ON THE SPECTRUM OF LIMIT MODELS

JEREMY BEARD AND MARCOS MAZARI-ARMIDA

ABSTRACT. We study the spectrum of limit models assuming the existence of a nicely behaved independence notion. Under reasonable assumptions, we show that all 'long' limit models are isomorphic, and all 'short' limit models are non-isomorphic.

Theorem. *Let* $\mathbf{K}$ *be a* $\aleph_0$-*tame abstract elementary class stable in* $\lambda \geq \mathrm{LS}(\mathbf{K})$ *with amalgamation, joint embedding and no maximal models. Suppose* $\underset{}{\downarrow}$ *is an independence relation on the models of size* λ *that satisfies uniqueness, extension, universal continuity,* $(\geq \kappa)$-*local character in some regular* $\kappa \leq \lambda$, *and non-forking amalgamation.*

Suppose $\delta_1, \delta_2 < \lambda^+$ *with* $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. *Then for any* $N_1, N_2, M \in \mathbf{K}_\lambda$ *where* N_l *is a* (λ, δ_l)-*limit model over* M *for* $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ over } M \iff \mathrm{cf}(\delta_1) \geq \kappa(\underset{}{\downarrow}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$$

Both implications in the conclusion have improvements. High cofinality limits are isomorphic without the $\aleph_0$-tameness assumption and assuming $\underset{}{\downarrow}$ is defined only on high cofinality limit models. Low cofinality limits are non-isomorphic without assuming non-forking amalgamation.

We show how our results can be used to study limit models in both abstract settings and in natural examples of abstract elementary classes.

CONTENTS

1. INTRODUCTION

Limit models (see Definition 2.4), originally introduced by Kolmann and Shelah as a surrogate for saturated models [KolSh96], have proved to be a key notion for extending the classification theory of first order model theory to abstract elementary classes (AECs). In particular, the theory developed around them has been used to prove various approximations of the main test question for AECs, *Shelah's categoricity conjecture* [She83], [She00, 6.13(3)], which proposes that

if an AEC is categorical in some large enough cardinal, then it is categorical in all large enough cardinals [She99], [Sh:h], [BGVV], [GrVan06b], [GrVan06c], [Vas18a], [Vas19], [ShVa24].

The required framework to study limit models has been weakened over time, from assuming categoricity in some higher cardinal [ShVi99], to λ-superstability and λ-symmetry [Van16b], and now even to strictly λ-stable very nice AECs [BoVan24].

For now, suppose $\mathbf{K}$ is a nice AEC (i.e. has a monster model), and is stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$. In this setting, all λ-limit models exist (see Fact 2.6). Since limit models play the role of saturated models to some extent, a key question that has been thoroughly studied is:

Question 1.1. *Suppose $\delta_1, \delta_2 < \lambda^+$ are limit ordinals. Suppose N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$. Is N_1 isomorphic to N_2 over M?*

Positive answers to Question 1.1 play an essential role in understanding superstability and stability in the context of AECs. The question of whether Question 1.1 holds for all δ_1, δ_2 seems to be a natural dividing line; in complete first order theories, or more generally tame AECs, this is equivalent to superstability [GrVas17]. Question 1.1 has also helped us better understand independence notions, see e.g. Lemma 3.23, [Sh:h, III.1.21], [Vas19, 2.8].

For nice AECs, Question 1.1 always has a positive answer whenever $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$ by a back and forth argument (see Fact 2.7). But in many settings, much more is true. In [ShVi99, 3.3.7], Shelah and Villaveces state that assuming a local character condition of λ-splitting and categoricity in some $\lambda' > \lambda$ with $\lambda' \geq \beth_{(2^{LS(\mathbf{K})})^+}$ (and still assuming the existence of a monster model for simplicity), Question 1.1 holds for all limit ordinals $\delta_1, \delta_2 < \lambda^+$. VanDieren found major gaps in [ShVi99] and over the course of several papers [Van02], [Van06], [Van13], [GVV16] with Grossberg and Villaveces, [Van16a], [Van16b] first VanDieren obtained a correct proof of [ShVi99, 3.3.7] (including fixing another gap found by Tapani Hyttinen, see [Van13]) and extended the arguments to AECs with λ-superstability and λ-symmetry (both of which follow from categoricity in any $\lambda' > \lambda$ [Vas17b, 4.8]).

In a strictly stable setting however, it is possible to have non-isomorphic limit models (even in the first order setting, see [GVV16, 6.1] and Theorem A.6). This naturally raises the question of what the structure of different isomorphism types of limit models look like in a more general setting.

In [BoVan24], Boney and VanDieren answer part of this question - they show that assuming $\mathbf{K}$ is λ-stable (but possibly not λ-superstable), and that non-splitting satisfies universal continuity, symmetry, and local character above some κ, all the limit models of regular length at least κ are isomorphic. So, 'long limit models are isomorphic' in this setting.

On the other hand, little has been said about the behavior of 'short' limit models. In [GVV16, 6.1], Grossberg, VanDieren and Villaveces noted that in the strictly λ-stable first order setting, the $(\lambda, \aleph_0)$-limit model is not isomorphic to the (λ, μ)-limit model for any regular cardinal $\mu \geq \kappa(T)$. In fact, in Appendix A, we generalise their argument to show that 'all short limits are non-isomorphic' in this setting. Unfortunately, this first order argument does not appear to immediately generalise to AECs.

In this paper, we find generalisations for both sides of this picture: criteria on an independence relation on long limit models that implies uniqueness of long limit models (Theorem 3.1), and criteria on an AEC that ensure the short limits are all distinct (Theorem 4.1). Both of these theorems are local in that they only use information about the models of cardinality λ.

Theorem 3.1. *Let $\mathbf{K}$ be an AEC stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$, with AP, JEP, and NMM in $\mathbf{K}_\lambda$. Let $\kappa < \lambda^+$ be a regular cardinal. Let $\underset{\smile}{\perp}$ be an independence relation on the $(\lambda, \geq \kappa)$-limit models of $\mathbf{K}$ that satisfies uniqueness, extension, non-forking amalgamation, $(\geq \kappa)$-local character, and $K_{(\lambda, \geq \kappa)}$-universal continuity* in $\mathbf{K}$.*

Let $\delta_1, \delta_2 < \lambda^+$ be limit ordinals where $\kappa \leq \mathrm{cf}(\delta_1), \mathrm{cf}(\delta_2)$. If $M, N_1, N_2 \in K_\lambda$ where N_l is a (λ, δ_l)-limit over M for $l = 1, 2$, then there is an isomorphism from N_1 to N_2 fixing M.

Moreover, if $N_1, N_2 \in K_\lambda$ where N_l is (λ, δ_l)-limit for $l = 1, 2$, then N_1 is isomorphic to N_2.

The proof of Theorem 3.1 relies on a notion called *towers*. Towers are increasing sequences of models with some level of independence along each level of the tower. They were originally introduced by Shelah and Villaveces in [ShVi99] in a form involving two systems of models and a list of singletons, and captured independence of the singletons with λ-splitting, which [Van02], [Van06], [Van13], [GVV16], [Van16a], [Van16b], and [BoVan24] also follow. Later, Vasey found a simplified presentation in the λ-superstable λ-symmetric case using a modified notion of tower [Vas19], which we use in this paper. Vasey's presentation used only a single system of models with a system of singletons, and used λ-non-forking rather than λ-non-splitting, which satisfies stronger properties (in particular, uniqueness, extension, and transitivity on limit models - λ-non-splitting is only known to satisfy weaker forms of these, see Fact 4.5).

Our argument is similar to that of Vasey [Vas19], but we assume our non-forking relation $\underset{\smile}{\perp}$ is defined only on $(\lambda, \geq \kappa)$-limit models, and that $\underset{\smile}{\perp}$ satisfies weaker forms of local character and continuity. Our proofs differ when these weakened forms are applied, and we must also take care to ensure all the models of our towers are $(\lambda, \geq \kappa)$-limit models. For example, since our notion of towers are not closed under unions of low cofinality chains, we adapt Vasey's tower extension lemma [Vas19, 16.17] to allow us to find extensions of any chain of towers, rather than just a single tower (see Proposition 3.24). Similar tweaks and workarounds appear in Lemma 3.22, Lemma 3.31, Proposition 3.35, and Lemma 3.38. Where the proofs differ less from [Vas19] we provide the details all the same for the convenience of the reader.

Theorem 3.1 is similar to [BoVan24] as they show high cofinality limit models are isomorphic in a different context. They use λ-non-splitting and towers similar to [ShVi99] as mentioned above, rather than an arbitrary independence relation with forking-like properties. While our assumptions on $\underset{\smile}{\perp}$ are stronger than the known properties of λ-non-splitting in their context (our relation has full extension, uniqueness, and transitivity, rather than the weaker forms λ-non-splitting is known to satisfy), our result is perhaps not far from a generalisation. Under their assumptions, λ-non-forking is an independence relation satisfying the assumptions of Theorem 3.1 provided you assume λ-non-forking satisfies uniqueness ([Leu24] attempts to prove uniqueness in the context of [BoVan24] - see Remark 2.30).

In a related setting, combining Theorem 3.1 with [Vas16b, §4, §5], we show that in μ-tame AECs where $\mu \geq \mathrm{LS}(\mathbf{K})$ with $\lambda \geq \mu^+$ a stability cardinal, assuming an appropriate symmetry property of μ-non-forking and universal continuity of μ-non-splitting, all $(\lambda, \geq \mu^+)$-limit models are isomorphic (see Corollary 3.46). The μ-non-forking relation is only known to behave well over μ^+-saturated models, so in particular over $(\lambda, \geq \mu^+)$-limit models. This and Example 3.13(3) use the full strength of our assumption that $\underset{\smile}{\perp}$ need only be defined over high cofinality limit models.

Now we move on to the 'short' limit side of the picture.

Theorem 4.1. *Let $\mathbf{K}$ be an AEC stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$, with AP, JEP, and NMM in $\mathbf{K}_\lambda$. Assume $\mathbf{K}$ is $\aleph_0$-tame. Let $\underset{\smile}{\perp}$ be an independence relation on $\mathbf{K}_\lambda$ that satisfies uniqueness, extension, universal continuity, and $(\geq \kappa)$-local character in some regular $\kappa \leq \lambda$.*

If $\mathrm{cf}(\delta_1) < \kappa(\underset{\smile}{\perp}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ and $\mathrm{cf}(\delta_1) \neq \mathrm{cf}(\delta_2)$, then the (λ, δ_1)-limit model is not isomorphic to the (λ, δ_2)-limit model.

The proof of Theorem 4.1 is surprisingly short. We show that for regular $\delta < \lambda^+$, if the (λ, δ)-limit model M is δ^+-saturated (which a (λ, δ')-limit model will be for all regular $\delta' > \delta$), then $\delta \geq \kappa(\underset{\smile}{\perp}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ (see Lemma 4.13). This uses the argument of [Vas22, 4.6], which allows us to find for any $p \in \mathbf{gS}(M)$ some $i < \delta$ such that p does not λ-split over M_i, where $\langle M_i : i \leq \delta \rangle$

witnesses that M is a (λ, δ)-limit model, and that non-splitting is 'close' to $\underset{\smile}{\,}$-non-forking (see Lemma 4.7). In Theorem 4.1 we assume that $\mathbf{K}$ is $\aleph_0$-*tame* - that is, types are equal if their restrictions to countable subsets are equal (see Definition 2.1). A problem left open is whether tameness can be avoided in Theorem 4.1.

A key technical step to prove Theorem 4.1 is to determine the relationship between λ-non-splitting, λ-non-forking and $\underset{\smile}{\,}$-non-forking. Due to this, we spend Subsection 4.1 studying how they interact. Among the results we obtain is a canonicity result for λ-non-forking for long limit models (see Theorem 4.10).

In Section 5, we present results which combine the results of Section 3 and 4 in natural set up. By combining Theorem 3.1 and Theorem 4.1, we show that for distinct regular cardinals $\delta_1, \delta_2 \leq \lambda$, the (λ, δ_1)-limit model and (λ, δ_2)-limit model are isomorphic exactly when $\delta_1, \delta_2 \geq \kappa(\underset{\smile}{\,}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ (see Theorem 5.1). Since $\kappa(\underset{\smile}{\,}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ eventually stabilises, we get that for all high enough λ (above $\beth_{\beth_{2^{\mathrm{LS}(\mathbf{K})_+}}}$), the isomorphism spectra of the λ-limit models are all the same (see Theorem 5.14 and Theorem 5.24). These results have the advantage that there are many examples of natural AECs that satisfy their hypotheses, and they can be used as *black boxes* when studying limit models in these cases; we demonstrate this in Section 6.

More precisely in Section 6, we showcase how to use these results to study limit models on the AECs of: modules, with embeddings; torsion abelian groups, with pure embeddings; and models of a complete first-order theory, with elementary embeddings. In particular, we derive as an immediate corollary a slight weakening of the main theorem of [Maz25]: that in the AEC of R-modules with embeddings, R is $(< \aleph_n)$-Noetherian but not $(< \aleph_{n-1})$-Noetherian if and only if for all stability cardinals $\lambda \geq \beth_{\beth_{(2^{\mathrm{card}(R)+\aleph_0})+}}$, there are exactly $n + 1$ non-isomorphic λ-limit models (this result is slightly weaker than [Maz25, 3.17] as the lower bound on λ is higher than in the original result).

The paper has six sections and an Appendix. Section 2 presents necessary background and some basic results (though many of the examples are not necessary to understand the main results). Section 3 deals with high cofinality limit models, and ends with the application in AECs with μ-tameness. Section 4 addresses low cofinality limit models. Section 5 presents 'general' results that give the full spectrum of limit models, including the 'black box' versions of the main theorem that can be applied for all large enough stable λ. Section 6 presents some applications of our results in natural AECs. Appendix A contains a short proof of the spectrum of limit models in the first order context.

2. Preliminaries and basic results

We assume some basic knowledge of abstract elementary classes (AECs), such as presented in [Bal09], aside from in Appendix A where only knowledge of first order model theory is necessary.

2.1. **Basic notions.** Typically we use $\mathbf{K} = (K, \leq_{\mathbf{K}})$ to denote an AEC, or sometimes an abstract class (an abstract class, or AC, is a class of models K in a fixed language closed under isomorphisms with a partial ordering $\leq_{\mathbf{K}}$ that respects isomorphisms). When we say $\mathbf{K}'$ is a sub-AC of $\mathbf{K}$, we mean $\mathbf{K}'$ is an AC, $K' \subseteq K$, and $\leq_{\mathbf{K}'} = \leq_{\mathbf{K}} \restriction (K')^2$. We use M, N for models, a, b for elements of models (always singletons), $|M|$ to denote the universe of model M, and $\|M\|$ for the cardinality of the universe of M. By $M \cong N$, we mean that M and N are isomorphic,

and by $M \cong_{M_0} N$, that M and N are isomorphic over M_0; that is, there is an isomorphism from M to N that fixes the substructure M_0 of M and N.

We use standard abuses of notation such as $a \in M$ as shorthand for $a \in |M|$. Given $M, N \in$ $\mathbf{K}$ where $M \leq_{\mathbf{K}} N$ and $a \in N$, $\mathbf{gtp}(a/M, N)$ denotes the (Galois) type of a over M in N. $\mathbf{gS}(M)$ is the set of types over singletons in M. Occasionally we use the more general notions of $\mathbf{gtp}(a/A, N)$ and $\mathbf{gS}(A; N)$, for $A \subseteq |N|$ where $N \in \mathbf{K}$, see for example [Vas16c, 2.16, 2.20]. Given types $p \in \mathbf{gS}(M)$ and $q \in \mathbf{gS}(N)$ with $M \leq_{\mathbf{K}} N$, we say q extends p or write $p \subseteq q$ if every realisation of q realises p (or equivalently, there exists a common realisation), defined similarly for $p \in \mathbf{gS}(A; N)$, $q \in \mathbf{gS}(B; N)$ where $A \subseteq B$.

We use $\alpha, \beta, \gamma, \delta$ for ordinals (typically γ, δ are limit), and λ, κ, μ for infinite cardinals (unless explicitly mentioned, cardinals will be infinite) - typically $\mathbf{K}$ will be stable in λ, and κ will be a regular cardinal with $\kappa < \lambda^+$. We usually use κ as a local character cardinal, as this is related to Shelah's $\kappa_r(T)$ for first order stable theories (see Definition A.2), and it mixes well with Vasey's notation in [Vas18b] (see Definition 2.32).

We also use the following standard notions. Let $\mathbf{K}$ be an AC. We say $\mathbf{K}$ satisfies the *amalgamation property* or *AP* if every span of models M_1, M_2 over M_0 in $\mathbf{K}$ may be amalgamated into some $N \in \mathbf{K}$. We say $\mathbf{K}$ satisfies the *joint embedding property* or *JEP* if any pair of models $M_1, M_2 \in \mathbf{K}$ can be embedded into some common model $N \in \mathbf{K}$. We say $\mathbf{K}$ has *no maximal models* or *NMM* if for every $M \in \mathbf{K}$, there exists $N \in \mathbf{K}$ where $M \leq_{\mathbf{K}} N$ and $M \neq N$.

Given a cardinal λ, $\mathbf{K}_\lambda$ denotes the sub-AC with underlying class $K_\lambda = \{M \in K : \|M\| = \lambda\}$, and with $\leq_{\mathbf{K}_\lambda}$ the restriction of $\leq_{\mathbf{K}}$ to K_λ. We say $\mathbf{K}$ has λ-AP if $\mathbf{K}_\lambda$ has AP. Similarly define λ-JEP and λ-NMM.

Tameness, identified by Grossberg and VanDieren in [GrVan06a], has proved to be a vital notion in the classification theory of AECs.

Definition 2.1. *Let $\mathbf{K}$ be an AEC and θ be a cardinal. We say $\mathbf{K}$ is $(< \theta)$-tame if for every $M \in \mathbf{K}$, and every $p, q \in \mathbf{gS}(M)$, if $p \restriction A = q \restriction A$ for all $A \subseteq |M|$ with $|A| < \theta$, then $p = q$.*
We say $\mathbf{K}$ is θ-tame if $\mathbf{K}$ is $(< \theta^+)$-tame.

Note that if $LS(\mathbf{K}) < \theta$ then the definition of $(< \theta)$-tame is equivalent when we replace $A \subseteq M$ with $M' \leq_{\mathbf{K}} N$. Also, if $\mathbf{K}$ is $(< \theta_1)$-tame and $\theta_1 < \theta_2$ then $\mathbf{K}$ is $(< \theta_2)$-tame. It is worth emphasizing that several of our results do not assume tameness.

2.2. **Limit Models.** We begin with a brief recounting of facts about limit models

Definition 2.2. *Let $\mathbf{K}$ be an AEC and $\lambda \geq LS(\mathbf{K})$.*

(1) Let $M_0, M \in K_\lambda$. We say M is universal over M_0 if for every $N \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} N$, there exists a $\mathbf{K}$-embedding $f : N \xrightarrow[M_0]{} M$. We sometimes write $M_0 \leq_{\mathbf{K}}^u M$.

(2) Let α be an ordinal, and let $\langle M_i : i < \alpha \rangle$ be a $\leq_{\mathbf{K}}$-increasing sequence of models in $\mathbf{K}$ (sometimes called a chain).
* (a) For $\gamma < \alpha$ limit, we say $\langle M_i : i < \alpha \rangle$ is continuous at γ if $M_\gamma = \bigcup_{i < \gamma} M_i$.*
* (b) We say $\langle M_i : i < \alpha \rangle$ is continuous if it is continuous at every limit $\gamma < \alpha$.*
* (c) We say $\langle M_i : i < \alpha \rangle$ is universal if for all $i < \alpha$ with $i + 1 < \alpha$, $M_i \leq_{\mathbf{K}}^u M_{i+1}$.*
* (d) We say $\langle M_i : i < \alpha \rangle$ is strongly universal if for all $i \in [0, \alpha)$, $\bigcup_{r < i} M_r \leq_{\mathbf{K}}^u M_i$.*

Remark 2.3. *We stated these definitions in terms of sequences indexed by ordinals for legibility, but they may be used also when α is replaced by an arbitrary well-ordering $(I, <)$ and the sequence by $\langle M_i : i \in I \rangle$. Also, if $\alpha = \delta + 1$, we may write $\langle M_i : i \leq \delta \rangle$ in place of $\langle M_i : i < \delta + 1 \rangle$.*

Definition 2.4. *Let $\mathbf{K}$ be an AEC and $\lambda \geq LS(\mathbf{K})$.*

(1) Let $M, N \in \mathbf{K}_\lambda$ and $\delta < \lambda^+$ a limit ordinal. We say N is a (λ, δ)-limit over M if there is a $\leq_{\mathbf{K}}$-increasing universal continuous chain $\langle M_i : i \leq \delta \rangle$ such that $M_0 = M$ and $M_\delta = N$ (in particular, $N = \bigcup_{i < \delta} M_i$).

   We say the sequence $\langle M_i : i \leq \delta \rangle$ is witnessing the limit, and that δ is the length of the limit.

(2) $N \in \mathbf{K}_\lambda$ is a (λ, δ)-limit model if it is a (λ, δ)-limit model over some $M \in K_\lambda$.

(3) $N \in \mathbf{K}_\lambda$ is a λ-limit model (over M) if it is a (λ, δ)-limit model (over M) for some limit ordinal $\delta < \lambda^+$. We sometimes omit λ when it is clear from context.

(4) Let $\kappa < \lambda^+$ be regular. We say $N \in \mathbf{K}_\lambda$ is a $(\lambda, \geq \kappa)$-limit model if N is a (λ, δ)-limit model for some regular $\delta \in [\kappa, \lambda^+)$.

(5) $K_{(\lambda, \geq \kappa)}$ is the class of all $(\lambda, \geq \kappa)$-limit models in $\mathbf{K}$.

Note in particular that if N is a λ-limit over M, then N is also universal over M. Also, the definition of (λ, δ)-limit model is equivalent if we only assume the sequence $\langle M_i : i \leq \delta \rangle$ is continuous at δ.

A lot is already known about when limit models exist and when they are isomorphic. The following are essentially [Sh:h, II.1.16.1(a)], but we cite [GrVan06a] as this provides a proof.

Fact 2.5 ([GrVan06a, 2.12]). *Let $\mathbf{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\langle M_i : i \leq \lambda \rangle$ is a $\leq_{\mathbf{K}}$-increasing sequence in K_λ continuous at λ such that M_{i+1} realises all types over M_i for every $i < \lambda$, then M_λ is universal over M_0.*

A corollary of this is:

Fact 2.6 ([GrVan06a, 2.9]). *Let $\mathbf{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is stable in λ with λ-AP and λ-NMM. Then for every $M \in K_\lambda$, there is $N \in K_\lambda$ universal over M.*

Moreover, under the same assumptions, for every $M \in K_\lambda$ and every limit $\delta < \lambda^+$ there is a (λ, δ)-limit model over M.

The following holds by a straightforward back and forth argument, first present in [She99].

Fact 2.7 ([ShVi99, 1.3.6]). *Let $\mathbf{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathbf{K})$. Suppose $\mathbf{K}$ has λ-AP, and that $\delta_1, \delta_2 < \lambda^+$ such that $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$. Suppose $M, N_1, N_2 \in K_\lambda$ where N_l is a (λ, δ_l)-limit over M for $l = 1, 2$. Then there is an isomorphism $f : N_1 \cong N_2$ fixing M.*

If in addition $\mathbf{K}$ has λ-JEP, then for any $N_1, N_2 \in K_\lambda$ where N_l is a (λ, δ_l)-limit for $l = 1, 2$, then $N_1 \cong N_2$.

So if $\mathbf{K}_\lambda$ is well behaved, all possible limit models exist, and by Fact 2.7 they are unique for a fixed cofinality of limit length. This means we can restrict to studying limits of infinite regular lengths.

Definition 2.8. *Let $\mathbf{K}$ be an AEC, μ an infinite cardinal. We say that a model $M \in \mathbf{K}$ is μ-saturated if for all $A \subseteq M$ with $|A| < \mu$, and $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{gS}(A; N)$, p is realised in M. We say M is saturated if M is $\|M\|$-saturated.*

Remark 2.9. *If $\mu > \mathrm{LS}(\mathbf{K})$, then M is μ-saturated if and only if for all $M_0 \leq_{\mathbf{K}} M$ with $\|M_0\| < \mu$, and all $p \in \mathbf{gS}(M_0)$, p is realised in M.*

The following is [GrVas17, 2.8(1)], but since they do not include a proof we provide one.

Fact 2.10. *Suppose $\mathbf{K}$ is an AEC with λ-AP, $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\delta \leq \lambda$ is regular, and M is a (λ, δ)-limit model. Then M is δ-saturated.*

Proof. Fix $\langle M_i : i < \delta \rangle$ witnessing that M is a (λ, δ)-limit model. Suppose $A \subseteq |M|$ with $|A| < \delta$, $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$, and $p \in \mathbf{gS}(A; N)$. Say $p = \mathbf{gtp}(a/A, N)$. Take $i < \delta$ such that $A \subseteq M_i$. Then $p \subseteq \mathbf{gtp}(a/M_i, N)$. Since $M_i \leq_{\mathbf{K}}^u M_{i+1} \leq_{\mathbf{K}} M$, $\mathbf{gtp}(a/M_i, N)$ is realised in M, and hence p is also realised in M. $\qquad\square$

2.3. **Independence relations.** Independence relations generalize first order non-forking to AECs. These have been thoroughly studied in recent years.

Definition 2.11. *Given an abstract class* $\mathbf{K}$, *a* weak independence relation *on* $\mathbf{K}$ *is a relation* $\underset{\smile}{\downarrow}$ *on tuples* (M_0, a, M, N), *where* $M_0, M, N \in \mathbf{K}$, $a \in N$, *and* $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$. *We write* $a \underset{M_0}{\overset{N}{\downarrow}} M$ *as a shorthand for* $\underset{\smile}{\downarrow}(M_0, a, M, N)$.

Definition 2.12. *An* independence relation $\underset{\smile}{\downarrow}$ *on an abstract class* $\mathbf{K}$ *is a weak independence relation that satisfies:*

(1) Invariance: *whenever* $a \underset{M_0}{\overset{N}{\downarrow}} M$ *holds, and* $f : N \cong N'$ *is an isomorphism, we have that*

$$f(a) \underset{f[M_0]}{\overset{N'}{\downarrow}} f[M].$$

(2) Monotonicity: *whenever* $a \underset{M_0}{\overset{N}{\downarrow}} M$ *holds and* $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$ *and* $a \in N_0$, *we have both* $a \underset{M_0}{\overset{N_0}{\downarrow}} M_1$ *and* $a \underset{M_0}{\overset{N'}{\downarrow}} M_1$ *(that is, we can shrink* M, *and we can shrink or grow* N, *so long as we preserve* $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$, $a \in N$).

(3) Base monotonicity: *whenever* $a \underset{M_0}{\overset{N}{\downarrow}} M$ *holds and* $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$, *then* $a \underset{M_1}{\overset{N}{\downarrow}} M$.

Definition 2.13. *Suppose* $\underset{\smile}{\downarrow}$ *is an independence relation over an abstract class* $\mathbf{K}$. *Let* $M_0 \leq_{\mathbf{K}} M$ *and* $p \in \mathbf{gS}(M)$. *We say* $p \underset{\smile}{\downarrow}$-*does not fork over* M_0 *if there exist* $N \in \mathbf{K}$ *and* $a \in N$ *such that* $p = \mathbf{gtp}(a/M, N)$ *and* $a \underset{M_0}{\overset{N}{\downarrow}} M$.

Remark 2.14. *By invariance and monotonicity,* $p \underset{\smile}{\downarrow}$-*does not fork over* M_0 *if and only if for every* $N \in \mathbf{K}$, $a \in N$ *such that* $p = \mathbf{gtp}(a/M, N)$, $a \underset{M_0}{\overset{N}{\downarrow}} M$. *That is, the choice of representatives of* p *do not matter.*

Nicely behaved independence relations typically satisfy a number of the following properties. Our results on isomorphism types of limit models will involve assuming existence of an independence relation satisfying different combinations of these (and some additional ones we will define later).

