WAECO Intro to AECs talk part 1

Based largely on material learned in Rami Grossberg's Model Theory II course.

<u>1st order logic</u>: All familiar with theory T in language L, study $(\mathrm{Mod}(T), \preceq)$ ← elementary substructure.

Call these <u>elementary classes</u>.

Lots of useful tools:
- compactness
- LS theorems ($\uparrow$ & $\downarrow$)
- Chain theorems ($\langle M_i : i < \delta \rangle$ cts & inc, then $\bigcup M_i \in \mathrm{Mod}(T)$, $M_j \preceq \bigcup M_i$, and $M_j \preceq N\ \forall j \Rightarrow \bigcup M_i \preceq N$)

But there are other logics & classes of interest, e.g.

(*) <u>Infinitary logic</u>: Models of some $\varphi \in L_{\infty,\omega}$ perhaps $(\mathrm{Mod}(\varphi), \preceq_F)$ where $\varphi \in L_{\lambda^+,\omega}$ (i.e. φ allows $\bigwedge$ and $\bigvee$ of size $< \lambda^+$), $|L| \leq \lambda$, and F a fragment of $L_{\lambda^+,\omega}$ containing φ with $|F| \leq \lambda$.

(i.e. $F \subseteq \mathrm{Form}(L_{\lambda^+,\omega})$ where
- F closed under f.o. connectives $\wedge, \vee, \neg, \forall x, \exists y$
- If $\varphi(\underline{x}) \in F$, $\underline{\tau}(\underline{y}) = \langle \tau_1(\underline{y}), \ldots, \tau_k(\underline{y}) \rangle$ where τ_i terms, then $\varphi(\underline{\tau}(\underline{y})) \in F$
- F is closed under subformulas.

so 'F is well enough closed')

(✱) $EC(T, \Gamma) = (\{M \models T : M \text{ omits all } q \in \Gamma\}, \preceq)$.

In these, compactness fails – so if we move to a more general framework that also covers these examples, we need to weaken our assumptions.

Definition: An <u>abstract class</u> (AC) is a pair $\underline{K} = (K, \leq_K)$ where

- K is a class of models in some language $L(K)$
- $\leq_K$ is a partial ordering on K refining submodel, i.e.

$$M \leq_K N \implies M \subseteq N$$

- <u>1st iso axiom</u>: K is closed under isomorphism: i.e. if $M \in K$, $N \cong M$, then $N \in K$.
- <u>2nd iso axiom</u>: $\leq_K$ respects isomorphism, i.e. given $M_1 \leq M_2$ and $M_2 \overset{f}{\cong} N$ then $f[M_1] \in K$ and $f[M_1] \leq N$

$$\begin{tikzcd} M_2 \arrow[r, "\simeq f"] & N \\ M_1 \arrow[u, "\leq_K"] \arrow[r, "f"'] & f[M_1] \arrow[u, "\leq_K"'] \end{tikzcd}$$

These assumptions are still too weak to capture much of the theory of the last two examples. So we have...

Definition: An <u>abstract elementary class</u> (AEC) is an AC $\underline{K} = (K, \leq_K)$ that also satisfies

- <u>Coherence</u>: If $M_1, M_2, M_3 \in K$ and $M_1 \leq_K M_3$, $M_2 \leq_K M_3$, and $M_1 \subseteq M_2$, then $M_1 \leq_K M_2$.

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- TV chain axioms: If $\langle M_i : i < \delta \rangle$ a $\leq_K$-inc cts $\leq_K$-chain of models in K where δ is limit, then
  - $\bigcup_{i<\delta} M_i \in K$
  - $M_j \leq_K \bigcup_{i<\delta} M_i$ for all $j < \delta$
  - If $M_j \leq_K N$ for all j, then $\bigcup_{i<\delta} M_i \leq_K N$.
- LS Axiom: There is an infinite cardinal $LS(K)$ such that $\forall M \in K\ \forall A \subseteq |M|\ \exists N \in K$ s.t. $N \leq_K M$, $A \subseteq |N|$, and $\|N\| \leq |A| + LS(K)$.

  We denote the least such cardinal by $LS(K)$.
- (sometimes) $\forall M \in K\ \ \|M\| \geq LS(K)$.

Examples:

- $(\mathrm{Mod}(T), \preceq)$, $LS(K) = |L(T)| + \aleph_0$
- $(\mathrm{Mod}(\varphi), \preceq_F)$ where $\varphi \in L_{\lambda^+,\omega}$, $|L|, |F| \leq \lambda$ then $LS(K) = \lambda$
- $EC(T, \Gamma)$, $LS(K) = |L(T)| + \aleph_0$
- Lots of specific algebraic examples:
  - groups with $\leq_K\ =\ \subseteq$

    or $\leq_K\ =$ 'pure subgroup'

    (i.e. $H \leq_K G \iff \forall n \in \omega\ \forall a \in H [G \models \exists x (a = x^n) \Rightarrow H \models \exists x (a = x^n)]$)
  - Locally finite groups
  - nil rings $(\forall x \in R\ \exists n \in \omega\ x^n = 0)$

    } with $\leq_K\ =\ \subseteq$, $LS(K) = \aleph_0$
  - modules over a ring with $\leq_K\ =\ \subseteq$, $LS(K) = |R| + \aleph_0$
  - Well orders of type $\leq \lambda^+$ with $\leq_K\ =$ 'initial segment', $LS(K) = \lambda$

– modules over a ring with $\le_K$ = 'pure submodule', $LS(K) = |R| + \aleph_0$

Non-examples: – well orders with $\le_K = \subseteq$ (chain breaks) not well founded

– well orders with $\le_K$ initial segment (LS axiom breaks).

