C*-Algebras and Representations

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1 Introduction

1.1 Motivation

The bounded linear operators $\mathcal{B}(H)$ on a Hilbert space H are an example of a Banach algebra with involution map taking a linear operator to its adjoint. Further, $\mathcal{B}(H)$ satisfies $\|T^*T\| = \|T\|^2$ for all $T \in \mathcal{B}(H)$. We take this as a motivating example of what we shall call a *C*-algebra*: a Banach algebra A with involution that satisfies the *C*-inequality*

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in A.$$

In fact, our motivating example and the appropriate substructures characterise the theory to an extent: for any C*-algebra A there is a Hilbert space H and an injective algebra homomorphism $\phi : A \to \mathcal{B}(H)$ satisfying $\phi(a^*) = (\phi(a))^*$ for all $a \in A$. We call the pair (H, ϕ) of such a Hilbert space and involution preserving algebra homomorphism a *representation* of the algebra A. More generally a $*$*-homomorphism* is an algebra homomorphism that respects involution, and a $*$-isomorphism is the bijective equivalent. Hence any C*-algebra is $*$-isomorphic to a *C*-subalgebra* of $\mathcal{B}(H)$ (explicitly $\phi(A)$) for some Hilbert H. This result linking all C*-algebras to our motivating example is called the *Gelfand-Naimark* theorem, proved in its original weaker form by Isreal Gelfand and Mark Naimark in their 1943 paper [1] and built upon over the next few decades to the form we will prove in this dissertation [2].

We give another example of a representation: consider for a locally compact topological space X the set $C_0(X)$ of continuous functions f on X that *vanish at infinity*, meaning the set $\{x \in X : |f(x)| \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$. This is a Banach algebra (it is closed in the space of bounded complex valued functions with the supremum norm) which we can give an involution by defining f^* as $f^*(x) = f(x)$ for all $f \in C_0(X)$ and $x \in X$, and as this respects the C*-identity it is a C*-algebra. Then the evaluation maps $\epsilon_t : C_0(X) \to \mathbb{C} = \mathcal{B}(\mathbb{C})$ given by $\epsilon_t(f) = f(t)1_{\mathcal{B}(\mathbb{C})} = f(t)$ for all $f \in C_0(X)$ make representations $(\mathbb{C}, \epsilon_t)$ for any $t \in X$. Since $\phi(A)$ is 1-dimensional, the only closed $\epsilon_t(A)$ invariant subspaces of $\mathcal{B}(\mathbb{C})$ are the trivial ones, $\{0\}$ and $\mathcal{B}(\mathbb{C})$. Such representations (H, ϕ) (with trivial closed $\phi(A)$ invariant subspaces) are referred to as *irreducible*. In fact we will see that up to the relevant notion of equivalence all irreducible representations of $C_0(X)$ are of the form $(\mathbb{C}, \epsilon_t)$.

1.2 Goals

The Gelfand-Naimark theorem demonstrates the benefits of representations when studying C*-algebras. In this dissertation we endeavour to explore their relationship; in particular, the properties of *irreducible representations*. After proving the Gelfand Representation result and Gelfand-Naimark theorem, we will explore concepts of irreducibility in representations and prove their equivalence in *Kadison's transitivity theorem*. We will explore the relationship between irreducible representations and *pure states*, before using this to rigorously define the spectrum and primitive spectrum of a C*-algebra, and exploring these structures through theorems and examples.

1.3 Prerequisites

We will assume the reader has some knowledge of functional analysis and algebras. In particular, the Riesz representation theorem, Hahn-Banach theorems and Banach-Alaoglu theorem are assumed, and some results concerning the spectrum $\sigma(T)$ of a bounded linear map $T \in \mathcal{B}(H)$ for Hilbert space H, which sometimes have analogous proofs in the C*-algebra case. These results may be found in [3], [4].

We also assume *Urysohn's lemma* from general topology (for a proof see [5]), as well as the Stone-Weierstrass theorem for locally compact Hausdorff topological spaces.

2 Fundamental C*-Algebra Results

2.1 The Gelfand Representation

Recall the Gelfand-Naimark theorem tells us that C*-algebras are characterised by the C*-subalgebras of $\mathcal{B}(H)$ for some Hilbert H. In the commutative case, we can do better. We consider the C*-algebra $C_0(X)$, where X is locally compact. Note that $C_0(X)$ is unital if and only if $\mathbb{1}_X \in C_0(X)$ (where $\mathbb{1}_X(t) = 1$ for all $t \in X$), if and only if X is compact. In that case $C_0(X) = C(X)$, the continuous functions on X. In fact, all abelian C*-algebras are in isometric $*$-isomorphism with $C_0(X)$ for some locally compact X via the *Gelfand Representation*. Toward the end of this dissertation, we will see that the spectrum of such a C*-algebra is in natural bijection with

X, and when X is compact and we endow the spectrum with the Jacobson topology even homeomorphic, unlike the messier non-commutitive cases.

The appropriate space $X = \Omega(A)$ for a particular abelian C*-algebra is called the *character space*, which is the set of non-zero $*$-homomorphisms $\tau : A \to \mathbb{C}$. We refer to the elements as *characters*. Since this is a subset of the closed unit ball B_{A^*} of the dual space A^*, we endow $\Omega(A)$ with the restriction of the weak* topology. It is easy to see that $\Omega(A) \cup \{0\}$ is weak* closed in B_{A^*}, and since B_{A^*} is weak* compact by the Banach-Alaoglu theorem (see [3] for a statement and proof) $\Omega(A) \cup \{0\}$ is compact. Therefore $\Omega(A)$ is locally compact. If A is unital, then we see that $\Omega(A)$ is itself closed, and so weak* compact. Hence it makes sense to consider $C_0(\Omega(A))$ in our context.

Theorem 2.1 (The Gelfand Representation)**.** Let A be a non-zero abelian C*-algebra. Then $\Omega(A)$ is non-empty, and for all $a \in A$ the *Gelfand transform*

$$\hat{a} : \Omega(A) \to \mathbb{C}, \quad \tau \mapsto \tau(a)$$

is a well defined element of $C_0(\Omega(A))$.

Furthermore the *Gelfand representation*

$$\Phi : A \to C_0(\Omega(A)), \quad a \mapsto \hat{a}$$

is an isometric $*$-isomorphism.

We remark that the *spectrum* $\sigma_A(a)$ or $\sigma(a)$ defined by

$$\sigma_A(a) = \{\lambda \in \Lambda : \lambda 1_A - a \text{ is not invertible in } A\}$$

of an element a of a unital C*-algebra A and the *radius* $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$ generalise from the $\mathcal{B}(H)$ concepts along with most of the results, with some extra work using the Hahn-Banach theorem to apply Liouville's theorem when proving non-emptiness of the spectrum. The concept can be further extended to non-abelian C*-algebras by adjoining a unit - the details can be found in [6, pp. 5-13, 38-40].

Definition 2.1. A *hermitian* or *self-adjoint* element a of a C*-algebra A is one that satisfies $a^* = a$. The set of self-adjoint elemtents of A is denoted A_{sa}.

Note a^*a is hermitian for all $a \in A$.

We state and assume the following result (omitting proof) that is necessary for our proof of the Gelfand Representation theorem.

Theorem 2.2. For all $a \in A$, $\|\hat{a}\| = r(a)$, and if a is hermitian $r(a) = \|a\|$.

A proof can be found in [6, pp. 14-15, 37]. Much of the work is in showing $\sigma(a) \cup \Delta = \hat{a}(\Omega(\tau))$ where Δ is $\{0\}$ for non-unital algebras and $\varnothing$ for unital algebras, which gives the first equality. The second part is deduced similarly to the $\mathcal{B}(H)$ case (details in [6, pp. 5-10, 37]).

Proof: The Gelfand Representation. The algebra homomorphism conditions are easy to check. Since $\Phi(a^*)(\tau) = \tau(a^*) = \tau(a)^* = (\Phi(a)(\tau))^*$ for all $a \in A, \tau \in \Omega(a)$, this is a $*$-homomorphism. Now for any $a \in A$ we have also $\|\Phi(a)\|^2 = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a^*a)\| = \|\widehat{a^*a}\| = r(a^*a) = \|a^*a\| = \|a\|^2$, making use of the C*-identity in both algebras and the applying then previous theorem (noting that a^*a is self-adjoint). Hence Φ is an isometry. So $\Phi(A)$ is a closed involution invariant subalgebra of $C_0(\Omega(A))$ that separates points of $\Omega(A)$. Further for any $\tau \in \Omega(A)$, τ is non-zero, so there exists $a \in A$ such that $\Phi(a)(\tau) = \tau(a) \neq 0$. Hence $\Phi(A)$ vanishes nowhere, and by the locally compact version of the Stone-Weierstrass theorem $\Phi(A) = \overline{\Phi(A)} = C_0(\Omega(A))$. $\square$

2.2 The Gelfand-Naimark Theorem

We return now to the general (non-abelian) case. As discussed, the classification result here is the *Gelfand-Naimark* theorem. The key idea behind the proof is to construct a representation of A, the *Gelfand-Naimark-Segal* or *GNS* representation mentioned previously, corresponding to certain linear functionals on A called *states*. It will turn out that the 'sum' of these, what we term the *universal representation*, is faithful, and hence provides a $*$-isomorphism onto the image.

Before we define the GNS construction, we need to set up the theory of positive elements.

Example 2.1. Consider again $C_0(X)$ for some locally compact topological space X. Note that any positive valued function $f \in C_0(X)$ is self-adjoint and has a square root $f^{1/2} \in C_0(X)$, defined by $f^{1/2}(t) = (f(t))^{1/2}$. Note that any positive valued function is of the form $f = g^*g$ for some $g \in C_0(X)$

(choose $g = f^{1/2}$, for example), and all such $g^*g \in C_0(X)$ are self-adjoint and positive valued.

We wish to generalise this notion of 'positive' elements to C*-algebras:

Definition 2.2. A *positive element* of a C*-algebra is an element $a \in A$ of the form $a = b^*b$ for some $b \in A$. Equivalently, a is self-adjoint and $\sigma(a) \subseteq \mathbb{R}_{\geqslant 0}$. We denote the positive elements of A by A^+.

We define a partial order $\leqslant$ on A_{sa} by $a \leqslant b$ if and only if $b - a$ is positive for $a, b \in A$.

The argument giving this equivalence may be found in [6, pp. 45-46].

Theorem 2.3. All positive elements of a C*-algebra A have a positive square root.

The proof may be found in [6], and relies on preservation of the spectrum under the Gelfand representation of the smallest C*-algebra containing a positive element, which is necessarily abelian. In the Gelfand representation of this C*-subalgebra we find the inverse as in the previous example and pass back into A.

We now move on to the Gelfand-Naimark theorem. First we must introduce some new concepts.

Note that $*$-homomorphisms preserve positive elements, since if $a \in A^+$ and $\phi : A \to B$ is a $*$-homomorphism of C*-algebras, then $a = b^*b$ for some $b \in A$ and $\phi(a) = \phi(b^*b) = (\phi(b))^*\phi(b) \in B^+$. But we can generalise this concept:

Definition 2.3. A linear operator $T : A \to B$ between C*-algebras A, B is *positive* if $T(A^+) \subseteq B^+$.

We denote the set of positive linear functionals $A \to \mathbb{C}$ by $S(A)$, and its the elements *states.*

Similar to positive elements, there is a natural order on positive linear functionals given by $\rho \leqslant \tau$ if and only if $\tau - \rho$ is positive, where ρ, τ are positive linear functionals.

Before we state the facts concerning positive linear functionals we will use to prove Gelfand-Naimark, we introduce *approximate units*, a useful tool for generalising arguments from unital algebras to the general setting.