Definition 2.15. *Given an independence relation* $\underset{\smile}{\downarrow}$ *on an abstract class* $\mathbf{K}$, κ *a regular cardinal, and* θ *any infinite cardinal, we say* $\underset{\smile}{\downarrow}$ *satisfies:*

(1) Uniqueness *if whenever* $M \leq_{\mathbf{K}} N$, *and* $q_1, q_2 \in \mathbf{gS}(N)$ *satisfy that* $q_1 \restriction M = q_2 \restriction M$ *and* $q_l \underset{\smile}{\downarrow}$-*does not fork over* M *for* $l = 1, 2$, *then* $q_1 = q_2$.

(2) Extension *if whenever* $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$, *and* $p \in \mathbf{gS}(M) \underset{\smile}{\downarrow}$-*does not fork over* M_0, *then there exists* $q \in \mathbf{gS}(N)$ *extending* p *such that* $q \underset{\smile}{\downarrow}$-*does not fork over* M_0.

(3) Transitivity *if whenever* $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ *and* $p \in \mathbf{gS}(N)$ *satisfies both that* $p \underset{\smile}{\downarrow}$-*does not fork over* M *and* $p \restriction M \underset{\smile}{\downarrow}$-*does not fork over* M_0, *then* $p \underset{\smile}{\downarrow}$-*does not fork over* M_0.

(4) Existence *if whenever* $M \in \mathbf{K}$ *and* $p \in \mathbf{gS}(M)$, $p \underset{\smile}{\downarrow}$-*does not fork over* M.

(5) κ-local character *if whenever* $\langle M_i : i < \kappa \rangle$ *is a* $\leq_{\mathbf{K}}^u$-*increasing sequence, and* $p \in \mathbf{gS}(M_\kappa)$ *where* $M_\kappa = \bigcup_{i < \kappa} M_i \in \mathbf{K}$, *then there exists* $i < \kappa$ *such that* $p \underset{\smile}{\downarrow}$-*does not fork over* M_i.

(6) $(\geq \kappa)$-local character *if* $\underset{\smile}{\downarrow}$ *satisfies* γ-*local character for each regular* $\gamma \geq \kappa$.

(7) strong $(< \kappa)$-local character *if for all $N \in K$ and $p \in \mathbf{gS}(N)$, there exists $M \leq_{\mathbf{K}} N$ with $\|M\| < \kappa$ and $p \underset{}{\downarrow}$-does not fork over M.*

(8) strong κ-local character *if $\underset{}{\downarrow}$ satisfies $(< \kappa^+)$-local character.*

(9) κ-universal continuity *if whenever $\langle M_i : i < \kappa \rangle$ is a $\leq_{\mathbf{K}}^u$-increasing sequence and $p \in \mathbf{gS}(M_\kappa)$ where $M_\kappa = \bigcup_{i < \kappa} M_i \in \mathbf{K}$, then provided that $p \upharpoonright M_i \underset{}{\downarrow}$-does not fork over M_0 for all $i < \kappa$, $p \underset{}{\downarrow}$-does not fork over M_0.*

(10) $(\geq \kappa)$-universal continuity *if $\underset{}{\downarrow}$ satisfies γ-universal continuity for each regular $\gamma \geq \kappa$.*

(11) Universal continuity *if $\underset{}{\downarrow}$ has $(\geq \aleph_0)$-universal continuity.*

(12) Non-forking amalgamation *if given $M_0, M_1, M_2 \in K$ and $a_1 \in M_1, a_2 \in M_2$, such that $M_0 \leq_{\mathbf{K}} M_l$ for $l = 1, 2$, then there exists $N \in \mathbf{K}$ with $M_0 \leq_{\mathbf{K}} N$ and $\mathbf{K}$-embeddings $f_l : M_l \underset{M_0}{\to} N$ such that $f_l(a_l) \overset{N}{\underset{M_0}{\downarrow}} f_{3-l}[M_{3-l}]$ for $l = 1, 2$ (that is, you can amalgamate M_1, M_2 such that the images of the a_l are independent of the 'opposite' model M_{3-l} over M_0).*

(13) $(< \theta)$-witness property (for singletons) *if whenever $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{gS}(N)$, if for all $A \subseteq N$ where $|A| < \theta$, there exists $N_0 \leq_{\mathbf{K}} N$ where $A \subseteq |N_0|$ and $M \leq_{\mathbf{K}} N_0$ such that $p \upharpoonright N_0 \underset{}{\downarrow}$-does not fork over M, then $p \underset{}{\downarrow}$-does not fork over M.*

(14) θ-witness property *if $\underset{}{\downarrow}$ has the $(< \theta^+)$-witness property.*

The following is essentially [She78, III.4.4].

Fact 2.16. *Let $\mathbf{K}$ be an AC with an independence relation $\underset{}{\downarrow}$. If $\underset{}{\downarrow}$ satisfies uniqueness and extension, then $\underset{}{\downarrow}$ satisfies transitivity.*

Proof. Say $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ and $p \in \mathbf{gS}(N)$ satisfies both that $p \underset{}{\downarrow}$-does not fork over M and $p \upharpoonright M \underset{}{\downarrow}$-does not fork over M_0. By extension, there is $q \in \mathbf{gS}(N)$ extending $p \upharpoonright M$ such that $q \underset{}{\downarrow}$-does not fork over M_0. By monotonicity, $q \underset{}{\downarrow}$-does not fork over M. By uniqueness, since p and $q \underset{}{\downarrow}$-do not fork over M and $p \upharpoonright M = q \upharpoonright M$, $p = q$. So $p \underset{}{\downarrow}$-does not fork over M_0 as desired. $\square$

Extension has a weaker formulation, but assuming uniqueness it is equivalent to our version.

Lemma 2.17. *Let $\mathbf{K}$ be an AC with an independence relation $\underset{}{\downarrow}$ satisfying uniqueness. Then extension is equivalent to saying that whenever $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{gS}(M) \underset{}{\downarrow}$-does not fork over M, then there is $q \in \mathbf{gS}(N)$ with $p \subseteq q$ and $q \underset{}{\downarrow}$-does not fork over M.*

Proof. The version from Definition 2.15 implies the second version by setting $M_0 = M$. For the reverse implication, suppose $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ with $p \in \mathbf{gS}(M)$ such that $p \underset{}{\downarrow}$-does not fork over M_0. Then $p \upharpoonright M_0 \underset{}{\downarrow}$-does not fork over M_0 by monotonicity. So there exists $q \in \mathbf{gS}(N)$ extending $p \upharpoonright M$ such that $q \underset{}{\downarrow}$-does not fork over M_0. By monotonicity, $q \upharpoonright M \underset{}{\downarrow}$-does not fork over M_0, so by uniqueness, $q \upharpoonright M = p$. Hence $p \subseteq q$ and $q \underset{}{\downarrow}$-does not fork over M_0 as desired. $\square$

Lemma 2.18. *Let $\mathbf{K}$ be an AC with an independence relation $\underset{}{\downarrow}$. If $\underset{}{\downarrow}$ satisfies the $(< \theta)$-witness property in some regular θ, then $\underset{}{\downarrow}$ satisfies $(\geq \theta)$-universal continuity.*

Proof. Suppose $\langle M_i : i \leq \gamma \rangle$ are in $\mathbf{K}$ for some regular $\gamma \geq \theta$, where $M_\gamma = \bigcup_{i < \gamma} M_i$. Suppose $p \in \mathbf{gS}(M_\gamma)$, and $p \upharpoonright M_i \underset{}{\downarrow}$-does not fork over M_0 for all $i < \gamma$.

We will verify the hypotheses of the $(< \theta)$-witnessing property for p and M_0. Fix $A \subseteq M_\gamma$ with $|A| < \theta$. Since γ is regular and $|A| < \gamma$, there exists $i < \gamma$ such that $A \subseteq M_i$. We know $M_0 \leq_{\mathbf{K}} M_i$, $A \subseteq |M_i|$, and $p \upharpoonright M_i \underset{}{\downarrow}$-does not fork over M_0. Therefore, by the $(< \theta)$-witnessing property, $p \underset{}{\downarrow}$-does not fork over M_0 as desired. $\square$

Remark 2.19. *Let* **K** *be an AC with an independence relation* $\underset{}{\perp}$. *If* $\underset{}{\perp}$ *satisfies strong* $(< \kappa)$-*local character then* $\underset{}{\perp}$ *satisfies* $(\geq \kappa)$-*local character.*

Two independence relations of interest are λ-*non-splitting* and λ-*non-forking*. The following definition follows [Van06, I.4.2], but both can be traced back to Shelah [She78].

Definition 2.20. *Let* **K** *be an AC with AP. Let* $M, N \in$ **K** *with* $M \leq_{\mathbf{K}} N$. *We say that* $p \in \mathbf{gS}(N)$ *splits over* M *if there exist* $N_1, N_2 \in$ **K** *with* $M \leq_{\mathbf{K}} N_l \leq_{\mathbf{K}} N$ *for* $l = 1, 2$, *and there is an isomorphism* $f : N_1 \underset{M}{\rightarrow} N_2$ *such that* $f(p \restriction N_1) \neq p \restriction N_2$.

We say p *does not split over* M *if no such* N_1, N_2, f *exist. If we require additionally that* $\|N_1\| = \|N_2\| = \lambda$, *we say* p λ-*splits over* M *and* p *does not* λ-*split over* M. *Note when restricted to models in* $\mathbf{K}_\lambda$, *non-splitting is the same as* λ-*non-splitting.*

We use $\underset{\mathrm{split}}{\perp}$ *to denote the independence relation given by* $a \underset{\underset{M_0}{\mathrm{split}}}{\overset{N}{\perp}} M$ *if an only if* $\mathbf{gtp}(a/M, N)$ *does not split over* M_0. *Similarly define* $\underset{\lambda-\mathrm{split}}{\perp}$ *for* λ-*splitting.*

The following definition follows [Vas16a, 4.2,3.8] (see also [Leu24, 4.1]).

Definition 2.21. *Let* $M, N \in \mathbf{K}_\lambda$. *We say that* $p \in \mathbf{gS}(N)$ *does not* λ-*fork over* M *if and only if there is* $M_0 \in \mathbf{K}_\lambda$ *such that* p *does not* λ-*split over* M_0 *and* $M_0 \leq_{\mathbf{K}}^u M$.

We define $\underset{\lambda-f}{\perp}$ *(the independence relation corresponding with* λ-*non-forking) as in Definition 2.20.*

Definition 2.22. *Let* $M, N \in \mathbf{K}_{\geq \lambda}$. *We say that* $p \in \mathbf{gS}(N)$ *does not* $(\geq \lambda)$-*fork over* M *if and only if there is* $M' \in \mathbf{K}_\lambda$ *where* $M' \leq_{\mathbf{K}} M$ *such that for all* $N' \in \mathbf{K}_\lambda$ *with* $M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$, *we have that* $p \restriction N'$ *does not* λ-*fork over* M'.

We define $\underset{(\geq \lambda)-f}{\perp}$ *(the independence relations corresponding with* $(\geq \lambda)$-*non-forking) as in Definition 2.20 and Definition 2.21.*

Remark 2.23. *All versions of non-splitting and non-forking satisfy invariance, monotonicity, and base monotonicity. Much more can be said under additional assumptions (see Example 2.28 and Example 2.29). Additionally,* $(\geq \lambda)$-*non-forking is the same as* λ-*non-forking when restricted to* $\mathbf{K}_\lambda$ *(use* $M' = M$ *and monotonicity).*

There are many well known classes **K** and relations $\underset{}{\perp}$ that satisfy some combination of the properties we just introduced. The following include our main examples of interest for this paper.

Example 2.24. *Suppose* T *is a* λ-*stable first order theory, where* $\lambda \geq |T|$. *Define* $\underset{}{\perp}$ *on* $(\mathrm{Mod}(T), \preccurlyeq)$ *by* $a \underset{M_0}{\overset{N}{\perp}} M$ *if and only if* $\mathbf{tp}(a/M, N)$ *does not fork over* M_0 *(in the usual sense). This satisfies invariance, monotonicity, base monotonicity, uniqueness, extension,* $(\geq \kappa)$-*local character in some* $\kappa \leq \lambda$, *universal continuity, non-forking amalgamation, and the* $(< \aleph_0)$-*witness property - see Lemma 6.11.*

Example 2.25. *In [LRV19, §3], the authors define a notion of a weakly stable independence relation from a categorical perspective on 'amalgams' (from here on we will call this a weakly stable independence relation in the sense of LRV). This can be viewed as a relation on 4-tuples of models. Given such a relation* $\underset{}{\perp}$, *they show it can be extended to a relation* $\overline{\underset{}{\perp}}$ *that allows the intermediate models to be replaced by arbitrary subsets (i.e. the relation* $A \underset{M_0}{\overset{N}{\overline{\perp}}} B$ *is defined for*

$M_0 \leq_{\mathbf{K}} N$, $A, B \subseteq N$) [LRV19, 8.2]. In [LRV19, §8] they show such relations satisfy (broader versions of) invariance, monotonicity, base monotonicity, existence, extension, uniqueness, and transitivity. It also has symmetry, which says that $A \underset{M_0}{\overset{N}{\cup}} B \iff B \underset{M_0}{\overset{N}{\cup}} A$.

Further, if $\cup$ is a stable independence relation in the LRV sense, rather than weakly stable, $\cup$ satisfies the $(< \theta)$-witness property in the sense of [LRV19, 8.7] (over models rather than singletons) in some cardinal θ (from here on we will call this the '$(< \theta)$-witness property in the LRV sense'), and has strong $(< \kappa)$-local character for some cardinal κ. If we restrict to singletons and models, $\overline{\cup}$ has many of the useful properties we listed - see Lemma 2.26.

Lemma 2.26. *Suppose $\cup$ is a weakly stable independence relation in the sense of LRV. Then the restriction $\overline{\cup}$ to singletons and models (that is, restrict to the case $a \underset{M_0}{\overset{N}{\cup}} M$) is an independence relation that satisfies invariance, monotonicity, base monotonicity, extension, uniqueness, and transitivity.*

Moreover:

(1) if $\overline{\cup}$ has strong $(< \kappa)$-local character, then $\overline{\cup}$ has $(\geq \kappa)$-local character

(2) if $\overline{\cup}$ satisfies the $(< \theta)$-witness property (for singletons), then the restriction does also, and if θ is regular, then $\overline{\cup}$ also satisfies $(\geq \theta)$-universal continuity.

Proof. Invariance, monotonicity, base monotonicity, extension, uniqueness, transitivity, and the $(< \theta)$-witness property all follow immediately from their versions in the $A \underset{M_0}{\overset{N}{\cup}} B$ case.

Next we address non-forking amalgamation. Suppose $M_0, M_1, M_2 \in K$ with $M_0 \leq_{\mathbf{K}} M_l$ for $l = 1, 2$ and $a_l \in M_l$ for $l = 1, 2$. Let $i_{0,l} : M_0 \to M_l$ be the identity maps for $l = 1, 2$. By existence, there are $N \in \mathbf{K}$ and $f_l : M_l \to N$ such that $\cup(i_{0,1}, i_{0,2}, f_1, f_2)$ - that is, $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$ and $f_1(M_1) \underset{f_1(M_0)}{\overset{N}{\cup}} f_2(M_2)$. Since $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$ and using invariance, we may assume $f_l \upharpoonright M_0 = \mathrm{id}_{M_0}$ by composing f_1, f_2 both with an extension of $(f_1 \upharpoonright M_0)^{-1}$ to N. By symmetry of $\cup$, we also have $f_2(M_2) \underset{M_0}{\overset{N}{\cup}} f_1(M_1)$. As $f_l(a) \in f_l(M_l)$, we have $f_l(a_l) \underset{M_0}{\overset{N}{\cup}} f_{3-l}(M_{2-l})$ for $l = 1, 2$; that is, $\mathbf{gtp}(f_l(a_l)/f_{3-l}(M_l), N)$ $\cup$-does not fork over M_0 for $l = 1, 2$ as desired.

For the moreover part (1), $(\geq \kappa)$-local character follows from Remark 2.19. For the moreover part (2), as before the $(< \theta)$-witness property for singletons holds for the restriction immediately. For $(\geq \theta)$-universal continuity, apply Fact 2.18. $\qquad\square$

Remark 2.27. *While they look similar, in general it is not obvious whether the $(< \theta)$-witness property (for singletons) is equivalent to the $(< \theta)$-witness property in the sense of LRV. That said, if $\mathbf{K}$ has intersections (that is, for all $N \in \mathbf{K}$ and $A \subseteq |N|$, $\bigcap\{M \in \mathbf{K} : A \subseteq M \leq_{\mathbf{K}} N\} \leq_{\mathbf{K}} N$) and $\overline{\cup}$ is a stable independence relation with the $(< \theta)$-witnessing property in the LRV sense, then $\overline{\cup}$ has the witness property (for singletons).*

Example 2.28 ([Vas, 13.16]). *If $\mathbf{K}$ is λ-superstable and has λ-symmetry, then λ-forking restricted to $\mathbf{K}_\lambda$ satisfies invariance, monotonicity, base monotonicity, uniqueness, extension, $(\geq \aleph_0)$-local character, universal continuity, and non-forking amalgamation.*

Example 2.29 ([Leu24, §4])**.** *Suppose* **K** *is an AEC with AP, JEP, and NMM, and there is* $\lambda \geq \mathrm{LS}(\mathbf{K})$ *such that*

(1) **K** *is* λ*-stable*

(2) **K** *is* λ*-tame*

(3) *when restricted to* $\mathbf{K}_\lambda$, $\underset{\lambda\text{-split}}{\downarrow}$ *satisfies universal continuity and* $(\geq \kappa)$*-local character for some regular* $\kappa \leq \lambda$.

(4) **K** *satisfies* λ*-symmetry*

Let $\downarrow$ *be the restriction of* $\underset{\lambda-f}{\downarrow}$ *to* $K_{(\lambda,\geq\kappa)}$. *Assuming that* $\downarrow$ *satisfies uniqueness,* $\downarrow$ *satisfies invariance, monotonicity, base monotonicity, uniqueness, extension,* $(\geq \kappa)$*-local character, universal continuity, and non-forking amalgamation.*

Remark 2.30. *Example 2.29 above is stated in [Leu24] without the uniqueness assumption (uniqueness is proved from the other hypotheses). Unfortunately, we located an error in the proof of uniqueness [Leu24, 4.5]. In the penultimate paragraph of the proof, it is claimed that the map* f *fixes* M_0, *but this is not true (in fact the image of* f *is a proper substructure of* M_0 *by NMM and* $f[M_{i+1}] = N^* \leq_{\mathbf{K}}^u M_0$). *We have not found a way to fix this issue. Note in the case* $\kappa = \aleph_0$, *uniqueness holds, as in that case* **K** *is* λ*-superstable and we fall into Example 2.28.*

Example 2.31 ([Vas16b, §4, §5])**.** *Let* **K** *be an AEC with AP, NMM, stable in* $\mu \geq \mathrm{LS}(\mathbf{K})$, *with* λ*-JEP and* μ*-tameness. Let* $\downarrow$ *be* $\underset{(\geq\mu)-f}{\downarrow}$ *restricted to models in* $\mathbf{K}_{\geq\mu^+}^{\mu^+\text{-sat}}$ *(that is, the* μ^+*-saturated models in* $\mathbf{K}_{\geq\mu^+}$ *ordered by* $\leq_{\mathbf{K}}$). *Then* $\downarrow$ *has many of the useful properties we listed - see Lemma 3.41.*

The following notation is similar to that of [Vas18b].

Definition 2.32. *Assume* **K** *is an AEC stable in* λ *and* $\downarrow$ *is an independence relation.*

(1) $\underline{\kappa}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \{\delta < \lambda^+ : \text{whenever } \langle M_i : i \leq \delta\rangle \text{ is a } \leq_{\mathbf{K}}^u\text{-increasing continuous chain in } \mathbf{K}_\lambda \text{ and } p \in \mathbf{gS}(M_\delta), \text{ then there is } i < \delta \text{ such that } p \downarrow\text{-does not fork over } M_i\}$

(2) $\underline{\kappa}^{\mathrm{wk}}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \{\delta < \lambda^+ : \text{whenever } \langle M_i : i \leq \delta\rangle \text{ is a } \leq_{\mathbf{K}}^u\text{-increasing continuous chain in } \mathbf{K}_\lambda \text{ and } p \in \mathbf{gS}(M_\delta), \text{ then there is } i < \delta \text{ such that } p \upharpoonright M_{i+1} \downarrow\text{-does not fork over } M_i\}$

(3) $\kappa(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \min\{\mu \leq \lambda : [\mu, \lambda^+) \cap Reg \subseteq \underline{\kappa}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)\}$ *when it exists, else* $\kappa(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \infty$.

Notation 2.33. *We will sometimes use the shorthand* $\kappa_\lambda^u(\downarrow)$ *to denote* $\kappa(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ *when the AEC* **K** *is unambiguous to avoid notational clutter in more complicated expressions.*

Remark 2.34. *If* **K** *is an AEC and* $\downarrow$ *is an independence relation on* $\mathbf{K}_\lambda$ *with extension, uniqueness, and universal continuity, then* $\underline{\kappa}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) \cap Reg = \underline{\kappa}^{\mathrm{wk}}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) \cap Reg$. *The* $\subseteq$ *direction follows from the same method as [BGVV, 11(1)], the* $\supseteq$ *direction follows from monotonicity. In particular, since* $\underline{\kappa}^{\mathrm{wk}}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ *is an interval of the form* $[\alpha, \lambda^+)$, *in this case* $\kappa(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \min(\underline{\kappa}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) \cap Reg)$.

Remark 2.35. *Our definition is inspired by the notion of* $\underline{\kappa}(\mathbf{K}_\lambda, <_{\mathbf{K}}^u)$ *from [Vas19, 3.8], and in fact for any AEC* **K**, $\underline{\kappa}(\mathbf{K}_\lambda, <_{\mathbf{K}}^u) = \underline{\kappa}(\underset{\text{split}}{\downarrow}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) \cup ([\lambda^+, \infty) \cap Reg)$. *The regular cardinals greater or equal to* λ^+ *give no new information, since no* $\leq_{\mathbf{K}}^u$*-increasing sequences of those lengths exist inside* $\mathbf{K}_\lambda$ *under* λ*-NMM. They are included in* $\underline{\kappa}(\mathbf{K}_\lambda, <_{\mathbf{K}}^u)$ *because the broader definition of* $\underline{\kappa}$ *in [Vas19, 2.2] allows for classes with arbitrarily large models, but our definition is more intuitive when restricted to* $\mathbf{K}_\lambda$.

3. Long limit models

Our goal in this section is to show that, in a very general setting, all the high cofinality limit models are the same:

Theorem 3.1. *Assume Hypothesis 3.7. Let $\delta_1, \delta_2 < \lambda^+$ be limit ordinals where $\kappa \leq \mathrm{cf}(\delta_1), \mathrm{cf}(\delta_2)$. If $M, N_1, N_2 \in K_\lambda$ where N_l is a (λ, δ_l)-limit over M for $l = 1, 2$, then there is an isomorphism from N_1 to N_2 fixing M.*

Moreover, if $N_1, N_2 \in K_\lambda$ where N_l is (λ, δ_l)-limit for $l = 1, 2$, then N_1 is isomorphic to N_2.

First we state the natural surrogate for universal continuity when an independence relation is only defined on a sub-AC of an AEC - specifically $K_{(\lambda, \geq \kappa)}$.

Definition 3.2. *Let $\mathbf{K}$ be an AEC, $\mathbf{K}'$ a sub-AC, and $\underset{\smile}{\perp}$ an independence relation on $\mathbf{K}'$.*

We say that $\underset{\smile}{\perp}$ has universal continuity* in $\mathbf{K}$ if and only if whenever δ is an ordinal and $\langle M_i : i \leq_{\mathbf{K}} \delta \rangle$ is a $\leq_{\mathbf{K}}^u$-increasing sequence in $\mathbf{K}'$ with $\bigcup_{i<\delta} M_i \leq_{\mathbf{K}} M$ for some $M \in \mathbf{K}'$, and $\langle p_i \in \mathbf{gS}(M_i) : i < \delta \rangle$ is an increasing sequence of types where $p_i \underset{\smile}{\perp}$-does not fork over M_0 for all $i < \delta$, then there is a unique $p_\delta \in \mathbf{gS}(\bigcup_{i<\delta} M_i)$ such that $p_i \subseteq p_\delta$ for all $i < \delta$.

We may omit in $\mathbf{K}$ when $\mathbf{K}$ is clear from context.

Remark 3.3. *In Definition 3.2, intuitively, p_δ is the unique $\underset{\smile}{\perp}$ non-forking type extending p_0 over $\bigcup_{i<\delta} M_i$ - the relation $\underset{\smile}{\perp}$ is not necessarily defined on this model, but if it could be extended, p_δ would be the only choice for the non-forking extension.*

Definition 3.4. *Let $\mathbf{K}$ be an AEC stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$, and $\kappa \leq \lambda$ be infinite and regular. Let $\underset{\smile}{\perp}$ be an independence relation on some sub-AC $\mathbf{K}'$ of $\mathbf{K}_\lambda$ where $K_{(\lambda, \geq \kappa)} \subseteq K'$ (normally $\mathbf{K}' = \mathbf{K}$ or $\mathbf{K}' = (K_{(\lambda, \geq \kappa)}, \leq_{\mathbf{K}})$).*

We say that $\underset{\smile}{\perp}$ has $K_{(\lambda, \geq \kappa)}$-universal continuity* in $\mathbf{K}$ if $\underset{\smile}{\perp}$ restricted to models in $K_{(\lambda, \geq \kappa)}$ has universal continuity* (in $\mathbf{K}$).

Notation 3.5. *We will often omit the 'in $\mathbf{K}$' part of the definition as in most cases $\mathbf{K}$ is clear from context.*

Remark 3.6. *The reason we use $K_{(\lambda, \geq \kappa)}$-universal continuity* rather than 'standard' universal continuity is to accommodate Example 2.29 and the setup of Subsection 3.4, where the relation behaves well on $K_{(\lambda, \geq \kappa)}$ but not on all models (or even all limit models). In a sense, this is the closest to continuity we can get when $\underset{\smile}{\perp}$ is only defined on $K_{(\lambda, \geq \kappa)}$. We formalise this to some degree in Lemma 3.11 and Lemma 3.12.*

Now we specify the conditions our non-forking relation needs to satisfy to apply Theorem 3.1.

Hypothesis 3.7. *Let $\mathbf{K}$ be an AEC stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$, with AP, JEP, and NMM in $\mathbf{K}_\lambda$. Let $\kappa < \lambda^+$ be a regular cardinal. Let $\underset{\smile}{\perp}$ be an independence relation on $(K_{(\lambda, \geq \kappa)}, \leq_{\mathbf{K}})$ that satisfies uniqueness, extension, non-forking amalgamation, $(\geq \kappa)$-local character, and $K_{(\lambda, \geq \kappa)}$-universal continuity* in $\mathbf{K}$.*

Remark 3.8. *If we made the same assumptions on a relation $\underset{\smile}{\perp}$ defined on all of $\mathbf{K}_\lambda$ (or any sub-AC $\mathbf{K}'$ with $K_{(\lambda, \geq \kappa)} \subseteq K' \subseteq K_\lambda$), the restriction to $K_{(\lambda, \geq \kappa)}$ satisfies Hypothesis 3.7. This is immediate for each property besides non-forking amalgamation - in that case, just note that the 'largest' model N can be replaced by a $(\lambda, \geq \kappa)$-limit model extending the original model.*

Remark 3.9. *For $M, N \in K_{(\lambda, \geq \kappa)}$, N is universal over M in $(K_{(\lambda, \geq \kappa)}, \leq_{\mathbf{K}})$ if and only if N is universal over M in $\mathbf{K}$, so for the properties involving $\leq_{(K_{(\lambda, \geq \kappa)}, \leq_{\mathbf{K}})}^u$ ($K_{(\lambda, \geq \kappa)}$-universal continuity*, local character) hold with the usual $\leq_{\mathbf{K}}^u$. Hence we can use $\leq_{\mathbf{K}}^u$ instead unambiguously when dealing with ACs $\mathbf{K}'$ with $K_{(\lambda, \geq \kappa)} \subseteq K' \subseteq K$.*

Before proving Theorem 3.1, we explore the assumptions in Hypothesis 3.7 and consider some examples that satisfy it.

