Model Theory

Varsity Practice 03/23/2025

1 Definitions

Loads of objects of interest to mathematicians can be framed as what we'll call a 'model' - that is, a set with structure given by relations (e.g. $<$, $\in$), functions ($+$, $\times$, $(-)^{-1}$), and constants (0, 1, e). Model theory is a framework for studying structures that share similar properties and how they interact, and for creating new structures with nifty querks.

1.1 Models

Definition 1. A *relation* R on a set A is some subset $R \subseteq A^n$ for some n. As a shorthand, we write $R(a_1, \ldots, a_n)$ instead of $(a_1, \ldots, a_n) \in R$.

You can think of a relation that takes in a tuple from A and outputs a 'true/false' value.

Example 2. The following are relations on $\mathbb{R}$:

1. $<$ (or if you prefer, $E_<$ is a relation on $\mathbb{R}$, where $E_< = \{(x, y) \in \mathbb{R}^2 : x < y\}$
2. $Q = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 4\}$

So think for example Q returns 'true' for input $(1, 2, 1)$, 'false' for $(1, 1, 0)$.

Definition 3. A model is a set with relations, functions, and constants. Formally, a model is a structure $M = (|M|, f_i^M, R_j^M, c_k^M)_{i \in I, j \in J, k \in K}$ where $|M|$ is a set (called the *universe*), for each $i \in I$ f_i^M is a function $f_i^M : |M|^n \to |M|$ for some $n \in \mathbb{N}$, for each $j \in J$ $R_j^M \subseteq |M|^n$ for some $n \in \mathbb{N}$, and for each $k \in K$ $c_k^M \in |M|$.

This looks pretty disgusting, so let's look at some examples:

Example 4.

1. The real numbers with $+, \cdot, <, 0, 1$:

$$M = (\mathbb{R}, +, \cdot, <, 0, 1)$$

Where $\mathbb{R}$ is the universe, $+$ and $\cdot$ are functions (with two inputs), $<$ is a relation, and 0 and 1 are constants.

2. The real numbers with only $<$:

$$M = (\mathbb{R}, <)$$

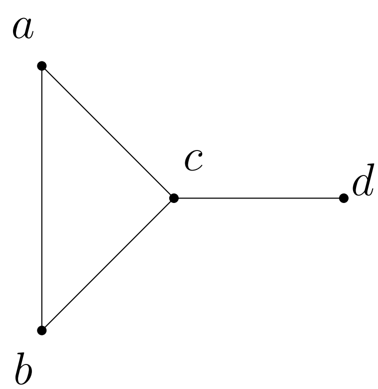
Here $<$ is again a relation. This model has somehow less information than Example 4.1 - we can only talk about the order using it's symbols. Note it's fine for a model not to not have any functions, relations, or constants.

3. The rational numbers with $+, \cdot, <, 0, 1$:

$$M = (\mathbb{Q}, +, \cdot, <, 0, 1)$$

Where $\mathbb{Q}$ is the universe, $+$ and $\cdot$ are functions (with two inputs), $<$ is a relation, and 0 and 1 are constants. Note the structure is more similar to Example 4.1 than Example 4.2, even though the previous two are both models over the same universe. In some sense, we can ask the same questions about Example 4.1 and Example 4.3 because we have the same symbols - more on this later.

4. Consider the graph below.



We can turn this into a model by giving the set $\{a, b, c, d\}$ a relation E saying x is related to y if and only if there is an edge from x to y:

$$M = (\{a, b, c, d\}, E)$$

where $E = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (c, d), (d, c)\}$ is a relation.

5. If you know what a group is, all groups are models: if G is a group with multiplication $\cdot$ and identity e, then

$$M = (G, \cdot, e)$$

is a model with underlying set G, 2-input function $\cdot$, and constant e. Similarly, rings, fields, vector spaces, and many other things mathematicians care about can be made into models. So, a lot of nice theorems in model theory immediately apply in all these settings.

1.2 Languages, formulas, sentences, and theories

When models have similar kinds of functions, relations, and constants, it makes more sense to compare them (see Example 4.3). Here's a way of formalising this:

Definition 5. A *language* is a list of symbols we can use for functions, relations, and constants. More precisely, it's a set of the form $L = \{f_i, R_j, c_k : i \in I, j \in J, k \in K\}$ is a list of symbols that we can use in a model.

We denote the language of a model by $L(M)$.

We normally use f_i, R_j, c_k for the symbols, and f_i^M, R_j^M, c_k^M for their interpretations in the models.

