Le plus court chemin.

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Soit G=(X;E) un graphe orienté et *d*: E→R une fonction qui associe à tout arc de G un poids réel. Souvent on parle de *distance*, mais, puisque il n'y a pas de restriction $d \geq 0$, elle peut s'interpréter aussi comme le coût ou le profit d'une décision. Pour chaque chemin Γ dans G on définit son poids

$$\ell(\Gamma) = \sum_{e \in \Gamma} d(e),$$

que l'on suppose nul si Γ ne comporte aucun arc.

Pour présenter le problème de recherche d'un chemin de poids optimum nous allons nous concentrer sur le problème du plus court chemin, car le problème de maximisation peut être résolu par le remplacement de la fonction d par $-d$.

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Cette recherche devrait permettre donc de trouver deux choses: la valeur minimale $l(\Gamma)$ et le chemin Γ qui réalise ce minimum. Si un plus court chemin d’un sommet x à un sommet y existe alors sa longueur sera appelée

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plus courte distance de x à y.

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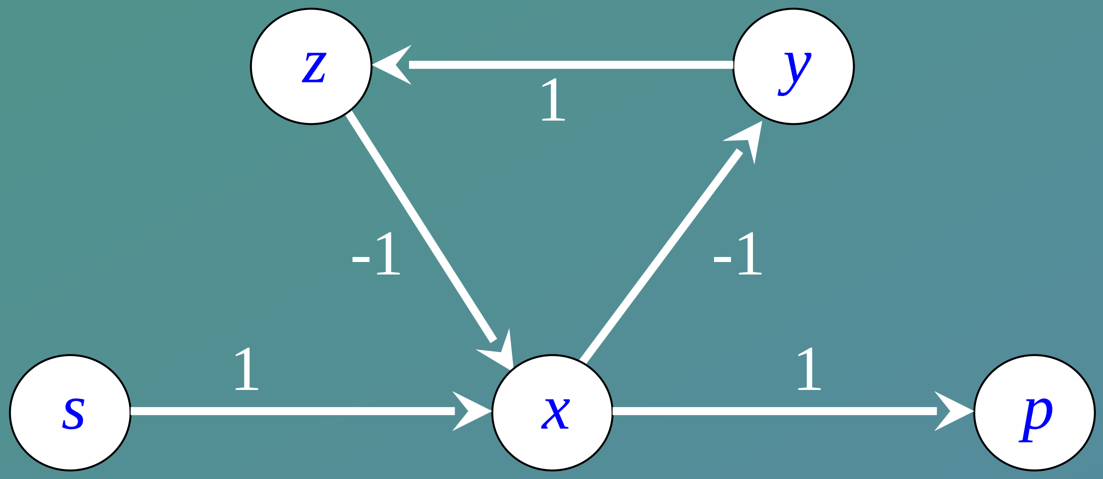
car la complexité des algorithmes peut en dépendre.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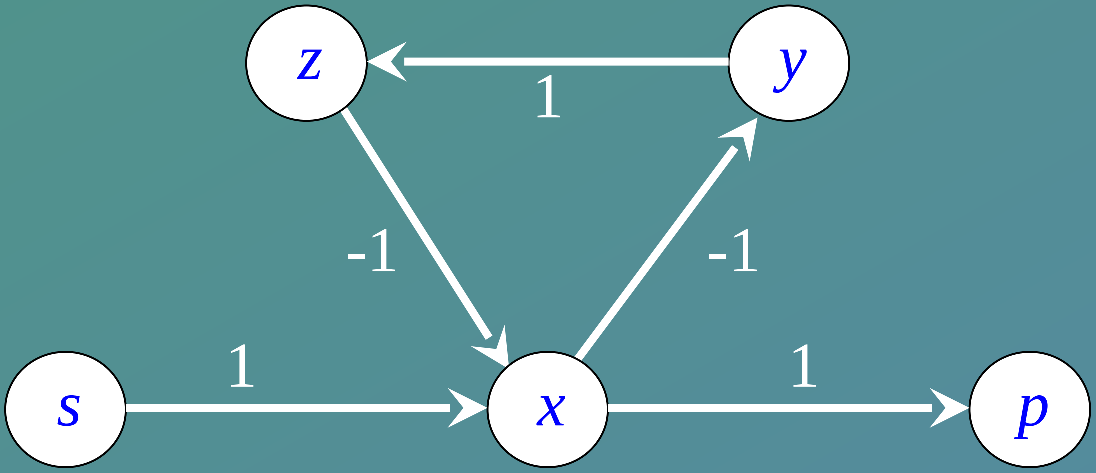
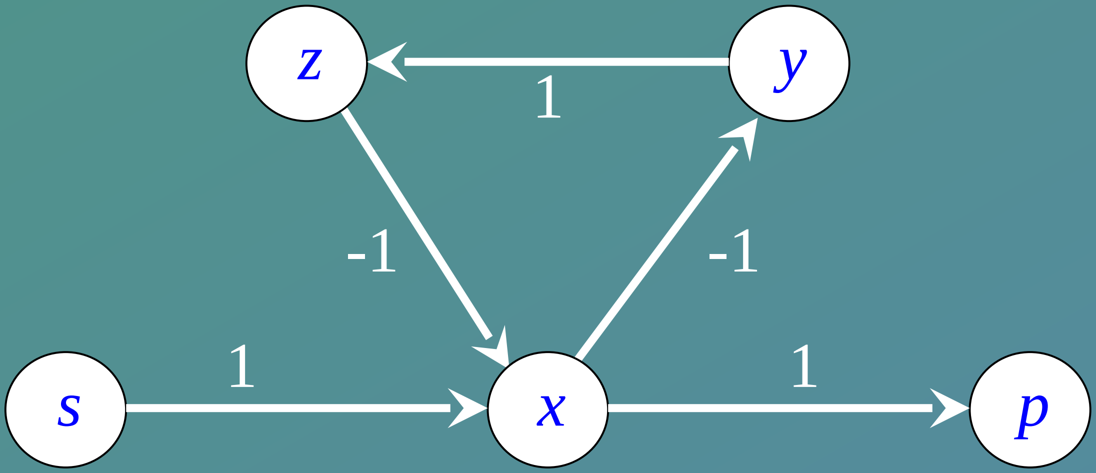


$l(s\text{-}x\text{-}p) = 2$



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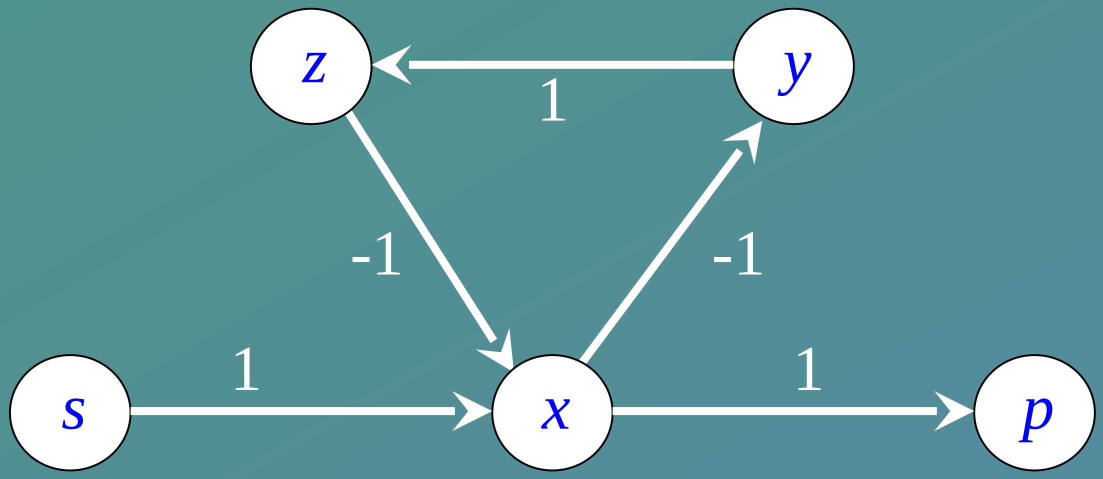

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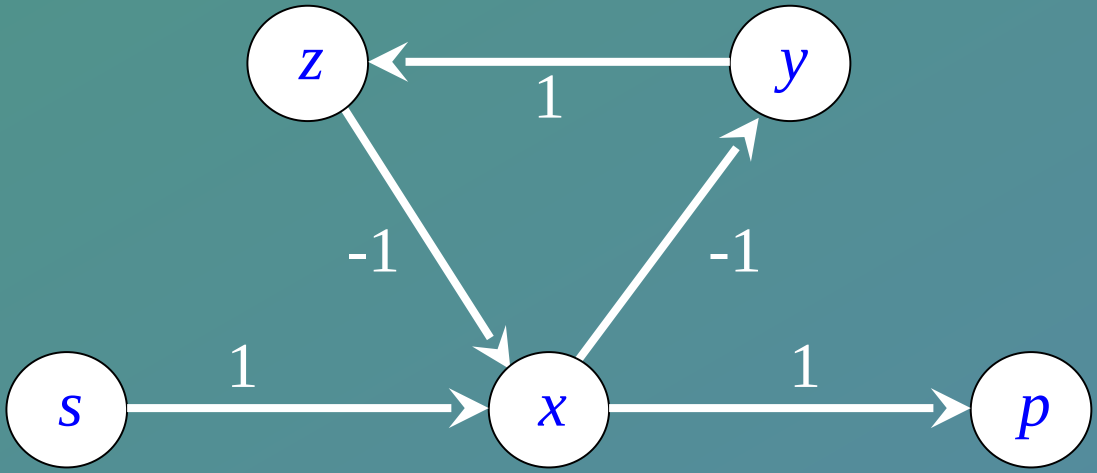

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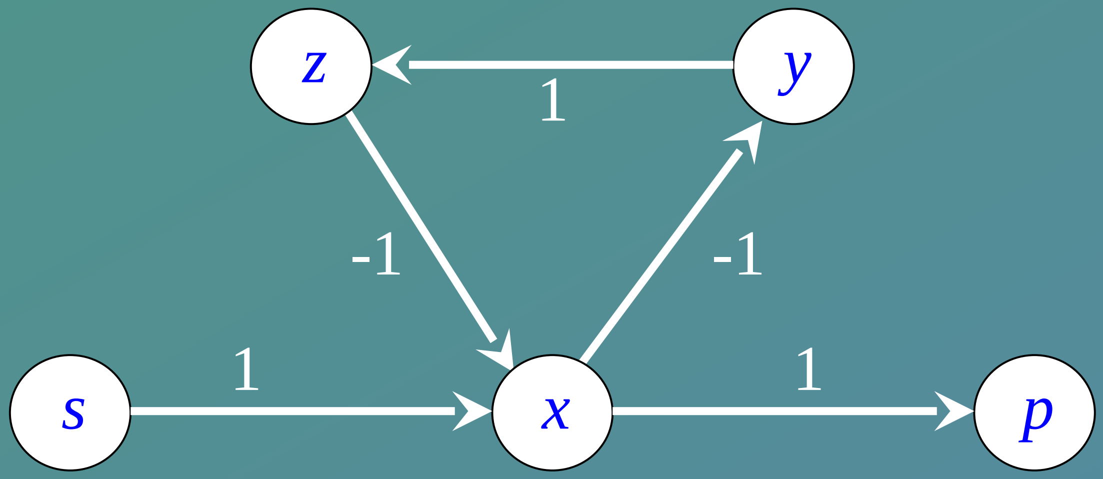

$$l(s\text{-}x\text{-}p) = 2$$

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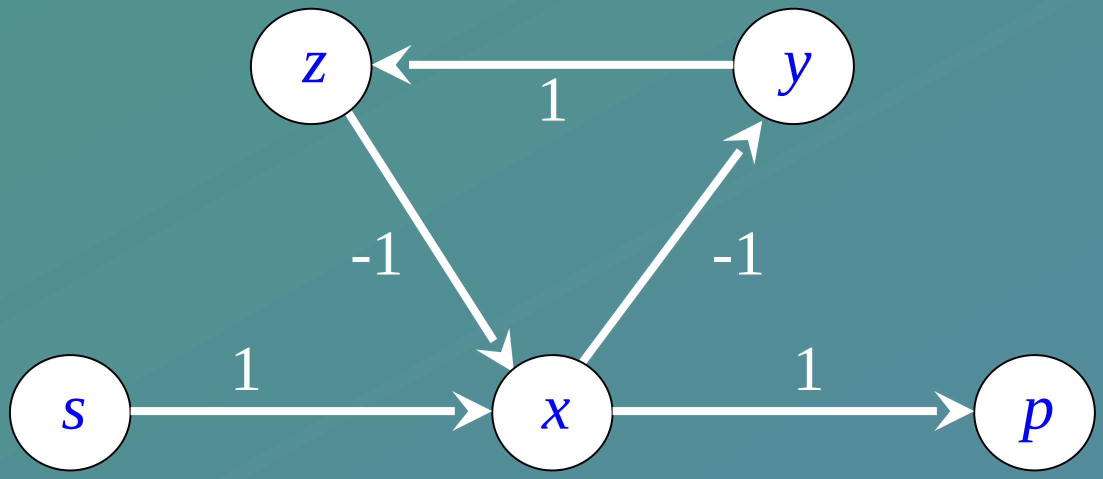
$$\ldots\ l \to -\infty$$



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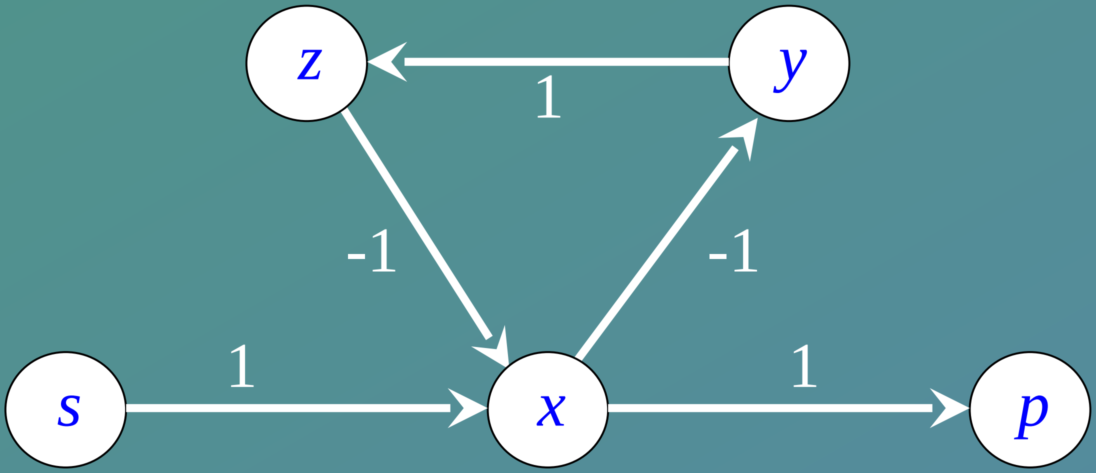
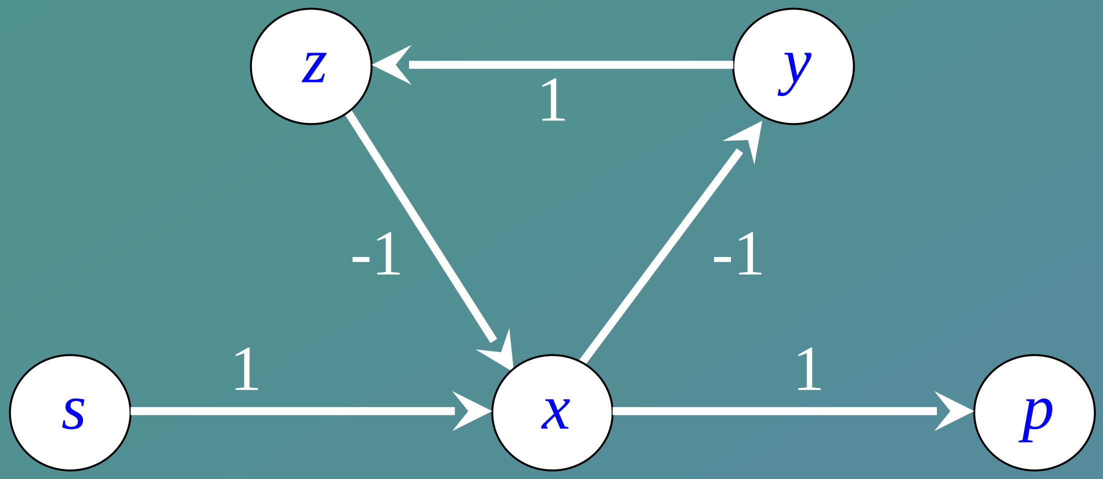


Ce phénomène est dû à l'existence d'un circuit C, dit *circuit absorbant*, pour lequel on a : $l\,(C\,) < 0$.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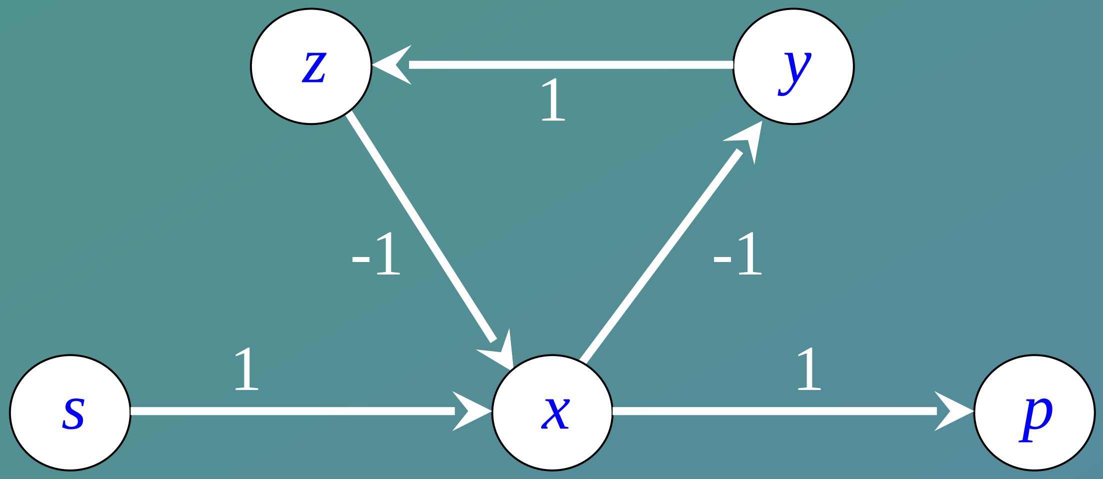

Pour éviter ce problème, on peut aussi considérer les problèmes (A'), (B') et (C') où l'on recherche à chaque fois le plus court chemin élémentaire. Ces problèmes ont toujours une solution, car il existe un nombre fini de chemins élémentaires. Bien que fini, ce nombre peut être exponentiel et nous ne connaissons pas de méthode efficace pour résoudre le problème du plus court chemin élémentaire en général. Par contre, il est évident qu'en l'absence de circuit absorbant, les problèmes de recherche d'un plus court chemin et celui du plus court chemin élémentaire sont équivalents, car le plus court chemin est obligatoirement élémentaire.

