Econometrics of Commodities and Asset Pricing

The Stochastic Behavior of Commodity Prices : Implications for Valuation and Hedging

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Contents

1 Introduction

The article aims to compare three different models involving a stochastic behavior of commodity prices. These models take into account mean reversion and are used to price futures contracts. Parameters estimation is realized through the Kalman Filter methodology. Estimated parameters are then used to analyze the implication of the models for the term structure of futures prices and volatilities beyond the observed contracts, and for heding contracts for future delivery. Implications of the models for capital budgeting decisions are analyzed in a last time.

The three models share a convenient characteristic: their structures conduce to closed form solutions for the prices of futures/forward contracts, simplying their comparison in terms of comparative statics and estimation. The common main difficulty is that the factors of these models are not directly observable. As the spot price of a commodity is sometimes very uncertain, associated futures contracts closest to maturity can be used as a proxy for the spot price, because their prices are more easily observed. When state variables are not directly observable but are generated by a Markov Chain, the so called Kalman Filter is to be used in order to estimate the parameters of the model, and the time series of the unobservable state variables.

Continuer l'introduction en résumant pages 924-925 qui abordent les autres problématiques de l'article. Proposer également une explication du filtre de Kalman.

- Rediger une introduction et la structure de l'article
- Rédiger la partie sur les modèles
- Rédiger un résumé sur le filtre de Kalman et lire la partie sur l'estimation

2 State of the art

- Vasicek (1977): used for the third model: stochastic interest rates with mean reversion.
- Real options approach o investment under uncertainty (For an excellent recent survey: Dixit and Pindick (1994)).
- The valuation of natural resource investment projects and the rule for determining when it is optimal to invest depend significantly on the stochastic process assumed for the underlying commodity price (Ingersell and Ross, 1992)
- Cox (1961,1962) for comparing non nested models and also Atkinson (1970), Pesaran and Deaton (1978) and Davidson and MacKinnon (1981).
- Brenan and Cew (1995), Culp and Miller (1994,1995), Edwards and Canter (1995) and Ross (1995) : feasability of hedging long term forward commitments in commodities with the existing short term futures contracts.

3 Models and estimation.

This section is dedicated to the presentation of the three models and their estimation procedure. Closed form formulas are given for futures contracts prices, making the estimation procedure easier. **A quick discussion comparing their characteristics is also provided**.

However, commodities present a specificity: spot prices are often hard to obtain because they are very uncertain. Hence, futures prices are used to play the role of a proxy, because they are more easily observable. This problem is common for the three models developed. The same problem occurs regarding the convenience yield: hence, two futures prices with different maturities are used to compute this convenience yield. Observing several exchanges also allows to determine the instantaneous interest rate, which is also unobservable.

The Kalman Filter [1] procedure is used to estimate the models, consisting in putting the model in a state space form. It should be applied to a time series of observable variables (futures prices for different maturities) linked to an unobservable vector of state variables (the spot price with or without the convenience yield). The link is called *measurement equation*. Regarding unobservable variables, they should be generated via *the transition equation*. In our case, dynamics of unobservable variables are given by stochastic differential equations. Hence, a good candidate for the transition equation is a discrete time version of the SDEs.

3.1 Model 1: Ornstein-Uhlenbeck type.

Model derivation.
Let us assume that the commodity price (S) is characterized by the following dynamics:

$$\frac{dS}{S} = \kappa(\mu - \ln S)dt + \sigma dz$$

Defining $X := \ln S$, Itô's Lemma gives [2]: $dX = \kappa(\alpha - X)dt + \sigma dz$ where $\alpha = \mu - \frac{\sigma^2}{2\kappa}$. Note that $\kappa > 0$ is the degree of mean reversion to the long run mean log price α. σ is a volatility parameter and z an increment of a standard Brownian motion. Under the convenient martingale measure, this dynamics can be rewritten: $dX = \kappa(\alpha^* - X)dt + \sigma dz^*$ where $\alpha^* = \alpha - \lambda$, λ stating for the risk premium. The associated mean and variance [3] are: $\mathbb{E}_0\left[X(T)\right] = e^{-\kappa T}X(0) + \left(1 - e^{-\kappa T}\right)\alpha^*$ and $V_0[X(T)] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T})$. Note that the spot price of the commodity at time T is log-normally distributed under the martingale measure as $X = \ln S$. Assuming a constant interest rate (hypothesis relaxed in Model 3), properties of the log-normal distribution imply:

$$F(S,T) = \mathbb{E}[S(T)] = e^{\mathbb{E}_0[X(T)] + \frac{1}{2}V_0[X(T)]} = \exp\left[e^{-\kappa T}\ln S + (1 - e^{-\kappa T})\alpha^* + \frac{\sigma}{4\kappa}(1 - e^{-2\kappa T})\right]$$

