Grand Model Theory & Set Theory Seminar (aka Pedro teaches maths)

The spectrum of limit models

Based on work with Marcos Mazari-Armida

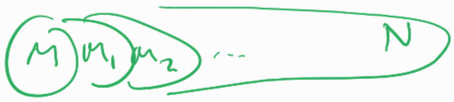
Let K be an AEC.

Definitions:

- Let $M, N \in K_\lambda$, N is universal $/M$ if for every $M' \in K_\lambda$ such that $M \leq_K M'$, there exists $f : M' \to N$ a K-embedding fixing M. Write $M \leq^u N$.

$$\begin{tikzcd} & M' \arrow[d, dotted, "\exists"] \\ M \arrow[ur] \arrow[r] & N \end{tikzcd}$$

- Let $M, N \in K_\lambda$, $\delta < \lambda^+$ limit. We say N is a (λ, δ)-limit over M if there is a chain $\langle M_i : i < \delta \rangle$ where
  - $M_0 = M$
  - $\bigcup_{i<\delta} M_i = N$
  - $\forall i < \delta$, M_{i+1} is universal $/M_i$.



Limits are a surrogate for saturated models - so when are they isomorphic?

Fact: If N_ℓ is a (λ, δ_ℓ)-limits over M for $\ell = 1, 2$, then if $cf(\delta_1) = cf(\delta_2)$, $N_1 \underset{M}{\cong} N_2$.

Sketch: Assume $\delta_\ell = cf(\delta_1) = \delta$.

$$\begin{tikzcd} & N_1^0 \arrow[r] \arrow[d, "f_0"'] & N_1^1 \arrow[r] \arrow[d, "f_1"'] & N_1^2 \arrow[r, dashed] \arrow[d, "f_2"'] & \cdots & N_1 \arrow[d, "\cong\ f"] \\ M \arrow[ur] \arrow[r] & N_2^0 \arrow[r] \arrow[ur, "g_1"] & N_2^1 \arrow[r] \arrow[ur, "g_2"] & N_2^2 & \cdots & N_2 \end{tikzcd}$$

What else can we say?

Definition: K is μ-tame if whenever $M \in K$, $p, q \in S(M)$, $p \neq q$, there exists $A \subseteq |M|$ such that $|A| \leq \mu$ and $p \restriction A \neq q \restriction A$.

Definition: A weak independence relation on K' is $\perp$ on tuples (M_0, a, M, N) where $M_0 \leq_{K'} M \leq_{K'} N$, $a \in N$.

Write $a \underset{M_0}{\overset{N}{\perp}} M$.

- An independence relation is a weak independence relation that satisfies
  - invariance: $\perp$ respects isomorphisms.
  - monotonicity: $a \underset{M_0}{\overset{N}{\perp}} M \Rightarrow a \underset{M_0}{\overset{N'}{\perp}} M'$

    so long as $M_0 \leq M' \leq M$, $N \leq N'$ or $N \geq N'$
  - base monotonicity: $a \underset{M_0}{\overset{N}{\perp}} M \Rightarrow a \underset{M_0'}{\overset{N}{\perp}} M$

    so long as $M_0 \leq M_0' \leq M$.

Def: $\perp$ indep rel.

We say $gtp(a/M,N)$ $\perp$-nf $/M_0$ if $a \underset{M_0}{\overset{N}{\perp}} M$.

(choice of a, M, N do not matter)

Def: $\perp$ indep rel, K regular. $\perp$ has

- uniqueness if $\forall M \leq N$ $\forall q_1, q_2 \in S(N)$ if q_1, q_2 $\perp$-nf $/M$ and $q_1 \upharpoonright M = q_2 \upharpoonright M$, then $q_1 = q_2$.

- extension if $\forall M_0 \leq M \leq N$ $\forall p \in S(M)$ $\downarrow$-nf$/M_0$
there exists $q \in S(N)$ $q \geq p$, q $\downarrow$-nf$/M_0$.

- $(\geq \kappa)$-local character if $\forall \delta < \lambda^+$ where $cf(\delta) \geq \kappa$,
$\forall \langle M_i : i < \delta \rangle$, $\forall p \in S(\bigcup_{i<\delta} M_i)$, there exists
$i < \delta$ s.t. p $\downarrow$-nf$/M_i$.

- continuity if whenever $\delta < \lambda^+$ limit, $\langle M_i : i < \delta \rangle$ chain,
$p \in S(\bigcup_{i<\delta} M_i)$ and $p \upharpoonright M_i$ $\downarrow$-nf$/M_0$,
then p $\downarrow$-nf$/M_0$.

- n.f.a.p. ?

Fact: uniqueness, ext., cty $\Rightarrow$ l.c. κ are upward closed.
Let $\kappa(\downarrow, K_\lambda, \leq^u) = \kappa^u_\lambda(\downarrow)$ be minimal.

$\lambda \geq LS(K)$

Theorem: (B., Mazari-Armida): Suppose K an AEC with AP, JEP, NMM in K_λ, stable in λ, K is $\aleph_0$-tame and $\downarrow$ an indep rel on K_λ with uniqueness, extension, $(\geq \kappa)$-l.c., cty, nf AP.

