the stiffness matrix on the space $\widetilde{\widetilde{W}}$ by $\widetilde{\widetilde{A}}$, and the scaled injection operator from $\widetilde{W}$ to $\widetilde{\widetilde{W}}$ by $\widetilde{\widetilde{R}}_D$. We also denote the jump operator on the space $\widetilde{\widetilde{W}}$ by $\widetilde{J}_D$, which computes the jump of the variables across subregion interface within each subdomain, and a discrete harmonic extension operator by $\widetilde{\mathscr{H}}$, which extends the jump across the subregion interface within each subdomain to the interior of subregions. Then a preconditioner for the partially subassembled problem (4) will be of the form $(\widetilde{\widetilde{R}}_D^T - \widetilde{\mathscr{H}}\widetilde{J}_D)\widetilde{\widetilde{A}}^{-1}(\widetilde{\widetilde{R}}_D - \widetilde{J}_D^T\widetilde{\mathscr{H}}^T)$. With the same analysis as for the two level BDDC algorithms in Theorem 3, we can establish the inequality in Assumption 2 for such an inexact BDDC solution of the partially subassembled problem and we can prove that $\widetilde{C}/\tilde{c}$ in Assumption 2 will be bounded by $C(1+\log(H_S/h))^2$, in the two-dimensional case represented in Fig. 2. If the exact subdomain Dirichlet problem solutions are also replaced by such a BDDC preconditioner within each subdomain, then an additional factor of $(1+\log(H_S/h))^2$ will be introduced. The resulting algorithm is similar but not the same as the three level BDDC methods of Tu [32–34]. Details on this work will be discussed in future work.

8. Numerical experiments

In our numerical experiments, a Poisson equation on a square domain with Dirichlet boundary conditions is discretized using bilinear elements on a uniform square mesh. The FETI-DP algorithm with a lumped preconditioner, the standard BDDC algorithm, and the preconditioners M_1^{-1}, M_2^{-1}, and M_3^{-1} are used to solve the discrete problem. We have implemented the standard BDDC algorithm in terms of the operator $\widetilde{R}_{D,\Gamma}^T\widetilde{S}_{\Gamma}^{-1}\widetilde{R}_{D,\Gamma}S_{\Gamma}$ for solving the interface problem. The preconditioned conjugate gradient method

Table 2
Condition number bounds and iteration counts for BDDC and $M_2^{-1}A$

H/h	# of Sub.	With edge aver.				Corners only			
		BDDC		$M_2^{-1}A$		BDDC		$M_2^{-1}A$	
		κ	it.	κ	it.	κ	it.	κ	it.
8	4×4	1.2	5	1.2	5	2.7	8	2.7	8
	8×8	1.2	5	1.2	5	3.0	10	3.0	9
	12×12	1.2	5	1.2	5	3.1	10	3.1	10
	16×16	1.2	5	1.2	5	3.1	10	3.1	10
	20×20	1.2	5	1.2	5	3.1	10	3.1	10
4	4×4	1.1	4	1.1	4	2.0	7	2.0	7
8		1.2	5	1.2	5	2.7	8	2.7	8
16		1.4	5	1.4	5	3.6	9	3.6	9
32		1.7	6	1.7	6	4.6	10	4.6	9

Table 1
Condition number bounds (κ) and iteration counts (it.) for FETI-DP (lumped) and $M_1^{-1}A$

H/h	# of Sub.	With edge aver.				Corners only			
		FETIDP		$M_1^{-1}A$		FETIDP		$M_1^{-1}A$	
		κ	it.	κ	it.	κ	it.	κ	it.
8	4×4	1.9	8	1.9	7	8.3	12	8.8	12
	8×8	2.0	8	2.0	7	10.8	19	11.3	17
	12×12	2.0	8	2.0	7	11.2	19	11.8	17
	16×16	2.0	8	2.0	7	11.3	19	11.6	17
	20×20	2.0	8	2.0	7	11.3	19	11.7	17
4	4×4	1.1	5	1.1	4	3.3	9	3.5	9
8		1.9	8	1.9	7	8.3	12	8.8	12
16		3.8	12	3.9	9	19.6	16	21.4	16
32		8.0	17	8.2	13	56.7	22	63.5	20

Table 3
Condition number bounds and iteration counts for $M_2^{-1}A$ and $M_3^{-1}A$ with three W-cycles

H/h	# of Sub.	With edge aver.				With edge mid.				Corners only			
		$M_2^{-1}A$		$3W_{22}$		$M_2^{-1}A$		$3W_{22}$		$M_2^{-1}A$		$3W_{22}$	
		κ	it.	κ	it.	κ	it.	κ	it.	κ	it.	κ	it.
8	4×4	1.2	5	1.2	5	1.7	6	1.7	6	2.7	8	2.7	8
	8×8	1.2	5	1.2	5	1.8	7	1.8	7	3.0	9	3.0	9
	12×12	1.2	5	1.2	5	1.8	7	1.8	7	3.1	10	3.0	9
	16×16	1.2	5	1.2	5	1.8	7	1.8	7	3.1	10	3.0	9
	20×20	1.2	5	1.2	5	1.8	7	1.8	7	3.1	10	3.0	9
4	4×4	1.1	4	1.1	4	1.3	5	1.3	5	2.0	7	2.0	7
8		1.2	5	1.2	5	1.7	6	1.7	6	2.7	8	2.7	8
16		1.4	5	1.4	5	2.2	7	2.2	7	3.6	9	3.5	9
32		1.7	6	1.6	6	2.8	8	2.7	8	4.6	9	4.5	10

is always used in the iterations, and it is stopped when the L_2-norm of the residual has been reduced by a factor 10^{-6}. In all the tables, the condition number bounds are estimated by using the smallest and largest eigenvalues obtained from the tridiagonal Lanczos matrix generated by the preconditioned conjugate gradient iterations.