Lemma 3.10. *Suppose $\mathbf{K}$ is an AEC, with sub-AC $\mathbf{K}'$ where $K_{(\lambda, \geq \kappa)} \subseteq K'$. Suppose $\underset{\smile}{\perp}$ is an independence relation on $\mathbf{K}'$ satisfying $(\geq \kappa)$-local character. If $M \in K_{(\lambda, \geq \kappa)}$ and $p \in \mathbf{gS}(M)$, then $p \underset{\smile}{\perp}$-does not fork over M.*

In particular, assuming Hypothesis 3.7, $\underset{\smile}{\perp}$ satisfies existence.

Proof. Let $\langle M_i : i \leq \delta \rangle$ witness that M is a $(\lambda, \geq \kappa)$-limit. Then by $(\geq \kappa)$-local character, $p \underset{\smile}{\perp}$-does not fork over M_i for some $i < \delta$. By base monotonicity, $p \underset{\smile}{\perp}$-does not fork over M. $\square$

The following lemmas show the relationship between full universal continuity and our surrogate $K_{(\lambda, \geq \kappa)}$-universal continuity* form.

Lemma 3.11. *Suppose $\mathbf{K}$ is an AEC with λ-AP, and $\underset{\smile}{\perp}$ is an independence relation on a sub-AC $\mathbf{K}'$ of $\mathbf{K}_\lambda$ where $K_{(\lambda, \geq \kappa)} \subseteq K'$ satisfying extension, uniqueness, and $K_{(\lambda, \geq \kappa)}$-universal continuity* in $\mathbf{K}$. Then $\underset{\smile}{\perp}$ has $(\geq \kappa)$-universal continuity.*

Proof. Suppose $\langle M_i : i < \delta \rangle$ is a $\leq^u_{\mathbf{K}}$-increasing chain in K', and $p \in \mathbf{gS}(\bigcup_{i<\delta} M_i)$ is such that $p \restriction M_i \underset{\smile}{\perp}$-does not fork over M_0. We must show $p \underset{\smile}{\perp}$-does not fork over M_0.

Note $\bigcup_{i<\delta} M_i \in K_{(\lambda, \geq \kappa)} \subseteq K'$. So by extension, there exists $q \in \mathbf{gS}(\bigcup_{i<\delta} M_i)$ such that $q \underset{\smile}{\perp}$-does not fork over M_0 and $q \restriction M_0 = p \restriction M_0$. By uniqueness, $p \restriction M_i = q \restriction M_i$ for $i < \delta$. For $i < \delta$, take $N_i \in K_{(\lambda, \geq \kappa)}$ such that $M_i \leq^u_{\mathbf{K}} N_i \leq^u_{\mathbf{K}} M_{i+1}$ (this is possible as $\langle M_i : i < \delta \rangle$ is $\leq^u_{\mathbf{K}}$-increasing). We now have $\langle N_i : i < \delta \rangle \leq^u_{\mathbf{K}}$-increasing where $p \restriction N_i = q \restriction N_i$ for all $i < \delta$ and $\bigcup_{i<\delta} N_i = \bigcup_{i<\delta} M_i$. So $p = q$ by $K_{(\lambda, \geq \kappa)}$-universal continuity* in $\mathbf{K}$. Therefore $p \underset{\smile}{\perp}$-does not fork over M_0 as desired. $\square$

Lemma 3.12. *Suppose $\mathbf{K}$ is an AEC with λ-AP, and $\underset{\smile}{\perp}$ is an independence relation on a sub-AC $\mathbf{K}'$ of $\mathbf{K}_\lambda$ where $K_{(\lambda, \geq \aleph_0)} \subseteq K'$ satisfying extension and uniqueness.*

Then $\underset{\smile}{\perp}$ satisfies universal continuity if and only if $\underset{\smile}{\perp}$ satisfies $K_{(\lambda, \geq \aleph_0)}$-universal continuity in $\mathbf{K}$.*

Proof. First, assume $\underset{\smile}{\perp}$ satisfies universal continuity. Suppose $\delta < \lambda^+$, $\langle M_i : i < \delta \rangle$ a $\leq^u_{\mathbf{K}}$-increasing sequence in $K_{(\lambda, \geq \aleph_0)}$, and $\langle p_i \in \mathbf{gS}(M_i) : i < \delta \rangle$ is an increasing sequence of types where $p_i \underset{\smile}{\perp}$-does not fork over M_0. Let $M_\delta = \bigcup_{i<\delta} M_i$. The result is trivial if δ is 0 or successor, so assume δ is limit. Since $M_\delta \in \mathbf{K}_{(\lambda, \geq \aleph_0)}$, $M \in \mathbf{K}'$, and we can take any $p \in \mathbf{gS}(M_\delta)$ extending p_0 which $\underset{\smile}{\perp}$-does not fork over M_0 by extension. By uniqueness, since $p \restriction M_i$ and p_i are both non-forking extensions of p_0, $p \restriction M_i = p_i$. And p is the unique extension of the p_i's: if also $q \in \mathbf{gS}(M_\delta)$ extends p_i for all $i < \delta$, universal continuity gives that $q \underset{\smile}{\perp}$-does not fork over M_0. $p \restriction M_0 = p_0 = q \restriction M_0$, so by uniqueness, $p = q$. Therefore p is the unique non-forking extension as desired.

The converse follows from Lemma 3.11 with $\kappa = \aleph_0$. $\square$

Our motivating examples satisfying Hypothesis 3.7 are the following:

Example 3.13.

(1) *Suppose $\underset{\smile}{\perp}$ is a stable independence relation in the LRV sense with universal continuity on an AEC $\mathbf{K}$, and $\lambda \geq \mathrm{LS}(\mathbf{K})$ is a stability cardinal where $\mathbf{K}_\lambda$ has JEP and NMM.*

Then the restriction of $\overline{\underset{\smile}{\perp}}$ to singletons and $(\lambda, \geq \kappa)$-limit models satisfies Hypothesis 3.7 with $\kappa = \kappa(\underset{\smile}{\perp}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$, by Lemma 2.26. Note that all the non-forking properties transfer down to this restriction by Lemma 3.8, and $K_{(\lambda, \geq \kappa)}$-universal continuity in $\mathbf{K}$ follows from universal continuity of $\underset{\smile}{\perp}$ and Lemma 3.12.*

(2) *Example 2.28 with $\kappa = \aleph_0$.*
(3) *Example 2.29 with $\kappa = \kappa$.*
(4) *Example 2.31 under mild assumptions- see Subsection 3.4.*

Remark 3.14. *In the setting of Example 3.13(2), the main result of this section (Theorem 3.1) is already known (proved originally in [Van16a], see also [Vas, 15.8]). This section is a generalisation of the method used in [Vas].*

Remark 3.15. *Of our examples, only Example 3.13(3) and Example 3.13(4) need $K_{(\lambda, \geq \kappa)}$-universal continuity* in $\mathbf{K}$ rather than regular universal continuity of non-forking. In the other cases, $\bigsmile$ is defined on all limit models, so we don't have to worry about whether $\bigsmile$ is well defined when taking unions of towers, and we could more closely mimic the approach of [Vas, 16.17]. In this sense, Example 3.13(3) and Example 3.13(4) (see Corollary 3.46) are our main examples that use the full strength of Theorem 3.1.*

3.1. **Towers.** We assume Hypothesis 3.7 throughout this subsection. Note since AP, JEP, and NMM hold in $\mathbf{K}_\lambda$ under Hypothesis 3.7, the Facts 2.6 and 2.7 apply, and $\bigsmile$ satisfies existence by Lemma 3.10.

In this subsection we start working towards a proof of Theorem 3.1. We begin by introducing a notion of *towers.*

Towers were introduced by Villaveces and Shelah in [ShVi99] in their attempt to prove uniqueness of limit models under certain assumptions involving categoricity. Their towers were comprised of two increasing chains of models and a list of singletons - the singletons' types over the big models did not λ-split over the smaller ones. This is also the version of tower used in [Van02], [Van06], [Van13], [GVV16], [Van16a], [Van16b], and [BoVan24]. In [Vas] and [Vas19], Vasey simplified the presentation of the argument using a different form of tower. First, he made use of λ-non-forking, which has stronger properties than λ-non-splitting on limit models. Also, he took out the models the types do not fork over, and instead captured the λ-non-forking of the singletons in the tower ordering. Our approach is heavily based off [Vas19], but with looser assumptions. The main differences are at the points that local character and universal continuity appear in Vasey's proof. In particular, in our setting unions of $\lhd$-increasing towers of length $< \kappa$ may not be towers, so we adapt Vasey's tower extension lemma [Vas19, 16.17] to allow us to find a towers that extend a chain of towers, rather than just a single tower (see Proposition 3.24). We will also fill in the details of some results given without proof, and generalise statements and proofs that involve our weakened assumptions.

Notation 3.16. *Let I be well ordered by $\leq_I$.*
   (1) *We use I^- to denote*
       • *I if I has 0 or limit order type (i.e. it has no final element)*
       • *$I \setminus \{i\}$ if I has successor order type where i is the final element of I.*
   (2) *We use $i +_I 1$ to denote the successor of $i \in I$ in the ordering $\leq_I$ of I, if it exists. When unambiguous we may write $i + 1$.*
   (3) *Given $r, s \in I$, we use $[r, s)_I$ to denote the interval of all $i \in I$ with $r \leq_I i <_I s$.*

That is, I^- removes the final element if it exists. When clear from context, we omit the subscript and write $<$ and $\leq$ for $<_I$ and $\leq_I$.

Our version of tower is almost the same as [Vas19, 5.4], differing only in that we require all models to be $(\lambda, \geq \kappa)$-limits rather than just limit models (so they are compatible with $\bigsmile$).

Definition 3.17. *A tower is a sequence $\mathcal{T} = \langle M_i : i \in I \rangle ^\frown \langle a_i : i \in I^- \rangle$ where*
   (1) *I is a well ordered set with $\text{otp}(I) < \lambda^+$*
   (2) *$\langle M_i : i \in I \rangle$ is a $\leq_{\mathbf{K}}$-increasing sequence of models in $K_{(\lambda, \geq \kappa)}$*

(3) for all $i \in I^-$, $a_i \in |M_{i+1}| \setminus |M_i|$.

Given such $I, \mathcal{T}$:

(1) We call I the index *of the tower $\mathcal{T}$.*

(2) If $I_0 \subseteq I$, then define $\mathcal{T} \restriction I_0 = \langle M_i : i \in I \in I_0 \rangle^\wedge \langle a_i : i \in (I_0)^- \rangle$.

(3) Given $i \in I$ limit, we say $\mathcal{T}$ is continuous *at i if $M_i = \bigcup_{r<i} M_r$.*

(4) $\mathcal{T}$ is universal *if $\langle M_i : i \in I \rangle$ is universal; that is, for all $i \in I^-$, $M_i \leq^u_{\mathbf{K}} M_{i+1}$.*

(5) $\mathcal{T}$ is strongly universal *if $\langle M_i : i \in I \rangle$ is strongly universal; that is, for all non-initial $i \in I$, $\bigcup_{r<i} M_r \leq^u_{\mathbf{K}} M_i$.*

Remark 3.18. *It is straightforward to see that given $M \in K_{(\lambda, \geq \kappa)}$, there is a strongly universal tower $\mathcal{T} = \langle M_i : i \in \alpha \rangle^\wedge \langle a_i : i \in \alpha^- \rangle$ where $M = M_0$ for any $\alpha < \lambda^+$ (or indeed any well ordering I with $\mathrm{otp}(I) < \lambda^+$): M_0 is given; if you have M_i take $\mathbf{gtp}(a_i/M_i, M'_{i+1})$ to be any non-algebraic type (which is possible by NMM) and take M_{i+1} to be any (λ, κ)-limit models over M'_{i+1}; at limits take M_i to be a (λ, κ)-limit over $\bigcup_{k<i} M_k$. Since our models are all in $K_{(\lambda, \geq \kappa)}$ rather than simply limit models, it is not obvious if continuous towers exist (that is, towers continuous at every limit $i \in I$). However we will be able to guarantee continuity at limits with high cofinality (e.g. take unions at these steps in the above construction, or use Lemma 3.31 and Proposition 3.35).*

Definition 3.19 ([Vas19, 5.7])**.** *Let $\mathcal{T} = \langle M_i : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$, $\mathcal{T}' = \langle M'_i : i \in I' \rangle^\wedge \langle a'_i : i \in (I')^- \rangle$ be towers. We define the* tower ordering *$\lhd$ by $\mathcal{T} \lhd \mathcal{T}'$ if and only if*

(1) $I \subseteq I'$.

(2) for all $i \in I$, M'_i is universal over M_i

(3) for all $i \in I^-$, $a'_i = a_i$

(4) for all $i \in I^-$, we have $\mathbf{gtp}(a_i/M'_i, M'_{i+_{I'}1}) \underset{\smile}{\quad}$-does not fork over M_i.

Remark 3.20. *$\lhd$ is a strict partial order - for transitivity of $\lhd$, (1) and (3) are clear, (2) follows from transitivity of $\leq^u_{\mathbf{K}}$, and (4) follows from transitivity of $\underset{\smile}{\quad}$-non-forking.*

In our setting, arbitrary unions of towers may no longer be towers - for example, the models of a union of ω many towers will be (λ, ω)-limit models, so this union may not be a tower if $\kappa \geq \aleph_1$. However, we can take unions of high cofinality, and these interact well with the tower ordering. Lemma 3.22 is our slight improvement on [Vas19, 5.10], and we'll sketch the details here:

Definition 3.21. *Suppose $\gamma < \lambda^+$ is a limit ordinal where $\mathrm{cf}(\gamma) \geq \kappa$ and $\langle \mathcal{T}^j : j < \gamma \rangle$ is a $\lhd$-increasing sequence of towers where $\mathcal{T}^j = \langle M^j_i : i \in I^j \rangle^\wedge \langle a^j_i : i \in (I^j)^- \rangle$ for $j < \gamma$, and $\bigcup_{j<\gamma} I^j$ is a well ordering. Define $\bigcup_{j<\gamma} \mathcal{T}^j = \langle M^\gamma_i : i \in I^\gamma \rangle^\wedge \langle a^\gamma_i : i \in (I^\gamma)^- \rangle$ where*

(1) $I^\gamma = \bigcup_{j<\gamma} I^j$

(2) for all $i \in I^\gamma$, $M^\gamma_i = \bigcup \{ M^j_i : j < \gamma \text{ such that } i \in I^j \}$

(3) for all $i \in (I^\gamma)^-$, a^γ_i is any a^j_i for which $i \in (I^j)^-$ (note the choice does not matter by the definition of the tower ordering).

Lemma 3.22. *Suppose $\gamma < \lambda^+$ is a limit ordinal where $\mathrm{cf}(\gamma) \geq \kappa$ and $\langle \mathcal{T}^j : j < \gamma \rangle$ is a $\lhd$-increasing sequence of towers where $\mathcal{T}^j$ is indexed by I^j for $j < \gamma$, and $\bigcup_{j<\gamma} I^j$ is a well ordering. Then $\bigcup_{j<\gamma} \mathcal{T}^j$ is a tower, and for all $k < \gamma$, $\mathcal{T}^k \lhd \bigcup_{j<\gamma} \mathcal{T}^j$*

Proof. Let $\mathcal{T}^\gamma = \bigcup_{j<\gamma} \mathcal{T}^j$, and say $\mathcal{T}^j = \langle M^j_i : i \in I^j \rangle^\wedge \langle a^j_i : i \in (I^j)^- \rangle$ for $j \leq \gamma$. Since $I^\gamma = \bigcup_{j<\gamma} I^j$ is a well ordering, and the other conditions are preserved by the unions of length δ, $\mathcal{T}^\gamma$ is a tower (in particular, as $\mathrm{cf}(\delta) \geq \kappa$, we have that M^γ_i is a $(\lambda, \geq \kappa)$-limit model for each i).

Suppose $k < \gamma$. We check the conditions for $\mathcal{T}^k \lhd \mathcal{T}^\gamma$ from Definition 3.19:

(1) $I^k \subseteq I^\gamma$ by the definition of I^γ.

(2) Let $i \in I^k$. Then M_i^{k+1} is universal over M_i^k. Since $M_i^{k+1} \leq_{\mathbf{K}} M_i^\gamma$, M_i^γ is also universal over M_i^k.

(3) For all $i \in (I^k)^-$, by definition $a_i^k = a_i^\gamma$.

(4) Let $i \in (I^k)^-$. Then $\mathbf{gtp}(a_i^\gamma / M_i^j, M_{i+_I j^1}^j) \perp$-does not fork over M_i^k for all $j \in [k, \gamma)$ by the definition of $\lhd$. By Lemma 3.11, $\mathbf{gtp}(a_i^\gamma / M_i^\gamma, M_{i+_{I^\gamma}1}^\gamma) \perp$-does not fork over M_i^k.

So $\mathcal{T}^k \lhd \mathcal{T}^\gamma$, as desired. $\qquad\square$

To prove the Theorem 3.1, we will construct a $(\delta_1 + 1)$-long $\lhd$-increasing sequence of towers, continuous at δ_1, where the towers' indexes contain a copy of $(\delta_2 + 1)$, and the final tower's models indexed by this copy will be $\leq_{\mathbf{K}}^u$-increasing and continuous at 'δ_2' (in the copy of $(\delta_2 + 1)$). This will guarantee the 'δ_2'th model in the final tower will be both a (λ, δ_1)-limit model and (λ, δ_2)-limit model.
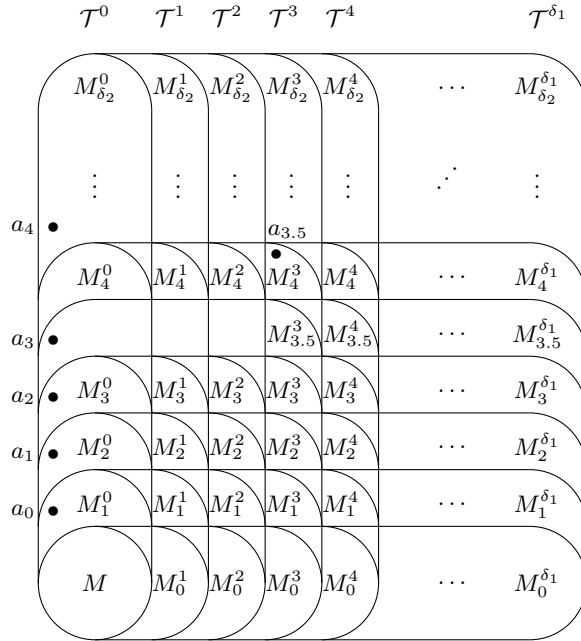
The above diagram captures the intuition of this construction - the jth column represents the tower $\mathcal{T}^j$, and the ith row consists of the models of the ith levels of each tower. The top row and final column will be continuous and universal, and hence witness that the largest model $M_{\delta_2}^{\delta_1}$ is both a (λ, δ_1) and (λ, δ_2)-limit model respectively. As we go further along the chain of towers, new levels will be inserted between the 'main' levels of the towers (see where row 3.5 is introduced in $\mathcal{T}^3$). This will become important when we examine *full towers*, and is why we index towers by arbitrary well orderings I instead of ordinals.

First we will show that strict extensions of towers exist. Towards that goal, we use the following fact, which is based on [Vas, 13.16(8)]. We will only need it for Proposition 3.24, Proposition 3.25, and Proposition 3.27, to verify the types of the a_i's in the extending towers are still non-algebraic.

Lemma 3.23 (Disjointness)**.** *If* $M, N \in K_{(\lambda, \geq \kappa)}$, $p \in \mathbf{gS}(N) \perp$-*does not fork over* M, *and* $p \upharpoonright M$ *is non-algebraic, then* p *is non-algebraic.*

Proof. Fix $\langle M_i : i < \delta \rangle$ witnessing that M is (λ, δ)-limit over M_0 where $\delta \geq \kappa$ is regular. Without loss of generality, for every $i < \delta$, M_i is a (λ, κ)-limit model. To see this, construct a sequence N_i where N_0 is (λ, κ)-limit and N_i is (λ, κ)-limit over $\bigcup_{i' < i} N_{i'}$ for all $0 < i < \delta$; the union of these models is a (λ, δ)-limit, hence isomorphic to M, so replace each M_i by the copy of N_i in M.

By $(\geq \kappa)$-local character of $\underset{\smile}{}$, there exists $i < \delta$ such that $p \restriction M \underset{\smile}{}$-does not fork over M_i. By transitivity, $p \underset{\smile}{}$-does not fork over M_i.

Find some N^* that is (λ, δ)-limit over N, hence over M_i also. By extension, there is $q \in \mathbf{gS}(N^*)$ extending $p \restriction M_i$ such that $q \underset{\smile}{}$-does not fork over M_i. By uniqueness, $q \restriction N = p$. By Fact 2.7, there is an isomorphism $f : N^* \underset{M_i}{\rightarrow} M$.

So $f(q) \in \mathbf{gS}(M)$ does not fork over M_i and $f(q) \restriction M_i = q \restriction M_i = p \restriction M_i$. By uniqueness, $f(q) = p \restriction M$. Since $p \restriction M$ is non-algebraic, and this property is preserved by isomorphism, q is non-algebraic, and therefore $p = q \restriction N$ is non-algebraic also as desired. $\square$

The following three results are based on [Vas, 16.17] and allow us to extend towers in various ways. The first is similar to [Vas, 16.17(1)] but differs in that we take an extension of a whole $\lhd$-increasing chain of towers rather than a single tower. This will be necessary as we cannot take arbitrary unions of towers under our assumptions, unlike [Vas].

Proposition 3.24. *Suppose $1 \leq \alpha < \lambda^+$ and $\langle \mathcal{T}^j : j < \alpha \rangle$ is a $\lhd$-increasing sequence of towers, where $\mathcal{T}^j = \langle M_i : i \in I^j \rangle^\wedge \langle a_i : i \in (I^j)^- \rangle$. Suppose $I = \bigcup_{j < \alpha} I^j$ is well-ordered. Then there exists a strongly universal tower $\mathcal{T}'$ indexed by I such that $\mathcal{T}^j \lhd \mathcal{T}'$ for each $j < \alpha$.*

Proof. For $i \in I$, use M_i^α to denote $\bigcup \{ M_i^j : j < \alpha, i \in I^j \}$, and let i_0 be minimal in I. Note M_i^α may not be in $K_{(\lambda, \geq \kappa)}$ if α is a limit, but if α is not a limit, say $\alpha = \beta + 1$, then $M_i^\alpha = M_i^\beta$. We define recursively on I a $\leq_{\mathbf{K}}$-increasing sequence of models $\langle N_i : i < \alpha \rangle$ in $K_{(\lambda, \geq \kappa)}$ and a $\subseteq$-increasing sequence of $\mathbf{K}$-embeddings $\langle f_i : i < \alpha \rangle$ such that

(1) for all $i \in I$, $f_i : M_i^\alpha \to N_i$
(2) $f_{i_0} = \mathrm{id}_{M_{i_0}^\alpha}$
(3) for all $i \in I$, N_i is universal over $f_i[M_i^\alpha]$
(4) for all $i \in I \setminus \{i_0\}$, N_i is universal over $\bigcup_{j < i} N_j$.
(5) for all $j < \alpha$ and $i \in (I^j)^-$, $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1}) \underset{\smile}{}$-does not fork over $f_i[M_i^j]$.

This is enough: Let $f = \bigcup_{i \in I} f_i : \bigcup_{i \in I} M_i^\alpha \to \bigcup_{i \in I} N_i$, and extend this to an isomorphism $g : M' \cong \bigcup_{i \in I} N_i$. Let $M_i' = g^{-1}[N_i]$ for each $i < \alpha$. Let $\mathcal{T}' = \langle M_i' : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$.

By (5), for all $j < \alpha$ and $i \in (I^j)^-$, $\mathbf{gtp}(a_i/M_i', M_{i+1}') \underset{\smile}{}$-does not fork over M_i^j. For each $i \in I^-$ there is $j < \alpha$ where $i, i +_I 1 \in I^j$ and $a_i \notin M_i^j$, so by Lemma 3.23 we have that $a_i \in |M_{i+1}'| \setminus |M_i'|$. So $\mathcal{T}'$ is a tower, and by (4) it is a strong limit tower. By (3), for all $j < \alpha$, for every $i \in I^j$, M_i' is universal over M_i^j. So $\mathcal{T}^j \lhd \mathcal{T}'$ as desired.

This is possible: For $\underline{i = i_0}$, we can take N_{i_0} (λ, κ)-limit over $M_{i_0}^\alpha$ (hence universal over M_{i_0} also), and $f_{i_0} = \mathrm{id}_{M_{i_0}^\alpha}$ as specified by (2).

For the $\underline{\text{successor step}}$, suppose we have $f_{i'} : M_{i'}^\alpha \to N_{i'}$ defined for $i' \leq i \in I$. Extend f_i to an embedding $\bar{f}_i : M_{i+1}^\alpha \to \bar{N}_i$ where $N_i \leq \bar{N}_i$ (this is possible by AP).

Let j_i be the least $j < \alpha$ such that $i \in (I^j)^-$. For $j \in [j_i, \alpha]$, let $q_j = \mathbf{gtp}(\bar{f}_i(a_i)/f_i[M_i^j], \bar{f}_i[M_{i+_I 1}^j])$. The tower ordering tells us that $q_j \underset{\smile}{}$-does not fork over $f_i[M_i^{j_i}]$ for $j \in [j_i, \alpha)$, so in particular these types all extend q_{j_i} and increase with j by uniqueness.

By extension, there is $p \in \mathbf{gS}(N_i)$ extending q_{j_i} that does not fork over $f_i[M_i^{j_i}]$. By uniqueness, $p \restriction f_i[M_i^j] = q_j$ for all $j \in [j_i, \alpha)$. If α is limit, by $K_{(\lambda, \geq \kappa)}$-universal continuity*, $p \restriction f_i[M_i^\alpha] = q_\alpha$ (note that $K_{(\lambda, \geq \kappa)}$-universal continuity* does not impose restrictions on the cofinality of α).

If α is not limit, then $\alpha = \beta + 1$ for some β, and we have $M_i^\alpha = \bigcup_{j<\alpha} M_i^j = M_i^\beta$, and $p \restriction f_i[M_i^\alpha] = q_\beta = q_\alpha$ by uniqueness. In either case, $p \restriction M_i^\alpha = q_\alpha$.

Say $p = \mathbf{gtp}(b/N_i, \hat{N}_i)$. Using $p \restriction f_i[M_i^\alpha] = q_\alpha$ and AP, there is an embedding $g_i : \bar{N}_i \to N'_{i+1}$ where $\hat{N}_i \leq_{\mathbf{K}} N'_{i+1}$ and $g_i(\bar{f}_i(a_i)) = b$. Let N_{i+1} be any (λ, κ)-limit over N'_{i+1}. Set $f_{i+1} = g_i \circ \bar{f}_i : M_{i+1}^\alpha \to N_{i+1}$.
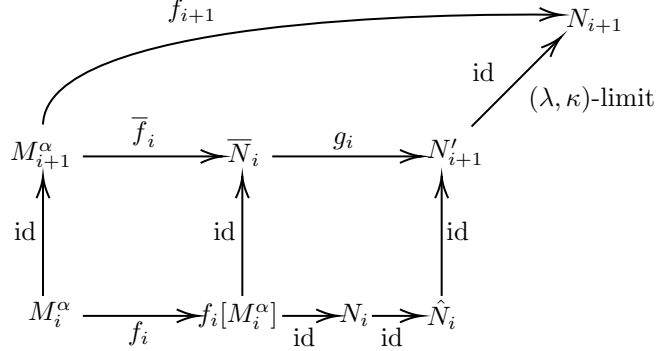
Note, f_{i+1} extends f_i, and (5) holds: since $g_i(\bar{f}_i(a_i)) = b$, we have $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1}) = \mathbf{gtp}(g_i(\bar{f}_i(a_i))/N_i, N_{i+1}) = \mathbf{gtp}(b/N_i, \hat{N}_i) = p$, which $\underset{\smile}{}$-does not fork over $f_i[M_i^{j_i}]$. Hence $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1}) \underset{\smile}{}$-does not fork over $f_i[M_i^j]$ for all $j \in [j_i, \alpha)$ by base monotonicity. So f_{i+1} fulfills the inductive hypothesis.

