Model Theory seminar talk P1 Based on work with Marcos Mazari-Armida. Similar to last talk.

Today's talk focuses on finding a general setup of $K, \perp$ to prove uniqueness of limit models above a certain threshold.

Work in AEC K, let $\lambda \geq LS(K)$.

Recall: • Given $M, N \in K_\lambda$, N is universal $/M$ ($M \leq^u N$) if $M \leq N$ and for all $N' \in K_\lambda$ s.t. $M \leq N'$, $\exists f : N' \xrightarrow[M]{} N$.

• If $\delta < \lambda^+$ limit and $M, N \in K_\lambda$, we say N is (λ, δ)-lim $/M$ if there exist $\langle M_i : i \leq \delta \rangle$ 3 $\leq$-inc cts and $\forall i < \delta$, $M_{i+1} \geq^u M_i$, and $M_0 = M$, $M_\delta = N$.

• $M \in K_\lambda$ is (λ, δ)-lim if (λ, δ)-lim over some $N \in K_\lambda$
• M is λ-lim, λ-lim$/M$ similar. We omit λ when clear.
• $\langle M_i : i < \delta \rangle$ is universal if $\forall i+1 < \delta$ $M_{i+1} \geq^u M_i$.
• $\langle M_i : i < \delta \rangle$ is strongly universal if $\forall 0 < j < \delta$ $M_j \geq^u \bigcup_{i<j} M_i$
• χ reg, M is $(\lambda, \geq \chi)$-lim if M is (λ, δ)-lim for cf$(\delta) \geq \chi$.

Fact: If K is λ-stable, universal / limit models exist.

Fact: If N_ℓ is (λ, δ_ℓ)-lim (over M) where $\delta_\ell < \lambda^+$ is lim for $\ell = 1, 2$, then $N_1 \underset{(M)}{\cong} N_2$

True more generally?

First AEC result of today's kind:

Theorem (Vandieren 16): Let K be an AEC with $\lambda \geq LS(K)$ where K is λ-superstable and has λ-symmetry. Then K has uniqueness of λ-limit models: If $M, N, N' \in K_\lambda$ and N, N' are both λ-lim$/M$, then $N \underset{M}{\cong} N'$.
(also if M, N both λ-lim, $M \cong N$.)

Vasey later gave a proof using λ-forking rather than λ splitting, which we base our method on heavily.

Improved by Boney & Vandieren:

<u>Thm</u>: Suppose K is λ-stable, has AP, JEP, NMM, has continuity of non-splitting, has λ-symmetry of non-splitting.

Let κ_{sp} be (minimal reg card s.t. $\forall$ reg $\alpha \geq \kappa$, $\forall \langle M_i : i \leq \alpha \rangle$ cts increasing chain, $\forall p \in S(M_\alpha)$ non-alg, $\exists i < \alpha$ p does not split $/ M_i$) loc. char card.

Then if $\delta_1, \delta_2 < \lambda^+$, $cf(\delta_1), cf(\delta_2) \geq \kappa_{sp}$, and N_l is (λ, δ_l)-lim $(/ M)$ for $l = 1, 2$,

$$N_1 \underset{M}{\cong} N_2$$

We aim for a similar result in a general framework.

<u>We want to include:</u>

- Stable independence relations $\perp$ in the sense of Lieberman, Rosický, and Vasey (Forking independence from the categorical point of view).
- λ-forking in Leung's setting (Forking independence from the categorical point of view) This is in part why we will restrict to $(\lambda, \geq \kappa)$-limits.

<u>Hypothesis</u>: K has AP, JEP, NMM. $\kappa \leq \lambda$ is regular, $\perp$ is a relation on triples (M_0, a, M, N) where M_0, M, N are $(\lambda, \geq\kappa)$-limit models and (write $a \underset{M_0}{\overset{N}{\perp}} M$) where $a \in |N|$. $\perp$ satisfies invariance, monotonicity, base monotonicity, uniqueness, existence, transitivity, and

- <u>Non-forking amalgamation</u>: If M_0, M_1, M_2 are $(\lambda, \geq\kappa)$-lim, $M_0 \leq M_1, M_2$, and $a_1 \in M_1$, $a_2 \in M_2$, then there exist N a $(\lambda, \geq\kappa)$-lim and $f_l : M_l \xrightarrow[M_0]{} N$ s.t. $f_l(a_l) \underset{M_0}{\overset{N}{\perp}} f_{3-l}(M_{3-l})$ for $l = 1, 2$.

• $(\geq\kappa)$-local character: If $\delta < \lambda^+$, $cf(\delta) \geq \kappa$, and $\langle M_i : i < \delta \rangle \leq_{K_{(\lambda, \geq\kappa)}}$ is u-incr, $M_\delta = \bigcup_{i<\delta} M_i$, then for all $p \in S(M_\delta)$ $\exists\, i < \delta$ s.t. p $\perp$-def / M_i (i.e. $\exists\, a, N$ $gtp(a/M_\delta, N) = p$, $a \underset{M_i}{\overset{N}{\perp}} M_\delta$)

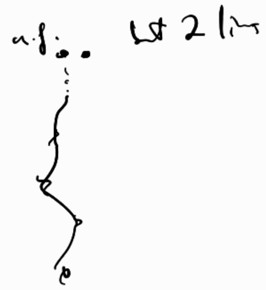
• $(\geq\kappa)$-weak continuity in λ: If $\delta < \lambda^+$ lim, $\langle M_i : i < \delta \rangle$ is a u-incr sequence of $(\lambda, \leq\kappa)$-limit models, and for $i < \delta$ $p_i \in S(M_i)$ satisfy p_i $\perp$-def / M_0 for $i < \delta$ and

$p_i \leq p_j$ for $i < j$

Then there exists a unique $q \in S(\bigcup_{i<\delta} M_i)$ such that $p_i \leq q$ for all $i < \delta$.

Note: Morally, q is the n.f. ext of p_0. But things get messy if we extend $\perp$ to these.

why do we need?



Fact: If $\perp$ def on all $(\lambda, \geq\aleph_0)$-lim, contin $\iff$ $(\geq\aleph_0)$ weak cty.

Fact: $\perp$ weak cty $\Rightarrow$ $\geq\kappa$-local cty.

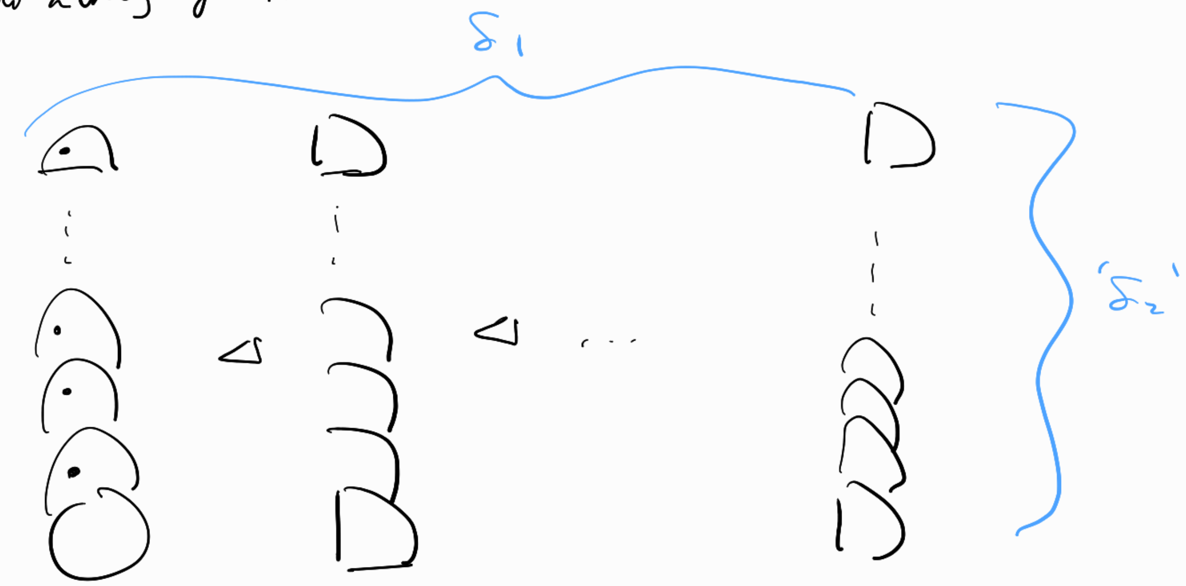
proof: If $\langle M_i : i < \delta \rangle$ $\leq^u$-inc of $(\lambda, \geq\kappa)$-lim models, $p \in S(\bigcup_{i<\delta} M_i)$ s.t. $p \restriction M_i$ $\perp$-def / M_0 for all i. Get $q \in S(\bigcup M_i)$ def / M_0, $\geq p \restriction M_0$. $q \restriction \bigcup M_i$ def / M_0, $q \restriction M_i = p \restriction M_i$ by uniq, so $p = q$ by $\geq\kappa$-weak cty. So p def / M_0. //

Fact: Both our previous examples fall into this setting for some $\kappa \leq \lambda$, as well as the f.o. case with $\perp$ as forking ($\kappa = \aleph_0$), and λ-forking if applicable.

