Some Optimization Methods for Optimal Muscular Force Response to Functional Electrical Stimulations based on Pontryagin-type Conditions and Observability

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joint work with T. Bakir (UBFC), B. Bonnard (INRIA & UBFC), L. Bourdin (Xlim)

Thematics and Contracts

Context : Electrical muscle stimulation : **force-fatigue model**

Aim : Optimize a pulses train w.r.t. some cost

Sampled-data control problem,

Pontryagin-type optimality conditions (open-loop control)

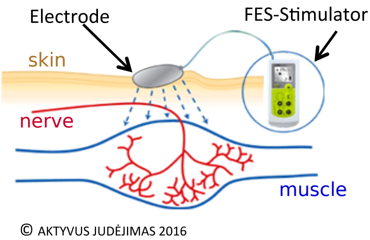
Theoritical works :
PGMO contract (09.2019–)
PEPS AMIES (12.2018–)

Sensitivity, Estimation, Model-Free Control, MPC,iPID (closed-loop control)

Electro-stimulation device

Industrial aspects :
CIFRE contract
UBFC & Segula Technologies (2020–2023)

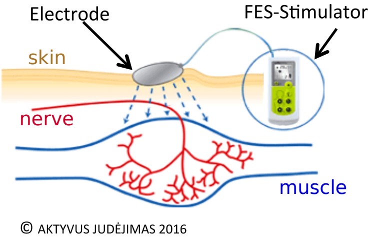
Muscular stimulation



Applications : Muscle strengthening, Mobility of paralyzed patients, Rehabilitation.
The protocols used in the applications are limited by

- fatigue analysis
- imprecision on the movements

Muscular stimulation



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The protocols used in the applications are limited by

- fatigue analysis
- imprecision on the movements

Industrial aim : Adjust automatically the stimulation parameters using control strategies based on muscle model to obtain better performance.

Mean : Change the intensity and/or frequency of the stimulations to control the force.

Ding et al. model [1]

FES input i. Dirac impulses $\boldsymbol{\delta}$ at times $t = 0, t_1, t_2, \ldots, t_N$.

$$i(t) = \sum_{i=1}^{N} R_i \eta_i \boldsymbol{\delta}(t - t_i), \quad \eta_i \in [0,1]$$

where

$$R_i := \begin{cases} 1, & \text{for } i = 0, \\ 1 + (\bar{R} - 1)\exp\left(-\dfrac{t_i - t_{i-1}}{\tau_c}\right), & \text{for } i = 1, \ldots, n, \end{cases}$$

takes into account the *tetanic* contraction.

1. J. Ding, A.S. Wexler and S.A. Binder-Macleod, *Development of a mathematical model that predicts optimal muscle activation patterns by using brief trains*, J. Appl. Physiol., **88** (2000) 917–925

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FES signal E_s.

$$E_s(t) = \frac{1}{\tau_c} \sum_{i=1}^{N} R_i \eta_i \boldsymbol{H}(t - t_i) \exp\left(-\frac{t - t_i}{\tau_c}\right)$$

$\boldsymbol{H}$: Heaviside

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The FES signal drives the evolution of the dynamics :

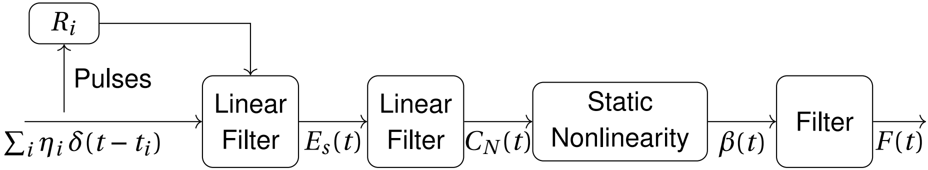
$$
\begin{aligned}
\dot{C}_N(t) &= -\frac{C_N(t)}{\tau_c} + E_s(t), \\
\dot{F}(t) &= -F(t)\ \boldsymbol{\gamma(t)} + A(t)\ \boldsymbol{\beta(t)}, \\
\dot{A}(t) &= -\frac{A(t) - A_{\text{rest}}}{\tau_{fat}} + \alpha_A F(t), \\
\dot{K}_m(t) &= -\frac{K_m(t) - K_{m,\text{rest}}}{\tau_{fat}} + \alpha_{K_m} F(t), \\
\dot{\tau}_1(t) &= -\frac{\tau_1(t) - \tau_{1,\text{rest}}}{\tau_{fat}} + \alpha_{\tau_1} F(t),
\end{aligned}
$$

where the Hill functions are given by

$$
\boldsymbol{\beta(t)} := \frac{C_N(t)}{K_m(t) + C_N(t)}, \ \text{and}\ \ \boldsymbol{\gamma(t)} := \frac{1}{\tau_1(t) + \tau_2 \beta(t)}.
$$

Constants of the model depend on the muscle

Summary of the model



Sampled-data control problem formulation

The dynamics can be written

$$\dot{x}(t) = f_1(x(t)) + f_2(t) \overbrace{\sum_{i=1}^{N} R_i \eta_i \, e^{\frac{t_i}{\tau_c}} \, H(t - t_i)}^{\text{piecewise constant}}$$

where f_1, f_2 are vector fields and $x = (C_N, F, A, K_m, \tau_1)$ is the state.

