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# Spectral Element Multigrid. I. Formulation and Numerical Results

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A variational spectral element multigrid algorithm is proposed, and results are presented for a one-dimensional Poisson equation on a finite interval. The key features of the proposed algorithm are as follows: the nested spaces and associated hierarchical bases are intra-element, resulting in simple data structures and rapid tensor-product sum-factorization evaluations; smoothing is effected by readily constructed and efficiently inverted (diagonal) Jacobi preconditioners; the technique is readily parallelized within the context of a medium-grained paradigm; and the (work-deflated) multigrid convergence rate $\bar{\rho}$ is bounded from above well below unity, and is only a weak function of the number of spectral elements $K$, the (large) order of the polynomial approximation, $N$, and the number of multigrid levels, $J$. Preliminary tests indicate that these convergence properties persist in higher space dimensions.

**KEY WORDS:** Elliptic; spectral element; $p$-type finite element; iterative methods; multigrid.

## 1. INTRODUCTION

Over the past decade multigrid methods have had tremendous impact on the computational efficiency of elliptic solvers (Brandt, 1977; Hackbusch and Trottenberg, 1982; Brand, Lemke, and Linden, 1986). The multigrid kernel of recursive high-wave number smoothing and coarse-grid correction results in an effectively resolution-independent convergence rate, which in turn translates into significant computational economies.

Multigrid techniques were initially applied primarily to finite difference techniques, but have more recently been cast in a geometrically flexible

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variational finite element context (Maitre and Musy, 1983; Bank and Douglas, 1985). However, despite the advantages of multigrid techniques for both finite-difference and finite element discretizations, application of multigrid techniques to (nonperiodic) spectral (Gottlieb and Orszag, 1977), spectral element (Patera, 1984; Rønquist and Patera, 1987a; Maday and Patera, 1987), and $p$-type finite element methods (Babuska and Dorr, 1981) has been rather limited. Zang, Wong, and Hussaini (1982a, 1982b) have applied multigrid techniques to multidimensional Chebyshev approximations; however, the success of their techniques is primarily due to the good preconditioning properties of $h$-type incomplete decompositions (Orszag, 1980; Deville and Mund, 1985; Phillips, 1987), and not to the multigrid process *per se*.

In this paper we describe an extension of variational $h$-type multigrid to an intra-element Legendre spectral element multigrid procedure. Somewhat suprisingly, we find that resolution-independent convergence rates can be achieved with only diagonal (Jacobi) smoothers, even for large-polynomial-order ("spectral") discretizations. These highly parallelizable Jacobi-based schemes are considerably simpler and more efficient than previous incomplete-decomposition spectral multigrid procedures, and inasmuch constitute a significant advance in treating the spectrum and condition number anomalies (Gottlieb and Lustman 1983; Weideman and Trefethen 1987) typically associated with spectral and $p$-type techniques.

The outline of the paper is as follows. In Section 2 we introduce the one-dimensional model Poisson problem to be considered, and describe the associated spectral element discretization. In Section 3, the intra-element multigrid algorithm is presented, and computational complexity estimates are given for general (multidimensional) problems. In Section 4 we demonstrate the resolution-independent convergence rate of the algorithm, both by direct calculation of the multigrid spectral radius and by observation of convergence histories. Lastly, in Section 5, we comment on preliminary multidimensional results and future work.

## 2. SPECTRAL ELEMENT DISCRETIZATION

We consider in this paper the one-dimensional "Poisson" equation,

$$-u_{xx} = f, \qquad x \in \Lambda \tag{2.1a}$$

with homogeneous Dirichlet boundary conditions

$$u(-1) = u(1) = 0 \tag{2.1b}$$

Here $\Lambda$ is the interval $x \in ]-1, 1[$. Equation (2.1) shares many of the

features of more complicated multidimensional elliptic and saddle (Stokes) systems (Maday and Patera, 1987; Rønquist and Patera, 1987b), yet is sufficiently simple to allow for clear illustration of the multigrid concepts of interest.

The basis for our numerical scheme is the variational form of (2.1): Find $u \in H_0^1(\Lambda)$ such that

$$a(u, v) = (f, v) \qquad \forall v \in H_0^1(\Lambda) \tag{2.2}$$

where

$$\forall \phi, \psi \in L^2(\Lambda) \qquad (\phi, \psi) = \int_\Lambda \phi(x)\, \psi(x)\, dx \tag{2.3a}$$

$$\forall \phi, \psi \in H_0^1(\Lambda) \qquad a(\phi, \psi) = \int_\Lambda \phi_x(x)\, \psi_x(x)\, dx \tag{2.3b}$$

The function spaces $L^2(\Lambda)$ and $H_0^1(\Lambda)$ are defined by $L^2(\Lambda) = \{v \mid \int_\Lambda v^2\, dx < \infty\}$, and $H_0^1(\Lambda) = \{v \mid v \in L^2(\Lambda),\ v_x \in L^2(\Lambda),\ v(-1) = v(1) = 0\}$.

The spectral element method (Patera, 1984; Rønquist and Patera, 1987a; Maday and Patera, 1987) proceeds by first specifying the discretization pair $h = (K, N)$, and breaking up the interval $\Lambda$ into $K$ equal subintervals (spectral elements),

$$\Lambda = \bigcup_{k=1}^{K} \Lambda_k$$

where $\Lambda_k$ is defined by $a_k \leqslant x \leqslant a_k + b$. We then choose for the approximation of the solution $u$ a subspace $X_h^N$ of $H_0^1(\Lambda)$ consisting of all piecewise high-order polynomials of degree $\leqslant N$,

$$X_h^N = Y_h^N \cap H_0^1(\Lambda) \tag{2.4a}$$

where

$$Y_h^N = \bigcup_{k=1}^{K} \mathbf{P}_N(\Lambda_k) \tag{2.4b}$$

and $\mathbf{P}_N(\Lambda_k)$ is the space of all polynomials of degree $\leqslant N$ on the interval $\Lambda_k$.