Which implies the useful equation:

$$\ln F(S,T) = \left[e^{-\kappa T}\ln S + (1 - e^{-\kappa T})\alpha^* + \frac{\sigma}{4\kappa}(1 - e^{-2\kappa T})\right]$$

Estimation procedure. From this last equation, the measurement equation writes down:

$$y_t := [\ln F(T_i)] = Z_t X_t + \varepsilon_t \qquad t = 1, ..., NT \quad i = 1, ..., N$$

Where $d_t = \left[\left(1 - e^{-\kappa T_i})\alpha^* + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T_i})\right)\right]$ and $Z_t = [e^{-\kappa T_i}]$ and ε_t is the uncorrelated error term serie given by $\mathbb{E}[\varepsilon_t] = 0$ and $V(\varepsilon_t) = H$. Note that these vectors are $N \times 1$ vectors.
The transition equation is given by discretizing $dX = \kappa(\alpha - X)dt + \sigma dz$:

$$X_t = c_t + Q_t X_{t-1} + \eta_t \iff X_t = \kappa\alpha\Delta t + (1 - \kappa\Delta t)X_{t-1} + \sigma\sqrt{\Delta t}\mathcal{N}(0,1) \qquad t = 1, ..., NT$$

With $c_t = \kappa\alpha\Delta t$ and $Q_t = 1 - \kappa\Delta t$. η_t is characterized by $\mathbb{E}[\eta_t] = 0$ and $V[\eta_t] = \sigma^2\Delta t$.

3.2 Model 2: a two-factor model.

Model estimation.
This model is a little bit more sophisticated as it also involves a stochastic dynamics for the convenience yield, defined as the flow of services accruing to the holder of the spot commodity but not to the owner of a futures contract. It is a two factor model (the spot price and the convenience yield). The factors follow a joint stochastic process:

$$\frac{dS}{S} = (\mu - \delta)dt + \sigma_1 dz_1 \qquad d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2$$

[1] Definition: the Kalman filter is a recursive procedure for computing the optimal estimator of the state variable at time t, based on the information available at time t, and it enables the estimate of the state vector to be continuously updated as new information becomes available. It should be specified that when the disturbance and the initial state vector are normal, the procedure enables the likelihood function to be calculated.

[2] Proof in Appendix 1.

[3] Proof in Appendix 1.

The increments of the Brownian motions z_1 and z_2 are correlated with: $dz_1 dz_2 = \rho dt$. Let us apply $X = \ln S$ and Itô to get $dX = (\mu - \delta - \frac{1}{2}\sigma_1^2 dt) + \sigma_1 dz_1$. This model considers the commodity as an asset that pays a stochastic dividend yield δ. As a result, the risk adjusted drift of the commodity is $r - \delta$. It should be noted that since convenience yield risk cannot be hedged, the risk-adjusted convenience yield process will have a market price of risk associated with it. Hence, under the equivalent martingale measures, one has:

$$\frac{dS}{S} = (r - \delta)dt + \sigma_1 dz_1^* \qquad d\delta = [\kappa(\alpha - \delta) - \lambda]dt + \sigma_2 dz_2^* \qquad dz_1^* dz_2^* = \rho dt$$

A closed form solution exists and is:

$$F(S, \delta, T) = S \exp\left[-\delta \frac{1 - e^{-\kappa T}}{\kappa} + A(T)\right] \iff \ln F(F, \delta, T) = \ln S - \delta \frac{1 - e^{-\kappa T}}{\kappa} + A(T)$$

$A(T)$ is a constant.

