On the spectrum of limit models

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Based on the paper of the same name, work with Marcos Mazari-Armida
[Beard and Mazari-Armida, 2025]

Overview

1. Preliminaries
2. Results
3. Applications

Preliminaries - Setting

For this talk, $\mathbf{K} = (K, \leq_{\mathbf{K}})$ is an abstract elementary class (AEC) - a class of models K with a substructure relation $\leq_{\mathbf{K}}$ which satisfies some of the nice properties of first order elementary substructure.

When T is a first order theory, $(\mathrm{Mod}(T), \preccurlyeq)$ satisfies these hypotheses with $\mathrm{LS}(\mathbf{K}) = |L(T)| + \aleph_0$, but the framework is much more general.

In this setting, we can generalise notions such as embeddings, types, stability, saturation, and more.

Preliminaries - Setting

We assume further that $\mathbf{K}$ has amalgamation, joint embedding, and no maximal models, and $\lambda \geq LS(\mathbf{K})$.

$$\begin{tikzcd}
 & N & \\
M_1 \arrow[ur, dashed, "f_1"] & & M_2 \arrow[ul, dashed, "f_2"'] \\
 & M_0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

AP

$$\begin{tikzcd}
 & M & \\
N_1 \arrow[ur, dashed, "f_1"] & & N_2 \arrow[ul, dashed, "f_2"']
\end{tikzcd}$$

JEP

$$\begin{tikzcd}
N \\
M \arrow[u, dashed, "\neq"']
\end{tikzcd}$$

NMM

$(\mathrm{Mod}(T), \preccurlyeq)$ satisfies these hypotheses if T is complete with infinite models (but again, our context is much broader).

Preliminaries - Universal extensions

$\mathbf{K}_\lambda$ is the class of models in $\mathbf{K}$ of size λ.

Definition

For $M, N \in \mathbf{K}_\lambda$, N is *universal* over M if $M \leq_{\mathbf{K}} N$ and whenever $N' \in \mathbf{K}_\lambda$ and $M \leq_{\mathbf{K}} N'$, there exists a $\mathbf{K}$-embedding $f : N' \to N$ fixing M. We write $M \leq^u_{\mathbf{K}} N$.

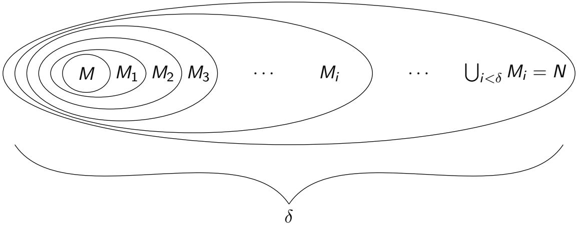
$$\begin{tikzcd}
& N \\
M \arrow[ur] \arrow[dr] & \\
& N' \arrow[uu, dashed, "f"']
\end{tikzcd}$$

Intuitively, this says N contains a copy of every superstructure N' of M of size λ.

Preliminaries - Limit models

Definition

Say $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\delta < \lambda^+$ is a limit ordinal. $N \in \mathbf{K}$ is a (λ, δ)-*limit model over* M if there exists a $\leq^u_{\mathbf{K}}$-increasing sequence $\langle M_i : i < \delta \rangle$ in $\mathbf{K}_\lambda$ where $M_0 = M$ and $N = \bigcup_{i<\delta} M_i$.



Fact (Existence of limit models)

If $\mathbf{K}$ *is* λ*-stable, then for any* $M \in \mathbf{K}_\lambda$*, there exists* $N \in \mathbf{K}_\lambda$ *universal over* M*. And for any* $\delta < \lambda^+$ *limit, there is a* (λ, δ)*-limit* N *over* M*.*

Preliminaries - Limit models

Question

Under what circumstances are two limit models isomorphic?

Fact (Uniqueness up to cofinality)

Suppose $\mathbf{K}$ *is* λ*-stable,* $\delta_1, \delta_2 < \lambda^+$ *are limits with* $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$ *and* N_l *is a* (λ, δ_l)*-limit model over* M *for* $l = 1, 2$*. Then* $N_1 \underset{M}{\cong} N_2$*.*

With this in mind, we may as well focus on when δ is a regular cardinal, and we can talk about 'the' (λ, δ)-limit model.

Preliminaries - Independence relations

Independence relations can give us a notion of non-forking of types that behaves similarly to first order non-forking.