The spectral element discretization corresponds to numerical quadrature of the variational form (2.2) restricted to the subspace $X_h^N$: Find $u_h \in X_h^N$ such that

$$a_{h,\mathrm{GL}}^N(u_h, v) = (f, v)_{h,\mathrm{GL}}^N \qquad \forall v \in X_h^N \tag{2.5}$$

where $(\cdot,\cdot)^N_{h,\mathrm{GL}}$ and $a^N_{h,\mathrm{GL}}(\cdot,\cdot)$ refer to Gauss–Lobatto quadrature of the inner products defined in (2.3a) and (2.3b), respectively,

$$(\phi, \psi)^N_{h,\mathrm{GL}} = \frac{b}{2} \sum_{k=1}^{K} \sum_{n=0}^{N} \rho^N_n \phi(\xi_{n,k})\, \psi(\xi_{n,k}) \tag{2.6a}$$

$$a^N_{h,\mathrm{GL}}(\phi, \psi) = \frac{b}{2} \sum_{k=1}^{K} \sum_{n=0}^{N} \rho^N_n \phi_x(\xi_{n,k})\, \psi_x(\xi_{n,k}) \tag{2.6b}$$

Here the $\xi_{n,k} = a_k + (\xi^N_n + 1)\, b/2$, $0 \leqslant n \leqslant N$, $1 \leqslant k \leqslant K$, are the locations of the local nodes $\{n, k\}$, and the $\xi^N_n$, $\rho^N_n$, $0 \leqslant n \leqslant N$, are the Gauss–Lobatto Legendre quadrature points and weights, respectively (Stroud and Secrest, 1966). It can be shown that the spectral element solution $u_h$ converges spectrally fast to the exact solution $u$ as $N \Rightarrow \infty$ for $K$ fixed, with *exponential* convergence obtaining for infinitely smooth data and solutions (Maday and Patera, 1987).

The last step in the actual implementation of (2.5) is representation of $u_h$ by a basis that reflects the sparsity and structure intrinsic to the piecewise-smooth space $X^N_h$ in (2.4). We choose an interpolant basis to represent $w_h \in X^N_h$,

$$w_h(x) = \bigcup_{k=1}^{K} \sum_{p=0}^{N} w^k_p h^N_p(r) \qquad x \in \Lambda_k \Rightarrow r \in \Lambda \tag{2.7a}$$

$$h^N_p \in \mathbf{P}_N(\Lambda), \qquad h^N_p(\xi^N_q) = \delta_{pq} \qquad \forall p, q \in \{0,\ldots, N\}^2 \tag{2.7b}$$

where $w^k_p = w_h(\xi_{p,k})$ is the value of $w_h$ at local node $\{p, k\}$, and $\delta_{pq}$ is the Kronecker-delta symbol. To honor the $H^1_0$ requirement and the essential boundary conditions (2.1b) we further require that

$$w^k_N = w^{k+1}_0 \qquad \forall k \in \{1,\ldots, K-1\} \tag{2.8a}$$

and

$$w^1_0 = w^K_N = 0 \tag{2.8b}$$

respectively.

In addition to a basis for $X^N_h$ it will also be convenient to represent functions (such as the inhomogeneity $f$) that are in $Y^N_h$ but not $X^N_h$. For $w_h \in Y^N_h$ we write

$$w_h(x) = \bigcup_{k=1}^{K} \sum_{p=0}^{N} w^{\mathbf{k}}_p h^N_p(r) \qquad x \in \Lambda_k \Rightarrow r \in \Lambda \tag{2.9}$$

with the $h_p^N$ defined as in (2.7b). As $w_h \in Y_h^N$ need not be in $H^1$ nor satisfy homogeneous boundary conditions (2.1b), we do not require the additional conditions (2.8). We will adopt the notation that local interpolants with element number superscript in bold (e.g., $w_p^{\mathbf{k}}$) will only be assumed to be in $Y_h^N$, whereas interpolants with nonbold element references (e.g., $w_p^k$) will be taken to be in $X_h^N$.

We now insert (2.7)–(2.9) into (2.5) and (2.6) to arrive at the final discrete matrix statement,

$$\sum_{k=1}^{K}{}'^{p} \sum_{q=0}^{N} A_{pq}^k u_q^k = \sum_{k=1}^{K}{}'^{p} \sum_{q=0}^{N} B_{pq}^k f_q^{\mathbf{k}} \tag{2.10}$$

where $f_q^{\mathbf{k}} = f(\xi_{q,k})$, and

$$A_{pq}^k = \frac{2}{b} \sum_{n=0}^{N} \rho_n^N D_{np} D_{nq} \qquad \forall p,q \in \{0,..., N\}^2 \tag{2.11a}$$

$$B_{pq}^k = \frac{b}{2} \rho_p^N \delta_{pq} \qquad \forall p,q \in \{0,..., N\}^2 \tag{2.11b}$$

$$D_{pq} = \frac{dh_q}{dr}(\xi_p^N) \qquad \forall p, q \in \{0,..., N\}^2 \tag{2.11c}$$

Here $\sum'^{p}$ denotes "direct stiffness" summation, in which contributions from local nodes $\{p, k\}$ that are physically coincident are summed [enforcing (2.8a)], and contributions from local nodes $\{p, k\}$ that correspond to domain boundary points (here $x = \pm 1$) are masked to zero [enforcing (2.8b)].

