Lesson 1
Trigonometry

The plane $\mathbb{R}^2$ is endowed with a orthonormal system $(O, \vec{i}, \vec{j})$.

1 Trigonometric functions

1.1 Definition

1.1.1 First properties

For all $x \in \mathbb{R}$,

$$\boxed{-1 \leq \cos(x) \leq 1 \text{ and } -1 \leq \sin(x) \leq 1}$$

Thales' theorem : for any $x \neq \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$,

$$\boxed{\tan(x) = \frac{\sin(x)}{\cos(x)}.}$$

Pythagora's theorem : for any $x \in \mathbb{R}$,

$$\boxed{\cos^2(x) + \sin^2(x) = 1.}$$

1.1.2 Functions

Trigonometric functions are defined as following :

$$\sin, \ \cos : \mathbb{R} \longrightarrow [-1, 1], \qquad \tan : \mathbb{R} \backslash \left\{ \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z} \right\} \longrightarrow \mathbb{R}.$$

For all real number x, one has

$$\sin(x + 2\pi) = \sin(x), \qquad \sin(x + 2\pi) = \sin(x),$$

and for all x in the well-defined set of tan,

$$\tan(x + \pi) = \tan(x).$$

Remark 1. We say that cos and sin are $2\pi-$**periodic** and tan is $\pi-$**periodic**.

1.1.3 Usual values

	$x = 0$	$x = \frac{\pi}{6}$	$x = \frac{\pi}{4}$	$x = \frac{\pi}{3}$	$x = \frac{\pi}{2}$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Many other values can be deduced from the usual values above and symetry properties.

1.2 Symmetries

Let $x \in \mathbb{R}$. One has :

$$\cos(-x) = \cos(x), \qquad \sin(-x) = -\sin(x).$$

Remark 2. We say that cos is en **even** function and sin is an **odd** function.

One has :

$$\cos(\pi + x) = -\cos(x), \qquad \sin(\pi + x) = -\sin(x).$$

One has :

$\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$	$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$
$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$	$\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$

2 Addition and duplication formula

2.1 Addition formula

These are the most important trigonometric formula : from them you can deduced all the others. Let $x, y \in \mathbb{R}$. One has :

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \qquad \sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y) \qquad \sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x)$$

Example 1.

$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}\cos(x) - \frac{\sqrt{3}}{2}\sin(x)$$

Example 2.

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Remark 3. To get the value of $\cos\left(x + \frac{\pi}{2}\right)$, we rather use symmetry (geometric) properties above.

2.2 Duplication formula

Let $x, y \in \mathbb{R}$. One has :

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

Example 3.

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{\sqrt{2} + 2}}{2}$$

Remark 4. The reserve of addition formula is also useful : to express some $\cos^2$ or $\sin^2$ as expression depending only on cos and sin without power. The process is called *linearize* et will be often used for the calculation of some primitive.

2.3 Other formula

Other formula can be deduced from addition formula. For example :

$$\begin{aligned}\cos(p)+\cos(q) &= 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\\ \sin(p)+\sin(q) &= \ldots\end{aligned}$$

3 Trigonometric equations

3.1 Resolution of $\cos(\ )=\cos(\ )$

The equation $\cos(u)=\cos(\alpha)$ with a given number α, can be rewritten as :

$$\boxed{u=\alpha+2k\pi,\quad k\in\mathbb{Z},\quad \text{ou}\quad u=-\alpha+2k\pi,\quad k\in\mathbb{Z}.}$$

Example 4. The set of solutions of $\cos(2x)=\frac{\sqrt{2}}{2}$ is :

$$S=\left\{\frac{\pi}{8}+k\pi,\ -\frac{\pi}{8}+k\pi, \text{with } k\in\mathbb{Z}\right\}$$

3.2 Resolution $\sin(\ )=\sin(\ )$

The equation $\sin(u)=\sin(\alpha)$ with a given number α, can be rewritten as :

$$\boxed{u=\alpha+2k\pi,\quad k\in\mathbb{Z},\quad \text{ou}\quad u=\pi-\alpha+2k\pi,\quad k\in\mathbb{Z}.}$$

Example 5. The set of solutions of $\sin(4x)=\frac{1}{2}$ is :

$$S=\left\{\frac{\pi}{24}+\frac{k}{2}\pi,\ \frac{5\pi}{24}+\frac{k}{2}\pi, \text{with } k\in\mathbb{Z}\right\}$$

3.3 Resolution $\cos(\ )=\sin(\ )$

There is no direct way to find the set of solutions of $\cos(u)=\sin(\alpha)$. We have to use symmetry formula to switch cos to sin **otherwise** to switch sin to cos. And then we find an equation like before.

Example 6. The set of solutions of $\cos(2x)=\sin(x)$is :

$$S=\left\{\frac{\pi}{2}+2k\pi,\ \frac{\pi}{6}+\frac{2k}{3}\pi, \text{with } k\in\mathbb{Z}\right\}$$

Remark 5. There is no direct way to solve $\cos(x)=2\sin(3x)$ because of the number 2 in front of $\sin()$.

Remark 6. Be careful of the following points when you solve a trigonometric equation :

1. You always have to change your equation into a same-type equation (without any constant in front of cos or sin !)
2. There always are two sorts of solutions, separated by a **or**.
3. You have to apply operations you do to the $+2k\pi$ at the end of each sorts.