At limit steps the proof is simpler as we do not have to worry about non-forking of the a_i's. Given $i \in I$ limit and $f_{i'}$ for all $i' < i \in I$ satisfying the inductive hypothesis, let $\hat{f}_i = \bigcup_{i'<i} f_{i'} : \bigcup_{i'<i} M_{i'}^\alpha \to \bigcup_{i'<i} N_{i'}$. Extend this to an embedding $f_i : M_i^\alpha \to N_i^0$ where $\bigcup_{i'<i} N_{i'} \leq_{\mathbf{K}} N_i^0$, using AP. Now find N_i a (λ, κ)-limit over N_i^0. Then $f_i : M_i^\alpha \to N_i$ satisfies the induction hypothesis. This completes the recursion, and the proof. $\square$

The following is the analogue in our context of [Vas, 16.17(2)]. It is the only place we will use non-forking amalgamation of $\underset{\smile}{}$. The proof is very similar to the last result - in fact, in [Vas] they are collected into a single proof. However we keep the results separate to avoid overcomplicating the statement - here we only need to extend a single tower.

We assume the index is an ordinal, but note it is also true (with some relabeling) for a general well ordered set I, as are all the following results where the index is an ordinal - we only use ordinals for notational convenience.

Proposition 3.25. *Suppose* $\mathcal{T} = \langle M_i : i < \beta \rangle^\wedge \langle a_i : i \in \beta^- \rangle$ *is a tower. Suppose in addition* $p \in \mathbf{gS}(M_0)$. *Then there exists a strongly universal tower* $\mathcal{T}' = \langle M_i' : i < \beta \rangle^\wedge \langle a_i : i \in \beta^- \rangle$ *and* $b \in M_0'$ *such that* $\mathcal{T} \lhd \mathcal{T}'$, $\mathbf{gtp}(b/M_0, M_0') = p$, *and for all* $i < \beta$, $\mathbf{gtp}(b/M_i, M_i') \underset{\smile}{}$-*does not fork over* M_0.

Proof. Let $N \in K_\lambda$ and $c \in N$ be such that $p = \mathbf{gtp}(c/M_0, N)$. As before, we recursively define a $\leq_{\mathbf{K}}$-increasing sequence of models $\langle N_i : i < \beta \rangle$ and a $\subseteq$-increasing sequence of $\mathbf{K}$-embeddings $\langle f_i : i < \beta \rangle$ such that

    (1) for all $i < \beta$, $f_i : M_i \to N_i$
    (2) $f_0 = \mathrm{id}_{M_0}$ (in particular, $M_0 \leq_{\mathbf{K}} N_0$), $N \leq_{\mathbf{K}} N_0$, and $c \in N_0$
    (3) for all $i < \beta$, N_i is universal over $f_i[M_i]$
    (4) for all $i \in [1, \beta)$, N_i is universal over $\bigcup_{r<i} N_r$
    (5) for all $i \in \beta^-$, $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1}) \underset{\smile}{}$-does not fork over $f_i[M_i]$
    (6) for all $i < \beta$, $\mathbf{gtp}(c/f_i[M_i], N_i) \underset{\smile}{}$-does not fork over M_0

This is enough: As before, extending $\bigcup_{i<\beta} f_i$ to an isomorphism $g : M' \cong \bigcup_{i<\beta} N_i$, setting $M_i' = g^{-1}[N_i]$ for $i < \beta$, and taking $\mathcal{T}' = \langle M_i' : i < \beta \rangle^\frown \langle a_i : i \in \beta^- \rangle$, we get $\mathcal{T}'$ is a tower and $\mathcal{T} \lhd \mathcal{T}'$ (just as in the last proof, with $\alpha = 1$). Further if we set $b = g^{-1}(c)$, we have $b \in M_0'$, $p = \mathbf{gtp}(b/M_0, M_0')$, and by condition (6), $\mathbf{gtp}(b/M_i, M_i') \downarrow$-does not fork over M_0 for all $i < \beta$ as desired.

This is possible: For $\underline{i = 0}$, we can take N_0 a (λ, κ)-limit over N (hence universal over M_0 also), and $f_0 = \mathrm{id}_{M_0}$ as specified.

For the $\underline{\text{successor step}}$, suppose we have defined N_i and f_i satisfying conditions (1)-(6), and must define f_{i+1} and N_{i+1}. By non-forking amalgamation of $\downarrow$, there exists $N_{i+1}' \in K_{(\lambda, \geq \kappa)}$ and $f_{i+1} : M_{i+1} \to N_{i+1}'$ such that $N_i \leq_{\mathbf{K}} N_{i+1}'$, $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1}')$ $\downarrow$-does not fork over $f_i[M_i]$, and $\mathbf{gtp}(c/f_{i+1}[M_{i+1}], N_{i+1})$ $\downarrow$.

Take N_{i+1} a (λ, κ)-limit over N_{i+1}', hence also over both $f_{i+1}[M_{i+1}]$ and N_i.

$$
\begin{array}{ccc}
& & N_{i+1} \\
f_{i+1} \nearrow & & \uparrow \mathrm{id} \\
& & (\lambda,\kappa)\text{-limit} \\
M_{i+1} & \xrightarrow{\ f_{i+1}\ } & N_{i+1}' \\
\mathrm{id} \uparrow & & \uparrow \mathrm{id} \\
M_i & \xrightarrow{\ f_i\ } & N_i
\end{array}
$$

By monotonicity, $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1})$ $\downarrow$-does not fork over $f_i[M_i]$, and $\mathbf{gtp}(c/f_{i+1}[M_{i+1}], N_{i+1})$ $\downarrow$-does not fork over $f_i[M_i]$. Since $\mathbf{gtp}(c/f_i[M_i], N_{i+1})$ $\downarrow$-does not fork over M_0 by the induction hypothesis and monotonicity, $\mathbf{gtp}(c/f_{i+1}[M_{i+1}], N_{i+1})$ $\downarrow$-does not fork over M_0 by transitivity. At this point we have shown N_{i+1} and $f_{i+1} : M_{i+1} \to N_{i+1}$ satisfy the relevant conditions (1)-(6).

For $\underline{\text{limit ordinals}}$ i, we have N_r and f_r for all $r < i$ and must construct N_i, f_i. Let $M_i^0 = \bigcup_{r<i} M_r$, $N_i^0 = \bigcup_{r<i} N_r$ and $f_i^0 = \bigcup_{r<i} f_r : M_i^0 \to N_i^0$. Let $g_i : M_i \to N_i'$ be any extension of f_i^0 with domain M_i and where $N_i^0 \leq_{\mathbf{K}} N_i'$ (this exists by AP). Take any $q \in \mathbf{gS}(g_i[M_i])$ extending p which $\downarrow$-does not fork over M_0. Since $\mathbf{gtp}(c/g_i[M_r], N_i^0)$ $\downarrow$-does not fork over M_0 for all $r < i$, by uniqueness $q \restriction g_i[M_r] = \mathbf{gtp}(c/g_i[M_r], N_i^0)$ for each $r < i$. By $K_{(\lambda, \geq \kappa)}$-universal continuity*, $q \restriction g_i[M_i^0] = \mathbf{gtp}(c/g_i[M_i^0], N_i^0)$. Let $\hat{N}_i \in K_\lambda$ and $d \in \hat{N}_i$ be such that $N_i' \leq_{\mathbf{K}} \hat{N}_i$ and $q = \mathbf{gtp}(d/g_i[M_i], \hat{N}_i)$. Then by type equality there is some $\bar{N}_i \in K_\lambda$ where $N_i' \leq_{\mathbf{K}} \bar{N}_i$ and some $h_i : \hat{N}_i \to \bar{N}_i$ such that h_i is the identity on M_i^0 and $h_i(d) = c$. Let N_i be a (λ, κ)-limit over $\bar{N}_i$.
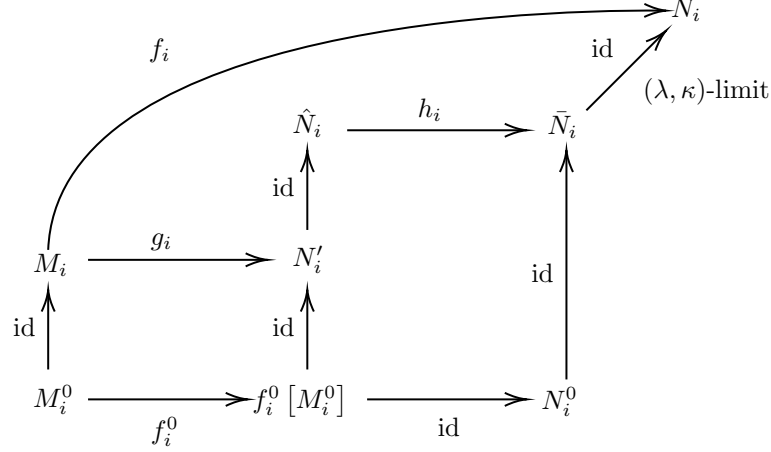
Then setting $f_i = h_i \circ g_i : M_i \to N_i$, we have that

$$\mathbf{gtp}(c/f_i[M_i], N_i) = \mathbf{gtp}(h_i(d)/h_i[g_i[M_i]], N_i) = \mathbf{gtp}(h_i(d)/h_i[g_i[M_i]], h_i[\hat{N}_i])$$

$$= h_i(\mathbf{gtp}(d/g_i[M_i], \hat{N}_i)) = h_i(q)$$

which $\underset{\smile}{\ }$-does not fork over $h_i[M_0] = M_0$, satisfying (6). Further, f_i extends $f_{i'}$ for all $i' < i$ as g_i extends f_i^0 and h_i fixes $f_i^0[M_i^0]$. As N_i is (λ, κ)-limit over $\bar{N}_i$, which contains both N_i^0 and $f_i[M_i]$, (3) and (4) are satisfied. So all the relevant conditions hold up to f_i. This completes the recursion, and the proof. $\square$

Remark 3.26. *In [Vas19], it is also assumed $\mathcal{T}$ is universal and continuous for Proposition 3.25. It appears this assumption is unnecessary. In fact for our proof of Theorem 3.1 we will need to apply it to possibly non-continuous towers, as the construction will only guarantee continuity of towers at ordinals of high cofinality.*

We need one final version of tower extension - the ability to complete a partial extension on an initial segment of the tower.

Proposition 3.27. *Suppose $\mathcal{T} = \langle M_i : i < \beta \rangle^\wedge \langle a_i : i \in \beta^- \rangle$ is a tower. Suppose there is $\gamma < \beta$ and $\mathcal{T}^* = \langle M_i^* : i < \gamma \rangle^\wedge \langle a_i : i \in \gamma^- \rangle$ indexed by γ such that $\mathcal{T} \restriction \gamma \lhd \mathcal{T}^*$. Then there exists a tower $\mathcal{T}' = \langle M_i' : i < \beta \rangle^\wedge \langle a_i : i \in \beta^- \rangle$ and some N, $g : \bigcup_{i < \beta} M_i \underset{\bigcup_{i<\gamma} M_i}{\longrightarrow} N$ such that $\mathcal{T} \lhd \mathcal{T}'$ and $g[\mathcal{T}'] \restriction \gamma = \mathcal{T}^*$ (where $g[\mathcal{T}'] = \langle g[M_i'] : i < \beta \rangle^\wedge \langle a_i : i \in \beta^- \rangle$).*

Proof. Without loss of generality, $\gamma \geq 1$ (the $\gamma = 0$ case is a just Proposition 3.24 with $\alpha = 1$). We again follow the blueprint of Proposition 3.24 and Proposition 3.25.

Similar to before, by recursion construct a $\leq_{\mathbf{K}}$-increasing sequence of models $\langle N_i : i < \beta \rangle$ and a $\subseteq$-increasing sequence of $\mathbf{K}$-embeddings $\langle f_i : i < \beta \rangle$ such that

(1) for all $i < \beta$, $f_i : M_i \to N_i$
(2) for $i < \gamma$, $N_i = M_i^*$ and $f_i = \mathrm{id}_{M_i}$
(3) for all $i < \beta$, N_i is universal over $f_i[M_i]$
(4) for all $i \in \beta^-$, $\mathbf{gtp}(f_{i+1}(a_i)/N_i, N_{i+1})$ $\underset{\smile}{\ }$-does not fork over M_i

Rather than just fixing f_0 to be the identity on M_0, we now ensure that all the f_i for $i < \gamma$ are the identity, so $\bigcup_{i < \beta} f_i$ will map $\mathcal{T} \restriction \gamma$ into $\mathcal{T}^*$. This determines the first γ steps of the construction. From there, proceed as in Proposition 3.24 (with $\alpha = 1$) for the remaining successor and limit $i < \beta$. After the construction is complete, as in Proposition 3.25 take $f = \bigcup_{i < \beta} f_i : \bigcup_{i < \beta} M_i \to \bigcup_{i < \beta} N_i$, extend to $g : M' \cong \bigcup_{i < \beta} N_i$, and take $M_i' = f^{-1}[N_i]$ for

$i < \beta$. As before take $\mathcal{T}' = \langle M_i' : i < \beta \rangle^\wedge \langle a_i : i < \beta^- \rangle$. We have again that $\mathcal{T} \lhd \mathcal{T}'$, and additionally $g(a_i) = a_i$ for $i \in \gamma^-$ and $g[M_i'] = M_i^*$ for $i < \gamma$, meaning $g[\mathcal{T}'] \upharpoonright \gamma = \mathcal{T}^*$ as claimed. $\qquad \square$

3.2. **Reduced towers and full towers.** We assume Hypothesis 3.7 throughout this subsection. In the previous subsection, we have shown we can build a chain of towers. We now focus on showing how we may guarantee that the final tower in the chain will be 'continuous at δ_2', and will be 'universal'. Note these properties will not necessarily be preserved by tower unions. So we will introduce stronger conditions that will be preserved by tower unions. First we address continuity with reduced towers.

Definition 3.28. *Let* $\mathcal{T} = \langle M_i : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$ *be a tower. We say* $\mathcal{T}$ *is* reduced *if for every tower* $\mathcal{T}' = \langle M_i' : i \in I' \rangle^\wedge \langle a_i : i \in (I')^- \rangle$ *such that* $\mathcal{T} \lhd \mathcal{T}'$, *for all* $r <_I s \in I$, *we have* $M_r' \cap M_s = M_r$.

Remark 3.29. *The definition is equivalent if we require that* $\mathcal{T}'$ *is indexed by I also, since the condition is true for* $\mathcal{T}'$ *if and only if it holds for* $\mathcal{T}' \upharpoonright I$.

Reduced towers behave well with high cofinality unions:

Lemma 3.30. *Suppose* $\mathrm{cf}(\delta) \geq \kappa$. *If* $\langle \mathcal{T}^j : j < \delta \rangle$ *is a* $\lhd$-*increasing chain of reduced towers with* $\mathcal{T}^j$ *indexed by* I^j, *and* $\bigcup_{j < \delta} I^j$ *is a well ordering, then* $\bigcup_{j < \delta} \mathcal{T}^j$ *is reduced.*

Proof. Set $\mathcal{T}^\delta = \bigcup_{j < \delta} \mathcal{T}^j$, and let $\mathcal{T}^j = \langle M_i^j : i \in I^j \rangle^\wedge \langle a_i : i \in (I^j)^- \rangle$ for all $j \leq \delta$. Suppose $\mathcal{T}' = \langle M_i : i \in I' \rangle^\wedge \langle a_i : i \in (I')^- \rangle$ is a tower and $\mathcal{T}^\delta \lhd \mathcal{T}'$. If $r < s \in I^\delta$, then

$$
\begin{aligned}
M_s^\delta \cap M_r' &= \left(\bigcup \{ M_s^j : j < \delta \text{ such that } j \in I^s \} \right) \cap M_r' \\
&= \bigcup \{ M_s^j \cap M_r' : j < \delta \text{ such that } j \in I^s \} \\
&= \bigcup \{ M_r^j : j < \delta \text{ such that } j \in I^s \} \qquad \text{since } \mathcal{T}^j \lhd \mathcal{T}' \text{ and } \mathcal{T}^j \text{ is reduced} \\
&= M_r^\delta
\end{aligned}
$$

Therefore $\mathcal{T}^\delta$ is reduced. $\qquad \square$

Lemma 3.31. *If* $\mathcal{T} = \langle M_i : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$ *is a tower, there exists a reduced tower* $\mathcal{T}' = \langle M_i : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$ *such that* $\mathcal{T} \lhd \mathcal{T}'$.

Proof. Suppose no such $\mathcal{T}'$ exists for contradiction. Then construct by recursion a $\lhd$-increasing sequence of towers $\langle \mathcal{T}^j : j < \lambda^+ \rangle$ where $\mathcal{T}^j = \langle M_i^j : i \in I \rangle^\wedge \langle a_i^j : i \in I^- \rangle$ for all $j < \lambda^+$ such that

(1) $\mathcal{T} \lhd \mathcal{T}^0$
(2) for all $j < \lambda^+$, there are $r < s \in I$ such that $M_r^{j+1} \cap M_s^j \neq M_r^j$.
(3) for all $j < \lambda^+$ such that $\mathrm{cf}(j) \geq \kappa$, $\mathcal{T}^j = \bigcup_{j' < j} \mathcal{T}^{j'}$

$\mathcal{T}^0$ can be found by Proposition 3.24, and $\mathcal{T}^j$ for limits j where $\mathrm{cf}(j) \geq \kappa$ are determined. For other limits, take any $\mathcal{T}^j$ where $\mathcal{T}^{j'} \lhd \mathcal{T}^j$ for all $j' < j$, which exists by Proposition 3.24. Finally, for successors, if $\mathcal{T}^j$ is given, since $\mathcal{T} \lhd \mathcal{T}^j$, $\mathcal{T}^j$ is not reduced. So there is $\mathcal{T}^{j+1} = \langle M_i^{j+1} : i \in I \rangle^\wedge \langle a_i^{j+1} : i \in I^- \rangle$ such that for some $r < s \in I$, $M_r^{j+1} \cap M_s^j \neq M_r^j$. This completes the construction.

We want this sequence of towers to be truly continuous for the argument we will use, but small unions of towers may not result in a tower (we could lose that $M_i^j \in K_{(\lambda, \geq \kappa)}$). Nevertheless, define N_i^j to be

- M_i^j if j is not a limit

- $\bigcup_{j' < j} M_i^{j'}$ if j is a limit (note this means $N_i^j = M_i^j$ if $\mathrm{cf}(j) \geq \kappa$).

Again, $\langle N_i^j : i \in I \rangle$ cannot necessarily be used to form a tower, but are close enough to the M_i^j's for the following argument to work (since $N_i^j \leq_{\mathbf{K}} M_i^j \leq_{\mathbf{K}} N_i^{j+1}$ for all $j < \delta$, and $N_i^j = M_i^j$ whenever $\mathrm{cf}(j) \geq \kappa$).

For notational convenience, let $N_i^{\lambda^+} = \bigcup_{j < \lambda^+} N_i^j$ for $i \in I$ and $N_I^j = \bigcup_{i \in I} N_i^j$ for $j < \lambda^+$. Note $N_i^{\lambda^+} = \bigcup_{j < \lambda^+} M_i^j$ also as $M_i^j \leq_{\mathbf{K}} M_i^{j+1} = N_i^{j+1}$ for all $i \in I$, $j < \lambda^+$, and for $\mathrm{cf}(j) \geq \kappa$, $N_I^j = \bigcup_{i \in I} M_i^j$.

Now define $C_i = \{j \in \lambda^+ : N_i^{\lambda^+} \cap N_I^j = N_i^j\}$ for all $i \in I$. It is straightforward to see these are closed in λ^+ (because we use the N_i^j's, which are continuous in j unlike the M_i^j's), and they are also unbounded in λ^+: given $j < \lambda^+$, we can construct an increasing continuous sequence $\langle j_n : n \leq \omega \rangle$ such that $j_0 = j$ and $N_i^{\lambda^+} \cap N_I^{j_n} \subseteq N_i^{j_{n+1}}$ for all $n < \omega$; then $j_\omega \in C_i$. So as $|I| < \lambda^+$, $\bigcap_{i \in I} C_i$ is closed and unbounded in λ^+. Since $S = \{j < \lambda^+ : \mathrm{cf}(j) = \kappa\}$ is stationary, $\left(\bigcap_{i \in I} C_i\right) \cap S$ is non-empty.

But if $j \in \left(\bigcap_{i \in I} C_i\right) \cap S$, for all $i \in I$ we have $N_i^{\lambda^+} \cap N_I^j = N_i^j$. Note that as $\mathrm{cf}(j) = \kappa$, $N_i^j = M_i^j$ for all $i \in I$. Hence $\left(\bigcup_{k < \lambda^+} M_i^k\right) \cap \left(\bigcup_{r \in I} M_r^j\right) = M_i^j$ for all $i \in I$. So in particular for all $r < s \in I$, we have $M_r^{j+1} \cap M_s^j = M_r^j$. This contradicts (2) of the construction. $\square$

Now we move towards showing reduced towers are continuous at all $i \in I$ with cofinality $\geq \kappa$. We begin with a lemma, which is our analogue of [Vas19, 5.18].

Lemma 3.32. *Suppose I is a well ordering, where $\mathrm{otp}(I) = \delta + 1$ for some limit ordinal $\delta < \lambda^+$ such that $\mathrm{cf}(\delta) \geq \kappa$. Let i_0, i_δ be the initial and final elements of I respectively. Let $\mathcal{T} = \langle M_i : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$ be a tower. Suppose there is $b \in M_{i_\delta}$ such that $\mathbf{gtp}(b/M_i, M_{i_\delta}) \underset{\smile}{\;}$-does not fork over M_{i_0} for all $i \in I$. Then there exists $\mathcal{T}' = \langle M_i' : i \in I \rangle^\wedge \langle a_i : i \in I^- \rangle$ such that $\mathcal{T} \lhd \mathcal{T}'$ and $b \in M_{i_0}'$.*

Proof. First note that without loss of generality, with some relabelling, we may assume $I = \delta + 1$, $i_0 = 0$, and $i_\delta = \delta$ respectively.

So assume $I = \delta + 1$, $i_0 = 0$, $i_\delta = \delta$. By applying Proposition 3.25 to $\mathcal{T} \upharpoonright \delta$ and $p = \mathbf{gtp}(b/M_0, M_\delta)$, there is a tower $\mathcal{T}^* = \langle M_i^* : i < \delta \rangle^\wedge \langle a_i : i < \delta \rangle$ and $b^* \in M_0^*$, such that $\mathcal{T} \upharpoonright \delta \lhd \mathcal{T}^*$, $\mathbf{gtp}(b/M_0, M_\delta) = \mathbf{gtp}(b^*/M_0, M_0^*)$, and $\mathbf{gtp}(b^*/M_i, M_i^*) \underset{\smile}{\;}$-does not fork over M_0 for each $i < \delta$. Since $\mathbf{gtp}(b/M_i, M_\delta)$ also $\underset{\smile}{\;}$-does not fork over M_0 and both of these types extend $p \in \mathbf{gS}(M_0)$, we have $\mathbf{gtp}(b/M_i, M_\delta) = \mathbf{gtp}(b^*/M_i, M_i^*)$ by uniqueness for each $i < \delta$. Since these types $\underset{\smile}{\;}$-do not fork over M_0, by $K_{(\lambda, \geq \kappa)}$-universal continuity*, $\mathbf{gtp}(b/\bigcup_{i < \delta} M_i, M_\delta) = \mathbf{gtp}(b^*/\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} M_i^*)$.

By type equality, there is some $M_\delta^\circ \in K_\lambda$ where $\bigcup_{i < \delta} M_i^* \leq_{\mathbf{K}} M_\delta^\circ$ and some $f : M_\delta \to M_\delta^\circ$ fixing $\bigcup_{i < \delta} M_i$ such that $f(b) = b^*$. Let M_δ^* be a (λ, κ)-limit over M_δ°. So $f : M_\delta \to M_\delta^*$, and there is some $g : M_\delta \cong M_\delta^*$ an isomorphism extending f. Then if we let $M_i' = g^{-1}[M_i^*]$ for all $i < \delta$, $\mathcal{T}' = \langle M_i' : i \leq \delta \rangle^\wedge \langle a_i : i < \delta \rangle$ is a tower, where $\mathcal{T} \lhd \mathcal{T}'$ since $\mathcal{T} \upharpoonright \delta \lhd \mathcal{T}^*$ and M_δ' is universal over M_δ, and $b = g^{-1}(b^*) \in g^{-1}[M_0^*] = M_0'$ as desired. $\square$

The following three results make up our generalisation of [Vas19, 5.19]. We fill in some details and split the proof into separate lemmas for clarity.

Lemma 3.33. *If $\mathcal{T} = \langle M_i : i < \alpha \rangle^\wedge \langle a_i : i \in \alpha^- \rangle$ is reduced and $\beta < \alpha$, then $\mathcal{T} \upharpoonright \beta$ is also reduced.*

Proof. Suppose $\mathcal{T} \upharpoonright \beta \lhd \mathcal{T}^*$. Using Proposition 3.27, there exists $\mathcal{T}' = \langle M_i' : i < \alpha \rangle^\wedge \langle a_i : i \in \alpha^- \rangle$ and $N, g : \bigcup_{i < \alpha} M_i' \underset{\bigcup_{i < \beta} M_i}{\longrightarrow} N$ such that $g[\mathcal{T}'] \upharpoonright \beta = \mathcal{T}$. Since $\mathcal{T}$ is reduced, we must have for any

$r < s < \beta$ that $M_s \cap M'_r = M_r$. In particular, for $r < s < \beta$, we have $M_s \cap M'_r = g[M_s \cap M'_r] = g[M_r] = M_r$. Therefore, $\mathcal{T} \restriction \beta$ is reduced as claimed. $\square$

Lemma 3.34. *Suppose $\mathcal{T} = \langle M_i : i < \alpha \rangle ^\wedge \langle a_i : i \in \alpha^- \rangle$ is a reduced tower. Then for all $\beta < \alpha$, $\mathcal{T} \restriction [\beta, \alpha)$ is reduced.*

Proof. Suppose $\mathcal{T} \restriction [\beta, \alpha) \lhd \mathcal{T}'$, where $\mathcal{T}' = \langle M'_i : i \in [\beta, \alpha) \rangle ^\wedge \langle a_i : i \in [\beta, \alpha)^- \rangle$. By Proposition 3.24 (with $\alpha = 1$), there exists a tower $\mathcal{T}^* = \langle M^*_i : i \leq \beta \rangle ^\wedge \langle a_i : i < \beta \rangle$ where $\mathcal{T} \restriction (\beta + 1) \lhd \mathcal{T}^*$.

Since $M_\beta \leq^u_{\mathbf{K}} M^*_\beta$, there exists a $\mathbf{K}$-embedding $f : M^*_\beta \underset{M_\beta}{\rightarrow} M'_\beta$. For $i < \beta$, set $M'_i = f[M^*_i]$.

Note that f fixes a_i for all $i < \beta$. Define a new tower $\mathcal{T}'' = \langle M'_i : i < \alpha \rangle ^\wedge \langle a_i : i \in \alpha^- \rangle$.

We claim that $\mathcal{T} \lhd \mathcal{T}''$. The conditions (1)-(3) of the definition of $\lhd$ (Definition 3.19) lift immediately from $\mathcal{T} \restriction (\beta+1) \lhd \mathcal{T}^*$ and $\mathcal{T} \restriction [\beta, \alpha) \lhd \mathcal{T}'$. So it remains to show that $\mathbf{gtp}(a_i/M'_i, M'_{i+1})$ $\underset{}{\downarrow}$-does not fork over M_i for all $i \in \alpha^-$.