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This falls into the **sampled-data control problem** where the controls are the amplitudes η_i, $i = 0, \ldots, N$ and the sampling times t_i, $i = 1, \ldots, N$.

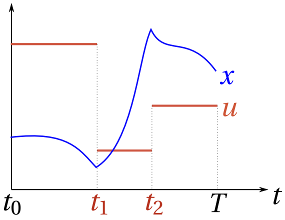
We may consider physical constraints :

$$\forall i = 0, \ldots, N, \; \eta_i \in [0, 1] \quad \text{and} \quad \underbrace{t_{i+1} - t_i}_{\text{Interpulse}} \geq \Delta.$$

Sampled-data control

$$\dot{x}(t) = f(x(t), u(t))$$

Non permanent control : we can change the value of the control **only a finite number of times**.
→ The state x is (absolutely) continuous while the control u is piecewise constant.



$0 = t_0 < t_1 < \ldots < t_N < T_{N+1} = T$ are **the N sampling times.**

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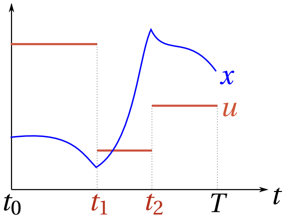

$0 = t_0 < t_1 < \ldots < t_N < T_{N+1} = T$ are **the N sampling times.**

N is fixed, t_i's are free

Sampled-data optimal control

Mayer formulation

$$\min \quad \varphi(x(T))$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f_1(x(t)) + f_2(t) \sum_{i=1}^{N} R_i \eta_i \, e^{\frac{t_i}{\tau_c}} \, H(t - t_i), & \\ x(0) = x_0, & \\ (\eta_0, \eta_1, \dots, \eta_N, t_1, \dots, t_N) \in \mathbb{R}^{2N+1}, & \\ \eta_i \in [0,1], & \forall i = 0, \dots, N, \\ t_0 = 0 < t_1 < t_2 < \dots < t_N < T = t_{N+1}, & \\ t_{i+1} - t_i \geq \Delta, & \forall i = 0, \dots, N, \end{cases}$$

Necessary optimality conditions

Recap : Permanent control case (Pontryagin,1962) [2]

$$\min_{u \in \mathcal{U}} \varphi(x(T)),$$

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

$\mathcal{U}$: Admissible controls = bounded measurable mappings.

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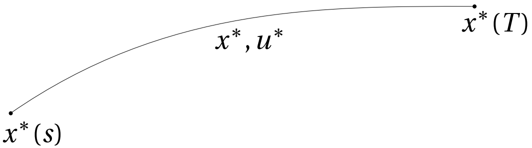

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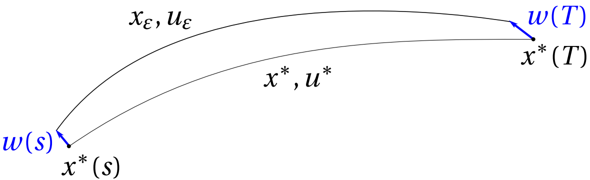

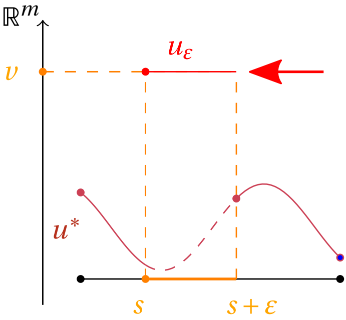

Necessary optimality conditions

- **L^1-perturbation :** $u_\varepsilon(t) := \begin{cases} v \in U \subset \mathbb{R}^m & \text{on } [s, s+\varepsilon[, \ (s \in [0, T[) \\ u^*(t) & \text{on } [s+\varepsilon, T[\end{cases}$

- Corresponding variation vector w s.t. : $x(t, u_\varepsilon) = x(t, u^*) + \varepsilon\, w(t) + o(\varepsilon)$

$$\dot{w}(t) = \nabla_x f(x^*(t), u^*(t))\, w(t),$$
$$w(s) = f(x^*(s), v) - f(x^*(s), u^*(s))$$

