WAECO Intro to AECs talk Part 2.

Throughout, $\mathbf{K} = (K, \leq)$ is an AEC.

In this talk we'll prove Shelah's presentation theorem for AECs as an example of a substantial theorem.

First, some technology:

<u>Definition</u>: • Suppose $I = \langle |I|, \leq_I \rangle$ is a partially ordered set.

- I is a <u>directed set</u> if $\forall s, t \in I \ \exists r \in I$ s.t. $s \leq_I r$ and $t \leq_I r$
- If I is directed set, $\langle M_i : i \in I \rangle$ is a <u>directed system</u> if $s \leq_I t \implies M_s \leq_K M_t$

  for all $s, t \in I$.

This gives ways to discuss 'chains' not of the form $\langle M_i : i < \delta \rangle$.

<u>Proposition</u>: If $\{M_s : s \in I\}$ is a directed system in $\mathbf{K}$, then setting $\bigcup_{s \in I} M_s = \left(\bigcup_{s \in I} |M_s|, \bigcup_{s \in I} f^{M_s}, c^{M_s}, \bigcup_{s \in I} R^{M_s} \right)_{f, c, R \in L(K)}$ any $s \in I$

- $\bigcup_{s \in I} M_s \in K$
- $M_{s_0} \leq_K \bigcup_{s \in I} M_s$ for all $s_0 \in I$
- If $M_s \leq_K N$ for all $s \in I$, $\bigcup_{s \in I} M_s \leq_K N$.

<u>Proof</u>: By induction on $|I|$.

For I finite, $\bigcup_{s \in I} M_s = M_{\max(I)}$, so easy.

<u>For $\|I\| = \aleph_0$</u>: List out $|I| = \{a_n : n \in \omega\}$.

Get $\{b_n : n \in \omega\}$ such that $a_0 = b_0$, $b_{n+1} \geq a_{n+1}$, b_n (possible as I is directed).

Then $\bigcup_{s\in I} M_s = \bigcup_{n\in\omega} M_{b_n}$.

The requirements all follow now from the TV chain axioms.

For $\|I\| > \aleph_0$: Find a resolution for I, ie $\langle I_\alpha : \alpha < \|I\| \rangle$ a continuous increasing chain where $I = \bigcup_{\alpha<\|I\|} I_\alpha$, $\|I_\alpha\| = \aleph_0 + |\alpha|$ (possible by repeated applications of LS axiom).

If $M_s \leq N$ for all s,

Then $$\bigcup_{s\in I} M_s = \bigcup_{\alpha<\|I\|} \Big(\underbrace{\bigcup_{s\in I_\alpha} M_s}_{\substack{\in K \text{ by IH} \\ \leq N \text{ by IH} \\ \geq M_{s_0} \text{ for all } s_0\in I}} \Big)$$

So $\bigcup_{s\in I} M_s$ $\in K$ by the TV chain axioms

$\leq N$ by the TV chain axioms

$\geq M_{s_0}$ by the TV chain axioms for all $s_0 \in I$. //

Fact: $\forall M \in K$ $\exists \{M_s : s\in I\}$ a directed system s.t. $\bigcup_{s\in I} M_s = M$ and $\|M_s\| = LS(K)$ for all $s\in I$.

proof: Let $I = \{ A \subseteq |M| : |A| < \aleph_0 \}$ ordered by subset.

I is directed $(A\cup B \supseteq A, B)$.

Define M_A by induction on $\|A\|$

- $M_\emptyset \leq M$ of size $LS(K)$ (by LS Ax)
- $M_A \leq M$ of size $LS(K)$ containing $\bigcup_{\substack{B\subsetneq A \\ |B|<|A|}} |M_B|$ (by LS Ax). //

Note: If you let $I' = {}^{<\omega}|M|$, and for $w\in I'$

$M'_w = M_{\mathrm{In}(w)}$

$\{M'_w : w\in I'\}$ is a directed system indexed by ${}^{<\omega}|M|$ instead.

