Taylor's Theorem & Taylor's Inequality

Recall Polynomial: $T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Remaineder Formula: $math$

Inequality: $|R_{n,0}(x)| \le \frac{|x|^{n+1}}{(n+1)!}$

Ex) Find $T_{2n+1,\,0}(x)$ for $\sin x$ and $T_{2n,\,0}(x)$ for $\cos x$. Also, find an expression for $|R_{n,\,0}(x)|$ for $\sin x$ and $\cos x$.

Solution: For $f(x) = \sin x,\ f(0) = 0$

$f'(x) = \cos x,\ f'(0) = 1$

$f''(x) = -\sin x,\ f''(0) = 0$

$f'''(x) = -\cos x,\ f'''(0) = -1$

$f''''(x) = \sin x,\ f''''(0) = 0$

Thus, the following pattern emerges: $f^{w}(0) = 0$ for n even, ±1 for n odd

therefor $\sin x \approx T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

repeat this for $f(x) = \cos x,\ f(0) = 1$

$f'(x) = -\sin x,\ f'(0) = 1$

$f''(x) = -\cos x,\ f''(0) = 0$

$f'''(x) = \sin x,\ f'''(0) = -1$

$f''''(x) = \cos x,\ f''''(0) = 0$

Thus, the following pattern emerges: $f^{w}(0) = 0$ for n odd, ±1 for n even

Short cut: $\cos x = \frac{d}{dx}(\sin x) \rightarrow$ take the derivative of $\sin x$'s equation above!

To find an expression for $|R_{n,0}(x)|$ for $\sin x$ and $\cos x$, observe that $\left|f^{n+1}(x)\right| \le 1$ for all n & x.

Thus, we obtain from Taylor's inequality that $|R_{n,0}(x)| \le \frac{|x|^{n+1}}{(n+1)!}$ (set $a = 0,\ k = 1$ in Taylor's Inequality.

Remark: For a fixed finite x value, $\lim_{n\to\infty} |R_{n,\,0}(x)| = 0$

Special Case:

Observe that for $n = 0$, Taylor's formula becomes the following:

$f(x) = T_{0,a}(x) + R_{0,a}(x)$ where $T_{0,a}(x) = f(a)$ and $R_{0,a}(x) = f'(c)(x-a)$

$\Rightarrow f'(c) = \frac{(f(x) - f(a))}{x-a}$ for c between a and x → mean value theorem

Applications of Taylor Polynomials include:

1. approximating functions by polynomials
2. approximating integrals
3. evaluating limits