Definition 2.4. An *approximate unit* of a C*-algebra is an net $(u_\lambda)_{\lambda\in\Lambda}$ of self-adjoint elements of A of norm at most 1 such that for all $a \in A$ $\lim_{\lambda\in\Lambda} au_\lambda = a$, or equivalently $\lim_{\lambda\in\Lambda} u_\lambda a = a$.

In fact, every C*-algebra has an approximate unit [6, pp. 77-79], which is why we can extend unital results as mentioned. A particularly interesting example is the Gelfand-Naimark theorem itself - the theorem as stated in this dissertation was proven for unital algebras by J. G. Glimm and R. V. Kadison in 1960, but this proof was extended to non-abelian algebras by B. J. Vowden in 1967 by applying approximate units in the place of the unit [2, p. 8].

Using approximate units we can derive the following results concerning linear functionals. The proofs are omitted for brevity but may be found in [6, pp. 88-90].

Theorem 2.4. Let A be a C*-algebra.

1. All positive linear functionals on A are bounded.
2. For any positive linear functional τ on A, for all $a \in A$

$$\tau(a^*) = \overline{\tau(a)} \text{ and } |\tau(a)|^2 \leqslant \|\tau\|\tau(a^*a)$$

These results show positive linear functionals are even closer to $*$-homomorphisms that we supposed; we are only missing the multiplicative condition.

Theorem 2.5. Let A be a C*-algebra

3. If τ is a bounded linear functional on A, the following are equivalent:
   - τ is positive
   - For any approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A, $\|\tau\| = \lim_{\lambda\in\Lambda} \tau(u_\lambda)$
   - There is some approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A such that $\|\tau\| = \lim_{\lambda\in\Lambda} \tau(u_\lambda)$
4. For any *normal* element $a \in A$, meaning $a^*a = aa^*$, there is a state τ such that $\|a\| = |\tau(a)|$

The proof of the Gelfand-Naimark theorem we give is constructive: we define an explicit representation as a 'sum' of representations, with the summand representations being constructed from states. The following theorem will be particularly important for the *Gelfand-Naimark-Segal* or *GNS* construction corresponding to a state, to ensure the associated Hilbert space is well defined. Again the proof is omitted, and can be found in [6, p. 90].

Theorem 2.6. Let A be a C*-algebra.

5. If τ is a positive linear functional on A then for any $a \in A$, $\tau(a^*a) = 0$ if and only if $\tau(ba) = 0$ for all $b \in A$

6. If $\tau \in S(A)$ then for all $a, b \in A$

$$\tau(b^*a^*ab) \leqslant \|a^*a\|\tau(b^*b)$$

Before we GNS construction, for clarity we will remark that any Hilbert space has a unique (up to linear isometric isomorphism) completion it embeds into. The construction of the norm completion can be found in [7], and that this corresponds to an extension of the inner product follows from continuity and the parallelogram law.

We will proceed now to describe the construction of the Gelfand-Naimark-Segal representation for a state τ of an arbitrary C*-algebra A.

Example 2.2. Let A be a C*-algebra. Let $N_\tau = \{a \in A : \tau(a^*a) = 0\}$. This is a left ideal since for $a \in N_\tau$, $b \in A$, $0 \leqslant \tau((ba)^*(ba)) = \tau(a^*b^*ba) \leqslant \|b^*b\|\tau(a^*a) = 0$ by (6) of the previous theorems, implying $ba \in N_\tau$. N_τ is also closed due to continuity of τ. We can define an inner product on A/N_τ by

$$\langle \cdot, \cdot \rangle : (A/N_\tau) \to \mathbb{C}, \quad \langle a + N_\tau, b + N_\tau \rangle = \tau(b^*a)$$

The definition of N_τ, (5) from the previous theorems and the definition of positive linear functional show this is positive definite. Sesquilinearity and conjugate symmetry follow from the properties of the involution map and $\tau(a^*) = \tau(a)^*$. Hence we can extend it uniquely to the Hilbert completion H_τ of $(A/N_\tau, \langle \cdot, \cdot \rangle)$.

We now define $\theta_\tau : A \to \mathcal{B}(A/N_\tau)$ by $\theta_\tau(a)(b + N_\tau) = ab + A/N_\tau$ for all $a, b \in A$. Then for all $a, b \in A$

$$\|\theta_\tau(a)(b + N_\tau)\|^2 = \langle ab, ab \rangle = \tau((ab)^*(ab)) = \tau(b^*a^*ab) \leqslant \|a^*a\|\tau(b^*b)$$

$$= \|a^*a\|\|b + N_\tau\| = \|a\|^2\|b + N_\tau\|$$

by (6) from the previous theorems, and taking the supremum over $\|b+N_\tau\| \leqslant 1$ we see $\|\theta_\tau(a)\| \leqslant \|a\|$. It is easy to see that θ_τ is a $*$-homomorphism. θ_τ has a unique extension to H_τ. Since the involution map is continuous, the extension of θ_τ to H_τ, which we denote ϕ_τ, is a $*$-homomorphism also.

We define (H_τ, ϕ_τ) to be the *Gelfand-Naimark-Segal representation*, or *GNS representation*, of τ.

Definition 2.5. If $\{(H_\lambda, \phi_\lambda) : \lambda \in \Lambda\}$ is a family of representations of a C*-algebra A, we define the *direct sum* to be (H, ϕ) where $H = \oplus_{\lambda\in\Lambda} H_\lambda$, and $\phi : A \to \mathcal{B}(H)$ is given by $\phi(a)(x) = \phi_\lambda(x)$ for all $x \in H_\lambda$ and extended linearly to H.

Definition 2.6. Let A be a C*-algebra. We define the *universal representation* of A to be the direct sum of the family of GNS representations of states of A, meaning the direct sum of $\{(H_\tau, \phi_\tau) : \tau \in S(A)\}$

Theorem 2.7 (Gelfand-Naimark)**.** For any C*-algebra A, the universal representation is faithful.

Proof: Gelfand-Naimark. Assume that there exists $a \in A$ such that $\phi(a) = 0$. We will show that $a = 0$. By Theorem 2.5(4) there exists $\tau \in S(A)$ that satisfies $\|a^*a\| = |\tau(a^*a)| = \tau(a^*a)$. Define $b = (a^*a)^{1/4}$. Note $\phi_\tau(b)^4 = \phi_\tau(a^*a) = 0$, and

$$0 = \|\phi_\tau(b)^4\| = \|(\phi_\tau(b)^2)^*\phi_\tau(b)^2\| = \|\phi_\tau(b)^2\|^2 = \|\phi_\tau(b)^*\phi_\tau(b)\|^2 = \|\phi_\tau(b)\|^4,$$

using the C*-algebra identity and that $\phi_\tau(b)$ is hermitian. Hence $\phi_\tau(b) = 0$. So $\|a\|^2 = \tau(a^*a) = \|\tau(b^2)\|^2 = \|bb + N_\tau\|^2 = \|\phi_\tau(b)(b + N_\tau)\|^2 = 0$. Therefore ϕ is faithful.

$\square$

Hence any C*-algebra A is $*$-isomorphic to its image under the universal representation, a $*$-subalgebra of $\mathcal{B}(H)$, where H is the Hilbert space associated with this representation.

At the crux of the proof is the use of states to construct the GNS representations. This construction will also serve as our link between irreducible representations and *pure states* - these are states $\rho \in S(A)$ with the property that if τ is a positive linear functional and $\tau \leqslant \rho$ then $\tau = t\rho$ for some $t \in [0, 1]$. Note in this case that if $\tau \in S(A)$ and $\tau \leqslant \rho$, then $\rho = \tau$, and so ρ doesn't 'contain' any other states. Considering this, the link to irreducible representations seems feasible.

3 Kadison's Transitivity Theorem

In this section we will prove Kadison's transitivity theorem, which says the following definitions of irreducibility are equivalent. While being useful for developing later theory, this result is valuable in its own right as it shows that irreducible representations pose stronger restrictions on their invariant subspaces than we might have expected.

3.1 Definitions

We say a subspace K of a Hilbert space H is *invariant* under a subset A of $\mathcal{B}(H)$ if every element of A maps K into K. In the case that $A \subseteq \mathcal{B}(H)$ is a C*-algebra, we say that A is *irreducible* or *acts irreducibly* on H if the only closed subspaces of H under A are H and $\{0\}$.

If H is a Hilbert space, we define the *strong topology* on $\mathcal{B}(H)$ to be the topology on $\mathcal{B}(H)$ generated by the seminorms $(p_x)_{x \in H}$ defined by

$$p_x : \mathcal{B}(H) \to \mathbb{R}_{\geqslant 0}, \quad u \mapsto \|u(x)\|$$

in the sense that it is the coursest topology that makes the maps $\mathcal{B}(H) \to \mathbb{R}_{\geqslant 0}$ defined by $u \mapsto p_x(u - v)$ continuous for all $v \in \mathcal{B}(H)$ and $x \in H$. This topology has basis

$$\{U(T_0, x_1, \ldots, x_n, \varepsilon) : T_0 \in \mathcal{B}(H), n \in \mathbb{N}, x_1, \cdots, x_n \in H, \varepsilon > 0\}$$

where $U(T_0, x_1, \ldots, x_n, \varepsilon) := \{T \in \mathcal{B}(H) : \|(T - T_0)(x_j)\| < \varepsilon \text{ for } j = 1, \ldots, n\}$ [8].

Note that a net $(v_\lambda)_{\lambda \in \Lambda}$ converges to v in the strong topology if and only if $\lim_{\lambda \in \Lambda} v_\lambda(x) = v(x)$ for all $x \in H$.

Remark 3.1. Using the inner product definition of weak convergence, it is easy to see that the strong topology is coarser than the weak topology on $\mathcal{B}(H)$, so the terminology makes sense.

The definition of an irreducible action of an algebra ties directly to the definition of (topologically) irreducible representations. From this point on we will carefully distinguish between the two notions about to be given. We also define the appropriate notion of equivalence for C*-algebras.

Definition 3.1. A representation (H, ϕ) of A is *(topologically) irreducible* if $\phi(A)$ acts irreducibly on $\mathcal{B}(H)$, meaning that the only closed vector subspaces invarient under $\phi(A)$ of $\mathcal{B}(H)$ are 0 and $\mathcal{B}(H)$.

We say a representation (H, ϕ) is *algebraically irreducible* if the only vector subspaces (not necessarily closed) invarient under $\phi(A)$ of $\mathcal{B}(H)$ are 0 and $\mathcal{B}(H)$.

We say two representations (H_1, ϕ_1) and (H_2, ϕ_2), are *unitarily equivalent* if there exists a unitary map $u : H_1 \to H_2$ such that $u\phi_1(a)u^* = \phi_2(a)$ for all $a \in A$.

Despite the algebraic form of irreducibility seeming more powerful, Kadison's transitivity theorem says that these notions are in fact equivalent.

Definition 3.2. For a subset C of an algebra A, we define the *commutant* C' of C to be the set of elements in A that commute with every element of C.

Note that $C \subseteq C''$ and $C' = C'''$ for any $C \subseteq A$, where A is an algebra. Note that C' is a closed subalgebra of A if A has a norm. Further if A is an involutive algebra then C' is a $*$-subalgebra.

3.2 The Transitivity Theorem

At last we have enough machinery and terminology to state and prove Kadison's transitivity theorem.

Theorem 3.1 (Kadison's Transitivity Theorem Version 1)**.** A representation is algebraically irreducible if and only if it is (topologically) irreducible.

In fact we prove the following, slightly stronger result, which is a little hard to decode at first glance.

