Applications of Taylor Polynomials

Recall Taylor's Formula: $f(x) = T_{n,a}(x) + R_{n,a}(x)$ where:

$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ and:

$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^n$ where c is between a and x

Recall Taylor's Inequality: $|R_{n,a}(x)| \leq \frac{K}{(n+1)!}|x-a|^{n+1}$ where $\left|f^{(n+1)}(c)\right| \leq K$

EX) Find $T_{2,0}(x)$ for $f(x) = tan^{-1}\left(x^2\right)$, on the interval [-1,1] with a=0, find a constant B such that $|f(x) - T_{2,0}(x)| \leq B|x|^3$

SOLUTION:

For $f(x) = tan^{-1}\left(x^2\right),\ \ f'(x) = \frac{2x}{1+x^4},\ \ f''(x) = \frac{\left(2-6x^4\right)}{\left(1+x^4\right)^2}$

$\Rightarrow f(0) = 0,\ \ f'(0) = 0,\ \ f''(0) = 2$

So: $T_{2,0}(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 = x^2$

By Taylor's Formula: $|f(x) - T_{2,0}(x)| = |R_{2,0}(x)|$ where $R_{2,0}(x) = \frac{f'''(c)}{3!}x^3$ and

$f'''(x) = \frac{8x^3\left(3x^4-5\right)}{\left(1+x^4\right)^3}$

$\rightarrow$ with c between 0 and x, since $x\epsilon[-1,1] \Rightarrow -1 \leq 0 \leq 1$

By Taylor's Inequality: $|R_{2,0}(x)| \leq \frac{K}{3!}|x|^3$ where $|f'''(c)| \leq K$

Thus, $|f(x) - T_{2,0}(x)| \leq \frac{K}{3!}|x|^3$ To determine the constant B we need to find K since $B = \frac{K}{3!}$

Now, $|f'''(c)| = \left|\frac{8c^3\left(3c^4-5\right)}{\left(1+c^4\right)^3}\right|$ we need to find the largest possible value for $|f'''(c)|$

To do this, we make use of properties of absolute value and the Triangle Inequality. Recall that $|a+b| \leq |a| + |b|$ which can be extended to: $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$.

Applying this we get:

$|f'''(c)| = \left|\frac{8c^3\left(3c^4-5\right)}{\left(1+c^4\right)^3}\right| = \frac{8|c|^3\left|3c^4-5\right|}{\left(1+c^4\right)^3} \leq \frac{8|c|^3\left|3c^4-5\right|}{(1)^3} \leq 8(1)^3\left[3|c|^4 + 1 - 51\right] \leq 64$ since:

$-1 \leq c \leq 1$

Thus, $K = 64$ and $|f(x) - T_{2,0}(x)| \leq \frac{64}{3!}|x|^3 \ \ \Rightarrow \ \ B = \frac{K}{3!} = \frac{64}{6} = \frac{32}{3}$

So:

$\left|tan^{-1}\left(x^2\right) - x^2\right| \leq \frac{32}{3}|x|^3$ for $x\epsilon[-1,1]$

Remark: When using Taylor's Inequality we don't need to find the max value of $\left|f^{(n+1)}(c)\right|$; an upper bound is all that is needed.

EX) Estimate $\int_0^1 \sin\left(x^3\right) dx$ with error less than $\frac{1}{1000}$.

SOLUTION:

Recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+1,\,0}(x)$ where:

$\left|R_{2n+1,\,0}(x)\right| \leq \frac{K|x|^{2n+3}}{(2n+3)!} \leq \frac{|x|^{2n+3}}{(2n+3)!}$ since $K = 1$ for $f(x) = \sin x$

Next, replace x by x^3 to get:

$\sin\left(x^3\right) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} + \ldots + \frac{(-1)^n x^{6n+3}}{(2n+1)!} + R_{2n+1,\,0}\left(x^3\right)$ and $\left|R_{2n+1,\,0}\left(x^3\right)\right| \leq \frac{|x|^{6n+9}}{(2n+3)!}$

Now, we need to find the smallest value of n such that:

$\int_0^1 \left|R_{2n+1,\,0}\left(x^3\right)\right| dx \leq \frac{1}{1000}$

$\Rightarrow \int_0^1 \frac{x^{6n+9}}{(2n+3)!} dx = \frac{1}{(2n+3)!(6n+10)} \leq \frac{1}{1000} \quad = \quad eq1$

By trial and error, $n = 1$ satisfies $eq1$

Thus, we can use the approximation $\sin\left(x^2\right) \approx x^3 - \frac{x^9}{3!}$ and

$\int_0^1 \sin\left(x^3\right) dx \approx \int_0^1 \left(x^3 - \frac{x^9}{3!}\right) dx = \frac{1}{4} - \frac{1}{10(3!)} = \frac{7}{30}$ to within 0.001 accuracy