Optimal control theory and the efficiency of the swimming mechanism of the Copepod Zooplankton

P. Bettiol * **B. Bonnard** ** **A Nolot** *** **J Rouot** ****

* *Laboratoire de Mathématiques Unité CNRS UMR 6205, Université de Bretagne Occidentale, 6, Avenue Victor Le Gorgeu, 29200 Brest, France (e-mail: piernicola.bettiol@univ-brest.fr)*
** *Inria Sophia Antipolis et Institut de Mathématiques de Bourgogne, 9 avenue Savary, 21078 Dijon, France (e-mail: bernard.bonnard@u-bourgogne.fr)*
*** *Institut de Mathématiques de Bourgogne, 9 avenue Savary, 21078 Dijon, France (e-mail: alice.nolot@gmail.com)*
**** *Inria Sophia Antipolis, 2004 route des lucioles, F-06902 Sophia Antipolis, France (e-mail: jeremy.rouot@inria.fr)*

Abstract: In this article, the model of swimming at low Reynolds number introduced by D. Takagi (2015) to analyze the displacement of an abundant variety of zooplankton is used as a testbed to analyze the motion of symmetric microswimmers in the framework of optimal control theory assuming that the motion occurs minimizing the energy dissipated by the fluid drag forces in relation with the concept of efficiency of a stroke. The maximum principle is used to compute periodic controls candidates as minimizing controls and is a decisive tool combined with appropriate numerical simulations using indirect optimal control schemes to determine the most efficient stroke compared with standard computations using Stokes theorem and curvature control. Also the concept of graded approximations in SR-geometry is used to evaluate strokes with small amplitudes providing a fixed displacement and minimizing the dissipated energy.

Keywords: Low Reynolds number, Copepod swimmer, SR-geometry, Periodic optimal control, Transversality conditions

1. INTRODUCTION

This article is entirely devoted to the analysis combining optimal control theory and sub-Riemannian (SR-) geometry of the swimming process of a variety of zooplankton observed by Takagi (2015) and modeled in the framework of swimming at low Reynolds number. See Fig.1 for the picture of the copepod (left) and the 2-link symmetric micro-robot swimmer to mimic the animal mechanism.

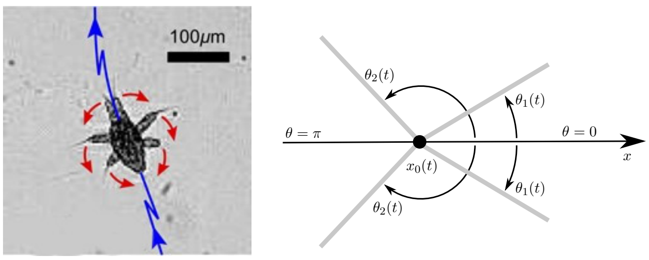

Fig. 1. *(left)* Observation of a zooplankton. *(right)* Sketch of the 2-link symmetric swimmer.

In micro-robot modeling, to produce the displacement along the line Ox_0, we use a pair of two symmetric links, with equal length normalized to $l = 1$, θ_1, θ_2 are the respective angles of the two links, and they satisfy the constraint $0 \leq \theta_1 \leq \theta_2 \leq \pi$.

Using the swimming model at low Reynolds number, we relate the speed of the displacement to the speed of the shape variable θ by the equation

$$\dot{x_0} = \frac{\sum_{i=1}^{2} \dot{\theta}_i \sin(\theta_i)}{\sum_{i=1}^{2}(1+\sin^2(\theta_i))}. \tag{1}$$

To parameterize the motion as a control system, one introduces the dynamics:

$$\dot{\theta}_1 = u_1, \quad \dot{\theta}_2 = u_2.$$

It provides a control system written as

$$\dot{q} = \sum_{i=1}^{2} u_i F_i(q)$$

with $q = (x_0, \theta)$, $\theta = (\theta_1, \theta_2)$. Moreover we have state constraints given by a triangle $\mathcal{T}$ in the shape variables: $\theta_i \in [0, \pi]$, $i = 1, 2$, and $\theta_1 \leq \theta_2$. u_1, u_2 are periodic controls producing strokes, which are closed curves in the θ-plane, and the reference problem can be analyzed in the framework of optimal control theory introducing a cost function. A choice of particular interest for the cost to minimize, in particular in relation with the concept of efficiency defined by Lightwill (1960), is the mechanical energy dissipated by the drag forces:

$$E = \int_0^T (\dot{q}^{\intercal} M(\theta) \dot{q}) \mathrm{d}t$$

where M is the matrix

$$M = \begin{pmatrix} 2-1/2(\cos^2\theta_1+\cos^2\theta_2) & -1/2\sin\theta_1 & -1/2\sin\theta_2 \\ -1/2\sin\theta_1 & 1/3 & 0 \\ -1/2\sin\theta_2 & 0 & 1/3 \end{pmatrix}$$