In Tables 1 and 2, we compare the condition number bounds and iteration counts for $M_1^{-1}A$ and $M_2^{-1}A$ with those of the FETI-DP algorithm with a lumped preconditioner and those of the BDDC algorithm, respectively. Two different sets of coarse level primal variables are used, either only the subdomain corner variables or both the subdomain corner and edge average variables. We see that both condition number bounds and iteration counts match very well for $M_1^{-1}A$ and the FETI-DP algorithm with a lumped preconditioner, and for $M_2^{-1}A$ and the BDDC algorithm. The convergence rates are always independent of the number of subdomains. With an increase of the subdomain problem size, the condition number bounds for $M_1^{-1}A$ and the lumped FETI-DP algorithm grow quickly, especially for the case when only subdomain corner variables are used as coarse level primal variables. Better condition number bounds are obtained for $M_2^{-1}A$ and the BDDC algorithm, where the subdomain Dirichlet problems are solved exactly; we see from Table 2 that the growth of the condition numbers is slow when the subdomain problem size increases.

Tables 3 and 4 show the condition number bounds and iteration counts for $M_3^{-1}A$, where inexact solvers are used, and they are also compared with those for $M_2^{-1}A$. We have chosen three different sets of coarse level primal variables:

- subdomain corner variables only;
- corner variables, variables at the middle points of the edges, and a variable at a center node of each subdomain mesh;
- corner variables, edge average variables, and a variable at a center node of each subdomain mesh.

Here we include the variable at a center node of each subdomain mesh in the coarse level primal variable set just for the convenience of multigrid implementation, so that the coarsest level mesh consists of uniform rectangular elements.

On each level of the V-cycle and W-cycle, two Gauss–Seidel iterations were used for both the pre and the posts-moothing steps, as represented by V_{22} and W_{22} in Tables 3 and 4. One V-cycle is always used for solving the subdomain Dirichlet problems. The $3W_{22}$, $2V_{22}$, and $1V_{22}$ in the tables, represent the use of three W-cycles, two

Table 4
Condition number bounds and iteration counts for $M_3^{-1}A$ with one and two V-cycles

H/h	# of Sub.	With edge aver.				With edge mid.				Corners only			
		$2V_{22}$		$1V_{22}$		$2V_{22}$		$1V_{22}$		$2V_{22}$		$1V_{22}$	
		κ	it.	κ	it.	κ	it.	κ	it.	κ	it.	κ	it.
8	4×4	1.3	5	2.0	8	1.6	6	1.6	7	2.4	8	1.9	8
	8×8	1.3	6	2.0	8	1.6	6	1.6	7	2.4	8	2.0	8
	12×12	1.3	6	2.0	8	1.7	7	1.6	7	2.4	8	2.1	8
	16×16	1.3	6	2.0	8	1.7	7	1.6	7	2.4	8	2.1	8
	20×20	1.3	6	2.0	8	1.7	7	1.6	7	2.4	8	2.1	8
4	4×4	1.1	4	1.3	6	1.3	5	1.3	6	1.9	7	1.7	7
8		1.3	5	2.0	8	1.6	6	1.6	7	2.4	8	1.9	8
16		1.5	6	2.1	8	1.9	7	1.8	7	2.8	9	2.1	8
32		1.8	7	7.9	13	2.2	7	1.9	7	3.1	9	2.3	8

V-cycles, and one V-cycle, respectively, for solving the partially subassembled problem in the algorithm. Table 3 shows that when $3W_{22}$ is used to solve the partially subassembled problem, the condition number bounds and iteration counts are the same as for the algorithm using exact solutions. If only one or two V-cycles are used, we can see from Table 4 that the changes of the condition number bounds and iteration counts are very small, for the cases where the coarse level degrees of freedom do not involve any edge averages. In fact better convergence rates are obtained when $1V_{22}$ was used to solve the partially subassembled problem than when it is solved exactly. For the case where the edge average degrees of freedom were part of the primal variable set, the use of $2V_{22}$ is sufficient to guarantee small changes in the convergence rates in all the experiments; we can see that the change of basis has only a very limited effect on the performance of multigrid V-cycles.

The CPU time for the algorithms which use multigrid has not been compared with those of the algorithms using direct solvers. Here we only consider the floating point operation counts (flops) required by the algorithms and show that, for problems of large size, using multigrid V-cycles in the algorithms will be less expensive than using exact solvers. Let us denote the size of the subdomain problems by N. When using a direct solver for solving the subdomain level problems, the factorization step is only implemented once in a preprocessing step of the algorithm reducing the rest of the work of an iteration to forward eliminations and backward substitutions. The best possible bounds for a two dimensional discrete Laplacian is given in [15, Section 8.1] and best possible bounds are also known for three dimensions, see [9]. (In our discussion, we will assume that an optimal ordering is used for the exact solvers.) In each iteration step, the forward eliminations and backward substitutions asymptotically require more flops, $O(N\log N)$ in two dimensions and $O(N^{4/3})$ in three dimensions, than $O(N)$ required by a multigrid V-cycle. In addition, the factorization step of a direct solver also requires $O(N^{3/2})$ flops in 2D and $O(N^2)$ in 3D. This shows that replacing the exact solvers in the BDDC algorithms by multigrid V-cycles, where essentially the same iteration count can be retained, will be more effective.

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