Theorem 3.2 (Kadison's Transitivity Theorem V2)**.** Let A be a non zero C*-algebra that acts irreducibly on Hilbert space H. Suppose $x_1, \ldots, x_n \in H$ and $y_1, \ldots, y_n \in H$ where the x_j are linearly independent. Then there exists $u \in A$ such that $u(x_j) = y_j$ for $j = 1, \ldots, n$. If there is a self-adjoint $v \in \mathcal{B}(H)$ satisfying $v(x_j) = y_j)$ for all $j = 1, \ldots, n$, then we may take u to be self-adjoint also. Further if $1_{\mathcal{B}(H)} \in A$ and v is unitary, then we may take u to be unitary of the form $u = e^{tw}$ for some hermitian $w \in A$.

To help unpack this, we note that the non trivial part of the latter two statements is essentially saying that if we assume there is a bounded operator with the properties described, then we can find an operator with the same properties in A.

In fact, we only need to prove the first statement about existence of u, and only in the case $n = 1$, to get version 1 of the transitivity theorem. However, our argument begins by proving the case when a hermitian v exists and using this to prove the more general case, and we may obtain the last statement with a little extra work. The proof is also not made much simpler by taking $n = 1$, so we do not impose this assumption.

Before we can prove Kadison's theorem, we state the following results that will be crucial for our argument - these involve a lot of work with *von Neumann algebras*, which are algebras A acting on Hilbert spaces which are equal to their *double commutant* A''. The proof of the following is omitted but may be found in [6, p. 113-121]

Theorem 3.3. Let H be a Hilbert space, and $A \subseteq \mathcal{B}(H)$ be a non-zero C*-algebra acting on H. Then the following conditions on A are equivalent:

1. A acts irreducibly on H
2. $A' = \mathbb{C}1_{\mathcal{B}(H)}$
3. A is strongly dense in $\mathcal{B}(H)$

In light of this, a representation (H, ϕ) is topologically irreducible if and only if $\phi(A)' = \mathbb{C}1_{\mathcal{B}(H)}$, if and only if $\phi(A)$ is strongly dense in $\mathcal{B}(H)$.

We will also need the following result of Kaplansky:

Theorem 3.4. Kaplansky Density Theorem

Let H be a Hilbert space, A a C*-subalgebra of $\mathcal{B}(H)$, and let B be the strong closure of A. Then A_{sa} is strongly dense in B_{sa}, and moreover the closed unit ball of A_{sa} is strongly dense in the closed unit ball of B_{sa}.

Again the proof is omitted, but the proof of a stronger form of Kaplansky's theorem appears in [6, pp. 129-132].

We are now in a position to prove Kadison's theorem. In an effort to make the proof readable, we break it into stages, beginning with the following lemma.

Lemma 3.1. For any Hilbert space H with elements $e_1, ..., e_n, y_1, ..., y_n \in H$ and e_j orthonormal, there exists $u \in \mathcal{B}(H)$ such that $u(e_j) = y_j$ for $j = 1, ..., n$ and $\|u\| \leqslant \sqrt{2n}\ \max\{\|y_1\|, ..., \|y_n\|\}$. In fact if there is $v \in \mathcal{B}(H)$ self-adjoint satisfying $v(e_j) = y_j$ for $j = 1, ..., n$ then we may take u to be self-adjoint also.

Note the non-trivial part of the second case is the norm condition on u. Before we prove this, we introduce some notation.

Definition 3.3. Let H be a Hilbert space, and $x, y \in H$. Then we define $(x \otimes y) : H \to H, \quad z \mapsto \langle z, y\rangle x$. Note $\|x \otimes y\| = \|x\|\|y\|$.

Proof. First we define $u = \sum_{i=1}^n y_j \otimes e_j$. It is easy to see that $u(e_j) = y_j$ for all appropriate j. For readability, define $M = \max\{y_1, \ldots, y_n\}$. Then $\|u(x)\| = \|\sum_{j=1}^n \langle x, e_j\rangle y_j\| \leqslant (\sum_{j=1}^n |\langle x, e_j\rangle| \|y_j\|) \leqslant (\sum_{j=1}^n |\langle x, e_j\rangle|^2 \|y_j\|^2)^{1/2} \leqslant \|x\|\sqrt{n}M$, where the final inequality comes from pythagorean theorem. Hence $\|u\| \leqslant \sqrt{n}M$

Now suppose there is a self-adjoint operator v on H such that $v(e_j) = y_j$. Then if we let $p = \sum_{j=1}^n e_j \otimes e_j$, we have $u = \sum_{j=1}^n v(e_j) \otimes e_j = v(\sum_{j=1}^n e_j \otimes e_j) = vp$. So if we define $u' = vp + pv - pvp$, then u' is hermitian and $u'(e_j) = vp(e_j) + pv(e_j) - pvp(e_j) = u(e_j) + pv(e_j) - pv(e_j) = y_j$ for all appropriate j. Finally note that

$$\begin{aligned}
\|u'\|^2 &= \|(u')^* u'\| = \|(vp + pv - pvp)(pv + vp - pvp)\| \\
&= \|(pv)^*(pv) + (pv - pvp)(vp - pvp) + vp(vp - pvp) + (pv - pvp)pv\| \\
&= \|(pv)^*(pv) + pv(1-p)(1-p)vp + (vpvp - vpvp) + (pvpv - pvpv)\| \\
&= \|(pv)^*(pv) + ((1-p)vp)^*(1-p)vp\| \\
&\leqslant \|pv\|^2 + \|(1-p)vp\|^2 \leqslant \|u^*\|^2 + \|(1-p)\|^2\|u\|^2 \leqslant 2\|u\|^2 \leqslant 2nM^2.
\end{aligned}$$

□

Note that if we disregard the condition that v is self-adjoint, then by the first paragraph we can take u such that $\|u\| \leqslant \sqrt{n}\ \max\{\|y_1\|, ..., \|y_n\|\}$

Proof. **Version 2**

We split into stages, which may seem a little out of order:

- Case 1: self-adjoint v exists
- Case 2: general case (no assumptions concerning existence of v)

- Case 3: $1_{\mathcal{B}(H)} \in A$ and v unital

Case 1:

We can assume x_j are orthonormal, since if we have the result in this case, we can apply it to an orthonormal basis $\{x'_1, \ldots, x'_n\}$ of $K := Cx_1 + \ldots + Cx_n$ and $\{y'_1, \ldots .y'_n\}$ where $y'_j := v(x'_j)$. We may assume also that $\|y_j\| \leqslant \sqrt{2n}$ for all appropriate j, since if we have the result in this case, we can consider $v' = v/\sqrt{2n}\max\{\|y_1\|, \ldots, \|y_n\|\}$ and $y''_j := y_j/\sqrt{2n}\max\{\|y_1\|, \ldots, \|y_n\|\}$ and solve this problem to obtain u', and then $u := u'\sqrt{2n}\max\{\|y_1\|, \ldots, \|y_n\|\}$ solves the original problem.

Define strong open sets $U_\varepsilon = \{u \in \mathcal{B}(H) : \max\|u(x_j)\| \leqslant \varepsilon\}$ for $\varepsilon > 0$ containing $0_{\mathcal{B}(H)}$. Since A is strongly dense in $\mathcal{B}(H)$ by Theorem 3.3 and the Kaplansky density theorem, the closed ball $B^c_{A_{sa}}$ in A_{sa} is strongly dense in the closed ball $B^c_{\mathcal{B}(H)_{sa}}$

Note $(w - U_\varepsilon) \cap B^c_{\mathcal{B}(H)_{sa}}$ is a strong open set for any $w \in B^c_{\mathcal{B}(H)_{sa}}$. So for any $\varepsilon > 0, w \in B^c_{\mathcal{B}(H)_{sa}}$ there exists $w' \in B^c_{A_{sa}}$ such that $w - w' \in U_\varepsilon$ and $\|w'\| \leqslant \|w\|$ (this means we can find $w' \in B^c_{A_{sa}}$ such that $w'(x_j)$ for appropriate j are as close as we want to y_j).

We now construct inductively two sequences of self-adjoint operators $v_k \in \mathcal{B}(H)$ and $u_k \in A$ such that $\|u_k\|, \|v_k\| \leqslant 2^{-k}$, $u_k - v_k \in U_{2^{-k-1}N^{-1}}$ $v_k(x_j)$ and $v_{k+1}(x_j) = (v_k - u_k)(x_j)$. If such sequences exist, we see that these conditions imply that $\sum_{k=0}^r u_k(x_j) = v_0(x_j) - v_{r+1}(x_j) = y_j - v_{r+1}(x_j)$, so with each iteration this sum up to r has 'error' v_{r+1} on the x_j. The norm conditions will ensure the sum converges and that $v_{r+1}(x_j)$ converges to zero.

First, by the previous lemma there exists $v_0 \in \mathcal{B}(H)$ such that $v_0(x_j) = y_j$ for all appropriate j and $\|v_0\| \leqslant 1$, since we restricted the the case where the y_j have the small enough norms to assert this. By the comments above, there exists $u \in A_{sa}$ that satisfies $\|u_0\| \leqslant 1$ and $u_0 - v_0 \in U_{1/2N}$, where we have taken $N = \sqrt{2n}$ for readability.

From here, suppose we have constructed $v_0, \ldots, v_r$ and $u_0, \ldots, u_r$ as describe above, meaning they are self-adjoint operators $v_k \in \mathcal{B}(H)$ and $u_k \in A$ such that $\|u_k\|, \|v_k\| \leqslant 2^{-k}$, $u_k - v_k \in U_{2^{-k-1}N^{-1}}$ and $v_k(x_j) = y_j$ for $k = 1, \ldots, r$ and $v_{k+1}(x_j) = (v_k - u_k)(x_j)$ for $k = 0, \ldots, r-1$ for all appropriate j. Then by the lemma again we can obtain self-adjoint $v_{r+1} \in \mathcal{B}(H)$ such that $v_{r+1}(x_j) = (v_r - u_r)(x_j)$ for all appropriate j, and $\|v_{r+1}\| \leqslant N\max_{1 \leqslant j \leqslant n}\|v_r(x_j) - u_r(x_j)\| \leqslant N(2^{r+1}N)^{-1} = 2^{-(r+1)}$. Hence by the comments above there exists $u_{r+1} \in A_{sa}$ with $\|u_{r+1}\| \leqslant 2^{-(r+1)}$ and $u_{r+1} - v_{r+1} \in U_{2^{-(r+2)}N^{-1}}$, and we find that $v_0, \ldots, v_{r+1}, u_0, \ldots, u_{r+1}$ still fulfil the required conditions. Hence by induction such a sequence as required exists.

As eluded to above, the sum $u := \sum_{k=0}^{\infty} u_k$ converges since $\sum_{k=0}^{\infty} \|u_k\| \leqslant \sum_{k=0}^{\infty} 2^{-k} = 2 \leqslant \infty$. As A_{sa} is closed, $u \in A_{sa}$. Note also

$$u(x_j) = \lim_{r\to\infty} \sum_{k=0}^{r} u_k(x_j) = \lim_{r\to\infty} v_0(x_j) - v_{r+1}(x_j) = y_j - \lim r \to \infty v_{r+1}(x_j) = y_j$$

for all appropriate j, since $\|v_k\| \to 0$ as $k \to \infty$. Hence this $u \in A_{sa}$ satisfies all the conditions we require.

Case 2: We drop all assumptions of the existence of v

We may assume the x_j for appropriate j are orthonormal and that $\|y_j\| \leqslant (2n)^{-1/2}$ for all appropriate j, reducing to this case by the same method as before.

