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Brunovsky's Canonical Form For Linear Dynamical Systems over Commutative Rings

Jose A. Hermida-Alonso*
Departamento de Matemáticas
Universidad de León
24071-León, Spain

M. Pilar Perez*
Departamento de Economia Aplicada
Facultad de C. Economicas
Universidad de Valladolid,
47011-Valladolid, Spain

and

Tomas Sanchez-Giralda*
Departamento de Algebra
Facultad de Ciencias
Universidad de Valladolid
47005-Valladolid, Spain

Submitted by Richard A. Brunaldi

ABSTRACT

This paper is devoted to studying the action of the feedback group on linear dynamical systems over a commutative ring R with unit. We characterize the class of m-input n-dimensional reachable linear dynamical systems $\Sigma = (F, G)$ over R that are feedback equivalent to a system $\Sigma_c = (F_c, G_c)$ with Brunovsky's canonical form. This characterization is obtained in terms of the minors of the matrices $\tilde{G}_i^{\Sigma} = (G, FG, \dots, F^{i-1}G)$ for $1 \leqslant i \leqslant n$.

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1. INTRODUCTION AND NOTATION

Throughout this paper R denotes a commutative ring with unit element. We consider an m-input n-dimensional linear dynamical system $\Sigma = (F, G)$ over R, where F and G are $n \times n$ and $n \times m$ matrices with entries in R, respectively. Assume the system Σ is reachable [i.e., the columns of the $n \times nm$ matrix $(G, FG, \dots, F^{n-1}G)$ generate R^n].

The linear dynamical system $\Sigma' = (F', G')$ is feedback equivalent to Σ when Σ can be transformed to Σ' by one element of the feedback group $\mathbf{F}_{n,m}$. For the reader's convenience we recall that $\mathbf{F}_{n,m}$ is the group generated by the following three types of transformations:

(1) $F \mapsto F' = PFP^{-1}$, $G \mapsto G' = PG$ for some invertible matrix P. This transformation is a consequence of a change of base in R^n, the state module.

(2) $F \mapsto F$, $G \mapsto G' = GQ$ for some invertible matrix Q. This transformation is a consequence of a change of base in R^m, the input module.

(3) $F \mapsto F' = F + GK$, $G \mapsto G$ for some $m \times n$ matrix K, which is called a feedback matrix.

When R is a field k, Brunovsky [2] gave the following classification result:

THEOREM 1.1. *Let $\Sigma = (F, G)$ be an m-input n-dimensional reachable linear dynamical system over k. Then there exist positive integers $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_s$ uniquely determined by Σ with $n = k_1 + k_2 + \cdots k_s$, such that Σ is feedback equivalent to $\Sigma_c = (F_c, G_c)$, where F_c is the block matrix*

$$F_c = \begin{pmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_S \end{pmatrix},$$

with E_i the $k_i \times k_i$ matrix

$$E_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and G_c is the block matrix

$$G_c = \begin{pmatrix} e_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_s & 0 & \cdots & 0 \end{pmatrix},$$

with e_i the $k_i \times 1$ matrix $e_i = (0, 0, \dots, 0, 1)'$, i.e.

$$G_c = \left(\begin{array}{c} \overbrace{\qquad\qquad\qquad}^{s \text{ columns}} \quad \overbrace{\qquad\qquad}^{m - s \text{ columns}} \\ \begin{array}{c|c|c|c|ccc} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \hline \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{array} \end{array}\right) \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_1 \textit{ rows} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_2 \textit{ rows} \\ \begin{array}{c} \\ \\ \\ \end{array} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_s \textit{ rows} \end{array}$$

The integers $k_1, k_2, \dots, k_s$ are called the Kronecker indices of Σ. They are a complete set of invariants for Σ by the action of the feedback group. Information over the feedback classification theorem can be found in [6] and [8].

Suppose that R is an arbitrary commutative ring and $\Sigma = (F, G)$ is an m-input n dimensional reachable linear dynamical system over R. In general, Σ is not feedback equivalent to a system $\Sigma_c = (F_c, G_c)$ where F_c and G_c are matrices as in the above theorem. For example, in [5] it is shown that if R is a principal ideal domain and Σ is a m-input 2-dimensional reachable linear dynamical system over R, the Σ is feedback equivalent to a system $\hat{\Sigma} = (\hat{F}, \hat{G})$ of the form

$$\hat{F} = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}, \qquad \hat{G} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \end{pmatrix},$$

where d and f are coprime.

We introduce the following:

DEFINITION 1.2. Let $\Sigma = (F, G)$ be an m-input n-dimensional reachable linear dynamical system over R. We shall say that Σ is a *Brunovsky system* if and only if Σ is feedback equivalent to a system of the form $\Sigma_c = (F_c, G_c)$.