Using (1) this amounts to minimize the quadratic form

$$E = \int_0^T (a_(\theta)u_1^2 + 2b(\theta)u_1u_2 + c(\theta)u_2^2)\mathrm{d}t$$

with

$$\begin{aligned} a &= \frac{1}{3} - \frac{\sin^2\theta_1}{2(2+\sin^2\theta_1+\sin^2\theta_2)}, \\ b &= -\frac{\sin\theta_1\sin\theta_2}{2(2+\sin^2\theta_1+\sin^2\theta_2)}, \\ c &= \frac{1}{3} - \frac{\sin^2\theta_2}{2(2+\sin^2\theta_1+\sin^2\theta_2)}. \end{aligned}$$

Denoting g the associated Riemannian metric in the θ-space, the optimal control problem can be rewritten as a sub-Riemannian problem

$$\dot{q} = \sum_{i=1}^2 u_i F_i(q), \qquad \min_{u(.)} \int_0^T g(\theta,\dot{\theta})\mathrm{d}t$$

with appropriate boundary conditions associated with periodic control

$$\theta(0) = \theta(T),$$

and with the triangle inequality constraints $0 \le \theta_1 \le \theta_2 \le \pi$. $x_0(T) - x_0(0)$ represents the displacement of a stroke and one can set $x_0(0) = 0$. According to Maupertuis principle, this is equivalent to minimize the length

$$L = \int_0^T g(\theta,\dot{\theta})^{1/2}\mathrm{d}t$$

and using the energy minimization point of view, the period T of a stroke can be fixed to $T = 2\pi$. We emphasize that the problem is equivalent to a minimal-time control by fixing the energy level $g = 1$ of the strokes.

From this point of view, the optimal control problem consists in computing the points of the sub-Riemannian sphere $S_{q_0}(r)$ formed by extremities of the minimizers starting from q_0 and with fixed length, and requiring that the optimal control is periodic. This is equivalent to fix the displacement $x_0(2\pi) - x_0(0)$ and to compute strokes minimizing the length.

The concept of geometric efficiency differs has a clear meaning in the SR-geometry context. Assuming $x_0(0) = 0$, the geometric efficiency is the ratio

$$\mathcal{E} = x_0(T)/L \tag{2}$$

where L is the length of the stroke producing the displacement $x_0(T)$, which is proportional, for a fixed g, to $\mathcal{E}' = x_0^2(T)/E$, which is introduced by Takagi (2015).

A further concept of efficiency can be similarly used, see for instance Chambrion et al (2014). It takes the ratio between the energy used to move the swimmer at constant speed $\bar{v}$ to produce the displacement $x_0(T)$ and the mechanical energy where the shape variables are hold on at $\theta(0)$, that is

$$Eff = \frac{|\bar{v}|^2 M_{11}(\theta(0))}{1/TE},\ M = (M_{ij}),\ \bar{v} = \frac{1}{T}\int_0^T \dot{x}_0\mathrm{d}t. \tag{3}$$

We have (taking $x_0(0) = 0$)

$$Eff \sim \frac{x_0^2(T)}{E} M_{11}(\theta(0))$$

If the concept of geometric efficiency is related to SR-geometry, the problem of maximizing the efficiency can be studied employing also techniques of optimal control. This will be the main achievement of this article.

The paper will be organized in two sections. The first section represents a geometric analysis, in relation with SR-geometry, and is devoted to the problem of computing optimal strokes with small amplitudes. The second section is a direct application of the maximum principle in the frame of periodic optimal control supplemented by second order optimality conditions and numerical simulations to compute strokes for the problem of maximizing different efficiencies.