For $i < \beta$, note $\mathbf{gtp}(a_i/M'_i, M'_{i+1}) = \mathbf{gtp}(a_i/M'_i, f[M^*_{i+1}])$ (since $M'_{i+1} = f[M_{i+1}]$ for $i+1 < \beta$, and if $i + 1 = \beta$ it follows from monotonicity of $\underset{}{\downarrow}$ as $f[M^*_{i+1}] \leq_{\mathbf{K}} M'_{i+1}$). So we have

$$\mathbf{gtp}(a_i/M'_i, M'_{i+1}) = \mathbf{gtp}(a_i/M'_i, f[M^*_{i+1}]) = \mathbf{gtp}(f(a_i)/f[M^*_i], f[M^*_{i+1}]) = f(\mathbf{gtp}(a_i/M^*_i, M^*_{i+1})).$$

Since $\mathbf{gtp}(a_i/M^*_i, M^*_{i+1})$ $\underset{}{\downarrow}$-does not fork over M_i by $\mathcal{T} \restriction (\beta + 1) \lhd \mathcal{T}^*$, $\mathbf{gtp}(a_i/M'_i, M'_{i+1})$ $\underset{}{\downarrow}$-does not fork over $f[M_i] = M_i$ by invariance.

For $i \in [\beta, \alpha)^-$, $\mathbf{gtp}(a_i/M'_i, M'_{i+1})$ $\underset{}{\downarrow}$-does not fork over M_i as $\mathcal{T} \restriction [\beta, \alpha) \lhd \mathcal{T}'$. So we have shown condition (4) of Definition 3.19 holds for all $i \in \alpha^-$, and $\mathcal{T} \lhd \mathcal{T}''$ as desired.

So, since $\mathcal{T}$ is reduced, for all $r, s < \alpha$ with $r \leq s$, we have $M_s \cap M'_r = M_r$. This holds in particular for $r, s \in [\beta, \alpha)$, so $\mathcal{T} \restriction [\beta, \alpha)$ is reduced as desired. $\square$

Proposition 3.35. *Suppose $\mathcal{T} = \langle M_i : i \in I \rangle ^\wedge \langle a_i : i \in I^- \rangle$ is a reduced tower, and $\delta \in I$ is limit in I with cofinality $\mathrm{cf}(\delta) \geq \kappa$. Then $\mathcal{T}$ is continuous at δ; that is, $M_\delta = \bigcup_{i < \delta} M_i$.*

Proof. Suppose for contradiction this is false. Let α be minimal such that there exists a well ordered set I with $\mathrm{otp}(I) = \alpha$, $\delta \in I$ with $\mathrm{cf}(\delta) \geq \kappa$, and a reduced tower $\mathcal{T} = \langle M_i : i \in I \rangle ^\wedge \langle a_i : i \in I^- \rangle$ that is not continuous at δ. As in the proof of Lemma 3.32, with some relabeling, we may assume $I = \alpha$.

Note by Lemma 3.33, $\mathcal{T} \restriction (\delta + 1)$ is also reduced and not continuous at δ. So by minimality, $\alpha = \delta + 1$.

Since $M_\delta \neq \bigcup_{i < \delta} M_i$, there is some $b \in |M_\delta| \setminus \bigcup_{i < \delta} |M_i|$. Using Proposition 3.24, there exists $\mathcal{T}^* = \langle M^*_i : i \leq \delta \rangle ^\wedge \langle a_i : i < \delta \rangle$ strongly universal such that $\mathcal{T} \lhd \mathcal{T}^*$.

As $\kappa \leq \mathrm{cf}(\delta)$, by $(\geq \kappa)$-local character there is $\beta < \delta$ such that $\mathbf{gtp}(b/\bigcup_{i < \delta} M^*_i, M^*_\delta)$ $\underset{}{\downarrow}$-does not fork over M^*_β. In particular, by monotonicity, $\mathbf{gtp}(b/M^*_i, M^*_{i+1})$ $\underset{}{\downarrow}$-does not fork over M^*_β for $i \in [\beta, \delta)$. Let $\mathcal{T}^{**} = \mathcal{T}^* \restriction [\beta, \delta]$. By Lemma 3.32 (using that $\mathrm{cf}(\mathrm{otp}([\beta, \delta))) = \mathrm{cf}(\delta) \geq \kappa$), there exists $\mathcal{T}' = \langle M'_i : i \in [\beta, \delta] \rangle ^\wedge \langle a_i : i \in [\beta, \delta) \rangle$ such that $\mathcal{T}^{**} \lhd \mathcal{T}'$ and $b \in M'_\beta$. Note $\mathcal{T} \restriction [\beta, \delta] \lhd \mathcal{T}^{**} \lhd \mathcal{T}'$. By Lemma 3.34, $\mathcal{T} \restriction [\beta, \delta]$ is reduced, so $M'_\beta \cap M_\delta = M_\beta$. But $b \in M'_\beta \cap M_\delta$ and $b \notin M_\beta$, a contradiction.

$\square$

Now we will recall the notion of full towers used in [Vas19], which will let us guarantee a tower contains universal chains.

Definition 3.36 ([Vas19, 5.20]). *Let $\mathcal{T} = \langle M_i : i \in I \rangle ^\wedge \langle a_i : i \in I^- \rangle$ be a tower, and $I_0 \subseteq I$. We say $\mathcal{T}$ is I_0-full if for every $i \in I_0$ and every $p \in \mathbf{gS}^{na}(M_i)$, there is $k \in [i, i +_{I_0} 1)_I$ such that $\mathbf{gtp}(a_k/M_k, M_{k+_I 1})$ is the $\underset{}{\downarrow}$-non-forking extension of p.*

The following remark motivates this definition. It describes how we will show our δ_1'th tower in the proof of Theorem 3.1 contains a universal sequence witnessing a (λ, δ_2)-limit model.

Remark 3.37. *If $\mathcal{T}$ is I_0-full, then in particular, by Fact 2.5, $M_{i+_{I_0}\lambda}$ is universal over M_i for all $i \in I_0$. Taking this further, if $\delta < \lambda^+$ is a limit ordinal, then $M_{i+_{I_0}\lambda\cdot\delta}$ is a (λ,δ)-limit model over M_i.*

The type extensions in Definition 3.36 are non-forking to make fullness work with high cofinality unions of towers - see the following lemma, which extends [Vas19, 5.24].

Lemma 3.38. *Suppose $\delta < \lambda^+$ is a limit ordinal and $\kappa \le \mathrm{cf}(\delta)$. Suppose $\langle \mathcal{T}^j : j \le \delta \rangle$ is a $\lhd$-increasing sequence of towers where $\mathcal{T}^\delta = \bigcup_{j<\delta} \mathcal{T}^j$ (in particular, $\bigcup_{j<\delta} I^j$ is a well ordering). Say $\mathcal{T}^j = \langle M_i^j : i \in I^j \rangle^\wedge \langle a_i : i \in (I^j)^- \rangle$ for $j \le \delta$. Suppose $I_0 \subseteq I^0$ and $\mathcal{T}^j$ is I_0-full for all $j < \delta$. Then $\mathcal{T}^\delta$ is I_0-full.*

Proof. Let $i \in I_0$ and $p \in \mathbf{gS}(M_i^\delta)$. Since $\langle M_i^j : j < \delta \rangle$ is a $\le_{\mathbf{K}}^u$-increasing sequence and $\kappa \le \mathrm{cf}(\delta)$, by $(\ge \kappa)$-local character there is $j < \delta$ such that $p \underset{\perp}{}$-does not fork over M_i^j.

Since $\mathcal{T}^j$ is I_0-full, there is $k \in [i, i +_{I_0} 1)_{I^j}$ such that $\mathbf{gtp}(a_k/M_k^j, M_{k+_{I^j}1}^j) \underset{\perp}{}$-does not fork over M_i^j and extends $p \restriction M_i^j$.

Additionally, from the definition of tower extension, $\mathbf{gtp}(a_k/M_k^\delta, M_{k+_I1}^\delta) \underset{\perp}{}$-does not fork over M_k^j. So transitivity of $\underset{\perp}{}$ gives that $\mathbf{gtp}(a_k/M_k^\delta, M_{k+_I1}^\delta) \underset{\perp}{}$-does not fork over M_i^j. By monotonicity, $\mathbf{gtp}(a_k/M_i^\delta, M_{i+_I1}^\delta) \underset{\perp}{}$-does not fork over M_i^j and extends $p \restriction M_i^j$, so by uniqueness of $\underset{\perp}{}$-non-forking extensions, $\mathbf{gtp}(a_k/M_i^\delta, M_{i+_I1}^\delta) = p$. Therefore $\mathbf{gtp}(a_k/M_k^\delta, M_{k+_I1}^\delta)$ extends p, and $\underset{\perp}{}$-does not fork over M_i^δ by base monotonicity, as desired. $\square$

Notation 3.39. *Given two well ordered sets I and J, $I \times J$ will denote the usual lexicographic ordering; that is, $(i,j) < (i',j')$ if and only if $(i <_I i'$ or $(i = i'$ and $j <_J j'))$. We will use the notationn $<_{\mathrm{lex}}$ when the ordering is ambiguous.*

The following is a slight improvement of [Vas19, 5.28], where a proof is not given - we include one for completeness, and relax the conditions on the limit ordinals, though the method appears to be the same.

Lemma 3.40. *Let I be a well-ordering, and $\alpha, \gamma < \lambda^+$ be limit ordinals with $\alpha < \gamma$ and $\mathrm{cf}(\gamma) = \lambda$. If $\mathcal{T}$ is a strongly universal tower indexed by $I \times \alpha$, then there is an $I \times \{0\}$-full tower $\mathcal{T}'$ indexed by $I \times \gamma$ such that $\mathcal{T}' \restriction (I \times \alpha) = \mathcal{T}$.*

Proof. Say $\mathcal{T} = \langle M_{(i,k)} : (i,k) \in I \times \alpha \rangle^\wedge \langle a_{(i,k)} : (i,k) \in I \times \alpha \rangle$. We define $M_{(i,k)}$ and $a_{(i,k)}$ for $(i,k) \in I \times [\alpha, \gamma)$ recursively by the following procedure.

Fix $i \in I$. Let $\langle p_k : k \in [\alpha, \gamma) \rangle$ be an enumeration of $\mathbf{gS}^{na}(M_{(i,0)})$, possibly with repetitions. Note this is possible by stability in λ and $\mathrm{cf}(\gamma) = \lambda$.

Define $M'_{(i,k)}$ and $a'_{(i.k)}$ for $k < \gamma$ by induction on k as follows:

(1) If $k < \alpha$, $M'_{(i,k)} = M_{(i,k)}$ and $a'_{(i.k)} = a_{(i,k)}$.

(2) If $k \ge \alpha$ is limit, then $M'_{(i,k)}$ is any (λ,κ)-limit model over $\bigcup_{l<k} M'_{(i,l)}$.

(3) If $k = l+1$, then let $\mathbf{gtp}(a'_{(i,l)}/M'_{(i,l)}, \hat{M}_{(i,l+1)})$ be a type extending p_l which $\underset{\perp}{}$-does not fork over $M_{(i,0)}$ (this is possible by existence and extension). Let $M'_{(i,l+1)}$ be a (λ,κ)-limit model over $\hat{M}'_{(i,l+1)}$. This determines $a'_{(i,l)}$ and $M'_{(i,k)}$.

As $\mathcal{T}$ is strongly universal, $M_{(i+1,0)}$ is universal over $\bigcup_{k\in\alpha} M_{(i,k)}$. We have $\bigcup_{k<\alpha} M_{(i,k)} = \bigcup_{k\in\alpha} M'_{(i,k)} \le_{\mathbf{K}} \bigcup_{k\in\gamma} M'_{(i,k)}$. So there exists $f : \bigcup_{k\in\gamma} M'_{(i,k)} \to M_{(i+1,0)}$ fixing $\bigcup_{k\in\alpha} M_{(i,k)}$. For $k \in [\alpha, \gamma)$, take $M_{(i,k)} = f[M'_{(i,k)}]$ and $a_{(i,k)} = f(a'_{(i,k)})$.

Invariance maintains the non-forking properties, so $\mathcal{T}' = \langle M_{(i,k)} : (i,k) \in I \times \gamma \rangle^\wedge \langle a_{(i,k)} : (i,k) \in I \times \gamma \rangle$ is a tower and $\mathcal{T}' \restriction (I \times \alpha) = \mathcal{T}$. Furthermore, $\mathcal{T}'$ is $I \times \{0\}$-full by condition (3) from the construction. $\square$

3.3. **The main result.** Finally we will restate and prove Theorem 3.1. The argument is similar to Vasey's proof of [Vas19, 2.7].

Theorem 3.1. *Assume Hypothesis 3.7. Let $\delta_1, \delta_2 < \lambda^+$ be limit ordinals where $\kappa \leq \mathrm{cf}(\delta_1), \mathrm{cf}(\delta_2)$. If $M, N_1, N_2 \in K_\lambda$ where N_l is a (λ, δ_l)-limit over M for $l = 1, 2$, then there is an isomorphism from N_1 to N_2 fixing M.*

Moreover, if $N_1, N_2 \in K_\lambda$ where N_l is (λ, δ_l)-limit for $l = 1, 2$, then N_1 is isomorphic to N_2.

Proof of Theorem 3.1. By Fact 2.7, it is enough to show that whenever $\delta_1, \delta_2 < \lambda^+$ are regular and $\kappa \leq \delta_1, \delta_2$, and $M \in K_\lambda$, there exists a model which is both (λ, δ_1)-limit model over M and a (λ, δ_2)-limit model over M. Note also it is enough to prove it for $M \in K_{(\lambda, \geq \kappa)}$, as each M has a (λ, κ)-limit M' over M, and a (λ, δ_l)-limit model over M' will also be (λ, δ_l)-limit over M. Fix such $\delta_1, \delta_2 \geq \kappa$ and $M \in K_{(\lambda, \geq \kappa)}$.

We will build a $\lhd$-increasing sequence of towers $\langle \mathcal{T}^j : j \leq \delta_1 \rangle$ and a $\leq$-increasing continuous sequence of limit ordinals $\langle \alpha_j : j \leq \delta_1 \rangle \subseteq \lambda^+$ such that

(1) for all $j < \delta_1$, $\mathcal{T}^j = \langle M_i^j : i \in (\delta_2 + 1) \times \lambda \times \alpha_j \rangle^\wedge \langle a_i^j : i \in (\delta_2 + 1) \times \lambda \times \alpha_j \rangle$
(2) $M = M_{(0,0,0)}^0$ (so all M_i^j contain M)
(3) for all $j < \delta_1$, $\mathcal{T}^{2j+2}$ is a reduced tower
(4) for all $j < \delta_1$, $\mathcal{T}^{2j+1}$ is a $((\delta_2 + 1) \times \lambda \times \{0\})$-full tower.
(5) $\mathcal{T}^{\delta_1} = \bigcup_{j < \delta_1} \mathcal{T}^j$ (which is valid as $\mathrm{cf}(\delta_1) \geq \kappa$).

This is possible: We proceed by recursion. Let $\alpha_0 = \omega$. By Remark 3.18, there is a tower $\mathcal{T}^0$ starting at M of length $(\delta_2 + 1) \times \lambda \times \alpha_0$.

For successors, given $\mathcal{T}^{2j}$ and α_{2j}, we do the next two steps. By Proposition 3.24, there is $\mathcal{T}_*^{2j}$ a strongly universal tower indexed by $(\delta_2 + 1) \times \lambda \times \alpha_{2j}$ such that $\mathcal{T}^{2j} \lhd \mathcal{T}_*^{2j}$. Let $\alpha_{2j+1} < \lambda^+$ be a limit ordinal greater than α_{2j} where $\mathrm{cf}(\alpha_{2j+1}) = \lambda$ (note this exists since such ordinals form an unbounded set in λ^+ by regularity). Then by Lemma 3.40 there exists a $(\delta_2 + 1) \times \lambda \times \{0\}$-full tower $\mathcal{T}^{2j+1}$ such that $\mathcal{T}^{2j+1} \upharpoonright (\delta_2 + 1) \times \lambda \times \alpha_{2j} = \mathcal{T}_*^{2j}$. In particular, $\mathcal{T}^{2j} \lhd \mathcal{T}^{2j+1}$. Let $\alpha_{2j+2} = \alpha_{2j+1}$. By Lemma 3.31, there exists a reduced tower $\mathcal{T}^{2j+2}$ indexed by $(\delta_2 + 1) \times \lambda \times \alpha_{2j+2}$ such that $\mathcal{T}^{2j+1} \lhd \mathcal{T}^{2j+2}$.

Finally, if $j < \delta_1$ is <u>limit</u>, let $\alpha_j = \bigcup_{j' < j} \alpha_{j'}$ and take $\mathcal{T}^j$ given by Proposition 3.24 such that for all $j' < j$, $\mathcal{T}^{j'} \lhd \mathcal{T}^j$.

$\mathcal{T}^{\delta_1}$ is given by (5). This completes the construction.

This is enough: Consider the final tower $\mathcal{T}^{\delta_1}$. Since $\langle \mathcal{T}^{2j+2} : j < \delta_1 \rangle$ is a $\lhd$-increasing sequence of reduced towers, $\mathcal{T}^{\delta_1}$ is reduced by Proposition 3.30. Hence it is continuous at $(\delta_2, 0, 0)$ (which has cofinality $\delta_2 \geq \kappa$ in $(\delta_2 + 1) \times \lambda \times \alpha_{\delta_1}$) by Proposition 3.35 - that is, $M_{(\delta_2, 0, 0)}^{\delta_1} = \bigcup_{k <_{\mathrm{lex}} (\delta_2, 0, 0)} M_k^{\delta_1} = \bigcup_{i < \delta_2} M_{(i, 0, 0)}^{\delta_1}$. Since $\langle \mathcal{T}^{2j+1} : j \leq \delta_1 \rangle$ is a $\lhd$-increasing sequence of $(\delta_2 + 1) \times \lambda \times \{0\}$-full towers, $\mathcal{T}^{\delta_1}$ is $(\delta_2 + 1) \times \lambda \times \{0\}$-full by Lemma 3.38. In particular for all $i < \delta_2 + 1$ and $i' < \lambda$, $M_{(i, i'+1, 0)}^{\delta_1}$ realises all types over $M_{(i, i', 0)}^{\delta_1}$, so by Fact 2.5, $M_{(i+1, 0, 0)}^{\delta_1}$ is universal over $M_{(i, 0, 0)}^{\delta_1}$ for all $i < \delta_2$. So $\langle M_{(i, 0, 0)}^{\delta_1} : i < \delta_2 \rangle$ is a $\leq_{\mathbf{K}}^u$-increasing chain with union $M_{(\delta_2, 0, 0)}^{\delta_1}$. Therefore $M_{(\delta_2, 0, 0)}^{\delta_1}$ is a (λ, δ_2)-limit over $M_{(0, 0, 0)}^{\delta_1}$, and in particular (as $M = M_{(0, 0, 0)}^0 \leq_{\mathbf{K}} M_{(0, 0, 0)}^{\delta_1}$)), a (λ, δ_2)-limit over M.

On the other hand, $M_{(\delta_2, 0, 0)}^{\delta_1} = \bigcup_{j < \delta_1} M_{(\delta_2, 0, 0)}^j$. $\langle M_{(\delta_2, 0, 0)}^j : j < \delta_1 \rangle$ is a $\leq_{\mathbf{K}}$-increasing universal chain by definition of the tower ordering, so $M_{(\delta_2, 0, 0)}^{\delta_1}$ is (λ, δ_1)-limit over $M_{(\delta_2, 0, 0)}^0$, and hence as before over M.

For the 'moreover' part, by the above there are some (λ, δ_l)-limit model N_l for $l = 1, 2$ which are isomorphic (over some M). Then Fact 2.7 implies all such limit models (over any models) are isomorphic. $\qquad\square$

3.4. **Tame AECs.** In this subsection we show how Theorem 3.1 can be used to show that in tame AECs with enough symmetry, there is a threshold above which all limit models at high enough stability cardinals are isomorphic (see Corollary 3.46). A key difference between the results of this subsection and the rest of the paper is that we do not assume the existence of a nice stable independence relation, but instead show that such a relation exists.

The following result follows [Vas16b, §4, §5] closely. However we have to do additional work to guarantee an approximation of 'full' universal continuity of non-forking: [Vas16b] does not assume μ-non-splitting satisfies universal continuity, and only guarantees $(\geq \kappa)$-universal continuity where κ is the local character cardinal. We need universal continuity* in $\mathbf{K}_{\geq \mu^+}$, a stronger condition. For this, we assume universal continuity of μ-non-splitting. The universal continuity* arguments are adapted from [Leu24, 4.4, 4.11].

Lemma 3.41. *Let* $\mathbf{K}$ *be an AEC with AP, NMM, stable in* $\mu \geq \mathrm{LS}(\mathbf{K})$, *and* μ-*tame. Assume* μ-*non-splitting satisfies universal continuity. Let* $\underset{(\geq\mu)-f}{\downarrow}$ *be* $\underset{(\geq\mu)-f}{\downarrow}$ *restricted to models in* $\mathbf{K}^{\mu^+\text{-sat}}_{\geq\mu^+}$ *(that is, the* μ^+-*saturated models in* $\mathbf{K}_{\geq\mu^+}$ *ordered by* $\leq_{\mathbf{K}}$*).*

Then $\downarrow$ *satisfies invariance, monotonicity, base monotonicity, extension, uniqueness , universal continuity* in* $\mathbf{K}_{\geq\mu^+}$, *and* $(\geq \mu^+)$-*local character.*

Proof. Invariance, monotonicity, and base monotonicity are clear from the definition. Extension, and uniqueness follow from [Vas16b, 5.9] and [Vas16b, 5.3] respectively with $\lhd = \leq^u_{\mathbf{K}}$. For extension, note the proof goes through even with algebraic types if you assume p $(\geq \mu)$-does not fork over M; this gives the weak form of extension from Lemma 2.17, and then we can apply Lemma 2.17). Note that μ^+-saturated models are μ^+-model homogeneous under our assumptions as $\mu^+ > \mathrm{LS}(\mathbf{K})$ (see [Vas16b, 2.11]), so these results may be applied.

By [Vas16b, 3.11, 4.11] we have a stronger form of $(\geq \kappa)$-local character for some minimal $\kappa \leq \mu^+$. To be precise, κ is the least regular cardinal such that for any increasing sequence $\langle M_i : i < \kappa \rangle$ where $M_i \in \mathbf{K}^{\mu^+\text{-sat}}_{\geq\mu^+}$ for all $i < \kappa$, and for all $p \in S(\bigcup_{i<\kappa} M_i)$, there exists $i < \kappa$ such that p $(\geq \mu)$-does not fork over M_i; and this property holds for all regular $\kappa' \geq \kappa$. This is stronger than $(\geq \kappa)$-local character of $\downarrow$ as it is possible that $\bigcup_{i<\kappa} M_i$ is no longer μ^+-saturated.

Now we prove universal continuity* of $\downarrow$ in $\mathbf{K}_{\geq\mu^+}$. Let $\langle M_i : i < \delta \rangle$ be an increasing sequence of μ^+-saturated models. Let $M_\delta = \bigcup_{i<\delta} M_i$. Suppose we have $p_i \in \mathbf{gS}(M_i)$ such that p_i does not μ-fork over M_0 for each $i < \delta$. Take a μ^+-saturated model $M^* \in \mathbf{K}$ with $M_\delta \leq^u_{\mathbf{K}} M^*$. By extension there is $p^* \in \mathbf{gS}(M^*)$ such that $p_0 \subseteq p^*$ for each $i < \delta$ and p does not $(\geq \mu)$-fork over M_0. By uniqueness, $p^* \upharpoonright M_i = p_i$. So $p_i \subseteq p^*$ for all $< \delta$. Let $p = p^* \upharpoonright M_\delta$. We must show p is the unique extension of the p_i's.

So suppose we have $q \in \mathbf{gS}(M_\delta)$ with $q \upharpoonright M_i = p_i$ for all $i < \delta$. We must show $p = q$. We go by cases.

Case 1: assume $\delta \geq \kappa$. By [Vas16b, 4.12], q does not $(\geq \mu)$-fork over M_0. This is true also of p, so by [Vas16b, 4.8] there exist $M_0^p, M_0^q \in \mathbf{K}_\mu$ where $M_0^p, M_0^q \leq_{\mathbf{K}} M_0$, p does not μ-split over M_0^p, and q does not μ-split over M_0^q (note this only requires that M_0 is μ^+-saturated, and not necessarily M_δ). Taking $M_0^0 \in \mathbf{K}_\mu$ with $M_0^p, M_0^q \leq_{\mathbf{K}} M_0^0 \leq_{\mathbf{K}} M_0$, we have p, q do not μ-split over M_0^0 by monotonicity. Then take $M_0^1 \leq_{\mathbf{K}} M_1$ in $\mathbf{K}_\mu$, universal over M_0^0 (this is possible as $M_0^0 \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} M_1$ and $\mathbf{K}$ is μ-stable). For every $N \in \mathbf{K}_\mu$ with $M_0^1 \leq_{\mathbf{K}} N \leq_{\mathbf{K}} M_\delta$, we have that $p \upharpoonright M_0^1 = q \upharpoonright M_0^1$, so by weak uniqueness of μ-non-splitting (Fact 4.5, using that $M_0^0 \leq^u_{\mathbf{K}} M_0^1 \leq_{\mathbf{K}} N$), we have $p \upharpoonright N = q \upharpoonright N$. This holds for all such N, so by μ-tameness, $p = q$ as desired.

Case 2: assume $\delta < \kappa$. Then $\delta \leq \mu$ in particular. By [Vas16b, 4.8] for each $i < \delta$ there is $N_i \in \mathbf{K}_\mu$ such that $N_i \leq_{\mathbf{K}} M_0$ and p_i does not μ-split over N_i. Take $N \in \mathbf{K}_\mu$ such that $N_i \leq_{\mathbf{K}} N \leq_{\mathbf{K}} M_0$ for each $i < \delta$. Take $N' \in \mathbf{K}_\mu$ such that $N \leq^u_{\mathbf{K}} N'$ and $N' \leq_{\mathbf{K}} M_1$.

Suppose that $N^* \in \mathbf{K}_\mu$ with $N' \leq_{\mathbf{K}} N^* \leq_{\mathbf{K}} M_\delta$. Take some $\leq_{\mathbf{K}}^u$-increasing sequence of models $\langle M_i' : i \in \delta \rangle$ in $\mathbf{K}_\mu$ where $N' \leq_{\mathbf{K}} M_i' \leq_{\mathbf{K}} M_{i+1}$ and $|M_i| \cap |N^*| \subseteq |M_i'|$ for all $i < \delta$ (this is possible as $M_i \leq_{\mathbf{K}}^u M_{i+1}$ and $\mathbf{K}$ is μ-stable). Let $M_\delta' = \bigcup_{i<\delta} M_i'$. Note that $N^* \leq_{\mathbf{K}} M_\delta'$. We have from monotonicity that $p \upharpoonright M_i' = q \upharpoonright M_i' = p_{i+1} \upharpoonright M_i'$ does not μ-split over N. So by universal continuity of μ-non-splitting, both $p \upharpoonright M_\delta'$ and $q \upharpoonright M_\delta'$ do not μ-split over N. As $N \leq_{\mathbf{K}}^u N' \leq_{\mathbf{K}} M_\delta'$ and $p \upharpoonright N' = q \upharpoonright N'$, by weak uniqueness of μ-non-splitting, $p \upharpoonright M_\delta' = q \upharpoonright M_\delta'$, and therefore $p \upharpoonright N^* = q \upharpoonright N^*$. This holds for all such N^*, so by μ-tameness, $p = q$ as desired. $\square$

Remark 3.42. *The above proof goes through if we take $\kappa = \mu^+$, rather than the minimal κ.*

Lemma 3.43. *Let $\mathbf{K}$ be an AEC, stable in $\mu \geq \mathrm{LS}(\mathbf{K})$ and stable also in $\lambda \geq \mu^+$, where $\mathbf{K}$ has AP, NMM, λ-JEP, and μ-tameness. Assume universal continuity of μ-non-splitting.*
Let $\underset{(\geq\mu)-f}{\bigcup}$ be $\bigcup$ restricted to $K_{(\lambda,\geq\mu^+)}$. Suppose $\bigcup$ satisfies non-forking amalgamation. Then $\bigcup$ satisfies Hypothesis 3.7 with $\kappa = \mu^+$ for any stability cardinal $\lambda \geq \mu^+$.