In Section 2 we associate to $\Sigma = (F, G)$ a set $\{M_i^{\Sigma}\}_{1 \leqslant i \leqslant n}$ of R-modules verifying:

(i) M_i^{Σ} is invariant under the action of feedback group for every i.
(ii) If Σ is a Brunovsky system, then M_i^{Σ} is free for every i.

Suppose that R is such that finitely generated projective R-modules are free. In Section 3 we prove the converse of statement (ii); that is, we characterize the Brunovsky systems.

We prove that $\{\text{rank } M_i^{\Sigma}\}_{1 \leqslant i \leqslant n}$ is a complete set of invariants for the feedback class of a Brunovsky system Σ. Moreover we include a determinantal method to obtain this set of invariants. In particular, when R is a field (the classical case) we have an alternative form to obtain the Kronecker indices of Σ.

2. BASIC RESULTS

Let $\Sigma = (F, G)$ be a m-input n-dimensional linear dynamical system over R. We first introduce some notation. We denote by L_i^{Σ} the submodule of R^n generated by the columns of the $n \times m$ matrix F^iG for $0 \leqslant i \leqslant n-1$. Also, by N_i^{Σ} we denote the submodule of R^n generated by the columns of the $n \times im$ matrix

$$\tilde{G}_i^{\Sigma} = (G, FG, \dots, F^{i-1}G),$$

for $1 \leqslant i \leqslant n$, that is,

$$N_i^\Sigma = L_0^\Sigma + L_1^\Sigma + \cdots + L_{i-1}^\Sigma.$$

We agree that $N_0^\Sigma = 0$. Recall that Σ is reachable if and only if $N_n^\Sigma = R^n$. We put $M_i^\Sigma = R^n/N_i^\Sigma$ for $1 \leqslant i \leqslant n$.

LEMMA 2.1. *Let $\Sigma = (F, G)$ and $\Sigma' = (F', G')$ be two m-input n-dimensional linear dynamical system over a commutative ring R. Then if Σ' is feedback equivalent to Σ, we have*:

(i) *N_i^Σ is isomorphic to $N_i^{\Sigma'}$ for $1 \leqslant i \leqslant n$.*
(ii) *M_i^Σ is isomorphic to $M_i^{\Sigma'}$ for $1 \leqslant i \leqslant n$.*

Proof. It is sufficient to prove the result when Σ' is obtained from Σ by one transformation of type (1), (2), or (3).

For a transformation of type (1) we have $F' = PFP^{-1}$ and $G' = PG$ for some invertible $n \times n$ matrix P, and hence $\tilde{G}_i'^{\Sigma'} = P\tilde{G}_i^\Sigma$ for $1 \leqslant i \leqslant n$. Let φ_P be the homomorphism defined, with respect to the standard base, by P. Then we have the commutative diagram with exact rows

$$\begin{tikzcd}
R^{im} \arrow[r, "\varphi_{\tilde{G}_i^\Sigma}"] \arrow[d, equal] & R^n \arrow[r] \arrow[d, "\varphi_P"] & R^n/N_i^\Sigma \arrow[r] \arrow[d, "\bar{\alpha}"] & 0 \\
R^{im} \arrow[r, "\varphi_{\tilde{G}_i'^{\Sigma'}}"] & R^n \arrow[r] & R^n/N_i^{\Sigma'} \arrow[r] & 0
\end{tikzcd}$$

where $\bar{\alpha}$ is the homomorphism induced by φ_P.

For a transformation of type (2) we have $F' = F$ and $G' = GQ$ for some invertible $m \times m$ matrix Q. Hence

$$\tilde{G}_i'^{\Sigma'} = \tilde{G}_i^\Sigma \tilde{Q}$$

for $1 \leqslant i \leqslant n$, where $\tilde{Q}$ is the $im \times im$ invertible block matrix

$$\tilde{Q} = \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{pmatrix}.$$

In this case $N_i^\Sigma = N_i^{\Sigma'}$ and $M_i^\Sigma = M_i^{\Sigma'}$ for $1 \leqslant i \leqslant n$.