2. A GEOMETRIC ANALYSIS OF THE PROBLEM IN THE FRAME OF SR-GEOMETRY

2.1 Geodesic computation

Consider the energy minimization problem

$$\dot{q} = \sum_{i=1}^2 u_i F_i(q), \quad \min_{u(.)} \int_0^T (a(q)u_1^2 + 2b(q)u_1u_2 + c(q)u_2^2)\mathrm{d}t$$

where the set of admissible controls $\mathcal{U}$ is the set of bounded measurable mapping valued in $\mathbb{R}^2$. We introduce the pseudo-Hamiltonian $H(q,p,u) = p\dot{q} + p_0(a(q)u_1^2 + 2b(q)u_1u_2 + c(q)u_2^2)$. According to the maximum principle (cf Vinter (2000)), minimizers are found among extremals curves, which are solutions of the following equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \frac{\partial H}{\partial u} = 0.$$

This leads to consider the following cases.

Normal case. Assume $p_0 \neq 0$ and it can be normalized to $p_0 = -1/2$. Corresponding controls are given by

$$u_1 = \frac{cH_1 - bH_2}{ab - b^2}, \qquad u_2 = \frac{aH_2 - bH_1}{ab - b^2}.$$

where H_1, H_2 are the Hamiltonian lifts of the vector fields F_1, F_2 and plugging such controls in H yields the *normal Hamiltonian*

$$H_n = \frac{1}{2(ac - b^2)}(aH_2^2 - 2bH_1H_2 + cH_1^2). \tag{4}$$

The corresponding solution $z = (q,p)$ are called *normal extremals.*

Abnormal case. If $p_0 = 0$, additional extremals $z = (q,p)$ appear and they are called *abnormal.* They are solutions of the implicit equations

$$H_1 = H_2 = 0. \tag{5}$$

If F, G are two smooth vector fields, the Lie bracket is computed as

$$[F,G](q) = \frac{\partial F}{\partial q}(q)G(q) - \frac{\partial G}{\partial q}(q)F(q)$$

and if $H_F = p \cdot F(q)$, $H_G = p \cdot G(q)$, the Poisson bracket is: $\{H_F, H_G\}(z) = \mathrm{d}H_F(\overrightarrow{H_G}) = p \cdot [F,G](q)$.

Differentiating twice (5) with respect to time, abnormal controls are given by

$$\begin{aligned} &H_1 = H_2 = \{H_1, H_2\} = 0, \\ &u_1\{\{H_1, H_2\}, H_1\} + u_2\{\{H_1, H_2\}, H_2\} = 0 \end{aligned} \tag{6}$$

and can be (generically) computed solving (6) provided one Poisson bracket $\{\{H_1, H_2\}, H_i\}$, $i = 1, 2$ is non zero.

Definition 1. The *exponential mapping* is, for fixed q_0 the map: $\exp_{q_0} : (t, p(0)) \mapsto \Pi(\exp t\overrightarrow{H_n}(z(0)))$ where Π is the standard projection: $\Pi : (q, p) \mapsto q$. A projection of an extremal is called a *geodesic*. It is called *strictly normal* if it is the projection of a normal extremal but not an abnormal one. A time t_c is a *conjugate time* if the exponential mapping is not of full rank at t_c and t_{1c} denotes the *first conjugate time* and $q(t_{1c})$ is called the *first conjugate point* along the reference geodesic $t \mapsto q(t)$.

Definition 2. Fixing q_0, the *wave front* $W(q_0, r)$ is the set of extremities of geodesics (normal or abnormal) with length r and the *sphere* $S(q_0, r)$ is the set of extremities of minimizing geodesics. The *conjugate locus* $C(q_0)$ is the set of first conjugate points of normal geodesics starting from q_0 and the cut locus $C_{cut}(q_0)$ is the set of points where geodesics cease to be optimal.

Definition 3. According to the previous definitions, a stroke is called (*strictly*) *normal* if it is a (strictly) normal geodesic with periodic control while an *abnormal stroke* is a piecewise smooth abnormal geodesic with periodic control.

2.2 Computation in the copepod case

One has:

$$F_i = \frac{\sin(\theta_i)}{\Delta}\frac{\partial}{\partial x_0} + u_i\frac{\partial}{\partial \theta_i},\ i = 1, 2.$$

with $\Delta = \sum_{i=1}^{2}(1 + \sin^2(\theta_i))$. We get

$$[F_1, F_2](q) = \tilde{f}(\theta_1, \theta_2)\frac{\partial}{\partial x_0}$$

with

$$\tilde{f}(\theta_1, \theta_2) = \frac{2\sin(\theta_1)\sin(\theta_2)(\cos(\theta_1) - \cos(\theta_2))}{\Delta^2}.$$

Furthermore,

$$[[F_1, F_2], F_i] = \frac{\partial \tilde{f}}{\partial \theta_i}\frac{\partial}{\partial x_0},\ i = 1, 2$$

and we have simple formulas to generate all Lie brackets.