Definition: • Let L, L_0 be languages where $L_0 \subseteq L$, T an L-theory, Γ a set of types in L.

$$PC(T, \Gamma, L_0) = \{M \upharpoonright L_0 : M \models T,\ M \text{ omits all } p \in \Gamma\}$$
$$= \{M \upharpoonright L_0 : M \in EC(T, \Gamma)\}$$

• We say an AEC is $PC_{\lambda,\mu}$ if $K = PC(T, \Gamma, L_0)$ for some T, Γ, L_0 where $|T| \leq \lambda$, $|\Gamma| \leq \mu$.

Theorem (Shelah's presentation theorem): Let K be an AEC where $LS(K) = \lambda$. Then K is $PC_{\lambda, 2^\lambda}$.

proof: let $L_0 = L(K)$. Let $L = L_0 \cup \{F_i^n : n < \omega, i < LS(K)\}$ where the F_i^n are new n-ary constant symbols.

Let $T = \{\forall x_0 \cdots x_{n-1}\ F_i^n(x_0, \dots, x_{n-1}) = x_i : n < \omega, i < n\}$.

Now to define Γ ...

For $M \models T$ and $\underset{\sim}{a} \in {}^{<\omega}|M|$

$$A_{\underset{\sim}{a}}^M := \{\tau^M(\underset{\sim}{a}) : \tau \text{ a term in } L\}$$

i.e. the outputs where the inputs are from a.

$$N_{\underset{\sim}{a}}^M := (A_{\underset{\sim}{a}}^M, F^M \upharpoonright A_{\underset{\sim}{a}}^M, R^M \upharpoonright A_{\underset{\sim}{a}}^M, c^M)_{F, R, c \in L}.$$

(maybe $N_{\underset{\sim}{a}}^M \notin K$ (or not even a model) but it's still well defined).

Let $p_{\underset{\sim}{a}}^M := \{\varphi(\underset{\sim}{x}) : \varphi(\underset{\sim}{x}) \text{ is quantifier free in } L,\ M \models \varphi(\underset{\sim}{a})\}$

Lemma: If $M_\ell \models T$, $\underset{\sim}{a}_\ell \in {}^{<\omega}|M|$ for $\ell = 1, 2$, then

$$p_{\underset{\sim}{a}_1}^M = p_{\underset{\sim}{a}_2}^M \iff \exists f : N_{\underset{\sim}{a}_1}^{M_1} \cong N_{\underset{\sim}{a}_2}^{M_2} \text{ s.t. } f(\underset{\sim}{a}_1) = \underset{\sim}{a}_2.$$

proof: $\Rightarrow$: If $d \in N_{\underline{a}}^{M_1}$, $d = \tau^{M_1}(\underline{a}_1)$ for some τ in L

set $f(d) = \tau^{M_2}(\underline{a}_2)$

As $p_{\underline{a}_1}^{M_1} = p_{\underline{a}_2}^{M_2}$, $M_1 \models \tau(\underline{a}_1) = \tau'(\underline{a}_1) \Leftrightarrow M_2 \models \tau(\underline{a}_2) = \tau'(\underline{a}_2)$

So f is well defined and bijective.

And $M_1 \models R(\underline{\tau}(\underline{a}_1)) \Leftrightarrow M_2 \models R(\underline{\tau}(\underline{a}_2))$

So f is an isomorphism. $//\Rightarrow$

$\Leftarrow$: Enough to show they satisfy the same atomic formulas.

f iso, $f(\underline{a}_1) = \underline{a}_2 \Rightarrow [M_1 \models \tau(\underline{a}_1) = \tau'(\underline{a}_1) \Leftrightarrow M_2 \models \tau(\underline{a}_1) = \tau'(\underline{a}_2)]$

and $\Rightarrow [M_1 \models R(\tau(\underline{a}_1)) \Leftrightarrow M_2 \models R(\tau(\underline{a}_2))]$

$//\Leftarrow$ $//$lem.