Proof. This is immediate from Lemma 3.41 and the fact that $K_{(\lambda,\geq\mu^+)} \subseteq \mathbf{K}_{\geq\mu^+}^{\mu^+\text{-sat}}$. Note that $K_{(\lambda,\geq\mu^+)}$-universal continuity* in $\mathbf{K}$ follows from the fact that $\bigcup$ satisfies universal continuity* in $\mathbf{K}_{\geq\mu^+}$. $\square$

The following definition is inspired by the formulation of λ-symmetry in [Vas19, 2.6].

Definition 3.44. *We say that $\mathbf{K}$ has μ-symmetry in $K_{(\lambda,\geq\mu^+)}$ if whenever $M \leq_{\mathbf{K}} N$ are both $(\lambda, \geq \mu^+)$-limit models, and $a, b \in N$, then the following are equivalent:*

(1) *There exist $M_b, N_b \in K_{(\lambda,\geq\mu^+)}$ with $M \leq_{\mathbf{K}} M_b \leq_{\mathbf{K}} N_b$ and $N \leq_{\mathbf{K}} N_b$ such that $b \in M_b$ and $\mathbf{gtp}(a/M_b, N_b)$ does not $(\geq \mu)$-fork over M*

(2) *There exist $M_a, N_a \in K_{(\lambda,\geq\mu^+)}$ with $M \leq_{\mathbf{K}} M_a \leq_{\mathbf{K}} N_a$ and $N \leq_{\mathbf{K}} N_a$ such that $a \in M_a$ and $\mathbf{gtp}(b/M_a, N_a)$ does not $(\geq \mu)$-fork over M*

Under our other assumptions, μ-symmetry in $K_{(\lambda,\geq\mu^+)}$ implies non-forking amalgamation of $(\geq \mu)$-non-forking in $K_{(\lambda,\geq\mu^+)}$.

Fact 3.45. *Let $\mathbf{K}$ be an AEC, stable in $\mu \geq \mathrm{LS}(\mathbf{K})$ and stable also in $\lambda \geq \mu^+$, where $\mathbf{K}$ has AP, NMM, λ-JEP, and μ-tameness. Assume universal continuity of μ-non-splitting.*
Let $\underset{(\geq\mu)-f}{\bigcup}$ be $\bigcup$ restricted to models in $K_{(\lambda,\geq\mu^+)}$. If $\mathbf{K}$ has μ-symmetry in $K_{(\lambda,\geq\mu^+)}$, then $\bigcup$ has non-forking amalgamation.

Proof. By the same method as [Vas, 16.2], which is based on [Sh:h, II.2.16], but replacing the use of λ-symmetry with μ-symmetry in $K_{(\lambda,\geq\mu^+)}$ (note $\bigcup$ satisfies all the other properties used by Lemma 3.43). $\square$

Corollary 3.46. *Let $\mathbf{K}$ be an AEC, stable in $\mu \geq \mathrm{LS}(\mathbf{K})$ and stable also in $\lambda \geq \mu^+$, where $\mathbf{K}$ has AP, NMM, λ-JEP, and μ-tameness. Assume universal continuity of μ-non-splitting. Suppose also that $\mathbf{K}$ has μ-symmetry in $K_{(\lambda,\geq\mu^+)}$ (or just that $(\geq \mu)$-non-forking restricted to models in $K_{(\lambda,\geq\mu^+)}$ satisfies non-forking amalgamation).*
Let $\delta_1, \delta_2 < \lambda^+$ be limit ordinals where $\mu^+ \leq \mathrm{cf}(\delta_1), \mathrm{cf}(\delta_2)$. If $M, N_1, N_2 \in K_\lambda$ where N_l is a (λ, δ_l)-limit over M for $l = 1, 2$, then there is an isomorphism from N_1 to N_2 fixing M.
Moreover, if $N_1, N_2 \in K_\lambda$ where N_l is (λ, δ_l)-limit for $l = 1, 2$, then N_1 is isomorphic to N_2.

Proof. Follows directly from Lemma 3.43, Fact 3.45, and Theorem 3.1. $\square$

4. Short limit models

Our goal in this section is to show that, in a very general setting, all the low cofinality limit models are non-isomorphic:

Theorem 4.1. *Assume Hypothesis 4.2. Suppose* **K** *is* $\aleph_0$-*tame. If* $\text{cf}(\delta_1) < \kappa(\bot, \mathbf{K}_\lambda, \leq^u_\mathbf{K})$ *and* $\text{cf}(\delta_1) \neq \text{cf}(\delta_2)$ *, then the* (λ, δ_1)-*limit model is not isomorphic to the* (λ, δ_2)-*limit model.*

We assume the following hypothesis throughout this section.

Hypothesis 4.2. *Let* **K** *be an AEC and* $\lambda \geq \text{LS}(\mathbf{K})$ *such that* **K** *is stable in* λ *and* $\mathbf{K}_\lambda$ *has AP, JEP, and NMM, and* $\bot$ *is an independence relation on* $\mathbf{K}_\lambda$ *satisfying invariance, monotonicity, base monotonicity, uniqueness, extension, universal continuity, and* $(\geq \kappa)$-*local character in some regular* $\kappa \leq \lambda$.

Remark 4.3. *By Remark 2.34, the minimal such* κ *is* $\kappa(\bot, \mathbf{K}_\lambda, \leq^u_\mathbf{K})$ *from Definition 2.32. In particular* $\kappa(\bot, \mathbf{K}_\lambda, \leq^u_\mathbf{K}) \leq \lambda < \infty$.

4.1. **Relating** $\bot$ **to splitting and** λ-**forking.** This short subsection studies how several independence relations interact with each other in a strictly stable set-up (assuming Hypothesis 4.2). It finishes with a canonicity result of independence relations for long limit models.

The following is essentially [BGKV16, 4.2] (see also [Vas, 14.1]). The method is due to Shelah [She78, Lemma III.1.9*].

Fact 4.4. *Let* $M \leq_\mathbf{K} N$ *in* $\mathbf{K}_\lambda$ *and* $p \in \mathbf{gS}(N)$. *If* p $\bot$-*does not fork over* M, *then* p *does not* λ-*split over* M.

Proof. Assume p $\bot$-does not fork over M. Suppose $N_1, N_2 \in \mathbf{K}_\lambda$ with $M \leq N_l \leq N$ for $l = 1, 2$. Suppose further that $f : N_1 \cong_M N_2$. By monotonicity and invariance, $f(p \restriction N_1) \in \mathbf{gS}(N_2)$ $\bot$-does not fork over M. By monotonicity $p \restriction N_2$ $\bot$-does not fork over M. Since $f(p \restriction N_1) \restriction M = (p \restriction N_2) \restriction M$, we have by uniqueness of $\bot$ that $f(p \restriction N_1) = p \restriction N_2$. Therefore p does not λ-split over M. $\qquad\square$

For our next result, we use the following fact.

Fact 4.5 ([Van06, I.4.12]). *(Weak uniqueness of splitting) Let* $M_0 \leq^u_\mathbf{K} M_1 \leq_\mathbf{K} M_2$ *all in* $\mathbf{K}_\lambda$. *If* $p, q \in \mathbf{gS}(M_2)$, $p \restriction M_1 = q \restriction M_1$, *and* p, q *do not* λ-*split over* M_0, *then* $p = q$.

The following argument has some similarities with the second half of [Vas, 14.1], and with [Leu24, 4.18].

Lemma 4.6. *Suppose* $L_1 \leq_\mathbf{K} L_2 \leq_\mathbf{K} M$ *are all in* $\mathbf{K}_\lambda$, L_2 *is a* $(\lambda, \geq \kappa^u_\lambda(\bot))-$*limit model over* L_1, *and* $p \in \mathbf{gS}(M)$. *If* p *does not* λ-*split over* L_1, *then* p $\bot$-*does not fork over* L_2.

Proof. Observe that $p \restriction L_2$ $\bot$-does not fork over L_2 by Lemma 3.10, . There exists $q \in \mathbf{gS}(M)$ such that q $\bot$-does not fork over L_2 and $q \restriction L_2 = p \restriction L_2$ by extension of $\bot$. Let $\langle M_i : i < \theta \rangle$ witness that L_2 is a $(\lambda, \geq \kappa^u_\lambda(\bot))$-limit model over L_1. By $(\geq \kappa^u_\lambda(\bot))$-local character, there is an $i < \theta$ such that $q \restriction L_2$ $\bot$-does not fork over M_i. Then q $\bot$-does not fork over M_i by transitivity of $\bot$. Therefore, q does not λ-split over M_i by Fact 4.4.

As $L_1 \leq_\mathbf{K} M_i$, p does not λ-split over M_i by base monotonicity of splitting. We showed earlier that q does not λ-split over M_i, and we know that $p \restriction L_2 = q \restriction L_2$, and L_2 is universal over M_i. Therefore, $p = q$ by weak uniqueness. As q $\bot$-does not fork over L_2, we have that p $\bot$-does not fork over L_2 as desired. $\qquad\square$

The following result is a partial converse of Fact 4.4.

Lemma 4.7. *Suppose $L_1 \leq^u_{\mathbf{K}} L_2 \leq_{\mathbf{K}} M$ are all in $\mathbf{K}_\lambda$, and $p \in \mathbf{gS}(M)$. If p does not λ-split over L_1, then $p \underset{\smile}{\perp}$-does not fork over L_2.*

Proof. Let $\langle M_i : i \leq \kappa^u_\lambda(\perp) \rangle$ be a $\leq^u_{\mathbf{K}}$-increasing continuous sequence with $M_0 = L_1$. By universality of L_2 over L_1, we may assume $M_i \leq_{\mathbf{K}} L_2$ for all $i \leq \kappa^u_\lambda(\perp)$. Then $p \underset{\smile}{\perp}$-does not fork over $M_{\kappa^u_\lambda(\perp)}$ by Lemma 4.6. Hence $p \underset{\smile}{\perp}$-does not fork over L_2 by base monotonicity. $\square$

Corollary 4.8. $\underline{\kappa}(\perp, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) = \underline{\kappa}(\underset{\text{split}}{\perp}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$. *In particular,* $\kappa(\underset{\text{split}}{\perp}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) = \kappa^u_\lambda(\perp)$.

Proof. $\underline{\kappa}(\perp, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) \subseteq \underline{\kappa}(\underset{\text{split}}{\perp}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ by Fact 4.4 (this inclusion also appears in [Vas18b, 3.9(2)]) and $\underline{\kappa}(\underset{\text{split}}{\perp}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) \subseteq \underline{\kappa}(\perp, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ by Lemma 4.7. $\square$

Corollary 4.9. *Suppose $L \leq_{\mathbf{K}} M$ are in $\mathbf{K}_\lambda$ and $p \in \mathbf{gS}(M)$. If p does not λ-fork over L, then $p \underset{\smile}{\perp}$-does not fork over L.*

Proof. Suppose $L \leq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$, and $p \in \mathbf{gS}(M)$ does not λ-fork over L. Then by the definition of λ-forking (Definition 2.21) there is $L_1 \in \mathbf{K}_\lambda$ such that $L_1 \leq^u_{\mathbf{K}} L$ and p does not λ-split over L_1. Hence $p \underset{\smile}{\perp}$-does not fork over L by Lemma 4.7. $\square$

The following result is the latest of a family of results dealing with canonicity of independence relations [BGKV16, 5.19] [Vas16a, 9.6], [Vas19, 2.5]. A key difference between our result and those just mentioned is that a priori we do not know if λ-non-forking has uniqueness.

Theorem 4.10. *Assume Hypothesis 4.2. Suppose $L \leq_{\mathbf{K}} M$ are in $\mathbf{K}_\lambda$, L is a $(\lambda, \geq \kappa^u_\lambda(\perp))$-limit model, and $p \in \mathbf{gS}(M)$. Then p does not λ-fork over L if and only if $p \underset{\smile}{\perp}$-does not fork over L.*

Proof. The forward implication follows from the Corollary 4.9. For the backward implication, suppose $p \underset{\smile}{\perp}$-does not fork over L. Let $\langle M_i : i \leq \theta \rangle$ witness that L is a $(\lambda, \geq \kappa^u_\lambda(\perp))$-limit model. Then by $(\geq \kappa^u_\lambda(\perp))$-local character there is $i < \theta$ such that $p{\upharpoonright}L \underset{\smile}{\perp}$-does not fork over M_i. Then $p \underset{\smile}{\perp}$-does not fork over M_i by transitivity of $\underset{\smile}{\perp}$. Hence p does not λ-split over M_i by Fact 4.4. As $M_i \leq^u_{\mathbf{K}} L$, it follows that p does not λ-fork over L. $\square$

Remark 4.11. *Corollary 4.10 provides a correct proof of [Leu24, 4.18] assuming Hypothesis 4.2 (see Remark 2.30).*

4.2. **Non-isomorphism results.** This subsection has the main result of the section. Recall that we are still assuming Hypothesis 4.2.

Part of the argument of the next Lemma is a special case of [Vas22, 4.6], when $\chi = \delta^+$ and $\theta = \delta$. We include that part of the argument for completeness.

Lemma 4.12. *Suppose that $\mathbf{K}$ is δ-tame for some regular $\delta < \lambda^+$. Suppose $\langle M_i : i \leq \delta \rangle$ is a $\leq^u_{\mathbf{K}}$-increasing continuous chain in $\mathbf{K}_\lambda$. If M_δ is δ^+-saturated and $p \in \mathbf{gS}(M_\delta)$, then there is an $i < \delta$ such that $p \underset{\smile}{\perp}$-does not fork over M_i.*

Proof. Observe that it is enough to show that p does not λ-split over M_i for some $i < \delta$ since then one has that $p \underset{\smile}{\perp}$-does not fork over M_{i+1} for some $i < \delta$ by Lemma 4.7.

Suppose for a contradiction that p λ-splits over M_i for every $i < \delta$. Then for each $i < \delta$, there exist $N^1_i, N^2_i \in \mathbf{K}_\lambda$ and $f_i : N^1_i \underset{M_i}{\cong} N^2_i$ such that $M_i \leq_{\mathbf{K}} N^l_i \leq_{\mathbf{K}} M_\delta$ for $l = 1, 2$ and $f_i(p \upharpoonright N^1_i) \neq p \upharpoonright N^2_i$.

For each $i < \delta$, by δ-tameness there exists $A_i \subseteq N^2_i$ such that $|A_i| \leq \delta$ and

$$(\dagger) \qquad\qquad f_i(p \upharpoonright N^1_i) \upharpoonright A_i \neq (p \upharpoonright N^2_i) \upharpoonright A_i$$

For each $i < \delta$, let $B_i = f_i^{-1}(A_i) \cup A_i$. Let $B = \bigcup_{i<\delta} B_i$. Note that $|B| \leq \delta$ as $|B_i| \leq \delta$ for all $i < \delta$.

Since M_δ is δ^+-saturated, there exists $b \in M$ such that $\mathbf{gtp}(b/B, M_\delta) = p \upharpoonright B$. As $\langle M_i : i \leq \delta \rangle$ is continuous, there is some $i < \delta$ such that $b \in M_i$. Let $g : M_\delta \cong N$ such that $f_i \subseteq g$ and $N_i^2 \leq_{\mathbf{K}} N$.

On one hand.

$$(p \upharpoonright N_i^2) \upharpoonright A_i = p \upharpoonright A_i = \mathbf{gtp}(b/A_i, M_\delta) = \mathbf{gtp}(b/A_i, N_i^2) = \mathbf{gtp}(b/A_i, N)$$

where the last two equalities follow from $b \in M_i \leq N_i^2 \leq_{\mathbf{K}} M_\delta$ and $N_i^2 \leq_{\mathbf{K}} N$.

On the other hand,

$$f_i(p \upharpoonright N_i^1) \upharpoonright A_i = f_i(p \upharpoonright f_i^{-1}(A_i)) = g(\mathbf{gtp}(b/f_i^{-1}(A_i), M_\delta)) = \mathbf{gtp}(g(b)/A_i, N) = \mathbf{gtp}(b/A_i, N)$$

where the last equality follows from the fact that $b \in M_i$ and $g \upharpoonright M_i = f_i \upharpoonright M_i = \mathrm{id}_{M_i}$.

Hence $(p \upharpoonright N_i^2) \upharpoonright A_i = f_i(p \upharpoonright N_i^1) \upharpoonright A_i$. This contradicts equation (†).   □

Lemma 4.13. *Suppose $\mu_1 < \mu_2 < \lambda^+$ are infinite regular cardinals and that $\mathbf{K}$ is μ_1-tame. If M_1 is a (λ, μ_1)-limit model, M_2 is a (λ, μ_2)-limit model and M_1 is isomorphic to M_2, then $\kappa_\lambda^u(\bot) \leq \mu_1$.*

Proof. It is enough to show that $\mu_1 \in \underline{\kappa}(\bot, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ by the minimality of $\kappa_\lambda^u(\bot)$. Let $\{N_i : i \leq \mu_1\}$ be a $\leq_{\mathbf{K}}^u$-increasing continuous chain in $\mathbf{K}_\lambda$ and $p \in \mathbf{gS}(N_{\mu_1})$.

As M_1 is isomorphic to M_2 and M_2 is a (λ, μ_2)-limit model, it follows from Fact 2.10 that M_1 is μ_2-saturated. So in particular M_1 is $(\mu_1)^+$-saturated. Since M_1 is a (λ, μ_1)-limit model, M_1 is isomorphic to N_{μ_1} by Fact 2.7. Hence N_{μ_1} is $(\mu_1)^+$-saturated. Then there is $i < \mu_1$ such that $p \bot$-does not fork over N_i by Lemma 4.12 as $\mathbf{K}$ is μ_1-tame. Therefore $\mu_1 \in \underline{\kappa}(\bot, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$.   □

Remark 4.14. *We only used μ_1-tameness in Lemma 4.13. However we can only use the lemma to understand the full spectrum of limit models if we assume $\aleph_0$-tameness.*

The main result of this section follows from the previous Lemma and the fact that if $\mathbf{K}$ is $\aleph_0$-tame then it is μ-tame for every infinite cardinal μ.

Theorem 4.1. *Assume Hypothesis 4.2. Suppose $\mathbf{K}$ is $\aleph_0$-tame. If $\mathrm{cf}(\delta_1) < \kappa(\bot, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ and $\mathrm{cf}(\delta_1) \neq \mathrm{cf}(\delta_2)$, then the (λ, δ_1)-limit model is not isomorphic to the (λ, δ_2)-limit model.*

Tameness plays a key role in the main result of this section. A key natural question is the following.

Problem 4.15. *Is Theorem 4.1 true without the tameness assumption?*

5. General results

The objective of this section is to combine the results of Section 3 and 4 in a natural set up. We hope that these results can be used as *black boxes* when studying limit models in natural abstract elementary classes. We showcase how this can be achieved in the next section.

5.1. **Main results.** The following result puts together the main results of the previous two sections. As such the result is a local result on limit models.

Theorem 5.1. *Let $\mathbf{K}$ be a $\aleph_0$-tame abstract elementary class stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$ with λ-AP, λ-JEP, and λ-NMM. Suppose $\bot$ is an independence relation on the models of size λ that satisfies uniqueness, extension, universal continuity, $(\geq \kappa)$-local character in some regular $\kappa \leq \lambda$, and non-forking amalgamation.*

Suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ over } M \iff \mathrm{cf}(\delta_1) \geq \kappa(\bot, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$$

Moreover, for any $N_1, N_2, \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \iff \mathrm{cf}(\delta_1) \geq \kappa(\bot, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$$

Proof. We can restrict ourselves to regular δ_1, δ_2 by Fact 2.7. Our assumption imply that $\mathbf{K}$ and $\bot$ satisfy Hypothesis 3.7 with $\kappa = \kappa(\bot, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ and Hypothesis 4.2. The conclusion of Theorem 4.1 gives both of the forward implications. The conclusion to Theorem 3.1 gives both of the reverse implications. $\square$

Remark 5.2. *The value of Theorem 5.1 is that we can fully understand the limit models locally just by computing $\kappa(\bot, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$.*

Remark 5.3. *Observe that in Theorem 5.1 one can substitute every occurrence of $\kappa(\bot, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ with $\kappa(\underset{\mathrm{split}}{\bot}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ by Corollary 4.8.*

Conjecture 1.1 of [BoVan24] follows directly from the previous result under the assumptions of Theorem 5.1.

Corollary 5.4. *Let $\mathbf{K}$ be a $\aleph_0$-tame abstract elementary class stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$ a regular cardinal with λ-AP, λ-JEP, and λ-NMM. Suppose $\bot$ is an independence relation on the models of size λ that satisfies uniqueness, extension, universal continuity, $(\geq \kappa)$-local character in some regular $\kappa \leq \lambda$, and non-forking amalgamation. . Let*

$$\Gamma = \{\alpha < \lambda^+ : cf(\alpha) = \alpha \text{ and the } (\lambda, \alpha)\text{-limit model is isomorphic to the } (\lambda, \lambda)\text{-limit model }\}.$$

Then $\Gamma = [\kappa(\bot, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}), \lambda^+) \cap \mathrm{Reg}$.

We know focus on obtaining a global result. The following hypothesis will be the main assumption for the rest of the section, but we will explicitly mention when it is being assumed.

Hypothesis 5.5. *Let $\mathbf{K}$ be an AEC.*

(1) *$\mathbf{K}$ has AP, JEP and NMM.*
(2) *$\mathbf{K}$ is a stable.*
(3) *$\bot$ is an independence relation on $\mathbf{K}$ that satisfies invariance, monotonicity, base monotonicity, uniqueness, extension, universal continuity, $(\geq \kappa)$-local character in some regular $\kappa \leq \mathrm{LS}(\mathbf{K})^+$, and non-forking amalgamation.*

The following is essentially [LRV19, 8.8], but the argument there has a minor error: if $A \not\supseteq M_0$ and $p \in \mathbf{gS}(A)$, then whether $p \bot$-forks over M_0 or not is not well defined. This is easy to correct - instead of using uniqueness on the types $p \restriction A$ and $q \restriction A$, apply it to $p \restriction M_0 \cup A$ and $q \restriction M_0 \cup A$, which are both $\bot$-non-forking. This still gives $p \restriction A = q \restriction A$. We provide the details in our context for convenience (note $M_0 \cup A$ is replaced by some model N_0 containing $|M_0| \cup A$ since we have not defined $\bot$-forking over types with domains that are not models).

Fact 5.6. *Let $\mathbf{K}$ be an AEC with an independence relation $\bot$ satisfying extension and uniqueness[1]. Suppose $\mathbf{K}$ is stable in $\lambda \geq \mathrm{LS}(\mathbf{K})$, and $\theta \leq \lambda$. If $\mathbf{K}$ is $(< \theta)$-tame, then $\bot$ has the $(< \theta)$-witness property. In particular if $\mathbf{K}$ is $(< \aleph_0)$-tame, then $\bot$ has the $(< \aleph_0)$-witness property*

---

[1]Note this all holds assuming Hypothesis 5.5.

Proof. Suppose $M \leq_{\mathbf{K}} N$ in $\mathbf{K}$ and $p \in \mathbf{gS}(N)$ satisfies that for all $A \subseteq |N|$ where $|A| < \theta$, there exists $N_0 \leq_{\mathbf{K}} N$ with $A \cup |M| \subseteq |N_0|$ and $p \restriction N_0 \mathop{\smile}\limits_{\textstyle\frown}$-does not fork over M. We must show p $\mathop{\smile}\limits_{\textstyle\frown}$-does not fork over M.

Using $A = \varnothing$ and monotonicity, $p \restriction M \mathop{\smile}\limits_{\textstyle\frown}$-does not fork over M. By extension, there is $q \in \mathbf{gS}(N)$ extending $p \restriction M$ such that $q \mathop{\smile}\limits_{\textstyle\frown}$-does not fork over M. It suffices to show $p = q$. By $(< \theta)$-tameness, it is enough to show $p \restriction A = q \restriction A$ for all $A \subseteq |N|$ where $|A| < \theta$.

So fix $A \subseteq |N|$ with $|A| < \theta$. By our hypothesis, there exists $N_0 \leq_{\mathbf{K}} N$ with $A \cup |M| \subseteq |N_0|$ and $p \restriction N_0 \mathop{\smile}\limits_{\textstyle\frown}$-does not fork over M. By monotonicity, $q \restriction N_0 \mathop{\smile}\limits_{\textstyle\frown}$-does not fork over M either. As $p \restriction M = q \restriction M$, uniqueness gives $p \restriction N_0 = q \restriction N_0$. In particular, $p \restriction A = q \restriction A$, as desired. $\square$

We show [Vas18b, 4.5] holds under Hypothesis 5.5 when non-splitting is replaced by $\mathop{\smile}\limits_{\textstyle\frown}$ for $\lambda \geq \mathrm{LS}(\mathbf{K})$. In fact, this result holds for $\mathop{\smile}\limits_{\textstyle\frown}$ under assumptions even weaker than Hypothesis 5.5, as we show now.

Lemma 5.7. *Let $\mathbf{K}$ be an AEC with an independence relation $\mathop{\smile}\limits_{\textstyle\frown}$ satisfying extension, uniqueness, and universal continuity* [2]. *Let $\mathbf{K}$ be stable in λ and μ with $\mathrm{LS}(\mathbf{K}) \leq \mu < \lambda$ and let $\theta \leq \mu$. If $\mathop{\smile}\limits_{\textstyle\frown}$ has the θ-witness property, then $\underline{\kappa}(\mathop{\smile}\limits_{\textstyle\frown}, \mathbf{K}_\mu, \leq_{\mathbf{K}}^u) \subseteq \underline{\kappa}(\mathop{\smile}\limits_{\textstyle\frown}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$. In particular, $\kappa(\mathop{\smile}\limits_{\textstyle\frown}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) \leq \kappa(\mathop{\smile}\limits_{\textstyle\frown}, \mathbf{K}_\mu, \leq_{\mathbf{K}}^u)$.*

Proof. Let $\delta \in \underline{\kappa}(\mathop{\smile}\limits_{\textstyle\frown}, \mathbf{K}_\mu, \leq_{\mathbf{K}}^u)$. We may assume δ is regular, as whether $\delta \in \underline{\kappa}(\mathop{\smile}\limits_{\textstyle\frown}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ is determined by $\mathrm{cf}(\delta)$. Suppose $\langle M_i : i \leq \delta \rangle$ is a $\leq_{\mathbf{K}}^u$-increasing continuous chain in $\mathbf{K}_\lambda$ and $p \in \mathbf{gS}(M_\delta)$.

Suppose for contradiction that $p \mathop{\smile}\limits_{\textstyle\frown}$- forks over M_i for all $i \leq \delta$. First we show we can assume without loss of generality that for each $i < \delta$, M_{i+1} is a (λ, μ^+)-limit over M_i. Taking $M_i' = M_i$ for $i = 0$ or limit, and M_{i+1}' to be a (λ, μ^+)-limit over M_i with $M_{i+1}' \leq_{\mathbf{K}} M_{i+1}$ (using universality of M_{i+1} over M_i), we see the sequence $\langle M_i' : i < \delta \rangle$ satisfies all the same conditions as M_i - that is, it is continuous, $M_\delta' = M_\delta$, and $p \mathop{\smile}\limits_{\textstyle\frown}$-forks over M_i' for $i < \delta$ by base monotonicity. Hence (replacing M_i with M_i') we may assume for $i < \delta$ that M_{i+1} is (λ, μ^+)-limit over M_i.

Now we show that without loss of generality this sequence witnesses failure of universal weak local character (this is similar to [BGVV, 11(1)]), i.e., that $p \restriction M_{i+1} \mathop{\smile}\limits_{\textstyle\frown}$-forks over M_i for all $i < \delta$.

