Applications of Taylor Polynomials 2

EX) For $f(x) = x^{\frac{1}{3}}$:

a) find $T_{2,8}(x)$

b) use Taylor's Inequality to find an upper bound on the error in using $T_{2,8}(x)$ to approx $10^{\frac{1}{3}}$

c) How accurate is the approx $f(x) \approx T_{2,8}(x)$ if $7 \leq x \leq 9$

Solution:

a)

$T_{2,8}(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2$

$f(8) = 8^{\frac{1}{3}} = 2,\ \ f'(x) = \frac{1}{3}x^{-\frac{2}{3}},\ \ f'(8) = \frac{1}{3}\left(8^{-\frac{2}{3}}\right) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}},\ \ f''(8) = -\frac{2}{9}\left(8^{-\frac{5}{3}}\right) = -\frac{1}{144}$

$\Rightarrow T_{2,8}(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

b)

Taylor's Ineq: $|R_{2,8}(x)| \leq \frac{K}{3!}|x-8|^3$ where $|f'''(c)| \leq K$ and $f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}$ with $8 < c < 10$

Since $f'''(x)$ is a decreasing function, $|f'''(c)| \leq |f'''(8)| = \frac{10}{27}\left(8^{-\frac{8}{3}}\right) = K$

so, $|R_{2,8}(x)| \leq \frac{10}{27}\left(8^{-\frac{8}{3}}\right)\left(\frac{1}{3!}\right)(10-8)^3 = \frac{5}{2592} \approx 0.0019$

NOTE $T_{2,8}(8) = 2.1528,\ \ 10^{\frac{1}{3}} = 2.1544 \ \ \Rightarrow \ \ \left|10^{\frac{1}{3}} - T_{2,8}(10)\right| = 0.0016 \ \ \Rightarrow$ within the upper bound of error (0.0019)

c)

On the interval $7 \leq x \leq 9,\ \ 7 < c < 9$

so, $|f'''(c)| \leq |f'''(7)| = \frac{10}{27}\left(7^{-\frac{8}{3}}\right) = K$, and so $|R_{2,8}(x)| \leq \frac{10}{27}\left(7^{-\frac{8}{3}}\right)\left(\frac{1}{3!}\right)(1)^3 \approx 0.00035$

Thus, $|f(x) - T_{2,8}(x)| \leq 0.00035$ on $[7,9]$

EX) Evaluate $\lim_{x \to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$ using Taylor Polynomials

Solution:

Since $x \to 0$ we begin by replacing $\sin x,\ \sin(2x)$ and e^x by their Maclaurin Polynomials.

Recall that $\sin x \approx x - \frac{x^3}{e!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Replace x by $2x$ to get: $\sin(2x) \approx (2x) - \frac{(2x)^3}{3!} + \ldots + \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

Then, $\lim_{x \to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$

$= \lim_{x \to 0} \frac{2\left[x - \frac{x^3}{3!} + \ldots\right] - \left[2x - \frac{8x^3}{3!} + \ldots\right]}{2\left[1 + x + \frac{x^2}{2!} + \ldots\right] - 2 - 2x - x^2}$

$= \lim_{x \to 0} \frac{-\frac{x^3}{3!} + \frac{4}{3}x^3 + \frac{1}{60}x^5 - \frac{4}{15}x^5 + \dots}{\frac{x^3}{3} + \frac{x^4}{12} + \dots}$

$= \lim_{x \to 0} \frac{x^3 - \frac{1}{4}x^5 + \dots}{\frac{x^3}{3} + \frac{x^4}{12} + \dots}$

$= \frac{1}{\frac{1}{3}}$

$math$

$= 3$

Note: take 3 terms, add if needed