Although in practice spectral element iterative procedures are based entirely on elemental calculations, it is convenient for notational and theoretical purposes to define global operators based on the element matrices (2.11). To this end, we define a global representation of a function $w_h \in X_h^N$ as $\mathbf{w}$, where $\mathbf{w}$ is the vector of values of $w_h$ at all physically distinct nodes (note $\mathbf{w}$ is not defined for functions that are only in $Y_h^N$). The transformation $w_p^k \Rightarrow \mathbf{w}$ can be thought of as a mapping from local node numbers $\{p, k\}$ to global node numbers $\{q\}_G$, in which all local node numbers that reference the same nodal position map to a unique $\{q\}_G$; for instance, $\{N, k\} = \{0, k+1\} = \{\cdot\}$. In terms of the global representation, we can rewrite (2.10) as

$$\mathbf{A}\boldsymbol{u} = \mathbf{g} \tag{2.12}$$

where

$$\mathbf{A} = \sum_{k=1}^{K}{}'^{pq} A_{pq}^{k} \qquad \forall p, q \in \{0,..., N\}^2 \tag{2.13a}$$

$$\mathbf{g} = \sum_{k=1}^{K}{}'^{p} \sum_{q=0}^{N} B_{pq}^{k} f_q^{\mathbf{k}} \qquad \forall p \in \{0,..., N\} \tag{2.13b}$$

## 3. MULTIGRID ALGORITHM

### 3.1. Nested Spaces

We first define nested spaces following the variational multigrid formulation described in Bank and Douglas (1985). In particular, we consider the series of spectral element spaces,

$$X_h^{N_j} = Y_h^{N_j} \cap H_0^1(\Lambda) \qquad \forall j \in \{1,..., J\} \tag{3.1a}$$

$$Y_h^{N_j} = \bigcup_{k=1}^{K} \mathbf{P}_{N_j}(\Lambda_k) \qquad \forall j \in \{1,..., J\} \tag{3.1b}$$

with $j = 1$ the coarsest mesh and $j = J$ the finest mesh. We will typically choose $N_{j+1} \cong 2N_j$. Note that the nested spaces $X_h^{N_j}$ correspond to *intra-element* multigrid, in that the $\Lambda_k$ are invariant. This intra-element multigrid concept preserves the natural hierarchy of the spectral element method: restriction and prolongation operators and associated data structures are easily and efficiently defined within the structured $\Lambda_k$; the tight coupling within the elements is consistent with a "native" medium-grained parallelism (McBryan and van de Velde, 1985; Fischer, Rønquist, Dewey, and Patera, 1988); the coarse ($j = 1$) grid is naturally provided as a low-order (e.g., $N_1 = 1$) finite element mesh defined on the macro-elemental "skeleton."

We now introduce the elemental and system matrices for the spectral element discretizations associated with the spaces $X_h^{N_j}$:

$$a_{h,\mathrm{GL}}^{N_j}(\cdot, \cdot) \text{ on } X_h^{N_j} \Rightarrow \mathbf{A}^j \qquad \forall j \in \{1,..., J\} \tag{3.2a}$$

$$\mathbf{A}^j = \sum_{k=1}^{K}{}'^{pq} A_{pq}^{j,k} \qquad \forall j \in \{1,..., J\}, \qquad \forall p, q \in \{0,..., N_j\}^2 \tag{3.2b}$$

and

$$(f, \cdot)_{h,\mathrm{GL}}^{N_j} \text{ on } X_h^{N_j}$$

$$\Rightarrow \sum_{k=1}^{K}{}'^{p} \sum_{q=0}^{N_j} B_{pq}^{j,k} f_q^{\mathbf{k}} \quad \forall j \in \{1,..., J\}, \qquad \forall p \in \{0,..., N_j\} \tag{3.3}$$

with the $A_{pq}^{j,k}$ and $B_{pq}^{j,k}$ defined analogously to (2.11).

Next, we require a smoothing operator that is effective in eliminating the high wavenumbers on a given mesh. To this end, we use a standard Jacobi smoother,

$$\mathbf{D}^j = \max_{\chi \in X_h^{N_j}} \left[ \frac{\chi^T \mathbf{A}^j \chi}{\chi^T \operatorname{diag}(\mathbf{A}^j)\, \chi} \right] \operatorname{diag}(\mathbf{A}^j) \tag{3.4}$$

[there should be no confusion with the interpolant derivative matrix in (2.11c) as the latter is defined only elementally]. Note that the choice of (3.4) rather than the incomplete Cholesky used in previous spectral multigrid investigations (Zang, Wong, and Hussaini, 1982a, b; Phillips, 1987) has many advantages: (3.4) is very easily formed and inverted, even in complex geometry; (3.4) is completely parallelizable, unlike the incomplete decomposition. Note that the maximum eigenvalue in (3.4) can be efficiently evaluated by a power iteration method; this work is readily amortized over future iterations, in particular for time-dependent problems or nested-elliptic Stokes solvers (Maday and Patera, 1987; Rønquist and Patera, 1987b).