Note for Leung's forking, we assume also uniqueness.

Main thm: Let $\delta_1, \delta_2 < \lambda^+$ be lim with $cf(\delta_1), cf(\delta_2) \geq \kappa$.
If $(M,) N_1, N_2 \in K_\lambda$, N_ℓ are (λ, δ_ℓ)-lim $(/M)$ for $\ell = 1, 2$,
then $N_1 \underset{(M)}{\cong} N_2$

Idea: Build a theory of towers



Notation: I a wellorder. Then I^- is I minus the first member if it has one, else $I^- = I$.

Defn: For I a wellorder, $\mathcal{T} = \langle M_i : i \in I \rangle ^\frown \langle a_i : i \in I^- \rangle$ is a tower if

(1) $\langle M_i : i \in I \rangle$ is a $\leq^K$-inc sequence of $(\lambda, \geq \kappa)$-lim models

(2) $\forall i \in I^-$, $a_i \in M_{i+1} \setminus M_i$.

- I is the index of $\mathcal{T}$
- If $I_0 \subseteq I$, $\mathcal{T} \upharpoonright I_0 = \langle M_i : i \in I_0 \rangle ^\frown \langle a_i : i \in (I_0)^- \rangle$.
- We say $\mathcal{T}$ is cts at i if $M_i = \bigcup_{j<i} M_j$.
- $\mathcal{T}$ is univ'l if $\forall i \in I$ $M_{i+1} \geq^u M_i$.
- $\mathcal{T}$ is strongly univ'l if $\forall i \in I$ $M_i \geq^u \bigcup_{j<i} M_j$.

Note! (strongly universal) towers exist.

Alt, continuity can be established when $cf(i) \geq \kappa$.

Definition: Given towers $\mathcal{T} = \langle M_i : i \in I \rangle^\frown \langle a_i : i \in I^- \rangle$,

$\mathcal{T}' = \langle M_i' : i \in I' \rangle^\frown \langle a_i' : i \in (I')^- \rangle$

define the tower ordering by $\mathcal{T} \lhd \mathcal{T}'$ iff

(1) $I \subseteq I'$

(2) $\forall i \in I \quad M_i \leq^u M_i'$

(3) $\forall i \in I^- \quad a_i = a_i'$

(4) $\forall i \in I^- \quad gtp(a_i / M_i', M_{i+1}') \ \text{⫝}\text{-}\mathrm{nf} / M_i$.

Note: $\lhd$ is a strict partial ordering of towers.

Lemma: Suppose $\gamma < \lambda^+$, $cf(\gamma) \geq \kappa$, $\langle \mathcal{T}^j : j < \gamma \rangle$ is $\lhd$-increasing seq of towers, $\mathcal{T}^j = \langle M_i^j : i \in I^j \rangle^\frown \langle a_i^j : i \in (I^j)^- \rangle$

Define (1) $I^\gamma = \bigcup_{j<\gamma} I^j$

(2) For $i \in I^\gamma$, $M_i^\gamma = \bigcup_{\substack{j<\gamma, \\ i \in I^j}} M_i^j$

(3) For $i \in (I^\gamma)^-$, $a_i^\gamma = a_i^j$ for any $j < \gamma$ where $i \in (I^j)^-$.

If I^γ is a well ordering, then

$\bigcup \mathcal{T}^j = \mathcal{T}^\gamma := \langle M_i^\gamma : i \in I^\gamma \rangle^\frown \langle a_i^\gamma : i \in (I^\gamma)^- \rangle$

is a tower, and for all $j < \gamma$,

$\mathcal{T}^j \lhd \mathcal{T}^\gamma$.

proof: Easy to check it's a tower.

If $j < r$, WTS $\mathcal{T}^j \triangleleft \mathcal{T}^r$.

(1) $I^j \subseteq I^r$

(2) $M_i^j \leq^u M_i^{j+1} \leq M_i^r$

(3) $i \in (I^j)^-$, $a_i^j = a_i^r$

(4) If $i \in (I^j)^-$ WTS $gtp(a_i^r / M_i^r, M^r_{i+1}) \perp\text{-}dnf / M_i^j$.

$gtp(a_i^r / M_i^{j'}, M^r_{i+1}) \perp\text{-}dnf / M_i^j$ for $j < j' < r$

$\Rightarrow gtp(a_i^r / M_i^r, M_{i+1}^r)$ dnf $/ M_i^{j'}$.

So $\mathcal{T}^j \triangleleft \mathcal{T}^r$. //

Lemma: (Disjointness): If M, N $(\lambda, \geq\kappa)$-lim models, $p \in S(N)$ $\perp$-dnf $/M$, and $p \restriction M$ is non-alg., then p is non-alg.

proof: Let $\langle M_i : i < \delta \rangle$ witness M is (λ, δ)-lim $/M_0$, $cf(\delta) \geq \kappa$.

Wlog M_i is (λ, κ)-lim model.

$$\left(M_i \leq^u M_{i+1} \leq M_{i+2} \right)$$
$$M_i \underset{(\lambda,\kappa)}{\leq} M'_{i+1} \nearrow M_{i+1}$$

By $(\geq\kappa)$-local char, $\exists i < \delta$ s.t. $p \restriction M$ $\perp$-dnf $/ M_i$.

By tameness, p $\not\perp$-dnf $/M_i$.

Let N^* (λ,δ)-lim mod $/N$.
By ext, get $q\in S(N^*)$ s.t. $q\restriction M_i = p\restriction M_i$, dnf $/M_i$.
Uniqueness $q\restriction N = p$.

$$M_i \leq M \leq N \leq N^*$$

So $\exists f : N^* \xrightarrow[M_i]{\cong} M$

Now $f(q)$ dnf $/M_i$, $f(q) \geq p\restriction M_i$.

so $f(q) = p\restriction M$.

As $p\restriction M_i$ is not alg, $f(q)$ not alg

$\Rightarrow q$ not alg. //

<u>Proposition:</u> Suppose $\alpha < \lambda^+$, $\langle \mathcal{T}^j : j<\alpha\rangle$ $\triangleleft$-inc seq of trees, $\mathcal{T}^j = \langle M_i^j : i\in I^j\rangle {}^\wedge \langle a_i : i \in (I^j)^-\rangle$.
Suppose $I^* = \bigcup_{j<} I^j$ is well ordered.
Then there exists $\mathcal{T}^*$, strongly unid indexed by I s.t. $\mathcal{T}^j \triangleleft \mathcal{T}^*$ for all $j<\alpha$.

<u>proof:</u> Let $M_i^\alpha := \bigcup_{j<\alpha} M_i^j$, let $i_0 \in I$ be minimal.

(Note M_i^α not $(\lambda,\geq\kappa)$-lim nec).

Define by recursion on $i \in I$ a $\leq_u$-inc seq of $(\lambda, \geq\kappa)$-limit models

$\langle N_i : i \in I \rangle$ and $\subseteq$-inc $\langle f_i : i < \alpha \rangle$ s.t.

(1) $\forall i \in I \; f_i : M_i^\alpha \to N_i$

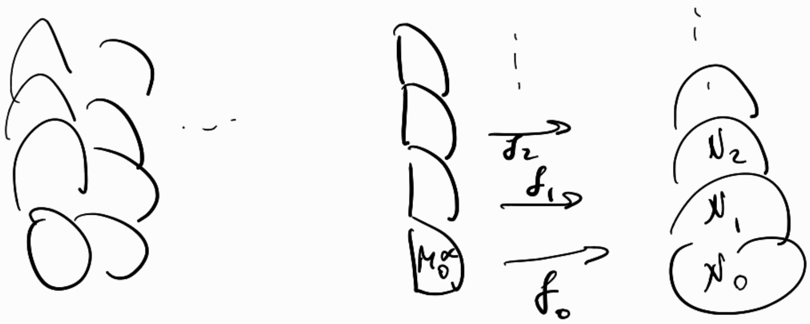
(2) $f_{i_0} = \mathrm{id}_{M^\alpha_{i_0}}$

(3) $\forall i \in I \quad f_i[M_i^\alpha] \leq^u N_i$

(4) $\forall i \in I \setminus \{i_0\}, \; \bigcup_{j<i} N_j \leq^u N_i$

(5) $\forall j < \alpha, \; i \in (I^j)^-, \quad \mathrm{gtp}(f_{i+1}(a_i)/N_i, N_{i+1})$

$\quad\quad \perp$-dnf $/ f_i[M_i^j]$.