Notation: • Given λ cardinal, $K_\lambda = \{ M \in K : \|M\| = \lambda \}$

• $I(K, \lambda) = |K_\lambda / \cong|$ $\quad (\le 2^{|L(T)| + \aleph_0 + \lambda})$

Question: What can $I(K, \lambda)$ look like?

Note $\{ \lambda \ge LS(K) : I(K,\lambda) \ne 0 \} = [LS(K), *)$ for some $* \in [LS(K), \infty)$ or $* = \infty$

by the LS axiom.

$* < \infty$ is possible, but

Fact (Shelah): If $* \ge \beth_{(2^{LS(K)})^+}$, then $* = \infty$.

Fact: There are examples of $* = \beth_\alpha$ for all $\alpha < \left(2^{LS(K)}\right)^+$.

Considering Morley's / Shelah's categoricity theorems,

Question: what can $\{ \lambda : I(K,\lambda) = 1 \}$ look like?

Big open problem:

Shelah's categoricity conjecture: If $\mu_0 = \beth_{(2^{LS(K)})^+}$ and $\exists \lambda_0 \ge \mu_0$ such that K is categorical in λ_0, then K is categorical in all $\lambda \ge \mu_0$.

or maybe use a different bound for μ_0 in terms of $LS(K)$.

Lifting basic notions to an AEC setting:

Rather than elementary embeddings ...

<u>Definition</u>: Given $M, N \in K$, $f : M \to N$ is a <u>K-embedding</u> iff $\exists N^* \leq_K N$ such that $f : M \cong N^*$ (i.e. $f : M \cong f[M] \leq_K N$)

Rather than types ...

<u>Definition</u>: K an AEC.

- $K^3 = \{(M, N, a) : M, N \in K,\ a \in N,\ M \leq N\}$.
- Given $(M_\ell, N_\ell, a_\ell) \in K^3$ for $\ell = 1, 2$

  $(M_1, N_1, a_1) \sim (M_2, N_2, a_2)$ iff $M_1 = M_2$ and $\exists N_3 \in K$ $\exists f_\ell : N_\ell \xrightarrow[M_1]{} N_3$ K-embeddings such that $f_1(a_1) = f_2(a_2)$

$$\begin{tikzcd} N_1 \arrow[r, "f_1"] & N_3 \\ M \arrow[u, "id"] \arrow[r, "id"'] & N_2 \arrow[u, "f_2"'] \end{tikzcd}$$

- $\approx$ is the <u>transitive closure</u> of $\sim$
- Given $(M, N, a) \in K^3$, $gtp(a/M, N) = [(M, N, a)]_{\approx}$.

In many cases, $\sim = \approx$.

<u>Definition</u>: Suppose K is an AC. We say K has the <u>amalgamation property</u> if whenever we have $f_\ell : M \to N_\ell$ for $\ell = 1, 2$, $\exists N_3$ $\exists g_\ell : N_\ell \to N_3$ such that $g_1 \circ f_1 = g_2 \circ f_2$

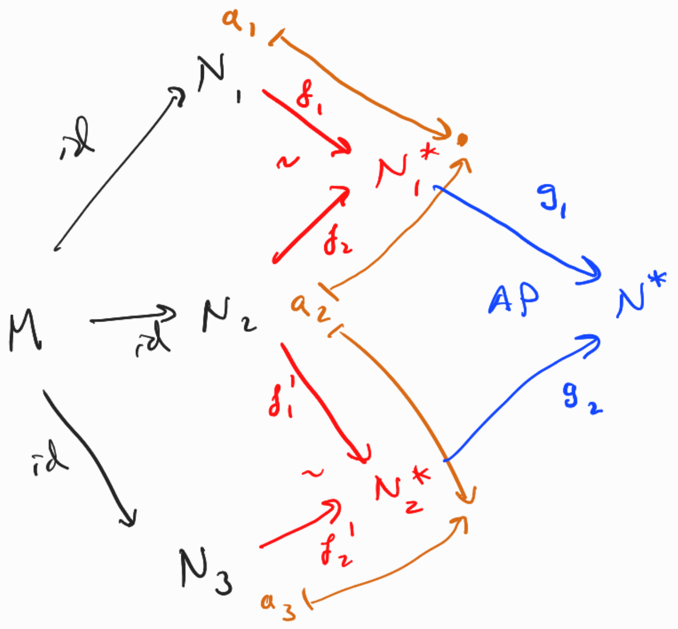
$$\begin{tikzcd} N_1 \arrow[r, "g_1"] & N_3 \\ M \arrow[u, "f_1"] \arrow[r, "f_2"'] & N_2 \arrow[u, "g_2"'] \end{tikzcd}$$

Proposition: If K has AP, then $\sim \; = \; \simeq$.

proof: Suppose $(M_1, N_1, a_1) \sim (M_2, N_2, a_2) \sim (M_3, N_3, a_3)$.