Algorithme de Bellman et Ford.

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Soit $\boldsymbol{G}=(X; E)$ un graphe orienté sans circuit, où l'ensemble des sommets $X=\{x_1, x_2,..., x_n\}$ a déjà été **trié** (c.-à-d. si $(x_i, x_j)\in E$ alors $i < j$), admettant une arborescence couvrante enracinée en $\mathbf{x_1}$; d: E→R une fonction-distance.

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$\pi_1 := 0; \quad A := \emptyset;$

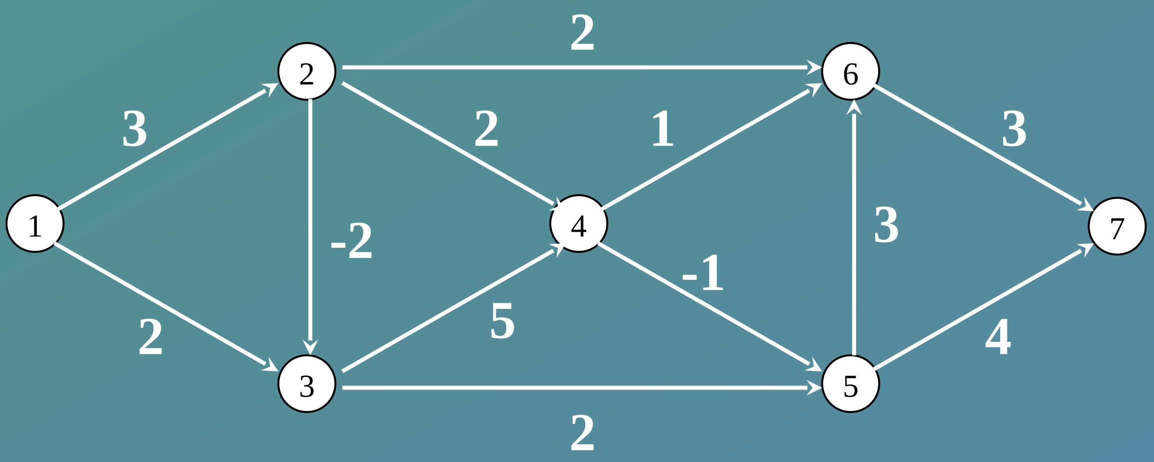
pour *i* ***de*** *2* ***à*** *n* ***faire***:

$\pi_i := \min\{ \pi_j + d(j, i) \; ; \; j \text{ tel que } (x_j, x_i)\in E\}$

$A := A \cup \{(x_j, x_i)\in E$, *tel que* $\pi_i = \pi_j + d(j, i)$

(j le plus petit indice avec cette propriété)}

fin pour



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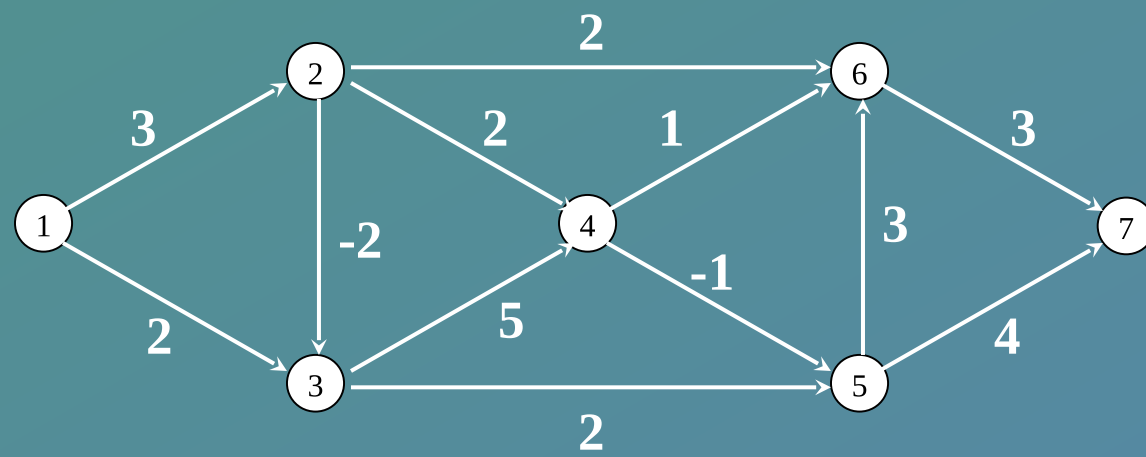
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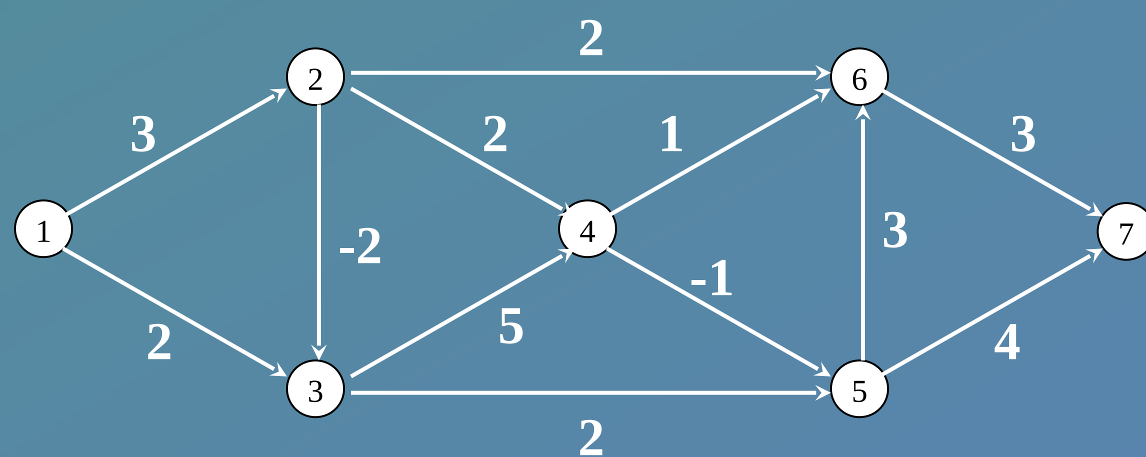
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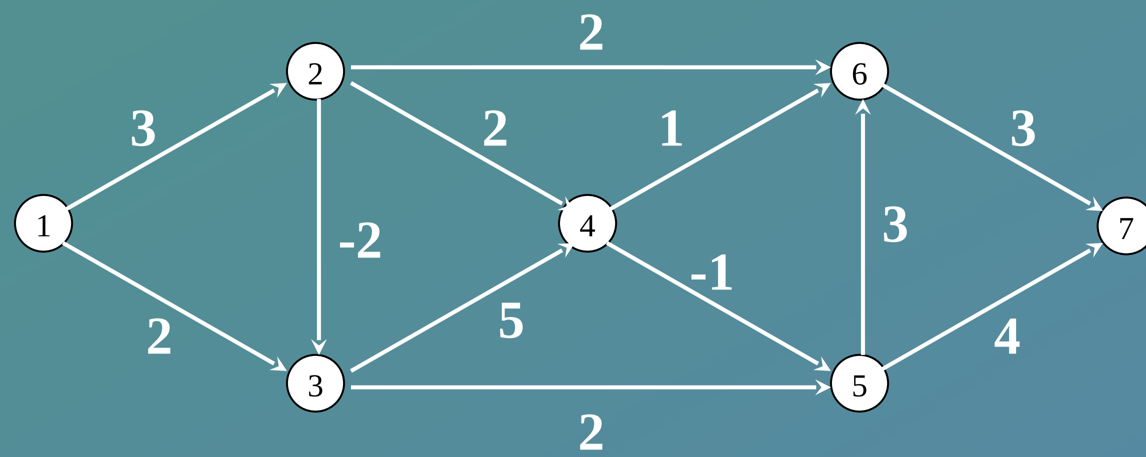

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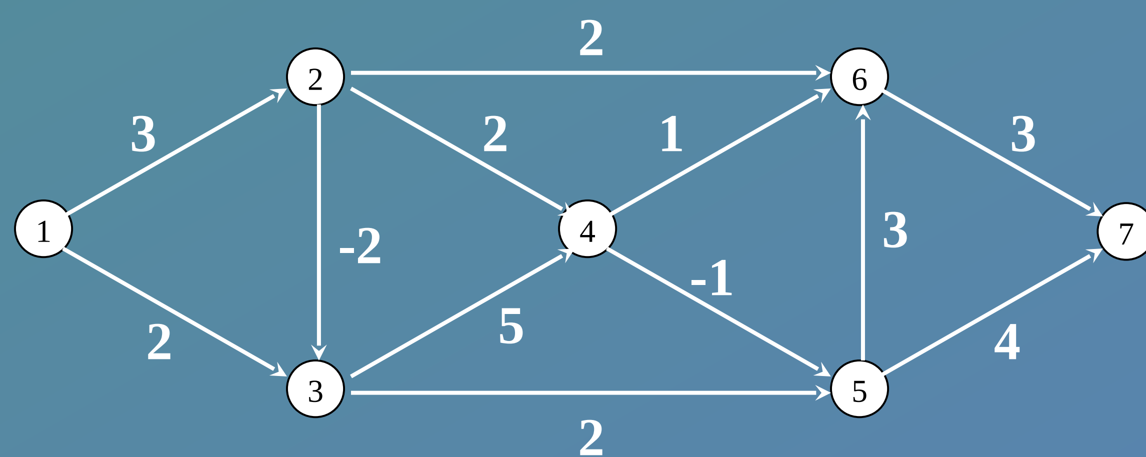

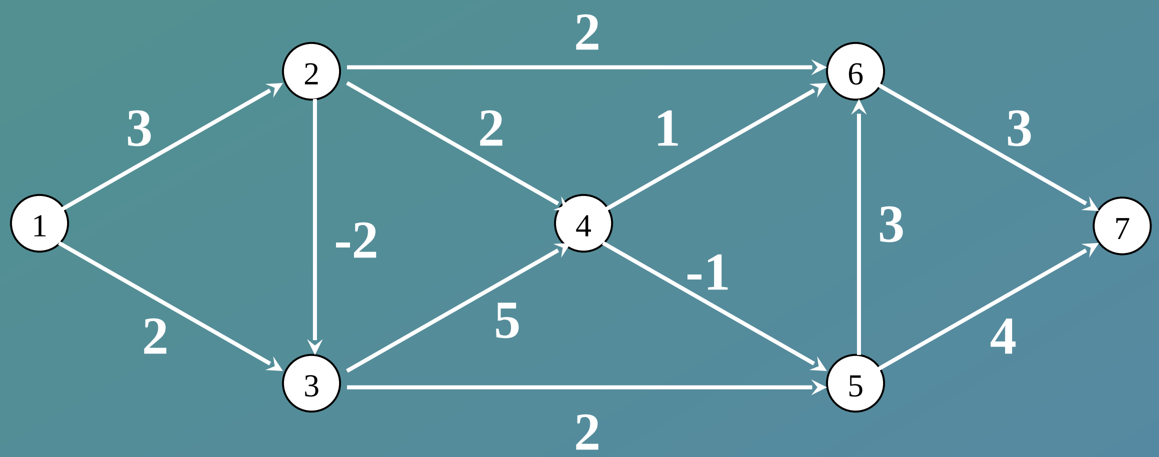

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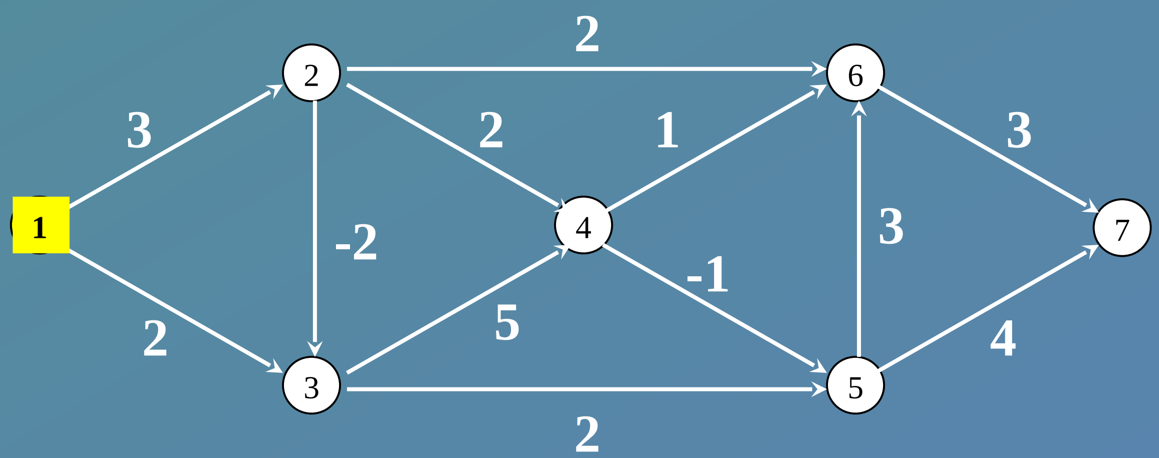

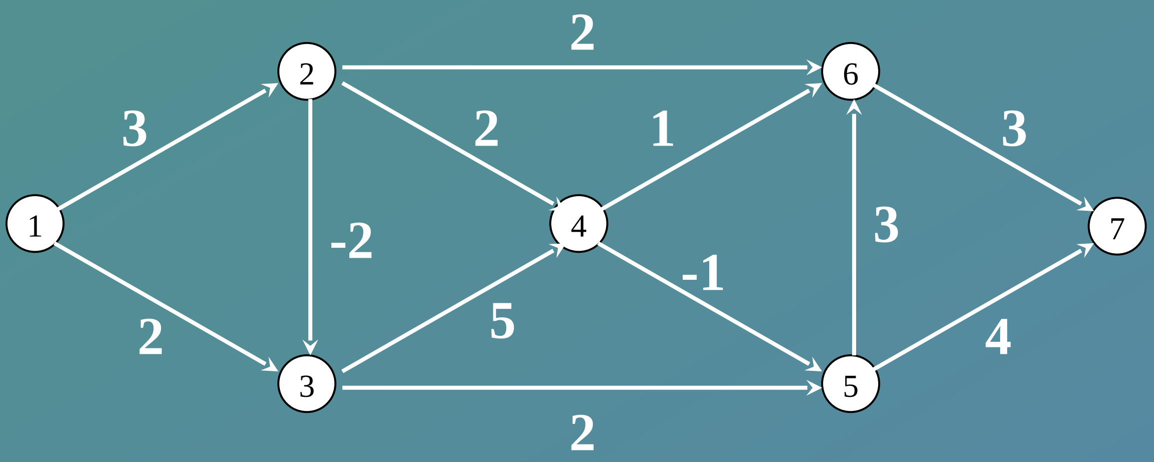

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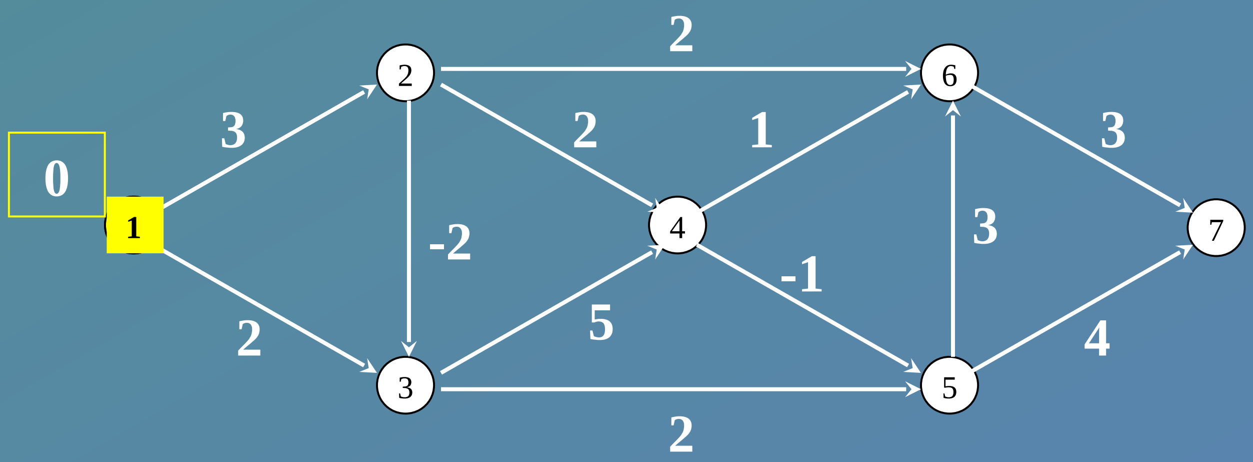

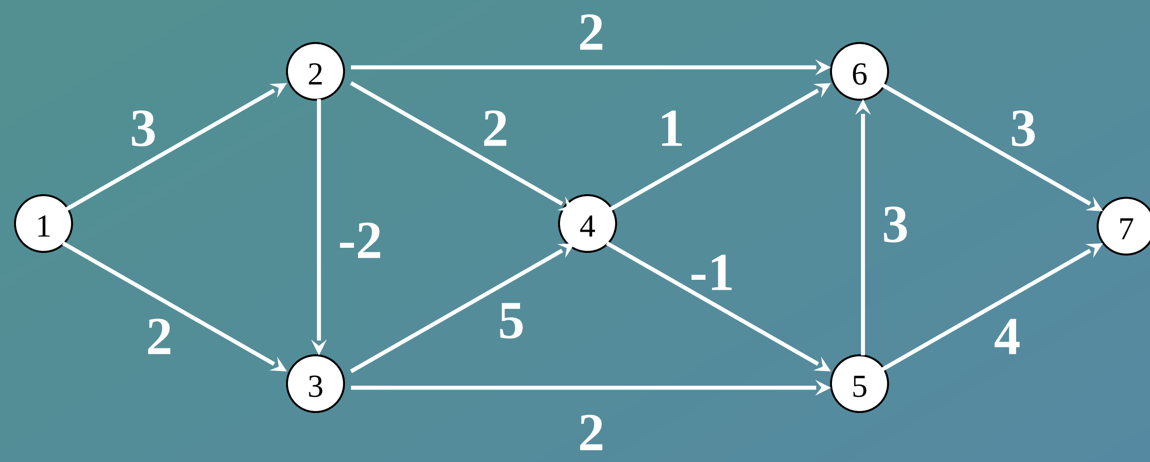

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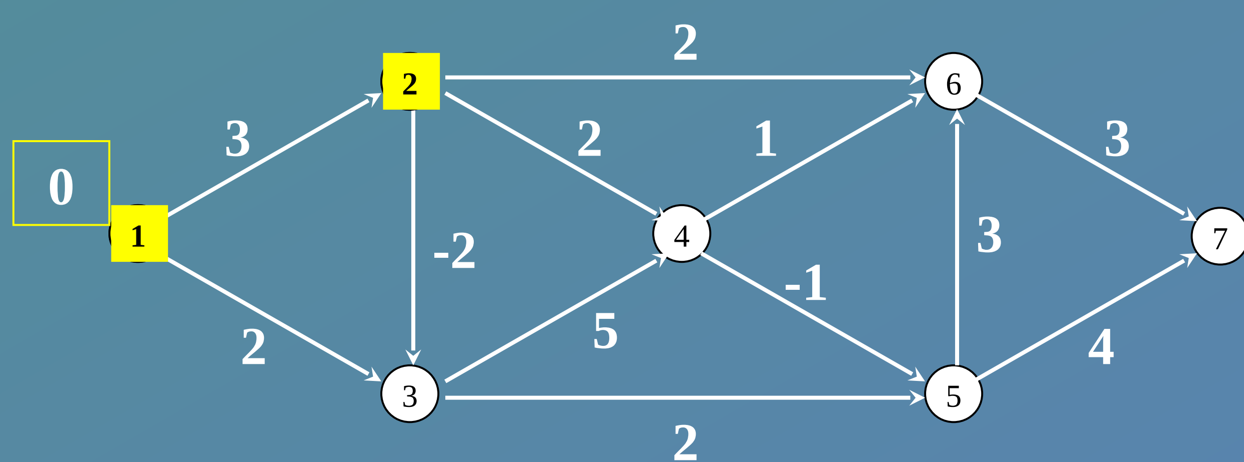