For type (3) we have $F' = F + GK$ and $G' = G$ for some $m \times n$ feedback matrix K. Since

$$(F + GK)^j G = F^j G + F^{j-1} GA_1 + \cdots + GA_j, \qquad 0 \leqslant j \leqslant n - 1,$$

where $A_1, A_2, \ldots, A_j$ are suitable $m \times n$ matrices, then we have $N_i^{\Sigma} = N_i^{\Sigma'}$ and $M_i^{\Sigma} = M_i^{\Sigma'}$ for $1 \leqslant i \leqslant n$. ∎

PROPOSITION 2.2. *Let $\Sigma = (F, G)$ be a Brunovsky system. Then for $1 \leqslant i \leqslant n$ the modules N_I^{Σ} and M_i^{Σ} are free.*

Proof. By the above result we can suppose that Σ is the canonical system Σ_c. In this case we have, with the notation of Theorem 1.1, that $N_i^{\Sigma_c}$ is the submodule of R^n generated by the columns of the matrix

$$\tilde{G}_{c_i}^{\Sigma_c} = \left(\begin{array}{cccc|cccc|c|cccc} e_1 & 0 & \cdots & 0 & E_1 e_1 & 0 & \cdots & 0 & \cdots & (E_1)^i e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 & 0 & E_2 e_2 & \cdots & 0 & \cdots & 0 & (E_2)^i e_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_s & 0 & 0 & \cdots & E_s e_s & \cdots & 0 & 0 & \cdots & (E_s)^i e_s \end{array}\right)$$

Consider the equality

$$(E_j)^p e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where 1 is placed in row $j - p$. Then the columns of the matrix $\tilde{G}_{c_i}^{\Sigma_c}$ are either zero or elements of the standard base for R^n. Consequently N_i^{Σ} and R^n / N_i^{Σ} are free for $1 \leqslant i \leqslant n$. ∎

The main result of this paper is related to the converse of the previous proposition. Next we include some ingredients which are used in the proof of the main result.

LEMMA 2.3. *Let $\Sigma = (F, G)$ be a m-input n-dimensional linear dynamical system over R such that the R-modules M_i^{Σ} are free for $1 \leqslant i \leqslant n$. Then the R-modules $N_i^{\Sigma}/N_{i-1}^{\Sigma}$ are projective for $1 \leqslant i \leqslant n$.*

Proof. Consider for every i, $1 \leqslant i \leqslant n$, the exact sequence of R-modules

$$0 \to N_i^{\Sigma}/N_{i-1}^{\Sigma} \to M_{i-1}^{\Sigma} \xrightarrow{\pi_i} M_i^{\Sigma} \to 0,$$

where π_i is the natural epimorphism and $M_0^{\Sigma} = R^n$. Since M_i^{Σ} is free, the exact sequence splits. Then $M_{i-1}^{\Sigma} \simeq M_i^{\Sigma} \oplus N_i^{\Sigma}/N_{i-1}^{\Sigma}$, and $N_i^{\Sigma}/N_{i-1}^{\Sigma}$ is projective because it is a summand of the free module M_{i-1}^{Σ}.

LEMMA 2.4. *Let $\Sigma = (F, G)$ be an m-input n-dimensional linear dynamical system over R. Then the homomorphisms*

$$\varphi_i : N_i^{\Sigma}/N_{i-1}^{\Sigma} \to N_{i+1}^{\Sigma}/N_i^{\Sigma},$$

$$x + N_{i-1}^{\Sigma} \mapsto F(x) + N_i^{\Sigma},$$

$$\Psi_i : L_i^{\Sigma} \to N_{i+1}^{\Sigma}/N_i^{\Sigma},$$

$$x \mapsto x + N_i^{\Sigma}$$

are surjective for $1 \leqslant i \leqslant n - 1$, where $x + N_i^{\Sigma}$ is the canonical image of x in the quotient module $N_{i+1}^{\Sigma}/N_i^{\Sigma}$.

Proof. The verification that φ_i and Ψ_i are surjective is a straightforward exercise which we leave to the reader. ■

Finally note that if

$$0 \to F' \xrightarrow{i} F \xrightarrow{\pi} F'' \to 0$$

is an exact sequence of free R-modules of finite rank, and $\mathscr{B}' = \{m'_i\}_{1 \leqslant i \leqslant n'}$ and $\mathscr{B}'' = \{m''_j\}_{1 \leqslant j \leqslant n''}$ are bases for F' and F'' respectively, then $\mathscr{B} = \{i(m'_i), m_j\}_{1 \leqslant i \leqslant n', 1 \leqslant j \leqslant n''}$ is a base for F, where m_j is an element of F such that $\pi(m_j) = m''_j$ for $1 \leqslant j \leqslant n''$.