Estimation procedure. Based on the same principle as before, the closed form solution conduces to the measurement equation:

$$y_t = [\ln F(T_i)] = d_t + Z_t[X_t, \delta_t]' + \varepsilon_t = [A(T_i)] + Z_t[X_t, \delta_t]' + \varepsilon_t \qquad t = 1, ..., NT \quad i = 1, ..., N$$

These vectors are $N \times 1$ at the exception of Z_t which is $N \times 2$. The error term ε_t is distributed with $\mathbb{E}(\varepsilon_t) = 0$ and $V(\varepsilon_t) = H$.
By discretizing the previous relations, we get the transition equation:

$$[X_t, \delta_t]' = c_t + Q_t[X_{t-1}, \delta_{t-1}]' + \eta_t \qquad t = 1, ..., NT$$

Where $c_t = \left[\left(\mu - \frac{\sigma_1^2}{2}\right)\Delta t, \kappa\alpha\Delta t\right]$ is a 2×1 vector.

$$\mathrm{Q_t} = \begin{vmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa\Delta t \end{vmatrix}$$

We also have η_t characterized by $\mathbb{E}[n_t] = 0$ and :

$$\mathrm{V}(\eta_\mathrm{t}) = \begin{vmatrix} \sigma_1^2\Delta t & \rho\sigma_1\sigma_2\Delta t \\ \rho\sigma_1\sigma_2\Delta t & \sigma_1^2\Delta t \end{vmatrix}$$

3.3 Model 3: a three-factor model with stochastic interest rates.

Model derivation.
This model is an extension of the second one, assuming the interest rate is stochastic (as in Vasicek Model). Under the convenient martingale measure, one has:

$$\frac{dS}{S} = (r - \delta)dt + \sigma_1 dz_1^* \qquad d\delta = \kappa(\hat{\alpha} - \delta)dt + \sigma_2 dz_2^* \qquad dr = a(m^* - r)dt + \sigma_3 dz_3^*$$

Where $\hat{\alpha} = \alpha - \lambda/\kappa$, $dz_1^* dz_2^* = \rho_1 dt$, $dz_2^* dz_3^* = \rho_2 dt$ and $dz_1^* dz_3^* = \rho_3 dt$.

The closed form solution of the model writes down:

$$F(S, \delta, r, T) = S \exp\left[\frac{-\delta(1 - e^{-\kappa T})}{\kappa} + \frac{r(1 - e^{-aT})}{a} + C(T)\right] \implies \ln F(S, \delta, r, T) = \ln S - \frac{\delta(1 - e^{-\kappa T})}{\kappa} + \frac{r(1 - e^{-aT})}{a} + C(T$$

Where $C(T)$ is a constant.
It is important to have in mind that interest rates are stochastic in this model: hence, futures prices are not equal to forward ones. The price (or present value) of a forward commitment is:

$$P(S, \delta, r, T) = S \exp\left[\frac{-\delta(1 - e^{-\kappa T})}{\kappa} + D(T)\right]$$

Dividing by the expression of zero-coupon from Vasicek gives the forward price. **Voir la partie sur les forward** **Estimation procedure.** A simplified version of the model is estimated. Instead of estimating the three stochastic processes simultaneously, we first estimate the parameters of the interest rate process and then use it to estimate the parameters of the spot price and convenience yield processes. A credible assumption is made: the parameters of the interest rate process are not affected by commodity future prices.

Once the interest rate parameters have been estimated, we can move on to the measurement equation:

$$y_t = [\ln F(T_i)] = d_t + Z_t[X_t, \delta_t]' + \varepsilon_t \qquad t = 1, ..., NT$$

Where $d_t = \left[\frac{r_t(1-e^{-aT_i})}{a} + C(T_i)\right]$, $i = 1, ..., N$, $N \times 1$ vector and $Z_t = \left[1, -\frac{1-e^{-\kappa T_i}}{\kappa}\right]$ is $N \times 2$ matrix. The $N \times 1$ error vector satisfies $\mathbb{E}(\varepsilon_t) = 0$ and $V(\varepsilon_t) = H$

4 Data and empirical results

This section aims to present the datasets used for estimating the three models. The estimation is realized according to the procedure developed in the last section. As far as the models present common characteristics (closed form formulas, Kalman filter procedure), a comparison can be made.

4.1 Data exploration

Some specifications. The models are tested using weekly observations of futures prices for oil, copper and gold. For each model, 5 futures contracts are used to compute the estimation (except for Enron oil data, where 10 contracts were used). Regarding indexes, it should be clear that i denotes a weekly observation, N the number of futures contracts and T the maturity. Note also that the interest rate data consisted in yields on 3-Month Treasury Bills.