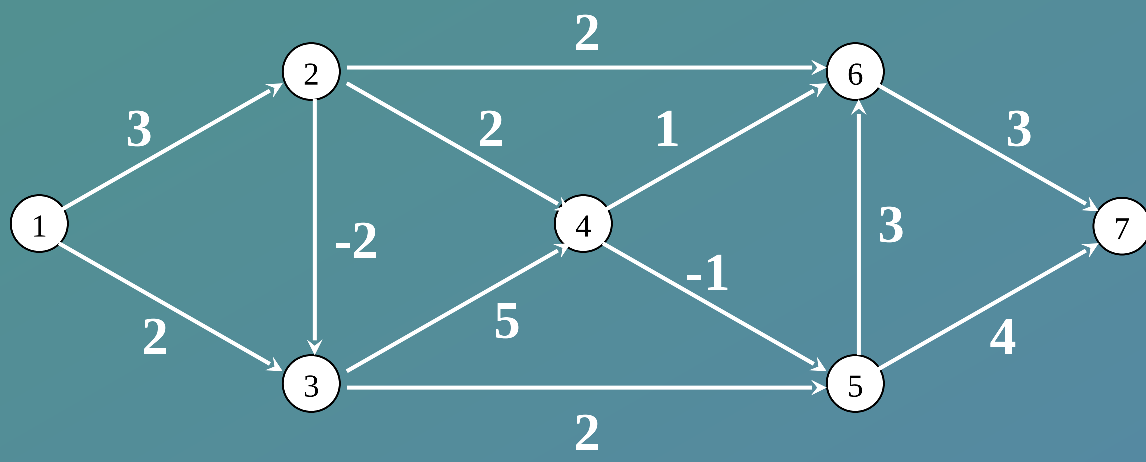

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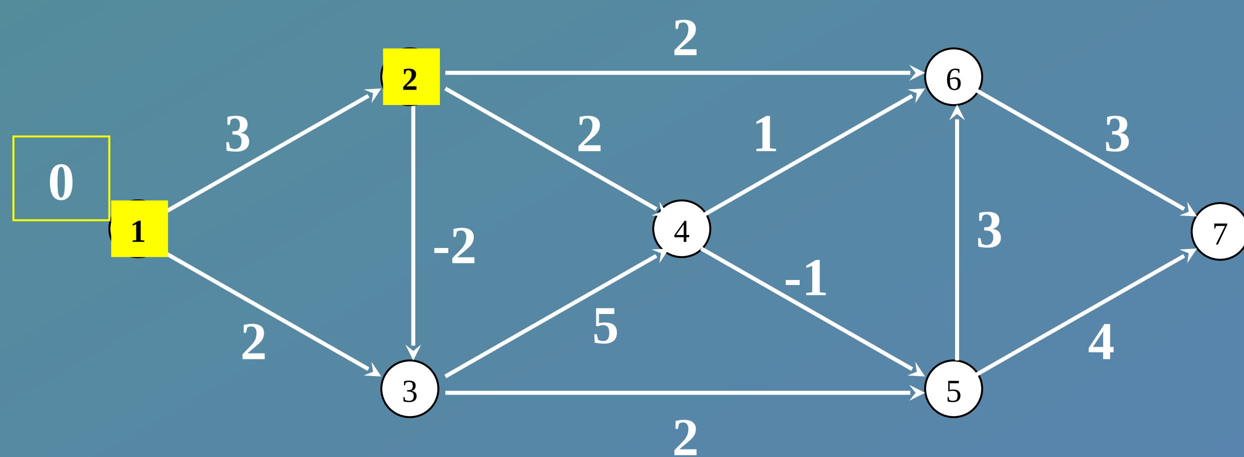

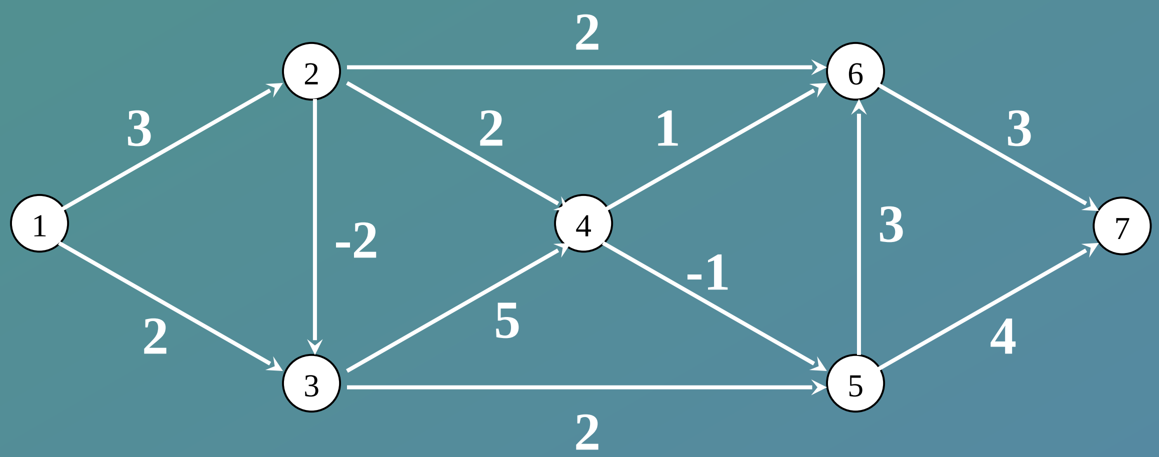

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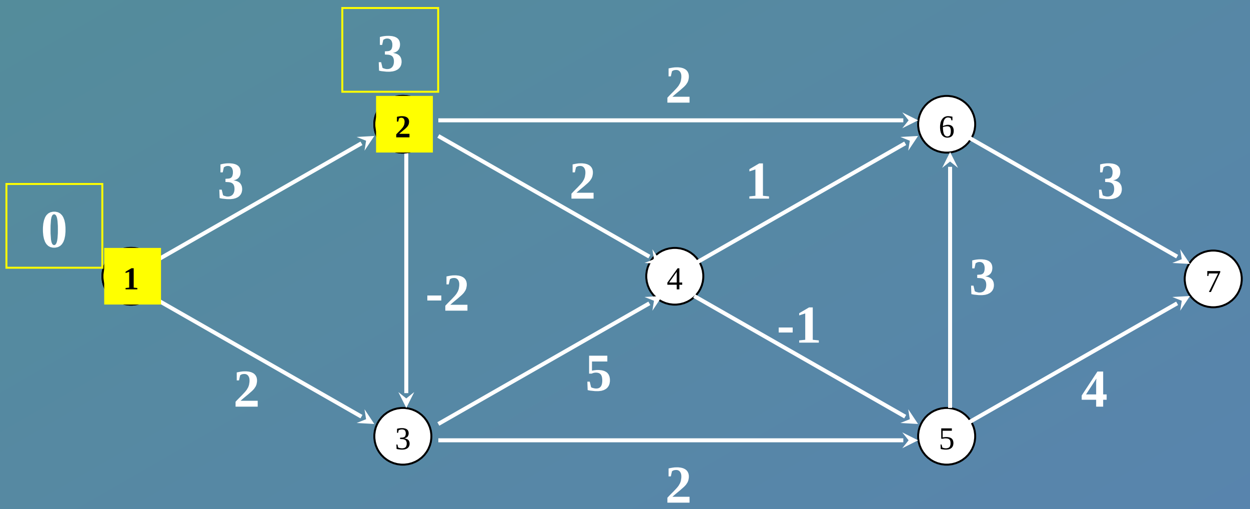

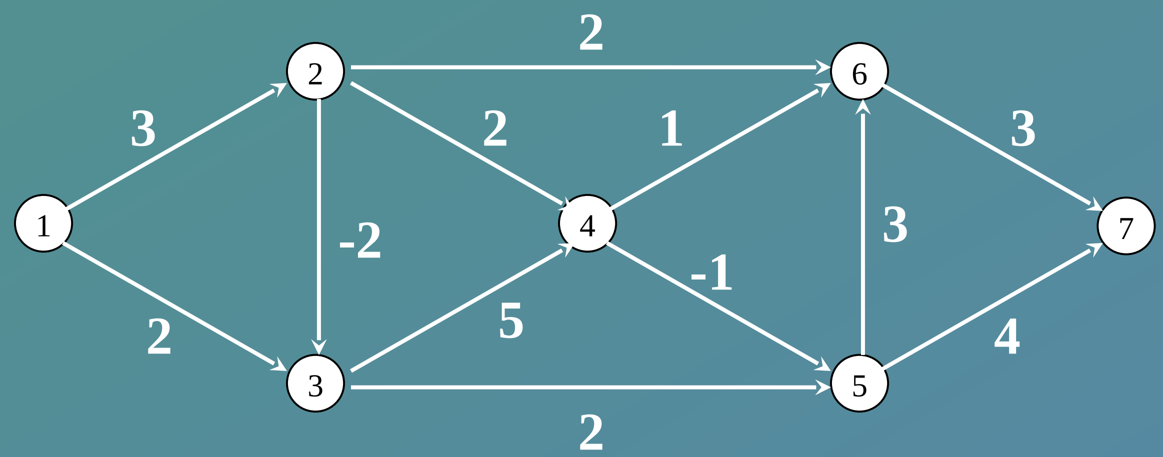

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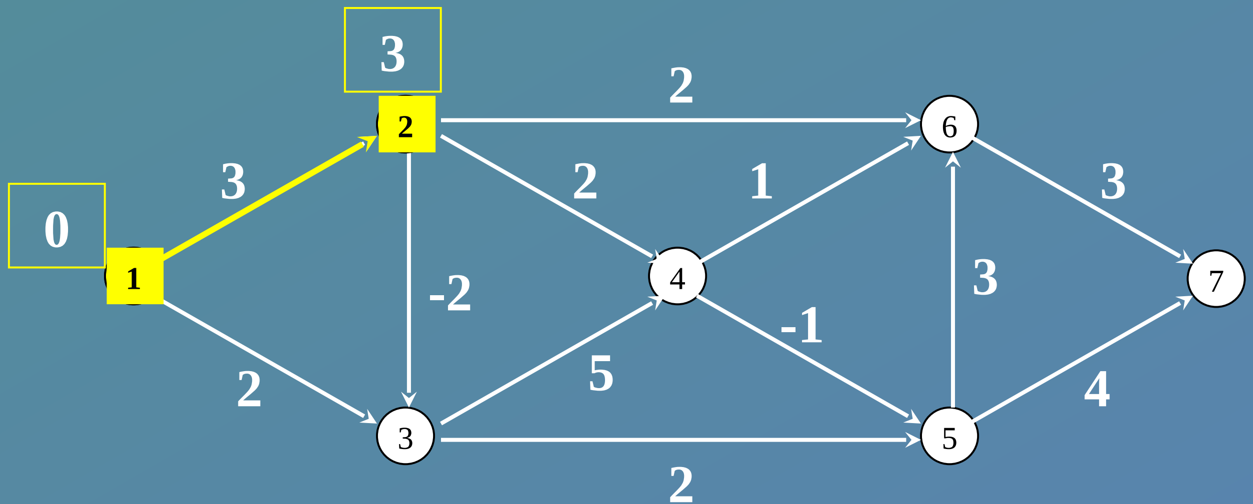

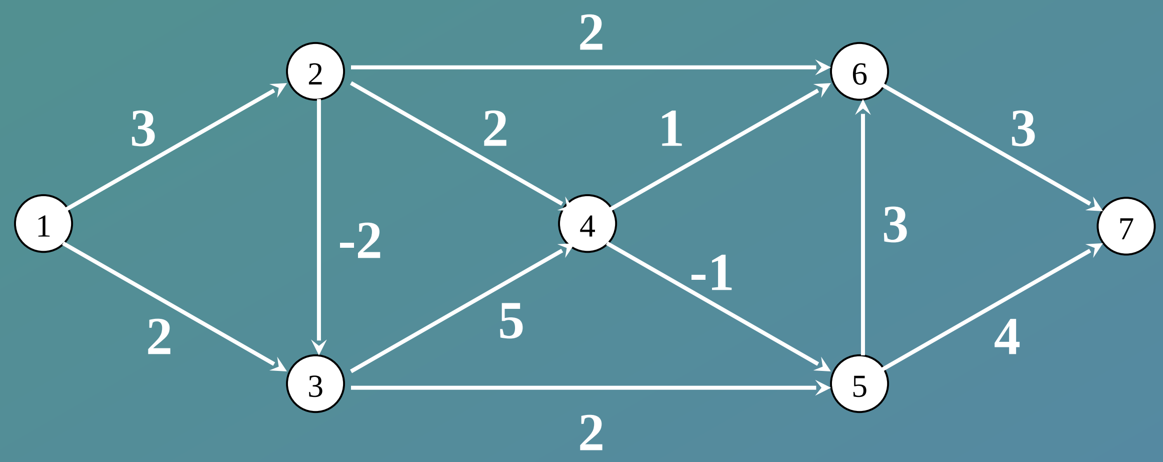

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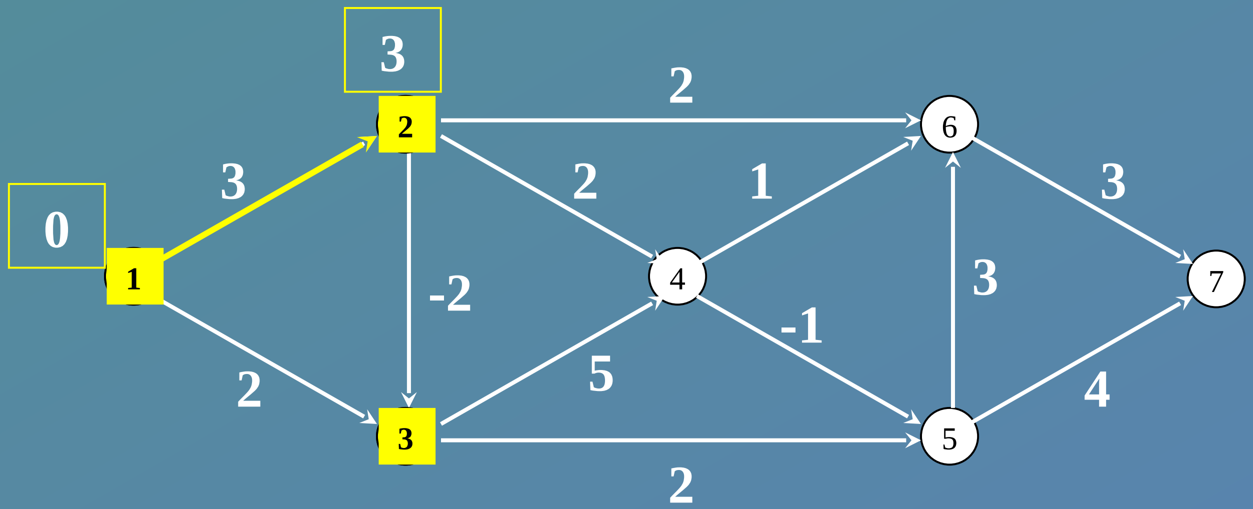

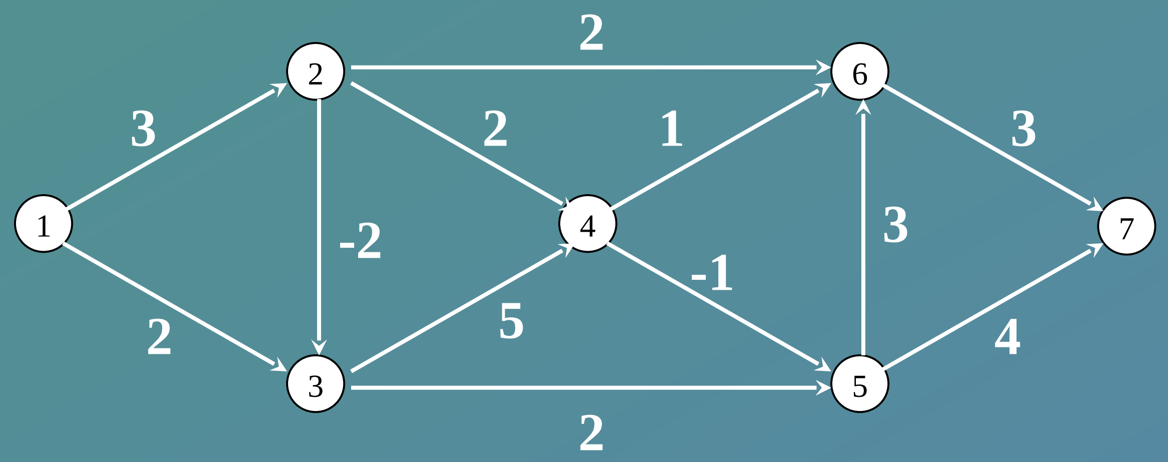

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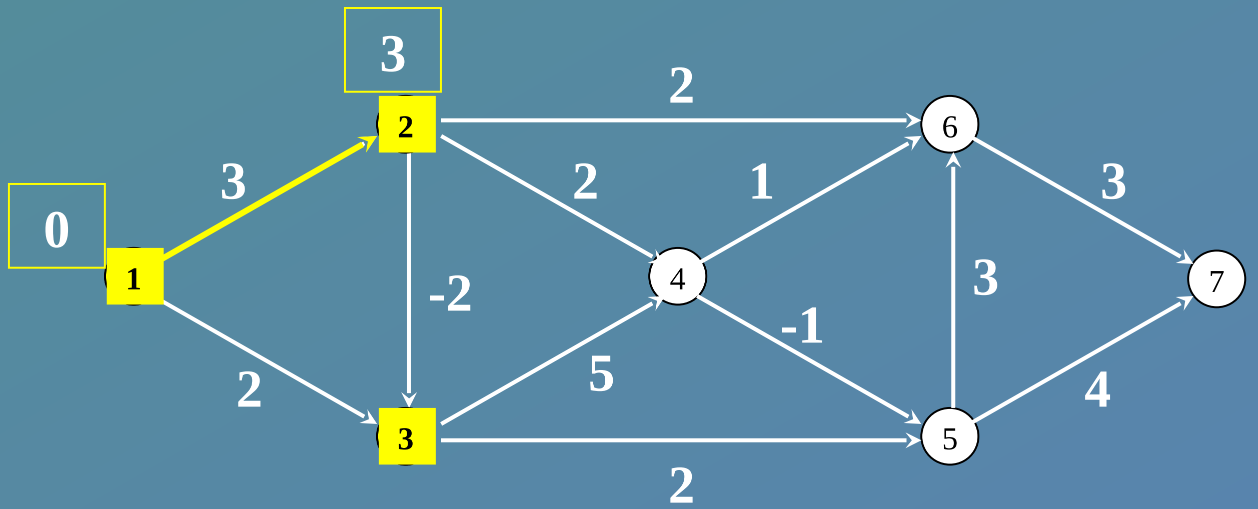