3. THE CHARACTERIZATION THEOREM

THEOREM 3.1. *Let R be a commutative ring such that finitely generated projective R-modules are free. Let $\Sigma = (F, G)$ be a m-input n-dimensional reachable linear dynamical system over R. Then the following statements are equivalent*:

(i) *Σ is a Brunovsky system.*
(ii) *For $1 \leqslant i \leqslant n$ the R-module M_i^Σ is free.*

Proof. Assume that statement (ii) holds. Since Σ is reachable, we have

$$N_0^\Sigma \subseteq N_1^\Sigma \subseteq \cdots \subseteq N_n^\Sigma = R^n.$$

Consequently there exists a positive integer p such that $N_{p-1}^\Sigma \neq R^n$ and $N_p^\Sigma = R^n$. Note that if $N_0^\Sigma = R^n$, then Σ is feedback equivalent to system $\hat{\Sigma} = (\hat{F}, \hat{G})$, where $\hat{F}$ is the zero $n \times n$ matrix and $\hat{G}$ is the matrix $\hat{G} = (\mathrm{Id}_n|0)$, where Id_n is the identity matrix of order n.

Since the modules M_i^Σ are free, by Lemma 2.3 it follows that $N_i^\Sigma/N_{i-1}^\Sigma$ is projective for $1 \leqslant i \leqslant p$. By hypothesis we have that $N_i^\Sigma/N_{i-1}^\Sigma$ is free for $1 \leqslant i \leqslant p$. Let s_i be the rank of $N_i^\Sigma/N_{i-1}^\Sigma$. Note that by Lemma 2.4 we have $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_p$.

We shall first prove that there exist elements

$$\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_{s_p}, \hat{g}_{s_p+1}, \ldots, \hat{g}_{s_2}, \hat{g}_{s_2+1}, \ldots, \hat{g}_{s_1}$$

of R^n such that the set

$$\mathscr{B} = \{$$

$$\begin{array}{llllllllll}
\hat{g}_1, & \ldots, & \hat{g}_{s_p}, & \hat{g}_{s_p+1}, & \ldots, & \hat{g}_{s_{p-1}}, & \ldots, & \hat{g}_{s_2}, & \hat{g}_{s_2+1}, \ldots, \hat{g}_{s_1}, \\
F\hat{g}_1, & \ldots, & F\hat{g}_{s_p}, & F\hat{g}_{s_p+1}, & \ldots, & F\hat{g}_{s_{p-1}}, & \ldots, & F\hat{g}_{s_2}, & \\
\vdots & & \vdots & \vdots & & \vdots & & & \\
F^{p-2}\hat{g}_1, & \ldots, & F^{p-2}\hat{g}_{s_p}, & F^{p-2}\hat{g}_{s_p+1}, & \ldots, & F^{p-2}\hat{g}_{s_{p-1}}, & & & \\
F^{p-1}\hat{g}_1, & \ldots, & F^{p-1}\hat{g}_{s_p}\} & & & & & &
\end{array}$$

is a base for R^n.

Let $g_1, g_2, \ldots, g_m$ be the columns vectors of the matrix G. Then $\{F^i g_1, F^i g_2, \ldots, F^i g_m\}$ is a system of generators of the R-module L_i for $1 \leqslant i \leqslant n-1$. Consequently

$$\{F^i g_1 + N_i^{\Sigma}, F^i g_2 + N_i^{\Sigma}, \ldots, F^i g_m + N_i^{\Sigma}\}$$

is a system of generators of the quotient module $N_{i+1}^{\Sigma}/N_i^{\Sigma}$.

Since $N_p^{\Sigma}/N_{p-1}^{\Sigma} = R^n/N_{p-1}^{\Sigma}$ is a free R-module of rank s_p, there exist elements of R^n

$$\hat{g}_i = \sum_{j=1}^{m} \lambda_{ij} g_j, \qquad 1 \leqslant i \leqslant s_p,$$

such that $\{F^{p-1}\hat{g}_1 + N_{p-1}^{\Sigma}, F^{p-1}\hat{g}_2 + N_{p-1}^{\Sigma}, \ldots, F^{p-1}\hat{g}_{s_p} + N_{p-1}^{\Sigma}\}$ is a base for the quotient module $N_p^{\Sigma}/N_{p-1}^{\Sigma}$. The exact sequence

$$0 \to N_{p-1}^{\Sigma} \to N_p^{\Sigma} = R^n \overset{\pi_p}{\to} N_p^{\Sigma}/N_{p-1}^{\Sigma} \to 0$$

splits because $N_p^{\Sigma}/N_{p-1}^{\Sigma}$ is a free R-module. Therefore

$$R^n \simeq R^n/N_{p-1}^{\Sigma} \oplus N_{p-1}^{\Sigma},$$

and hence a base for R^n is obtained with the elements $F^{p-1}\hat{g}_1, F^{p-1}\hat{g}_2, \ldots, F^{p-1}\hat{g}_{s_p}$ and a base for the free R-module N_{p-1}^{Σ}.

