Applications of Taylor Polynomials

Recall Taylor's Formula: $f\left(x
ight) = T_{n,a}\left(x
ight) + R_{n,a}\left(x
ight)$ where:

$$T_{n,a}\left(x
ight)=f\left(a
ight)+f'\left(a
ight)\left(x-a
ight)+rac{f''(a)}{2!}\left(x-a
ight)^{2}+...+rac{f^{(n)}(a)}{n!}\left(x-a
ight)^{n}$$
 and: $R_{n,a}\left(x
ight)=rac{f^{(n+1)}(c)}{(n+1)!}\left(x-a
ight)^{n}$ where c is between a and x

Recall Taylor's Inequality:
$$|R_{n,a}\left(x
ight)| \ \leq \ rac{K}{(n+1)!} \left|x-a
ight|^{n+1}$$
 where $\left|f^{(n+1)}\left(c
ight)
ight| \leq K$

EX) Find $T_{2,0}\left(x\right)$ for $f\left(x\right)=tan^{-1}\left(x^2\right)$, on the interval [-1,1] with a=0, find a constant B such that $\left|f\left(x\right)-T_{2,0}\left(x\right)\right|\leq B\left|x\right|^3$

SOLUTION:

For
$$f\left(x\right)=tan^{-1}\left(x^{2}\right),\;\;f'\left(x\right)=rac{2x}{1+x^{4}},\;\;f''\left(x\right)=rac{\left(2-6x^{4}\right)}{\left(1+x^{4}\right)^{2}}$$
 $\Rightarrow f\left(0
ight)=0,\;\;f'\left(0
ight)=0,\;\;f''\left(0
ight)=2$ So: $T_{2,0}\left(x
ight)=f\left(0
ight)+f'\left(0
ight)+rac{f''\left(0
ight)}{2!}x^{2}=x^{2}$

By Taylor's Formula:
$$|f\left(x
ight)-T_{2,0}\left(x
ight)|=|R_{2,0}\left(x
ight)|$$
 where $R_{2,0}\left(x
ight)=rac{f'''\left(c
ight)}{3!}x^3$ and $f'''\left(x
ight)=rac{8x^3\left(3x^4-5
ight)}{(1+x^4)^3}$

ightarrow with c between 0 and x, since $x\epsilon\left[-1,1
ight] \Rightarrow -1 \leq 0 \leq 1$

By Taylor's Inequality: $|R_{2,0}\left(x
ight)|\leqrac{K}{3!}\left|x
ight|^{3}$ where $|f'''\left(c
ight)|\leq K$

Thus, $|f\left(x
ight)-T_{2,0}\left(x
ight)|\leq rac{K}{3!}\left|x
ight|^{3}$ To determine the constant B we need to find K since $B=rac{K}{3!}$

Now,
$$|f'''\left(c
ight)|=\left|rac{8c^3\left(3c^4-5
ight)}{(1+c^4)^3}
ight|$$
 we need to find the largest possible value for $|f'''\left(c
ight)|$

To do this, we make use of properties of absolute value and the Triangle Inequality. Recall that $|a+b| \leq |a|+|b|$ which can be extended to: $|a_1+a_2+...+a_n| \leq |a_1|+|a_2|+...+|a_n|$.

Applying this we get:

$$|f'''\left(c
ight)| = \left|rac{8c^3\left(3c^4-5
ight)}{(1+c^4)^3}
ight| = rac{8|c|^3\left|3c^4-5
ight|}{(1+c^4)^3} \leq rac{8|c|^3\left|3c^4-5
ight|}{(1)^3} \leq 8\left(1
ight)^3\left[3\left|c
ight|^4+1-51
ight] \leq 64 ext{ since:} \ -1 \leq c \leq 1$$

Thus,
$$K=64$$
 and $\left|f\left(x
ight)-T_{2,0}\left(x
ight)
ight|\leq rac{64}{3!}\left|x
ight|^{3}\ \Rightarrow\ B=rac{K}{3!}=rac{64}{6}=rac{32}{3}$

So:

$$\left|tan^{-1}\left(x^{2}
ight)-x^{2}
ight|\leqrac{32}{3}\left|x
ight|^{3}$$
 for $x\epsilon\left[-1,1
ight]$

Remark: When using Taylor's Inequality we don't need to find the max value of $|f^{(n+1)}(c)|$; an upper bound is all that is needed.

EX) Estimate $\int_0^1 \sin(x^3) dx$ with error less than $\frac{1}{1000}$.

SOLUTION:

Recall that
$$\sin x = x - rac{x^3}{3!} + rac{x^5}{5!} + ... + rac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+1,\,0}\left(x
ight)$$
 where:

$$|R_{2n+1,\,0}\left(x
ight)|\leqrac{K|x|^{2n+3}}{(2n+3)!}\leqrac{|x|^{2n+3}}{(2n+3)!}$$
 since $K=1$ for $f\left(x
ight)=\sin x$

Next, replace x by x^3 to get:

$$\sin\left(x^3
ight) = x^3 - rac{x^9}{3!} + rac{x^{15}}{5!} + ... + rac{(-1)^n x^{6n+3}}{(2n+1)!} + R_{2n+1,\,0}\left(x^3
ight)$$
 and $\left|R_{2n+1,\,0}\left(x^3
ight)
ight| \leq rac{|x|^{6n+9}}{(2n+3)!}$

Now, we need to find the smallest value of n such that:

$$egin{aligned} \int_0^1 \left| R_{2n+1,\,0}\left(x^3
ight)
ight| dx & \leq rac{1}{1000} \ \Rightarrow \int_0^1 rac{x^{6n+9}}{(2n+3)!} dx & = rac{1}{(2n+3)!(6n+10)} \leq rac{1}{1000} \end{aligned} = eq1 \end{aligned}$$

By trial and error, n=1 satisfies eq1

Thus, we can use the approximation $\sin\left(x^2
ight) pprox x^3 - rac{x^9}{3!}$ and

$$\int_0^1\sin\left(x^3
ight)dxpprox \int_0^1\left(x^3-rac{x^9}{3!}
ight)dx=rac{1}{4}-rac{1}{10(3!)}=rac{7}{30}$$
 to within 0.001 accuracy