This is enough: Set $f := \bigcup_{i\in I} f_i : \bigcup_{i \in I} M_i^\alpha \to \bigcup_{i \in I} N_i$.

Extend f to $g : M^* \cong \bigcup_{i \in I} N_i$.

Now set $M_i^* = g^{-1}[N_i]$ for $i \in I$.

Then $\mathfrak{y}^* = \langle M_i^* : i \in I^* \rangle ^\frown \langle a_i : i \in (I^*)^- \rangle$

works.

(disjointness gives $a_i \in M^*_{i+1} / M^*_i$)

(3), (5) gives $\mathfrak{y} \triangleleft \mathfrak{y}^*$, (4) gives $\mathfrak{y}^*$ str. unif.

This is possible: $\underline{i=0}$: Take $N_{i_0} \succeq^u M^\alpha_{i_0}$.

Let $f_{i_0}: M^\alpha_{i_0} \to N_{i_0}$ be inclusion.

$\underline{i+1}$: Say $i+1 \in I^*$, N_i, f_i given.

$$\begin{tikzcd}
M^\alpha_{i+1} \arrow[r, "f'_{i+1}"] \arrow[rrr, bend left, "f_{i+1}"] & N'_{i+1} \arrow[r, "g_i"] & N''_{i+1} \arrow[r, "id", "(\lambda,\kappa)\text{-lim}"'] & N_{i+1} \\
M^\alpha_i \arrow[u, "id"] \arrow[r, "f_i"'] & f_i[M^\alpha_i] \arrow[u, "id"] \arrow[ur, phantom, "\text{AP}"] \arrow[r, "id"'] & N_i \arrow[r, "id"'] & \tilde{N}_{i+1} \arrow[ul, "id"]
\end{tikzcd}$$

type eq

$g_i(f'_{i+1}(a_i)) = c_i$

Ext

$c \in \tilde{N}_{i+1}$

$gtp(c_i / N_i, \tilde{N}_{i+1})$ dnf $/ M^0_i$

$\geq gtp(f'_{i+1}(a_i) / f_i[M^0_i], N'_{i+1})$

uni gens $\Rightarrow$ $gtp(c_i / N_i, \tilde{N}_{i+1})$

$\geq gtp(f'_{i+1}(a_i) / f_i[M^j_i], N'_{i+1})$

for $j < \alpha$

weakly $\Rightarrow$

$gtp(c_i / N_i, \tilde{N}_{i+1}) \geq$

$gtp(f_{i+1}(a_i) / f_i[M^\alpha_i], N_{i+1})$.

$gtp(f_{i+1}(a_i) / N_i, N_{i+1})$ dnf $/ f_i[M^\alpha_i]$

$$\begin{matrix} f_i[M^\alpha_i] & \subseteq & & & \\ & & N''_{i+1} & \leq^u & N_{i+1}. \\ N_i & \subseteq & & & \end{matrix}$$

$\underline{i \text{ limit}}$: let $f_i^0 = \bigcup_{i'<i} f_{i'} : \bigcup_{i'<i} M^\alpha_{i'} \longrightarrow \bigcup_{i'<i} N_{i'}$

$$\begin{tikzcd}
M^\alpha_i \arrow[r, "f'_i"] \arrow[rr, bend left, "f_i"] & N'_i \arrow[r, "(\lambda,\kappa)\text{-lim}"'] & N_i \\
\bigcup_{i'<i} M^\alpha_{i'} \arrow[u, "id"] \arrow[r, "f^0_i"'] & \bigcup_{i'<i} N_{i'} \arrow[u, "id"'] \arrow[ul, phantom, "\text{AP}"] &
\end{tikzcd}$$

$$\begin{matrix} f_i[M_i] & \subseteq & & & \\ & & N'_i & \leq^u & N_i. \\ \bigcup_{i'<i} N_{i'} & \subseteq & & & \end{matrix}$$

//

Propn: Suppose $\mathcal{T} = \langle M_i : i < \beta \rangle ^\wedge \langle a_i : i < \beta^- \rangle$ a tower indexed by β, $p \in S(M_0)$.

Then there exists $\mathcal{T}^* = \langle M_i^* : i < \beta \rangle ^\wedge \langle a_i : i < \beta^- \rangle$, strongly univ indexed by β, and $b \in M_0^*$, s.t. $\mathcal{T} \triangleleft \mathcal{T}^*$,

$p = \text{gtp}(b / M_0, M_0^*)$, and $b \underset{M_0}{\overset{M_i^*}{\perp}} M_i$ for all $i < \beta$.

Sol: As before, construct $\langle N_i : i < \beta \rangle$, $\langle f_i : i < \beta \rangle$ inc, c (λ,κ)-limits

s.t.
(1) $f_i : M_i \to N_i$
(2) f_0 is inclusion
(3) $f_i[M_i] \leq^u N_i$
(4) $\bigcup_{i' < i} N_{i'} \leq^u N_i$
(5) $\text{gtp}(f_i(a_i) / N_i, N_{i+1})$ dnf $/ f_i[M_i]$
(6) $c \in N_0$ and $\text{gtp}(c / f_i[M_i], N_i)$ dnf $/ M_0$.

Enough: $f = \bigcup f_i$

$g \supseteq f$, $g : M^* \xrightarrow{\cong} \bigcup_{i<\beta} N_i$

$M_i^* = g^{-1}[M_i]$.

$b = g^{-1}(c)$.

$\mathcal{T}^* = \langle M_i^* : i < \beta \rangle ^\wedge \langle a_i : i < \beta^- \rangle$

(6) $\Rightarrow$ $c \underset{M_0}{\overset{N_i}{\perp}} f_i[M_i] \xRightarrow{\text{inv}} b \underset{M_0}{\overset{M_i^*}{\perp}} M_i$.

__Possible:__

__$i=0$__ Get N_0, c s.t. $p = gtp(c/M_0, N_0)$ where N_0 is (λ,κ)-lim.

f_0 is inclusion $f_0 : M_0 \to N_0$.

__$i+1$:__

$$
\begin{tikzcd}
 & & & N_{i+1} \\
 & g_{i+1}(a_i) & & \\
M_{i+1} \arrow[r, "g_{i+1}"] \arrow[rrruu, bend left, "f_{i+1}"] & g_{i+1}[M_{i+1}] \arrow[r, "\substack{NF\\ AP}"'] & N'_{i+1} \arrow[ru, "(\lambda,\kappa)\text{-lim}"'] & \\
M_i \arrow[u, "id"] \arrow[r, "f_i"'] & f_i[M_i] \arrow[r] \arrow[u, "id"] & N_i \arrow[u, "id"'] & \ni c
\end{tikzcd}
$$

$g_{i+1}(a_i) \underset{f_i[M_i]}{\overset{N'_{i+1}}{\perp}} N_i$, $c \underset{f_i[M_i]}{\overset{N'_{i+1}}{\perp}} g_{i+1}[M_{i+1}]$

IM $\Rightarrow$ $c \underset{M_0}{\overset{N'_i}{\perp}} f_i[M_i]$, so transitivity $\Rightarrow$ $c \underset{M_0}{\overset{N'_{i+1}}{\perp}} g_{i+1}[M_{i+1}]$

__i lim:__ $M_i^0 = \bigcup_{i'<i} M_{i'}$, $N_i^0 = \bigcup_{i'<i} N_{i'}$, $f_i^0 = \bigcup_{i'<i} f_{i'} : M_i^0 \to N_i^0$

$$
\begin{tikzcd}
 & & & & N_i \\
 & & \circledast\ \overset{d_p}{\hat{N}_i} \arrow[r, "h_i"] & \bar{N}_i \arrow[ru, "(\lambda,\kappa)\text{-lim}"'] & \\
M_i \arrow[rr, "g_i"] \arrow[rrrruu, bend left, "f_i"] & & N_i^* \arrow[u, "id"] & & \\
M_i^0 \arrow[u, "id"] \arrow[r, "f_i^0"'] & f_i^0[M_i^0] \arrow[rr, "id"] \arrow[ru, "id"'] & & N_i^0 \arrow[uu, "id"'] & \ni c
\end{tikzcd}
$$

AP

type eq

$h_i(d) = c$

Let $q \in S(g_i[M_i]) \geq p$ dnf $/M_0$.