We then need to define a restriction operator that takes a residual on $X_h^{N_j}$ and projects it onto $X_h^{N_{j-1}}$. To effect this projection we follow the usual procedure of representing the lower-order basis in terms of the higher-order basis, which gives the elemental operator

$$R_{pq}^{j,k} = h_p^{N_j}(\xi_q^{N_{j+1}}), \qquad \forall j \in \{1,\ldots, J-1\}, \qquad \forall k \in \{1,\ldots, K\},$$
$$\forall p \in \{0,\ldots, N_j\}, \qquad \forall q \in \{0,\ldots, N_{j+1}\} \tag{3.5}$$

Given the elemental residuals $\hat{r}_p^{j+1,\mathbf{k}} \in Y_h^{N_{j+1}}$

$$\hat{r}_p^{j+1,\mathbf{k}} = w_p^{j+1,k} - \sum_{q=0}^{N_{j+1}} A_{pq}^{j+1,k} z_q^{j+1,k} \qquad \forall k \in \{1,\ldots, K\},$$
$$\forall p \in \{0,\ldots, N_{j+1}\} \tag{3.6a}$$

for some data $w_p^{j+1,k}$ and iterate $z_q^{j+1,k}$, the restricted global residual (which will enter as the inhomogeneity at the next multigrid level) is then calculated as

$$\mathbf{r}^j = \sum_{k=1}^{K}{}' \sum_{q=0}^{N_{j+1}} R_{pq}^{j,k} \hat{r}_q^{j+1,\mathbf{k}} \qquad \forall p \in \{0,\ldots, N_j\} \tag{3.6b}$$

It may not be readily apparent from (3.6b) that if $\mathbf{r}^{j+1} = \sum_k' \hat{r}_p^{j+1,\mathbf{k}} = \mathbf{0}$ then $\mathbf{r}^j = \mathbf{0}$, which is certainly a necessary condition for multigrid convergence. However, from the simple property of the restriction operator,

$$R_{p0}^{j,k} = \delta_{p0} \qquad \forall p \in \{0,\ldots, N_j\} \tag{3.7a}$$

$$R_{pN_{j+1}}^{j,k} = \delta_{pN_j} \qquad \forall p \in \{0,\ldots, N_j\} \tag{3.7b}$$

it follows that (3.6b) is equivalent to the global statement

$$\mathbf{r}^j = \mathbf{R}^j \mathbf{r}^{j+1} \tag{3.8}$$

where

$$\mathbf{R}^j = \sum_{k=1}^{K}{}' {}^{pq} \frac{1}{1+(\delta_{0q}+\delta_{N_{j+1}q})} R_{pq}^{j,k} \qquad \forall p \in \{0,\ldots,N_j\}, \quad \forall q \in \{0,\ldots,N_{j+1}\} \tag{3.9}$$

We conclude from (3.8) that if $\mathbf{r}^{j+1} = \sum_k' {}^p \hat{r}_p^{j+1,\mathbf{k}} = \mathbf{0}$, then the restriction defined by (3.6b) will also vanish.

Lastly, we require a prolongation operator that, given a $\mathbf{z}^j \in X_h^{N_j}$, extends the function $\mathbf{z}^j$ onto $X_h^{N_{j+1}}$. It can readily be seen from the definition of (3.5) that the correct choice for prolongation is

$$z_p^{j+1,k} = \sum_{q=0}^{N_j} R_{qp}^{j,k} z_q^{j,k} \qquad \forall k \in \{1,\ldots,K\}, \qquad \forall p \in \{0,\ldots,N_{j+1}\} \tag{3.10}$$

which is again defined on an elemental basis. Note that it is by virtue of (3.7) that if $\mathbf{z}^j \in X_h^{N_j}$ then $\mathbf{z}^{j+1} \in X_h^{N_{j+1}}$. The global statement of (3.10) is given by

$$\mathbf{z}^{j+1} = (\mathbf{R}^j)^T \mathbf{z}^j \tag{3.11}$$

where it is seen that, as is usually the case in variational multigrid, the prolongation operator is the transpose of the restriction operator.

### 3.2. Multigrid *V*-Cycle

We now consider a simple *V*-cycle for solution of

$$\mathbf{A}^J \mathbf{u}^J = \mathbf{g}^J \tag{3.12}$$

which corresponds to (2.12) on the finest mesh $X_h^{N_J}$. We first set the *V*-cycle iteration counter to $l=0$, and choose an initial guess $u^{J,(0)}$. At the highest level, we then perform

$$* \ \mathrm{MG}(J, \mathbf{u}^{J,(l)}, \mathbf{g}^J, m)$$

$$\mathbf{u}^{J,(l+1)} = \mathbf{u}^{J,(l)}$$

$$l = l+1; \quad \text{go to } *$$

Here $\mathrm{MG}(j, \mathbf{z}, \mathbf{w}, m)$ is our multigrid algorithm [a simplified form of the procedure in Bank and Douglas (1985)], and $2m$ is the number of smoothings on level $j$ per *V*-cycle.

The MG algorithm is readily defined in terms of the operators of Section 3.1. Although in practice all operations are performed elementally (as described in the next section), for succinctness we describe the algorithm in terms of global operators. Any operators or vectors for which the multigrid level is not indicated may be assumed to be at level $j$.

*Algorithm* $MG(j, \mathbf{z}, \mathbf{w}, m)$:
If $j = 1$ solve $\mathbf{Az} = \mathbf{w}$
If $j \neq 1$

(a) Form residual: $\mathbf{r} = \mathbf{w} - \mathbf{Az}$

(b) Smooth: $\mathbf{z}^{(0)} = \mathbf{z}$, $\mathbf{r}^{(0)} = \mathbf{r}$

$$\left.\begin{aligned} \mathbf{Dz}^{(n+1)} &= \mathbf{Dz}^{(n)} + \mathbf{r}^{(n)} \\ \mathbf{r}^{(n+1)} &= \mathbf{w} - \mathbf{Az}^{(n+1)} \end{aligned}\right] n = 0,\ldots, m-1$$

$$\mathbf{z} = \mathbf{z}^{(m)}, \qquad \mathbf{r} = \mathbf{r}^{(m)}$$

(c) Restrict residual: $\mathbf{w}^{j-1} = \mathbf{R}^{j-1}\mathbf{r}^{j}$

(d) $\mathbf{e}^{j-1} = \mathbf{0}$; $MG(j-1, \mathbf{e}^{j-1}, \mathbf{w}^{j-1}, m)$