By Lemma 2.4 the exact sequence

$$0 \to \operatorname{Ker}\varphi_{p-1} \to N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma} \overset{\varphi_{p-1}}{\to} N_p^{\Sigma}/N_{p-1}^{\Sigma} \to 0$$

splits because $N_p^{\Sigma}/N_{p-1}^{\Sigma}$ is free. Since $\varphi_{p-1}(F^{p-2}\hat{g}_i + N_{p-2}^{\Sigma}) = F^{p-1}\hat{g}_i + N_{p-1}^{\Sigma}$ for $1 \leqslant i \leqslant s_p$, it follows that a base for $N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma}$ is formed by the elements $F^{p-2}\hat{g}_i + N_{p-2}^{\Sigma}$, $1 \leqslant i \leqslant s_p$, and a base for the free R-modulo $\operatorname{Ker}\varphi_{p-1}$ of rank $s_{p-1} - s_p$. Now, since $\{F^{p-2}\hat{g}_i + N_{p-2}^{\Sigma}\}_{1 \leqslant i \leqslant m}$ is a system of generators of the quotient module $N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma}$, it follows that there exist elements $\hat{g}_{s_p+1}, \ldots, \hat{g}_{s_{p-1}}$ of R^n such that

$$\Big\{F^{p-2}\hat{g}_1 + N_{p-2}^{\Sigma}, \ldots, F^{p-2}\hat{g}_{s_p} + N_{p-2}^{\Sigma},$$

$$F^{p-2}\hat{g}_{s_p+1} + N_{p-2}^{\Sigma}, \ldots, F^{p-2}\hat{g}_{s_{p-1}} + N_{p-2}^{\Sigma}\Big\}$$

is a base for the quotient $N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma}$.

Considering that the exact sequence

$$0 \to N_{p-2}^{\Sigma} \to N_{p-1}^{\Sigma} \to N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma} \to 0$$

splits, it follows that

$$R^n \simeq R^n/N_{p-1}^{\Sigma} \oplus N_{p-1}^{\Sigma} \simeq R^n/N_{p-1}^{\Sigma} \oplus N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma} \oplus N_{p-2}^{\Sigma}.$$

The elements of R^n given by

$$\Big\{F^{p-1}\hat{g}_1, \ldots, F^{p-1}\hat{g}_{s_p}, F^{p-2}\hat{g}_1, \ldots, F^{p-2}\hat{g}_{s_p}, F^{p-2}\hat{g}_{s_p+1}, \ldots, F^{p-2}\hat{g}_{s_{p-1}}\Big\}$$

can be completed to a base for R^n with a base for N_{p-2}^{Σ}. Iterating this process, we obtain the decomposition

$$R^n \simeq R^n/N_{p-1}^{\Sigma} \oplus N_{p-1}^{\Sigma}/N_{p-2}^{\Sigma} \oplus \cdots \oplus N_2^{\Sigma}/N_1^{\Sigma} \oplus N_1^{\Sigma}$$

and the desired base $\hat{\mathscr{B}}$ for R^n.

For $1 \leqslant i \leqslant s_1$ we put

$$k_i = \max\Big\{j \in \mathbf{Z}_+ | F^{j-1}\hat{g}_i \in \hat{\mathscr{B}}\Big\},$$

so we have

$$\begin{aligned} k_1 = k_2 = \cdots &= k_{s_p} (= p), \\ k_{s_p+1} = k_{s_p+2} = \cdots &= k_{s_{p-1}}, \\ &\vdots \\ k_{s_2+1} = k_{s_2+2} = \cdots &= k_{s_1}, \end{aligned}$$

and, in particular $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{s_p} \geqslant \cdots \geqslant k_{s_1}$ and $n = \sum_{t=1}^{s_1} k_t$.

Next, using the same technique that R. E. Kalman used in [6] for the classical case (i.e. when R is a field), we prove that Σ is a Brunovsky system.