Oil data sets. It covers the period from 1/2/85 to 2/17/95 over which nine futures contracts were available, denoted F_i (F_1 being the closest to maturity and so on.). The contracts have a fixed maturity date: hence, time to maturity is to be evolving as time runs. A first set ot test was created, including $F_1, ..., F_5$ contracts. An aim of the study is to analyze investment decisions in real assets with much longer maturities. As a consequence, a second test set was used including $F_1, F_5, F_9, F_{13}, F_{17}$ contracts from 1/2/90 to 2/17/19. A third test set was used mixing contracts from the first data set and dates from the second one, in order to neutralize potential conclusions due to the timestamp or the nature of the contract.

Copper data set. For this commodity, futures contracts started trading in 1988. Observations are realized from 7/29/88 to 6/13/95 using only F_1, F_2, F_3, F_5, F_7 contracts.

Gold data sets. The same methodology as the one used for oil was applied from 1/2/85 to 6/13/95 including $F_1, F_3, F_6, F_9, F_{11}, F_{13}, F_{18}$ contracts.

Enron Capital and Trade Resources data. They made available some proprietary historical crude oil forward price curves from 1/15/93 to 5/16/96. Main advantage: the longer maturities of the contracts. Main disadvantage: we do not know how were the crude oil forward curves constructed.

4.2 Empirical results.

⋆⋆ **Model 1.** We are interested in the estimation of the parameters of $dX = \kappa(\alpha^* - X)dt + \sigma dz^*$.

$\star$ **Oil.** A general observation, common to the four data sets can be made: the speed of adjustment is highly significant and the market price of risk in not significantly different from zero. When we compare same contracts but different time periods, we find that the shortest period has much stronger mean reversion (0.7 as opposed to 0.3 for the whole period). Other parameters are quite similar. Regarding predictions, one can say that the average error is small, in general. But the absolute standard deviation is much more important. One of the main observation is that the state variable (the log of the spot price) is very close to the log of the price of the futures contracts closest to maturity. It is a great empirical result. When comparing different futures contracts with the same time period, we observe that the main effect of extending the maturity of the contracts is to reduce the mean reversion parameter (from 0.7 to 0.4). As a consequence, this empirical evidence implies that, for long term oil investment, the relevant futures contracts would be much longer in maturity.
$\star$ **Copper and gold.** Regarding copper, estimators are quite similar than the ones obtained for oil. Main patterns: there is a strong and significant mean reversion (around 0.37) and the market price of risk is positive, but not very different from zero. Regarding gold, the model could not be fitted to the data available. As a consequence, there is no detectable mean reversion in gold prices for the period considered.

$\star\star$ **Model 2.** We are interested in the estimation of the parameters of $dS = (\mu - \delta)Sdt + \sigma_1 S dz_1$ and $d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2$. In this model, the interest rate r is a parameter. It was calibrated on the average interest rate over the period.

$\star$ **Oil.** For each data set, the speed of adjustment coefficient (κ) in the convenience yield equation and the correlation coefficient (ρ) between the two factors are large and very significant. Other parameters are positive but not always significant at standard levels. Once again, we observe a close relation between the log spot price and the lof of the F1 contract (the closest one to maturity). Regarding prediction errors, both the mean error and the average absolute deviation are smaller than those obtained in model 1.
$\star$ **Copper and gold.** The conclusion is the same as for oil regarding κ and ρ. It should be noted that the average convenience yield is high as we are estimating instantaneous convenience yields. Results for gold are different than the ones for commercial commodities. The mean reversion in the convenience yield and the correlation between the state variables are significant only in the later period and of muck smaller magnitude. For each case, the total expected return, average convenience yield and market price of risk are not significant.

$\star\star$ **Model 3.** Note that the parameters for the interest rate equation were obtained by using 3-month Treasury Bill yields. The equation is: $dr = a(m^* - r)dt + \sigma_3 dz_3^*$. The parameter a was set to 0.2. The parameter m^* was computed so that the the infinite maturity discount yield be 7 percent. The instantaneous correlations were estimated by the correlations obtained using weekly data between the 3-month Treasury Bill yield and the values of these state variables obtained from model 2. The unique correlation parameter statistically different from 0 is the one between the interest rate and the convenience yield for copper and gold.