Example 6. 1. In Examples 4.1 and 4.3, the models use the same language $L = \{+, \cdot, <, 0, 1\}$. Technically since we use those as the symbols, we should have called them $+^M, \cdot^M, <^M, 0^M, 1^M$ in Example 4, but life is short.

2. In Example 4.2, we use language $\{<\}$.

3. In Example 4.4, we use language $\{E\}$. You might call this the *language of (undirected) graphs*, since we can describe every graph as with our example.

4. In Example 4.5, we use language $\{\cdot, e\}$, the *language of groups*.

Definition 7. If $L_0 \subseteq L(M)$, then $M \upharpoonright L_0$ is the restriction of M to L_0. That is, you only remember the function, relations, and constants from L_0. For example, $(\mathbb{R}, +, \cdot, <, 0, 1) \upharpoonright \{<\} = (\mathbb{R}, <)$.

As we saw above, $(\mathbb{R}, +, \cdot, <, 0, 1)$ and $(\mathbb{Q}, +, \cdot, <, 0, 1)$ use the same language, and they also believe similar things. For example, both believe that $x + y = y + x$ for all x and y. Note we only need the symbols from the common language to say this. In model theory, we are interested in what statements like these we can say hold in different structures. Formally, these statements are formulas and sentences.

Definition 8. A *term* in language L is an expression formed from variables, constants, and functions.

Think of terms as all the ways we can combine symbols to get things inside a model. For example, $f(x, g(c, y, z))$ is a term if f, g are functions in L, c is a constant, and x and y are variables - the output is still in the model. If $L = \{+, \cdot, <, 0, 1\}$, this could be $(x + y) \cdot 1$ or $1 + (1 + (1 + (1 + 1)))$.

Question 1.1. Which of the following are terms in L (where x, y are variables, c a constant symbol, and $+, -$ functions in L):

1. $x + y < x \cdot y$
2. $g(x, y) \cdot c$ where g is a function in L
3. $E(x, y) + z$ where E is a relation in L

Definition 9. A *formula* is any way we can combine terms with logical symbols ($\vee$ (or), $\wedge$ (and), $\neg$ (not), $\implies$ (implies), $\forall$ (for all), $\exists$ (there exists) with terms from L, $=$, and relation symbols from L to get a 'true/false statement'. An important distinction is that the quantifiers are over the whole model - that is, $\forall x \varphi(x)$ should be interpreted as meaning that all $x \in |M|$ (the universe of the model), $\varphi(x)$ holds.

The set of all formulas in L is $Fml(L)$.

Example 10. For example,

1. $x = 0$, where x is a variable and 0 is a constant
2. $E(x, y)$ where E is a relation
3. $1 < 0$

4. $x = y \wedge y = z \wedge z = c \wedge c \neq x$ where x, y, z are variables and c is constant
5. $\forall x \forall y(x < y \implies \exists z(x < z \wedge z < y))$
6. $\forall a \forall b \forall c((a + b) + c) = (a + (b + c))$

The system built in this way is often called *first order logic*. It is worth noting that, while this is a very expressive setup (we can say a lot with such formulas), there are certain things we cannot express. For example, there is no formula for 'there exist infinitely many x'. However, we can say 'there exist at least n x' for any $n \in \mathbb{N}$ - see Questions 2.1 and 2.2. And the more expressive logics tend not to behave as nicely as first order logic - for example, the compactness theorem normally fails to hold (see the next section).

Definition 11. Formally, a *sentence* is a formula with no *free* variables, where a variable is free if it does not appear under a quantifier (e.g. x is free in $\forall y(0 < x \wedge x < y)$, but y is not as it only appears in a chunk with $\forall$ in front of it). More informally, is a formula that always has a true or false value in a model - that is, there are no variables that haven't been quantified over.

The set of all sentences in L is $Sent(L)$.

For example, $x < 1 \wedge x = 0$ is not a sentence, since whether it is true could depend on what value x is - but $\forall x(x < 1 \wedge x = 0)$ is a sentence, since by quantifying over x, in any model it is either true or false.

Question 1.2. Which of the formulas from Example 10 are sentences?

Definition 12. Let φ be a sentence in the language of M. If φ is true in M, we write $M \models \varphi$, and say M *models* φ. If M does not model φ, we write $M \not\models \varphi$.

Question 1.3. True or false:

1. $(\mathbb{R}, +, \cdot, <, 0, 1) \models \forall x \forall y(x < y \implies \exists z(x < z \wedge z < y))$
2. $(\mathbb{Z}, +, \cdot, <, 0, 1) \models \forall x \forall y(x < y \implies \exists z(x < z \wedge z < y))$

Definition 13. We say a set of sentences $T \subseteq Sent(L)$ is *consistent* if there is at least one (non-empty) model M with language L such that for every $\varphi \in T$, $M \models \varphi$. In this case, we can call T a *theory*.