Since $\{F^{k_i}\hat{g}_1 + N^{\Sigma}_{k_i}, \ldots, F^{k_i}\hat{g}_{s_{k_i+1}} + N^{\Sigma}_{k_i}\}$ is a base for $N^{\Sigma}_{k_i+1}/N^{\Sigma}_{k_i}$, it follows that there exist elements α_{ik_il} of R such that

$$F^{k_i}\hat{g}_i + N^{\Sigma}_{k_i} = \sum_{l=1}^{s_{k_i+1}} \alpha_{ik_il}\left(F^{k_i}\hat{g}_l + N^{\Sigma}_{k_i}\right)$$

and hence

$$F^{k_i}\hat{g}_i - \sum_{l=1}^{s_{k_i+1}} \alpha_{ik_il}F^{k_i}\hat{g}_l \in N^{\Sigma}_{k_i}, \qquad 1 \leqslant i \leqslant s_1.$$

Because $\{F^{k_i-1}\hat{g}_1, \ldots, F^{k_i-1}\hat{g}_{s_{k_i}}, \ldots, \hat{g}_1, \ldots, \hat{g}_{s_1}\}$ is a base for $N^{\Sigma}_{k_i}$ it follows that there exist elements α_{irl} of R such that

$$F^{k_i}\hat{g}_i - \sum_{l=1}^{s_{k_i+1}} \alpha_{ik_il}F^{k_i}\hat{g}_l$$
$$= \sum_{l=1}^{s_1} \alpha_{i_0l}\hat{g}_l + \sum_{l=1}^{s_2} \alpha_{i1l}F\hat{g}_l + \cdots + \sum_{l=1}^{s_{k_i}} \alpha_{i\,k_i-1\,l}F^{k_i-1}\hat{g}_l. \tag{1}$$

or

$$F^{k_i}\left(\hat{g}_i - \sum_{l=1}^{s_{k_i+1}} \alpha_{ik_il}\hat{g}_l\right)$$
$$= \sum_{l=1}^{s_1} \alpha_{i0l}\hat{g}_l + \sum_{l=1}^{s_2} \alpha_{i1l}F\hat{g}_l + \cdots + \sum_{l=1}^{s_{k_i}} \alpha_{i\,k_i-1\,l}F^{k_i-1}\hat{g}_l. \tag{2}$$

We put

$$\begin{cases} \tilde{g}_i = \hat{g}_i & \text{for} \quad 1 \leqslant i \leqslant s_p, \\ \tilde{g}_i = \hat{g}_i - \displaystyle\sum_{l=1}^{s_{k_i+1}} \alpha_{ik_il}\hat{g}_l & \text{for} \quad s_p + 1 \leqslant i \leqslant s_1. \end{cases} \tag{3}$$

Note that, by construction, we have

$$\hat{g}_i = \tilde{g}_i + \sum_{l=1}^{s_{k_i+1}} \beta_{il}\tilde{g}_l, \qquad s_p + 1 \leqslant i \leqslant s_1, \tag{4}$$

and hence

$$F^r\hat{g}_i = F^r\tilde{g}_i + \sum_{l=1}^{s_{k_i+1}} \beta_{il}F^r\tilde{g}_l, \qquad s_p + 1 \leqslant i \leqslant s_1, \quad 1 \leqslant r \leqslant p-1. \tag{5}$$

These equalities and (3) give that the set

$$\begin{aligned}\tilde{\mathscr{B}} = \{&\tilde{g}_1, \ldots, \tilde{g}_{s_p}, \tilde{g}_{s_p+1}, \ldots, \tilde{g}_{s_{p-1}}, \ldots, \tilde{g}_{s_2}, \tilde{g}_{s_2+1}, \ldots, \tilde{g}_{s_1},\\ &F\tilde{g}_1, \ldots, F\tilde{g}_{s_p}, F_{\tilde{g}_{s_p+1}}, \ldots, F\tilde{g}_{s_{p-1}}, \ldots, F\tilde{g}_{s_2},\\ &\vdots \qquad \vdots \qquad \vdots \qquad \vdots\\ &F^{p-2}\tilde{g}_1, \ldots, F^{p-2}\tilde{g}_{s_p}, F^{p-2}\tilde{g}_{s_p+1}, \ldots, F^{p-2}\tilde{g}_{s_{p-1}},\\ &F^{p-1}\tilde{g}_1, \ldots, F^{p-1}\tilde{g}_{s_p}\}\end{aligned}$$

is also a base for R^n. (Note that the matrix which relates the change of the base $\hat{\mathscr{B}}$ to $\tilde{\mathscr{B}}$ is a lower triangular matrix with the elements of the main diagonal are equal to 1).