$gtp(c/g_{i'}[M_{i'}], N_i^*)$ dnf $/M_0$ for all $i'<i$, so

$q \geq gtp(c/g_{i'}[M_{i'}], N_i^*)$ by uniqueness.

By $(\geq\kappa)$-weak cty, $q \restriction g_i[M_i^0] = gtp(c/g_i[M_i^0], N_i^*)$.

Say $q = \mathrm{gtp}(d/g_i[M_i], \hat{N}_i)$ where $\hat{N}_i \geq N_i^*$. ⊛

$\mathrm{gtp}(c / f_i[M_i], N_i)$ dnf / M_0

$$f_i[M_i] \leq \bar{N}_i \leq^u N_i.$$
$$N_i^0 \leq$$

//

What if we already have a partial extension? We can 'twist' it into a full one.

<u>Proposition</u>: Suppose $\mathcal{T} = \langle M_i : i \in \beta \rangle ^\frown \langle a_i : i \in \beta^- \rangle$ a tower, and $\gamma < \beta$, and there is $\mathcal{T}^* = \langle M_i^* : i < \gamma \rangle ^\frown \langle a_i : i \in \gamma \rangle$ s.t. $\mathcal{T} \upharpoonright \beta \triangleleft \mathcal{T}^*$. Then there exists strongly univ $\mathcal{T}' = \langle M_i' : i < \beta \rangle ^\frown \langle a_i : i < \beta^- \rangle$ and some $N \in K$,

$$g : \bigcup_{i<\beta} M_i \xrightarrow[\bigcup_{i<\gamma} M_i]{} N \quad \text{s.t.} \quad \mathcal{T} \triangleleft \mathcal{T}' \text{ and } g[\mathcal{T}'] \upharpoonright \gamma = \mathcal{T}^*.$$

<u>proof</u>: If $\gamma = 0$, this is a special case of other thm. So wlog $\gamma \geq 1$.

Do as before: get $\langle N_i : i < \beta \rangle$, $\langle f_i : i < \beta \rangle$ s.t.

(1) $f_i : M_i \to N_i$

(2) for $i < \gamma$, $N_i = M_i^*$, f_i is inclusion.

(3) $f_i[M_i] \leq^u N_i$

(4) $\bigcup_{i'<i} N_{i'} \leq^u N_i$

(5) $\mathrm{gtp}(f_i(a_i) / N_i, N_{i+1})$ dnf / $f_i[M_i]$

<u>Finish</u>: As before, $g : M' \cong \bigcup N_i$, $M_i' = g^{-1}[N_i]$

Now $g[\mathcal{T}'] \upharpoonright \gamma = \mathcal{T}^*$.

Pos limit: N_i, f_i given for $i < \gamma$.

For $i \geq \gamma$, use construction from first case ($\alpha = 1$). //

To make the final tower continuous at δ_2:

Definition: A tower $\mathcal{T} = \langle M_i : i \in I \rangle^\frown \langle a_i : i \in I^- \rangle$ is reduced if for all $\mathcal{T}' = \langle M_i' : i \in I' \rangle^\frown \langle a_i : i \in (I')^- \rangle$,

if $\mathcal{T} \triangleleft \mathcal{T}'$ and $i_1 < i_2 \in I$,

$$\left.\begin{aligned} M_{i_1}' \cap M_{i_2} = M_{i_1} \end{aligned}\right\} *$$

Remark: Equivalently, we could enforce $\mathcal{T}'$ is indexed by I, as $*$ holds for $\mathcal{T}'$ $\iff$ holds for $\mathcal{T}' \upharpoonright I$.

Wts
- reduced extensions exist
- unions of reduced towers are reduced
- reduced towers are cts at cf $\geq \kappa$.

Proposition: If $\mathcal{T} = \langle M_i : i \in I \rangle^\frown \langle a_i : i \in I^- \rangle$ a tower, then $\exists \mathcal{T}'$ indexed by I s.t. $\mathcal{T} \triangleleft \mathcal{T}'$ and $\mathcal{T}'$ is reduced.

proof: Suppose no $\mathcal{T}'$ exists for ⋇.

Then construct $\langle \mathcal{T}^j : j < \lambda^+ \rangle$ $\triangleleft$-inc

where $\mathcal{T}^j = \langle M_i^j : i \in I \rangle^\frown \langle a_i : i \in I^- \rangle$ and

(1) $\mathcal{T} \triangleleft \mathcal{T}^0$

(2) $\forall j < \lambda^+ \; \exists\, i_j < i_j' \in I$ s.t.
$$M^{j+1}_{i_j} \cap M^j_{i_j'} \neq M$$

Possibilities: $j = 0$ by last time

$j+1$: $\mathcal{T}^j$ is not reduced

j lim, $cf(j) < \kappa$: $\mathcal{T}^{j'} \triangleleft \mathcal{T}^j$ by last time

j lim, $cf(j) \geq \kappa$: $\mathcal{T}^j = \bigcup_{j' < j} \mathcal{T}^{j'}$

Let $N_i^j = \begin{cases} M_i^j & j \text{ not limit} \\ \bigcup_{j'<j} M_i^{j'} & j \text{ is lim.} \end{cases}$

so N_i^j may not be $(\lambda, \geq\kappa)$-lim.

Let $N_i^{\lambda^+} = \bigcup_{j<\lambda^+} N_i^j = \bigcup_{j<\lambda^+} M_i^j$, $N_I^j = \bigcup_{i\in I} N_i^j$.

Let $C_i = \{ j < \lambda^+ : N_i^{\lambda^+} \cap N_I^j = N_i^j \}$.

C_i is closed & unbounded in λ^+ (closed easy, for unbdd, given j_0 take j_{n+1} s.t. $\underbrace{N_i^{\lambda^+}}_{\text{union of}} \cap N_I^{j_n} \subseteq N_i^{j_{n+1}}$ them, $j_\omega = \bigcup j_{n+1}$)

So $\bigcap_{i\in I} C_i$ is closed unbdd. Let $j \in \bigcap_{i\in I} C_i$ have $cf(j)=\kappa$.

Then

$$M_i^j = N_i^j = N_i^{\lambda^+} \cap N_I^j$$
$$= \left(\bigcup_{j'<\lambda^+} M_i^{j'}\right) \cap \left(\bigcup_{i'\in I} N_{i'}^j\right)$$
$$= \left(\bigcup_{j'<\lambda^+} M_i^{j'}\right) \cap \left(\bigcup_{i'\in I} N_{i'}^j\right)$$
$$\supseteq M_{i_j}^{j+1} \cap M_{i'_j}^j$$

※ //

Lemma: If $\langle \gamma^j : j<\delta \rangle$ is a $\triangleleft$-inc seq of reduced towers, $\bigcup \gamma^j$ a tower, & $cf(\delta) \geq \kappa$, then $\gamma^\delta = \bigcup \gamma^j$ is reduced.

proof: If $\gamma^\delta \triangleleft \gamma^*$, $\gamma^j \triangleleft \gamma^*$, so for $i < i' \in I^\delta$,

$$M_{i'}^\delta \cap M_i^* = \bigcup_{j<\delta} M_{i'}^j \cap M_i^*$$
$$= \bigcup_{j<\delta} M_i^j \quad \text{as } \gamma^j \triangleleft \gamma^*$$
$$= M_i^\delta .$$

//

Propn: Suppose $\mathcal{T} = \langle M_i : i \in \beta \rangle ^\frown \langle a_i : i \in \beta^- \rangle$ a tower, and $\gamma < \beta$, and there is $\mathcal{T}^* = \langle M_i^* : i < \gamma \rangle ^\frown \langle a_i : i \in \gamma \rangle$ s.t. $\mathcal{T} \restriction \beta \lhd \mathcal{T}^*$. Then there exists strongly univ $\mathcal{T}' = \langle M_i' : i < \beta \rangle ^\frown \langle a_i : i < \beta^- \rangle$ and some N etc,

$$g : \bigcup_{i<\beta} M_i \xrightarrow[\bigcup_{i<\gamma} M_i]{} N \quad \text{s.t.} \quad \mathcal{T} \lhd \mathcal{T}' \text{ and } g[\mathcal{T}'] \restriction \gamma = \mathcal{T}^*.$$

proof: If $\gamma = 0$, this is a special case of other thm. So wlog $\gamma \geq 1$.