Definition

For our purposes, an *independence relation on* $\mathbf{K}$ is a relation $\mathop{\smile\!\!\!\!\!\!|}$ on tupples (M_0, a, M, N) where $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$, $a \in N$ (written $a \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M$) satisfying invariance and monotonicity.

If $p \in \mathbf{gS}(M)$ where $M_0 \leq_{\mathbf{K}} M$, we say p $\mathop{\smile\!\!\!\!\!\!|}$-does not fork over M_0 if there exist $N \geq_{\mathbf{K}} M$ and $a \in N$ such that $p = \mathbf{gtp}(a/M, N)$ and $a \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M$.

Similar to first order, we can define notions of uniqueness, extension, (universal) continuity, $(\geq \kappa)$-local character, and non-forking amalgamation (which follows from symmetry).

In some sense, the existence of a nice independence relation means our AEC has nice structure.

Results - Main result

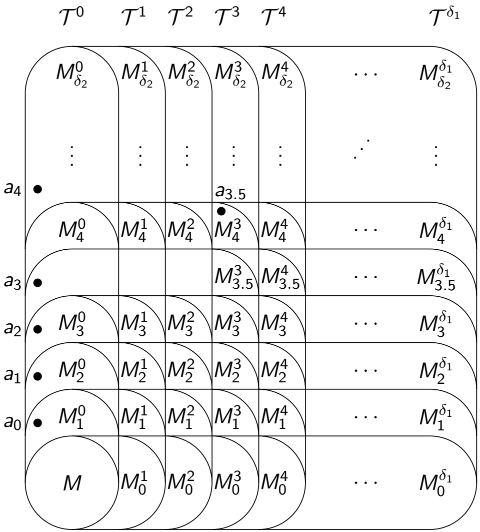
Theorem (B., Mazari-Armida)

Suppose $\mathbf{K}$ *is an AEC stable in* $\lambda \geq \mathrm{LS}(\mathbf{K})$, *with amalgamation, joint embedding, and no maximal models, and* $\mathbf{K}$ *is* $\aleph_0$*-tame. Suppose* $\perp$ *is an independence relation on* $\mathbf{K}_\lambda$ *that satisfies uniqueness, extension, (universal) continuity, non-forking amalgamation, and* $(\geq \kappa)$*-universal local character in some regular* $\kappa < \lambda^+$*. Let* $\kappa^u_\lambda(\perp)$ *be the least such* κ*.*

Suppose $N_1, N_2, M \in \mathbf{K}_\lambda$, *and* $\mu_1 < \mu_2 < \lambda^+$ *are regular, and* N_l *is a* (λ, μ_l)*-limit model over* M *for* $l = 1, 2$*. Then* $N_1 \underset{M}{\cong} N_2$ *if and only if* $\mu_1 \geq \kappa^u_\lambda(\perp)$*.*

$$\overbrace{\aleph_0 \qquad \aleph_1 \qquad \aleph_2 \qquad \aleph_3 \qquad \cdots \qquad \bullet \qquad \kappa^u_\lambda(\perp) \qquad \bullet \qquad \bullet \qquad \cdots}^{\mathrm{REG} \cap \lambda^+}$$

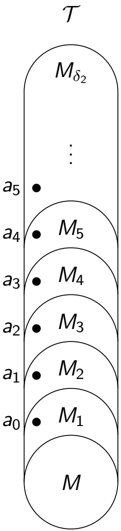
Results - Long limits are isomorphic



To prove long limit models are isomorphic, we build a matrix of models, similar to [Vasey, 2019]. The method involves developing a theory of *towers*.

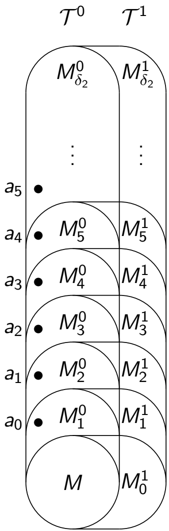
Compared to previous results, our constructions widen the assumptions to allow general independence relations, possibly with $\kappa^u_\lambda(\perp) > \aleph_0$, and possibly defined only on $(\lambda, \geq \kappa)$-limit models. $\aleph_0$-tameness is not needed for this argument.