By the equalities (2) and (3) we have

$$F^{k_i}\tilde{g}_i = \sum_{r=0}^{k_i-1}\left(\sum_{l=1}^{s_{r+1}} \alpha_{irl}F^r\hat{g}_l\right), \qquad 1 \leqslant i \leqslant s_1, \tag{6}$$

and hence, by (4) and (5), there exist elements $\gamma_{irl} \in R$ such that

$$F^{k_i}\tilde{g}_i = \sum_{r=0}^{k_i-1}\left(\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^r\hat{g}_l\right) \qquad 1 \leqslant i \leqslant s_1, \tag{7}$$

We now consider the following elements of R^n:

$$\begin{aligned} e_{ik_i} &= \tilde{g}_i \qquad \text{for} \quad 1 \leqslant i \leqslant s_1,\\ e_{it} &= F^{k_i-t}\tilde{g}_i - \sum_{r=t}^{k_i-1}\left(\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^{r-t}\tilde{g}_l\right) \end{aligned} \tag{8}$$

for $\quad 1 \leqslant i \leqslant s_1, \quad 1 \leqslant t \leqslant k_i - 1.$

The set $\mathscr{B} = \{e_{11}, e_{12}, \ldots, e_{1k_1}, e_{21}, e_{22}, \ldots, e_{2k_2}, \ldots, e_{s_1 1}, e_{s_1 2}, \ldots, e_{s_1 k_{s_1}}\}$ is a base for R^n. The matrix of change from the base $\tilde{\mathscr{B}}$ to $\mathscr{B}$ is again a lower triangular matrix with the elements of the main diagonal equal to 1.

By (7) and (8) we have the equalities

$$F(e_{i1}) = F\left(F^{k_i-1}\tilde{g}_i - \sum_{r=1}^{k_i-1}\left(\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^{r-1}\tilde{g}_l\right)\right)$$

$$= F^{k_i}\tilde{g}_i - \sum_{r=1}^{k_i-1}\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^r\tilde{g}_l = \sum_{r=0}^{k_i-1}\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^r\tilde{g}_l - \sum_{r=1}^{k_i-1}\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^r\tilde{g}_l$$

$$= \sum_{l=1}^{s_1} \gamma_{i0l}\tilde{g}_l = \sum_{l=1}^{s_1} \gamma_{i0l}e_{lk_i} \qquad \text{for} \quad 1 \leqslant i \leqslant s_1,$$

$$F(e_{it}) = F^{k_i-t+1}\tilde{g}_i - \sum_{r=t}^{k_i-1}\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^{r-t+1}\tilde{g}_l$$

$$= F^{k_i-(t-1)}\tilde{g}_i - \sum_{r=t-1}^{k_i-1}\sum_{l=1}^{s_{r+1}} \gamma_{irl}F^{r-(t-1)}\tilde{g}_l + \sum_{l=1}^{s_t} \gamma_{i\,t-1\,l}\tilde{g}_l$$

$$= e_{i\,t-1} + \sum_{l=1}^{s_t} \gamma_{i\,t-1\,l}e_{lk_l} \qquad \text{for} \quad 1 \leqslant i \leqslant s_1, \quad 2 \leqslant t \leqslant k_i - 1,$$

and

$$F(e_{ik_i}) = F\tilde{g}_i = F\tilde{g}_i - \sum_{l=1}^{s_{k_i}} \gamma_{i\,k_i-1\,l}\tilde{g}_l + \sum_{l=1}^{s_{k_i}} \gamma_{i\,k_i-1\,l}\tilde{g}_l$$

$$= e_{i\,k_i-1} + \sum_{l=1}^{s_{k_i}} \gamma_{i\,k_i-1\,l}e_{lk_l} \qquad \text{for} \quad 1 \leqslant i \leqslant s_1.$$

Here F has the following form with respect to the base $\mathscr{B}$:

$$F' = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s_1} \\ E_{21} & E_{22} & \cdots & E_{2s_1} \\ \vdots & \vdots & \ddots & \vdots \\ E_{s_11} & E_{s_12} & \cdots & E_{s_1s_1} \end{pmatrix}$$

where for $i \neq j$, E_{ij} is a $k_i \times k_j$ matrix of the form

$$E_{ij} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \gamma_{j0i} & \gamma_{j1i} & \gamma_{j2i} & \cdots & \gamma_{j\,k_j-1\,i} \end{pmatrix}$$

and E_{ii} is a $k_i \times k_i$ matrix of the form

$$E_{ii} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \gamma_{i0i} & \gamma_{i1i} & \gamma_{i2i} & \cdots & \gamma_{i\,k_i-1\,i} \end{pmatrix}$$

On the other hand, since N_1^{Σ} is free, the exact sequence

$$0 \to \operatorname{Ker}\theta_G \to R^m \overset{\theta_G}{\to} N_1^{\Sigma} \to 0$$

splits, where θ_G is the homomorphism defined by G with respect to the standard bases for R^m and R^n respectively. Therefore there exists a base $\{u_i\}_{1 \leqslant i \leqslant m}$ for R^m such that $\theta_G(u_i) = e_{ik_i}$ for $1 \leqslant i \leqslant s_1$ and $\theta_G(u_i) = 0$ for $s_1 + 1 \leqslant i \leqslant m$.