By the lemma there exists $v \in \mathcal{B}(H)$ with $\|v\| \leqslant 1$ and $v(x_j) = y_j$. We may decompose v into the real and imaginary hermitian parts $v = v_R + iv_I$ (where $v_R := \frac{1}{2}(v + v^*), v_I := \frac{1}{2i}(v - v^*)$). By appying Case 1 to v_R, v_I, we obtain u_R, u_I in A which are self-adjoint, satisfying $u_R(x_j) = v_R(x_j)$ and $u_I(x_j) = v_I(x_j)$ for all appropriate j. By setting $u = u_R + iu_I \in A$ we see that $u(x_j) = u_R(x_j) + u_I(x_j) = v_R(x_j) + v_I(x_j) = v_(x_j) = y_j$, and u satisfies the required conditions.

Case 3:

Consider $K = \operatorname{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Extending $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ to bases $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ of K, we define a unitary operator on K by $v_0(x_j) = y_j$. There is an orthonormal basis $e_1, \ldots, e_m$ making v_0 diagonal, say $v_0(e_j) = \lambda_j$. Then $|\lambda_j| = 1$, so $\lambda_j = e^{it_j}$ for some $t_j \in \mathbb{R}$. Then $w' = \sum_{j=1}^{m} t_j e_j \otimes e_j \in \mathcal{B}(H)$ is hermitian and $w'(e_j) = t_j e_j$. By case 1 there exists $w \in A_{sa}$ satisfying $w(e_j) = t_j e_j$. Then u^{iw} is unitary and $u(e_j) = e^{it_j} = v_0(e_j)$, so $u(x_j) = y_j$ as required for appropriate j.

$\square$

Proof. **Version 1**

The forwards direction is trivial.

Conversely, suppose (H, ϕ) is a topologically irreducible representation of a C*-algebra A, and that K is a non-zero $\phi(A)$ invarient subspace (without imposing the condition K is closed) such that $\{0\} \lneqq K \leqslant H$. Fix some $x \neq 0$ in K and let $y \in H$ be arbitrary. Then by Kadison's transitivity theorem version 2 (taking $n = 1$) there exists an operator $u \in \phi(A)$ such that $u(x) = y$. As K is $\phi(A)$ invarient, $y \in K$. Therefore $K = H$, and as the only $\phi(A)$ invarient subspaces of H are $\{0\}$ and H, (H, ϕ) is algebraically irreducible.

$\square$

In light of Kadison's transitivity theorem, we will hereafter refer to both topologially and algebraically irreducible representations simply as *irreducible* representations.

4 Representations and States

We now discuss the links between states and representations. In order to give some motivation, we introduce new concepts of spectra C*-algebra briefly and explain why some theory of states will give these definitions rigour.

4.1 Spectra

Let A be C*-algebra. Let $[(H, \phi)]$ be the unitary equivalence class of (H, ϕ) (note this is not a set). We want to define spectra in the following, seemingly naive, way:

The *spectrum* of A is $\hat{A} = \{[(H, \phi)] : (H, \phi) \text{ is non-zero and irreducible}\}$.

The *primitive spectrum* of A is $\check{A} = \{\ker \phi : (H, \phi) \text{ irreducible}\}$. There are natural surjective maps

$$\begin{aligned} Irr(A) &\longrightarrow \quad \hat{A} \quad \longrightarrow \check{A} \\ (H, \phi) &\longmapsto [(H, \phi)] \longmapsto \ker \phi. \end{aligned}$$

The reader may have spotted some issues with these definitions - are either of them truly sets? And surely the spectrum can not even be a class, since it is a collection of equivalence classes (which in this case are strictly not sets). Given that we will later wish to give these spectra topologies, it is vital they are in some sense set like.

To this end, we will show that all irreducible represenions are unitary equivalent to the GNS representation some pure state.

Note that any equivalence classes of these is a subset of $PS(A)$ and hence is also a set. Then although the collection $\hat{A}$ is not a set, it is in 'bijection' with the unitary equivalence classes of the GNS representations of pure states, denoted for now by $\hat{A}_{ps}$, which is truly a set. The bijection is the obvious one:

$$\hat{A}_{ps} \to \hat{A}, \qquad [(H_\tau, \phi_\tau)]_{ps} \mapsto [(H, \tau)]$$

where the subscript ps denotes that this is the equivalence class within the set of pure state GNS representations only. Note this also means that

$$\hat{A}_{ps} \to \check{A}, \qquad [(H_\tau, \phi_\tau)]_{ps} \mapsto \ker(\phi_\tau)$$

is a surjection, and so $\check{A}$ is a set.

While we cannot technically place a topology on $\hat{A}$, we can define a topology on the set of unitary equivalence classes of pure states and think of $\hat{A}$ as inheriting this. To summarise, this means $\check{A}$ is a set, and we can treat $\hat{A}$ as a set and place a topology on it, while in the back of our minds remembering that this is strictly a topology on $\hat{A}_{ps}$.

Although this motivates the following section, the close relationship between representations and states, in particular between irreducible representations and pure states, ties back to our overarching theme of representations and justifies investigation for its own merits. We will return to spectra afterwards.

4.2 States, Pure States and Cyclic Representations

Definition 4.1. We say $x \in H$ is *cyclic* for algebra $A \subseteq \mathcal{B}(H)$ acting on Hilbert space H if the closure of $A\{x\} = \{a(x) : a \in A\}$ is H.

Suppose (H, ϕ) is a representation of C*-algebra A. We say a $x \in H$ is a *cyclic vector* if x is cyclic for $\phi(A)$. If such x exists, we say (H, ϕ) is a cyclic *representation*.

Recall for the following lemma the definition of N_τ for a state τ on C*-algebra A, as defined in the GNS construction. We also introduce a new concept:

Definition 4.2. If A acts on Hilbert space H, we say the action is *non-degenerate* if $[AH] = H$, where $AH = \text{span}\{a(x) : a \in A, x \in H\}$ and $[AH]$ denotes the closure. We say a representation (H, ϕ) of A is *non-degenerate* if $\phi(A)$ acts non-degenerately on H.

Since $[AH]$ is A invariant, it is clear that all non-zero algebras acting irreducibly act non-degeneratly. Note if A acts non-degenerately on Hilbert

H and has approximate unit $(u_\lambda)_{\lambda\in\Lambda}$, then $x = \lim_{\lambda\in\Lambda} u_\lambda x$ for all $x \in H$ by density of AH and continuity of the u_λ, so (u_λ) converges to $1_{\mathcal{B}}(H)$. In particular, if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for A, then $(\phi(u_\lambda))_{\lambda\in\Lambda}$ is an approximate unit for $\phi(A)$, and hence converges to $1_{\mathcal{B}(H)}$.

Hence if A acts non-degenerately on H, then for every $x \in H\backslash\{0\}$ there exists $a \in A$ such that $a(x) \neq 0$, since $x = \lim_{\lambda\in\Lambda} u_\lambda x$. In fact, the converse is true, and this equivalent definition perhaps gives better motivation for the terminology.

Lemma 4.1. Let A be a C*-algebra, and $\tau \in S(A)$. Then there exists a unique $x_\tau \in N_\tau$ that satisfies $\tau(a) = \langle a + N_\tau, x_\tau\rangle$ for all $a \in A$. In fact this x_τ is a unit cyclic vector for (H_τ, ϕ_τ), and for all $a \in A$ we have $\phi_\tau(a)x_\tau = a+N_\tau$.

Proof. First, define a linear functional ρ_0 on A/N_τ, given by $\rho_0(a + N_\tau) = \tau(a)$. This is norm decreasing as $\|\tau\| = 1$. Since A/N_τ is dense in H_τ, we may extend this to a norm decreasing linear functional ρ on H_τ. By the Riesz representation theorem there exists a unique $x_\tau \in H_\tau$ such that $\rho(y) = \langle y, x_\tau\rangle$ for all $y \in H_\tau$. Then for all $a \in A$, $\tau(a) = \rho(a+N_\tau) = \langle a+N_\tau, x_\tau\rangle$. Uniqueness of such x_τ follows easily.

Now, for any $a, b \in A$ we have $\langle b + N_\tau, \phi_\tau(a)x_\tau\rangle = \langle \phi(a^*)(b + N_\tau), x_\tau\rangle = \langle a^*b + N_\tau, x_\tau\rangle = \tau(a^*b) = \langle b + N_\tau, a + N_\tau\rangle$. By density of A/N_τ in H_τ, $\langle y, \phi_\tau(a)x_\tau\rangle = \langle y, a + N_\tau\rangle$ for all $y \in H_\tau$. So by non-degeneracy of the inner product on H_τ, $\phi_\tau(a)x_\tau = a + N_\tau$ for all $a \in A$. Hence $a + N_\tau \in \phi_\tau(A)x_\tau$ for all $a \in A$, so $A/N_\tau \subseteq \phi_\tau(A)x_\tau$, and $\phi_\tau(A)x_\tau$ is dense in H_τ. So x_τ is a cyclic vector for (H_τ, ϕ_τ).

In particular, this means that $\phi_\tau(A)$ acts non-degenerately on H_τ. Let $(u_\lambda)_{\lambda\in\Lambda}$ be any approximate unit of A. Then since ϕ_τ is a $*$-homomorphism, $(\phi_\tau(u_\lambda))_{\lambda\in\Lambda}$ is an approximate unit for $\phi_\tau(A)$. By non-degeneracy, this converges to $1_{\mathcal{B}(H)}$. Therefore

$$\begin{aligned}\|x_\lambda\|^2 = \langle x_\tau, x_\tau\rangle = \lim_{\lambda\in\Lambda}\langle \phi(u_\lambda)x_\tau, x_\tau\rangle &= \lim_{\lambda\in\Lambda}\langle \phi_\tau(u_\lambda)x_\tau, x_\tau\rangle \\ &= \lim_{\lambda\in\Lambda} \tau(u_\lambda) = \|\tau\| = 1,\end{aligned}$$

since τ is positive. Hence x_τ is unital.

□

Definition 4.3. In the set up of the previous lemma, we call the cyclic vector x_τ constructed in the previous proof the *canonical cyclic vector* for (H_τ, ϕ_τ).

Recall a sesquilinear form $\sigma : H^2 \to \mathbb{C}$ on vector space H is *hermitian* if and only if $\sigma(x, y) = \overline{\sigma(y, x)}$ for all $x, y \in H$, and *positive* if and only if $\sigma(x, x) \geqslant 0$ for all $x \in X$. In fact, if H is a Hilbert space and $u \in \mathcal{B}(H)$, then the sesquilinear form $\sigma_u : H^2 \to \mathbb{C}$ given by $\sigma_u(x, y) = \langle u(x), y\rangle$ is hermitian if and only if u is, and positive if and only if u is. The hard part is in showing that if σ_u is positive then u is positive also, but this follows by showing for all $\lambda < 0$, $u - \lambda 1_{\mathcal{B}(H)}$ satisfies $\|u - \lambda 1_{\mathcal{B}(H)}(x)\| \geqslant |\lambda|\|x\|$ for all $x \in H$ and is surjective, so is invertible, and hence as u is hermitian by the first statement the spectrum is contained in $\mathbb{R}_{\geqslant 0}$.

Lemma 4.2. If A is a C*-algebra, $\tau \in S(A)$ and $v \in \phi_\tau(A)'$ satisfies $0_{\mathcal{B}(H_\tau)} \leqslant v \leqslant 1_{\mathcal{B}(H_\tau)}$, then $\rho(a) := \langle \phi_\tau(a) v x_\tau, x_\tau\rangle$ defines a positive linear functional on A such that $\rho \leqslant \tau$.

Proof. Note that for any $a \in A$, $\rho(a) = \langle \phi_\tau(a) v x_\tau, x_\tau\rangle = \langle v\phi_\tau(a)x_\tau, x_\tau\rangle = \langle v(a + N_\tau), x_\tau\rangle$ and $\tau(a) = \langle a + N_\tau, x_\tau\rangle$, and therefore we have $(\tau - \rho)(a) = \langle (1_{\mathcal{B}(H_\tau)} - v)(a + N_\tau), x_\tau\rangle$.