Example 14. 1. Let $L = \{<\}$ where $<$ is a 2-input relation symbol. Then

$$\begin{aligned} T_{lo} = \{&\forall x(\neg(x < x)), \\ &\forall x \forall y(x < y \implies \neg(y < x)), \\ &\forall x \forall y((x < y \vee y < x) \vee x = y), \\ &\forall x \forall y \forall z((x < y \wedge y < z) \to x < z)\} \end{aligned}$$

is the theory of linear orderings. It is consistent, as $(\mathbb{R}, <) \models T_{lo}$. We call models of this theory *linear orders*.

2. Let $L = \{\cdot, e\}$ where $\cdot$ is a 2-input function symbol and e a constant symbol (the language of groups). Then

$$\begin{aligned} T_{gr} = \{&\forall x \forall y \forall z((x \cdot y) \cdot z) = (x \cdot (y \cdot z)), \\ &\forall x(x \cdot e = x \wedge e \cdot x = x), \\ &\forall x \exists y(x \cdot y = e \wedge y \cdot x = e)\} \end{aligned}$$

is a theory - the theory of groups. It is consistent - any group models T_{gr}, for example $(\mathbb{R}, +, 0)$.

3. Let $L = \varnothing$. Then $T = \{\exists x(\neg(x = x))\}$ is not a theory, since in any model M, we always have $x = x$ for any $x \in |M|$.

Question 1.4. Which of the following are theories?

1. $T = \{x = 0\}$
2. $T = \{\exists x(\neg(x + 0 = x))\}$
3. $T = T_{lo} \cup \{\forall x \forall y(x < y \to \exists z(x < z < y))\}$

Definition 15. If M is a model, the *theory of* M is $Th(M) = \{\varphi \in Sent(L(M)) : M \models \varphi\}$.

Note $Th(M)$ is always a theory, as M is a model of it.

Question 1.5. Can you find an example of φ in $Th((\mathbb{R}, +, \cdot, <, 0, 1))$, but not in $Th((\mathbb{Q}, +, \cdot, <, 0, 1))$?

2 Compactness

Now we have the main definitions sorted, let's introduce one of the most powerful tools we can use in model theory. The compactness theorem can help us prove the existence of models with desirable properties by proving weird sets of sentences are consistent.

Theorem 16 (The compactness theorem). Let $T \subseteq Sent(L)$. Then T is consistent if and only if for all finite $T_0 \subseteq T$, T_0 is consistent.

We won't prove this today, but for those interested, there are several proofs, all of which rely in some way on the axiom of choice. One version involves proving that a theory is inconsistent if and only if there is a semantic 'proof' of a false sentence. Since a proof only needs finitely many statements, it uses finitely many formulas, so there is a finite subset of T that is inconsistent.

Another approach involves a way of joining models together called an 'ultraproduct'. If you take $M_{T_0} \models s$ for each finite $T_0 \subseteq T$, you can show that an ultraproduct of the models M_{T_0} will be a model of T.

But let's first give an example of how the compactness theorem can be used.

Proposition 17. Prove there is a model M modeling $Th((\mathbb{N}, +, \cdot, 0, 1))$ where there is $x \in |M|$ such that $M \models \overline{n} < x$ for all $n \in \mathbb{N}$ (where $\overline{n} = 1^M + 1^M + \cdots + 1^M$ with the sum n times).

Proof. Let $L' = \{+, \cdot, 0, 1, c\}$, and $T' = Th((\mathbb{N}, +, \cdot, 0, 1)) \cup \{1 < c, 1+1 < c, 1+1+1 < c, \dots\}$. If T_0 is a finite subset of T', then T_0 is a finite subset of $Th((\mathbb{N}, +, \cdot, 0, 1)) \cup \{1 < c, 1+1 < c, \dots, \overline{n} < c\}$ for some n. Let $c^N = n+1$, and let $N = (\mathbb{N}, +, \cdot, 0, 1, c^N)$. Then $N \models T_0$. So by the compactness theorem, T' is consistent.

Now take $M' \models T'$ and let $M = M' \restriction \{+, \cdot, 0, 1\}$. Then $M \models Th((\mathbb{N}, +, \cdot, 0, 1))$, and $M' \models \overline{n} < c$. So take $x = c^{M'}$, and we are done. □

Proposition 18. Suppose T has models of arbitrarily large finite size. Then T has an infinite model.