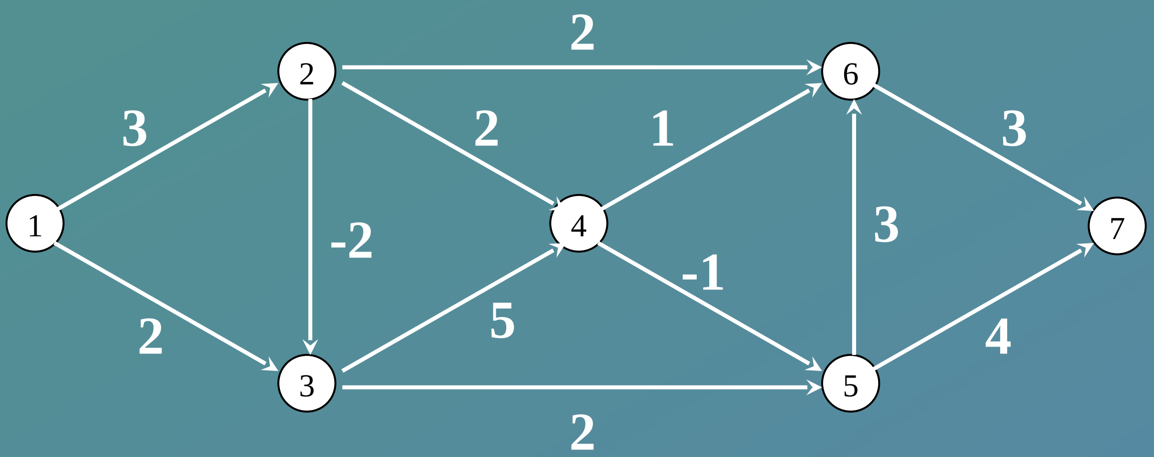

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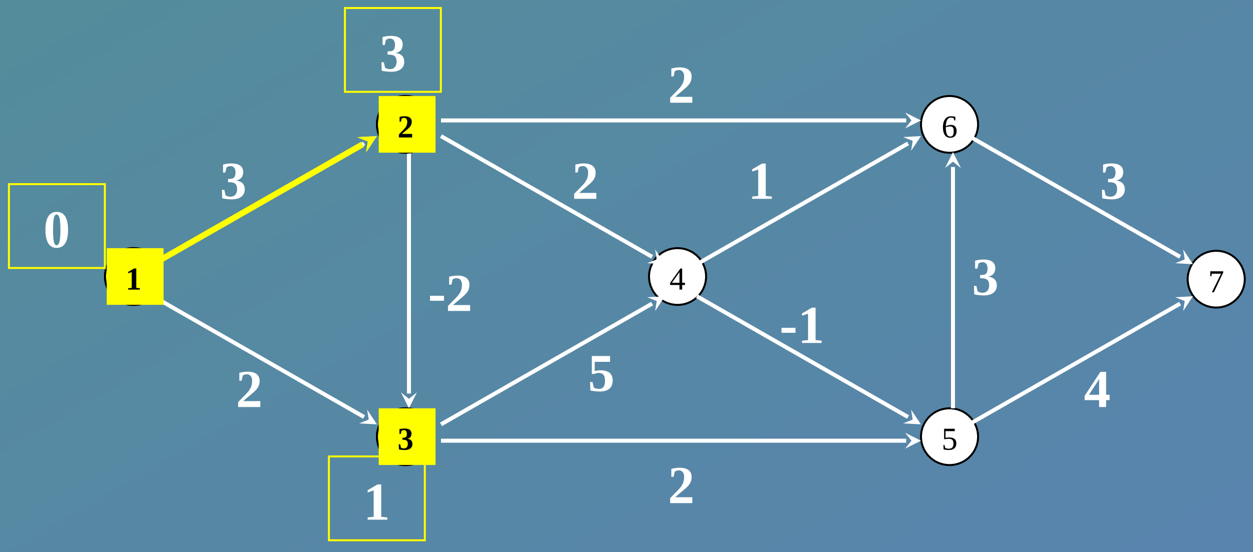

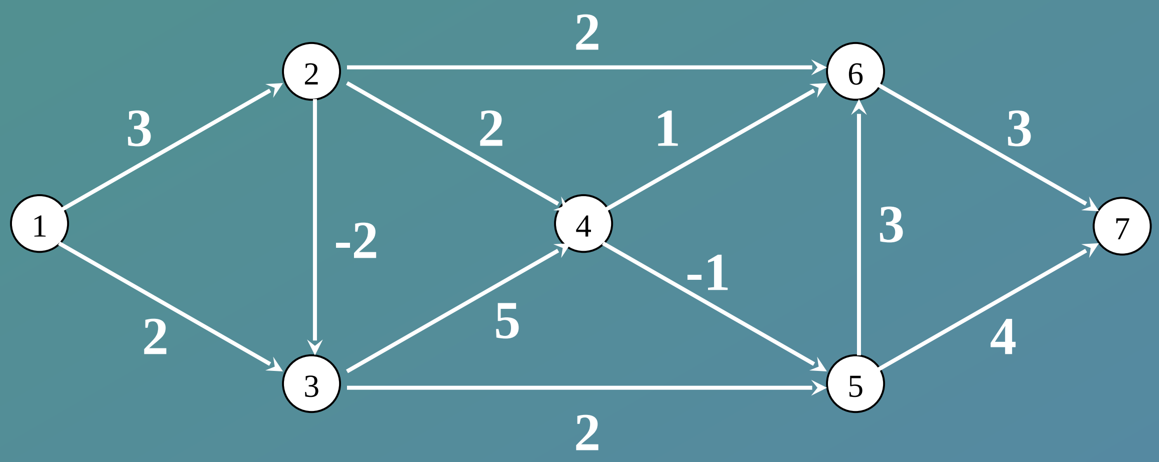

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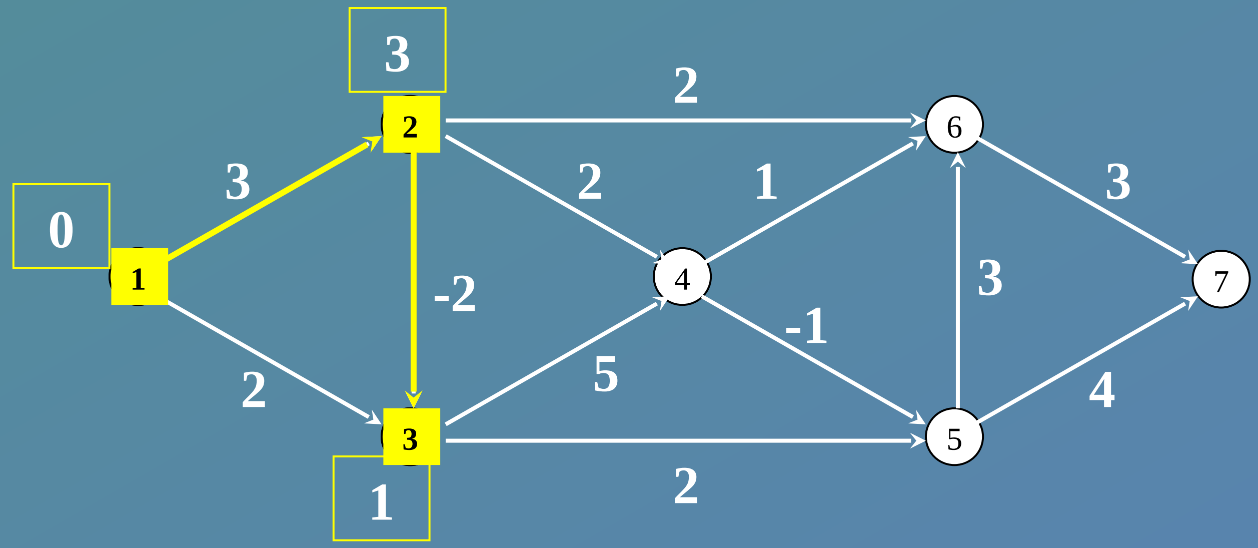

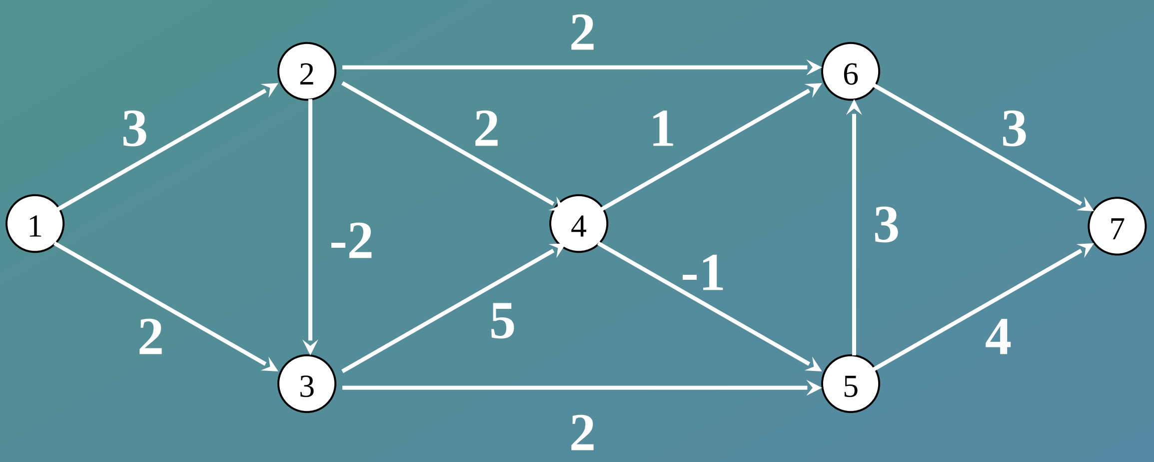

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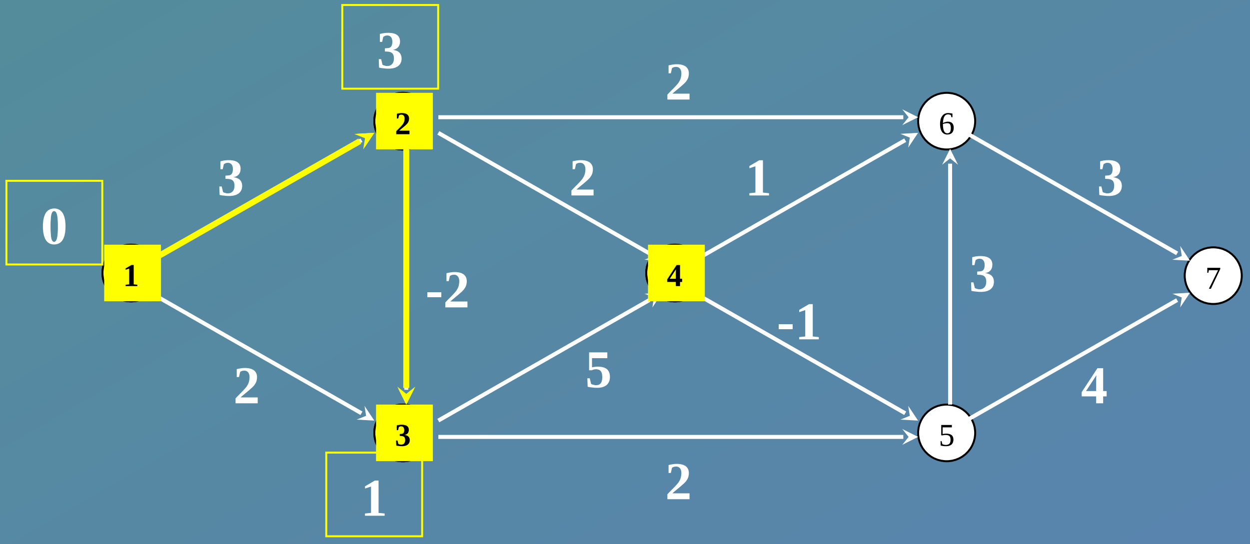

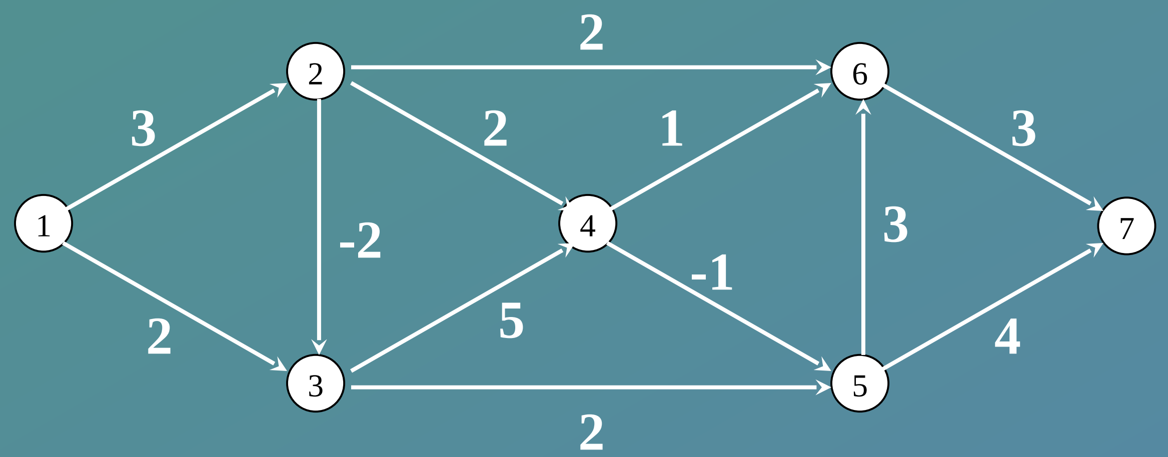

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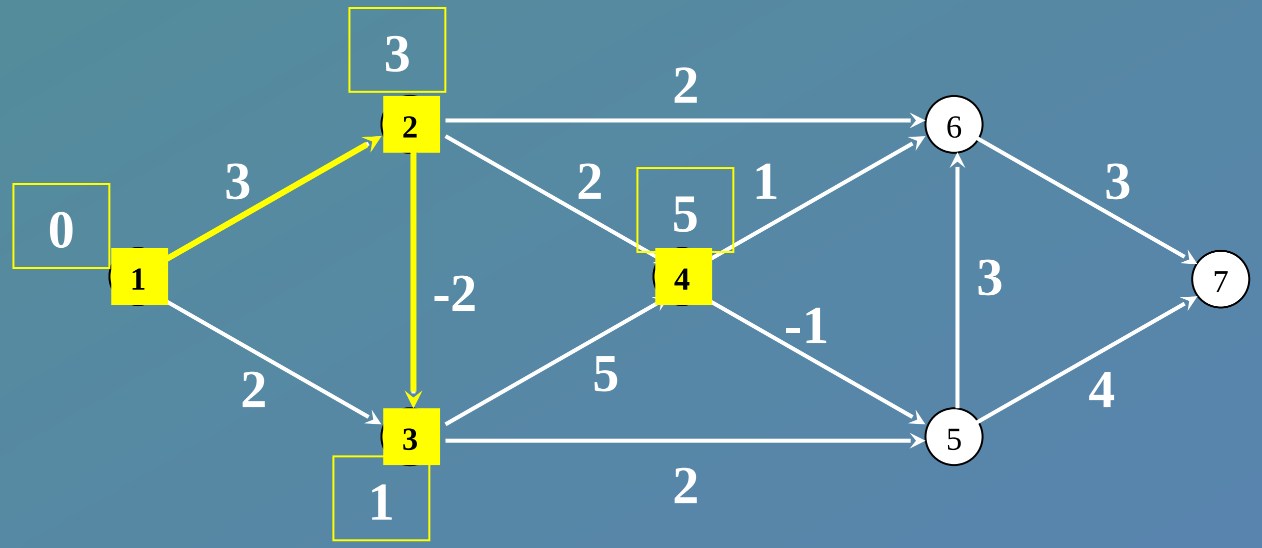

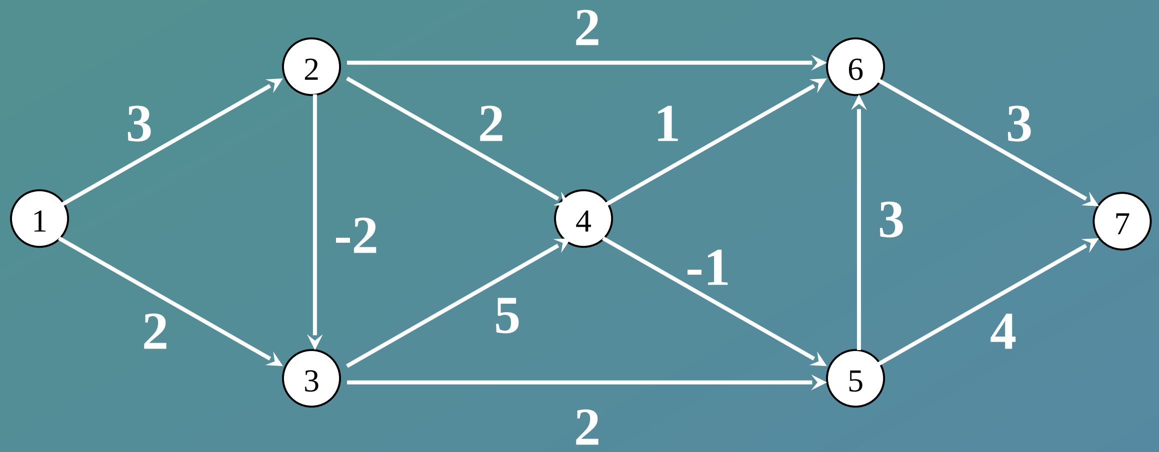

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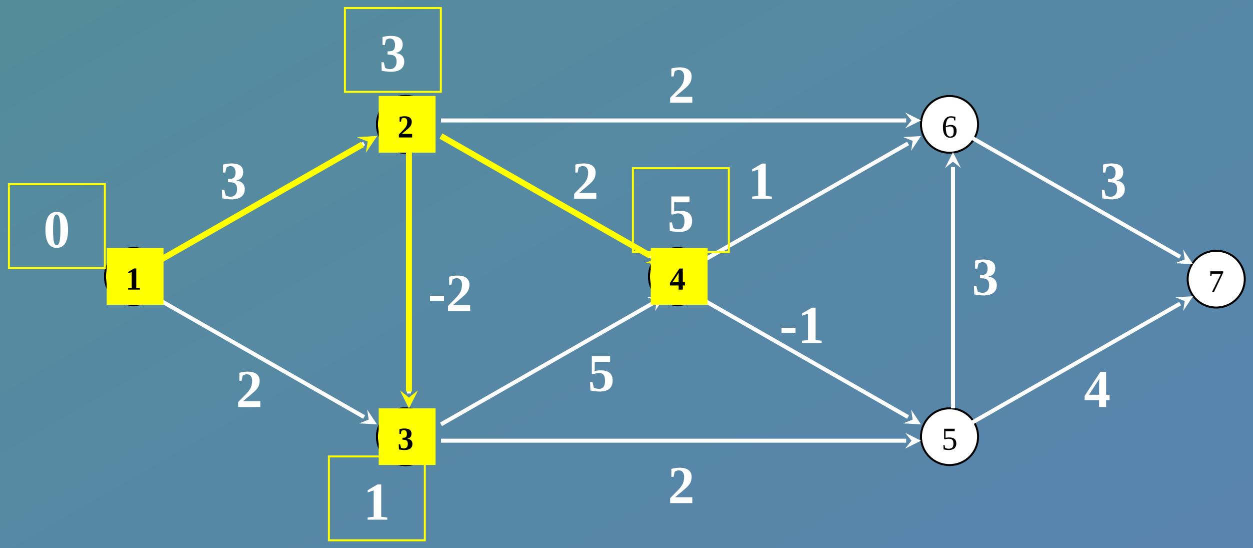