Denote by $\Phi(\cdot,\cdot)$ the state-transition matrix of $\nabla_x f(x^*, u^*)$:

$$w(T) = \Phi(T, s)\, w(s).$$

From optimality of (x^*, u^*),

$$0 \quad \leq \quad \frac{\varphi(x_\varepsilon(T)) - \varphi(x^*(T))}{\varepsilon} \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \Big\langle \nabla\varphi(x^*(T)), w(T) \Big\rangle$$

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Introducing the co-state vector $p(t)$ s.t. :

$$\begin{aligned} \dot{p}(t) &= -\nabla_x f(x^*(t), u^*(t))^\top p(t), \\ p(T) &= -\nabla\varphi(x^*(T)). \end{aligned}$$

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$$p(T) = -\nabla\varphi(x^*(T)).$$

Using $w(T) = \Phi(T, s)\ w(s)$ and $p(s) = \Phi(T, s)^\top\ p(T)$ we finally get :

$$\boxed{\forall v \in U, \quad \left\langle p(s), f(x^*(s), v) - f(x^*(s), u^*(s)) \right\rangle \leq 0}$$

which is the so-called *maximization condition of the Pontryagin maximum principle.*

Necessary optimality conditions

Non Permanent control case (Bourdin, Trélat, 2016)[2].

$$\min_{u\in\mathscr{U}} \varphi(x(T)),$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

$\mathscr{U}$: Admissible controls = piecewise constant mappings.
Let x^* a reference optimal trajectory associated to u^*.

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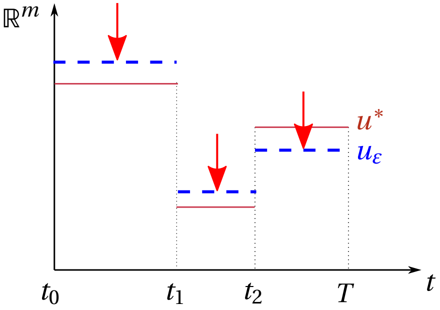

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Let x^* a reference optimal trajectory associated to u^*.

- $\boldsymbol{L^\infty}$**-perturbation :** $u_\varepsilon := u^* + \varepsilon(\xi - u^*)$ (ξ is valued in U has the same sampling times as u^*).
- This time, the corresponding variation vector w satisfies :

$$\dot{w} = \nabla_x f(x^*, u^*)\ w + \nabla_u f(x^*, u^*)\ (\xi - u^*),$$
$$w(0) = 0$$

hence,

$$w(T) = \int_0^T \Phi(T,s)\ \nabla_u f(x^*(s), u^*(s))\ (\xi(s) - u^*(s))\,\mathrm{d}s.$$

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Using it, together with $0 \leq \langle \nabla\varphi(x^*(T)), w(T)\rangle$, yield

$$\int_0^T \left\langle p(s), \nabla_u f(x^*(s), u^*(s))\ (\xi(s) - u^*(s))\right\rangle \mathrm{d}s \leq 0.$$

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Finally, taking $\xi = \boldsymbol{v} \in U$ over $[t_i^*, t_{i+1}^*[$ and $\xi(t) := u^*(t)$ elsewhere, we get

$$\boxed{\left\langle \int_{t_i^*}^{t_{i+1}^*} \nabla_u H(x^*(s), p(s), u_i^*)\,\mathrm{d}s\,,\ \boldsymbol{v} - u_i^* \right\rangle \leq 0,}$$

for all $\boldsymbol{v} \in U$ and all $i = 0, \ldots, N$, where u_i^* corresponds to the value of u^* over the interval $[t_i, t_{i+1}[$.

Remarks

- Same weaker maximization condition than the **discrete Pontryagin maximum principle** (Boltyanskii, 1978) [3]
- Generalization to **time scale** (Bourdin, Trélat, 2013)
- Another proof with different approach by Dmitruk and Kaganovich (2011) [4]

3. V.G. Boltyanskii, *Optimal control of discrete systems*, John Wiley & Sons, New York-Toronto, Ont., 1978.
4. A.V. Dmitruk, A.M. Kaganovich. *Maximum principle for optimal control problems with intermediate constraints*, Comput. Math. Model., 22(2) :180–215, 2011.