Do as before: get $\langle N_i : i < \beta \rangle$, $\langle f_i : i < \beta \rangle$ s.t.

(1) $f_i : M_i \to N_i$

(2) for $i < \gamma$, $N_i = M_i^*$, f_i is inclusion. (used to be f_0 inclusion)

(3) $f_i[M_i] \leq^u N_i$

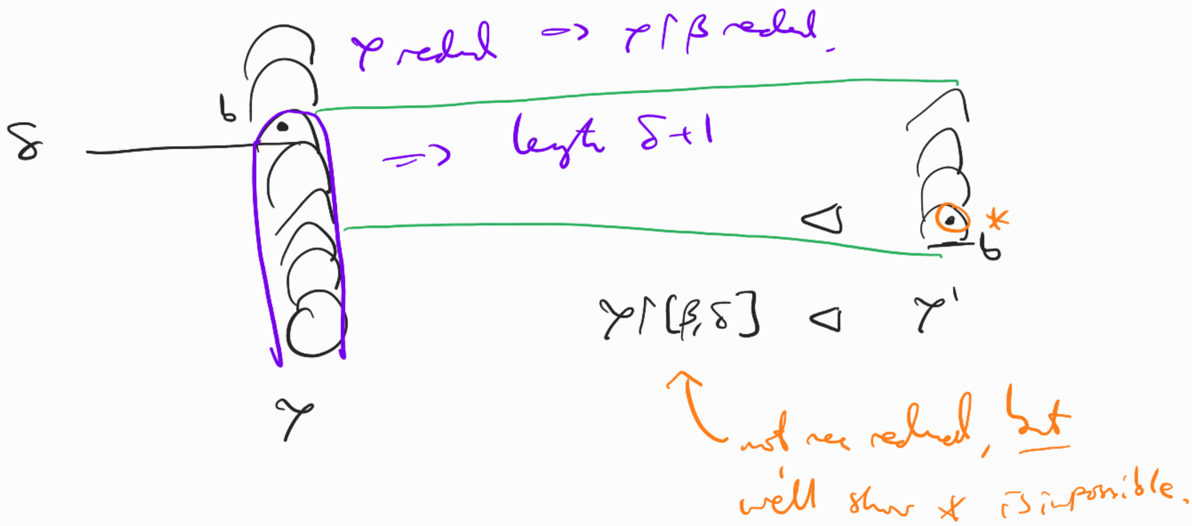
(4) $\bigcup_{i' < i} N_{i'} \leq^u N_i$

(5) $g\text{tp}(f_i(a_i)/N_i, N_{i+1})$ dnf $/ f_i[M_i]$

Finish: As before, $g : M' \cong \bigcup N_i$, $M_i' = g^{-1}[N_i]$

Now $g[\mathcal{T}'] \restriction \gamma = \mathcal{T}^*$.

For each δ: get tower of min length contradicting this.



Lemma: If $\gamma = \langle M_i : i < \alpha \rangle ^\frown \langle a_i : i < \alpha^- \rangle$ is reduced, $\beta < \alpha$, then $\gamma \restriction \beta$ is reduced.

proof: Say $\gamma \restriction \beta \triangleleft \gamma^*$.
By thm, $\exists\, \gamma' \triangleright \gamma$, $g : \bigcup M_i' \xrightarrow[\bigcup_{i<\beta} M_i]{} N$ s.t.

$$g[\gamma' \restriction \beta] = \gamma^*.$$

So for $i < i' < \beta$,

$$M_{i'} \cap M_i^* = g[M_{i'} \cap M_i'] = g[M_i] = M_i.$$

//

Lemma: Suppose $\delta < \lambda^+$ lim, $\gamma = \langle M_i : i \leq \delta \rangle ^\frown \langle a_i : i < \delta \rangle$ a tower (indexed by $(\delta+1)$), and $b \in M_\delta$ satisfies $b \underset{M_0}{\overset{M_\delta}{\perp}} M_i$ for all i. Then $\exists\, \gamma' = \langle M_i' : i \leq \delta \rangle ^\frown \langle a_i : i < \delta \rangle$ s.t. $\gamma \triangleleft \gamma'$ and $b \in M_0'$.

proof: Apply prop to $\gamma \restriction \delta$ and $p = \text{gtp}(b / M_0, M_\delta)$. There is $\gamma^* \triangleright \gamma$ indexed by δ, $b^* \in M_0'$ s.t. $b^* \underset{M_0}{\overset{M_i^*}{\perp}} M_i$.

Uniqueness gives $\text{gtp}(b / M_i, M_\delta) = \text{gtp}(b^* / M_i, M_i^*)$ for all i.

By weak cty, $gtp(b/\bigcup_{i<\delta} M_i, M_\delta) = gtp(b^*/\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} M_i^*)$.

By type eq, $\exists M_\delta^*$, (λ,κ)-lim

$$\begin{tikzcd} M_\delta \arrow[r, "f"] & M_\delta^* \\ \bigcup M_i \arrow[u, "id"] \arrow[r, "id"] & \bigcup M_i^* \arrow[u, "id"] \end{tikzcd}$$

$f(b) = b^*$.

Extend f to iso $\bar{f} : \bar{M}_\delta \longrightarrow M_\delta^*$

Let $M_i' = \bar{f}^{-1}[M_i^*]$ for $i \leq \delta$.

$\mathcal{T}' = \langle M_i' \longrightarrow \rangle^\frown \langle a_i : \longrightarrow \rangle \rhd \mathcal{T}$, $b \in M_0'$. //

Lemma: Suppose $\delta < \lambda^+$ limit, $\mathcal{T} = \langle M_i : i \leq \delta \rangle^\frown \langle a_i : i < \delta \rangle$ reduced, $b \in M_\delta \setminus \bigcup_{i<\delta} M_i$. Then there is no pair of $\beta < \delta$ and $\mathcal{T}'$ indexed by $[\beta, \delta]$ s.t. $\mathcal{T} \upharpoonright [\beta, \delta] \lhd \mathcal{T}'$ and $b \in \bigcup_{i \in [\beta,\delta)} M_i'$

proof: Suppose for contradiction such β, $\mathcal{T}'$ exist.

Fix $i_0 \in [\beta, \delta)$ s.t. $b \in M_{i_0}'$.

Lemma: Suppose $\mathcal{T} = \langle M_i : i < \alpha \rangle^\frown \langle a_i : i < \alpha^- \rangle$ is a reduced tower. Then for all $\beta < \alpha$, $\mathcal{T} \upharpoonright [\beta, \alpha)$ is reduced.

proof: Say $\mathcal{T} \upharpoonright [\beta, \alpha) \lhd \mathcal{T}' = \langle M_i' : i \in [\beta, \alpha) \rangle^\frown \langle a_i : \rightarrow$

By tower ext prop, $\exists \mathcal{T}^* \rhd \mathcal{T} \upharpoonright (\beta + 1)$

$M_\beta \leq^u M_\beta'$, so $\exists f : M^*_\beta \xrightarrow[M_\beta]{} M_\beta'$.

For $i < \beta$, set $M_i' = f[M_i^*]$.

Let $\mathcal{T}'' = \langle M_i' : i < \alpha \rangle^\frown \langle a_i : i < \alpha^- \rangle$.

For $i < \beta$, $\quad a_i \underset{M_i}{\overset{M_{i+1}^*}{\downarrow}} M_i^* \implies a_i \underset{M_i}{\overset{M_{i+1}'}{\downarrow}} M_i'$ $\left(\begin{array}{l} a_i \in M_\beta \\ \Rightarrow f(a_i) = a_i \end{array} \right)$

For $\beta \leq i < \alpha$, $\quad a_i \underset{M_i}{\overset{M_{i+1}'}{\downarrow}} M_i'$ by $\mathcal{T}' \rhd \mathcal{T}$.

So $\mathcal{T}''$ a tower, $\mathcal{T} \lhd \mathcal{T}''$

$\implies \forall \beta \leq i_1 < i_2 < \alpha$ (as $\mathcal{T}$ reduced)

$M_{i_2} \cap M_{i_1}' = M_{i_1}$ as desired. //

<u>Prop</u>: Suppose $\mathcal{T} = \langle M_i : i \in I \rangle^\frown \langle a_i : i \in I^- \rangle$ is reduced, $\delta \in I$ where $cf_I(\delta) \geq \kappa$. Then $\mathcal{T}$ is cts at δ, i.e.

$$M_\delta = \bigcup_{i<\delta} M_i.$$

<u>proof</u>: Wlog $I = \alpha$. Suppose false for ~~α~~.