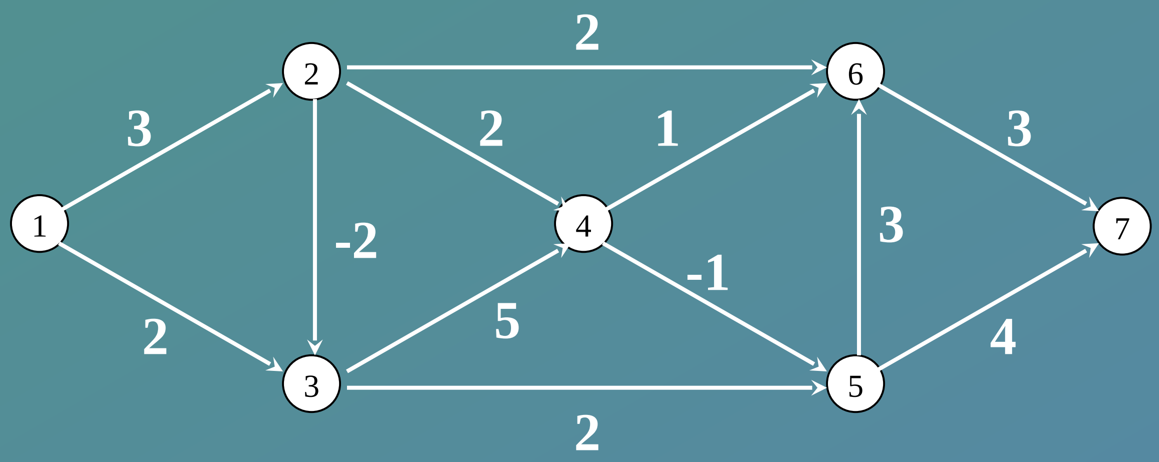

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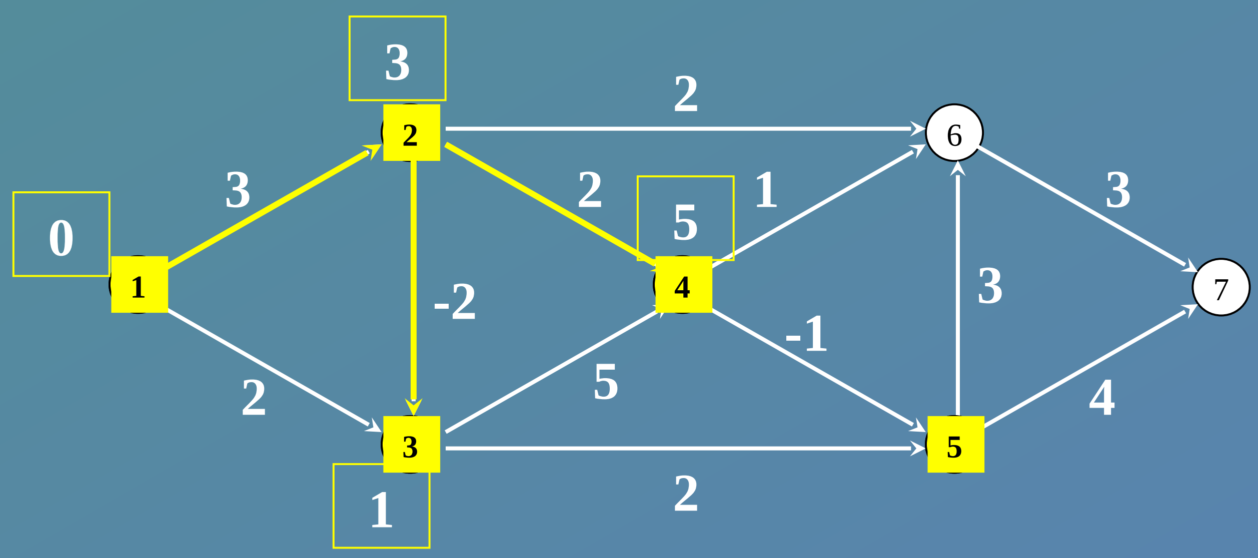

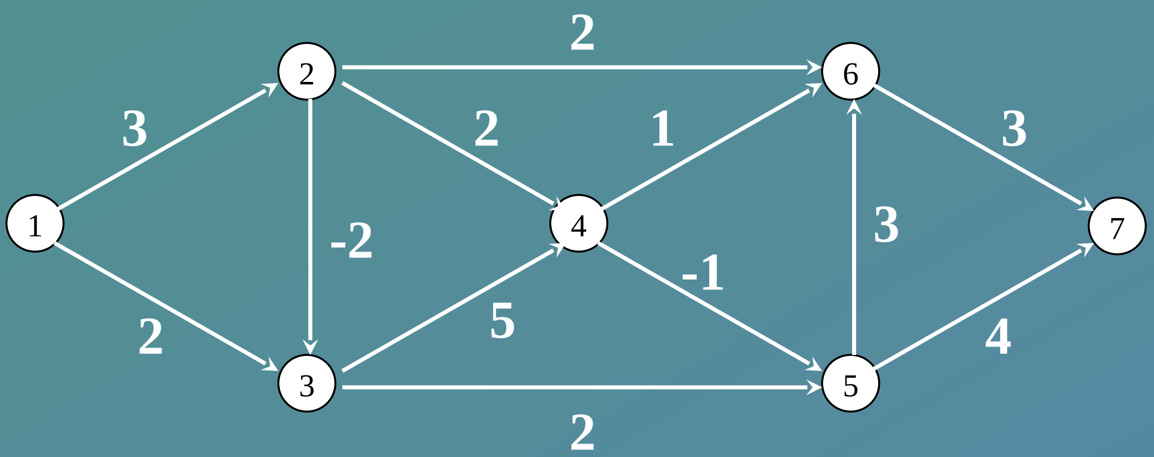

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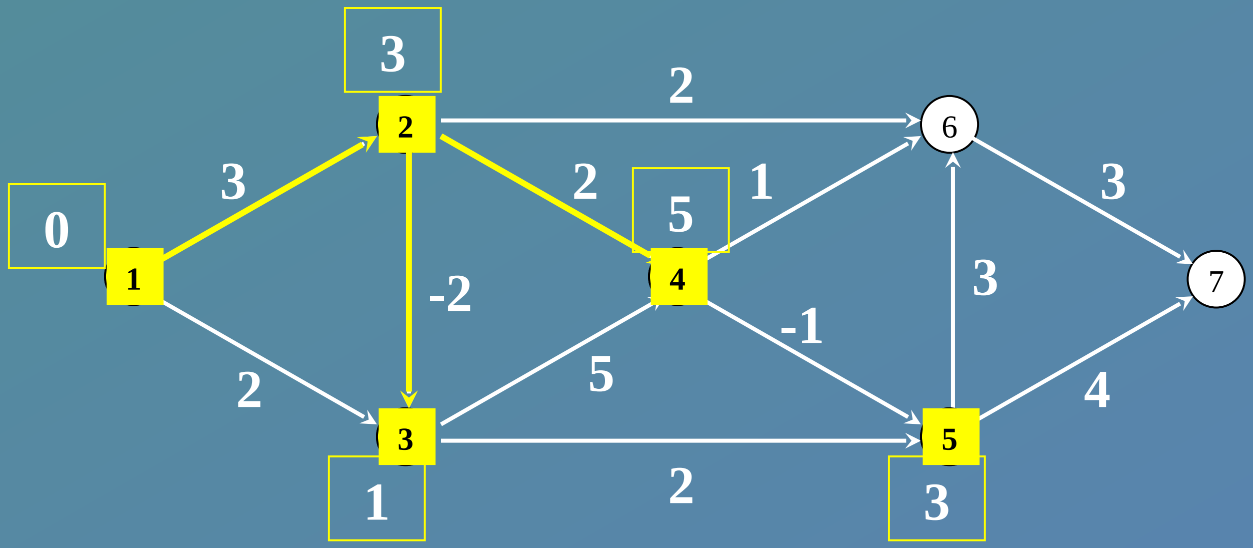

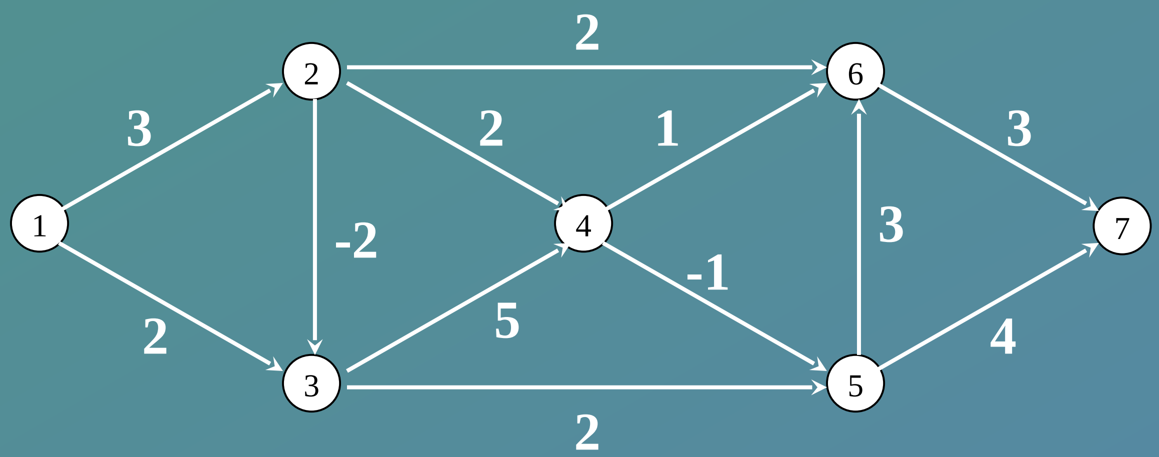

$\pi_1 := 0$

$\pi_2 := \pi_1 + d(1, 2) = 0 + 3 = 3$

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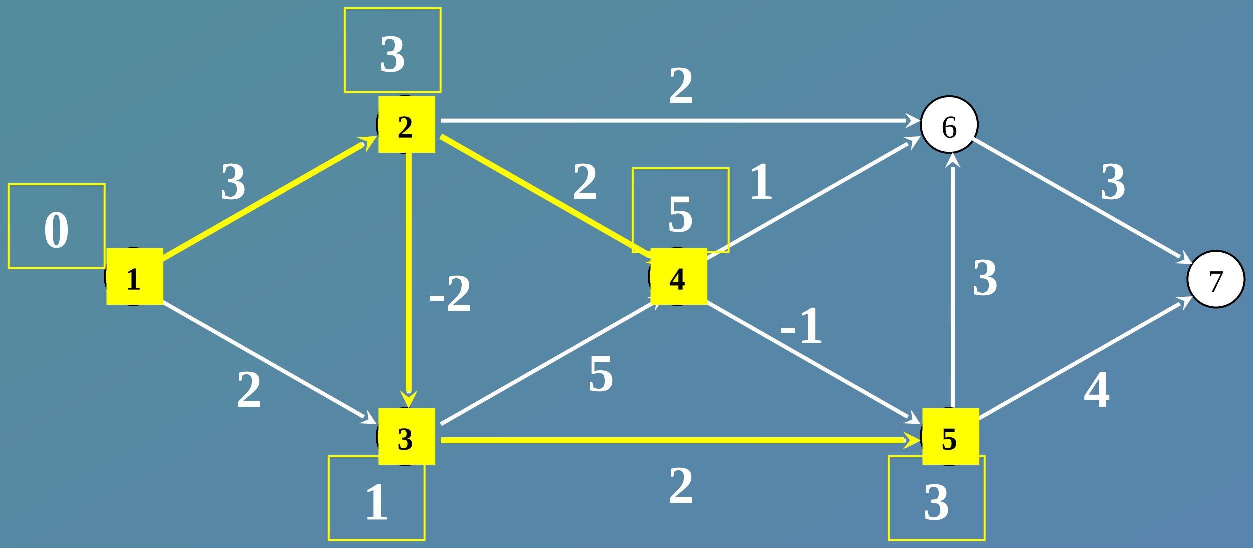

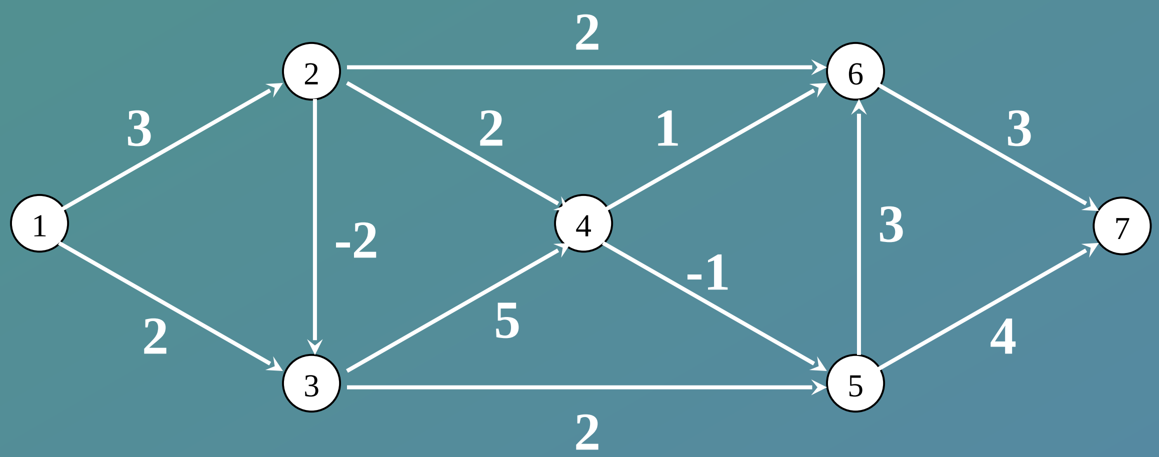

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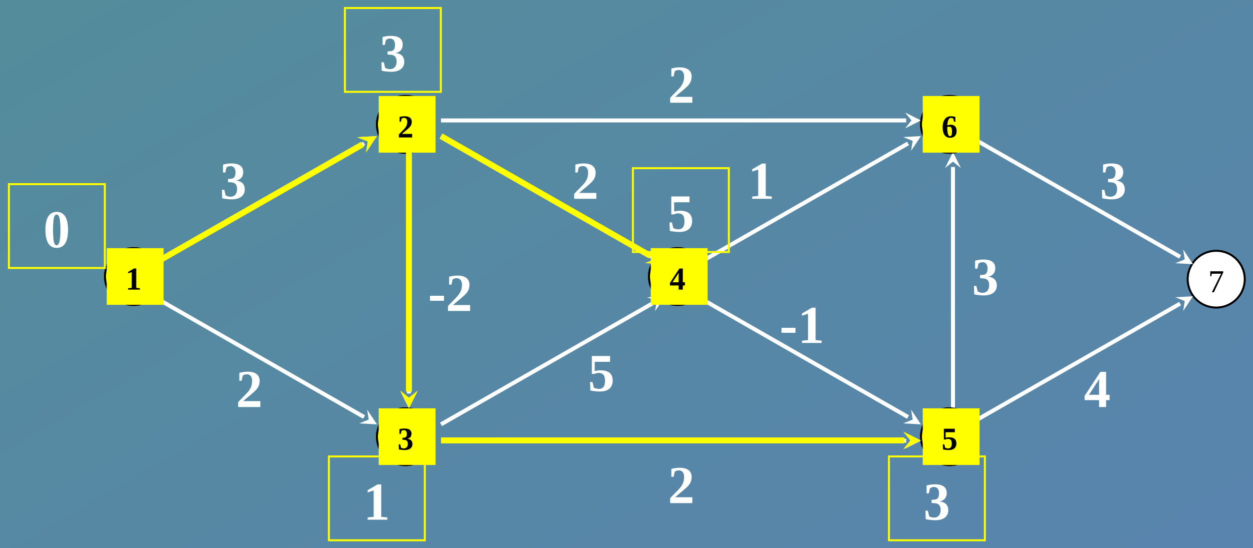