(e) Prolong correction: $\mathbf{z} = \mathbf{z} + (\mathbf{R}^{j-1})^{T}\mathbf{e}^{j-1}$

(f) Form residual: $\mathbf{r} = \mathbf{w} - \mathbf{Az}$

(g) Smooth: $\mathbf{z}^{(0)} = \mathbf{z}$, $\mathbf{r}^{(0)} = \mathbf{r}$

$$\left.\begin{aligned} \mathbf{Dz}^{(n+1)} &= \mathbf{Dz}^{(n)} + \mathbf{r}^{(n)} \\ \mathbf{r}^{(n+1)} &= \mathbf{w} - \mathbf{Az}^{(n+1)} \end{aligned}\right] \begin{aligned} n &= 0,\ldots, m-1 \\ n &= 0,\ldots, m-2 \end{aligned}$$

$$\mathbf{z} = \mathbf{z}^{(m)}$$

(h) End.

The MG algorithm is readily implemented using recursive programming techniques.

### 3.3. Computational Complexity

Our initial tests indicate that multidimensional spectral element multigrid performs in a similar fashion to our one-dimensional model problem, and we therefore present operation counts for a general (rectilinear-element) problem in $\mathbf{R}^{d}$. As regards computational complexity the salient modifications to the one-dimensional spectral element procedure described in the previous sections are as follows (Maday and Patera, 1987). First, the $\mathbf{P}_N$ in $X_h^{N_j}$ in (3.1) now refer to polynomials of order less than or equal to $N$ in *each* of the $d$ space dimensions. Second, the one-dimensional Gauss–Lobatto quadrature is replaced by a $d$-dimensional tensor-product Gauss–Lobatto quadrature rule. Third, the bases for the multidimensional

spaces are taken to be tensor-product extensions of (2.7); for instance, in $\mathbf{R}^3$, $w_h \in X_h$ is given by

$$w_h(x, y, z) = \bigcup_{k=1}^{K} \sum_{l=0}^{N} \sum_{p=0}^{N} \sum_{q=0}^{N} w_{lpq}^k h_l^N(r)\, h_p^N(s)\, h_q^N(t) \tag{3.13}$$

where $r$, $s$, and $t$ are the local coordinates corresponding to $x$, $y$, and $z$, respectively.

The reason for choosing the tensor product spaces, quadratures, and bases is to allow for rapid evaluation of the discrete Laplacian and restriction operators by sum factorization. The discrete Laplacian in $\mathbf{R}^3$ on mesh $j$ is given by

$$\sum_k{}' \sum_{lpq} \tilde{A}_{\alpha\beta\gamma lpq}^{j,k} u_{lpq}^{j,k} = \sum_k{}^{K\prime} \sum_{l}^{N_j} \sum_{p}^{N_j} \sum_{q}^{N_j} \{ A_{\alpha l}^{j,k} B_{\beta p}^{j,k} B_{\gamma q}^{j,k} + B_{\alpha l}^{j,k} A_{\beta p}^{j,k} B_{\gamma q}^{j,k} + B_{\alpha l}^{j,k} B_{\beta p}^{j,k} A_{\gamma q}^{j,k} \}\, u_{lpq}^{j,k}$$
$$\forall \alpha\beta\gamma \in \{0, \ldots, N_j\}^3 \tag{3.14a}$$

The sum (3.14a) appears to require $O(KN^{2d})$ operations, but in fact can be evaluated by sum factorization (Orszag, 1980) in

$$W_A^j = K(2d)\, N_j^{d+1} \tag{3.14b}$$

operations. (For simplicity we include only multiplications in our operation count.) Note that we have included in (3.14b) only the leading-order terms; for instance, direct stiffness summation requires only $O(N_j^{d-1})$ operations, which is generally negligible compared to the elemental matrix multiplies. [The factor of 2 in (3.14b) is, in fact, present only in nonrectilinear geometry; however, we retain it here for generality.]

The multidimensional restriction (and hence prolongation) operator is treated in a similar fashion to the Laplacian. That is, we write the operator in tensor-product sum-factorized form (here in $\mathbf{R}^3$),

$$\sum_k{}' \sum_{lpq} \tilde{R}_{\alpha\beta\gamma lpq}^{j,k} \hat{r}_{lpq}^{j+1,\mathbf{k}} = \sum_{k=1}^{K}{}' \left\{ \sum_{l=0}^{N_{j+1}} R_{\alpha l}^{j,k} \left[ \sum_{p=0}^{N_{j+1}} R_{\beta p}^{j,k} \left( \sum_{q=0}^{N_{j+1}} R_{\gamma q}^{j,k} \hat{r}_{lpq}^{j+1,\mathbf{k}} \right) \right] \right\}$$
$$\forall \alpha\beta\gamma \in \{0, \ldots, N_j\}^3 \tag{3.15a}$$

where the $R_{pq}^{j,k}$ are defined in (3.5). The leading-order operation count for restriction of the residual from mesh $j$ to mesh $j-1$ is then

$$W_R^j = K \left[ \sum_{n=1}^{d} \left(\tfrac{1}{2}\right)^n \right] N_j^{d+1} \tag{3.15b}$$

where we have again neglected the $O(N_j^{d-1})$ direct stiffness summation. Note that the work to evaluate the restriction is less than that for the Laplacian, as the size of the problem shrinks as each directional sum is performed.