Let α be minimal s.t. $\exists\ \mathcal{T}$ indexed by α, not cts at δ. Fix $\mathcal{T}$.

So $\alpha \geq \delta + 1$. Note $\mathcal{T} \upharpoonright (\delta+1)$ is reduced, not cts at $\mathcal{T}$.

So $\alpha = \delta + 1$.

$M_\delta \neq \bigcup_{i<\delta} M_i$, so $\exists\ b \in M_\delta \setminus \bigcup M_i$.

Construct $\mathcal{T}^j = \langle M_i^j : i \leq \delta \rangle^\frown \langle a_i : i < \delta \rangle$ $\lhd$-inc

for $j \leq \delta$ s.t. $\mathcal{T} \lhd \mathcal{T}^0$ by tower ext.

Let $\mathcal{T}^* = \langle M_i^i : i \leq \delta \rangle^\frown \langle a_i : i < \delta \rangle$.

Easy to check $\mathcal{T} \lhd \mathcal{T}^*$, $\mathcal{T}^*$ is ~~u~~ normal

$(M_i^i \leq^u M_i^{i+1} \leq M_{i+1}^{i+1})$.

By $(\geq\kappa)$-locality, as $cf(\delta) \geq \kappa$, $\exists \beta < \delta$ s.t.

$$gtp(b / \bigcup_{i<\delta} M_i^i, M_\delta^\delta) \text{ dnf} / M_\beta^\beta.$$

By monotonicity, $\qquad b \underset{M_\beta^\beta}{\overset{M_{i+1}^{i+1}}{\perp}} M_i^i$.

Let $\mathcal{T}^{**} = \mathcal{T}^{**} \restriction [\beta, \delta]$.

By last lemma $\exists\, \mathcal{T}' = \langle M_i' : i \in [\beta,\delta] \rangle ^\frown \langle a_i : i \in [\beta, \delta) \rangle$

s.t. $\mathcal{T}^{**} \lhd \mathcal{T}'$, $b \in M_\beta'$.

But $\mathcal{T} \restriction [\beta, \delta]$ is reduced and

$\mathcal{T} \restriction [\beta,\delta] \lhd \mathcal{T}^{**} \lhd \mathcal{T}'$

$$\Rightarrow \quad \underset{\overset{\cup}{\delta}}{M_\delta} \cap M_\beta' = \underset{\not\ni \delta}{M_\beta} \qquad ⨳ . \qquad //$$

Now to ensure the final tower is 'universal'.
In fact, we will add rows to the towers as we go, and make sure the towers are universal when restricted to the 'main' levels.

To formalise this:

Definition: Let $\mathcal{T} = \langle M_i : i \in I \rangle ^\frown \langle a_i : i \in I^- \rangle$ be a tower, $I_0 \subseteq I$. We say $\mathcal{T}$ is I_0-full if for every $i \in I_0$,

$\forall p \in S^{na}(M_i)\ \exists k \in [i, i+_{I_0} 1)_I$ s.t.

$$gtp(a_k / M_k, M_{k+1}) \geq p \text{ dnf} / M_i.$$

Fact: If $\langle M_i : i < \lambda \rangle$ $\leq$-inc, $\forall p \in S(M_i)$ realised in M_{i+1},

then $\bigcup_{i<\lambda} M_i \geq^u M_0$.

Remark: If $\mathcal{T}$ is I_0-full, then $\forall i \in I_0$, $M_{i+_{I_0}\lambda}$ is universal $/M_i$.

So $M_{i+_{I_0}\lambda\cdot\delta}$ is a (λ,δ)-limit $/M_i$.

Lemma: Suppose $\langle \mathcal{T}^j : j \leq \delta \rangle$ is a $\triangleleft$-inc seq of towers, $\mathcal{T}^j = \langle M_i^j : i \in I^j \rangle ^\frown \langle a_i : i \in (I^j)^- \rangle$, and $\mathcal{T}^\delta = \bigcup \mathcal{T}^j$.

If $I_0 \subseteq I^0$ and $\mathcal{T}^j$ is I_0-full for all $j < \delta$, then $\mathcal{T}^\delta$ is I_0-full also.

proof: Say $i \in I_0$, $p \in S(M_i^\delta)$.

By $(\geq \kappa)$-l.c., $\exists\, j < \delta$ s.t. p dnf $/M_i^j$.

$\mathcal{T}^j$ is I_0 full, so $\exists\, k \in [i, i+_{I_0} 1)_{I^j}$ s.t.

$gtp(a_k / M_k^j, M_{k+1}^j)$ dnf $/M_i^j$.

Note $gtp(a_k / M_k^\delta, M_{k+1}^\delta)$ dnf $/M_k^j$ as $\mathcal{T}^j \triangleleft \mathcal{T}^\delta$

So by transitivity $gtp(a_k / M_k^\delta, M_{k+1}^\delta)$ dnf $/M_i^j$.

By uniqueness, $gtp(a_k / M_i^\delta, M_{k+1}^\delta) = p$.

So k is as desired. //

Lemma: Say I a w.o., $\alpha, \gamma < \lambda^+$ limit where $\alpha < \gamma$, $cf(\gamma) = \lambda$.
If $\mathcal{T}$ is a strongly universal tower indexed by $I \times \alpha$, then there exists an $I \times \{0\}$-full tower $\mathcal{T}'$ indexed by $I \times \gamma$ s.t.

$$\mathcal{T}' \restriction (I \times \alpha) = \mathcal{T}$$

proof: Say $\mathcal{T} = \langle M_{(i,k)} : (i,k) \in I \times \alpha \rangle ^\frown \langle a_{(i,k)} : (i,k) \in I \times \alpha \rangle$

Fix $i \in I$. Define $M_{(i,k)}, a_{(i,k)}$ for $k \in [\alpha, \gamma)$ as follows:

Let $\overset{na}{S}(M_{(i,0)}) = \{ p_k : k \in [\alpha, \gamma) \}$

(possible as $cf(\gamma) = \lambda$, K λ-stable).

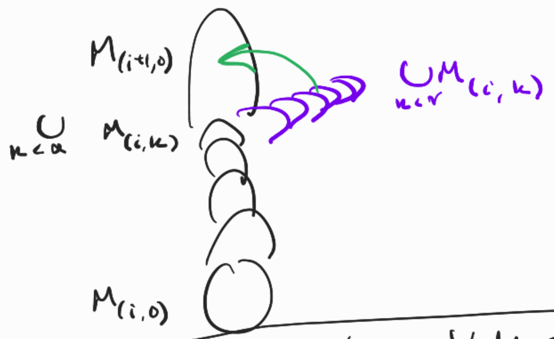
Define $M_{(i,k)}, a_{(i,k)}$ for $k \in \gamma$ by

- $k \geq \alpha$ limit: $M_{(i,k)}$ is (λ, κ)-lim $/ \bigcup_{l<k} M_{(l,k)}$
- $k = l+1 > \alpha$: $gtp(a_{(i,l)} / M_{(i,l)}, \hat{M}_{(i,l)}) \geq p_k$,

  def $/ M_{(i,0)}$

  $M_{(i,l+1)}$ is (λ, κ)-lim $/ \hat{M}_{(i,l)}$.

$M_{(i+1,0)}$ is univ $/ \bigcup_{k<\alpha} M_{(i,k)}$, so wlog $M_{(i,k)} \leq M_{(i+1,0)}$ //



Thm: If $cf(\delta_1), cf(\delta_2) \geq \kappa$, then $\forall M \in K_\lambda$ $\forall N_1, N_2$ (λ, δ_1)-lim, (λ, δ_2)-lim $/ M$ resp, $N_1 \underset{M}{\cong} N_2$.

Proof of main theorem: Let $M \in K_\lambda$. ETS $\exists N$ $(\lambda, \delta_1), (\lambda, \delta_2)$-lim $/ M$.

Wlog M is (λ, κ)-lim.