Application to the force-fatigue model

Theorem

If $(\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)$ **is optimal**, *then there exists* p *satisfying the co-state equation and the transversality condition.*

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Theorem

If $(\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)$ **is optimal**, *then there exists p satisfying the co-state equation and the transversality condition.*
Moreover, the necessary conditions are :

(i) *the inequality*

$$\left(\int_{t_i^*}^{T} p_1(s) b(s)\,\mathrm{d}s\right) \tilde{\eta}_i \leq 0,$$

for all $i = 0, \ldots, n$ and all admissible perturbation $\tilde{\eta}_i$ of η_i^* ;

(ii) *and the inequality*

$$\boldsymbol{NC_i} := \Bigg(- p_1(t_i^*) b(t_i^*) G(t_{i-1}^*, t_i^*) \eta_i^* + b(-t_i^*) \eta_i^* \int_{t_i^*}^{T} p_1(s) b(s)\,\mathrm{d}s \\ + b(-t_i^*)(\bar{R} - 1)\eta_{i+1}^* \int_{t_{i+1}^*}^{T} p_1(s) b(s)\,\mathrm{d}s \Bigg) \tilde{t}_i \leq 0,$$

for all $i = 1, \ldots, n$ and all admissible perturbation $\tilde{t}_i$ of t_i^*.

Numerical methods

Three numerical schemes :

1. *Open-loop control*.
   Direct methods : not based on necessary optimality conditions.
   Indirect methods :
   - Shooting algorithm to solve the *boundary value problem* coming from the necessary conditions
   - Newton-like algorithm to find a zero of the shooting function
   - Direct method to give an initialization
   - Adapted integration scheme (stiff dynamics).

2. *Closed-loop control*. **Adaptive control algorithms** where the fatigue is estimated by a non-linear observer.

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2. *Closed-loop control*. **Adaptive control algorithms** where the fatigue is estimated by a non-linear observer.

$\Longrightarrow$ *Complementaries of the methods* &

open-loop : compute a pulses train to reach the maximal force (T $\sim$ 1s),

closed-loop : stabilization near a reference force with rest and stimulation periods (T $\gg$ 10s)

Direct method

Idea.

Sampled-data optimal control problem	$\Longleftrightarrow$	Finite-dimensional optimization problem

Direct method

Idea.

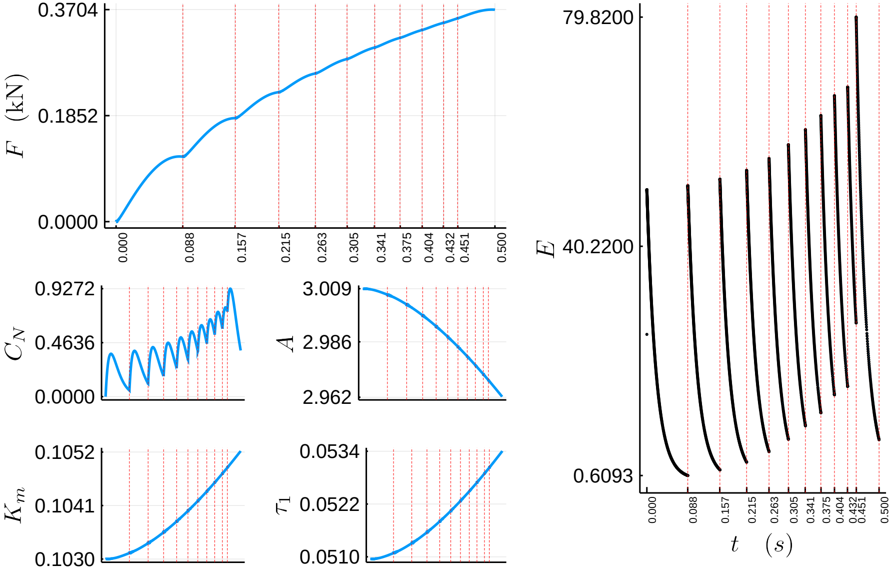
Sampled-data optimal control problem $\Longleftrightarrow$ Finite-dimensional optimization problem

Method. Transform the optimal control problem in a nonlinear finite-dimensional optimization problem (NLP) via discretization in time of the state.
t_i, $i = 1, \ldots, N$ are the optimization variables of the NLP.

Algorithms

- primal-dual interior point algorithm
- derivatives are computed by automatic differentiation.

$\Rightarrow$ robust w.r.t. initialization, **handle constraints on the state/control**, in general less precise than indirect methods.



Direct method : $\max\limits_{t_i} F(T),\ N = 10,\ 10ms \leq t_{i+1} - t_i,\ i = 0, \ldots, N.$

Indirect method.

Exploit the **geometric structure** of the solutions via the necessary conditions. *Preliminary results :* relax the inequalities in the optimality conditions to obtain a boundary value problem.

$\Longrightarrow$ Fast convergence and high accuracy/precision.