Since $0_{\mathcal{B}(H_\tau)} \leqslant v \leqslant 1_{\mathcal{B}(H_\tau)}$, the sesquilinear form $\sigma_{(1-v)}$ as defined above is positive. So

$$\begin{aligned}(\tau - \rho)(a^*a) = \langle (1_{\mathcal{B}(H_\tau)} - v)(a + N_\tau), x_\tau\rangle &= \langle \phi_\tau(a^*)(1_{\mathcal{B}(H_\tau)} - v)(a + N_\tau), x_\tau\rangle \\ = \langle (1_{\mathcal{B}(H_\tau)} - v)(a + N_\tau), \phi_\tau(a)x_\tau\rangle &= \langle (1_{\mathcal{B}(H_\tau)} - v)(a + N_\tau), a + N_\tau\rangle \\ &= \sigma_{(1_{\mathcal{B}(H_\tau)} - v)}(a + N_\tau, a + N_\tau) \geqslant 0,\end{aligned}$$

and since $A^+ = \{a^*a \in A : a \in A\}$ this implies $\rho \leqslant \tau$.

□

In fact the converse is true:

Lemma 4.3. For any positive linear functional ρ on C*-algebra A and $\tau \in S(A)$ such that $\rho \leqslant \tau$, there exists a unique $v \in \phi_\tau(A)'$ such that $\rho(a) = \langle \phi_\tau(a) v x_\tau, x_\tau\rangle$, where τ is the canonical cyclic vector for (H_τ, ϕ_τ), and moreover $0_{\mathcal{B}(H_\tau)} \leqslant v \leqslant 1_{\mathcal{B}(H_\tau)}$.

Proof. Define a map $\sigma_0 : (A/N_\tau)^2 \to \mathbb{C}$ by $(a + N_\tau, b + N_\tau) \mapsto \rho(b^*a)$. To see this is well defined, note that if $a_1, a_2, b_1, b_2 \in A$ and $(a_1 + N_\tau, b_1 + N_\tau) = (a_2 + N_\tau, b_2 + N_\tau)$, then

$$\tau((a_1 - a_2)^*(a_1 - a_2)) = 0 \text{ and } \tau((b_1 - b_2)^*(b_1 - b_2)) = 0,$$

and as $(a_1 - a_2)^*(a_1 - a_2)$ and $(b_1 - b_2)^*(b_1 - b_2)$ are positive elements and $\tau - \rho, \rho$ are positive

$$\rho((a_1 - a_2)^*(a_1 - a_2)) = 0 \text{ and } \rho((b_1 - b_2)^*(b_1 - b_2)) = 0.$$

By Theorem 2.6(5), $\rho((b_1 - b_2)^* a_1) = 0$ and $\rho((b_2^*(a_1 - a_2)) = 0$. So $\rho(b_1^* a_1) - \rho(b_2^* a_2) = \rho((b_1 - b_2)^* a_1) + \rho((b_2^*(a_1 - a_2)) = 0$, and hence σ_0 is well defined.

It is also a sesquilinear form, by an identical approach used in the GNS construction. Since ρ is positive, so is σ_0. Hence

$$\begin{aligned} |\sigma_0(a + N_\tau, b + N_\tau)|^2 \leqslant \sigma_0(a, a)\sigma_0(b, b) &= \rho(a^*a)\rho(b^*b) \\ &\leqslant \tau(a^*a)\tau(b^*b) = \|a + N_\tau\| \|b + N_\tau\| \end{aligned}$$

where the first inequality follows from Cauchy-Schwarz for positive sesquilinear forms, and so $\|\sigma_0\| \leqslant 1$. By density of A/N_τ in H_τ we may extend σ_0 to a sesquilinear form σ on H_τ such that $\|\sigma\| \leqslant 1$. By the Riesz representation theorem for sesquilinear forms (see [4, p. 155-156] for details) there exists $v \in \mathcal{B}(H_\tau)$ such that $\sigma(x, y) = \langle v(x), y\rangle$ for all $x, y \in H_\tau$ where $\|v\| = \|\sigma\| \leqslant 1$. Note that $\rho(b^*a) = \sigma(a + N_\tau, b + N_\tau) = \langle v(a + N_\tau), b + N_\tau\rangle$, and hence $\langle v(a + N_\tau), a + N_\tau\rangle \geqslant 0$ for all $a \in A$. By density of A/N_τ in H_τ, σ is positive, so v is positive by the previous lemma. Applying the Gelfand representation to the C*-subalgebra generated by v and $1_{\mathcal{B}(H_\tau)}$ we see also that $v \leqslant 1_{\mathcal{B}(H_\tau)}$.

Suppose now that $a, b, c \in A$. Then

$$\begin{aligned} \langle \phi_\tau(a)v(b + N_\tau), c + N_\tau\rangle &= \langle v(b + N_\tau), \phi_\tau(a)^*(c + N_\tau)\rangle \\ &= \langle v(b + N_\tau), a^*c + N_\tau\rangle = \rho(c^*ab) \\ &= \langle v(ab + N_\tau), c + N_\tau\rangle \\ &= \langle v\phi_\tau(a)(b + N_\tau), c + N_\tau\rangle. \end{aligned}$$

Once again by density of A/N_τ in H_τ, this implies that $v\phi(a) = \phi(a)v$ for every $a \in A$, and hence $v \in \phi(A)'$.

If $a, b \in A$, then

$$\begin{aligned} \rho(b^*a) = \langle v(a + N_\tau), b + N_\tau\rangle = \langle v(\phi_\tau(a)x_\tau, \phi_\tau(b)x_\tau\rangle &= \langle \phi(b^*)v\phi_\tau(a)x_\tau, x_\tau\rangle \\ &= \langle v\phi_\tau(b^*a)x_\tau, x_\tau\rangle. \end{aligned}$$

Thus, if we take $(u_\lambda)_{\lambda\in\Lambda}$ to be any approximate unit, then $\lim_{\lambda\in\Lambda} \rho(u_\lambda a) = \lim_{\lambda\in\Lambda}\langle v\phi_\tau(u_\tau a)x_\tau, x_\tau\rangle = \langle v\phi_\tau(a)x_\tau, x_\tau\rangle$, and v satisfies all the required properties.

Now to prove uniqueness; suppose that $w \in \phi_\tau(A)'$ satisfies $\rho(a) = \langle \phi_\tau(a) w x_\tau, x_\tau \rangle$ for all $a \in A$. Then we have

$$\begin{aligned}\langle w(a+N_\tau), b+N_\tau\rangle &= \langle \phi_\tau(b^*) w \phi_\tau(a) x_\tau, x_\tau\rangle = \langle \phi_\tau(b^* a) w x_\tau, x_\tau\rangle = \phi(b^* a) \\ &= \langle \phi_\tau(b^* a) v x_\tau, x_\tau\rangle = \langle \phi_\tau(b^*) v \phi_\tau(a) x_\tau, x_\tau\rangle = \langle v(a+N_\tau), b+N_\tau\rangle,\end{aligned}$$

and once again by density of A/N_τ in H_τ this implies $v = w$.

□

We use this to obtain our first major result concerning the link between pure states and irreducible representations:

Theorem 4.1. If A is a C*-algebra and $\tau \in S(A)$, then τ is pure if and only if (H_τ, ϕ_τ) is irreducible.

Proof. If $\tau \in S(A)$ and (H_τ, ϕ_τ) is irreducible and $\rho \leqslant \tau$, then as $\rho(a) = \langle \phi_\tau(a) v x_\tau, x_\tau\rangle$ for some $v \in \phi(A)' = \mathbb{C}1_{\mathcal{B}(H_\tau)}$ where $0_{\mathcal{B}(H_\tau)} \leqslant v \leqslant 1_{\mathcal{B}(H_\tau)}$, there is some $t \in [0,1]$ such that $v = t1_{\mathcal{B}(H_\tau)}$. So $\rho(a) = \langle \phi_\tau(a) v x_\tau, x_\tau\rangle = t\langle \phi_\tau(a) x_\tau, x_\tau\rangle = t\tau(a)$. Hence τ is pure.

Conversely, suppose $\tau \in PS(A)$. Let $v \in \phi_\tau(A)'$ be such that $0_{\mathcal{B}(H_\tau)} \leqslant v \leqslant 1_{\mathcal{B}(H_\tau)}$. Then by Lemma 4.2 the function $\rho : A \to \mathbb{C}$ defined by $\rho(a) = \langle \rho_\tau(a) v x_\tau, x_\tau\rangle$ is a positive linear functional satisfying $\rho \leqslant \tau$. Since τ is pure, there exists $t \in [0,1]$ such that $\rho = t\tau$. So

$$\langle \phi_\tau(a) v x_\tau, x_\tau\rangle = \rho(a) = t\tau(a) = t\langle a+N_\tau, x_\tau\rangle = \langle t\phi_\tau(a) x_\tau, x_\tau\rangle.$$

Using this, we can show that for all $a, b \in A$ we have

$$\begin{aligned}\langle v(a+N_\tau), b+N_\tau\rangle &= \langle v\phi_\tau(a) x_\tau, \phi_\tau(b) x_\tau\rangle = \langle \phi_\tau(b^* a) v x_\tau, x_\tau\rangle \\ &= \langle t\phi_\tau(b^* a) x_\tau, x_\tau\rangle = \langle \phi_\tau(b)^* t\phi_\tau(a) x_\tau, x_\tau\rangle \\ &= \langle t\phi_\tau(a) x_\tau, \phi_\tau(b) x_\tau\rangle = \langle t(a+N_\tau), b+N_\tau\rangle.\end{aligned}$$

By non-degeneracy and density of A/N_τ in H_τ, $v = t1_{\mathcal{B}(H_\tau)}$.

Note that any C*-algebra is spanned by its positive elements: for any $a \in A$, setting $b = 1/2(a + a^*)$ and $c = 1/2i(a - a^*)$ $a = b + ic$ where b, c are hermitian. Further for any hermitian $b \in A$ we can pass to the Gelfand representation of the smallest C*-algebra containing b, which is abelian since b is self-adjoint. Then if we let $|b|$ be the element of this smallest algebra with Gelfand transform $|\hat{b}|$, we find that $|b| + b$ and $|b| - b$ are in A^+, and $b = 1/2((|b| + b) - (|b| - b)) \in \operatorname{span} A^+$. Hence $\operatorname{span} A^+ = \operatorname{span} A_{sa} = A$.

Therefore $\phi_\tau(A)'$ is spanned by its positive elements, and we see that $\phi_\tau(A)' = \mathbb{C}1_{\mathcal{B}(H_\tau)}$, and by Theorem 3.3 (H_τ, ϕ_τ) is irreducible.

$\square$

Before our next intermediate lemma, we note that cyclic representations are non-degenerate: if (H, ϕ) is a cyclic representation with cyclic vector x, then $[\phi(A)H] \supseteq \overline{\phi(A)x} = H$. Therefore $[\phi(A)H] = H$ and (H, ϕ) is non-degenerate.

We will soon see that all irreducible representations are cyclic. Because of this, the following result will be useful when comparing irreducible representations in terms of unitary equivalence, and bring us closer to our goal of defining the spectra of a C*-algebra rigorously.

Lemma 4.4. Let A be a C*-algebra with representations (H_1, ϕ_1) and (H_2, ϕ_2), with cyclic vectors x_1, x_2 respectively. Then there exists a unitary operator $u : H_1 \to H_2$ satisfying $x_2 = u(x_1)$ and $u\phi_1(a) = \phi_2(a)u$ for all $a \in A$ if and only if $\langle \phi_1(a)x_1, x_1 \rangle = \langle \phi_2(a)x_2, x_2 \rangle$ for all $a \in A$.