$\star$ **Oil and copper.** Results are close to the ones obtained in model 2, meaning that the parameters of the processes for the spot price and convenience yield seem to be robust to the specification of the interest rate process. It does not mean, however, that the value of a futures contract is insensitive to the interest rate used in the computation.
$\star$ **Gold.** Results suggest that a mean reversion in prices induced by a mean-reverting convenience yield does not seem to hold for gold.

4.3 Comparing the models.

Specifications. The comparison is realized by using three data sets: the long term oil futures data, the copper futures data and the Enron oil forward data. Regarding fit performance, the model 1 is very often unable to correctly describe the data. We also observe that the term structure of futures prices implied by Models 2 and 3 for the long-term oil data and the copper data are sometimes indistinguishable and that they are always very close to each other. But this does not say that these models have the same implications for futures prices with longer observations than the observable ones.

Statistical tests. We would like to compare statistically the performances of the models. As far as they are not nested, such a comparison is not easy. In order to compare the relative performance of the models, we performed two types of tests. The first were cross-section tests using all the available futures prices which were not used in the estimation of the parameters of the models. The second were time series tests using the last 50 observations in the sample. For both tests, the metrics used was the Root-Mean-Square-Error (RMSE). If $\widehat{\theta}$ is an estimator of the parameter θ, then: $\text{RMSE}(\widehat{\theta}) = \sqrt{\mathbb{E}(\widehat{\theta} - \theta)^2}$. Taking the empirical counterpart is the solution to obtain the RMSE over a sample of observations.

Results. Regarding cross-section tests, models 2 and 3 outperform model 1 in all the situations. The comparison between models 2 and 3 is not so clear, as model 2 outperforms model 3 for oil and vice-versa for copper. Using the longer maturity Enron oil data, model 3 is the winner. Regarding time-series tests, models 2 and 3 also outperform model 1. And there is still a competition between model 1 and 2.

5 Models implications

The structure of the models proposed in section 3 implies some specificities, developed throughout this section.

5.1 Futures returns volatility

As specified before, our models share common characteristics. We will observe that our models imply volatilities independent of the state variables, depending only on time to maturity of futures contracts. Regarding model 1, applying Itô's Lemma to $F(S,T) = \exp\left[e^{-\kappa T}\ln S + (1-e^{-\kappa T})\alpha^* + \frac{\sigma}{4\kappa}(1-e^{-2\kappa T})\right]$, one obtains the following volatility structure:

$$\sigma_F^2(T) = \sigma^2 e^{-2\kappa T} \implies \sigma_F^2(\infty) = 0$$

Concerning model 2, the structure of the volatility writes down:

$$\sigma_F^2(T) = \sigma_1^2 + \sigma_2^2\frac{(1-e^{-\kappa T})^2}{\kappa^2} - 2\rho\sigma_1\sigma_2\frac{(1-e^{-\kappa T})}{\kappa} \implies \sigma_F^2(\infty) = \sigma_1^2 + \frac{\sigma_2^2}{\kappa^2} - \frac{2\rho\sigma_1\sigma_2}{\kappa}$$

Finally, the volatility from model 3 is a linear combination of σ_1, σ_2,σ_3, κ, T and the correlation terms. As time to maturity becomes very large, the volatility converges towards:

$$\sigma_F^2(\infty) = \sigma_1^2 + \frac{\sigma_2^2}{\kappa^2} + \frac{\sigma_3^2}{a^2} - \frac{2\rho_1\sigma_1\sigma_2}{\kappa} + \frac{2\rho_3\sigma_1\sigma_3}{a} - \frac{2\rho_2\sigma_2\sigma_3}{a\kappa}$$

When comparing these volatilities to the volatility of the data, we observe that both models 2 and 3 fit the data very well. Regarding model 1, it always implies volatilities that are smaller than the volatility of the data with the following specificity: the difference is smaller for midmaturities and increases when the maturity of futures contracts increases.

5.2 Long maturity Futures Contracts

The previous section was dedicated to the implications of the models developed in terms of volatility. This section is dedicated to the implications with respect to price as time to maturity goes to infinity. For the range of observed contracts (i.e. short maturities), models 2 and 3 give very similar values but they diverge as maturity increases. Once again, the futures prices from model 1 converge towards a fixed value, so the rate of change in price as maturity increases goes to 0. This rate is calculated by taking the derivative of the futures contracts with respect to T and then dividing the whole by the value of the futures contracts. For model 2, we obtain:

$$\frac{1}{T}\frac{\partial F}{\partial T}(T \longrightarrow \infty) = r - \widehat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\rho\sigma_1\sigma_2}{\kappa}$$

Using the estimated parameters, the rate is 2,71 percent per year for oil futures and 0,85 percent per year for copper futures.