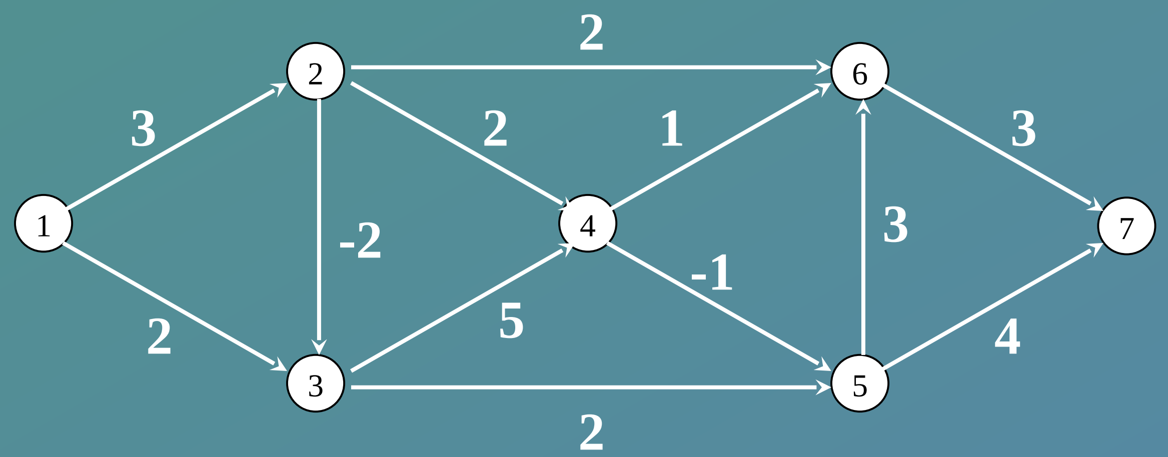

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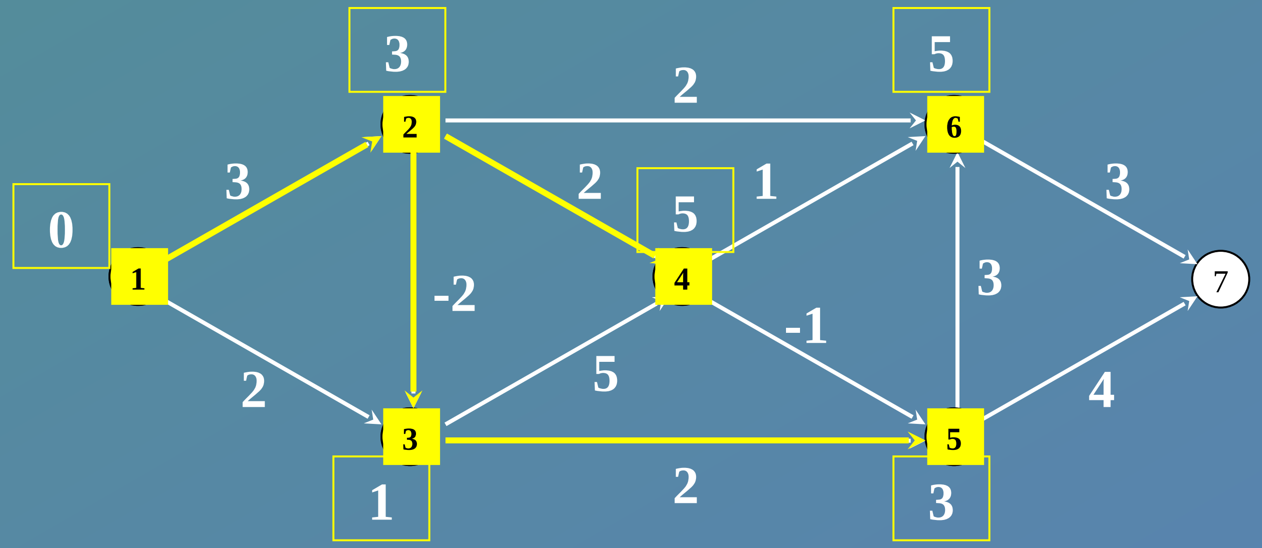

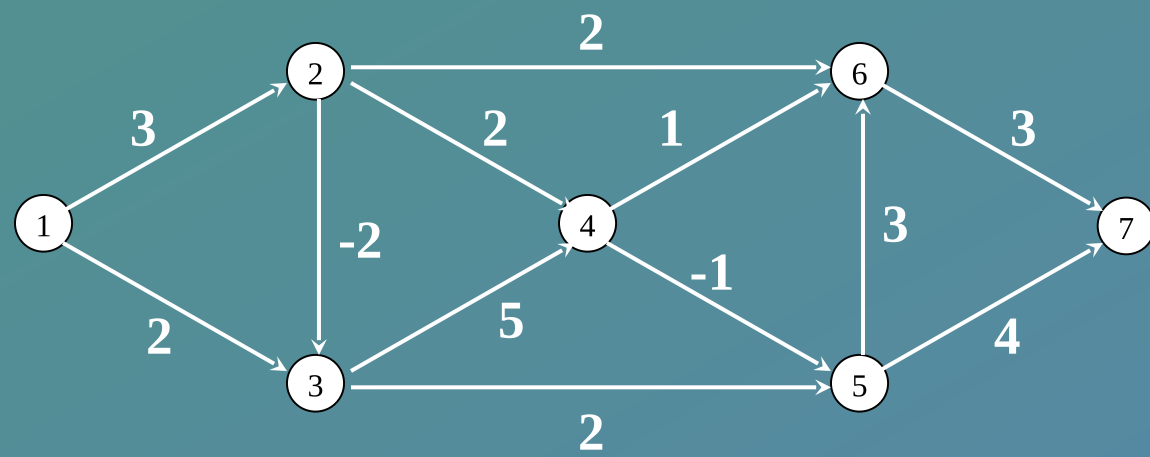

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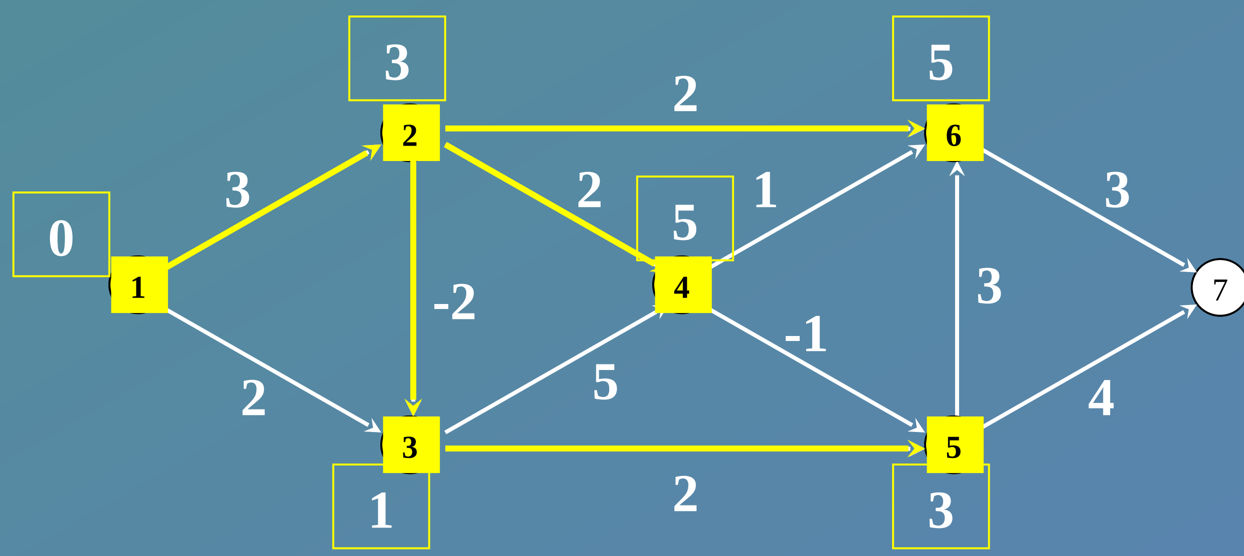

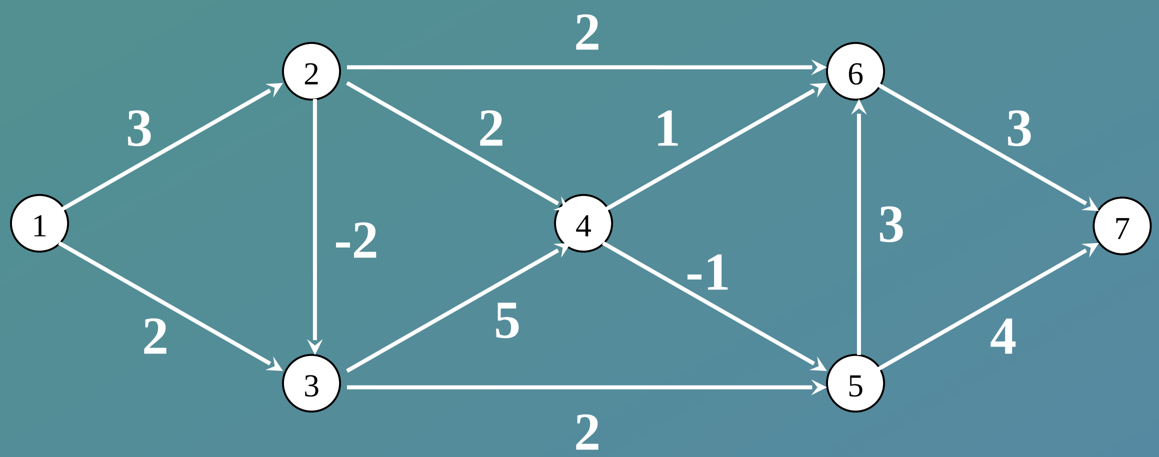

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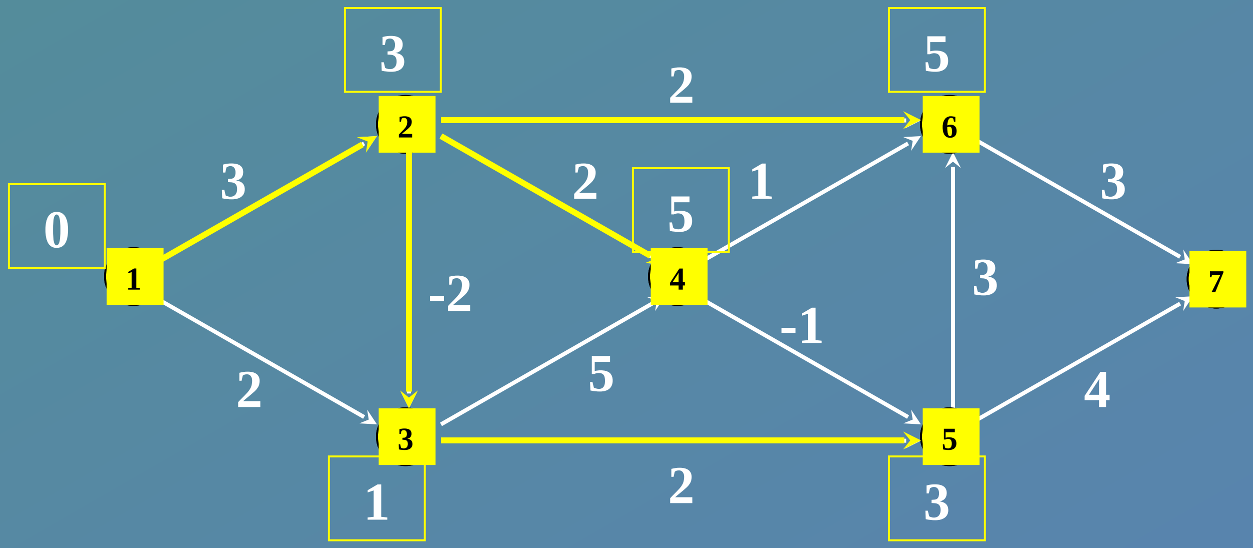

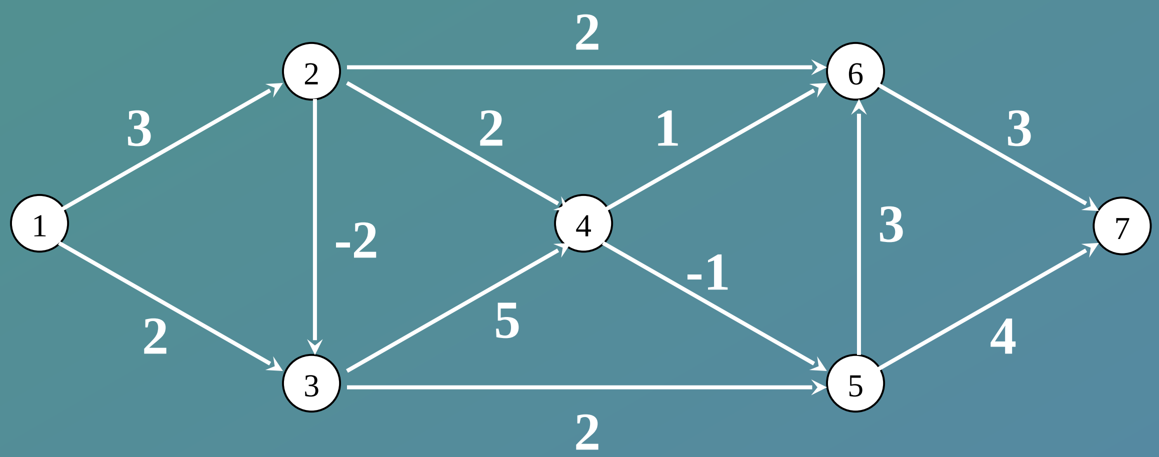

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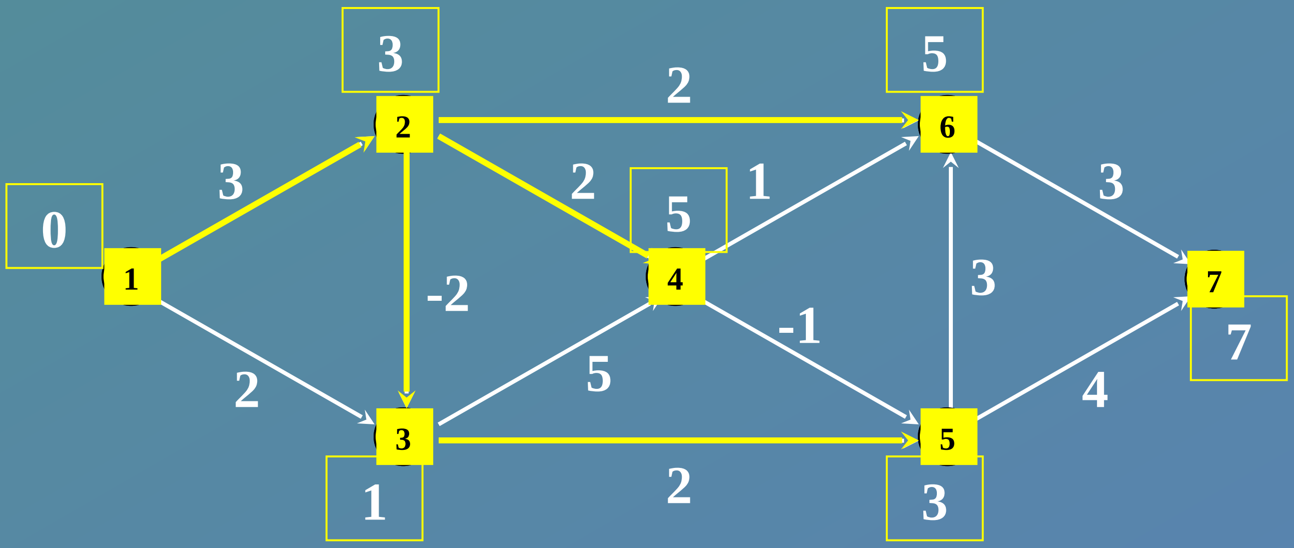

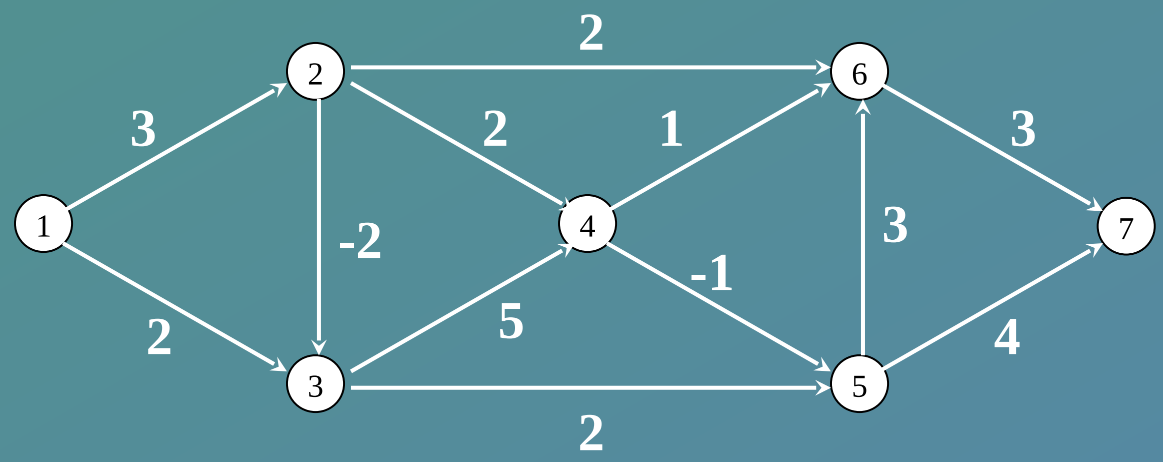

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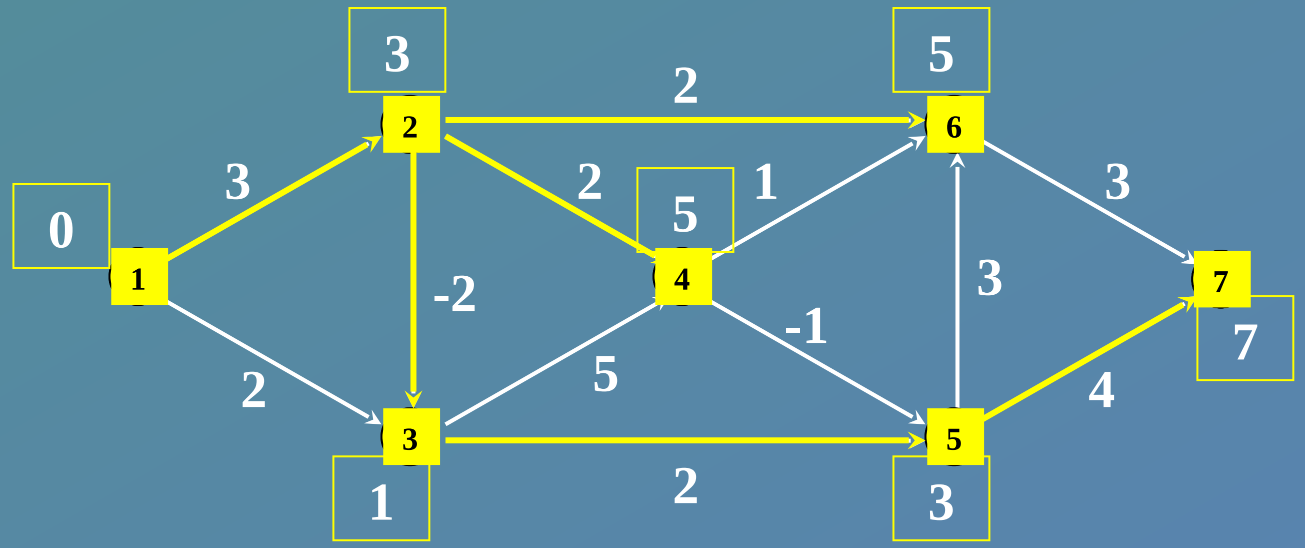