Proof. If such a unitary exists, then $\langle \phi_1(a)x_1, x_1 \rangle = \langle u(\phi_1(a)x_1), u(x_1) \rangle = \langle u\phi_1(a)u^* x_2, x_2 \rangle = \langle \phi_2(a)x_2, x_2 \rangle$, which gives the forward direction.

Conversely, suppose that $\langle \phi_1(a)x_1, x_1 \rangle = \langle \phi_2(a)x_2, x_2 \rangle$ for all $a \in A$. We note that, $\|\phi_1(a)x_1\|^2 = \langle \phi_1(a^*a)x_1, x_1 \rangle = \langle \phi_2(a^*a)x_2, x_2 \rangle = \|\phi_2(a)x_2\|^2$, meaning $\phi_1(a)x_1 = 0$ if and only if $\phi_2(a)x_2 = 0$. From these observations we deduce that the map $u_0 : \phi_1(A)x_1 \to H_2$ given by $u_0(\phi_1(a)x_1) = \phi_2(a)x_2$ is well defined, linear and isometric.

As x_1 is cyclic for (H_1, ϕ_1), there is a unique extension of u_0 to a map $u : H_1 \to H_2$. Then $u(H_1) = [\phi_2(A)x_2] = H_2$, and hence u is a surjective linear isometry, meaning it is unitary.

For any $a, b \in A$ we have $u\phi_1(a)(\phi_1(b)x_1) = u\phi_1(ab)x_1 = \phi_2(ab)x_2 = \phi_2(a)u(\phi_1(b)x_1)$. By density of $\phi_1(A)x_1$ in H_1, $u\phi_1(a) = \phi_2(a)u$ for all $a \in A$. Then $\phi_2(a)ux_1 = u\phi_1(a)x_1 = \phi_2(a)x_2$ as $u|_{\phi_1(A)x_1} = u_0$. Hence $\phi_2(a)(u(x_1) - x_2) = 0$ for all $a \in A$. By non-degeneracy of the action of $\phi_2(A)$ on H_2, we must have $u(x_1) = x_2$, and we are done.

$\square$

To bring irreducible representations back into the picture, we have the following lemmas, which will show how irreducible representations relate to cyclic representations and, crucially, pure states.

Lemma 4.5. For any irreducible representations (H, ϕ) of a C*-algebra A, every $x \in H\backslash\{0\}$ is a cyclic vector for (H, ϕ).

Proof. First note that since ϕ is non-zero, there is some $y \in H$ such that $\phi(A)y$ is non-zero. Then since $[\phi(A)y]$ is non-zero and $\phi(A)$ invariant, $[\phi(A)y] = H$ by irreducibility of (H_ϕ), and in fact $[\phi(A)H] = H$. Hence (H, ϕ) is non-degenerate.

So if we take an arbitrary element $x \in H\backslash\{0\}$, the space $[\phi(A)x]$ is non-zero and $\phi(A)$ invariant. Hence $[\phi(A)x] = H$ by irreducibility of (H_ϕ), and so x is a cyclic vector for (H, ϕ).

□

Lemma 4.6. Let A be a C*-algebra, and $\tau \in S(A)$. Then

1. If A is abelian, then τ is pure if and only if τ is a character
2. If A is abelian, then $PS(A) = \Omega(A)$

Proof. (1) Assume A is abelian and τ is pure. Then $\phi_\tau(A) \subseteq \phi_\tau(A)' = \mathbb{C}1_{\mathcal{B}(H_\tau)}$. So if $a, b \in A$ there exist $s, t \in \mathbb{C}$ such that $\phi_\tau(a) = s1_{\mathcal{B}(H_\tau)}$ and $\phi_\tau(b) = t1_{\mathcal{B}(H_\tau)}$. Then

$$\begin{aligned}\tau(ab) = \langle \phi_\tau(ab)x_\tau, x_\tau\rangle = st\langle x_\tau, x_\tau\rangle = \langle sx_\tau, x_\tau\rangle\langle tx_\tau, x_\tau\rangle \\ = \langle \phi_\tau(a)x_\tau, x_\tau\rangle\langle \phi_\tau(b)x_\tau, x_\tau\rangle = \tau(a)\tau(b),\end{aligned}$$

and hence τ is multiplicative and so a character on A.

Conversely, suppose τ is a character, and ρ is some positive linear functional such that $\rho \leqslant \tau$. Supposing that $\tau(a) = 0$ where $a \in A$, then $\tau(a^*a) = \tau(a^*)\tau(a) = 0$ and $\tau(a^*a) - \rho(a^*a) \geqslant 0$ forces $\rho(a^*a) = 0$ also. Hence $|\rho(a)|^2 \leqslant \|\rho\|\rho(a^*a) = 0$ by Theorem 2.4(2), and we see that $\ker(\tau) \subseteq \ker(\rho)$. Since τ and ρ are linear maps into $\mathbb{C}$, they have kernels of codimension at most 1, and by basic linear algebra there exists $t \in \mathbb{C}$ such that $\rho = t\tau$. Since τ is non-zero, there exists $a \in A$ such that $\tau(a) = 1$, and so $\tau(a^*a) = 1$. Hence $0 \leqslant \rho(a^*a) \leqslant \tau(a^*a) = 1$. Since $\rho(a^*a) = t\tau(a^*a) = t$, $t \in [0, 1]$. So τ is a pure state.

(2) By (1), we only need to prove that all characters are states, for which it only remains to show that if τ is a character that $\|\tau\| = 1$. To this end we set an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A. Then $\|\tau\| = \lim_{\lambda\in\Lambda} \tau(u_\lambda^2) = (\lim_{\lambda\in\Lambda} \tau(u_\lambda))^2 = \|\tau\|^2$, since $(u_\lambda^2)_{\lambda\in\Lambda}$ is an approximate unit also. But τ is a character, so non-zero, and therefore $\|\tau\| = 1$ as required.

□

Lemma 4.7. Let A be a C*-algebra with representation (H, ϕ). Suppose (H, ϕ) has unit cyclic vector $x \in H$. Then $\tau : A \to \mathbb{C}$ given by $\tau(a) = \langle \phi(a)x, x \rangle$ is a state on A, and furthermore (H, ϕ) is unitary equivalent to (H_τ, ϕ_τ). Moreover, if (H, ϕ) is irreducible then $\tau \in PS(A)$.

Proof. If $a \in A$, then $\phi(a^*a) = \phi(a)^*\phi(a)$ is positive, and so the sesquilinear form $\sigma_{\phi(a^*a)}$ (as defined in Lemma 4.3) is positive. Hence $\tau(a^*a) = \langle \phi(a^*a)x, x \rangle = \sigma_{\phi(a^*a)}(x, x) \geqslant 0$. Since $A^+ = \{a^*a \in A : a \in A\}$, τ is a positive linear functional.

Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of A. Since (H, ϕ) is cyclic, it is non-degenerate. Hence $(\phi(u_\lambda))_{\lambda \in \Lambda}$ converges strongly to $1_{\mathcal{B}(H)}$. So $\|\tau\| = \lim_{\lambda \in \Lambda} \tau(u_\lambda) = \lim_{\lambda \in \Lambda} \langle \phi(u_\lambda)x, x \rangle = \langle x, x \rangle = 1$ by Theorem 2.5(3), and therefore $\tau \in S(A)$.

Note $\langle \phi(a)x, x \rangle = \tau(a) = \langle \phi_\tau(a)x_\tau, x_\tau \rangle$ for all $a \in A$. This means by Lemma 4.4 that (H, ϕ) and (H_τ, ϕ_τ) are unitarily equivalent. And in the case that (H, ϕ) is irreducible, (H_τ, ϕ_τ) is also since irreducibility is preserved by unitary equivalence, and by Theorem 4.1 this implies that $\tau \in PS(A)$.

□

Since all irreducible representations have unit cyclic vectors, this shows that all irreducible representations are unitary equivalent to (H_τ, ϕ_τ) for some pure state $\tau \in PS(A)$. Hence, for the reasons discussed at the beginning of this section, the spectra of a C*-algebra are well defined.

5 Spectra and the Jacobson Topology

We now return to studying spectra of C*-algebras and the Jacobson topology. One reason for this construction is simply to place more structure on the irreducible representations. In general, the spectrum of a C*-algebra A is studied rather than its primitive spectrum because it in some sense contains ‘more information’ that the primitive specrum [9, pp. 121]. By this we refer to the natural surjection $\Phi : \hat{A} \to \check{A}$ given by $\Phi([(H, \phi)]) = \ker \phi$. One benefit of the primitive spectrum $\check{A}$ is in the natural topology that we will define on it, the Jacobson topology, which the spectrum $\hat{A}$ inherits via the natural surjection. We note however that in the examples explored in this document, the primitive spectrum and spectrum of our chosen examples of C*-algebras will generally coincide, in the sense that the natural surjection is a bijection.

5.1 Non-emptiness of Spectra

Before defining the Jacobson topology, we remark that for any non-zero C*-algebra A, the set $PS(A)$ is non-empty, and hence the primitive spectrum $\check{A}$ and spectrum $\hat{A}$ are non-empty. The proof is not within the scope of this dissertation. It involves an application of the *Krein-Milman theorem*, and a proof can be found in [10].

5.2 The Jacobson Topology

We begin by defining the *Jacobson topology* on the the primitive spectrum.

Definition 5.1. For $F \subseteq \check{A}$, we define the *kernel* of F as $\ker(F) = \bigcap_{t \in F} t$ where we take this intersection to be $\check{A}$ if $F = \varnothing$.

For $I \subseteq A$, the *hull* of I is defined as $\mathrm{hull}(I) = \{t \in \check{A} : I \subseteq t\}$

The *Jacobson topology* on $\check{A}$ is the topology with closure operation $\bar{F} = \mathrm{hull}(\ker(F))$.

The Jacobson topology is also referred to as the *hull-kernel topology*, due to the definition of its closure operation. We will show that the topology is well defined, but for this we need the following proposition. First, we define *primitive* and *prime* ideals.

Definition 5.2. Let A be a C*-algebra. A *primitive ideal* I of A is an ideal of the form $I = \ker(\phi)$ for some non-zero irreducible A representation (H, ϕ).

We say that an ideal I of A is *prime* if whenever J_1, J_2 are closed ideals of A such that $J_1 J_2 \subseteq I$, then $J_1 \subseteq I$ or $J_2 \subseteq I$.

Note this justifies our name for the primitive spectrum of a C*-algebra.

The following theorem gives an equivalent concept of primitive ideal. We omit the proof for brevity, but note it can be found in [6, pp. 156-157].

Theorem 5.1. For any modular left ideal L of an algebra A, there is a largest ideal I of A contained in L which is equal to $\{a \in A : aA \subseteq L\}$. We mean largest in the sense that if $J \subseteq L$ is an ideal of A, then $J \subseteq I$. Further, an ideal is primitive if and only if it is the largest ideal I contained in some L as above.

Note if A is an abelian algebra, then any modular left ideal L is also an ideal, and hence $I = L$ by the maximality condition.

For another example, let A be a C*-algebra and $\tau \in PS(A)$. Then the largest ideal associated to N_τ in the sense of the previous theorem is $\ker(\phi_\tau)$, since $\ker(\phi_\tau) = \{a \in A : \phi_\tau(a)(A/N_\tau) = \{0\}\} = \{a \in A : aA \subseteq N_\tau\}$.

Proposition 5.1. If A is a C*-algebra, all its primitive ideals are prime.

