Forking independence from a categorical point of view.

Based on paper of the same name by Lieberman, Rosický, Vasey.

Normal model theory independence relation on models: relation $\mathop{\smile\!\!\!\!\!\!|}\,$ on

$$M_0 \;\begin{matrix} \leq M_1 \leq \\ \leq M_2 \leq \end{matrix}\; M_3 \qquad\qquad M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$$

'M_1, M_2 independent / M_0 (in M_3)'.

One perspective

$$\begin{tikzcd} M_1 \arrow[r, "id"] & M_3 \\ M_0 \arrow[u, "id"] \arrow[r, "id"'] \arrow[ur, phantom, "\mathop{\smile\!\!\!\!\!\!|}"] & M_2 \arrow[u, "id"'] \end{tikzcd}$$

the diagram is '$\mathop{\smile\!\!\!\!\!\!|}$ - independent.'

$\mathop{\smile\!\!\!\!\!\!|}$ satisfies nice properties – invariance, monotonicity, base monotonicity, transitivity, uniqueness, extension, symmetry, etc.

We want to extend this to

- Arbitrary categories (not just classes of models)
- Arbitrary maps

What can we preserve?

Work in a fixed category K.

Definition:

$$\begin{tikzcd} M_1 \arrow[r, "g_1"] & M_3 \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\circlearrowleft"] & M_2 \arrow[u, "g_2"'] \end{tikzcd}$$

(g_1, g_2) an amalgam of (f_1, f_2).

(f_1, f_2) a span in K.

(f_1, f_2, g_1, g_2) an amalgamation diagram.

Interested in $\perp(f_1, f_2, g_1, g_2)$ on amalgam diagrams.
To be sensible, 'identical' amalgams should be the same under $\perp$.

__$\sim^*$ on amalgam diagrams__ : $(f_1, f_2, g_1^a, g_2^a) \sim^* (f_1, f_2, g_1^b, g_2^b)$

iff $\exists\ g^a, g^b, N$ s.t.

$$\begin{tikzcd}
& N^b \arrow[rr, "g_b"] & & N \\
M_1 \arrow[ur, "g_1^b"] \arrow[rr, "g_1^a"'] & & N^a \arrow[ur, "g_a"'] & \\
& M_2 \arrow[uu, "g_2^b"] \arrow[ur, "g_2^a"'] & & \\
M_0 \arrow[uu, "f_1"] \arrow[ur, "f_2"'] & & &
\end{tikzcd}$$

commutes.

$\sim$ means thinking M_1, M_2 are 'smooshed together' over M_0 is basically the same.

$\sim$ = transitive cl. of $\sim^*$.

__Defn__: An __independence relation__ on K is a relation on amalg. tuples that is closed under $\sim$.

$\perp(f_1, f_2, g_1, g_2) \iff$

$$\begin{tikzcd}
M_1 \arrow[r, "g_1"] \arrow[dr, phantom, "\perp"] & N \\
M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"']
\end{tikzcd}$$

__Def__: $\perp$ is __invt__ if invariant under 'isos of amalg diagrams'

$$\begin{tikzcd}
M_1 \arrow[r, "g_1"] \arrow[dr, phantom, "\perp"] \arrow[ddrr, bend right, "h_1", "\cong"'] & N \arrow[ddrr, "h_3", "\cong"'] & & \\
M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ddrr, bend right, "h_0", "\cong"'] & M_2 \arrow[u, "g_2"'] \arrow[ddrr, bend right, "h_2", "\cong"'] & & \\
& & M_1' \arrow[r, "g_1"] \arrow[dr, phantom, "\perp"] & N' \\
& & M_0' \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2' \arrow[u, "g_2"']
\end{tikzcd}$$

then

Def: $\hat{\perp}$ dual of $\perp$

$$\hat{\perp}(f_1, f_2, g_1, g_2) \iff \perp(f_2, f_1, g_2, g_1)$$

$\perp$ is symmetric iff $\hat{\perp} = \perp$.

$$\begin{tikzcd} \cdot \arrow[r, "g_1"] & \cdot \\ \cdot \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\perp"] & \cdot \arrow[u, "g_2"'] \end{tikzcd} \Rightarrow \begin{tikzcd} \cdot \arrow[r, "g_2"] & \cdot \\ \cdot \arrow[u, "f_2"] \arrow[r, "f_1"'] \arrow[ur, phantom, "\perp"] & \cdot \arrow[u, "g_1"'] \end{tikzcd}$$

Def: $\perp$ has the existence property if $\forall (f_1, f_2)$ spans, $\exists (g_1, g_2)$ amalg. s.t. $\perp(f_1, f_2, g_1, g_2)$

$$\begin{tikzcd} M_1 \arrow[r, "g_1"] & N \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\perp"] & M_2 \arrow[u, "g_2"'] \end{tikzcd}$$

Lemma: If $\perp$ has existence and

$$\begin{tikzcd} M_1 \arrow[r, "g_1"] & M_3 \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"'] \end{tikzcd}$$

commutes & one of f_1, f_2 is an iso, then the diagram is $\perp$.