Definition 4. A point q_0 is called a Darboux or *contact point* if at q_0, F_1, F_2 and $[F_1, F_2]$ are linearly independent and a *Martinet point* if F_1, F_2 and $[F_1, F_2]$ are coplanar but at least one $i = 1, 2$, $[[F_1, F_2], F_i] \notin \text{span}\{F_1, F_2\}$.

According to this terminology and Lie brackets computations, we have

Proposition 5. 1) All interior points of the triangle $\mathcal{T}$: $0 \le \theta_1 \le \theta_2 \le \pi$ are contact points.

2) The sides of the triangle (vertices excluded) are Martinet points and the triangle is a (piecewise smooth) abnormal stroke.

Geometric comments. Hence the observed stroke by Takagi (2015) corresponds to the policy:

$\theta_2 : 0 \to \pi,\ \theta_1 : 0 \to \pi,\ \theta_i : \pi \to 0,\ i = 1, 2$ with $\theta_1 = \theta_2$.

where the copepod swimmer follows the triangle boundary of the physical domain corresponding to the unique abnormal stroke.

Moreover it has a nice geometric interpretation using Stokes' theorem and curvature control methods.

Lemma 6. One has:

1) $$\oint \sum_{i=1}^{2} \frac{\sin(\theta_i)}{\Delta} \mathrm{d}\theta_i = \int \left[\frac{\partial}{\partial \theta_2}\left(\frac{\sin(\theta_1)}{\Delta}\right) - \frac{\partial}{\partial \theta_1}\left(\frac{\sin(\theta_2)}{\Delta}\right)\right] \mathrm{d}\theta_1 \wedge \theta_2 = \int \mathrm{d}\omega.$$

2) The points where $\mathrm{d}\omega = 0$ are precisely the abnormal triangle, and $\mathrm{d}\omega < 0$ in the interior domain and $\mathrm{d}\omega > 0$ in the exterior.

2.3 SR-classification in dimension 3 and strokes with small amplitude for the copepod swimmer

The contact case. The crucial results applicable to our model come from Alaoui et al. (1996). Consider a standard SR-problem (D, g), where Δ is a distribution and g is a SR-metric, near a point $q_0 \in \mathbb{R}^3$ identified with 0, one has:

- *Heisenberg-Brockett nilpotent model.* The nilpotent model (of order -1) is the so-called Heisenberg-Brockett model where (D, g) is defined by the orthonormal frame: $D = \text{span}\{\hat{F}, \hat{G}\}$

$$\hat{F} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \hat{G} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

with $q = (x, y, z)$ and the graduation 1 for x, y and 2 for z, forming a set of privileged coordinates.

Using this gradation, the normal form of order 0 is similar and the generic model is given by the normal form of order 1

$$F = \hat{F} + yQ(w)\frac{\partial}{\partial z},\ G = \hat{G} - xQ(w)\frac{\partial}{\partial z},$$

$w = (x, y)$ and Q is quadratic in w, $Q = \alpha x^2 + 2\beta xy + \gamma y^2$, where α, β, γ are parameters.

We introduce $\tilde{Q}(w) = (1 + Q(w))$ and one can write

$$F = \frac{\partial}{\partial x} + y\tilde{Q}\frac{\partial}{\partial z},\ G = \frac{\partial}{\partial y} - x\tilde{Q}\frac{\partial}{\partial z}.$$

Computing, one has

$$[F, G] = (2\tilde{Q} + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y})\frac{\partial}{\partial z}$$

and using Euler formula

$$[F, G] = 2(1 + 2Q)\frac{\partial}{\partial z} = f(w)\frac{\partial}{\partial z}.$$

Geodesics equations. We use Poincaré coordinates associated with the frame $(F, G, \frac{\partial}{\partial z})$:

$$H_1 = p \cdot F, \quad H_2 = p \cdot G, \quad H_1 = p \cdot \frac{\partial}{\partial z}$$

and the normal Hamiltonian is

$$H_n = \frac{1}{2}(H_1^2 + H_2^2).$$

One has

$\dot{H}_1 = \mathrm{d}H_1(\overrightarrow{H_n}) = \{H_1, H_2\}H_2 = (p \cdot [F, G])H_2 = p_z f(w) H_2$

with $p_z = H_3$ is constant.

$$\dot{H}_2 = \mathrm{d}H_2(\overrightarrow{H_n}) = -\{H_1, H_2\}H_1 = -p_z f(w) H_1.$$

Since

$$f(w) = 2 + O(|w|^2)$$

and for p_z non zero, one can introduce the parametrization

$$\mathrm{d}s = p_z f(w)\mathrm{d}t. \tag{7}$$

Denoting by $'$ the derivative with respect to s, we get

$$H_1' = H_2, \qquad H_2' = -H_1.$$

Hence we deduce

Lemma 7. In the s-parameter, the normal controls are solutions of the linear pendulum equation $H_1'' + H_1 = 0$ and are trigonometric functions.