We construct recursively a $\leq$-inc seq of towers $\langle \mathcal{T}^j : j \leq \delta_1 \rangle$ and a $\leq$-inc seq of limit ordinals $\langle \alpha_j : j \leq \delta_1 \rangle \leq \lambda^+$

such that (1) $\forall j < \delta_1$

$\mathcal{T}_j = \langle M_i^j : i \in (\delta_2+1) \times \lambda \times \alpha_j \rangle ^\frown \langle a_i^j : i \in (\delta_2+1) \times \lambda \times \alpha_j \rangle$

(2) $M = M^0_{(0,0,0)}$

(3) $\forall j < \delta_1$, $\mathcal{T}^{2j+2}$ is reduced

(4) $\forall j < \delta_1$, $\mathcal{T}^{2j+1}$ is a $((\delta_2+1) \times \lambda \times \{0\})$-full tower

(5) $\mathcal{T}^{\delta_1} = \bigcup_{j<\delta_1} \mathcal{T}^j$

This is possible: $j=0$: $\alpha_0 = \omega$, $\mathcal{T}^0$ any str. univ. tower indexed by $((\delta_2+1) \times \lambda \times \omega)$ with $M^0_{(0,0,0)} = M$.

$2j+1$: let $\mathcal{T}^{2j}_* \rhd \mathcal{T}^{2j}$, $\mathcal{T}^{2j}_*$ str. univ. indexed as $\mathcal{T}^{2j}$.

let $\alpha_{2j+1} = \alpha_{2j} + \lambda$

let $\mathcal{T}^{2j+1}$ be $((\delta_2+1) \times \lambda \times \{0\})$-full, indexed by $((\delta_2+1) \times \lambda \times \alpha_{2j+1})$, where

$\mathcal{T}^{2j+1} \upharpoonright ((\delta_2+1) \times \lambda \times \alpha_{2j}) = \mathcal{T}^{2j}_*$

$\mathcal{T}^{2j} \lhd \mathcal{T}^{2j+1}$

$2j+2$: $\alpha_{2j+2} = \alpha_{2j+1}$, $\mathcal{T}^{2j+2} \rhd \mathcal{T}^{2j+1}$ is reduced.

j lim, $cf(j) < \kappa$: $\mathcal{T}^j$ extends $\langle \mathcal{T}^{j'} : j' < j \rangle$

j lim, $cf(j) \geq \kappa$. $\mathcal{T}^j = \bigcup_{j'<j} \mathcal{T}^{j'}$. // possible.

This is enough: $\langle M^j_{(\delta_2,0,0)} : j < \delta_1 \rangle$ witness

$M^{\delta_1}_{(\delta_2,0,0)}$ is (λ, δ_1)-lim $/ M^0_{(\delta_2,0,0)} \geq M$.

$\mathcal{T}^{\delta_1}$ is reduced, so $M^{\delta_1}_{(\delta_2,0,0)} = \bigcup_{i<\delta_2} M^{\delta_1}_{(i,0,0)}$.

$\mathcal{T}^{\delta_1}$ is $((\delta_2+1) \times \lambda \times \{0\})$-full, so

$M^{\delta_1}_{(i,k+1,0)}$ realizes all types over $M^{\delta_1}_{(i,k,0)}$

So $M^{\delta_1}_{(i+1,0,0)} \geq^u M^{\delta_1}_{(i,0,0)}$ for all $i < \delta_2$

So $M^{\delta_1}_{(\delta_2,0,0)}$ is (λ, δ_2)-lim $/ M^{\delta_1}_{(0,0,0)} \geq M$.

Hence $M^{\delta_1}_{(\delta_2,0,0)}$ is both (λ, δ_1)- and (λ, δ_2)-lim $/M$.

//

The end :)

OR NOT!

Part 4: Short limit models are distinct.

(*)

Hypothesis: K an AEC, $\lambda \geq LS(K)$, K_λ λ-stab, K_λ has AP, and $\perp$ an independence relation on K_λ with invariance, mon., basemon., uniqueness, ext., cty, κ-loc. in sure $\kappa \in \lambda^+ \cap \mathrm{Reg}$

$p \in S(\bigcup M_i)$ M_i $\leq^u$-inc $\exists i < \kappa$ p $\perp$-dnf$/M_i$

(Boney or easy) **Defn**: Given $\perp$ on K_λ,

$$\underline{\kappa}(\perp, K_\lambda, \leq^u) = \underline{\kappa}^u_\lambda(\perp) := \{\kappa < \lambda^+ \cap \mathrm{Reg} : \perp \text{ has } \kappa\text{-univ loc char}\}$$

$$\kappa(\perp, K_\lambda, \leq^u) := \kappa^u_\lambda(\perp) := \begin{cases} \min \underline{\kappa}(\perp, K_\lambda, \leq^u) & \underline{\kappa} \neq \emptyset \\ \infty & \underline{\kappa} = \emptyset \end{cases}$$

Fact: Under (*), the set

$$\underline{\kappa}(\perp, K_\lambda, \leq^u) = \{\kappa < \lambda^+ \cap \mathrm{Reg} : \perp \text{ has } \kappa\text{-univ loc char}\}$$

is an end segment of $\lambda^+ \cap \mathrm{Reg}$ (weak ulc is clearly, use weak ucl = ucl under cty).

So $\forall \mu \in [\kappa^u_\lambda(\perp), \lambda^+) \cap \mathrm{Reg}$,

μ is a univ loc char cardinal for $\perp$.

(Van 06)

Recall: If $M \leq N$ in K, $p \in S(N)$, then $\underline{p\ \lambda\text{-splits}/M}$ if $\exists N_1, N_2 \in K$ s.t. $\|N_\ell\| = \lambda$ and $M \leq N_\ell \leq N$ for $\ell = 1,2$ and $\exists f : N_1 \cong_M N_2$ st $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$.

Else p $\underset{\text{(does not split)}}{\underline{\lambda\text{-dns}/M}}$.

(Van 06) p $\underset{\text{(does not fork)}}{\underline{\lambda\text{-dnf}/M}}$ if $\exists M_0 \in K_\lambda$ st. $M_0 \leq^u M$ and p λ-dns$/M_0$.

well defined in λ superstable K.

Fact: Under ⊛, if $M \leq N \in K_\lambda$, $p \in S(N)$, and $p \perp\text{-dnf}/M$, then p λ-dns$/M$.

proof: Say $M \leq N_\ell \leq N$, $f : N_1 \xrightarrow[M]{\cong} N_2$.

Then $f(p \upharpoonright N_1)$, $p \upharpoonright N_2$ both extend $p \upharpoonright M$, $\perp$-dnf$/M$ by invariance, monotonicity.

So $f(p \upharpoonright N_1) = p \upharpoonright N_2$.

Hence p λ-dns$/M$. //

Other direction? Close to canonicity of forking.

Fact: (VanDieren): If K has AP, NMM in λ:
Weak uniqueness: If $M_0 \leq^u M_1 \leq M_2 \in K_\lambda$, $p, q \in S(M_2)$, $p \upharpoonright M_1 = q \upharpoonright M_1$,
p, q dns $/M_0$, then $p = q$.

Lemma: Assume ⊛. Suppose $L_1 \leq L_2 \leq M$ and L_2 is $(\lambda, \geq\kappa)$-lim $/L_1$,
and $p \in S(M)$. Then p $\downarrow$-dnf $/L_2$.

proof: By ext of $\downarrow$, there is $q \in S(M)$ s.t. $q \upharpoonright L_2 = p \upharpoonright L_2$,
q $\downarrow$-dnf$/L_2$.

WTS $p = q$.

Say $\langle M_i : i \leq \delta \rangle$ witness L_2 lim$/L_1$. Then
by $(\geq\kappa)$-l.c., $\exists\, i < \delta$ s.t. q $\downarrow$-dnf$/M_i$.
By monotonicity, p dns $/M_i$
last Lemma $\Rightarrow$ q dns$/M_i$
weak!
$M_i \leq^u L_2 \leq M$ $\Longrightarrow$ $p = q$. //
needed for $(\lambda, \geq\kappa)$-lim

Assume ⊛
Corollary 1: If $L_1 \leq^u L_2 \leq M$ all in K_λ, $p \in S(M)$,
p dns $/L_1$, then p $\downarrow$-dnf $/L_2$.

proof: Get $\langle M_i : i \leq \kappa \rangle$ $\leq^u$-inc cts s.t.
(1) $M_0 = L_1$
(2) $M_i \leq L_2$ for all i.
By last lemma, p $\downarrow$-dnf $/M_\kappa$. By mon, p $\downarrow$-dnf$/L_2$.
//

Cor 2: Assume ⊛: If $L \leq M$ in K_λ, $p \in S(M)$ dnf$/L$.
Then p $\perp$-dnf$/L$.

proof: If p dnf$/L$, $\exists L_0 \leq^u L$ st. p dns$/L_0$.
So p $\perp$-dnf$/L$ by cor 1. //

Lemma: Assume ⊛, and that K is $\aleph_0$-tame. (⊛ can come from syntactic types.)
Let $\delta < \lambda^+$. Suppose $\langle M_i : i \leq \delta \rangle \leq K_{\lambda,\beta}$ $\leq^u$-inc and cts.
If M_δ is $|\delta|^+$-saturated, and $p \in S(M_\delta)$, then $\exists i < \delta$ st.
p $\underset{\text{Vanny}}{\perp}$-dnf $/M_i$.

proof: **Claim:** there exists $i < \delta$ st. p dns $/M_i$.

proof of claim: Else for all i: there exists N_i^1, N_i^2, $f_i : N_i^1 \xrightarrow[M_i]{} N_i^2$
st. $M_i \leq N_i^1, N_i^2 \leq N$ and
$f_i(p \restriction N_i^1) \neq p \restriction N_i^2$.