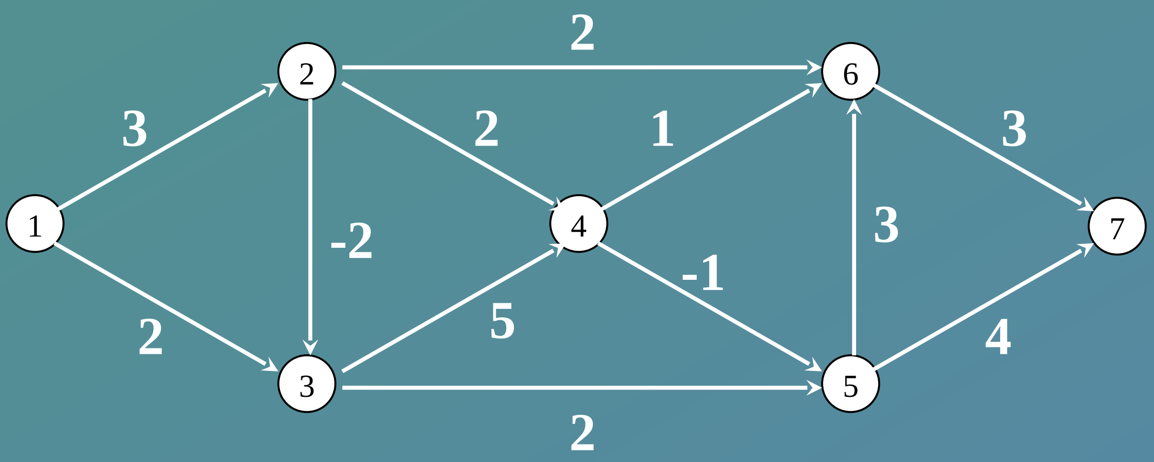
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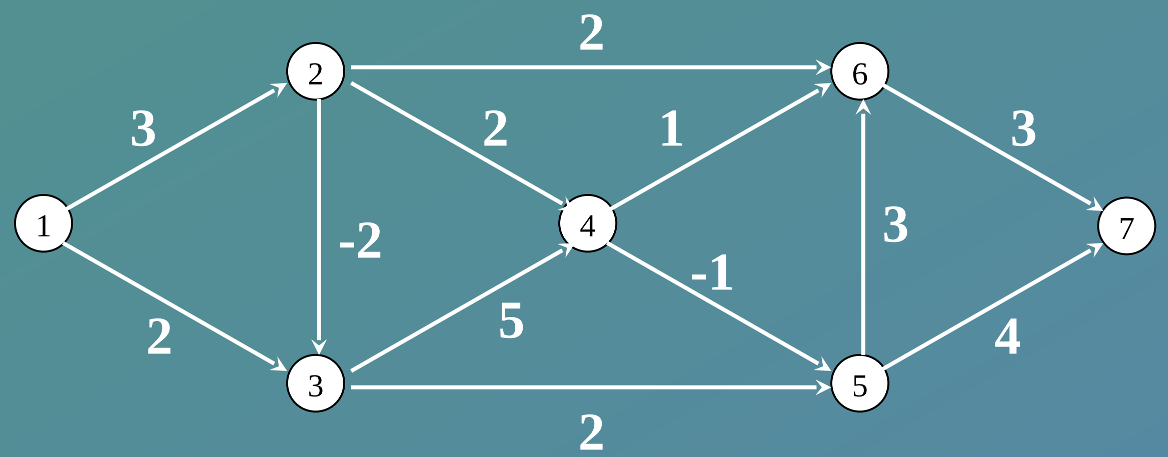
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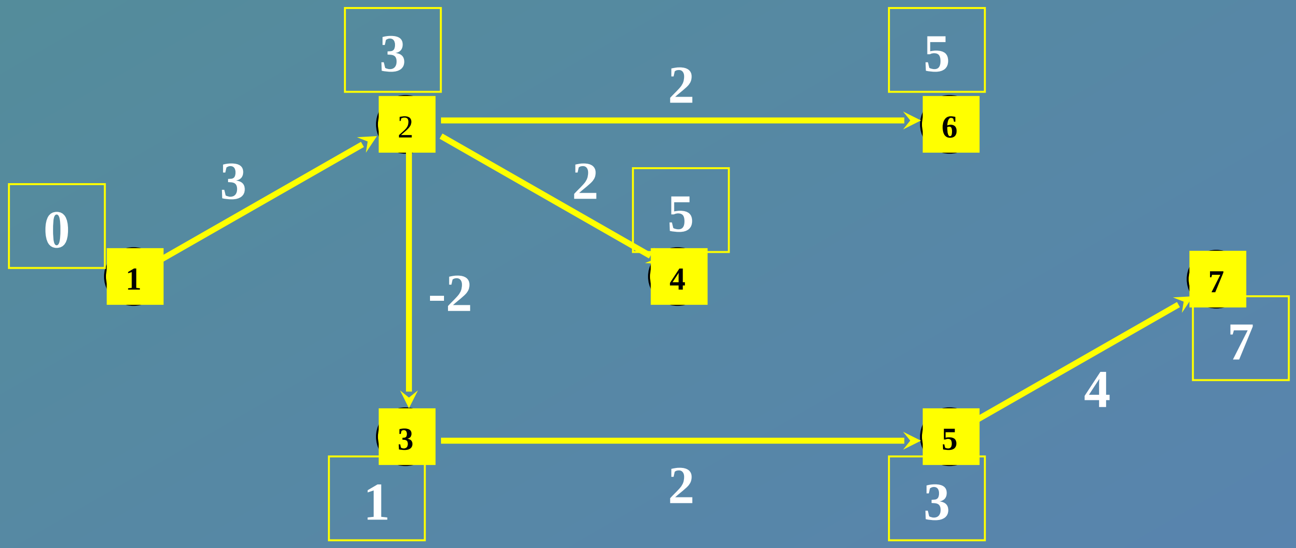
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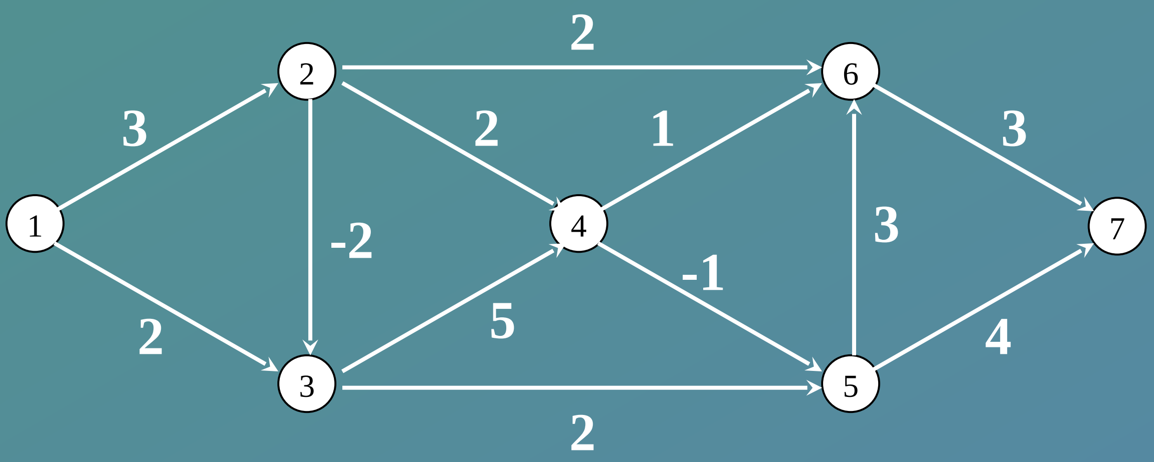

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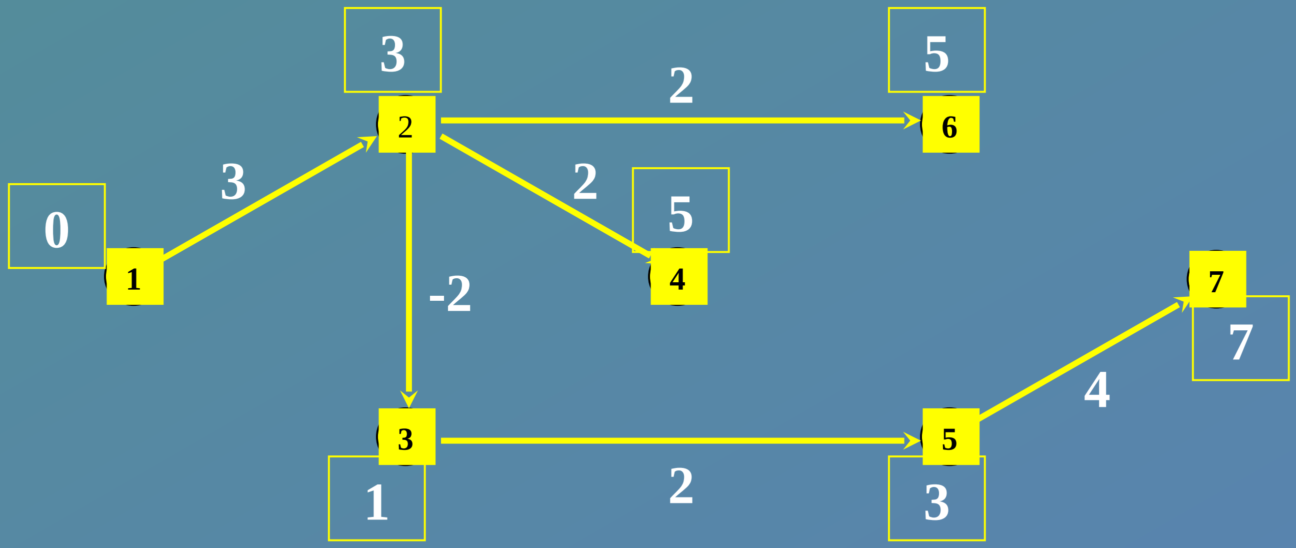