A more precise analysis requires higher order Lie brackets computations. Note that, in particular, the relations

$$[[F,G],F] = \frac{\partial f}{\partial x}\frac{\partial}{\partial z}, \quad [[F,G],G] = \frac{\partial f}{\partial y}\frac{\partial}{\partial z}.$$

The remaining equations to be integrated using (7) are

$$x' = \frac{H_1}{p_z f(w)}, \quad y' = \frac{H_2}{p_z f(w)}, \quad z' = \frac{(H_1 y - H_2 x)\tilde{Q}}{p_z f(w)}. \tag{8}$$

The solution can be estimated using the following expansion, associated with the gradation, setting

$$x = \varepsilon X, \quad y = \varepsilon Y, \quad z = \varepsilon^2 Z,$$
$$p_x = p_X, \quad p_y = p_Y, \quad p_z = p_Z/\varepsilon.$$

so that the Darboux form is homogeneous.

Denoting $Q = (X, Y, Z)$, $P = (P_X, P_Y, P_Z)$ the solution can be obtained in the expansion

$$\begin{aligned} X(t) &= X_0(t) + \varepsilon X_1(t) + o(\varepsilon), \\ Y(t) &= Y_0(t) + \varepsilon Y_1(t) + o(\varepsilon), \\ Z(t) &= Z_0(t) + \varepsilon Z_1(t) + o(\varepsilon). \end{aligned} \tag{9}$$

Clearly by identification one gets that $t \mapsto (X_0(t), Y_0(t), Z_0(t))$ is the solution obtained by the (nilpotent) Heisenberg-Brockett model and similarly for the higher order expansions.

The Martinet case. The analysis at a Martinet point q_0 not belonging to the vertices of the triangle is more intricate and can be analyzed using the results of Bonnard et al. (2003).

Generic model at a Martinet point q_0 identified with 0. There exist local coordinates $q = (x, y, z)$ such that the SR-geometry is given by (D, g) with

- $D = \mathrm{span}\{F, G\}$ and $F = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}$, $G = \frac{\partial}{\partial y}$ where $q = (x, y, z)$ are graded coordinates with respective weights $(1, 1, 3)$.
- The metric q is of the form

$a(q)\mathrm{d}x^2 + c(q)\mathrm{d}y^2$ and we have

- Nilpotent model (flat Martinet case): $a = c = 1$.
- Generic model:

$$a = (1 + \alpha y)^2 \sim 1 + 2\alpha y \text{ (order zero)}$$
$$c = (1 + \beta x + \gamma y)^2 \sim 1 + 2\beta x + 2\gamma y \text{ (order zero)}$$

where α, β, γ are parameters.

Geodesic equations. We introduce the orthonormal frame

$$F_1 = \frac{F}{\sqrt{a}}, \quad F_2 = \frac{G}{\sqrt{c}}, \quad F_3 = \frac{\partial}{\partial z}$$

and denoting $H_i = p \cdot F_i$, the normal Hamiltonian is given by $H_n := 1/2(H_1^2 + H_2^2)$.

We parameterize by arc-length: $H_1^2 + H_2^2 = 1$, $H_1 = \cos\chi$, $H_2 = \sin\chi$ and $H_3 = p_z = \lambda$ constant, assuming $\lambda \neq 0$. Hence the geodesic equations become

$$\begin{aligned} &\dot{x} = \frac{\cos\chi}{\sqrt{a}}, \ \dot{y} = \frac{\sin\chi}{\sqrt{b}}, \ \dot{z} = \frac{y^2\cos\chi}{2\sqrt{a}}, \\ &\dot{\chi} = \frac{1}{\sqrt{ac}}(y\lambda - \alpha\cos\chi - \beta\sin\chi) \end{aligned} \tag{10}$$

We introduce the new parameterization

$$\sqrt{ac}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}s} \tag{11}$$

and denoting by ϕ' the derivative of a function ϕ with respect to s, we get the equations

$$y' = \sin\chi(1 + \alpha y), \quad \chi' = (y\lambda - \alpha\cos\chi - \beta\sin\chi)$$

and the second order differential equation

$$\chi'' + \lambda\sin\chi + \alpha^2\sin\chi\cos\chi - \alpha\beta\sin^2\chi + \beta\chi'\cos\chi = 0. \tag{12}$$

As a consequence, we obtain the following result.