proof: Proof for f_1.

$$\begin{tikzcd} & M_3' \arrow[rr] & & N \\ M_1 \arrow[ur, "h_1"] \arrow[rr, "g_1"] & & M_3 \arrow[ur, "AP"'] & \\ M_0 \arrow[u, "f_1 \cong"] \arrow[r, "f_2"'] \arrow[u, phantom, shift left=6, "\perp"] & M_2 \arrow[uu, "h_2"'] \arrow[ur, "g_2"'] & & \end{tikzcd}$$

$\left(\begin{tikzcd} M_3' \arrow[r] & N \\ M_1 \arrow[u] \arrow[r] & M_3 \arrow[u] \end{tikzcd} \quad \text{amalgamates as } M_1 \text{ is basically } M_0 \longrightarrow M_2 \right).$

Def: $\perp$ has *uniqueness* if whenever $\perp(f_1, f_2, g_1, g_2)$, $\perp(f_1, f_2, g_1', g_2')$, then $(f_1, f_2, g_1, g_2) \sim (f_1, f_2, g_1', g_2')$.

$$\begin{tikzcd}
 & & M_3 & \\
M_1 \arrow[urr, "g_1"] \arrow[rrr, "g_1'"] & & & M_3' \\
M_0 \arrow[u, "f_1"] \arrow[rr, "f_2"'] & & M_2 \arrow[uu, "g_2"] \arrow[ur, "g_2'"'] &
\end{tikzcd}$$

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Def: $\perp$ is *right transitive* if

$$\begin{tikzcd}
M_1 \arrow[r, "f_{13}"] \arrow[dr, phantom, "\perp"] & M_3 \arrow[r, "f_{35}"] \arrow[dr, phantom, "\perp"] & M_5 \\
M_0 \arrow[u, "f_{01}"] \arrow[r, "f_{02}"'] & M_2 \arrow[u, "f_{23}"] \arrow[r, "f_{24}"'] & M_4 \arrow[u, "f_{45}"']
\end{tikzcd}
\implies
\begin{tikzcd}
M_1 \arrow[r, "f_{35} \circ f_{13}"] \arrow[dr, phantom, "\perp"] & M_5 \\
M_0 \arrow[u, "f_{01}"] \arrow[r, "f_{24} \circ f_{02}"'] & M_4 \arrow[u, "f_{45}"']
\end{tikzcd}$$

Note: Sym gives 'left trans' $\quad \boxed{} \implies \square$

left + right = trans.

Prop: $\perp$ has existence, right trans $\implies$ $\perp$ is invariant.

proof: Say

$$\begin{tikzcd}
M_1 \arrow[r, "g_1"] & M_3 & & M_1' \arrow[r, "g_1'"] & M_3' \\
M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"'] & & M_0' \arrow[u, "f_1'"] \arrow[r, "f_2'"'] & M_2' \arrow[u, "g_2'"']
\end{tikzcd}$$

with $M_1 \cong M_1'$, $M_3 \cong M_3'$, $M_0 \cong M_0'$, $M_2 \cong M_2'$.

$$\begin{tikzcd}
M_1' \arrow[r, "\cong"] \arrow[rrr, bend left, "f_1'"] \arrow[dr, phantom, "\perp"] & M_1 \arrow[r, "g_1"] \arrow[dr, phantom, "\perp"] & M_3 \arrow[r, "\cong"] \arrow[dr, phantom, "\perp"] & M_3' \\
M_0' \arrow[u, "f_1'"] \arrow[r, "\cong"'] \arrow[rrr, bend right, "f_2'"'] & M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"] \arrow[r, "\cong"'] & M_2' \arrow[u]
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
M_1' \arrow[r, "g_1'"] \arrow[dr, phantom, "\perp"] & M_3' \\
M_0' \arrow[u, "f_1'"] \arrow[r, "f_2'"'] & M_2' \arrow[u, "g_2'"']
\end{tikzcd}$$

By ext + $\cong$ iso

Last ingredient: Category theory stuff & accessible categories

Defn: Given $\overset{|}{\smile}$ right transitive with existence, K_{NF} is the category

Objects: $f : M_1 \to M_2$ from K.

Morphisms: A morphism from $f : M_1 \to M_2$ to $g : N_1 \to N_2$ is a pair $(h_1 : M_1 \to N_1,\ h_2 : M_2 \to N_2)$ s.t.

$$\begin{tikzcd} M_2 \arrow[r, "h_2"] & N_2 \\ M_1 \arrow[u, "f_1"] \arrow[r, "h_1"'] \arrow[ur, phantom, "\overset{|}{\smile}"] & N_1 \arrow[u, "g_1"'] \end{tikzcd}$$

(so a subcat of K^2, cat of morphisms).

Note: RT gives composition, existence gives identity.

Defn: $\overset{|}{\smile}$ is stable if $\overset{|}{\smile}$ satisfies symmetry, transitivity, existence, uniqueness, and K_{NF} is accessible.