With respect to the bases $\{u_i\}_{1 \leqslant i \leqslant m}$ for R^m and $\mathscr{B}$ for R^n, the matrix G takes the following form:

$$
G_c = \begin{array}{c}
\begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad}^{s \text{ columns}} & \overbrace{\qquad\qquad\qquad}^{m\text{-}s \text{ columns}} \end{array} \\
\left(\begin{array}{c|c|c|c|ccc}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\hline
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0
\end{array}\right)
\end{array}
\begin{array}{l}
\\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_1 \text{ rows} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_2 \text{ rows} \\
\begin{array}{c} \\ \\ \\ \end{array} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_s \text{ rows}
\end{array}
$$

Considering the $m \times n$ matrix K with entries in R defined by

$$
K = \begin{pmatrix}
-\gamma_{101} & \cdots & -\gamma_{1\,k_i-1\,1} & \cdots & -\gamma_{s_1 01} & \cdots & -\gamma_{s_i\,k_{s_1}-1\,1} \\
-\gamma_{102} & \cdots & -\gamma_{1\,k_1-1\,2} & \cdots & -\gamma_{s_1 02} & \cdots & -\gamma_{s_1\,k_{s_1}-1\,2} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-\gamma_{10s_1} & \cdots & -\gamma_{1\,k_1-1\,s_1} & \cdots & -\gamma_{s_1 0 s_1} & \cdots & -\gamma_{s_1\,k_{s_1}-1\,s_1} \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix},
$$

we have that

$$
F_c = F' + G_c K = \begin{pmatrix}
E_1 & 0 & \cdots & 0 \\
0 & E_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E_{s_1}
\end{pmatrix},
$$

where E_i is the $k_i \times k_i$ matrix

$$E_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for $1 \leqslant i \leqslant s_1$. ■

REMARK 3.2. For a system Σ of the class of Brunovsky systems, the parameter set $\{k_1, k_2, \dots, k_{s_1}\}$ (or equivalently $\{s_1, s_2, \dots, s_p\}$) constitutes an independent and complete set of invariants for the feedback class of Σ.

REMARK 3.3. Next we answer the following questions:

(a) When does statement (ii) of Theorem 3.1 hold?

(b) Does there exist a practical method to obtain the feedback invariants $\{s_i\}$ associated to a Brunovsky system Σ?

Consider the free presentation of M_i^{Σ}:

$$R^{im} \xrightarrow{\theta_{\tilde{G}_i^{\Sigma}}} R^n \to M_i^{\Sigma} \to 0,$$

where $\theta_{\tilde{G}_i^{\Sigma}}$ is the homomorphism defined by $\tilde{G}_i^{\Sigma}$ with respect to the standard bases for R^{im} and R^n respectively. Suppose that R is such that finitely generated projective R-modules are free. By [7, p. 122], M_i^{Σ} is a free module of rank n_i if and only if

$$\mathscr{U}_{\mu}\left(\tilde{G}_i^{\Sigma}\right) = \begin{cases} R & \text{for } 0 \leqslant \mu \leqslant n - n_i, \\ 0 & \text{for } \mu > n - n_i. \end{cases}$$

where $\mathscr{U}_{\mu}(\tilde{G}_i^{\Sigma})$ denotes the μth determinantal ideal of $\tilde{G}_i^{\Sigma}$ [i.e., $\mathscr{U}\mu(\tilde{G}_i^{\Sigma})$ is the ideal of R generated by all the $\mu \times \mu$ minors of $\tilde{G}_i^{\Sigma}$].

Finally we have the equalities

$$\begin{aligned} s_i &= \operatorname{rank}\left(N_i^{\Sigma}/N_{i-1}^{\Sigma}\right) = \operatorname{rank} N_i^{\Sigma} - \operatorname{rank} N_{i-1}^{\Sigma} \\ &= \left(n - \operatorname{rank} M_i^{\Sigma}\right) - \left(n - \operatorname{rank} M_{i-1}^{\Sigma}\right) \\ &= \operatorname{rank} M_{i-1}^{\Sigma} - \operatorname{rank} M_i^{\Sigma} = n_{i-1} - n_i \end{aligned}$$

for $1 \leqslant i \leqslant p$, and where $M_0^{\Sigma} = R^n$. Consequently, the positive integers $n_1, n_2, \ldots, n_p$ are a complete set of invariants for the feedback class of Σ. Moreover this set of invarinats can be obtained by an algorithmic method applied to the matrices F and G of Σ.

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