So done.

Finally, set $T' = \{p_{\underline{a}}^M : N_{\underline{a}}^M \upharpoonright L_0 \notin K$ or $\exists \underline{b} \subseteq \underline{a}$ $N_{\underline{a}}^M \upharpoonright L_0 \not\leq_K N_{\underline{b}}^M \upharpoonright L_0\}$.

Claim: $K = PC(T, T', L_0)$.

proof: $\supseteq$: Say $M \in EC(T, T')$.

Then $\{N_{\underline{a}}^M \upharpoonright L_0 : \underline{a} \in {}^{<\omega}|M|\}$ is a directed system in K

(since M omits T').

So $M \upharpoonright L_0 = \bigcup_{\underline{a} \in {}^{<\omega}|M|} (N_{\underline{a}}^M \upharpoonright L_0) \in K$

$\subseteq$: Say $M \in K$. We need an expansion M' of M to L

s.t. $M' \in EC(T, T')$.

By the remark, there is a directed system $\{M_{\underline{a}} : \underline{a} \in {}^{<\omega}|M|\}$

where $\|M_{\underline{a}}\| = LS(K)$.

Define $F_i^n(\underline{a})$ where $lh(\underline{a}) = n$ by:

- Set $\langle b_i : i < LS(K) \rangle = |M_{\underline{a}}|$ where $\langle b_0, \dots, b_{n-1} \rangle = \underline{a}$
- Set $F_i^n(\underline{a}) = b_i$.

Now, set $M' = (M, F_i^n)_{i < LS(K),\, n < \omega}$

$M' \models T$ by choice of F_i^n.

Claim: M' omits T'.

proof: Say $\underline{a} \in M'$ realises $p_{\underline{b}}^{M^2}$

As $\underline{a} \models p_{\underline{a}}^{M'}$, there $\exists$ $f : N_{\underline{a}}^{M'} \cong N_{\underline{b}}^{M^2}$

As $|N_{\underline{a}}^{M'}| = |M_{\underline{a}}|$, $N_{\underline{a}}^{M'} \restriction L_0 \in K$, so $N_{\underline{b}}^{M^2} \restriction L_0 \in K$ by 1st iso axiom.

Say $\underline{c} \subsetneq \underline{b}$. So there $\exists$ $\underline{d} \subsetneq \underline{a}$ st $f(\underline{d}) = \underline{c}$.

So $p_{\underline{d}}^{M'} = p_{\underline{c}}^{M^2}$. Hence $g : N_{\underline{d}}^{M'} \cong N_{\underline{c}}^{M^2}$ where $g(\underline{d}) = \underline{c}$.

Then $g \subseteq f$, and

$$\begin{tikzcd}
N_{\underline{a}}^{M'} \arrow[r, "f"] & N_{\underline{b}}^{M^2} \\
N_{\underline{d}}^{M'} \arrow[u, "id"] \arrow[r, "\cong", "g"'] & N_{\underline{c}}^{M^2} \arrow[u, "id"']
\end{tikzcd}$$

By the 2nd iso axiom, $N_{\underline{c}}^{M_2} \leq_K N_{\underline{b}}^{M_2}$.

So $p_{\underline{b}}^{M^2} \notin T'$. Hence M' omits T'.

So $M \in PC(T, T', L_0)$. // claim // Shelah.

In fact, $M \leq_K N \iff \exists M', N' \in EC(T, T')$ s.t.

$M' \subseteq N'$, $M' \upharpoonright L_0 = M$, $N' \upharpoonright L_0 = N$.

So

<u>Corollary</u>: $|\{K : LS(K) = \lambda\}/\sim| \leq 2^{(2^\lambda)}$

2^λ choices of L, L_0

2^λ choices of T

$2^{(2^\lambda)}$ choices of T'.

END OF PART II.