Armed with the above definitions we can now readily evaluate the work required per $V$-cycle on mesh $j$ as

$$W_V^j = \{(2m+1)\, W_A^j + 2W_R^j + \cdots) \qquad \forall j \in \{2,\ldots, J\} \tag{3.16}$$

where the first term derives from steps (a), (b), (f), and (g) of MG($j, \cdot, \cdot, m$), while the second term derives from steps (c) and (e). We have again neglected lower-order terms in $N_j$, such as direct stiffness summations which are $O(KN_j^{d-1})$, and inversions of the diagonal smoother which are $O(KN_j^d)$. Note that in previous spectral multigrid schemes the work associated with the incomplete decomposition smoother could not be neglected, in particular on nonserial architectures.

To arrive at a simple expression for the total work per $V$-cycle we make three approximations. First, we note that for all $m \geqslant 1$ and all $d \geqslant 1$, $2W_R^j < \frac{1}{6}(2m+1)\, W_A^j$, and we therefore neglect the restriction and prolongation work. Second, we note that for spectral schemes the coarse-mesh work is relatively less important than for $h$-type methods; $W_V^{j-1}/W_V^j \cong (N_{j-1}/N_j)^{d+1} \cong (1/2)^{d+1}$, which is reasonably small for $d=1$, and very small for $d=3$. We therefore neglect the work on meshes $2 \leqslant j \leqslant J-1$. Lastly, we neglect the exact-solution work on the coarsest mesh $j=1$ to arrive at

$$W_V \cong (2m+1)\, W_A^J \tag{3.17}$$

The validity of the last assumption is discussed further in the next section.

## 4. CONVERGENCE RATE OF THE MULTIGRID ALGORITHM

### 4.1. $J=2$ Multigrid Spectral Radius

We consider here the spectral radius of the multigrid iteration matrix corresponding to the $J=2$ (two-grid) algorithm applied to our one-dimensional Poisson problem. Defining the iteration error $\mathbf{e}^{(l)}$ as

$$\mathbf{e}^{(l)} = \mathbf{u}^J - \mathbf{u}^{J,(l)} \tag{4.1}$$

it can readily be shown (Bank and Douglas, 1985) that for the $J=2$ algorithm

$$\mathbf{L}^T\mathbf{e}^{(l+1)} = \mathbf{M}\mathbf{L}^T\mathbf{e}^{(l)} \tag{4.2a}$$

where

$$\mathbf{M} = (\mathbf{I} - \mathbf{L}^T[\mathbf{D}^J]^{-1}\mathbf{L})^m \cdot (\mathbf{I} - \mathbf{L}^T[\mathbf{R}^{J-1}]^T[\mathbf{A}^{J-1}]^{-1}\mathbf{R}^{J-1}\mathbf{L}) \cdot (\mathbf{I} - \mathbf{L}^T[\mathbf{D}^J]^{-1}\mathbf{L})^m \tag{4.2b}$$

All operators in (4.2) have been defined previously save $\mathbf{L}$, $\mathbf{L}^T$, which are defined by the Cholesky decomposition (or any other choice of square root) of the positive definite symmetric matrix $\mathbf{A}^J$,

$$\mathbf{A}^J = \mathbf{L}\mathbf{L}^T \tag{4.3}$$

It follows from (4.2) that

$$|||\mathbf{e}^{(l+1)}||| \leqslant \rho(\mathbf{M})\, |||\mathbf{e}^{(l)}||| \tag{4.4a}$$

where $|||\mathbf{e}^{(l)}|||$ is the $\mathbf{A}^J$-norm of $\mathbf{e}^{(l)}$,

$$|||\mathbf{e}^{(l)}||| = (\mathbf{e}^{(l)T}\mathbf{A}^J\mathbf{e}^{(l)})^{1/2} \tag{4.4b}$$

and $\rho(\mathbf{M})$ is the spectral radius of $\mathbf{M}$. Lastly, from (3.17) we know that the total work is best represented in terms of number of $\mathbf{A}^J$ matrix-vector products, and we thus define a new iteration counter $l' = (2m+1)\,l$, for which

$$|||\mathbf{e}^{(l'+1)}||| \leqslant \bar{\rho}\, |||\mathbf{e}^{(l')}||| \tag{4.5a}$$

where

$$\bar{\rho} = [\rho(\mathbf{M})]^{1/(2m+1)} \tag{4.5b}$$

The "work-deflated" spectral radius $\bar{\rho}$ allows us to compare various multigrid schemes while keeping the computational work constant.

In Table I we present $\bar{\rho}$ for various $m$, $K$, and $N_J$ (with $N_2 \cong 2N_1$ in all cases). Also indicated is the condition number of $\mathbf{A}^J$, $\kappa$, for each $h = (K, N_J)$. The most important fact to note from Table I is that the multigrid convergence rate $\bar{\rho}$ is *bounded from above well below unity*, and is *sensibly independent of K and* $N_J$. Furthermore, there is an optimal number of smoothings which appears to be on the order of $m = 2$ or $m = 3$. These conclusions are somewhat tempered for the $K = 1$ "pure spectral" case: here there is a more noticeable increase in $\bar{\rho}$ for larger $N_J$ (although there is clearly an asymptote below unity), and there is no optimal number of smoothings.

The results of Table I indicate that, despite the existence of high-wavenumber outliers (Weideman and Trefethen, 1987) in the spectral element operator $\mathbf{A}$ [resulting in a condition number $\kappa$ that scales as

$\kappa \sim O(K^2N^3)$], the diagonal preconditioner is nevertheless sufficient to smooth all the modes on $X_h^{N_{j+1}} \cap \bar{X}_h^{N_j}$ [$\bar{X}$ denotes the complement of $X$ in $H_0^1(\Lambda)$], and ensure that the definition of "coarse" on $X_h^{N_{j+1}}$ and "fine" on $X_h^{N_j}$ is, in some sense, the same.