Multiple shooting method : $(n + 2nN + N)$ unknowns :

$$p(0), \quad Z_i = (x(t_i), p(t_i)),\ i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N).$$

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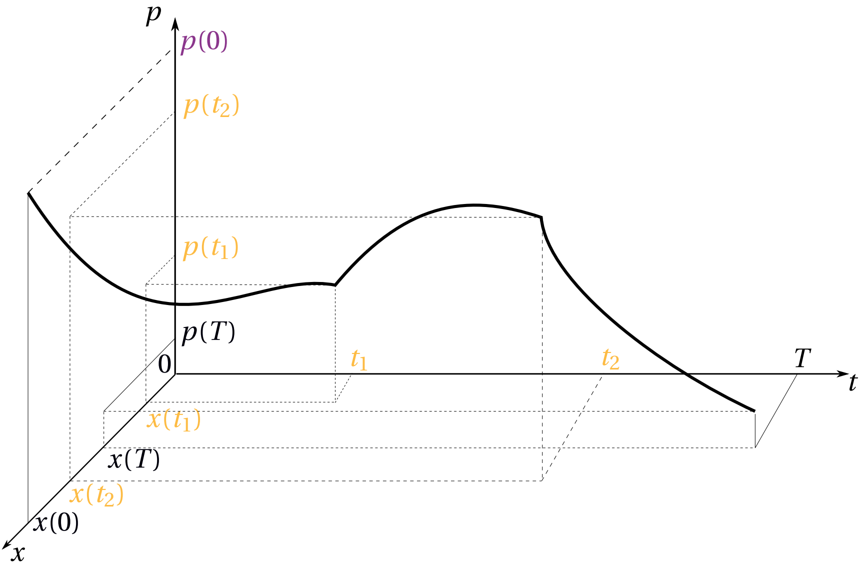

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Shooting function. Find a zero of the function $\boldsymbol{S}(p_0, Z_1, \ldots, Z_N, \sigma)$ so that

- the initial condition $\boldsymbol{x(0) = x_0}$,
- the continuity conditions $\boldsymbol{Z_i^-} = \boldsymbol{Z_i^+}$, $i = 1, \ldots, N$,
- the necessary conditions $\boldsymbol{NC_i \leq 0}$ $i = 1, \ldots, N$,

are satisfied.

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are satisfied.

Shooting algorithm. Sensitive to initialization.

Initialization : compute a solution $(\tilde{x}, \tilde{u})$ with a direct method, by continuation or by approximation.

Starting from $(\tilde{x}(T), p(T))$ (where $p(T) = -\nabla\varphi(\tilde{x}(T))$ is known), **integrate backward the co-state dynamics** to obtain $p(0)$.

Tools : Julia's libraries :

- Extended precision for float (`ArbNumerics.jl`)
- **Stiff** numerical integrator (`DifferentialEquations.jl`)

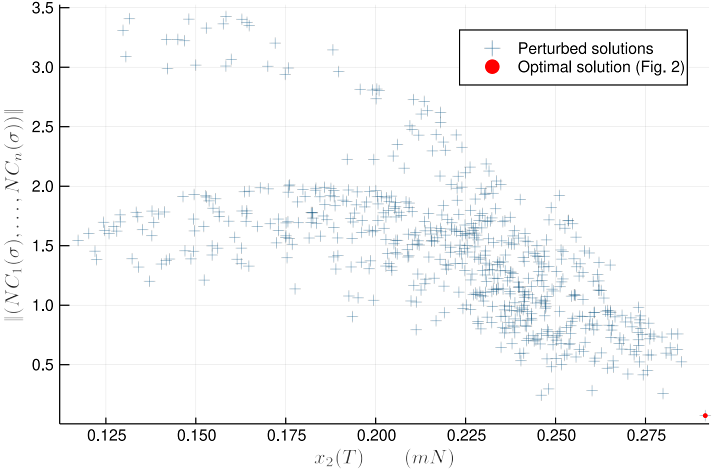


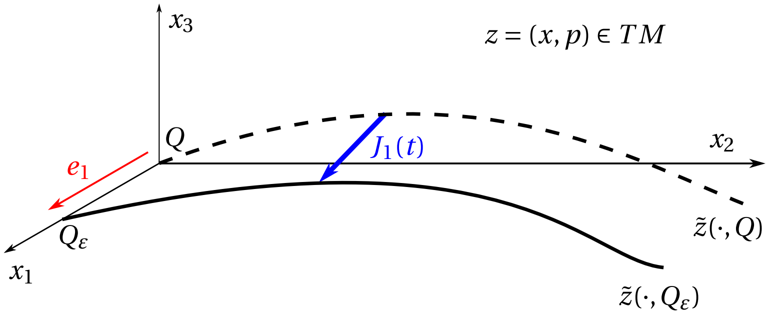
FIGURE – Quality of the optimal solution computed with multiple shooting with respect to its perturbations. The quality is measured from the necessary conditions and the value of the cost.