Results - Long limits are isomorphic



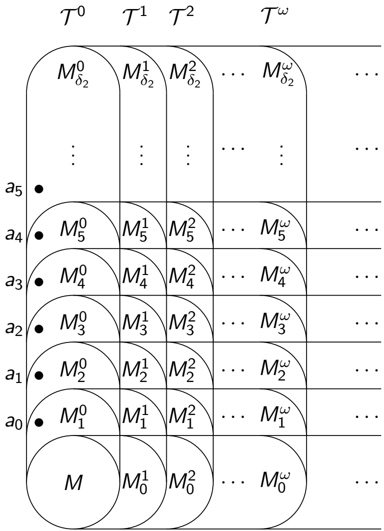
A tower $\mathcal{T}$ is a $\leq_{\mathbf{K}}$-increasing sequence of $(\lambda, \geq \kappa^u_\lambda(\downfree))$-limit models M_i with named constants $a_i \in M_{i+1} \setminus M_i$. It is not hard to see towers of any length $< \lambda^+$ exist.

Results - Long limits are isomorphic



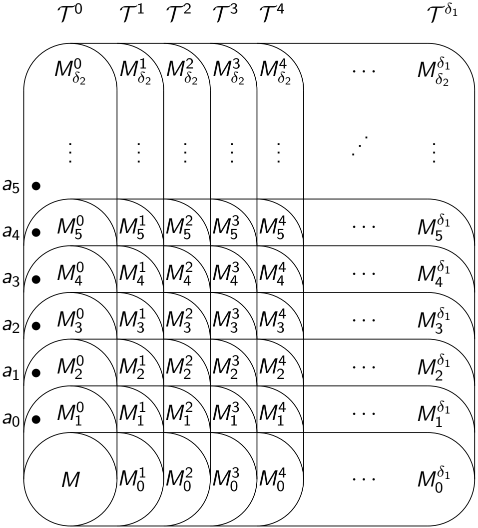
We define an ordering on towers $\mathcal{T} \lhd \mathcal{T}'$. One condition is that models in the same row must be universal extensions of previous towers. Also, we demand that the types of the a_i $\perp$ do not fork over models in the smaller tower. You can show that such extensions exist.

Results - Long limits are isomorphic



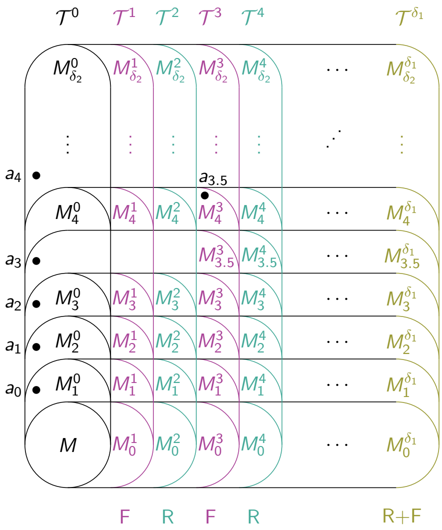
Unions of low cofinality towers may not be towers. So we also need to be able to extend arbitrary $\lhd$-sequences of towers.

Results - Long limits are isomorphic



To make the top row continuous, we need to ensure we can take unions of cofinality $\geq \kappa^u_\lambda(\downfree)$ chains of towers.

Results - Long limits are isomorphic



Continuity and universality of towers are not preserved by unions, so we introduce 'stronger' notions, *reduced* and *full* towers. This does require us to add 'extra' rows as we add new towers.

Every tower has a reduced extension, and unions of reduced towers are reduced. The same is true of full towers. So we can intersperse them in the construction, making the final tower reduced and full, so (essentially) universal and continuous at δ_2.

Results - Short limits are non-isomorphic

We prove the contrapositive: that if two limit models of different cofinalities are isomorphic, then both cofinalities are 'high'. So suppose $\mu_1 < \mu_2 < \lambda^+$ are regular and the (λ, μ_1)-limit model and (λ, μ_2)-limit model are isomorphic. We will sketch why $\mu_1 \geq \kappa_\lambda^u(\downfree)$.

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Step 1: $\mathop{\smile\!\!\!\!\!\!|}$ is 'close' to a relation called λ-*non-splitting*. This is enough to guarantee $\kappa^u_\lambda(\mathop{\smile\!\!\!\!\!\!|}\,) = \kappa^u_\lambda(\underset{\lambda-\mathrm{split}}{\mathop{\smile\!\!\!\!\!\!|}})$.