Proposition 8. The generic case projects, up to a time reparameterization, onto a two dimensional equation (12), associated with a generalized dissipative pendulum depending on the parameters α, β only.

Geometric application.

- The flat case is $\alpha = \beta = \gamma = 0$ and corresponds to the standard pendulum.
- In the generic case we have two subcases
  - · $\beta = 0$ and (12) is integrable using elliptic functions.
  - · $\beta \neq 0$, due to dissipation, we are in the non-integrable case.

Application to the copepod. In the integrable case: $\beta = 0$ models of periodic strokes with elliptic functions with modulus k are

- $k = 0$: circles
- $k \simeq 0.65$: eight shape (Bernoulli lemniscates). Note that $\beta = 0$ is not a stable case, moreover the triangle constraint $0 \leq \theta_1 \leq \theta_2 \leq \pi$ is not taken into account.

3. A POWERFUL APPROACH USING OPTIMAL CONTROL THEORY AND NUMERICAL SIMULATIONS

In this section, the problem is analyzed using the maximum principle applied for optimal control with periodic controls (cf Vinter (2000)) and complemented by necessary second order optimality conditions corresponding to the concept of conjugate point.

3.1 The maximum principle with periodic controls

The crucial point is the existence of a maximum principle suitable to analyze the problem of maximizing different concept of efficiencies. The system and the energy are written in the extended state space with $\bar{q} = (q, q^0)$ and the corresponding dynamic

$$\dot{q} = \sum_{i=1}^{2} u_i F_i(q) = F(q, u),\ u \in \mathbb{R}^2$$

$$\dot{q}^0 = \sum_{i=1}^{2} u_i^2,\ q^0(0) = 0.$$

and the problem is to minimize a cost of the form

$$\min_{u(.)} h(\bar{q}(0), \bar{q}(2\pi))$$

where the end-point conditions are of the form $(\bar{q}(0), \bar{q}(2\pi)) \in C$, where $C \subset \mathbb{R}^3 \times \mathbb{R}^3$ is a given closed set.

We denote $\bar{p} = (p, p_0)$ the extended adjoint vector. The pseudo- Hamiltonian takes the form

$$H(\bar{q}, \bar{p}, u) = \sum_{i=1}^{2} u_i H_i + p_0(u_1^2 + u_2^2)$$

and $H_i = p \cdot F_i(q)$.

From the maximum principle (cf Vinter (2000)), an optimal control pair (q, u) satisfies the following necessary conditions that we split into two distinct parts

Standard conditions.

$$\dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}}, \quad \dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}}, \quad \frac{\partial H}{\partial u} = 0.$$

Transversality conditions.

$$(\bar{p}(0), -\bar{p}(2\pi)) \in \lambda \nabla h(\bar{q}(0), \bar{q}(2\pi)) + N_C(\bar{q}(0), \bar{q}(2\pi)) \quad (13)$$

where N_C is the (limiting) normal cone to the (closed) set C, $(\bar{p}, \lambda) \neq 0$, $\lambda \geq 0$.

Application. $q = (x_0, \theta_1, \theta_2)$

- Maximizing the geometric efficiency with periodic condition

  $$\theta(0) = \theta(2\pi), \quad h = -\frac{x_0^2(2\pi)}{E}$$

  where E is the mechanical energy

  $$E = \int_0^{2\pi} (u_1^2 + u_2^2)\mathrm{d}t.$$

  In this case we deduce the periodicity condition on $p_\theta = (p_{\theta_1}, p_{\theta_2})$ dual to θ:

  $$p_\theta(0) = p_\theta(2\pi), \quad (14)$$

  to produce a *smooth stroke* in the normal case $p_0 \neq 0$.
  Moreover (p_x, p_0) at the final point have to be collinear to the gradient of the set $h(x_0, x^0) = c$ where $x^0 = u_1^2 + u_2^2$.
- Maximizing an efficiency depending on $\theta(0)$, with periodic conditions:

  $$\theta(0) = \theta(2\pi), \quad h = -\frac{m(\theta(0))x_0^2(2\pi)}{E}$$

  where m is a chosen smooth function.
  In this case, (14) becomes:

  $$p_\theta(0) - p_\theta(2\pi) = \lambda \frac{\partial h}{\partial \theta(0)} \quad (15)$$

  hence producing a *jump* of the adjoint vector.