By $\aleph_0$-tameness, there is $A_i \subseteq |N_i^2|$ st. $|A_i| \leq \aleph_0$
$$f_i(p \restriction N_i^1) \restriction A_i \neq (p \restriction N_i^2) \restriction A_i.$$
(i.e. $\forall A_i \leq L \leq N_i^2$ $f_i(p \restriction N_i^1) \restriction L \neq p \restriction L$.)

Let $B_i = f_i^{-1}(A_i) \cup A_i$.

Let $B = \bigcup_{i<\delta} B_i$ $\qquad |B| \leq |\delta| \cdot \aleph_0 = |\delta|$

$|B| \leq |\delta| < |\delta|^+$, so $\exists b \in M_\delta$ st. $\mathrm{gtp}(b/B, M_\delta) = p \restriction B$.

$b \in M_i$ for some $i < \delta$.

$$\mathrm{gtp}(b/A_i, M_{i+1}) = \mathrm{gtp}(f_i(b)/A_i, M_{i+1}) = f_i(\mathrm{gtp}(b/f_i^{-1}(A_i), M_{i+1})) = f_i(p \restriction f_i^{-1}(A_i))$$
f_i fixes M_i
$$\neq p \restriction A_i = \mathrm{gtp}(b/A_i, M_{i+1})$$
$\times$ //claim.

So there is $i < \delta$ s.t. p dns $/ M_i$.

$M_i \leq^u M_{i+1} \leq M_\delta$

So Cor 1 $\Longrightarrow$ p $\perp$-dnf $/ M_{i+1}$. //

⊛ Seems like you can get away with δ-tameness even.

Theorem: Suppose ⊛ and K is $(<\aleph_0)$-tame. Assume $\mu_1 < \mu_2 < \lambda^+$ are regular.
If M_1 is (λ, μ_1)-lim, M_2 is (λ, μ_2)-lim, and $M_1 \cong M_2$, then $\kappa^u_\lambda(\perp) \leq \mu_1$.

proof: By def of $\kappa^u_\lambda(\perp)$, ETS $\mu_1 \in \underline{\kappa}(\perp, K_\lambda, \leq^u)$.

Suppose $\langle N_i : i \leq \mu_1 \rangle$ $\leq^u$-incr cts sequence of models, $p \in S(N_{\mu_1})$.

Then $N_{\mu_1} \cong M_1 \cong M_2$.

So N_{μ_1} is μ_2-saturated

$\mu_2 > \mu_1 \Rightarrow N_{\mu_1}$ is μ_1^+-saturated.

By Lemma, $\exists i < \mu_1$ st p $\perp$-dnf $/ N_i$. //Thm.

Corollary: Suppose ⊛ and K is $(<\aleph_0)$-tame.

If $\alpha, \beta < \lambda^+$ limit, $cf(\alpha) < \kappa^u_\lambda(\perp)$ and $cf(\alpha) \neq cf(\beta)$, then the (λ, α)-limit model is not isomorphic to the (λ, β)-limit model. //

Corollary: Suppose $\circledast$, (K an AEC $\lambda \geq LS(K)$, K_λ has AP, and $\perp$ an independence relation on K_λ with invariance, mon., base mon., uniqueness, ext., cty.)

AND

$(\geq \kappa)$ - univ. loc. char. in some $\kappa \leq \lambda$ regular, and $\kappa^u_\lambda(\perp)$ is minimal as above.

AND

K is $(<\aleph_0)$-tame, and K_λ has JEP, NMM, and $\perp$ has non-f. AP.

Then the (λ, μ)-limit models for $\mu \in \left(\kappa^u_\lambda(\perp) \cap \mathrm{Reg}\right) \cup \{\kappa^u_\lambda(\perp)\}$ are non-isomorphic, and cover all isomorphism types.

Examples: Such $\perp$ exist in the following K. Bar (1), they all arise from finding a stable independence relation $\perp$ in the LRV sense, then extending to arb. sets $\perp$ $\left(A \overset{N}{\underset{M_0}{\perp}} B\right)$, and restricting to $a \overset{N}{\underset{M_0}{\perp}} M$.

(1) f.o. stable theories

(2) R-modules with embeddings

(3) $\quad$ " $\quad$ pure embeddings

(4) Torsion Abelian groups with embeddings

(5) $\quad$ " $\quad$ pure embeddings

(6) s-torsion modules with pure embeddings

$$\begin{tikzcd} N_1 \arrow[r] & Q \\ M \arrow[u] \arrow[r] \arrow[ur, phantom, "\perp"] & N_2 \arrow[u] \end{tikzcd} \iff \begin{tikzcd} & & Q \\ N_1 \arrow[r] \arrow[urr] & P \arrow[ur] & \\ M \arrow[u] \arrow[r] & N_2 \arrow[u] \arrow[uur] & \end{tikzcd}$$

K-emb

pushout in larger category.

Other notables: Forget $(<\aleph_0)$.

We showed (under ⊛)
- If $M \leq N$, $p \in S(N)$, then p λ-def$/M \Rightarrow p \downarrow$-def$/M \Rightarrow p$ λ-def$/M$
- If $L_1 \leq^u L_2 \leq N$, $p \in S(N)$, then p λ-def$/L_1 \Rightarrow p \downarrow$-def$/L_2$

So such $\downarrow$ are 'between' split/fork,
and 'close' to splitting.

Take this further:

Assume ⊛

Lemma: $\kappa(\underset{split}{\downarrow}, K_\lambda, \leq^u) = \kappa(\downarrow, K_\lambda, \leq^u) =: \kappa^u_\lambda(\downarrow)$

proof: $\kappa(\underset{split}{\downarrow}, K_\lambda, \leq^u) \leq \kappa(\downarrow, K_\lambda, \leq^u)$

as given $\langle M_i : i \leq \delta \rangle$, $p \in S(M_\delta)$,
p $\downarrow$-def$/M_i \Rightarrow p$ dns$/M_i$

$\kappa(\underset{split}{\downarrow}, K_\lambda, \leq^u) \geq \kappa(\downarrow, K_\lambda, \leq^u)$

as given $\langle M_i : i \leq \delta \rangle$, $p \in S(M_\delta)$,
p dns$/M_i \Rightarrow p$ $\downarrow$-def$/M_{i+1}$. //

Fact: If K is λ-stab, λ-tame, then $\kappa^u_\lambda(\underset{split}{\downarrow}) \leq \lambda$.

Under ⊛, if K is λ-tame, then

Corollary: $\kappa^u_\lambda(\downarrow) \leq \lambda$.

Proof: $\lambda \in \kappa^u_\lambda(\underset{split}{\downarrow}) = \kappa^u_\lambda(\downarrow)$. //

So in the $(<\aleph_0)$-tame case, this is as above.

Now say $\perp$ is defined on all of K (not just K_x);
$\perp$ satisfies $\circledast$ for each $\lambda \in Stab(K)$. $\Bigg\}$ $\circledast$

Definition: $\underline{\mathcal{X}}^u(\perp) = \bigcup_{\lambda \in Stab(K)} \underline{K}^u_\lambda(\perp)$

$\mathcal{X}^u(\perp) = \min \underline{\mathcal{X}}^u(\perp) = \min_{\lambda \in Stab} K^u_\lambda(\perp)$.

(may be ∞).

Assume $\circledast$

Corollary: $\underline{\mathcal{X}}^u(\perp) = \underline{\mathcal{X}}^u(\perp_{split})$

$\mathcal{X}^u(\perp) = \mathcal{X}(\perp_{split})$

Assume K is $LS(K)$-tame

Fact: $\mathcal{X}^u(\perp_{split}) = \aleph_0 \iff K$ is λ-superstable in some λ

Assume $\circledast$, K is $LS(K)$-tame. Then

Cor: K is superstable in some $\lambda \iff \mathcal{X}^u(\perp) = \aleph_0$.