On the spectrum of limit models

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Based on work with Marcos Mazari-Armida [Beard and Mazari-Armida, 2025]

Overview

1. Preliminaries
2. Results
3. Applications

Preliminaries - Setting, universal models

For this talk, $\mathbf{K} = (K, \leq_{\mathbf{K}})$ is an abstract elementary class (AEC) with amalgamation, joint embedding, and no maximal models, and $\lambda \geq LS(\mathbf{K})$.
$\mathbf{K}_\lambda$ is the set of models in $\mathbf{K}$ of size λ.

Definition

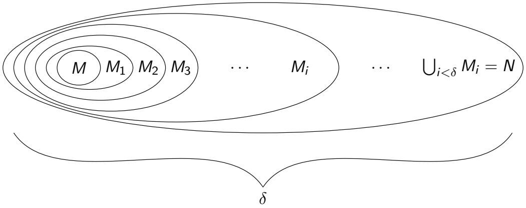
For $M, N \in \mathbf{K}_\lambda$, N is *universal* over M if $M \leq_{\mathbf{K}} N$ and whenever $N' \in \mathbf{K}_\lambda$ and $M \leq_{\mathbf{K}} N'$, there exists a $\mathbf{K}$-embedding $f : N' \to N$ fixing M. We write $M \leq^u_{\mathbf{K}} N$.

$$\begin{tikzcd}
 & N \\
M \arrow[ur] \arrow[dr] & \\
 & N' \arrow[uu, dashed, "f"']
\end{tikzcd}$$

Preliminaries - Limit models

Definition

Say $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\delta < \lambda^+$ is a limit ordinal. $N \in \mathbf{K}$ is a (λ, δ)-*limit model over* M if there exists a $\leq^u_{\mathbf{K}}$-increasing sequence $\langle M_i : i < \delta \rangle$ where $M_0 = M$ and $N = \bigcup_{i<\delta} M_i$.



- If κ is regular and $\mathrm{cf}(\delta) \geq \kappa$, we say N is a $(\lambda, \geq \kappa)$-limit model over M.

Preliminaries - Limit model

Fact (Existence of limit models)

If $\mathbf{K}$ is λ-stable, then for any $M \in \mathbf{K}_\lambda$, there exists $N \in \mathbf{K}_\lambda$ universal over M. And for any $\delta < \lambda^+$ limit, there is a (λ, δ)-limit N over M.

Question

Under what circumstances are two limit models isomorphic?

Fact (Uniqueness up to cofinality)

Suppose $\mathbf{K}$ is λ-stable, $\delta_1, \delta_2 < \lambda^+$ are limits with $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$ and N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$. Then $N_1 \underset{M}{\cong} N_2$.

With this in mind, we may as well focus on when δ is a regular cardinal.

Preliminaries - Independence relations

Before we state the theorem, we briefly introduce *independence relations*.

Definition

Let $\mathbf{K}'$ be a sub-AC of $\mathbf{K}$. An *independence relation on* $\mathbf{K}$ is a relation on triples (M_0, a, M, N) where $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ and $a \in N$ (written $a \underset{M_0}{\overset{N}{\downarrow}} M$) that satisfies *invariance*, *monotonicity*, and *base monotonicity*.

If $p \in \mathbf{gS}(M)$ where $M_0 \leq_{\mathbf{K}} M$, we say p $\downarrow$-does not fork over M_0 if there exist $N \geq_{\mathbf{K}} M$ and $a \in N$ such that $p = \mathbf{gtp}(a/M, N)$ and $a \underset{M_0}{\overset{N}{\downarrow}} M$.

Similarly to first order, we can define notions of uniqueness, extension, (universal) continuity, $(\geq \kappa)$-local character, and non-forking amalgamation.

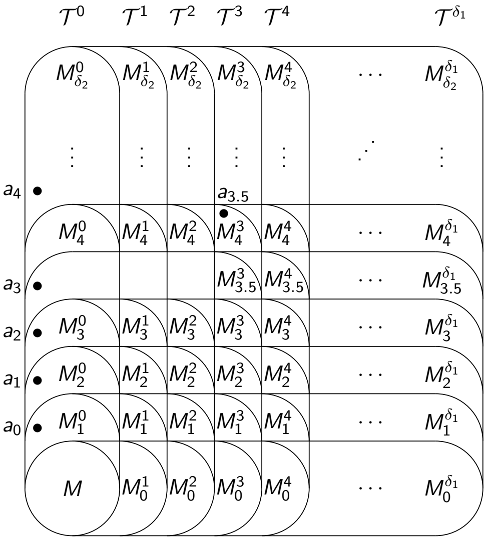
Though non-forking is an obvious choice in first order complete stable theories, such don't exist in general in stable AECs.