Closed-loop algorithm

- **Sensitivity analysis :** select the relevant fatigue variable for estimation
- **Detectability :** construct an observer to estimate the chosen fatigue variable
- Adaptive control algorithm (MPC) based on the observer

Sensitivity analysis

Let $\tilde{z}(\cdot, Q) = (x(\cdot), p(\cdot))$ a reference extremal associated to u and starting at $Q \in TM$.
$H(x, p, u) = p \cdot f(x, u)$: Hamiltonian of the system : $\dot{x} = f(x, u)$,
$\overrightarrow{H}(z, u)$: Hamiltonian vector field evaluated along the extremal $z(\cdot)$.



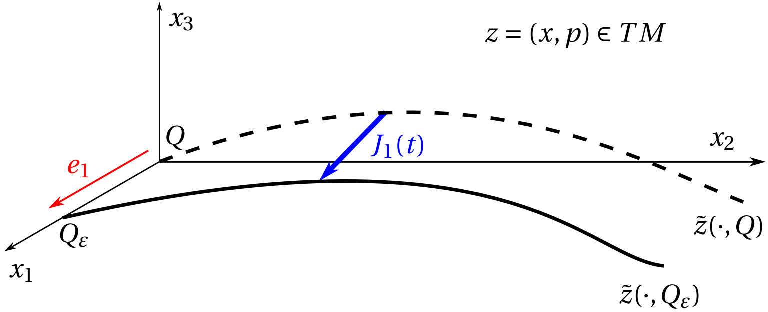
Sensitivity analysis

Definition (Jacobi fields)

The Jacobi equation is

$$\dot{\delta} z(t) = \frac{\partial}{\partial z}\overrightarrow{H}(z(t), u(t))\,\delta z(t)$$

The **Jacobi fields associated to** x_i**-variation** $i = 1, \ldots, n$ *are the solutions* $J_i(t)$, $i = 1, \ldots, n$ *with* $J_i(0) = e_i$, $i = 1, \ldots, n$ *where* $(e_i)_i$ *is the* $\mathbb{R}^n \times \mathbb{R}^n$ *canonical basis.*



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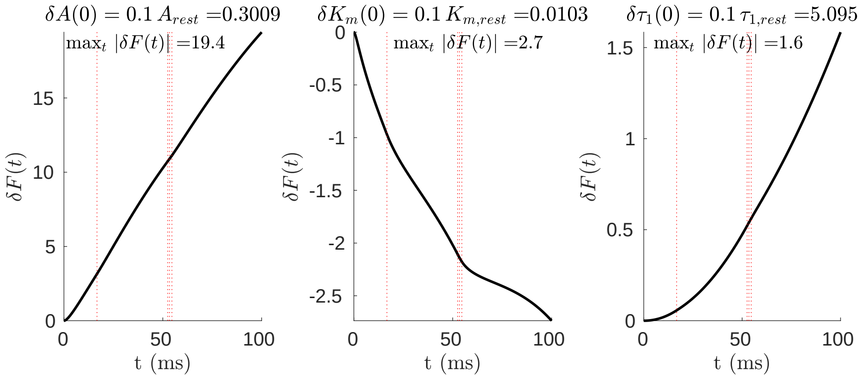
Definition (Sensitivity)

The sensitivity of the **fatigue variables** $\boldsymbol{x_i}$, $i = 3, 4, 5$ **w.r.t. the force** *is defined by*

$$\max_{t \in [0,T]} |\Pi_F(J_i(t))|, \quad i = 3, 4, 5 \quad (n = 5)$$

where Π_F *is the projection* $z \to x_2$ *(on the force variable).*



Sensitivity analysis. Time evolution of the Jacobi fields component $\delta F(\cdot)$.

The fatigue variable A is the most relevant for the given extremal

Observability characterization

$$(S) \qquad \begin{cases} \dot{x} = f(x) + u\,g(x) \\ y = x_2 = F \end{cases}$$

: *force is measured, fatigue is estimated*

High gain nonlinear observer *(Gauthier et al., 1992)*[5]

5. Gauthier J.P., Hammouri H., Othman S, *A simple observer for nonlinear systems - applications to a bioreactors*, IEEE Transactions on Automatic Control, **37** (1992) 875-880

Observability characterization

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High gain nonlinear observer *(Gauthier et al., 1992)*

Theorem

(S) *is uniformly observable for any input iff* (S) *is* **diffeomorphic** *to a system of the form*

$$\dot{z} = \tilde{f}(z) + u\,\tilde{g}(z)$$

where

$$\tilde{f}(z) = \begin{pmatrix} z_2 \\ \vdots \\ z_{n-1} \\ k(z) \end{pmatrix} \quad \text{and} \quad \tilde{g}(z) = \begin{pmatrix} \tilde{g}_1(z_1) \\ \tilde{g}_2(z_1, z_2) \\ \vdots \\ \tilde{g}_n(z_1, \ldots, z_n) \end{pmatrix}$$