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Step 2: The union of any $\leq^u_{\mathbf{K}}$-increasing sequence $\langle M_i : i < \mu_1 \rangle$ is μ_1^+-saturated.

Step 3: We have μ_1-local character of non-splitting, so $\mu_1 \geq \kappa^u_\lambda(\mathop{\smile\!\!\!\!\!\mid}_{\lambda-\text{split}})$. *This is where the $\aleph_0$-tameness assumption is used.*

Step 1 and Step 3 combined give the conclusion.

Results - Global version

If we assume we have a relation satisfying our conditions at all λ, the cardinal $\kappa_\lambda^u(\mathop{\smile}\limits^{\vert})$ stabilises for high enough stable λ to a value $\chi(\mathop{\smile}\limits^{\vert}, \mathbf{K}, \leq_{\mathbf{K}}^u)$. This gives the following:

Theorem (B., Mazari-Armida)

Assume $\mathop{\smile}\limits^{\vert}$ is defined on $\mathbf{K}$ and has all the properties from before, with $\kappa \leq \mathrm{LS}(\mathbf{K})$. Let $\lambda \geq \beth_{(2^{LS(\mathbf{K})})^+}$ such that $\mathbf{K}$ is stable in λ.
Suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ over } M \iff \mathrm{cf}(\delta_1) \geq \chi(\mathop{\smile}\limits^{\vert}, \mathbf{K}, \leq_{\mathbf{K}}^u)$$

Applications - First order stable theories

Given first order complete stable theory T, non-forking is a relations satisfying our hypotheses in $\mathbf{K} = (\mathrm{Mod}(T), \preccurlyeq)$ with $\kappa^u_\lambda(\mathop{\smile\!\!\!\!\!\!|}) = \kappa_r(T) \leq |T|$ for all stable $\lambda \geq |L(T)| + \aleph_0$. Thus we have:

Theorem

Let T be a first order complete stable theory. Then for every stable $\lambda \geq |T|$, and any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \textit{ is isomorphic to } N_2 \textit{ over } M \iff \mathrm{cf}(\delta_1) \geq \kappa_r(T)$$

Though the $\Leftarrow$ implication was known, the $\Rightarrow$ implication was previously unexplored.

Applications - Tame AECs

In fact, in nice μ-tame AECs, Vasey showed in [Vasey, 2016] the existence of an independence relation with many nice properties. With a little work, combining this with our 'long limit' result gives the uniqueness of $(\lambda, \geq \mu^+)$-limit models for all stable $\lambda \geq \mu^+$.

Theorem

Let $\mathbf{K}$ *be an AEC, stable in* $\mu \geq \mathrm{LS}(\mathbf{K})$ *and stable also in* $\lambda \geq \mu^+$*, where* $\mathbf{K}$ *has AP, NMM,* λ*-JEP, and* μ*-tameness. Assume universal continuity of* μ*-non-splitting. Suppose also that* $\mathbf{K}$ *has* μ*-symmetry on* $(\lambda, \geq \mu^+)$*-limit models.*
Let $\delta_1, \delta_2 < \lambda^+$ *be limit ordinals where* $\mu^+ \leq \mathrm{cf}(\delta_1), \mathrm{cf}(\delta_2)$*. If* $M, N_1, N_2 \in K_\lambda$ *where* N_l *is a* (λ, δ_l)*-limit over* M *for* $l = 1, 2$*, then there is an isomorphism from* N_1 *to* N_2 *fixing* M*.*

Note we only require nice structural properties of the AEC - no independence relation is assumed to exist.

Applications - Algebraic AECs

There are several examples of algebraic AECs satisfying the assumptions of our theorem, but let's focus on one.

Let $\mathbf{K}^{R-\text{mod}}$ be the AEC of modules over a fixed ring R with $\subseteq$ (or $\leq_p$, pure substructure). In this case, $\chi(\downarrow, \mathbf{K}^{R-\text{mod}}, \leq^u_{\mathbf{K}}) = \gamma_r(R)$. In fact, considering when these are $\aleph_0$, we have

Theorem (Originally [Mazari-Armida, 2021])

The following are equivalent:

1. *R is Noetherian*
2. *for any $\lambda \geq \beth_{(2^{\aleph_0})^+}$, all λ-limit models in $\mathbf{K}^{R-\text{mod}}$ are isomorphic*

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Thanks for listening!