Regarding model 3, the associated rate is:

$$\frac{1}{T}\frac{\partial F}{\partial T}(T \longrightarrow \infty) = m^* - \widehat{\alpha} + \frac{\sigma_2^2}{2k^2} - \frac{\rho_1\sigma_1\sigma_2}{\kappa} + \frac{\sigma_3^2}{2a^2} + \frac{\rho_3\sigma_1\sigma_3}{a} - \frac{\rho_2\sigma_2\sigma_3}{\kappa a}$$

It gives higher values: 4.19 percent per year for oil futures and 2.7 percent per year for copper futures. Such a difference with model 2 can give important difference for futures prices with ten years to maturity. Now we want to check what are the implications of getting estimators from short term data in terms of pricing futures/forward prices with more important maturities, using these parameters. Such a task has been done in the paper thanks to Enron oil data. One can say that model 2 always outperforms model 1. But it is not the case for model 3. This gives the confirmation of the importance of the interest rate process parameters in Model 3 for the valuation of long terms contracts.

5.3 Hedging contracts for future delivery

A popular issue is the feasibility of hedging long term forward commitments in commodities with the existing short term futures contracts.

Heging rule: the sensitivity of the present value of the commitment with respect to each one of the underlying factors must equal the sensititvity of the portfolio of futures contracts used for hedging the commitment with respect to the same factors. As a consequence, the number of factors in the model used for the hedge is equal to the number of factors in the model used. In the following, w_i denotes a long position with maturity t_i.

Hedging in model 1:

$$w_1 F_S(S, t_1) = e^{-rT} F_S(S, T) \implies w_1 = e^{-rT}\frac{F_S(S,T)}{F_S(S;T_1)}$$

Hedging in model 2:

$$\begin{cases} w_1 F_S(S,\delta,t_1) + w_2 F_S(S,\delta,t_2) = e^{-rT} F_S(S,\delta,T) \\ w_1 F_\delta(S,\delta,t_1) + w_2 F_\delta(S,\delta,t_2) = e^{-rT} F_\delta(S,\delta,T) \end{cases}$$

Hedging in model 3:

$$\begin{cases} w_1 F_S(S,\delta,r,t_1) + w_2 F_S(S,\delta,r,t_2) + w_3 F_S(S,\delta,r,t_3) = P_S(S,\delta,r,T) \\ w_1 F_\delta(S,\delta,r,t_1) + w_2 F_\delta(S,\delta,r,t_2) + w_3 F_\delta(S,\delta,r,t_3) = P_\delta(S,\delta,r,T) \\ w_1 F_r(S,\delta,r,t_1) + w_2 F_r(S,\delta,r,t_2) + w_3 F_r(S,\delta,r,t_3) = P_r(S,\delta,r,T) \end{cases}$$

5.4 Investment under uncertainty

Issue: as commodity prices are stochastic, it has important implications for the valuation of projects related to the prices of those commodities and for the determination of the optimal investment rule, meaning the commodity price above which it is optimal to undertake the project immediately. This section analyses a simple investment and evaluate it using the models developed through the article and two other benchmark procedures. Benchmarks are the traditional DCF and a real option approach based on the assumption that commodity prices follow a geometric random walk (hence, neglecting mean reversion). The net present value of the project (over the ten years involved) once the investment has been deduced is:

$$\text{NPV} = \sum_{T=1}^{10} P(r,T,\cdot) - C\sum_{T=1}^{10} B(r,T) - K$$

Where $P(r,T,\cdot)$ is the present value of the commodity to be received at time T when the interest rate is r. $B(r,T)$ is the zero coupon of maturity T. K denotes the initial investment cost. Let us present the different approaches right now.

(A) Discounted Cash Flow Criteria. This approach reflects that the expected net cash flows are discounted at a rate that reflects the risk of these cash flows.