Theorem (B., Mazari-Armida)

Suppose $\mathbf{K}$ *is an AEC stable in* $\lambda \geq \mathrm{LS}(\mathbf{K})$, *with amalgamation, joint embedding, and no maximal models, and* $\mathbf{K}$ *is* $\aleph_0$*-tame. Suppose* $\mathop{\smile\!\!\!\!\!\mid}$ *is an independence relation on* $\mathbf{K}_\lambda$ *that satisfies uniqueness, extension, continuity, non-forking amalgamation, and* $(\geq \kappa)$*-universal local character in some regular* $\kappa < \lambda^+$*. Let* $\kappa(\mathop{\smile\!\!\!\!\!\mid}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ *be the least such* κ.

Suppose $N_1, N_2, M \in \mathbf{K}_\lambda$*, and* $\mu_1 < \mu_2 < \lambda^+$ *are regular, and* N_l *is a* (λ, μ_1)*-limit model over* M *for* $l = 1, 2$*. Then* $N_1 \underset{M}{\cong} N_2$ *if and only if* $\mu_1 \geq \kappa(\mathop{\smile\!\!\!\!\!\mid}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$.

Results - Long limits are isomorphic



To prove long limit models are isomorphic, we build a theory of *towers* similar to [Vasey, 2019]. Our constructions widen the assumptions to allow general independence relations, possibly with $\kappa(\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\vert$}}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) > \aleph_0$, and possibly defined only on $(\lambda, \geq \kappa)$-limit models. $\aleph_0$-tameness is not needed for this argument.

Results - Short limits are non-isomorphic

To prove short limit models are non-isomorphic, we show that $\mathop{\smile\!\!\!\!\!\!\vert}$ is 'close' to a relation called λ-non-splitting.

Using $\aleph_0$-tameness, we can show that if a (λ, μ_1)-limit model and a (λ, μ_2)-limit model are isomorphic with $\mu_1 < \mu_2$ both regular, then we have μ_1-local character of λ-non-splitting. This is where $\aleph_0$-tameness is used.

This implies μ_1-local character of $\mathop{\smile\!\!\!\!\!\!\vert}$, since $\mathop{\smile\!\!\!\!\!\!\vert}$ and λ-non-splitting are 'close' to each other. So $\mu_1 \geq \kappa(\mathop{\smile\!\!\!\!\!\!\vert}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$.

As isomorphic limit models are long, we must have that the short limit models are non-isomorphic.

If we assume we have a relation satisfying our conditions at all λ, the cardinal $\kappa(\mathop{\smile}\limits^{|}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ stabilises for high enough stable λ to a value $\chi(\mathop{\smile}\limits^{|}, \mathbf{K}, \leq^u_{\mathbf{K}})$. This gives the following:

Theorem (B., Mazari-Armida)

Assume $\mathop{\smile}\limits^{|}$ is defined on $\mathbf{K}$ and has all the properties from before, with $\kappa \leq \mathrm{LS}(\mathbf{K})$. Let $\lambda \geq \beth_{(2^{LS(\mathbf{K})})^+}$ such that $\mathbf{K}$ is stable in λ.

Suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ over } M \iff \mathrm{cf}(\delta_1) \geq \chi(\mathop{\smile}\limits^{|}, \mathbf{K}, \leq^u_{\mathbf{K}})$$

Applications - First order stable theories

Given first order complete stable theory T, non-forking is a relations satisfying our hypotheses in $\mathbf{K} = (\mathrm{Mod}(T), \preccurlyeq)$ with $\kappa(\smile, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) = \kappa_r(T) \leq |T|$ for all stable $\lambda \geq LS(\mathbf{K})$. Thus we have:

Theorem

Let T be a first order complete stable theory. Then for every stable $\lambda \geq |T|$, and any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \textit{ is isomorphic to } N_2 \textit{ over } M \iff \mathrm{cf}(\delta_1) \geq \kappa_r(T)$$

Though the $\Leftarrow$ implication was known, the $\Rightarrow$ implication was previously unexplored.

In fact, in nice μ-tame AECs, Vasey showed in [Vasey, 2016] the existence of an independence relation with many nice properties. With a little work, our 'long limit' result gives the uniqueness of $(\lambda, \geq \mu^+)$-limit models for all $\lambda \geq \mu^+$.

Applications - Algebraic AECs

There are several examples of algebraic AECs satisfying the assumptions of our theorem, but let's focus on one.

Let $\mathbf{K}^{R-\text{mod}}$ be the AEC of modules over a fixed ring R with $\subseteq$ (or $\leq_p$, pure substructure). In this case, $\chi(\perp, \mathbf{K}^{R-\text{mod}}, \leq^u_{\mathbf{K}}) = \gamma_r(R)$. In fact, considering when these are $\aleph_0$, we have

Theorem

The following are equivalent:

1. *R is Noetherian*
2. *for any $\lambda \geq \beth_{(2^{\aleph_0})^+}$, all λ-limit models in $\mathbf{K}^{R-\text{mod}}$ are isomorphic*

References

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Thanks for listening!