3.2 Second-order necessary optimality condition

It is the standard necessary optimality condition related to existence of conjugate point (cf Bonnard et al. (2003)).

Proposition 9. Let $(x_0(t), \theta(t))$, $t \in [0, 2\pi]$ be a strictly normal stroke. Then a necessary optimality condition is the non existence of conjugate time $t_c \in]0, 2\pi[$.

It can be checked numerically using the `HamPath` code.

3.3 Applications: numerical simulations

We present a sequence of simulations using the `HamPath` software. These are based on our computations for the $\int_0^{2\pi}(u_1^2 + u_2^2)\mathrm{d}t$ cost.

- Fig.2: Different kind of normal strokes: simple loop, limaçon and eight and computation of conjugate points. Only the simple loops are candidates to be optimal strokes.
- There is a one-parameter family of simple strokes, each of them associated with a different energy. The corresponding efficiency is represented in Table 1 and compared with the efficiency of the abnormal stroke, producing the maximum efficiency value. We have numerically checked that it corresponds to the stroke with the transversality condition provided by the maximum principle (13), cf Fig.6.
- As in Chambrion et al (2014), we consider a cost depending upon $\theta(0)$, namely

  $$h = -\frac{x_0^2(2\pi)m(\theta(0))}{E}$$

  where $m(\theta_1(0)) = 2 - \cos^2(\theta_1(0))$. In Fig.7 is illustrated the corresponding optimal solution satisfying the transversality conditions (13), it is a non smooth stroke and it can be compared to the previous smooth solution of Fig.6.

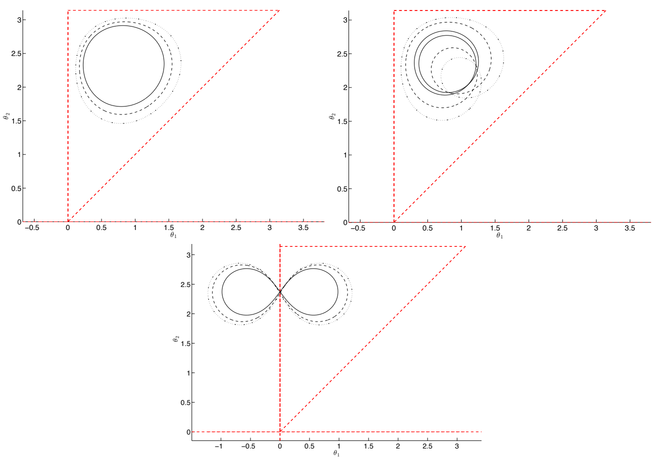

Fig. 2. One parameter family of simple loops, limaçons and Bernoulli lemniscates normal strokes

ACKNOWLEDGEMENTS

J. Rouot is supported by the French Space Agency CNES, R&T action R-S13/BS-005-012 and by the région Provence-Alpes-Côte d'Azur.

Types of γ	$x_0(T)$	$L(\gamma)$	$x_0(T)/L(\gamma)$
Simple loops	2.000×10^{-1}	4.946	4.043×10^{-2}
	2.100×10^{-1}	5.109	4.110×10^{-2}
	2.701×10^{-1}	7.850	3.452×10^{-2}
Normal stroke, Fig.6	$\mathbf{2.169 \times 10^{-1}}$	**5.180**	$\mathbf{4.187 \times 10^{-2}}$
Abnormal	2.742×10^{-1}	10.73	2.555×10^{-2}
Limaçon	2.000×10^{-1}	6.147	3.253×10^{-2}
Eight	2.000×10^{-1}	6.954	3.307×10^{-2}

Table 1. Value of the geometric efficiency for abnormal solution and different normal strokes for the Copepod swimmer.