Proof. Let L be the language of T. Let c_i be a new constant symbol for $i \in \mathbb{N}$. Let $L' = L \cup \{c_i : i \in N\}$, and $T' = T \cup \{c_i \neq c_j : i \neq j \in \mathbb{N}\}$. If we can show that T' is consistent, we will be done: if $M' \models T'$, note $c_i^M \neq c_j^M$ for $i \neq j$ as $M \models c_i \neq c_j$. So $\{c_i^M : i \in \mathbb{N}\}$ is an infinite subset of $|M'|$, meaning $|M'|$ itself is infinite. Then $M' \restriction L$ is an infinite model of T.

To see T' is consistent, we will show all finite subsets are consistent. Let $T_0 \subseteq T'$ be finite. Then there is some $n \in \mathbb{N}$ such that $T' \subseteq T \cup \{c_i \neq c_j : i < j < n\}$. By our assumption, there is some $M \models T$ such that $\|M\| \geq n$. Take $\{c_i^M : i < n\}$ to be any list of n distinct elements. Then $M' = (M, c_i^M)_{i<n}$ is a model of T_0. □

2.1 Questions

Question 2.1. Find a sentence that says 'M has size at least n' for each $n \in \mathbb{N}$.

Question 2.2. Is there a sentence that says 'M is infinite' in any language? *Hint: consider Proposition 18.*

Question 2.3. Prove there is a model M that models $Th(\mathbb{R}, +, -, <, 0, 1)$ and has an element x such that $M \models 0 < x < \frac{1}{\overline{n}}$ for all $n \in \mathbb{N}$ (where $\overline{n} = 1^M + 1^M + \cdots + 1^M$ with the sum n times). This is an example of a non-Archimedean field.

Question 2.4. Prove that any set A can be linearly ordered - that is, there is $<$ a relation on A such that $(A, <) \models T_{lo}$ from Example 14.

Question 2.5 (Harder)**.** A group $M = (G, \cdot, e)$ is *periodic* if for each $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = e$. We call the least $n > 0$ such that $g^n = e$ the *degree* of g. Prove there is no sentence φ in the language of groups $\{\cdot, e\}$ such that a group models φ if and only if it is periodic. *Hint: try using compactness on the theory of a periodic group with arbitrarily large degrees.*

Question 2.6 (Harder)**.** A linear order $(A, <)$ is a *well-ordering* if for all $S \subseteq A$, there is a minimum element of S; that is, there is $s_0 \in S$ such that for all $s \in S$, $s = s_0$ or $s_0 < s$. Prove that there is no theory T in language $\{<\}$ such that $M \models T$ if and only if M is a well ordering

Question 2.7 (Tricky indeed)**.** The following might require some understanding of infinite cardinalities.

Fact 19. If L is countable, and T is a theory in L with an infinite model, then T has a countable model.

Two models M, N with language L are *isomorphic* if there is a bijection $f : |M| \to |N|$ that preserves the functions, relations, and constants of M and N (e.g. $f(c^M) = c^N$, $R^M(a, b)$ for $a, b \in |M|$ implies $R^N(f(a), f(b))$, and so on). Essentially this says the structures are identical, up to renaming the elements.

Prove that there are an uncountable number models $M \models Th(\mathbb{N}, +, \cdot, 0, 1)$ that are pairwise non-isomorphic.

Hint: for each $S \subseteq \{p \in \mathbb{N} : p \text{ is prime}\}$*, find a model* M *with* $x_S \in |M|$ *such that* p *divides* $x_S \iff p \in S$*. Are all these models non-isomorphic? If not, are enough of them? How many such* S *are there?*

Question 2.8 (Very tricky indeed)**.** Prove that there is only one model of the following theory, up to isomorphism.

$$
\begin{aligned}
L_{DLO} = L_{lo} \cup \{&\forall x \forall y (x < y \to \exists z (x < z < y)),\\
&\forall x \exists z (z < x),\\
&\forall y \exists z (y, z)\}
\end{aligned}
$$

Hint: The compactness theorem is not needed for this problem.
This is the theory of dense linear orderings without endpoints*, that is, linear orderings that are dense (between any two points is another), and have no least or greatest element. A great example is* $(\mathbb{Q}, <)$.
Maybe start by fixing countable models M *and* N *of* L_{DLO}*. A good start might be to see how to make an injection from* M *into* N *in a way that preserves the order. Maybe list out all of* $|M| = \{a_1, a_2, \dots\}$ *and see if you can find a method to map* a_1*, then* a_2*, and so on one at a time into* N *without messing up the order. It will depend on how* a_n *fits into the first* $a_1, \dots, a_{n-1}$*. A picture might help! If you can see that, how can you modify the construction to make sure it's a surjection as well? It might help to know that this is called a 'back and forth' construction!*