Applications of Taylor Polynomials 2

EX) For
$$f\left(x
ight)=x^{rac{1}{3}}$$
:

- a) find $T_{2,8}\left(x
 ight)$
- b) use Taylor's Inequality to find an upper bound on the error in using $T_{2.8}\left(x
 ight)$ to approx $10^{\frac{1}{3}}$
- c) How accurate is the approx $f\left(x
 ight)pprox T_{2,8}\left(x
 ight)$ if $7\leq x\leq 9$

Solution:

a)

$$egin{aligned} T_{2,8}\left(x
ight) &= f\left(8
ight) + f'\left(8
ight)\left(x-8
ight) + rac{f''\left(8
ight)}{2!}\left(x-8
ight)^2 \ &f\left(8
ight) &= 8^{rac{1}{3}} = 2, \;\; f'\left(x
ight) = rac{1}{3}x^{-rac{2}{3}}, \;\; f'\left(8
ight) = rac{1}{3}\left(8^{-rac{2}{3}}
ight) = rac{1}{12} \ &f''\left(x
ight) = -rac{2}{9}x^{-rac{5}{3}}, \;\; f''\left(8
ight) = -rac{2}{9}\left(8^{-rac{5}{3}}
ight) = -rac{1}{144} \ &\Rightarrow T_{2,8}\left(x
ight) = 2 + rac{1}{12}\left(x-8
ight) - rac{1}{288}\left(x-8
ight)^2 \end{aligned}$$

b)

Taylor's Ineq:
$$|R_{2,8}\left(x\right)| \leq \frac{K}{3!} \, |x-8|^3$$
 where $|f'''\left(c\right)| \leq K$ and $f'''\left(x\right) = \frac{10}{27} x^{-\frac{8}{3}}$ with $8 < c < 10$ Since $f'''\left(x\right)$ is a decreasing function, $|f'''\left(c\right)| \leq |f'''\left(8\right)| = \frac{10}{27} \left(8^{-\frac{8}{3}}\right) = K$ so, $|R_{2,8}\left(x\right)| \leq \frac{10}{27} \left(8^{-\frac{8}{3}}\right) \left(\frac{1}{3!}\right) \left(10-8\right)^3 = \frac{5}{2592} \approx 0.0019$ NOTE $T_{2,8}\left(8\right) = 2.1528, \ \ 10^{\frac{1}{3}} = 2.1544 \ \Rightarrow \ \ \left|10^{\frac{1}{3}} - T_{2,8}\left(10\right)\right| = 0.0016 \ \Rightarrow$ within the upper bound of error (0.0019)

c)

On the interval
$$7 \leq x \leq 9, \ \ 7 < c < 9$$
 so, $|f'''\left(c\right)| \leq |f'''\left(7\right)| = \frac{10}{27}\left(7^{-\frac{8}{3}}\right) = K$, and so $|R_{2,8}\left(x\right)| \leq \frac{10}{27}\left(7^{-\frac{8}{3}}\right)\left(\frac{1}{3!}\right)\left(1\right)^3 \approx 0.00035$ Thus, $|f\left(x\right) - T_{2,8}\left(x\right)| \leq 0.00035$ on $[7,9]$

EX) Evaluate $\lim_{x \to 0} rac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$ using Taylor Polynomials

Solution:

Since x o 0 we begin by replacing $\sin x, \; \sin \left(2x
ight)$ and e^x by their Maclaurin Polynomials.

Recall that
$$\sin x pprox x - rac{x^3}{e!} + ... + rac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Replace
$$x$$
 by $2x$ to get: $\sin{(2x)} pprox (2x) - rac{(2x)^3}{3!} + ... + rac{(-1)^n(2x)^{2n+1}}{(2n+1)!}$

$$e^x pprox 1 + x + rac{x^2}{2!} + rac{x^3}{3!} + ... + rac{x^n}{n!}$$

Then,
$$\lim_{x o 0} rac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$$

$$= \lim_{x \to 0} \frac{2\left[x - \frac{x^3}{3!} + \ldots\right] - \left[2x - \frac{8x^3}{3!} + \ldots\right]}{2\left[1 + x + \frac{x^2}{2!} + \ldots\right] - 2 - 2x - x^2}$$

$$\begin{split} &= \lim_{x \to 0} \frac{-\frac{x^3}{3!} + \frac{4}{3}x^3 + \frac{1}{60}x^5 - \frac{4}{15}x^5 + ...}{\frac{x^3}{3} + \frac{x^4}{12} + ...} \\ &= \lim_{x \to 0} \frac{x^3 - \frac{1}{4}x^5 + ...}{\frac{x^3}{3} + \frac{x^4}{12} + ...} \\ &= \frac{1}{\frac{1}{3}} \\ &math \\ &= 3 \end{split}$$

Note: take 3 terms, add if needed