Computation.

$$\dot{x}(t) = \beta^m(t)\, f_1(x(t), E_s(t)) = f(x(t), E_s(t)), \qquad m \in \mathbb{N}$$
$$y(t) = h(x(t))$$

Change of variables.

$$\begin{aligned} \varphi : \quad & \mathbf{\Omega} \to \mathbb{R}^n \\ & x \mapsto \big(h(x), \mathscr{L}_f h(x), \mathscr{L}_f(\mathscr{L}_{f_1} h)(x), \ldots\big)^{\mathsf{T}} \end{aligned}$$

where $\mathscr{L}_f(h)(x)$: Lie derivative of h w.r.t. f at the point x.

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where $\mathscr{L}_f(h)(x)$: Lie derivative of h w.r.t. f at the point x.
Under the action of φ, the dynamics becomes

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{pmatrix} = \beta^m \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k(u, z) \end{pmatrix}.$$

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Theorem

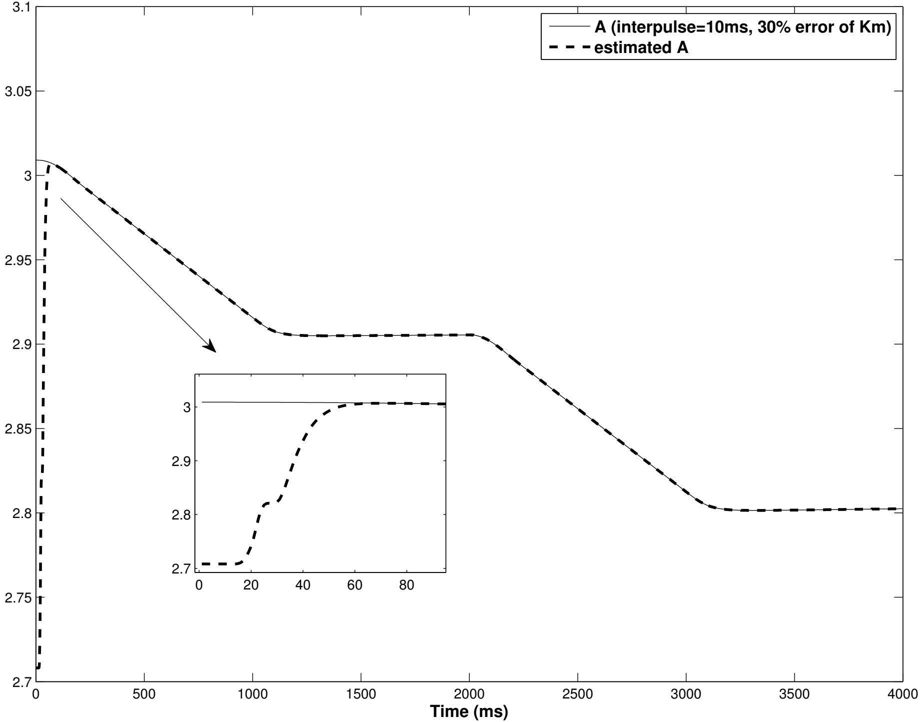
Under technical assumptions, the observer

$$\dot{\hat{z}}(t) = \beta(t)^m A \hat{z}(t) - \beta(t)^m S_\theta{}^{-1} C^T (C\hat{z}(t) - y(t))$$

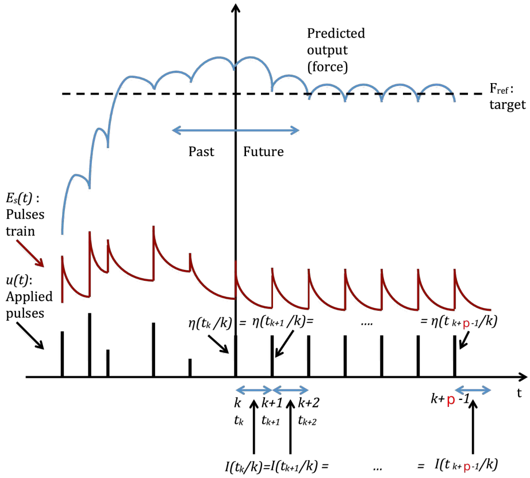
where $C = (1, 0, \ldots, 0)$ *and* S_θ *is the solution of the Lyapunov equation :*

$$\theta S_\theta + A^T S_\theta + S_\theta A - C^T C = 0$$

is convergent exponentially on $\mathbb{R}^n$.