Construct a continuous $<$-increasing sequence of ordinals $\langle i_j : j \leq \delta \rangle$ such that $i_j < \delta$ for $j < \delta$, and $p \restriction M_{i_{j+1}} \mathop{\smile}\limits_{\textstyle\frown}$-forks over M_{i_j}. To do this, set $i_0 = 0$, at limits take $i_j = \sup_{j' < j} i_{j'}$ (possible by regularity of δ), and for successors, since $p \mathop{\smile}\limits_{\textstyle\frown}$-forks over M_{i_j+1}, by applying universal continuity to p and the sequence $\langle M_i : i > i_j + 1 \rangle$, there is $i_{j+1} > i_j + 1$ such that $p \restriction M_{i_{j+1}}$ $\mathop{\smile}\limits_{\textstyle\frown}$-forks over M_{i_j+1}. Setting $M_0'' = M_0$, $M_j'' = M_{i_j}$ for $j < \delta$ limit, and $M_j'' = M_{i_j+1}$ for $j < \delta$ successor, we have that $\langle M_j'' : j < \delta \rangle$ is $\leq_{\mathbf{K}}^u$-increasing continuous with $p \restriction M_{j+1}'' \mathop{\smile}\limits_{\textstyle\frown}$-forking over M_j'' and M_{j+1}'' is (λ, μ^+)-limit over M_j'' for all $j < \delta$ (this is why we used $i_j + 1$ rather than i_j, which may not be a successor). Thus without loss of generality we now have (replacing M_i by M_i'') that $p \restriction M_{i+1} \mathop{\smile}\limits_{\textstyle\frown}$-forks over M_i for all $i < \delta$.

We now construct $\langle N_i : i \leq \delta \rangle$, a $\leq_{\mathbf{K}}^u$-increasing continuous sequence in $\mathbf{K}_\mu$, such that for all $i < \delta$,

    (1) $N_i \leq_{\mathbf{K}} M_i$
    (2) $p \restriction N_{i+1} \mathop{\smile}\limits_{\textstyle\frown}$-forks over N_i

This is possible: We may take N_0 to be any $N_0 \leq_{\mathbf{K}} M_0$ in K_μ, which satisfies requirement (1). At limit i, take unions (which preserves requirements (1) and (2)).

For successors, given N_i, we know $p \restriction M_{i+1} \mathop{\smile}\limits_{\textstyle\frown}$-forks over N_i by base monotonicity. By the θ-witness property, there is $A \subseteq |M_{i+1}|$ where $|A| \leq \theta$ such that for all $N \leq_{\mathbf{K}} M_{i+1}$ with

---

[2]Note this all holds assuming Hypothesis 5.5.

$A \cup |N_i| \subseteq N$, $p \restriction N \mathrel{\underset{}{\downarrow}}$-forks over N_i. Let $\langle M_i^j : j < \mu^+ \rangle$ witness that M_{i+1} is (λ, μ^+)-limit over M_i. As $|A| \leq \theta \leq \mu$ and $\|N_i\| \leq \mu$, there is some $j < \mu^+$ such that $A \cup |N_i| \subseteq |M_i^j|$. As M_i^{j+1} is universal over M_i^j, there is $N_{i+1} \in K_\mu$ where $N_i \leq_{\mathbf{K}}^u N_{i+1} \leq_{\mathbf{K}} M_{i+1}$ and $A \subseteq N_{i+1}$. By our choice of A, we know $p \restriction N_{i+1} \mathrel{\underset{}{\downarrow}}$-forks over N_i, so this N_{i+1} is as required. This completes the construction.

This is enough: This sequence $\langle N_i : i \leq \delta \rangle$ along with $p \restriction N_\delta$ shows that $\delta \notin \underline{\kappa}^{\mathrm{wk}}(\downarrow, \mathbf{K}_\mu, \leq_{\mathbf{K}}^u)$. But this contradicts Remark 2.34. $\qquad\square$

Remark 5.8. *A couple of results in this section will use results of [Vas18b] which assume that splitting has universal continuity. When we use those results we will assume that the AEC is $(< \aleph_0)$-tame. As universal continuity of splitting follows from $(< \aleph_0)$-tameness (by [MaYa24, 2.5(1)] and [MVY, 3.2]) we can use those results.*

In those results, instead of assuming $(< \aleph_0)$-tame one could assume $\aleph_0$-tameness and work in $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})^+}$ instead of in $\mathbf{K}$ (see Remark 5.26 for more details).

The notation of the following two definitions is similar to that of [Vas18b, 4.6].

Definition 5.9. *Assume $\downarrow$ is an independence relation.*
   (1) $\mathrm{Stab}(\mathbf{K}) = \{\lambda \geq \mathrm{LS}(\mathbf{K}) : \mathbf{K} \text{ is stable in } \lambda\}$
   (2) $\underline{\chi}(\downarrow, \mathbf{K} \leq_{\mathbf{K}}^u) = \bigcup_{\lambda \in \mathrm{Stab}(\mathbf{K})} \underline{\kappa}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$
   (3) $\chi(\downarrow, \mathbf{K}, \leq_{\mathbf{K}}^u) = \min(\underline{\chi}(\downarrow, \mathbf{K} \leq_{\mathbf{K}}^u) \cap Reg)$ if it exists or ∞ otherwise.

Remark 5.10. *Our definitions differ slightly in character from [Vas18b, 4.6]. In particular $\chi(\underset{\mathrm{split}}{\downarrow}, \mathbf{K}, \leq_{\mathbf{K}}^u)$ is defined differently to $\chi(\mathbf{K})$ from [Vas18b, 4.6], but they are the same assuming Hypothesis 5.5 and $(< \aleph_0)$-tameness by [Vas18b, 2.9, 4.5].*

Corollary 5.11. *Assume Hypothesis 5.5. If $\mathbf{K}$ is stable in $\lambda \geq \mathrm{LS}(\mathbf{K})^+$, then $\underline{\kappa}(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \underline{\kappa}(\underset{\mathrm{split}}{\downarrow}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ and $\kappa(\downarrow, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u) = \kappa(\underset{\mathrm{split}}{\downarrow}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$. In particular, if $\mathbf{K}$ is $(< \aleph_0)$-tame then $\chi(\downarrow, \mathbf{K}, \leq_{\mathbf{K}}^u) = \chi(\underset{\mathrm{split}}{\downarrow}, \mathbf{K}, \leq_{\mathbf{K}}^u)$.*

Proof. The first part follows from Corollary 4.8. The *in particular* follows from the result and [Vas18b, 4.5], which says that $\kappa(\underset{\mathrm{split}}{\downarrow}, \mathbf{K}_\lambda, \leq_{\mathbf{K}}^u)$ is weakly decreasing with λ, and Lemma 5.7. $\quad\square$

Definition 5.12. *Assume Hypothesis 5.5. Let $\theta(\mathbf{K}, \downarrow)$ be the least stability cardinal $\theta \geq \mathrm{LS}(\mathbf{K})$ such that $\kappa(\downarrow, \mathbf{K}_\mu, \leq_{\mathbf{K}}^u) = \kappa(\downarrow, \mathbf{K}_\theta, \leq_{\mathbf{K}}^u)$ for every stability cardinal $\mu \geq \theta$, or ∞ if no such cardinal exists.*

The following follows from Fact 5.6 and Lemma 5.7.

Corollary 5.13. *Assume Hypothesis 5.5. If $\mathbf{K}$ is $\aleph_0$-tame, then $\theta(\mathbf{K}, \downarrow) < \infty$ and $\chi(\downarrow, \mathbf{K}, \leq_{\mathbf{K}}^u) = \kappa(\downarrow, \mathbf{K}_{\theta(\mathbf{K}, \downarrow)}, \leq_{\mathbf{K}}^u)$.*

The following result is a global version of Theorem 5.1.

Theorem 5.14. *Assume Hypothesis 5.5. Suppose $\mathbf{K}$ is $\aleph_0$-tame.*
   Let $\lambda \geq \theta(\downarrow, \mathbf{K})$ such that $\mathbf{K}$ is stable in λ.
   Suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ over } M \iff \mathrm{cf}(\delta_1) \geq \chi(\downarrow, \mathbf{K}, \leq_{\mathbf{K}}^u)$$

   Moreover, for any $N_1, N_2, \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \iff \mathrm{cf}(\delta_1) \geq \chi(\downarrow, \mathbf{K}, \leq_{\mathbf{K}}^u)$$

Proof. As $\lambda \geq \theta(\perp, \mathbf{K})$, it follows that $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u}) = \kappa(\perp, \mathbf{K}_{\lambda}, \leq_{\mathbf{K}}^{u})$. The result then follows from Theorem 5.1. $\square$

Remark 5.15. *The value of Theorem 5.14 is that we only need to compute $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u})$ and $\theta(\mathbf{K}, \perp)$ to understand the λ-limit models for large enough λ.*

Remark 5.16. *Observe that in Theorem 5.14 one can substitute every occurrence of $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u})$ with $\chi(\underset{\text{split}}{\perp}, \mathbf{K}, \leq_{\mathbf{K}}^{u})$ by Corollary 4.8.*

5.2. **Toward applications.** If one wants to simply apply Theorem 5.14 as a *black box* to understand limit models in a natural AEC, it still has the shortfall that one needs to calculate both $\theta(\perp, \mathbf{K})$ and $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u})$. In this subsection we focus on simplifying these calculations.

We focus first on $\theta(\perp, \mathbf{K})$.

Definition 5.17 ([Vas18b, 4.9]). *Assume Hypothesis 5.5. $\lambda'(\mathbf{K})$ is the least stability cardinal $\lambda' \geq \text{LS}(\mathbf{K})$ such that $\kappa(\underset{\text{split}}{\perp}, \mathbf{K}_{\mu}, \leq_{\mathbf{K}}^{u}) = \kappa(\underset{\text{split}}{\perp}, \mathbf{K}_{\lambda'}, \leq_{\mathbf{K}}^{u})$ for every stability cardinal $\mu \geq \lambda'$, or ∞ if no such cardinal exists.*

Remark 5.18. *This is a little different from how $\lambda'(\mathbf{K})$ is defined in [Vas19, 4.9], to accommodate for us not including the regular cardinals $\geq \lambda^{+}$ in our definition of $K_{(\lambda, \geq \kappa)}$, but it is equivalent under Hypothesis 5.5 and $\aleph_0$-tameness by [Vas19, 4.5].*

Lemma 5.19. *Assume Hypothesis 5.5. If $\mathbf{K}$ is $(< \aleph_0)$-tame, then $\theta(\mathbf{K}, \perp) < \beth_{\beth_{(2^{LS(\mathbf{K})})^{+}}}$.*

Proof. We show that $\theta(\mathbf{K}, \perp) \leq \lambda'(\mathbf{K})^{+}$. This is enough as $\lambda'(\mathbf{K}) < \beth_{\beth_{(2^{LS(\mathbf{K})})^{+}}}$ by [Vas18b, 11.3]. Observe that we can use [Vas18b, §11] by Remark 5.8.

Suppose $\lambda'(\mathbf{K}) \geq \text{LS}(\mathbf{K})^{+}$, then for every $\mu \geq \lambda'(\mathbf{K})$, $\kappa(\underset{\text{split}}{\perp}, \mathbf{K}_{\mu}, \leq_{\mathbf{K}}^{u}) = \kappa(\underset{\text{split}}{\perp}, \mathbf{K}_{\lambda'}, \leq_{\mathbf{K}}^{u})$ by definition of $\lambda'(\mathbf{K})$. Hence $\kappa(\perp, \mathbf{K}_{\mu}, \leq_{\mathbf{K}}^{u}) = \kappa(\perp, \mathbf{K}_{\lambda'}, \leq_{\mathbf{K}}^{u})$ by Corollary 5.11 for every $\mu \geq \lambda'(\mathbf{K})$. Hence $\theta(\perp, \mathbf{K}) \leq \lambda'(\mathbf{K})$.

If $\lambda'(\mathbf{K}) = \text{LS}(\mathbf{K})$, then $\lambda'(\mathbf{K})^{+} = \text{LS}(\mathbf{K})^{+}$, so the argument of the previous paragraph shows that $\kappa(\perp, \mathbf{K}_{\mu}, \leq_{\mathbf{K}}^{u}) = \kappa(\perp, \mathbf{K}_{\lambda'(\mathbf{K})^{+}}, \leq_{\mathbf{K}}^{u})$ for all $\mu \geq \lambda'(\mathbf{K})^{+}$. Hence $\theta(\perp, \mathbf{K}) \leq \lambda'(\mathbf{K})^{+}$. $\square$

Remark 5.20. *In fact, the $\lambda'(\mathbf{K}) \geq \text{LS}(\mathbf{K})^{+}$ case in the previous lemma shows that in this case, $\lambda'(\mathbf{K}) = \theta(\perp, \mathbf{K})$, since $\kappa(\underset{\text{split}}{\perp}, \mathbf{K}_{\mu}, \leq_{\mathbf{K}}^{u}) = \kappa(\underset{\text{split}}{\perp}, \mathbf{K}_{\lambda'}, \leq_{\mathbf{K}}^{u})$ for all $\lambda \geq \text{LS}(\mathbf{K})^{+}$ and by definition both $\lambda'(\mathbf{K})$ and $\theta(\perp, \mathbf{K})$ are the λ at which this stabilises.*

Problem 5.21. *Find a better upper bound for $\theta(\mathbf{K}, \perp)$ or show that the result of Lemma 5.19 is sharp.*

We now focus on $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u})$.

Lemma 5.22. *Assume Hypothesis 5.5. Suppose $\mathbf{K}$ is $(< \aleph_0)$-tame. Assume ρ is an infinite cardinal and consider the following two statements:*

(a) For every $\lambda \geq 2^{LS(\mathbf{K})}$, if $\mathbf{K}$ is stable in λ, then $\lambda^{<\rho} = \lambda$.
(b) For every $\lambda \geq 2^{LS(\mathbf{K})}$, if $\lambda^{<\rho} = \lambda$, then $\mathbf{K}$ is stable in λ.

The following hold:

(1) If Statement (a) holds, then $\rho \leq \chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u})$.
(2) If Statement (b) holds and ρ is regular , then $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u}) \leq \rho$.
(3) If Statements (a) and (b) hold and ρ is regular, then $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u}) = \rho$.
(4) If Statement (b) holds and ρ is singular, then $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u}) \leq \rho^{+}$.
(5) If Statements (a) and (b) hold and ρ is singular, then $\chi(\perp, \mathbf{K}, \leq_{\mathbf{K}}^{u}) = \rho^{+}$.

Proof.

(1) This is essentially [Vas18b, 11.2] with the observation that $\chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}}) = \chi(\underset{\text{split}}{\perp}, \mathbf{K}, \leq^u_{\mathbf{K}}) =$ $\chi(\mathbf{K})$ by Corollary 5.11 and Remark 5.10.

(2) Let $\lambda = \beth_{\rho+\rho}(\theta)$ where $\rho+\rho$ is the ordinal given by ordinal arithmetic, $\theta = \theta(\mathbf{K}, \perp) + \chi_0$, and χ_0 is the cardinal given in [Vas18b, 4.10]. The precise value of χ_0 is not important, we only need to notice that $\lambda > \chi_0$.

Observe that $\lambda^{<\rho} = \lambda$ as $\text{cf}(\lambda) = \rho$ and ρ is regular. Hence $\mathbf{K}$ is stable in λ by Statement (b).

We show that $\mathbf{K}$ is stable in unboundably many cardinals below λ. Let $\mu < \lambda$, then there is an ordinal $\alpha \geq \rho$ such that $\mu \leq \beth_{\alpha+1}(\theta)$. Observe that

$$\beth_{\alpha+1}(\theta)^{<\rho} \leq \beth_{\alpha+1}(\theta)^{\rho} = 2^{\beth_\alpha(\theta)\rho} = 2^{\beth_\alpha(\theta)} = \beth_{\alpha+1}(\theta).$$

Hence $\beth_{\alpha+1}(\theta)^{<\rho} = \beth_{\alpha+1}(\theta)$ and $\mathbf{K}$ is stable in $\beth_{\alpha+1}(\theta) < \lambda$ by Statement (b).

Therefore, $\text{cf}(\lambda) = \rho \in \underline{\kappa}(\underset{\text{split}}{\perp}, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ by [Vas18b, 4.11]. Hence $\rho \in \underline{\kappa}(\perp, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}})$ by Corollary 5.11. Then $\kappa(\perp, \mathbf{K}_\lambda, \leq^u_{\mathbf{K}}) \leq \rho$ by minimality of κ. Hence $\chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}}) \leq \rho$ as $\lambda \geq \theta(\mathbf{K}, \perp)$ and Corollary 5.13.

(3) Follows directly from (1) and (2).

(4) The argument given in (2) works if one changes ρ by ρ^+ as if $\lambda^{<\rho^+} = \lambda$ then $\lambda^{<\rho} = \lambda$.

(5) Follows directly from (1) and (4).

$\square$

Remark 5.23. *The value of Theorem 5.22 is that it allows us to compute $\chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}})$ directly from the stability spectrum of the AEC.*

We present a revised version of Theorem 5.14 which serves better as a *black box* in applications (see next section). However in the case where $\theta(\mathbf{K}, \perp)$ can be computed, Theorem 5.14 gives a better bound (see Lemma 6.11).

Theorem 5.24. *Assume Hypothesis 5.5. Suppose $\mathbf{K}$ is $(< \aleph_0)$-tame.*
. Let $\lambda \geq \beth_{(2^{LS(\mathbf{K})})^+}$ such that $\mathbf{K}$ is stable in λ.

Suppose $\delta_1, \delta_2 < \lambda^+$ with $\text{cf}(\delta_1) < \text{cf}(\delta_2)$. Then for any $N_1, N_2, M \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model over M for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ over } M \iff \text{cf}(\delta_1) \geq \chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}})$$

Moreover, for any $N_1, N_2, \in \mathbf{K}_\lambda$ where N_l is a (λ, δ_l)-limit model for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \iff \text{cf}(\delta_1) \geq \chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}})$$

Proof. Since $\theta(\perp, \mathbf{K}) < \beth_{(2^{LS(\mathbf{K})})^+}$ by Lemma 5.19, the result follows from Theorem 5.14. $\square$

Remark 5.25. *This removes the need to calculate $\theta(\perp, \mathbf{K})$, but restricts which λ the result applies to. One can calculate $\chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}})$ using Theorem 5.22.*

Remark 5.26. *In fact, we can replace $(< \aleph_0)$-tameness with $\aleph_0$-tameness if we increase the lower bound for λ. Note we only needed the stronger form of tameness to guarantee universal continuity of splitting in $\mathbf{K}_\lambda$. But for $\lambda \geq LS(\mathbf{K})^+$, universal continuity of λ-non-splitting in $\mathbf{K}_\lambda$ follows from universal continuity of $\perp$ using Fact 4.4, Fact 4.5, and Lemma 4.7, universal continuity of $\perp$, and weak transitivity of λ-non-splitting [Vas16a, 3.7]. Note we need $\lambda \geq LS(\mathbf{K})^+$ as Lemma 4.7 relies on $(\geq \kappa)$-local character of $\perp$ for some $\kappa \leq \lambda$.*

Then by the same reasoning as before, but working in $\mathbf{K}_{\geq LS(\mathbf{K})^+}$ rather than $LS(\mathbf{K})$, we deduce the statement of Theorem 5.24 but with $(< \aleph_0)$-tameness, $\beth_{(2^{LS(\mathbf{K})})^+}$, and $\chi(\perp, \mathbf{K}, \leq^u_{\mathbf{K}})$

replaced by $\aleph_0$-*tameness,* $\beth\beth_{(2^{(LS(\mathbf{K})^+)})^+}$, *and* $\chi(\mathop{\smile}\limits^{\textstyle\perp}, \mathbf{K}_{\geq LS(\mathbf{K})^+}, \leq^u_{\mathbf{K}})$ *respectively. Note that under the hypotheses of Theorem 5.24,* $\chi(\mathop{\smile}\limits^{\textstyle\perp}, \mathbf{K}, \leq^u_{\mathbf{K}}) = \chi(\mathop{\smile}\limits^{\textstyle\perp}, \mathbf{K}_{\geq LS(\mathbf{K})^+}, \leq^u_{\mathbf{K}})$.

6. Some applications

The objective of this section is to present some applications of the results obtained in this paper. In particular, we will show how the results of the previous section can be used to understand the limit models of some natural AECs.

6.1. **Checking the assumptions.** Although Hypothesis 5.5 is a natural *theoretical* assumption, in applications independence relations in the sense of LRV [LRV19] (see Example 2.25) arise more naturally.

The following Lemma, which relies on the notions introduced in Example 2.25 and follows almost immediately from Lemma 2.26, shows when Hypothesis 5.5 follows from the existence of a stable independence relation in the sense of LRV. We include the result because we expect this result to be useful when studying limit models in natural abstract elementary classes. We showcase how to use this result later in this section.

Lemma 6.1. *Suppose* $\mathbf{K}$ *is an AEC with JEP, NMM, and* $\mathbf{K}$ *has a weakly stable independence relation in the sense of LRV [LRV19] with strong* $\mathrm{LS}(\mathbf{K})$-*local character. Then* $\mathbf{K}$ *satisfies Hypothesis 5.5 with the possible exception of universal continuity.*

Moreover, if $\mathbf{K}$ *is* $(< \aleph_0)$-*tame, then* $\mathbf{K}$ *satisfies Hypothesis 5.5.*

Furthermore, if $\mathbf{K}$ *is* $(< \aleph_0)$-*tame, for each stability cardinal* $\lambda \geq \mathrm{LS}(\mathbf{K})^+$, $\mathbf{K}$ *satisfies the hypothesis of Theorem 5.1.*

Proof. Observe that if $\lambda^{LS(\mathbf{K})} = \lambda$, then $\mathbf{K}$ is stable in λ (see for example [LRV19, 8.15] for an idea of the proof). The rest of the main statement all follows from Lemma 2.26.

For the moreover part, note that by $(< \aleph_0)$-tameness and Fact 5.6, $\mathop{\smile}\limits^{\textstyle\perp}$ has the $(< \aleph_0)$-witnessing property, and therefore universal continuity (the only property missing from the first part) by Lemma 2.18.

For the furthermore part, observe that the hypothesis of Theorem 5.1 are a local analogue of Hypothesis 5.5 so the result follows from the previous two parts. $\square$

6.2. **Modules with embeddings.** We showcase how our results can be used to understand the limit models in the abstract elementary class of modules with embeddings. Most of the results we present in this subsection were originally obtained in [Maz25] using algebraic methods and were one of the key motivations to write this paper.

Denote by $\mathbf{K}^{R\text{-}\mathrm{Mod}}$ the abstract elementary class of modules with embeddings. Observe that the language of this AEC is $\{+, -, 0\} \cup \{r\cdot : r \in R\}$ where $r\cdot$ is a unary function for every $r \in R$ and that $LS(\mathbf{K}^{R\text{-}\mathrm{Mod}}) = \mathrm{card}(R) + \aleph_0$.

Lemma 6.2. $\mathbf{K}^{R\text{-}Mod}$ *satisfies Hypothesis 5.5 and is* $(< \aleph_0)$-*tame.*

Proof. $\mathbf{K}^{R\text{-}\mathrm{Mod}}$ has joint embedding and no maximal models by for example [Maz21a, 3.1]. $\mathbf{K}^{R\text{-}\mathrm{Mod}}$ has a weakly stable independence relation in the sense of LRV with strong $(\mathrm{card}(R) + \aleph_0)$-local character by [MaRo, 3.9, 3.10, 3.11]. As $\mathbf{K}^{R\text{-}\mathrm{Mod}}$ is $(< \aleph_0)$-tame by [Maz21a, 3.3], it follows that $\mathbf{K}^{R\text{-}\mathrm{Mod}}$ satisfies Hypothesis 5.5 by Lemma 6.1. $\square$

Definition 6.3 ([Ekl71]). *Let* R *be a ring. Define* $\gamma(R)$ *as the minimum infinite cardinal such that every left ideal of* R *is* $(< \gamma(R))$-*generated. Let* $\gamma_r(R) = \gamma(R)$ *if* $\gamma(R)$ *is regular and* $\gamma(R)^+$ *if* $\gamma(R)$ *is singular.*

Observe that for every ring, $\gamma(R) \leq \mathrm{card}(R) + \aleph_0$.

Fact 6.4 ([Maz21b, 3.8])**.** *Assume $\lambda \geq (\mathrm{card}(R) + \aleph_0)^+$. $\mathbf{K}^{R\text{-}Mod}$ is stable in λ if and only if $\lambda^{<\gamma(R)} = \lambda$.*

The following result is new.

Lemma 6.5. $\chi(\perp, \mathbf{K}^{R\text{-}Mod}, \leq^u_{\mathbf{K}^{R\text{-}Mod}}) = \gamma_r(R)$.

Proof. $\mathbf{K}^{R\text{-}Mod}$ satisfies the assumptions of Lemma 5.22 by Lemma 6.2 with $\rho = \gamma(R)$ by Fact 6.4. The result then follows from Lemma 5.22. $\qquad\square$

Lemma 6.6. *Let $\lambda \geq \beth_{\beth_{(2^{\mathrm{card}(R)+\aleph_0})^+}}$ such that $\mathbf{K}^{R\text{-}Mod}$ is stable in λ.*

Suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2 \in \mathbf{K}_\lambda$ (and $M \in \mathbf{K}_\lambda$) where N_l is a (λ, δ_l)-limit model (over M) for $l = 1, 2$,

$$N_1 \text{ is isomorphic to } N_2 \text{ (over } M) \iff \mathrm{cf}(\delta_1) \geq \gamma_r(R)$$

Proof. The result follows directly from Lemma 6.5, Lemma 6.2 and Theorem 5.24. $\qquad\square$

Remark 6.7. *Lemma 6.6 is slightly weaker than the original result [Maz25, 3.22] as the original result has $\lambda \geq \gamma(R)^+$ instead of $\lambda \geq \beth_{\beth_{(2^{\mathrm{card}(R)+\aleph_0})^+}}$.*

It is possible that Lemma 6.6 could be improved to obtain the bound $\gamma(R)^+$ using Theorem 5.14, but in order to do that one would need to show that $\theta(\mathbf{K}^{R\text{-}Mod}, \perp) = (\mathrm{card}(R) + \aleph_0)^+$. We can actually calculate $\theta(\mathbf{K}^{R\text{-}Mod}, \perp)$ but using the original result [Maz25, 3.22] (see Remark 6.10).

Recall that for an integer $n \geq 0$, a ring R is $(< \aleph_n)$-noetherian if $\gamma(R) \leq \aleph_n$.

Theorem 6.8. *Let $n \geq 0$ be an integer. The following are equivalent.*
  (1) R is left $(< \aleph_n)$-noetherian but not left $(< \aleph_{n-1})$-noetherian.[3]
  (2) $\mathbf{K}^{R\text{-}Mod}$ has exactly $n + 1$ non-isomorphic λ-limit models for every $\lambda \geq \beth_{\beth_{(2^{\mathrm{card}(R)+\aleph_0})^+}}$
      such that $\mathbf{K}^{R\text{-}Mod}$ is stable in λ.

Proof. $(1) \Rightarrow (2)$: It follows from the assumption on the ring that $\gamma(R) = \aleph_n$. Then the $(\lambda, \aleph_s)$-limit models for $0 \leq s \leq n$ are all the λ-limit models up to isomorphisms by Lemma 6.6. Hence there are exactly $n + 1$ non-isomorphic λ-limit models.

$(2) \Rightarrow (1)$: Assume for the sake of contraction that (1) fails. Then either $\gamma(R) > \aleph_n$ or $\gamma(R) \leq \aleph_{n-1}$. Doing a similar argument to that of $(1) \Rightarrow (2)$ one can show that there are either $n + 2$ non-isomorphic λ-limit models or less than n non-isomorphic λ-limit models for $\lambda = \beth_{\beth_{(2^{\mathrm{card}(R)+\aleph_0})^+}}$. This is clearly a contradiction. $\qquad\square$

Remark 6.9. *Theorem 6.8 is also slightly weaker than the original result [Maz25, 3.17] as the original result has $\lambda \geq (\mathrm{card}(R) + \aleph_0)^+$ instead of $\lambda \geq \beth_{\beth_{(2^{\mathrm{card}(R)+\aleph_0})^+}}$.*

Remark 6.10. *Our results can also be used to find $\theta(\mathbf{K}^{R\text{-}Mod}, \perp)$ in a roundabout way. Let θ_0 be the least stability cardinal with $\theta_0 \geq (\mathrm{card}(R) + \aleph_0)^+$. Observe that $\theta_0 \leq 2^{\mathrm{card}(R)+\aleph_0}$ by Fact 6.4.*

Theorem 5.1 and [Maz25, 3.22] give the 'cut offs' of where limit models go from non-isomorphic to isomorphic as $\kappa(\perp, \mathbf{K}^{R\text{-}Mod}_{\theta_0}, \leq_{\mathbf{K}})$ and $\gamma_r(R)$ respectively. That is, for regular $\mu_1 < \mu_2 < \theta_0^+$, the (θ_0, μ_1)-limit model and (θ_0, μ_2)-limit model are isomorphic if and only if $\mu_1 \geq \kappa(\perp, \mathbf{K}^{R\text{-}Mod}_{\theta_0}, \leq^u_{\mathbf{K}})$, again if and only if $\mu_1 \geq \gamma_r(R)$. Therefore $\kappa(\perp, \mathbf{K}^{R\text{-}Mod}_{\theta_0}, \leq^u_{\mathbf{K}}) = \gamma_r(R)$. As $\chi(\perp, \mathbf{K}^{R\text{-}Mod}, \leq^u_{\mathbf{K}}) = \gamma_r(R)$ by Fact 6.5, we must have $\theta(\mathbf{K}^{R\text{-}Mod}, \perp) \leq \theta_0$.