(B) Constant convenience yield. This model is a real options to valuation with a constant convenience yield model. The risk-adjusted process for the spot price is $dS = S[(r-c)dt + \sigma dz^*]$ where c denotes this constant convenience yield. Here, the net present value becomes:

$$\text{NPV}(S) = S\sum_{T=1}^{10} e^{-cT} - C\sum_{T=1}^{10} e^{-rT} - K = S\beta_1 - \beta_2$$

And the option to invest solves:

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + (r-c)SV_S - rV = 0 \quad \text{such that} \quad V(s) \geq \max[S\beta_1 - \beta_2, 0]$$

The solution is:

$$V(S) = (S^*\beta_1 - \beta_2)\left(\frac{S}{S^*}\right)^d \qquad S^* = \frac{d\beta_2}{\beta_1(d-1)} \qquad d = \frac{1}{2} - \frac{r-c}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r-c}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

S^* being the commodity price above which it is optimal to invest in the project.

(C) Mean reverting spot price: model 1. We replace the net present value by discounting the futures/-forward prices obtained in the model. The resulting PDE is solved numerically.

(D) Stochastic convenience yield: model 2. The convenience yield is also stochastic in this model. Hence, the value of the option to invest is of the form $V(S,\delta)$ and satisfies a PDE solved numerically.

(E) Stochastic convenience yield and interest rate: model 3. The value of the option to invest is of the form $V(S,\delta,r)$ and satisfies a PDE solved numerically.

6 Appendixes

6.1 Model 1: Derivating formulas.

6.1.1 Dynamics for $X := \ln S$

Recall that:

$$\frac{dS}{S} = \kappa(\mu - \ln S)dt + \sigma dz$$

With $X := \ln S = f(S)$, applying Itô's Lemma gives:

$$dX = \frac{\partial f}{\partial x}(S)dS + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(S)d[S] = [\kappa(\mu - \ln S)dt + \sigma dz] - \frac{\sigma^2}{2}dt = \left[\kappa(\mu - X) - \frac{\sigma^2}{2}\right]dt + \sigma dz = \kappa(\alpha - X)dt + \sigma dz$$

Where $\alpha = \mu - \frac{\sigma^2}{2\kappa}$

6.1.2 Closed form solution for $X := \ln S$

Under the convenient martingale measure, X follows the following dynamics: $dX = \kappa(\alpha^* - X)dt + \sigma dz^*$. Define $f(X_t, t) = X_t e^{\kappa t}$. Then, using Itô and integration by parts, we get: $df = e^{\kappa t}\kappa\alpha^* dt + \sigma e^{\kappa t} dW_t$ and:

$$X_t e^{\kappa t} = X_0 + \int_0^t e^{\kappa s}\kappa\alpha^* ds + \sigma \int_0^t e^{\kappa s} dW_s = X_0 + \alpha^* \left[e^{\kappa t} - 1\right] + \sigma \int_0^t e^{\kappa s} dW_s$$

Hence the following closed form solution, very convenient for calculating the mean and the variance:

$$X_t = X_0 e^{-\kappa t} + \alpha^* \left[1 - e^{-\kappa t}\right] + \sigma \int_0^t e^{-\kappa(t-s)} dz$$

6.1.3 Mean and variance for $X := \ln S$

Calculating the mean:

$$\mathbb{E}_0[X(T)] = e^{-\kappa T} X(0) + \left(1 - e^{-\kappa T}\right)\alpha^*$$

Because the expectation of a stochastic integral (i.e. with respect to a Brownian motion) is nill. Let $a(\cdot)$ denote a deterministic function. Then, if W denotes a standard Brownian motion, over a grid $t_0 < t_1 < ... < T$:

$$\int_0^T a(s) dW_s = \sum_{i=0}^{n-1} a_{t_i}(W_{t_{i+1}} - W_{t_i})$$

Hence:

$$\mathbb{E}\left[\int_0^T a(s) dW_s\right] = \mathbb{E}\left[\sum_{i=0}^{n-1} a_{t_i}(W_{t_{i+1}} - W_{t_i})\right] = \sum_{i=0}^{n-1} a_{t_i} \underbrace{\mathbb{E}\left[W_{t_{i+1}} - W_{t_i}\right]}_{0}$$

Because $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $s \leq t$.

Calculating the variance:

$$V_0[X(T)] = \sigma^2 V\left(\int_0^T e^{\kappa(T-s)} dW_s\right) = \underbrace{\sigma^2 \int_0^T e^{-2\kappa(T-s)} ds}_{\text{Isometry}} = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa T}\right]$$