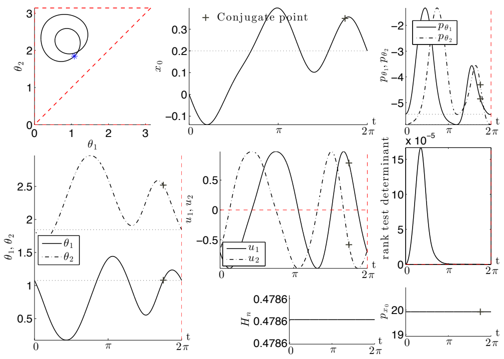

Fig. 3. Normal stroke of the Copepod swimmer with limaçon shape. The first conjugate point is computed (indicated by the cross)

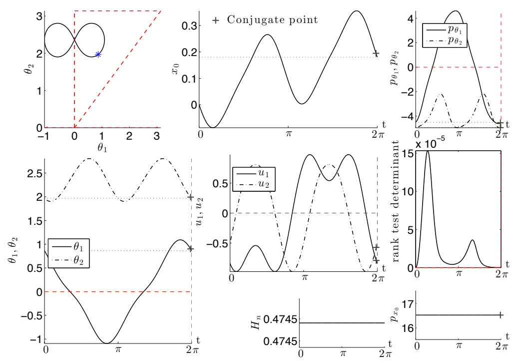

Fig. 4. Normal stroke of the Copepod swimmer with eight shape. The first conjugate point is computed (indicated by the cross)

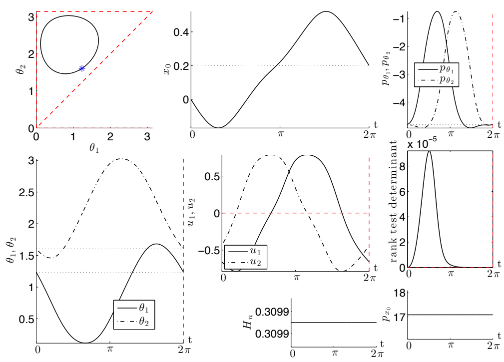

Fig. 5. Normal stroke of the Copepod swimmer with simple loop shape. There is no conjugate point on $]0, 2\pi[$.

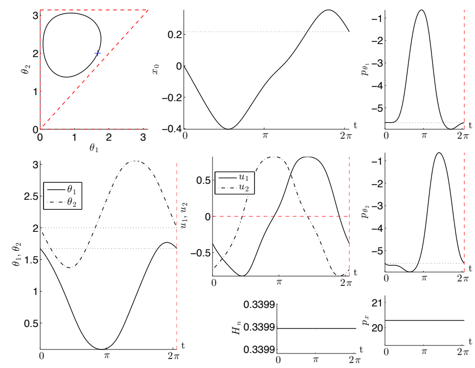

Fig. 6. Normal stroke of the Copepod swimmer for the geometric efficiency, obtained by the transversality conditions of the maximum principle (13).

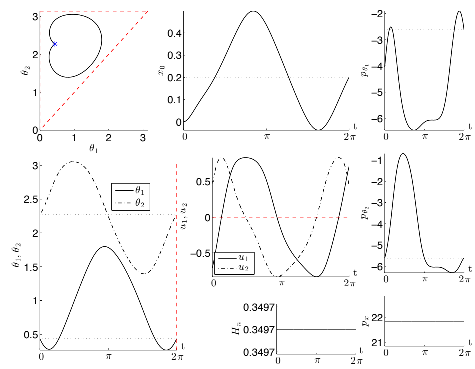

Fig. 7. Normal stroke of the Copepod swimmer for the efficiency depending upon the initial angle $\theta(0)$, obtained by the transversality conditions of the maximum principle (13)

REFERENCES

El-H.C. Alaoui, J.P. Gauthier, I. Kupka. Small sub-Riemannian balls on R^3. J. Dynam. Control Systems. **2**, 3 359–421 (1996)

B. Bonnard, M. Chyba. Singular trajectories and their role in control theory. Mathématiques & Applications **40**, Springer-Verlag, Berlin (2003)

R.W. Brockett. Control theory and singular Riemannian geometry. New directions in applied mathematics (Cleveland, Ohio, 1980) Springer, New York-Berlin (2016) pp.11–27.

T. Chambrion, L. Giraldi, A. Munnier. Optimal strokes for driftless swimmers: A general geometric approach. (Submitted, 2014, hal-00969259)

P.H. Lenz, D. Takagi, D.K. Hartline. Choreographed swimming of copepod nauplii. Journal of The Royal Society Interface 12, no 112 (2015)

M.J. Lighthill. Note on the swimming of slender fish. *J. Fluid Mech.* **9**, 305–317 (1960)

J. Lohéac, J.-F. Scheid. Time optimal swimmers and Brockett integrator. (2015) hal-01164561

D. Takagi. Swimming with stiff legs at low Reynolds number. Phys. Rev. E 92. (2015)

R.B. Vinter, Optimal control. *Systems & Control: Foundations & Applications* (2000) xviii–507.