## 4.2. $J > 2$ Convergence Histories

Our work estimate (3.17) assumes that the exact-solution work on the $j = 1$ mesh is negligible. This will clearly only be true if $N_1 \sim O(1)$, which in

**Table I.** Table of the Work-Deflated $J = 2$ Multigrid Matrix Spectral Radius $\bar{\rho}$ for Various Values of $K$, $N_J$, and Smoothings $m$[a]

| $m \backslash N_2 \mid N_1$ | 8 \| 4 | 12 \| 6 | 16 \| 8 | 19 \| 10 | 41 \| 19 |
|---|---|---|---|---|---|
| | | | $K = 1$ | | |
| 1 | 0.745 | 0.775 | 0.788 | 0.772 | 0.839 |
| 2 | 0.702 | 0.736 | 0.752 | 0.733 | 0.810 |
| 3 | 0.685 | 0.720 | 0.737 | 0.717 | 0.798 |
| 4 | 0.675 | 0.711 | 0.728 | 0.708 | 0.791 |
| 5 | 0.669 | 0.706 | 0.723 | 0.703 | 0.787 |
| 10 | 0.657 | 0.694 | 0.712 | 0.691 | 0.778 |
| $\kappa$ | 35 | 103 | 232 | 381 | 3630 |
| | | | $K = 4$ | | |
| 1 | 0.759 | 0.779 | 0.790 | 0.773 | |
| 2 | 0.718 | 0.741 | 0.754 | 0.734 | |
| 3 | 0.701 | 0.725 | 0.739 | | |
| 4 | 0.709 | 0.720 | 0.730 | | |
| 5 | 0.727 | 0.733 | 0.738 | | |
| 10 | 0.791 | 0.788 | 0.787 | | |
| $\kappa$ | 1151 | 3665 | 8469 | 14023 | |
| | | | $K = 8$ | | |
| 1 | 0.760 | 0.779 | 0.790 | | |
| 2 | 0.719 | 0.741 | 0.754 | | |
| 3 | 0.702 | 0.726 | 0.739 | | |
| 4 | 0.710 | | | | |
| 5 | 0.731 | | | | |
| 10 | 0.794 | | | | |
| $\kappa$ | 4603 | 14622 | 33828 | | |

[a] The condition number of the **A** matrix, $\kappa$, is also given for each case. The convergence rate is extremely insensitive to $h = (K, N_J)$, in particular for $K > 1$ and moderate $N_J$.

turn requires that several multigrid levels be used, $J > 2$. In this section we demonstrate that the convergence rates of Table I for $J = 2$ are also indicative of performance for $J > 2$. In particular, we present convergence histories for a problem (2.1) in which the inhomogeneity $f$ is chosen such that the solution $u$ is given by

$$u = \tfrac{1}{10} e^{8(x-1)} \sin(10\pi x) \tag{4.6}$$

Note that $u$ contains sufficient high and low wave numbers so that neither Jacobi iteration nor coarse-grid solution alone is sufficient to provide rapid convergence. All results presented are for the particular (but representative) discretization $K = 8$, $N_J = 12$ (with $N_{j+1} = 2N_j$).

In Fig. 1 we plot $|||\mathbf{e}^{(l')}|||$ as a function of $l'$ for the $J = 2$ and $J = 3$ multigrid algorithms; for the purposes of convergence histories $|||\mathbf{e}^{(l')}|||$ is interpreted as the error after each fine mesh smoothing. We make two comments concerning Fig. 1. First, the spectral radius $\bar{\rho}$ for the $J = 2$ case is a very good predictor of actual performance. Second, the $J = 3$ convergence history is only slightly worse than the $J = 2$ case, suggesting that our

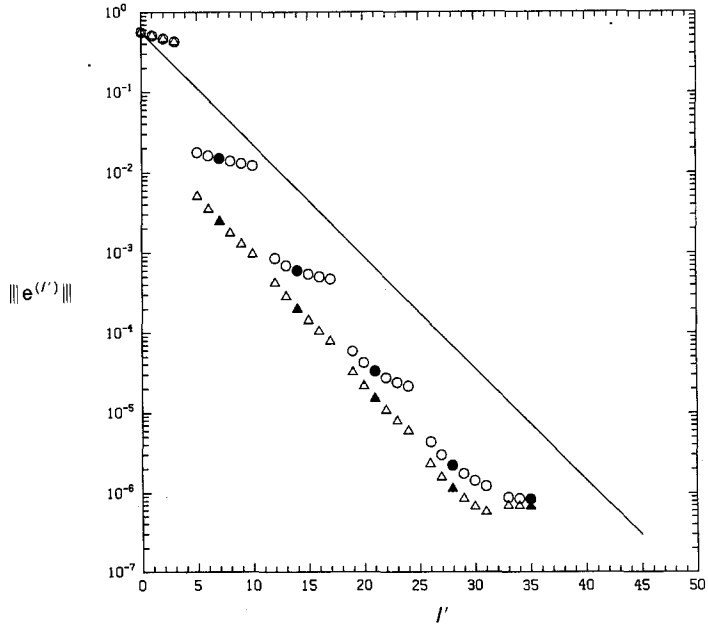


**Fig. 1.** A convergence history plot of $|||\mathbf{e}^{(l')}|||$ vs. $l'$ for the $J = 2$ (△) and $J = 3$ (○) multigrid schemes with $K = 8$, $N_J = 12$, and $m = 3$ smoothings. Here $|||\mathbf{e}^{(l')}|||$ is interpreted as the iteration error after each fine-mesh smoothing; the solid symbols (▲, ● for $J = 2, 3$, respectively) indicate the end of each $V$-cycle. The solid line represents an upper bound on the $J = 2$ iteration error based on $\bar{\rho} = 0.726$.

multigrid convergence rates will, indeed, be insensitive to the number of multigrid levels for moderate $J$.