Proof. Suppose I is a primitive ideal and J_1, J_2 are closed ideals such that $J_1 J_2 \subseteq I$. Then $I = \ker(\phi_\tau) = N_\tau$ for some $\tau \in PS(A)$, and so $J_1 J_2 \subseteq N_\tau$. Suppose $J_2 \nsubseteq N_\tau$. Then there is some $a \in J_2$ such that $\phi_\tau(a)$ is non-zero, and so $\phi_\tau(J_2)x_\tau$ is a non-zero ϕ_τ invariant vector subspace of H_τ. By Kadison's transitivity theorem and irreducibility of (H_τ, ϕ_τ), $\phi_\tau(J_2)x_\tau = H_\tau$. Therefore there exists $a_0 \in J_2$ such that $\phi_\tau(a_0)x_\tau = x_\tau$. For all $b \in J_1$ we have $b + N_\tau = \phi_\tau(b)x_\tau = \phi(ba_0)x_\tau = ba_0 + N_\tau = 0$ since $J_1 J_2 \subseteq I = N_\tau$, and so $b \in N_\tau$. Therefore $J_1 \subseteq N_\tau$, and by maximality of I $J_1 \subseteq I$ as required.

□

Now we can prove the Jacobson topology is well defined:

Proposition 5.2. The closure operation used in the definition of the Jacobson topology on $\check{A}$, where A is a C*-algebra, does define a topology on $\check{A}$, and the closed sets are exactly those of the form $\mathrm{hull}(I)$ where $I \subseteq A$.

Our argument is based on the proof given in [9, p. 122].

Proof. We prove this by showing the *Kuratowski axioms* of a closure operation, which may be found in [11], hold for the operator $\mathcal{C} : \mathcal{P}(\check{A}) \to \mathcal{P}(\check{A})$ defined by $\mathcal{C}(F) = \mathrm{hull}(\ker(F))$.

1. $\mathcal{C}(\varnothing) = \mathrm{hull}(\ker(\varnothing)) = \mathrm{hull}(\check{A}) = \varnothing$

2. Fix $t \in F$. Then $\ker(F) \subseteq t$ by definition of ker, and so $t \in \mathrm{hull}(\ker(F))$ $= \mathcal{C}(F)$. Therefore $F \subseteq \mathcal{C}(F)$.

3. Suppose $F = \mathrm{hull}(I)$ for some set $I \subseteq A$. We will show $\mathcal{C}(F) = F$, which will force $\mathcal{C}(\mathcal{C}(F')) = \mathcal{C}(F')$ for all $F' \subseteq \check{A}$.

   If $t \in \mathcal{C}$ then $\ker(F) \subseteq t$. But since if $x \in I$ then for all $t' \in \mathrm{hull}(I)$ we have $x \in t'$ and $x \in \ker(\mathrm{hull}(I))$. So $I \subseteq \ker(F) = \ker(\mathrm{hull}(I))$.

4. We want to show that $\mathcal{C}(F_1) \cup \mathcal{C}(F_2) = \mathcal{C}(F_1 \cup F_2)$ for all $F_1, F_2 \subseteq \check{A}$. Suppose $t \in \mathcal{C}(F_1) \cup \mathcal{C}(F_2)$. Without loss of generality, $t \in \mathcal{C}(F_1)$. So $t \supseteq \ker(F_1) \supseteq \ker(F_1 \cup F_2)$, and so $t \in \mathcal{C}(F_1 \cup F_2)$ as required.

For the converse, suppose $t \in \mathcal{C}(F_1 \cup F_2)$. Then $t \supseteq \ker(F_1 \cup F_2) = \ker(F_1) \cap \ker(F_2) = (\ker(F_1))(\ker(F_2))$, since $\ker(F_1), \ker(F_2)$ are closed ideals. By Proposition 5.2, t is prime, so either $\ker(F_1)$ or $\ker(F_2)$ is contained in t. Hence $\mathcal{C}(F_1) \cup \mathcal{C}(F_1) = \mathcal{C}(F_1 \cup F_2)$.

So the operator $\mathcal{C}$ satisfies the Kuratowski axioms for a closure operation, and the Jacobson topology is well defined. Since we have also shown in (3) that $\mathcal{C}(\text{hull}(I)) = \text{hull}(I)$ for any $I \subseteq A$, the set of closed sets in the Jacobson topology is $\{\text{hull}(I) : I \subseteq A\}$

$\square$

Let A be a C*-algebra. The Jacobson open sets are in order-preserving correspondence with the closed ideals of A, via the maps I defined by $I(U) = \ker(\check{A} \backslash U)$ for Jacobson open sets U with inverse map G given by $G(J) = \check{A} \backslash \text{hull}(J)$ for any closed ideal J [6, pp. 153-157]. This is another example of how the Jacobson topology relates to the algebra it is defined from.

Definition 5.3. The *Jacobson topology* on $\hat{A}$ is the weakest topology that makes the natural surjection continuous.

At this point, we might ask ourselves: what kinds of Jacobson topologies can we obtain? The following lemma shows that we can impose some (rather weak) conditions.

Lemma 5.1. For any C*-algebra A, the Jacobson topology is T_0.

Proof. Let $(H_1, \phi), (H_2, \psi)$ are representations of C*-algebra A such that $\ker \phi \neq \ker \psi$. Suppose $\ker \phi \in \overline{\{\ker \psi\}}$ and $\ker \psi \in \overline{\{\ker \phi\}}$. Noting that $\overline{\{\ker \psi\}} = \text{hull}(\ker \phi) = \{t \in \check{A} : \ker_\phi \subseteq t\}$, this means $\ker \phi \subseteq \ker \psi$. But by a symmetrical argument, $\ker \psi \subseteq \ker \phi$, and so $\ker \phi = \ker \psi$, a contradiction. $\square$

However, it will turn out we cannot demand much more. In particular, they need not be Hausdorff. In our examples the spectrum and primitive spectrum coincide (meaning canonical surjection is bijective), which significantly simplifies the problem.

5.3 Examples

Example 5.1. Abelian C*-algebras

Let A be an abelian C*-algebra. Recall the pure states of A are precisely the characters of A. If $\tau, \rho \in \Omega(A)$ and $(H_\tau, \phi_\tau), (H_\rho, \phi_\rho)$ are unitary equivalent, multiplying by a constant if needed we can assume this sends x_τ to x_ρ. Then by Lemma 3.3 $\tau = \rho$.

If we have two irreducible representations with equal kernels unitary equivalent to (H_τ, ϕ_τ) and (H_ρ, ϕ_ρ) where $\tau, \rho \in PS(A)$, then $N_\tau = \ker(\phi_\tau) = \ker(\phi_\rho) = N_\rho$ by the remark after Theorem 5.1 and commutitivity. Hence $\tau = \rho$.

Therefore the natural surjection $\Phi : \hat{A} \to \check{A}$ is a bijection, and so is the map $\Omega(A) \to \check{A}$ given by $\tau \mapsto \ker(\phi_\tau)$.

After the following example, we will be able to show these maps are also homeomorphisms in the unital case.

Example 5.2. C(X) for compact Hausdorff X

We first state without proof the following result:

Theorem 5.2. *X is homeomorphic to $\Omega(C(X))$ with the inherited weak* topology via $t \mapsto \varepsilon_t$, where $\varepsilon_t(f) = f(t)$ for all $f \in C(X)$.*

Our aim now is to show that the topology agrees with the Jacobson topology derived from the surjection $X \to \widecheck{C(X)}$ given by $t \mapsto \ker(\varepsilon_t)$. From this we can see that for these three sets $X, \Omega(C(X)), \widecheck{C(X)}$ with natural bijections, their natural topologies make these bijections homeomorphisms.

We know from the commutative example and the above theorem that

$$X \to \Omega(C(X)) \to \widecheck{C(X)}, \quad t \mapsto \varepsilon_t \mapsto \ker \varepsilon_t$$

are bijections.

If $F = \{\ker \varepsilon_t : t \in S\}$ then

$$\bigcap_{T \in F} T = \bigcap_{t \in S} \ker \varepsilon_t = \{f \in C(X) : f|_S = 0\} = \{f \in C(X) : f|_{\bar{S}} = 0\}$$

First we note that $\bigcap_{T \in F} T = \{f \in C(X) : f|_{\bar{S}} = 0\} = \bigcap_{t \in \bar{S}} \ker \varepsilon_t \subseteq \ker t$ for all $t \in \bar{S}$. Conversely if $s \notin \bar{S}$, by Urysohn's lemma there exists $f \in C(X)$ such that $f_{\bar{S}} = 0$ and $f(s) = 1$, which is in $\bigcap_{T \in F} T$ and not $\ker \varepsilon_s$. Therefore

$\ker \varepsilon_s \in \bar{F} \iff \bigcap_{T \in F} T \subseteq \ker \varepsilon_s \iff s \in \bar{S}$. So $\bar{F} = \{\ker \varepsilon_s : s \in \bar{S}\}$, and the closures correspond under the bijection given.

Note that in light of the last two results, the Jacobson topology of a unital abelian C*-algebra is homeomorphic to the character space with its inherited weak* topology.

In the following example, we'll need a result from [12, pp. 210-211]. The proof is not difficult, but we omit it for brevity.

Lemma 5.2. Hence all non-zero irreducible representations $\phi : M_n \to \mathcal{B}(H)$ is unitarily equivalent to the identity operator $M_n \to M_n$ for any $n \in \mathbb{N}$.

We will also need the representations of $\mathbb{C}^n$, given C*-algebra structure by coordinate-wise multiplication and involution. With some work involving the inner product, we see that the only irreducible representations up to unitary equivalence are the projection maps, and these are not equivalent to one another (this is immediate after considering their kernels).

The following example demonstrates the range of Jacobson topologies possible is wider than the previous example might suggest - in particular, we cannot guarantee Hausdorffness:

Example 5.3. A non-Hausdorff example

Let X be a non-empty compact Hausdorff space with $x_0 \in X$ a limit point. Here we consider a certain C*-subalgebra A of

$$C(X, M_2) = \{\left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{2}2 \end{smallmatrix}\right) : f_{ij} \in C(X) \text{ for } i, j \in \{1, 2\}\},$$

given by

$$A = \{f \in C(X, M_2) : f(x_0) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \text{ for some } \lambda, \mu \in \mathbb{C}\}.$$

We first note that the centre contains a $*$-isomorphic copy of $C(X)$ as a subalgebra via the map $\Phi : C(X) \to A$ given by $f \mapsto f1_{M_2}$. Suppose (H, ϕ) is irreducible. Then $\phi(\Phi(f)) = \phi(f1_{M_2}) \in \phi(A)' = \mathbb{C}1_H$, since a $*$-homomorphic image of the centre must be contained in the commutant of the image. This is nonzero as $\phi(1_{M_2}) = 0$ implies that $\phi(a) = \phi(1_{M_2})\phi(a) = 0$ for all $a \in A$. So $(H, \phi \circ \Phi)$ is a non-zero irreducible representation of $C(X)$. By the previous example there exists $t \in X$ such that $\phi(f1_{M_2}) = \phi \circ \Phi(f) = f(t)1_{\mathcal{B}(H)}$.

What we have essentially shown here is that on the centre of the algebra A, which is equal to $\Phi(C(X))$, the representation is evaluation at a point, just as in the previous example.

