'Pedro teaches maffs' talk on equivalent definition of superstability.

Theorem: Let T be a complete stable f.o. theory. TFAE:

(1) T is superstable (i.e. $\exists \lambda$ s.t. T is stable in all $\mu \geq \lambda$)

(2) T is stable in all $\mu \geq 2^{|T|}$

(3) $\kappa(T) = \aleph_0$

(4) T does not have the tree property, i.e. there are no $\{\varphi_n(\underline{x}, \underline{y}_n) : n < \omega\}$ and $\{a_\eta : \eta \in {}^{\leq\omega}\lambda\} \subseteq \mathfrak{C}$ s.t. $(\mathfrak{C} \models \varphi_n[\underline{a}_\eta, \underline{a}_\nu] \iff \nu < \eta)$ for all $\nu \in {}^{<\omega}\lambda, \eta \in {}^{\omega}\lambda$.

(5) T is stable and $D^n[\underline{x} = \underline{x}, L(T), \infty] < \infty$

(6) $\forall \lambda$ a union of an inc chain of λ-sat models is λ-saturated.

$\underline{(1) \Leftrightarrow (2) \Leftrightarrow (3)}$: Recall

<u>Def:</u> $\kappa(T) = \min\{\kappa : \forall A \subseteq \mathfrak{C}\ \forall p \in S(A)\ \exists B \subseteq A\ |B| < \kappa$ and p dnf $/B\}$

$=$ least κ such that there is no $\subseteq$ increasing cts sequence of A_i for $i \leq \kappa$ and $p \in S(A_\kappa)$ st. for all $i < \kappa$ $p \restriction A_{i+1}$ forks (strongly splits) $/A_i$.

$\overline{(\text{for any } \lambda \geq \aleph_0)}$

$\overset{\lambda}{=}$ least κ st. there is no $\langle A_i : i < \kappa \rangle$ $\subseteq$-inc cts with $I_i = \{\underset{\sim}{a}^i_j : j < \lambda\} \subseteq A_{i+1}$ indisc $/A_i$, $p \in S(A_\kappa)$ and $\varphi_i(\underset{\sim}{x}, \underset{\sim}{a}^i_0) \in p$ st. $\forall j > 0$ $\neg\varphi_i(\underset{\sim}{x}, \underset{\sim}{a}^i_j) \notin p$ and $\exists n_i < \omega$ st. $\{\varphi_i(\underset{\sim}{x}, \underset{\sim}{a}^i_j) : j < \lambda\}$ is n_i-inconsistent.

<u>Fact:</u> T stable $\Rightarrow$ $\kappa(T) \leq |T|^+$.

<u>Lemma 1:</u> If $2^{|T|} \leq \mu = \mu^{<\kappa(T)}$, then T is μ-stable.

<u>proof:</u> Else there is A where $|A| \leq \mu < |S(A)|$.

By def of $\kappa(T)$ for all $p \in S(A)$ there is $B_p \subseteq A$ st. p dnf $/B_p$, $|B_p| < \kappa(T)$.

$\#$ possible B_p is $\mu^{<\kappa(T)} = \mu$.

So there is $B \subseteq A$ st. $|\{p : B_p = B\}| > \mu$.

$\kappa(T) \leq |T|^+ \Rightarrow |B| \leq |T|$.

Let $B \subseteq |M|$, $\|M\| = |T|$.

So $\mu < |\{p \in S(A) : B_p = B\}|$

$\le |\{p \in S(A) : p \text{ def}/B\}|$

$\le |\{p \in S(|M| \cup A) : p \text{ def}/B\}|$ ($p \in S(|M|)$ stationary)

$= |\{p \in S(M) : p \text{ def}/B\}|$

$\le |S(M)| \le 2^{|T|} \le \mu$ //L1.

<u>Lemma 2:</u> If $2^{|T|} \le \mu < \mu^{<\kappa(T)}$ then T is not stable in μ.

<u>proof:</u> $\mu \ge \kappa(T)$ so there is $\kappa < \kappa(T)$ minimal such that $\mu < \mu^\kappa$.

Note $\mu^{<\kappa} = \mu$.

As $\kappa < \kappa(T)$, there are $\langle A_i : i \le \kappa \rangle$, $p \in S(A_\kappa)$,

$I_i = \{\underline{a}^i_j : j < \mu\} \subseteq A_{i+1}$ indesc $/A_i$, $\varphi_i(\underline{x}, \underline{a}^i_0)$

s.t. $\neg \varphi_i(\underline{x}, \underline{a}^i_j) \in p$ for $j > 0$.