So either $\theta(\mathbf{K}^{R\text{-}Mod}, \perp) = \theta_0$, or $\theta(\mathbf{K}^{R\text{-}Mod}, \perp) = \mathrm{LS}(\mathbf{K}^{R\text{-}Mod}) = \mathrm{card}(R) + \aleph_0$.

---

[3]If $n = 0$ this should be understood as R is left $(< \aleph_0)$-noetherian, i.e., R is left noetherian.

6.3. **Beyond modules with embeddings.** We briefly present two additional applications of our results.

Let T be a complete first-order theory, denote by $\kappa_r(T)$ the least regular cardinal greater or equal to $\kappa(T)$ (see Definition A.2). Let $\underset{f}{\downarrow}$ denote first-order non-forking.

Lemma 6.11. *Let T be a complete first-order theory and $\mathbf{K}^T$ the abstract elementary class of models of T with elementary embeddings.*

(1) $\kappa_r(T) = \chi(\underset{f}{\downarrow}, \mathbf{K}^T_\lambda, \preccurlyeq^u)$

(2) *If $\lambda \geq |T|$ is stable, then* $\kappa_r(T) = \kappa(\underset{f}{\downarrow}, \mathbf{K}^T_\lambda, \preccurlyeq^u)$

(3) $\theta(\underset{f}{\downarrow}, \mathbf{K}^T) = |T|$

(4) *For every stability cardinal $\lambda \geq |T|$, suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2 \in \mathbf{K}^T_\lambda$ (and $M \in \mathbf{K}^T_\lambda$) where N_l is a (λ, δ_l)-limit model (over M) for $l = 1, 2$,*

$$N_1 \text{ is isomorphic to } N_2 \text{ (over } M) \iff \mathrm{cf}(\delta_1) \geq \kappa_r(T)$$

Proof. Let $\mathfrak{C}$ be the monster model of T. First we show that $\mathbf{K}^T$ is $(< \aleph_0)$-tame and that $\mathbf{K}^T$ (with $\underset{f}{\downarrow}$) satisfies Hypothesis 5.5. $(< \aleph_0)$-tameness follows from the fact that first order types are determined by formulas over only finitely many elements of the model. $\underset{f}{\downarrow}$ satisfies invariance, monotonicity, base monotonicity, uniqueness, extension, existence, and $(\geq |T|)$-local character by [She78, Chapter III] (in particular, strong $(< |T|)$-local character follows from [She78, III.3.3]). Universal continuity holds because of $(< \aleph_0)$-tameness, Fact 5.6, and Lemma 2.18. For non-forking amalgamation, suppose $M_0, M_1, M_2 \models T$ where $M_0 \preccurlyeq M_l$ and $a_l \in M_l$ for $l = 1, 2$. Let $p = \mathrm{tp}(M_1/M_0)$. By existence, p does not fork over M_0, so by extension, there is $q \in \mathbf{gS}(M_2)$ such that $p \subseteq q$ and q does not fork over M_0. Let $f_1 : \mathfrak{C} \to \mathfrak{C}$ be an automorphism fixing M_0 where $f(M_1) \models q$, and $f_2 : \mathfrak{C} \to \mathfrak{C}$ be the identity. Then $\mathrm{tp}(f_1[M_1]/f_2[M_2]) = q$ does not fork over M_0, and by symmetry of non-forking [Kim, 2.5] $\mathrm{tp}(f_2[M_2]/f_1[M_1])$ does not fork over M_0. Therefore by monotonicity, $\mathrm{tp}(f_l(a_l)/f_{3-l}[M_{3-l}])$ does not fork over M_0 for $l = 1, 2$, and non-forking amalgamation holds.

(1) Since $\mathbf{K}^T$ is stable in $\lambda \geq 2^{|T|}$ if and only if $\lambda^{<\kappa_r(T)} = \lambda$ by [She78, III.5.5]. The result follows from Lemma 5.22 with $\rho = \kappa_r(T)$.

(2) By their definitions, $\kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$ and $\kappa_r(T)$ differ only in that the A_i in Definition A.2 are restricted to be models for $\kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$, so $\kappa_r(T) \geq \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$. By Lemma 5.7, the definition of $\chi(\underset{f}{\downarrow}, \mathrm{Mod}(T), \preccurlyeq^u)$, and (1), $\kappa_r(T) \leq \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$. So $\kappa_r(T) = \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$.

(3) Follows immediately from (2)

(4) Follows directly from Theorem 5.14.

$\square$

Remark 6.12. *Lemma 6.11(1) was originally obtained in [Vas18b, 4.18]. We have an alternative proof of a slight weakening of Lemma 6.11(4) in the appendix (Corollary A.7).*

Remark 6.13. *By using Theorem 5.14 rather than Theorem 5.24, we have that Lemma 6.11(4) applies to all $\lambda \geq |T|$, rather than all $\lambda \geq \beth_{\beth_{(2^{|T|})^+}}$.*

Let G, H be abelian groups, recall that G is a pure subgroup of H if divisibility is preserved between G and H.

Lemma 6.14. *Let $\mathbf{K}^{Tor}$ be the AEC of torsion abelian groups with pure embeddings.*

    (1) $\chi(\underset{\sim}{\perp}, \mathbf{K}^{Tor}, \leq^u_{\mathbf{K}^{Tor}}) = \aleph_1$.

    (2) For every stability cardinal $\lambda \geq \beth_{\beth_{(2^{\aleph_0})^+}}$, there are exactly 2 non-isomorphic limit models.

Proof. First observe that $\mathbf{K}^{Tor}$ satisfies Hypothesis 5.5 and is $(< \aleph_0)$-tame by [Maz23b, 4.1], [Maz23a, 4.12, 4.14] , [Maz23b, 4.3], and Lemma 6.1.

    (1) Since $\mathbf{K}^{Tor}$ is stable in λ if and only if $\lambda^{\aleph_0} = \lambda$ by [Maz21a, 5.5]. The result follows from Lemma 5.22 with $\rho = \aleph_1$.

    (2) Follows directly from Theorem 5.24.

<div align="right">□</div>

Remark 6.15. *Lemma 6.14 is a weakening of [Maz23b, 5.7]. Observe that for $\lambda < \beth_{\beth_{(2^{\aleph_0})^+}}$ we know that there at least 2 λ-limit models by Lemma 6.14(1), Lemma 5.7 and Theorem 5.1.*

Appendix A. First order results

In writing this paper, the first author found a simpler proof of our main result in the first order case, which we present in this appendix. The argument develops the method from [GVV16, 6.1], but rather than showing only that the (λ, ω)-limit model is not isomorphic to any of the $(\lambda, \geq \kappa_r(T))$-limit models, we show in fact all (λ, δ)-limit models are non-isomorphic for $\delta \leq \kappa_r(T)$. First we present some known machinery from first order non-superstable theories.

For this section, fix a complete first order theory T stable in $\lambda \geq |T|$. Denote its monster model by $\mathfrak{C}$.

Definition A.1 ([She78, Definition VII 3.2(A)]). *We say $\{(\varphi_a, m_\alpha) : \alpha < \beta, \alpha \text{ is a successor ordinal } \}$ with $\{\mathbf{a}_\eta : \eta \in {}^{\leq\beta}\theta\}$ is a* uniform β-tree *provided that*

    (1) φ_α is a formula and $m_\alpha \in \omega$ for all successor $\alpha < \beta$, and θ is an infinite ordinal

    (2) $\mathfrak{C} \models \varphi_\alpha(\mathbf{a}_\eta, \mathbf{a}_{\eta\restriction\alpha})$ for all successor $\alpha < \beta$ and all $\eta \in {}^\beta\theta$

    (3) For all α such that $\alpha+1 < \beta$, $\eta \in {}^\alpha\theta$, and $\nu \in {}^\beta\theta$, $\mathbf{a}_\nu$ realises $< m_{\alpha+1}+1$ of the formulas $\varphi_{\alpha+1}(\mathbf{x}, \mathbf{a}_{\eta^\wedge i})$ where $i < \theta$

Note if a uniform β-tree exists for one (infinite) θ, by compactness there is a uniform β-tree for any other θ with the same φ_α's and m_α's. Also note for zero and limit α, it doesn't matter what $\mathbf{a}_\nu$ is for all $\nu \in {}^\alpha\theta$.

Recall the following:

Definition A.2 ([She78, III.3.1]). *Given a first order complete theory T, $\kappa(T)$ is the least κ such that for all $\langle A_i : i < \kappa \rangle$ $\subseteq$-increasing sequences of sets, and all $p \in \mathbf{gS}(A_\kappa)$ (types of any finite length), there exists $i < \kappa$ such that p does not fork over A_i. If no such κ exists, $\kappa(T) = \infty$.*
    $\kappa_r(T)$ is the first regular $\kappa \geq \kappa(T)$ which is regular (so $\kappa(T)$ or $\kappa(T)^+$), or ∞ if $\kappa(T) = \infty$.

Shelah's original definition was slightly different, but they are equivalent by a short argument using [She78, III.3.2] and monotonicity.

Observe that for μ regular, $\mu < \kappa(T)$ if and only if $\mu < \kappa_r(T)$.

Fact A.3 ([She78, VII 3.7(2)]). *Suppose T is not superstable. If $\beta < \kappa(T)$, then T has a uniform β-tree where $m_\alpha = 1$ for all $\alpha < \beta$ successors.*

Lemma A.5 is a weakening of Theorem 3.1. We include it for those primarily interested in the first order context since it avoids most of the complex machinery established in Section 3. First, we note that unions of long chains of saturated models are saturated.

Fact A.4 ([She78, III.3.11]). *If $\langle M_i : i < \delta \rangle$ is an elementary chain of λ-saturated models where $\mathrm{cf}(\delta) \geq \kappa(T)$, then $\bigcup_{i<\delta} M_i$ is λ-saturated.*

Lemma A.5. *If $\kappa_r(T) \leq \mu < \lambda^+$ and μ is regular, then any (λ, μ)-limit model is saturated. Hence if $\kappa_r(T) \leq \mu_1 \leq \mu_2 < \lambda^+$, where μ_1 and μ_2 are both regular, then the (λ, μ_1)-limit model and the (λ, μ_2)-limit model are isomorphic.*

Proof. Let M be a (λ, μ)-limit model. By uniqueness of limit models of a fixed cofinality (Fact 2.7), there we can assume $M = \langle M_i : i < \mu \rangle$ where for all $i < \mu$, M_{i+1} is saturated and universal over M_i (such extensions exist under stability - universal extensions exist by Fact 2.6, and a saturated model extending this exists by [She78, III.3.12]). Then $M = \bigcup_{i < \mu} M_{i+1}$, and hence is saturated by Fact A.4.

The latter part of the statement follows from [She78, I 1.11], which says that saturated models of the same cardinality are isomorphic. $\square$

Theorem A.6 is a weakening of Theorem 4.2, but avoids the theory of relating non-forking to λ-non-splitting.

Theorem A.6. *Let T be a λ-stable, not superstable complete theory where $\lambda \geq |T|$. Let μ_1, μ_2 be regular infinite cardinals where $\mu_1 < \kappa_r(T)$ and $\mu_1 < \mu_2 < \lambda^+$. Then the (λ, μ_1)-limit model and (λ, μ_2)-limit model are not isomorphic.*

Proof of Theorem A.6. We will show that the (λ, μ_1)-limit is not μ_2-saturated. Since the (λ, μ_2)-limit is μ_2-saturated, this is enough.

Let $\theta = (2^\lambda)^+$. Since $\mu_1 < \kappa_r(T)$ is regular, $\mu_1 < \kappa(T)$, so by Fact A.3 there exist a uniform μ_1-tree $\{(\varphi_a, m_\alpha) : \alpha < \mu_1, \alpha \text{ successor}\}$ and $\{\mathbf{a}_\eta : \eta \in {}^{\leq \mu_1}\theta\}$ where $m_\alpha = 1$ for all α. Since $m_\alpha = 1$ for all $\alpha < \mu_1$ successor, the uniform μ_1-tree condition implies that for all $\eta \in {}^{\mu_1}\theta, \alpha < \mu_1, i < \theta$,

$$(1) \qquad\qquad \mathfrak{C} \models \varphi_{\alpha+1}(\mathbf{a}_\eta, \mathbf{a}_{(\eta\restriction\alpha)^\wedge i}) \iff \eta(\alpha) = i$$

Now we construct an increasing continuous sequence of models $\langle M_\alpha : \alpha < \mu_1 \rangle$, an increasing continuous sequence of maps $\langle \eta_\alpha : \alpha < \mu_1 \rangle$ in ${}^{\leq \mu_1}\theta$, and another (not increasing or continuous) sequence of maps $\langle \nu_\alpha; \alpha < \mu_1 \rangle$ in ${}^{\leq \mu_1}\theta$ such that for all $\alpha < \mu_1$

(1) $M_{\alpha+1}$ is universal over M_α and $\|M_\alpha\| = \lambda$
(2) $\eta_\alpha, \nu_\alpha \in {}^\alpha\theta$, and $\mathbf{a}_{\eta_\alpha}, \mathbf{a}_{\nu,\alpha} \in M_{\alpha+1}$
(3) $\eta_{\alpha+1} \restriction \alpha = \nu_{\alpha+1} \restriction \alpha = \eta_\alpha$, and $\eta_{\alpha+1}(\alpha) \neq \nu_{\alpha+1}(\alpha)$
(4) $\operatorname{tp}(\mathbf{a}_{\eta_\alpha}/M_\alpha) = \operatorname{tp}(\mathbf{a}_{\nu_\alpha}/M_\alpha)$ when α is a successor.

For some intuition, think of $\eta = \eta_{\mu_1}$ as the main 'branch' up the tree, and the $\mathbf{a}_{\nu_\alpha}$ as 'offshoots' or 'thorns' of the main branch that believe the same sentences up to M_α. Note $\mathbf{a}_\eta$ believes the formulas φ_α along the main branch, but on none of the thorns, as $m_\alpha = 2$ for all α. The type that encapsulates this 'branch with thorns' will be the one we show is omitted in M_{μ_1}.

This is possible: For $\alpha = 0$, let M_0 be any model of size λ, and let $\eta_0 = \nu_0 = \langle\rangle$.

For limit α, take $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$, $\eta_\alpha = \bigcup_{\beta < \alpha} \eta_\beta$, and $\nu_\alpha \in {}^\alpha\theta$ (the choice of ν_α doesn't matter).

For successor $\alpha = \beta + 1$, suppose we have defined $M_\beta, \eta_\beta, \nu_\eta$. Take M_α any model universal over M_β containing $\mathbf{a}_{\eta_\beta}$ and $\mathbf{a}_{\nu_\beta}$. Now $|S(M_\beta)| \leq 2^\lambda < \theta$ (in fact we could have used $\theta = \lambda^+$ by stability in λ). So there exist $i \neq j < \theta$ such that $\operatorname{tp}(\mathbf{a}_{\eta_{\beta^\wedge i}}/M_\beta) = \operatorname{tp}(\mathbf{a}_{\eta_{\beta^\wedge j}}/M_\beta)$. Let $\eta_\alpha = \eta_{\beta^\wedge i}$ and $\nu_\alpha = \eta_{\beta^\wedge j}$. This completes the construction.

This is enough: Now, let $M = \bigcup_{\alpha < \mu_1} M_\alpha$ and $\eta = \bigcup_{\alpha < \mu_1} \eta_\alpha$. Note that M is a (λ, μ_1)-limit model. Let $p = \{\varphi_{\alpha+1}(\mathbf{x}, \mathbf{a}_{\eta_{\alpha+1}}) \wedge \neg\varphi_{\alpha+1}(\mathbf{x}, \mathbf{a}_{\nu_{\alpha+1}}) : \alpha < \mu_1\}$. By Equation (1), $\mathbf{a}_\eta \models p$, so p is consistent. This type has domain size at most $\mu_1 < \mu_2$, so it is enough to show M fails to realise p.

Suppose for contradiction that p is realised in M, say $\mathbf{b} \in M$ and $\mathbf{b} \models p$. Then there is some $\alpha < \mu_1$ such that $\mathbf{b} \in M_{\alpha+1}$. As $\mathbf{b} \models p$, $\mathfrak{C} \models \varphi_{\alpha+1}(\mathbf{b}, \mathbf{a}_{\eta_{\alpha+1}})$ and $\mathfrak{C} \models \neg\varphi_{\alpha+1}(\mathbf{b}, \mathbf{a}_{\nu_{\alpha+1}})$. But

$\mathbf{b} \in M_{\alpha+1}$ and $\mathrm{tp}(\mathbf{a}_{\eta_{\alpha+1}}/M_{\alpha+1}) = \mathrm{tp}(\mathbf{a}_{\nu_{\alpha+1}}/M_{\alpha+1})$, which contradicts condition (4). Hence the (λ, μ_1)-limit model is not μ_2-saturated as claimed, and therefore not isomorphic to the (λ, μ_2)-limit model by Fact 2.10. $\qquad \square$

Corollary A.7. *Let T be a λ-stable complete theory where $\lambda \geq |T|$. Suppose T is λ-stable where $\lambda \geq |T|$. Suppose $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) < \mathrm{cf}(\delta_2)$. Then for any $N_1, N_2 \models T$ where N_l is a (λ, δ_l)-limit model for $l = 1, 2$,*

$$N_1 \text{ is isomorphic to } N_2 \iff \mathrm{cf}(\delta_1) \geq \kappa_r(T)$$

Proof. If T is superstable, then $\kappa_r(T) = \aleph_0$ by [She78, 3.8(2)], so by Fact A.5 all limit models are isomorphic as desired. If T is not superstable, the result is immediate from Theorem A.6 and Fact A.5. $\qquad \square$

Corollary A.8. *Let T be a λ-stable, not superstable complete theory where $\lambda \geq |T|$. If $\kappa_r(T) = \aleph_\alpha \leq \lambda$, there are exactly $|\alpha| + 1$ limit models.*

Proof. Note that there are $|\alpha| + 1$ regular cardinals $\leq \aleph_\alpha$ for every ordinal α. The theorem shows that the limit models of these lengths are all distinct. On the other hand, all (λ, μ)-limit models for $\mu \geq \kappa_r(T)$ regular are isomorphic to the $(\lambda, \kappa_r(T))$-limit model by Fact A.5. $\qquad \square$

Remark A.9. *Theorem 5.1 is stronger than Corollary A.7 in two ways: it applies to non-elementary classes (see Section 6), and it shows the bi-implication holds also if we specify that N_1, N_2 are limits over some base model M, and replace $N_1 \cong N_2$ by $N_1 \underset{M}{\cong} N_2$.*

Remark A.10. *The method of [GVV16, 6.1] proved that the (λ, ω)-limit model isn't $\aleph_1$-saturated, whereas the (λ, μ)-limits are λ-saturated for $\mu \geq \kappa(T)$, to establish there were non-isomorphic limit models. Essentially the argument here is the same, but we replace ω and λ by arbitrary regular $\mu_1 < \mu_2 \leq \kappa(T)$. The trade-off is this forces us to use Fact A.3 as stated ([GVV16] uses a stronger version of uniform ω-tree that is easily constructed from the normal kind, but this is not possible as far as we can tell for $\beta > \omega$), but the weaker form of uniform β-tree is enough.*

Remark A.11. *Shelah constructs uniform β-trees in [She78, VII 3.7(2)] (i.e. the formulas φ_α and constants $\mathbf{a}_\eta$) from first order dividing. It is not clear that this argument can be translated into an AEC setting using Galois types rather than syntactic types.*

Remark A.12. *Corollary A.7, together with Theorem 5.1 applied to $\underset{f}{\downarrow}$, give an alternative proof of Lemma 6.11(2). Note that $\kappa_r(T) \leq \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$ if and only if the $(\lambda, \kappa_r(T))$-limit model and $(\lambda, \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u))$-limit model are isomorphic by Corollary A.7. But this holds if and only if $\kappa_r(T) \geq \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$ by Theorem 5.1. Since either $\kappa_r(T) \leq \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$ or $\kappa_r(T) \geq \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$, both must be true. Therefore $\kappa_r(T) = \kappa(\underset{f}{\downarrow}, \mathrm{Mod}(T)_\lambda, \preccurlyeq^u)$.*

References

[Bal09]   John Baldwin, *Categoricity*, American Mathematical Society (2009).

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BoVan24] Will Boney and Monica VanDieren, *Limit Models in Strictly Stable Abstract Elementary Classes*, Mathematical Logic Quarterly **70** (2024), 438–453.

[BGVV]   Will Boney, Rami Grossberg, Monica M. VanDieren, Sebastien Vasey, *Superstability from Categoricity in Abstract Elementary Classes*, Annals of Pure and Applied Logic Volume **168**, Issue 7, July 2017, Pages 1383-1395

[Ekl71]   Paul Eklof, *Homogeneous universal modules*, Mathematica

[EkSa71] Gabriel Sabbagh and Paul Eklof, *Model-completion and modules*, Annals of Mathematical Logic **2** (1971), 251–295.

[GrVan06a] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GrVan06b] Rami Grossberg and Monica VanDieren, *Shelah's categoricity conjecture from a successor for tame abstract elementary classes*, Journal of Symbolic Logic **71** (2006), no. 2, 553—568.

[GrVan06c] Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 2, 181–201.

[GVV16] Rami Grossberg, Monica VanDieren and Andres Villaveces, *Uniqueness of limit models in classes with amalgamation*, Mathematical Logic Quarterly **62** (2016), 367–382.

[GrVas17] Rami Grossberg and Sebastien Vasey, *Equivalent definitions of superstability in tame abstract elementary classes*, The Journal of Symbolic Logic **82** (2017), no. 4, 1387 – 1408.

[Kim]   Byunghan Kim, *Forking in Simple Unstable Theories*, Journal of the London Mathematical Society, **57** (1998), 257-267.

[KolSh96] Oren Kolman and Saharon Shelah, *Categoricity of Theories in $L_{\omega,\kappa}$ when κ is a measurable cardinal. Part 1*, Fundamenta Mathematicae **151** (1996), 209–240.

[Leu24]   Samson Leung, *Stability results assuming tameness, monster model and continuity of non-splitting*, The Journal of Symbolic Logic **89** (2024), Issue 1, 383–425.

[LRV19] Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Forking independence from the categorical point of view*, Advances in Mathematics **346** (2019), 719–772.

[Maz21a] Marcos Mazari-Armida, *A model theoretic solution to a problem of László Fuchs*, Journal of Algebra **567** (2021), 196–209.

[Maz21b] Marcos Mazari-Armida, *Superstability, noetherian rings and pure-semisimple rings*, Annals of Pure and Applied Logic **172** (2021), no. 3, 102917 (24 pages).

[Maz23a] Marcos Mazari-Armida, *Some stable non-elementary classes of modules*, The Journal of Symbolic Logic, **88** (2023), no. 1, 93– 117.

[Maz23b] Marcos Mazari-Armida, *A note on torsion modules with pure embeddings*, Notre Dame Journal of Formal Logic **64** (2023), Vol. 4, 407–424.

[Maz25]   Marcos Mazari-Armida, *On limit models and parametrized Noetherian rings*, Journal of Algebra **669** (2025), 58 – 74.

[MaRo]   Marcos Mazari-Armida and Jiri Rosicky, *Relative injective modules, superstability and noetherian categories*, Journal of Mathematical Logic, Online Ready, 2450027. URL: https://doi.org/10.1142/S0219061324500272

[MVY]   Marcos Mazari-Armida, Sebastien Vasey, and Wentao Yang, *On Stability and Existence of Models in Abstract Elementary Classes*, Preprint. URL: https://arxiv.org/abs/2406.15263

[MaYa24] Marcos Mazari-Armida and Wentao Yang, 'Building Models in Small Cardinals in Local Abstract Elementary Classes', The Journal of Symbolic Logic (2024), pp. 1–11. doi:10.1017/jsl.2024.32.

[She78]   Saharon Shelah, *Classification Theory and the Number of Non-Isomorphic Models, Second Edition*, North-Holland Publishing Company (1978).

[She83]   Saharon Shelah, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), 212–240.

[She99]   Saharon Shelah, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261 – 294.

[She00]   Saharon Shelah, *On what I do not understand (and have something to say), model theory*, Math. Japonica **51** (2000), 329 –377.

[Sh:h]   Saharon Shelah, *Classification Theory for Abstract Elementary Classes*, vol. 1 & 2, Mathematical Logic and Foundations, no. 18 & 20, College Publications (2009).

[ShVa24] Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, J. Eur. Math. Soc. 26 (2024), no. 7, 2301—2372.

[ShVi99] Saharon Shelah and Andres Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1-3):1-25 (1999)

[Van02]   Monica VanDieren, *Categoricity and stability in abstract elementary classes*, ProQuest LLC, Ann Arbor, MI (2002), 122 pp.

[Van06]   Monica VanDieren, *Categoricity in abstract elementary classes with no maximal models*, Annals Pure Applied Logic **141** (2006), 108–147.

[Van13]   Monica VanDieren, *Erratum to "Categoricity in abstract elementary classes with no maximal models"*, Annals Pure Applied Logic **164** (2013), no. 2, 131–133.

[Van16a] Monica VanDieren, *Symmetry and the union of saturated models in superstable abstract elementary classes* Annals Pure Applied Logic **167** (2016), no. 4, 395–407.

[Van16b] Monica VanDieren, *Superstability and symmetry*, Annals of Pure and Applied Logic **167** (2016), no. 12, 1171-1183.

[Vas16a] Sebastien Vasey, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] Sebastien Vasey, *Forking and Superstability in Tame AECs*, The Journal of Symbolic Logic, **Volume 81**, Issue 1, (2016), 357-383

[Vas16c] Sebastien Vasey, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), nos. 3-4, 562–592.

[Vas17b] Sebastien Vasey, *Saturation and solvability in abstract elementary classes with amalgamation*, Archive for Mathematical Logic 56 (2017), 671-690.

[Vas18a] Sebastien Vasey, *Shelah's eventual categoricity conjecture in tame AECs with primes*, Mathematical Logic Quarterly **64** (2018), nos. 1–2, 25–36.

[Vas18b] Sebastien Vasey, *Toward a stability theory of tame abstract elementary classes*, Journal of Mathematical Logic **18** (2018), no. 2, 1850009. Quarterly 64 (2018), nos. 1–2, 25–36

[Vas19]   Sebastien Vasey, *The categoricity spectrum of large abstract elementary classes*, Selecta Mathematica **23** (2019), no. 5, 65 (51 pages).

[Vas22]   Sebastien Vasey, *On categoricity in succesive cardinals*, Journal of Symbolic Logic **87**(2022) Issue 2, 545–563.

[Vas]   Sebastien Vasey, *Model theory for abstract elementary classes*, Lecture notes, 80 pages. URL: http://people.math.harvard.edu/ sebv/aec-spring-2018/aec-lecture-notes_04_26_2018.pdf .

Email address: jbeard@andrew.cmu.edu
URL: https://jeremypbeard.github.io

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA, USA

Email address: marcos_mazari@baylor.edu
URL: https://sites.baylor.edu/marcos_mazari/

DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, WACO, TEXAS, USA