Letting $I_t = \{a \in C(X, M_2) : a(t) = 0\}$, we can show $I_t \subseteq \ker\phi$ as follows: Suppose $a(t) = 0$ and $\varepsilon > 0$. By continuity, there exists some open neighbourhood $U \subseteq X$ of t such that $|a(s)| < \varepsilon$ for all $s \in U$. By Urysohn's lemma there exists a continuous function $f : X \to [0,1]$ such that $f(s) = 0$ for $s \in X \backslash U$ and $f(t) = 1$, since X being Hausdorff implies that $\{t\}$ and $X \backslash U$ are disjoint and closed. Then $\|a(s) - f(s)a(s)\| < \varepsilon$ for all $s \in X$, meaning $\|a - (f1_{M_2})a\| \leqslant \varepsilon$, and so $\|\phi(a) - \phi(f1_{M_2})\phi(a)\| = \|\phi(a - (f1_{M_2})a)\| \leqslant \|\phi\|\|a - (f1_{M_2})a\| \leqslant \varepsilon$. But since $\phi(f1_{M_2}) = 0$, this means $\|\phi(a)\| \leqslant \varepsilon$ for all $\varepsilon > 0$. Hence $\phi(a) = 0$, and our claim has been proven.

We define a representation $(H, \tilde{\phi})$ where $\tilde{\phi} : A/I_t \to H$ by $\tilde{\phi}(a+I_t) = \phi(a)$, and note that this is irreducible since $\tilde{\phi}(A/I_t) = \phi(A) = \mathbb{C}1_{\mathcal{B}(H)}$. At this stage the cases diverge. However in each our stratedgy is the same: we define $B_t = \{a(t) : a \in A\}$ and a $*$-isomorphism ψ_t by $\psi_t(a) = a(t)$ from A_t onto its image B_t. From this we can build an irreducible representation of B_t, namely $(H, \tilde{\phi} \circ \psi_t^{-1})$. Then noting ϕ is of the form $\phi = \tilde{\phi} \circ \psi_t^{-1} \circ \epsilon_t$, where we now take $\epsilon_t : A \to M_2$ to be $\epsilon_t(a) = a(t)$ (note the use of ϵ_t rather that ε_t), and provided we understand the C*-algebra B_t enough to find the irreducible representations, $\tilde{\phi} \circ \psi_t^{-1}$ is unitarily equivalant to one. Then we can hopefully work out from ϕ's equivalence to $\tilde{\phi} \circ \psi_t^{-1} \circ \epsilon_t$ what form ϕ can take, and then go on to compute the topology.

Case 1: $t \in X \backslash \{x_0\}$

For all $t \in X \backslash \{x_0\}$ we have $B_t = M_2$ and $\psi_t(a) = a(t)$. By the previous lemma, the irreducible representation $\tilde{\phi} \circ \psi_t^{-1}$ is unitary equivalent to the identity $id_2 : M_2 \to M_2$, say by unitary $U : H \to M_2$, by which we mean that $\tilde{\phi} \circ \psi_t^{-1}(m) = U id_2(m) U^*$ for all $m \in M_2$. Then for all $f \in A$ we have $\phi(f) = (\tilde{\phi} \circ \psi_t^{-1})(\epsilon_t(f)) = U id_2(\epsilon_t(f)) U^* = U \epsilon_t(f) U^*$, and hence ϕ is unitary equivalent to ϵ_t.

Note that such ϵ_t are not unitary equivalent, and have disjoint kernels - for $s \neq t$, the sets $\{s\}$ and $\{t\}$ are closed and disjoint, so by Urysohn's lemma there is a function $a : X \to [0,1]$ with $a(s) = 0$, $a(t)$, and then $\epsilon_s(1_{M_2}a) = 0 \neq 1 = \epsilon_t(1_{M_2}a)$. The representation $(\mathbb{C}^2, \epsilon_t)$ is also always irreducible, since for all $m \in M_2$ $m = \epsilon_t(\tilde{m}) \in \epsilon_t$, where $\tilde{m}(s) = m$ for all $s \in X$, and so $\epsilon_t(A)' = M_2' = \mathbb{C}1_{M_2}$.

Case 2: $t = x_0$

For $t = x_0$ the argument is different. We have $B_{x_0} = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) : \lambda, \mu \in \mathbb{C}\}$.

This is $*$-isomorphic to $\mathbb{C}^2$ via the transformation $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \mapsto (\lambda, \mu)$. By the comments proceeding this example, the distinct irreducible representations on $\mathbb{C}^2$ up to unitary equivalence are the two projections $(\mathbb{C}, \pi_1)$ and $(\mathbb{C}, \pi_2)$ where $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ are given by $\pi_j(x_1, x_2) = x_j$ for $j = 1, 2$. Recovering these in B_{x_0} gives us unique distinct (up to unitary equivalence) irreducible representations $(\mathbb{C}, \xi_1)$, $(\mathbb{C}, \xi_2)$ defined by $\xi_1\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) = \lambda$ and $\xi_2\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) = \mu$ that are representatives of the two unitary equivalence classes of irreducible representations of B_{x_0}. If $\tilde{\phi} \circ \psi_t^{-1}(m) = U\xi_j(m)U^*$ for all $m \in B_{x_0}$ where $U : H \to B_{x_0}$ is unitary, then for all $f \in A$ we have $\phi(f) = (\tilde{\phi} \circ \psi_t^{-1})(\epsilon_t(f)) = U\xi_j(\epsilon_t(f))U^*$. This means ϕ is unitary equivalent to τ_1 or τ_2, where $\tau_j : A \to \mathbb{C}$ are defined for $f = \left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right) \in A$ by $\tau_j(f) = \xi(\epsilon_1(f)) = \xi(f(1)) = f_{jj}(1)$ for $j = 1, 2$. These are non-zero, and irreducible, since they are 1-dimensional. They are not unitary equivalent to each other - for example the constant matrix $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in A$ is in $\ker(\tau_2)$ and not in $\ker(\tau_1)$. And finally they are not unitary equivalent to the other unitary equivalence classes $[(M_n, \epsilon_t)]$ for $t \in [0, 1)$ by the same method as the $t \in X \backslash \{x_0\}$ case - if $f : X \to [0, 1]$ has $f(t) = 0$ and $f(x_0) = 1$, then $\epsilon_t(f 1_{M_2}) = 0 \neq 1 = \tau_j(f 1_{M_2})$, so they have distinct kernels and are not unitarily equivalent. These arguments also show that the representations $(\mathbb{C}^2, \epsilon_t)$ for $t \in X \backslash \{x_0\})$ and $(\mathbb{C}, \tau_j$ for $j = 1, 2$ have distinct kernels.

Hence if we define $Y = (X \backslash \{x_0\}) \cup \{x_1, x_2\}$ where x_1, x_2 are distinct elements not in X, then the maps maps $Y \to \hat{A}$ and $Y \to \check{A}$ defined by $t \mapsto [(\mathbb{C}^n, \epsilon_t)]$ for $t \in X \backslash \{x_0\}$ and $x_j \mapsto [(\mathbb{C}, \tau_j)]$ for $j = 1, 2$ and $t \mapsto \ker(\epsilon_t)$ for $t \in X \backslash \{x_0\}$ and $x_j \mapsto \ker(\tau_j)$ for $j = 1, 2$ respectively are well defined bijections, and we also see that the canonical surjection $\hat{A} \to \check{A}$ is a bijection.

We will now define a topology on Y to make this map a homeomorphism. If $V \subseteq X$ we use the notation

$$V(x_j) = \begin{cases} (V \backslash \{x_0\}) \cup \{x_j\} & x_0 \in V \\ V & x_0 \notin V \end{cases}$$

for $j = 1, 2$. We give Y the topology with open sets $V_1(x_1) \cup V_2(x_2)$ where V_1, V_2 are open in X. A quick check shows this is a well defined topology.

For $S \subseteq X \backslash \{x_0\}$, to avoid confusion we write $\bar{S}^Y$ for the closure in Y and $\bar{S}^X$ for the closure in X. In light of our definition of Y, we rename $\tau_j = \epsilon_{x_j}$. To see the map $Y \to \check{A}$ defined by $t \mapsto \ker(\epsilon_t)$ for $t \in Y$ is a homeomorphism, as before it suffices to show $\overline{\{\ker \epsilon_t : t \in N\}} = \{\ker \epsilon_t : t \in \bar{N}^Y\}$ for any $N \subseteq Y$. The left hand side equals $\{\ker \epsilon_t : t \in M\}$ where $M = \{s : \bigcap_{t \in N} \ker \epsilon_t \subseteq \ker \epsilon_s\}$, and so it suffices to prove $M = \bar{N}^Y$.

Clearly $\bigcap_{t\in N} \ker \epsilon_t \supseteq \bigcap_{t\in \bar{N}^Y} \ker \epsilon_t$. Suppose $t \in \bar{N}^Y \notin N$. Firstly, if $t = x_j$ for some j, then $x_0 \in \bar{N}^X$. Suppose $a \in \bigcap_{s\in N} \ker \epsilon_s$. Then $a(s) = 0$ for $s \in N$, so by continuity $a(x_0) = 0$, and hence $a \in \ker \epsilon_{x_j} = \ker \epsilon_t$. So $\bigcap_{s\in N} \ker \epsilon_s \subseteq \ker \epsilon_t$. The other case is $t \in \bar{N}^X$, and again if $a \in \bigcap_{s\in N} \ker \epsilon_s$ then $a(s) = 0$ for all $s \in N$ implies by continuity of a that $a(t) = 0$ and so $a \in \ker \epsilon_t$. So $\bigcap_{s\in N} \ker \epsilon_s \subseteq \ker \epsilon_t$. Hence $\bigcap_{t\in N} \ker \epsilon_t \subseteq \bigcap_{t\in \bar{N}^Y} \ker \epsilon_t$, and therefore $\bigcap_{t\in N} \ker \epsilon_t = \bigcap_{t\in \bar{N}^Y} \ker \epsilon_t$. This shows that $M \subseteq \bar{N}_Y$.

For the converse, suppose $s \notin \bar{N}$. If $s \in X\backslash\{x_0\}$ we obtain (by Urysohn's lemma) some $f \in C(X)$ such that $f(t) = 0$ for $t \in \overline{(N \cap X)}^X \cup \{1\}$ and $f(s) = 1$ and note $f1_{M_2} \in \bigcap_{t\in N} \ker \epsilon_t \backslash \ker \epsilon_s$ so $s \notin M$.

If $s = x_1$, use $F = \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)$ where $f : X \to [0,1]$ satisfies $f(t) = 0$ for $t \in \overline{(N \cap X)}^X$ and $f(x_0) = 1$ (noting this is reasonable since $x_1 \notin \bar{N}$ implies $x_0 \notin \overline{(N \cap X)}^X$) to get $F \in (\bigcap\limits_{t\in N} \ker \epsilon_t)\backslash \ker \epsilon_s$, so $s \notin M$.

If $s = x_2$, use $F = \left(\begin{smallmatrix} 0 & 0 \\ 0 & f \end{smallmatrix}\right)$ where $f : X \to [0,1]$ satisfies $f(t) = 0$ for $t \in \overline{(N \cap X)}^X$ and $f(x_0) = 1$ to get $F \in (\bigcap\limits_{t\in N} \ker \epsilon_t)\backslash \ker \epsilon_s$, so $s \notin M$.

So $M = \bar{N}^Y$. Hence the topologies coincide under the bijections $Y \to \hat{A}$, $Y \to \check{A}$, and these are homeomorphisms.

Remark 5.1. This demonstrates that the Jacobson topology need not be Hausdorff (or T_2). To see this, note if $X = [0,1]$ and $x_0 = 0$ then the sequence $(1/n)_{n\in\mathbb{N}}$ converges to x_1 and x_2, and recall that in Hausdorff spaces sequences converge to at most one limit. In fact, Jacobson topologies need not even be T_1 spaces, meaning topological spaces X where every singleton set $\{x\}$ for $x \in X$ is closed (note that A is T_1). An example of a C*-algebra with non T_1 spectrum can be found in [13].

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