Define F_η elementary for $\eta \in {}^{\le\kappa}\mu$ s.t. $\forall \eta$

(1) $\operatorname{Dom} F_\eta = \bigcup \{\underline{a}^i_j : j < \mu,\ i < \operatorname{len}(\eta)\}$

(2) $\rho \le \eta \Rightarrow F_\rho \subseteq \eta$

(3) If $\operatorname{len}(\eta) = i$, then $\forall j$

(A) $F_{\eta^\frown\langle j\rangle}(\underline{a}^i_0) = F_{\eta^\frown\langle 0\rangle}(\underline{a}^i_j)$

(B) $F_{\eta^\frown\langle j\rangle}(\underline{a}^i_j) = F_{\eta^\frown\langle 0\rangle}(\underline{a}^i_0)$

(C) For $\alpha \ne 0$ and $\alpha \ne j$, $F_{\eta^\frown\langle j\rangle}(\underline{a}^i_\alpha) = F_{\eta^\frown\langle 0\rangle}(\underline{a}^i_\alpha)$

(only hard at $i+1$, take $F_{\eta^\frown\langle 0\rangle}$ any ext, take $F_{\eta^\frown\langle j\rangle}$ as def by (3).)

Let $B = \bigcup \{F_\eta(\underset{\sim}{a}^i_j) : \eta \in {}^{<\kappa}\mu,\ i < \lg(\eta),\ j < \mu\}$

$|B| \leq {}^{<\kappa}\mu \cdot \mu \cdot \mu = \mu.$

Take $q_\eta \in S(B) \supseteq F(p \upharpoonright \operatorname{dom} F_\eta)$

<u>Claim</u>: $\eta \neq \nu \in \mu^\kappa \Rightarrow q_\eta \neq q_\nu$.

[If $\eta \neq \nu$, and i min s.t. $\underset{\substack{\shortparallel \\ j_1}}{\eta(i)} \neq \underset{\substack{\shortparallel \\ j_2}}{\nu(i)}$, $\eta \upharpoonright i = s$
then

$j_1 \neq 0$: $F_\eta(a^i_0) = F_{s\frown j_1}(a^i_0)$
$= F_{s\frown 0}(a^i_{j_1})$

$F_\nu(a^i_{j_1}) = F_{s\frown j_2}(a^i_{j_1})$
$= F_{s\frown 0}(a^i_{j_1})$

So $\varphi(x, a^i_0) \in p \Rightarrow \varphi(x, F_\eta(a^i_0)) \in q_\eta$

$\neg\varphi(x, a^i_{j_1}) \in p \Rightarrow \varphi(x, \underset{\substack{\shortparallel \\ F_\nu(a^i_{j_1})}}{F_\eta(a^i_0)}) \notin q_\nu$.

$j_1 = 0$ $F_\eta(a^i_{j_1}) = F_{s\frown j_1}(a^i_{j_1})$
$= F_{s\frown 0}(a^i_0)$
$= F_{s\frown j_2}(a^i_0)$
$= F_\nu(a^i_0)$

$\neg\varphi(x, \underset{\sim}{a}^i_{j_1}) \in p \Rightarrow \neg\varphi(x, F_\eta(a^i_{j_1})) \in p_\eta$

$\varphi(x, \underset{\sim}{a}^i_0) \in p \Rightarrow \varphi(x, F_\eta(a^i_{j_1})) \in p_\nu$]

So $|S(B)| \geq \mu^{\kappa} > \mu \geq |B|$

So T is not μ-stable. // L2.

<u>Corollary</u>[1]: If $2^{|T|} \leq \mu$, then T is stable in μ iff $\mu = \mu^{<\kappa(T)}$.

<u>Corollary</u>[2]: $(1) \Leftrightarrow (2) \Leftrightarrow (3)$

<u>proof</u>: $(2) \Rightarrow (1)$ easy

$(3) \Rightarrow (2)$ For $\mu \geq 2^{|T|}$, $\mu^{<\kappa(T)} = \mu^{\aleph_0} = \mu$,

so μ-stable

$(1) \Rightarrow (3)$ If (3) fails, then for any α where

$\mu = \aleph_{\alpha+\omega} \geq 2^{|T|}$,

$$\mu^{<\kappa(T)} \geq \mu^{\aleph_0} > \mu \quad \text{as } cf(\mu) = \aleph_0.$$

So not μ-stable.

Arb large μ, so <u>not</u> superstable. // Cor 2.