What is an accessible category? It captures the idea that every $M \in \mathcal{C}$ can be decomposed into 'small' chunks

(think $M = \bigcup_{\sigma \in {}^{<\omega}|M|} M_\sigma$ where $\|M_\sigma\| = LS(K)$).

(Fact: up to category equivalence, μ-AECs are the same as accessible categories whose morphisms are all mono).

Def: • I poset is λ-directed if $\forall J \subseteq I$ s.t. $|J| < \lambda$, J has an upper bound in I

• In cat K, a λ-directed colimit is a colimit

$$\langle N_i \longrightarrow N \;:\; i \in I \rangle \quad \text{of a system}$$
$$\langle N_i : i \in I \rangle, \quad \langle N_i \xrightarrow{\phi_{ij}} N_j \;:\; i < j \in I \rangle$$

where I is λ-directed.

• $M \in K$ is λ-presentable if for any $\langle N_i \xrightarrow{\phi_i} N : i \in I \rangle$ λ-dir colimit, for any $f : M \longrightarrow N$,

$$f = \phi_i f_i \text{ for some } f_i : M \to N_i .$$

(in an AEC, M is λ-pres $\iff \|M\| < \lambda$.)

• K is λ-accessible if it has λ-dir colimits and contains a set S of λ-pres objects s.t. every $M \in K$ is a λ-dir colimit of objects in S.

(in an AEC, S = one of each model size $< \lambda$ up to iso.

think $\quad M = \bigcup_{\sigma \in {}^{<\omega}|M|} M_\sigma$)

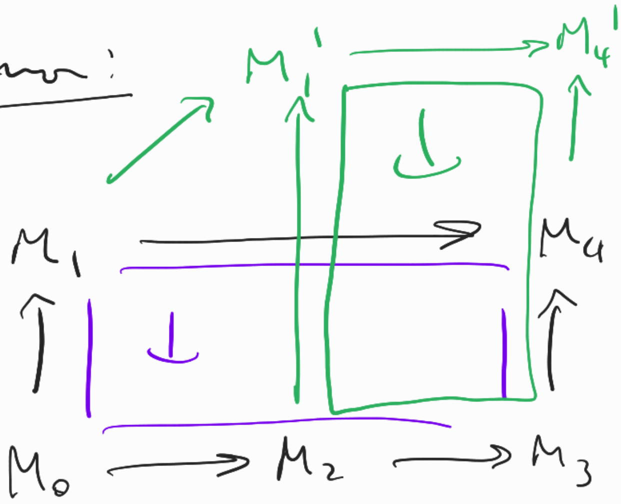
Fact: $\mathop{\smile}\limits^{|}$ stable $\Rightarrow$ $\mathop{\smile}\limits^{|}$ has (right) monotonicity, base monotonicity, K is accessible, ...

RMon:

$$\begin{tikzcd}
M_1 \arrow[rr] & & M_4 \\
M_0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\mathop{\smile}\limits^{|}"] & M_2 \arrow[r] \arrow[ur, phantom, "\mathop{\smile}\limits^{|}"] & M_3 \arrow[u]
\end{tikzcd}$$

$\mathop{\smile}\limits^{|} \Rightarrow \mathop{\smile}\limits^{|}$

R - base amr :



$\perp \implies \exists \otimes \perp$

E.g., (1) $K = (F\text{Vec sp}, \subseteq)$

$$M_0 \subseteq M_1, M_2 \subseteq M_3$$

$$M_1 \underset{M_0}{\overset{M_3}{\perp}} M_2 \iff M_1 \cap M_2 = M_0$$

(2) $K = (ACF_p, \subseteq)$

$M_1 \underset{M_0}{\overset{M_3}{\perp}} M_2 \iff M_0 \subseteq M_1, M_2 \subseteq M_3$ and $\forall A \subseteq |M_1|$ finite $\forall a \in |M_1|$ transcendence degree of a/AM_2 $= \operatorname{trdeg}_{AM_0}(a)$.

(3) F.o. forking

(4) $K = (Gr, \subseteq_{\text{full subgraph}})$

$M_1 \underset{M_0}{\overset{M_3}{\perp}} M_2 \iff M_0 \subseteq M_1, M_2 \subseteq M_3$, $M_1 \cap M_2 = M_0$, and no edges from M_3 join $M_1 \setminus M_0$ and $M_2 \setminus M_0$.

. . .

Remark: Given $\mathop{\smile\!\!\!\!\!\!|}$ on μ-AEC

define $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} B$ for all $A, B \subseteq N$ by

$$A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} B \iff \exists M_1, M_2 \text{ s.t. } A \subseteq M_1,\ B \subseteq M_2,\ M_1 \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$$

Define $p \in S(A)$, $M_1 \subseteq A$,

p $\mathop{\smile\!\!\!\!\!\!|}$-def $/ M_0 \iff \exists b$ $p = \text{gtp}(b/A, N)$, $b \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} A$.

it satisfies all the properties as we expect them.

$\mathop{\smile\!\!\!\!\!\!|}$ satisfies • inv, mon, base mon, sym, existence,

trans, & uniqueness

(more or less for basic props)

• Some local char, μ-w.p.

(from accessible).