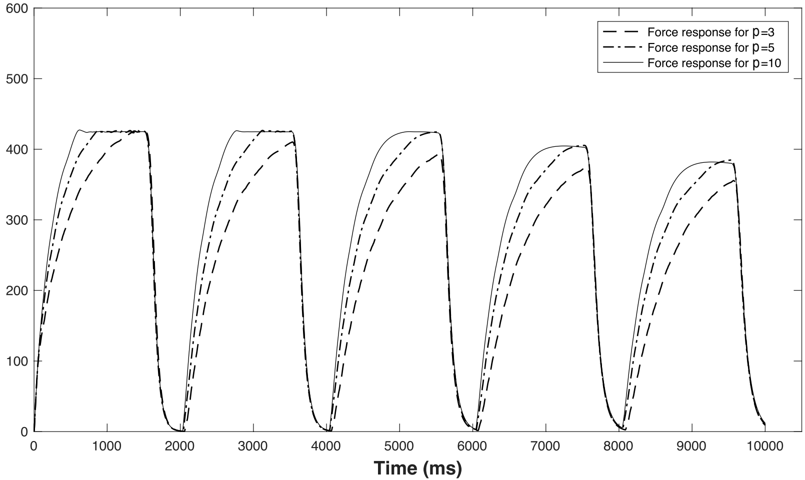

A (interpulse=10ms, 30% error of Km)
estimated A
Time (ms)
3.1
3.05
3
2.95
2.9
2.85
2.8
2.75
2.7
0
500
1000
1500
2000
2500
3000
3500
4000
20
40
60
80

(Nonlinear) Model Predictive Control algorithm

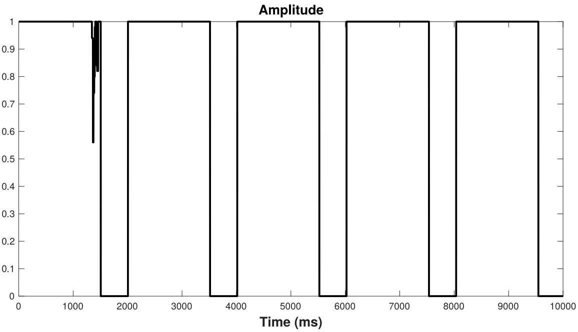


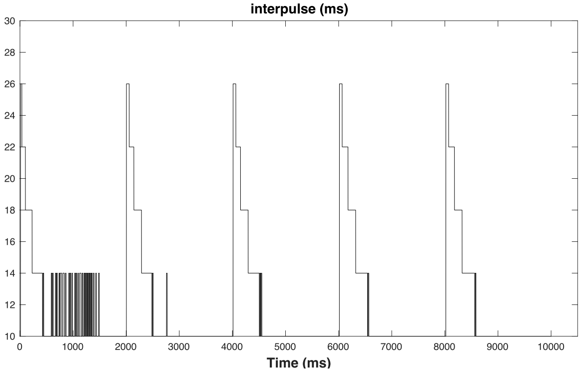
At time $t = t_k$, the fatigue is not known : use the observer $\hat{A}$ to estimate it in the optimization on a horizon of size $\boldsymbol{p}$.

Stabilization near a force of reference F_{ref} : we minimize $\left|\int_0^T F(s)\,\mathrm{d}s - F_{\text{ref}}\right|$.



Evolution of the force for $F_{\text{ref}} = 425$N and different horizon ($p = 3, 5, 10$) .


Amplitude
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
Time (ms)


interpulse (ms)
30
28
26
24
22
20
18
16
14
12
10
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
Time (ms)

Future works

- **Time-scale context** : theoretical works (free sampling times),
- **Software development** to handle first order optimality conditions in the sampled case with variational differential inequality,
- Optimality conditions in the sampled-data case with **state constraints** (related to the industrial contract),
- Number N of sampling times **not fixed**,
- **Geometric study** : direct computation of the **derivative of the exponential function** (Baker-Campbell-Hausdorff), second order necessary optimality conditions (conjugate points).
- **Industrial project :** couple optimization techniques with **estimations** of the variables and **parameters** (characterizing the muscle), observability $\rightarrow$ **iPID controller** , robustness with respect to noise.

1. Bakir T., Bonnard B., Bourdin L., Rouot J., *Pontryagin-Type Conditions for Optimal Muscular Force Response to Functional Electric Stimulations*, J Optim Theory Appl (2020) 184 :581.

2. Bakir T., Bonnard B., Rouot J., *A case study of optimal input-output system with sampled-data control : Ding et al. force and fatigue muscular control model*, Networks and Heterogeneous Media, AIMS-American Institute of Mathematical Sciences, **14** (1) (2019) pp.79–100.