$\Longleftrightarrow (5) \Longleftrightarrow (4)$

Def: **Degree** $D[p, \Delta, \lambda]$ given by

type $\subseteq L(T)$ $\in \mathrm{ON} \cup \{\infty\}$

- $D[p, \Delta, \lambda] \geq 0$ when p consistent
- $D[p, \Delta, \lambda] \geq \delta$ for δ limit if $D[p, \Delta, \lambda] \geq \alpha$ for all $\alpha < \delta$
- $D[p, \Delta, \lambda] \geq \alpha + 1 \iff \forall \mu < \lambda, \forall r \subseteq p$ finite, $\exists q \supseteq r, n < \omega, \psi(\underline{x}, \underline{y}) \in \Delta, \underline{a}_i : i \leq \mu$ s.t.

  (i) $D[q \cup \{\psi(\underline{x}, \underline{a}_i)\}, \Delta, \lambda] \geq \alpha \quad \forall i$

  (ii) $\{\psi(\underline{x}, \underline{a}_i) : i \leq \mu\}$ is n-contradictory (or n-inconsistent) over q, i.e. $\forall w \subseteq \mu, |w| = n$

  $$\models \neg \exists x \bigwedge_{i \in w} \psi(\underline{x}, \underline{a}_i) \wedge \bigwedge q$$

D has finite character, monotonicity, the ultrametric property, extension, etc.

Fact: $D[p, \Delta, \mu^+] \geq \mu$ for $\mu = |T|^+$

$$\iff D[p, \Delta, \infty] = \infty$$

$(4) \Rightarrow (5)$:

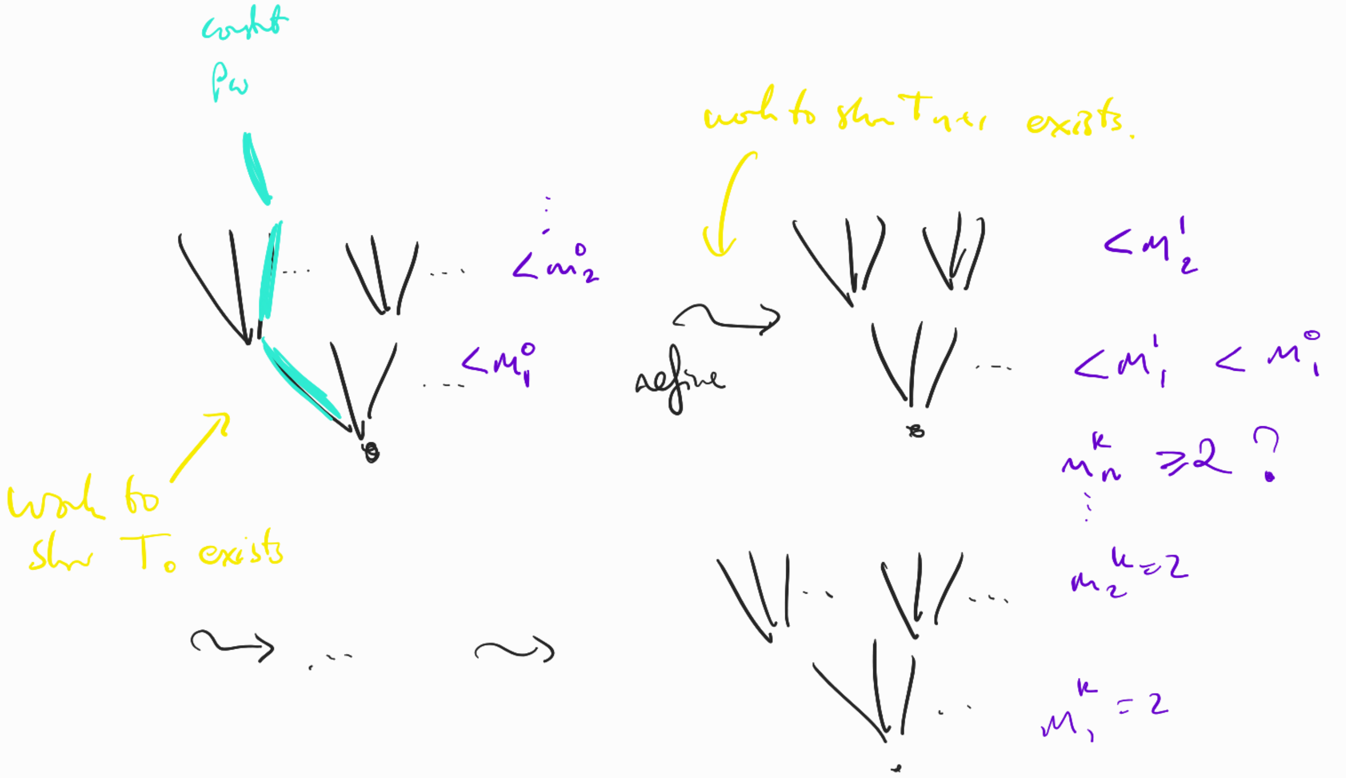
Sketch: Assume $D[p, \Delta, \mu^+] \geq \mu$. (i.e. $\neg(5)$)

Build by induction on $k < \omega$ $\varphi^k_n, n^k_n \in \omega$ s.t.