Suppose N_l (λ, δ_l)-lim $/M$ for $l = 1, 2$, $cf(\delta_1) < cf(\delta_2)$. Then

$$N_1 \cong_M N_2 \iff cf(\delta_1) \geq \kappa^u_\lambda(T).$$

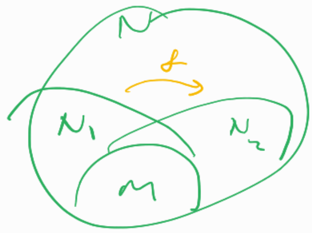
$\Leftarrow$ proof is long and scary.

$\Rightarrow$ We might manage today.

Definition: $M, N \in K_\lambda$, $p \in S(N)$. p λ-splits $/M$ if there exist $M \leq N_\ell \leq N$ $(\ell = 1, 2)$, $f : N_1 \cong_M N_2$

s.t. $f(p \restriction N_1) \neq f(p \restriction N_2)$.

p does different things on small copies inside N.



Else, p λ-dns $/M$.

Fact: λ-dns is an independence rel with weak uniqueness:

If $M_0 \leq^u_K M \leq N$, in K_λ, $q_1, q_2 \in S(N)$ λ-dns$/M_0$, $q_1 \restriction M = q_2 \restriction M$, then $q_1 = q_2$.

Lemma^1: If p $\perp$-dnf$/M$, p λ-dns$/M$.

($f(p \restriction N_1)$, $p \restriction N_2 \in S(N_2)$, $\perp$-dnf$/M$ by mon, $=$ on M, so uniqueness of $\perp$ $\Rightarrow f(p \restriction N_1) = p \restriction N_2$.)

Lemma^2: If $L_1 \leq L_2 \leq M$ in K_λ, L_2 a $(\lambda, \geq \kappa^u_\lambda(\perp))$-lim $/L_1$, $p \in S(M)$, and p λ-dns $/L_1$, then p $\perp$-dnf$/L_2$.

proof: By $(\geq \kappa)$-l.c. there is $L_1 \leq^u L_{1.5} \leq^u L_2 \leq M$

$p \restriction L_2$ $\perp$-dnf$/L_{1.5}$

Cont $q \in S(M) \geq p \restriction L_2$, $\perp$-dnf$/L_{1.5}$.

By last lemma & mon, p, q λ-dns$/L_{1.5}$

So $p = q$ by weak !ness

By mon, p $\perp$-dnf$/L_2$. $/\!/ L_2$

L3: Can replace $(\lambda, \geq \kappa)$-lim by universal.

$$L_1 \underset{(\lambda,\geq\kappa)}{\leq} L_2^* \overset{u}{\leq} L_2 \leq M$$

Prop: Say K is δ-tame for regular $\delta < \lambda^+$.

If $\langle M_i : i \leq \delta \rangle$ a chain where $M_\delta = \bigcup_{i<\delta} M_i$, $p \in S(M_\delta)$

and M_δ is δ^+-saturated, then $\exists i < \delta$ $p \perp$-def$/M_i$.

proof: ETS p λ-dns$/M_i$ some i

$\Rightarrow p$ $\perp$-def$/M_{i+1}$.

So say p λ-s$/M_i$ for all $i < \delta$.

So there are $N_i^1, N_i^2 \leq N_i$, $f_i : N_i^1 \underset{M_i}{\cong} N_i^2$ s.t.

$f_i(p \upharpoonright N_i^1) \neq p \upharpoonright N_i^2$

Cont $A_i \subseteq |N_i^2|$ s.t. $|A| \leq \delta$, $f_i(p \upharpoonright N_i^1) \upharpoonright A_i \neq p \upharpoonright A_i$.

Let $B = \bigcup_{i<\delta} A_i \cup f_i^{-1}[A_i]$.

$|B| \leq \delta$. Since M_δ is δ^+-sat, there is $b \in |M_\delta|$ where $b \models p \upharpoonright B$.

Say $b \in M_{i_0}$, $i_0 < \delta$. Let $i = i_0 + 1$. Then

$$f_i(p \upharpoonright N_i^1) \upharpoonright A_i = f_i(\mathrm{gtp}(b / f_i^{-1}[A_i], M_\delta)) \upharpoonright A_i = \mathrm{gtp}(f_i(b)/A_i, M_\delta)$$
$$= \mathrm{gtp}(b/A_i, M_\delta) = p \upharpoonright A_i \quad \text{※} \quad //$$

Corollary: If $N_1 \underset{M}{\cong} N_2$, then $\mathrm{cf}(\delta_1) \geq \kappa^u(\perp)$.

Proof: ETS $\perp$ has δ_1-l.c., $\mathrm{cf}(\delta_1) = \delta_1$.

Say $\langle M_i : i \leq \delta_1 \rangle$ universal, $M_{\delta_1} = \bigcup_{i<\delta_1} M_i$.

$p \in S(M_{\delta_1})$.

Then $M_{\delta_1} \cong N_1 \cong N_2 \implies M_{\delta_1}$ (λ, δ_2)-lim $\implies \delta_2$-sat

$\implies \delta_1^+$-sat.

So by prop, $\exists i < \delta_1$ pcl-def$/M_i$. //