Let $M = M_1 = M_2 = M_3$.

sketch:



$$\begin{tikzcd}
 & N_1 \arrow[dr, "g_1 \circ f_1"] & \\
M \arrow[ur] \arrow[dr] & & N^* \\
 & N_3 \arrow[ur, "g_2 \circ f_2'"'] &
\end{tikzcd}$$

$$g_1 \circ f_1(a_1) \underset{\sim \text{def}}{=} g_1 \circ f_2(a_2) \underset{\text{AP}}{=} g_2 \circ f_1'(a_2) \underset{\sim \text{def}}{=} g_2 \circ f_2'(a_3)$$

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Definition: Let K be an AEC, $M \in K$.

$S(M) = \{ gtp(a/M, N) : M \leq_K N, \; a \in |N| \}$

Fact: $|S(M)| \leq 2^{\|M\|}$

(wlog $\|N\| = \|M\|$ by LS Ax & def of $\sim$,

so $\|M\|$ choices for $|N| \setminus |M|$,

$\leq 2^{\|M\|}$ choices for interpreting $L(M)$ in N,

$\leq \|M\|$ choices for A.)

Definition: If $p = gtp(a/M, N)$, we say p is realised in N.

Definition: M is λ-saturated if $\forall M_0 \leq M$ s.t. $\|M_0\| < \lambda$, $\forall p \in S(M_0)$, p is realised in M.

M is saturated if $\|M\|$-saturated.

These still behave as we expect.

Proposition: Suppose K is an AEC, $\lambda > LS(K)$, $K_{<\lambda}$ has AP, and $M \in K_\lambda$. Then

$$M \text{ is saturated} \iff \underbrace{\forall N \leq M\ \forall N' \geq_K N \text{ s.t. } \|N'\| < \|M\|\ \exists f : N' \xrightarrow[N]{} M}_{M \text{ is model homogeneous.}}$$

proof ideas: Like 1st order, but be careful.

$$\begin{tikzcd} & \text{m-h} \arrow[dl, Rightarrow] & \iff & \forall \mu < \lambda \quad M \text{ is m-h for } \|N\| = \|N'\| = \mu \\ \text{sat} \arrow[urrr, bend right, "\text{like 1st order}"'] & & & \end{tikzcd}$$

like 1st order, but there is no notion of (M,N)-maps in AECs, (all sets are models), so we have all maps between 'larger' models.

Other important notions:

Definition: We say an AC K has the joint embedding property (JEP) iff $\forall M_1, M_2 \in K\ \exists N \in K\ \exists f_\ell : M_\ell \to N$.

$$\begin{tikzcd} & N & \\ M_1 \arrow[ur] & & M_2 \arrow[ul] \end{tikzcd}$$

Fact: $(\mathrm{Mod}(T), \preceq)$ has JEP $\iff$ T is complete.

($\Rightarrow$ if not complete, get $M_1 \not\equiv M_2$

$\Leftarrow$ use monster model, or large saturated model, as N).

Definition: We say an AC K has <u>no maximal models</u> (NMM) if

$\forall M \in K\ \exists N \in K \quad M \precneq_K N$.

Fact: If K has AP, JEP, NMM, then K has monster models similar to 1st order logic, i.e. $\forall \mu > LS(K)\quad \exists M^* \in K$ s.t.

- All models of card $< \mu$ embed into M^*
- M^* is μ-m.h. (i.e. m.h holds for $\|N\| = \|N'\| < \mu$)
- $\forall N, N' \leq_K M^*$ s.t. $\|N\| = \|N'\| < \mu$, $\forall f : N \cong N'$ f extends to an automorphism of M^*.

Note: Existence of such M^* recovers AP, JEP, NMM

proof: JEP, NMM easy

<u>AP:</u>

$$\begin{tikzcd} N_1 & & N_2 \\ & M \arrow[ul, "id"] \arrow[ur, "f"'] & \end{tikzcd}$$

$M, N_1 \leq M^*$ wlog

get $g : N_2' \cong N_2$ extending f. By μ-m.h.,

$\exists h : N_2' \xrightarrow[M]{} M^*$

then

$$\begin{tikzcd} N_1 \arrow[r, "id"] & M^* \\ M \arrow[u, "id"] \arrow[r, "f"'] & N_2 \arrow[u, "h \circ g^{-1}"'] \end{tikzcd}$$

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Note: - With a monster model, types $gtp(a/M,N)$ over small sets become orbits under automorphisms fixing M

- The notion of types agrees with the first order notion: if $a_\ell \in N_\ell \geq_K M_\ell$.

$tp(a/M_1,N_1) = tp(b/M_1,N_1) \iff gtp(a_1/M_1,N_1) = gtp(a_2/M_2,N_2)$.

END OF PART 1