In Fig. 2 we compare the convergence history of the $J=3$ multigrid scheme with several (admittedly straw-man) competitors, namely, pure (smoothing) Jacobi and unpreconditioned conjugate gradient iteration. The savings due to multigrid are impressive even for this relatively small problem; given the insensitivity of $\bar{\rho}$ to resolution, savings should become increasingly dramatic as the problem size grows.

### 4.3. Conclusions on Convergence Rate and Computational Effort

We first compare our spectral element results with past work on variational finite element multigrid, and nonvariational spectral multigrid. As regards the former, we appear to achieve multigrid convergence rates comparable to those predicted and observed for variational $h$-type finite element techniques, even though the proofs of Bank and Douglas (1985) do not directly apply to our "nonuniform" spectral approximations. As regards the latter, we achieve convergence rates (even for $K=1$)

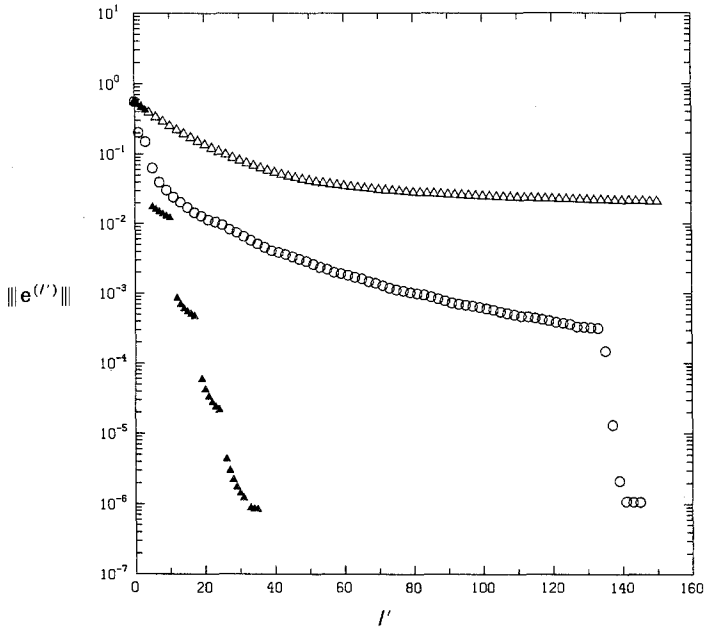


**Fig. 2.** A convergence history plot of $\|\mathbf{e}^{(l')}\|$ vs. $l'$ for the $J=3$ multigrid scheme with $m=3$ smoothings (▲), as compared to (damped) Jacobi iteration (△), and unpreconditioned conjugate gradient iteration (○). The final rapid convergence of the conjugate gradient iteration is due to the fact that all of the modes of the system have been exhausted.

comparable to those of Zang, Wong, and Hussaini (1982a, b), but we do so without the computational complexity associated with incomplete-decomposition smoothers. Whether simple Jacobi smoothing will also be effective for "nonvariational" (e.g., Chebyshev) spectral multigrid is not yet known; however, the requisite numerical experiments are simple to perform [and are, in fact, suggested in Zang, Wong, and Hussaini (1982a)].

From the results of Table I for $J=2$, and Fig. 1 for $J>2$, we conclude that for virtually any spectral element discretization $h=(K, N_J)$ the multigrid work-deflated convergence rate is approximately constant, and the computational effort is roughly $O(KN_J^{d+1})$. The one caveat to this conclusion is our weakly motivated assumption in Section 3.3 that the coarse-mesh ($j=1$) work can be neglected; in fact, this will only be true for $K$ that are in some sense small. For instance, if we effect an $h-p$ refinement ($K \Rightarrow \infty$, $N \Rightarrow \infty$) (Babuska and Dorr, 1981), rather than a $p$ refinement ($K$ fixed, $N \Rightarrow \infty$), it is probable that our $O(KN_J^{d+1})$ estimate will be overly optimistic unless *inter*-element multigrid is considered.

If follows from these estimates of computational complexity that if the solution to a partial differential equation is sufficiently smooth, and if the discretization error required is sufficiently small, then spectral element methods will be more efficient of computational resources than low-order $h$-type alternatives. The key contribution of the multigrid algorithm is that it successfully addresses the conditioning problem that typically plagues iterative solution of spectral discretizations.

## 5. FUTURE WORK

It is clear that the results in this paper should be viewed as preliminary until more tests have been performed on multidimensional problems in complex geometry. In particular, the viability of the Jacobi preconditioner for the "fuller" multidimensional operators must be verified. However, initial tests in two space dimensions indicate that, although $\bar{\rho}$ for two-dimensional problems ($d=2$) is slightly larger than $\bar{\rho}(d=1)$, $\bar{\rho}(d=2)$ is nevertheless bounded well below unity, and shares the resolution independence of the one-dimensional case. We therefore tentatively conclude that in higher dimensions the results of the current paper will be only slightly modified.

Once the multidimensional behavior is verified, it remains to perform the usual multigrid parametric investigations to determine the effect of non-constant coefficients, aspect ratio, and solution smoothness on convergence rate. Lastly, it is important to place the algorithm on firm theoretical ground by appropriate extension of variational multigrid theory; this work is, in fact, complete, and will appear as Part II of this paper (Maday and Munoz, 1988).

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