(1) $\forall k \in \omega\ \exists\, \underset{\sim}{a}_\eta,\ \eta \in {}^{<\omega}\mu$ s.t.

(i) $\forall \eta \in {}^{\omega}\mu \quad p_\eta = p \cup \{\varphi_n^k(\underset{\sim}{x}, \underset{\sim}{a}_{\eta\restriction n}) : 0 < n < \omega\}$ is consistent

(ii) $\forall \eta \in {}^{n}\mu,\ n < \omega,\ w \subseteq \mu,\ |w| \geq m^k_{n+1}$,
$\{\varphi^k_{n+1}(\underset{\sim}{x}, \underset{\sim}{a}_{\eta^\frown i}) : i \in w\}$ is inconsistent.



Eventually you get tree property by compactness. // (4) $\Rightarrow$ (5)

$(\neg 4)$

(1) $\Rightarrow$ (4): Assume (1). If $\neg 4$, we have the tree property.

By (1) $\Leftrightarrow$ (3), there is λ s.t. $\lambda < \lambda^{\aleph_0}$, T λ-stable.

Get a $\lambda^{\leq\omega}$-tree φ_n $n \in \omega$, $\underset{\sim}{a}_\eta$ $\eta \in {}^{\leq\omega}\lambda$.

Then let $A = \bigcup_{\eta \in {}^{<\omega}\lambda} \underset{\sim}{a}_\eta$, $p_\eta = tp(\underset{\sim}{a}_\eta / A)$.

$\eta \neq \nu \Rightarrow p_\eta \neq p_\nu$ as if $\eta\restriction\alpha = \nu\restriction\alpha$, $\eta(\alpha) \neq \nu(\alpha)$,

$\varphi(x, \underset{\sim}{a}_{\eta\restriction(\alpha+1)}) \in p_\eta$, So $\neg$ ——— $\in p_\nu$

by $m_n = 2$.

$|A| = \lambda < \lambda^{\leq\omega} \leq |S(A)|$ $\times\!\!\times$

$(5) \Rightarrow (1)$ Fact: T stable, then

$$D[p, L, \infty] = R[p, L, \infty]$$
$$= R[p, L, (2^{|T|})^{++}]$$

So set $\lambda = (2^{|T|})^{++}$

Let $A \subseteq \mathbb{C}$, For all $p \in S(A)$, let
$q_p \subseteq p$ finite st. $R[q_p, L, \lambda] = R[p, L, \lambda]$.

Def of $R \Rightarrow$ for all q, $|\{p \in S(A) : q_p = q\}| \leq 2^{|T|}$.
(else $R[q] \geq R[q]+1$.)

So $|S(A)| \leq |\{q_p : p \in S(A)\}| \cdot 2^{|T|}$
$$\leq |A| \cdot |T| \cdot 2^{|T|}$$
$$= |A| \cdot 2^{|T|}$$

So T is stable in all $\lambda \geq 2^{|T|}$.
i.e. (1). // $(5) \Rightarrow (1)$.

(7) $\forall \lambda$, any union of λ-sat. models is λ-sat

(8) Uniqueness of limit models in all stable λ

(9) Uniqueness of limit models in some λ

(1) $\Rightarrow$ (7) by general result:

If $\mathrm{cf}(\delta) \geq \kappa(T)$, M_i λ-sat $\Rightarrow \bigcup_{i<\delta} M_i$ λ-sat.

(7) $\Rightarrow$ (8) limit models unique up to $\mathrm{cf}(\delta)$

All λ-sat.

($M_i \stackrel{u}{\leq} M_{i+1}$, M_i all λ-sat.)

(8) $\Rightarrow$ (9) easy

(9) $\Rightarrow$ (1) You can show $\neg(3) \Rightarrow \mu_1 < \mu_2$,

then (λ, μ_1)-lim is not μ_2 sat

— μ_2 ——— is ———

(10) T has a superlimit model

(11) T is (λ, μ)-solvable in high λ

$\vdots$

Fin