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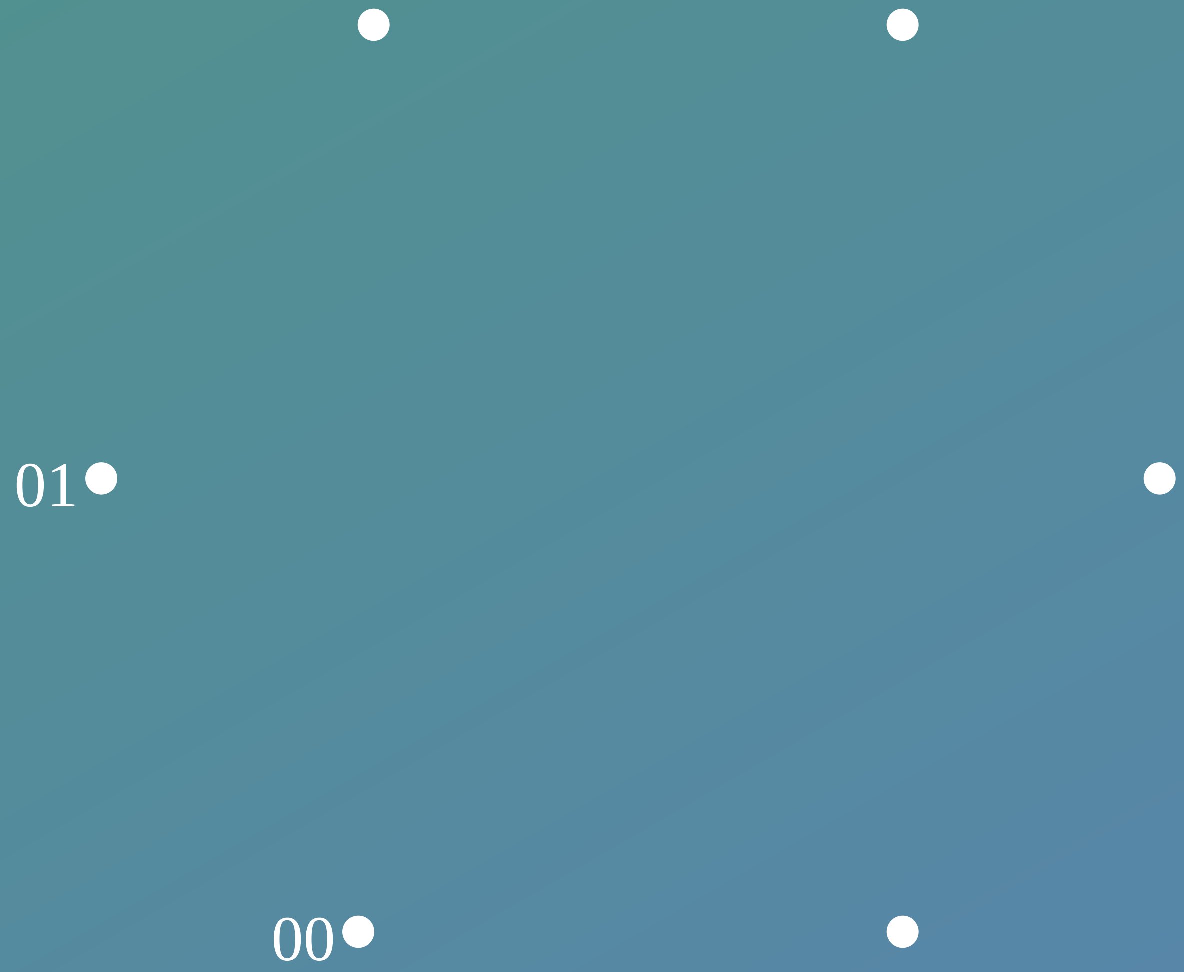
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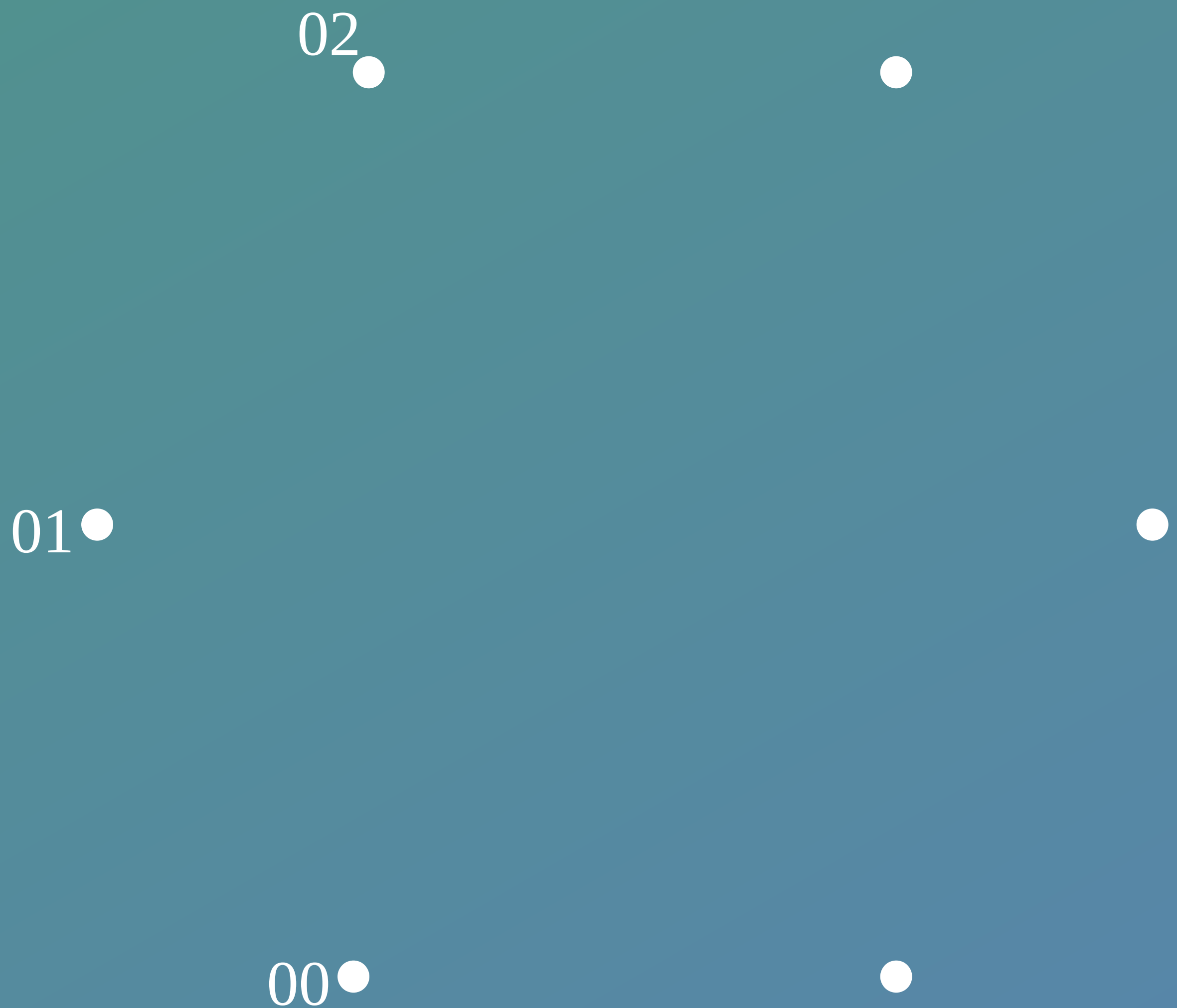
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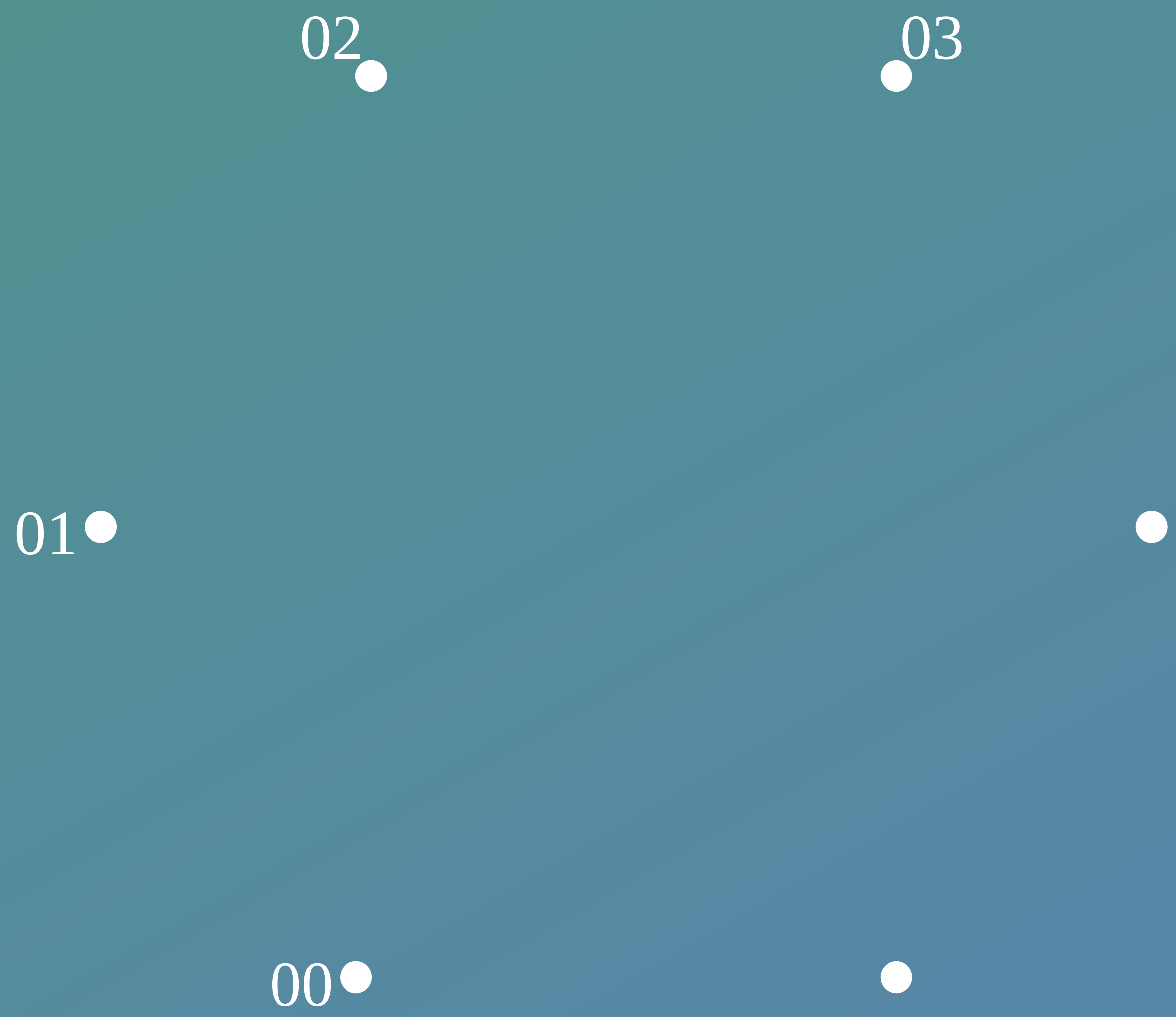
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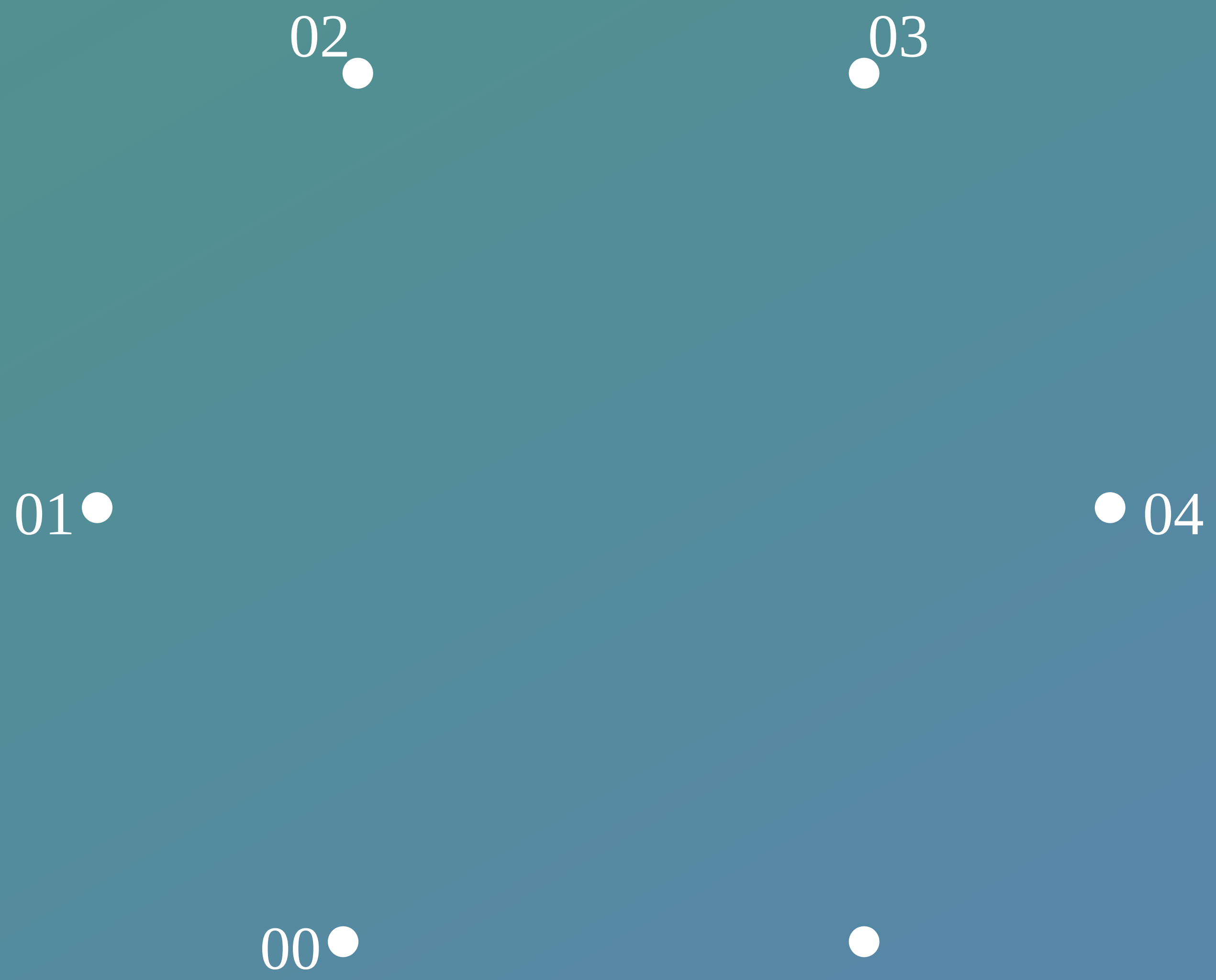
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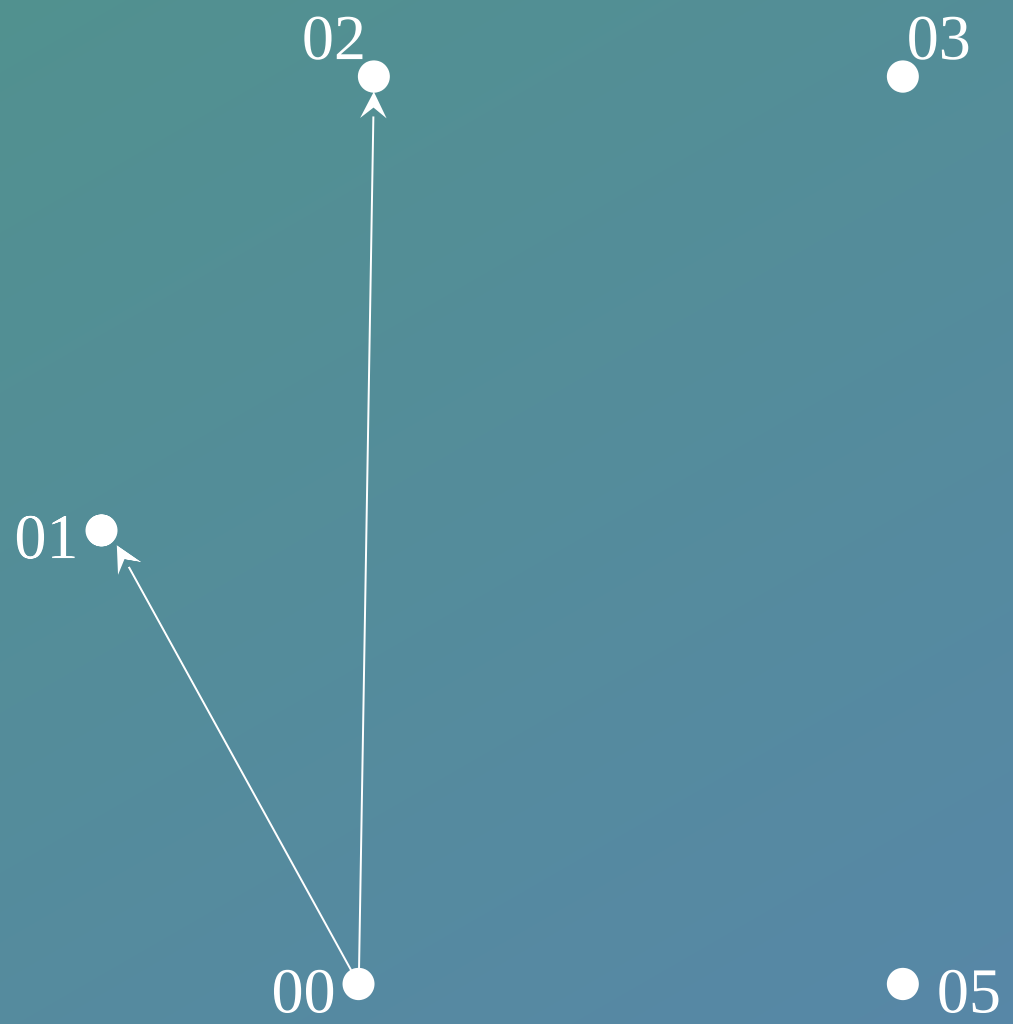
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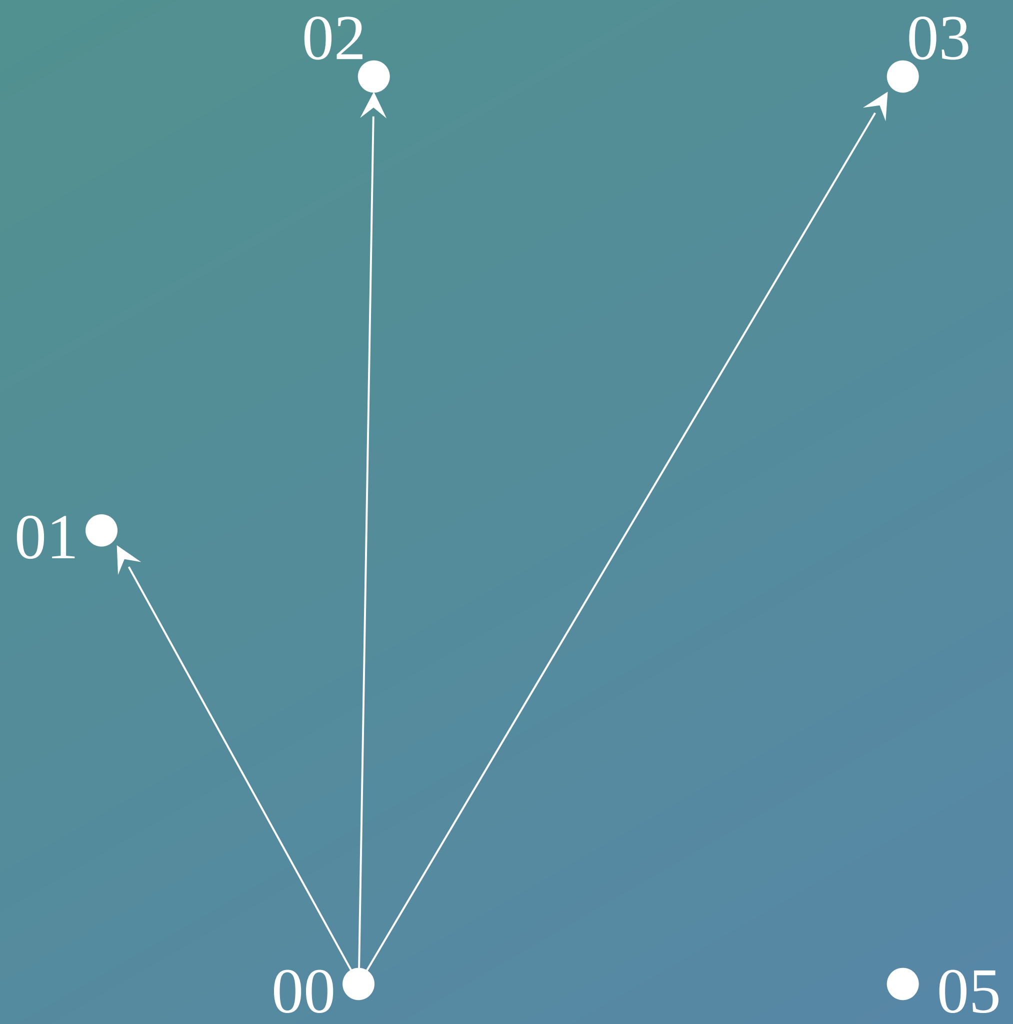
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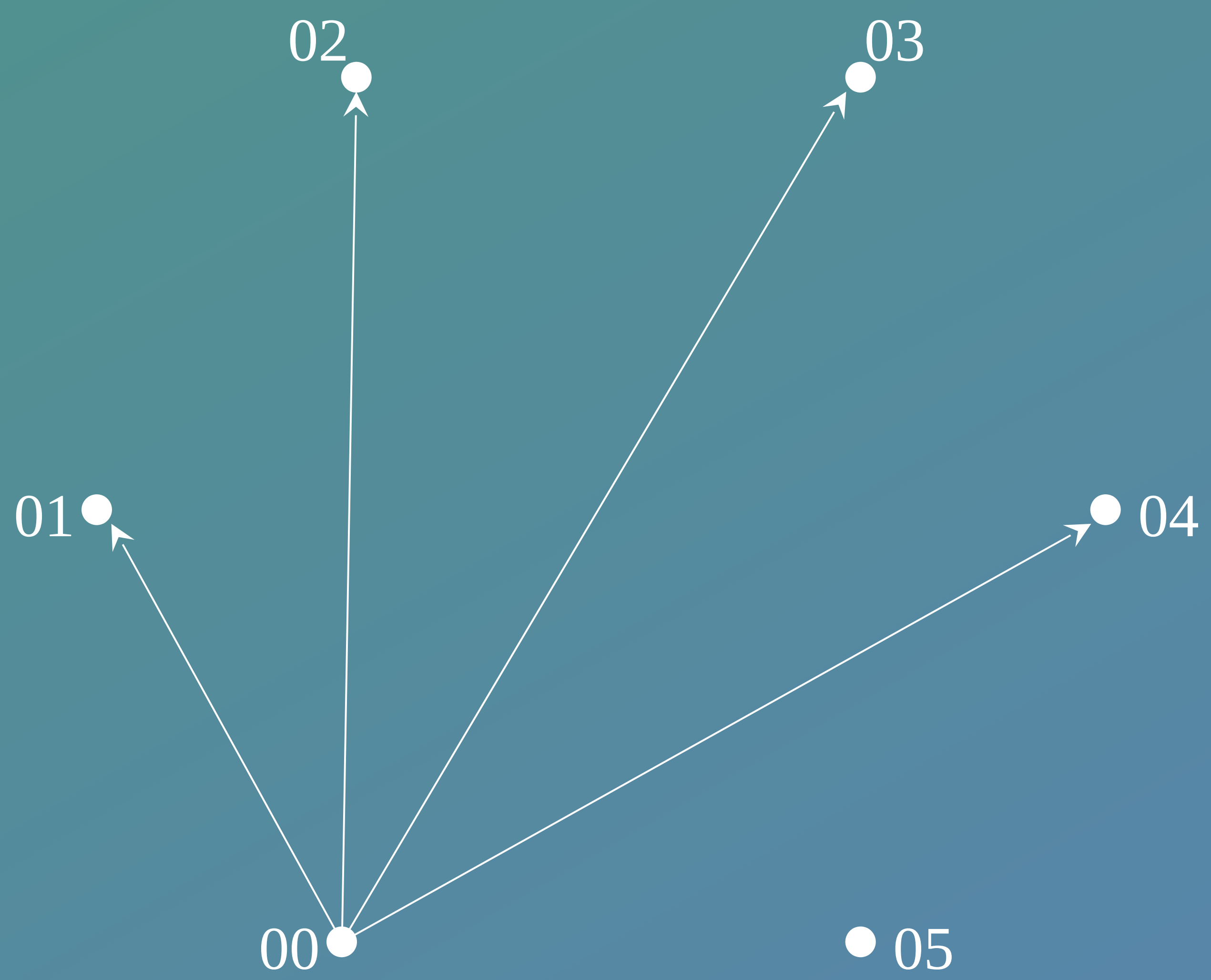
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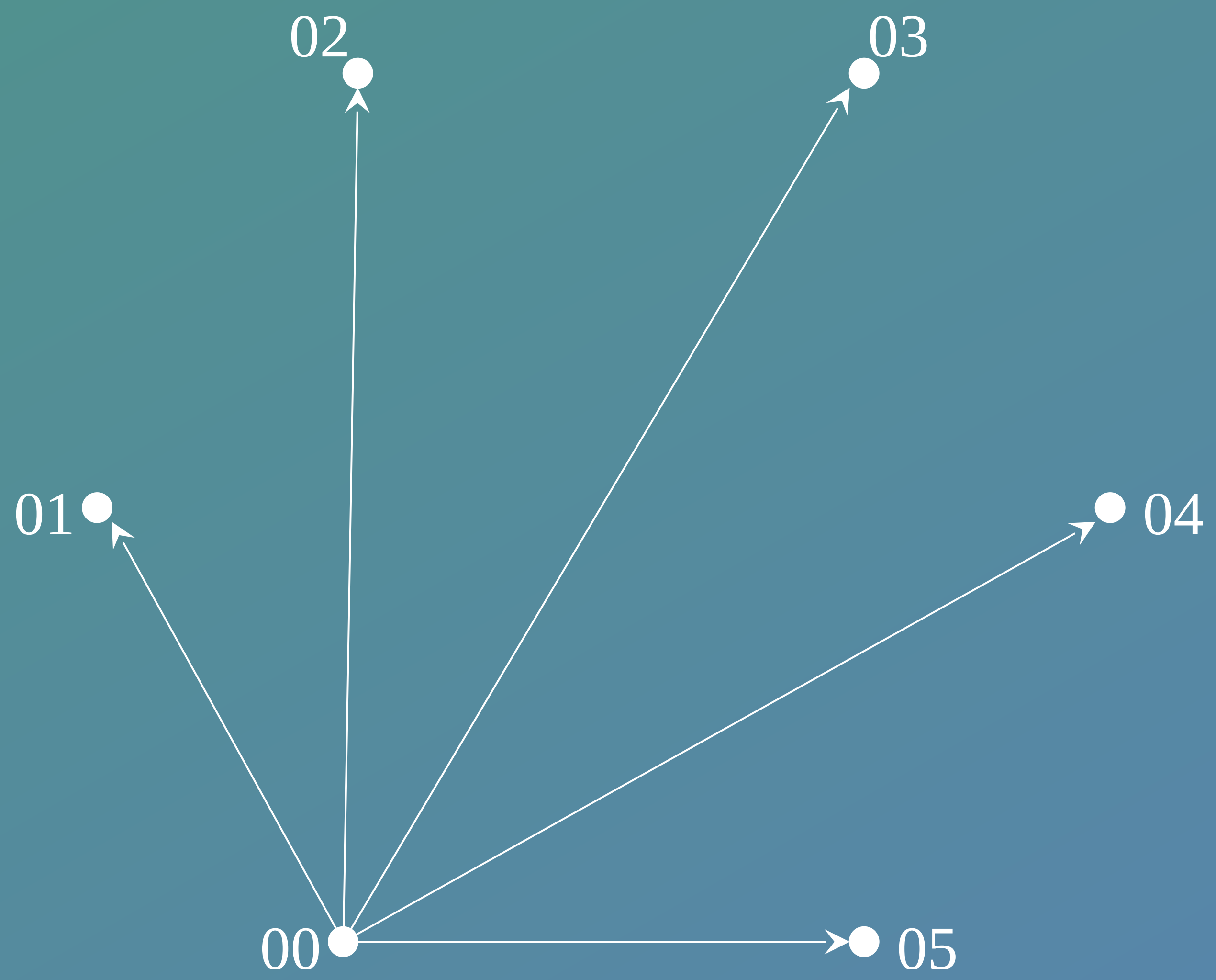
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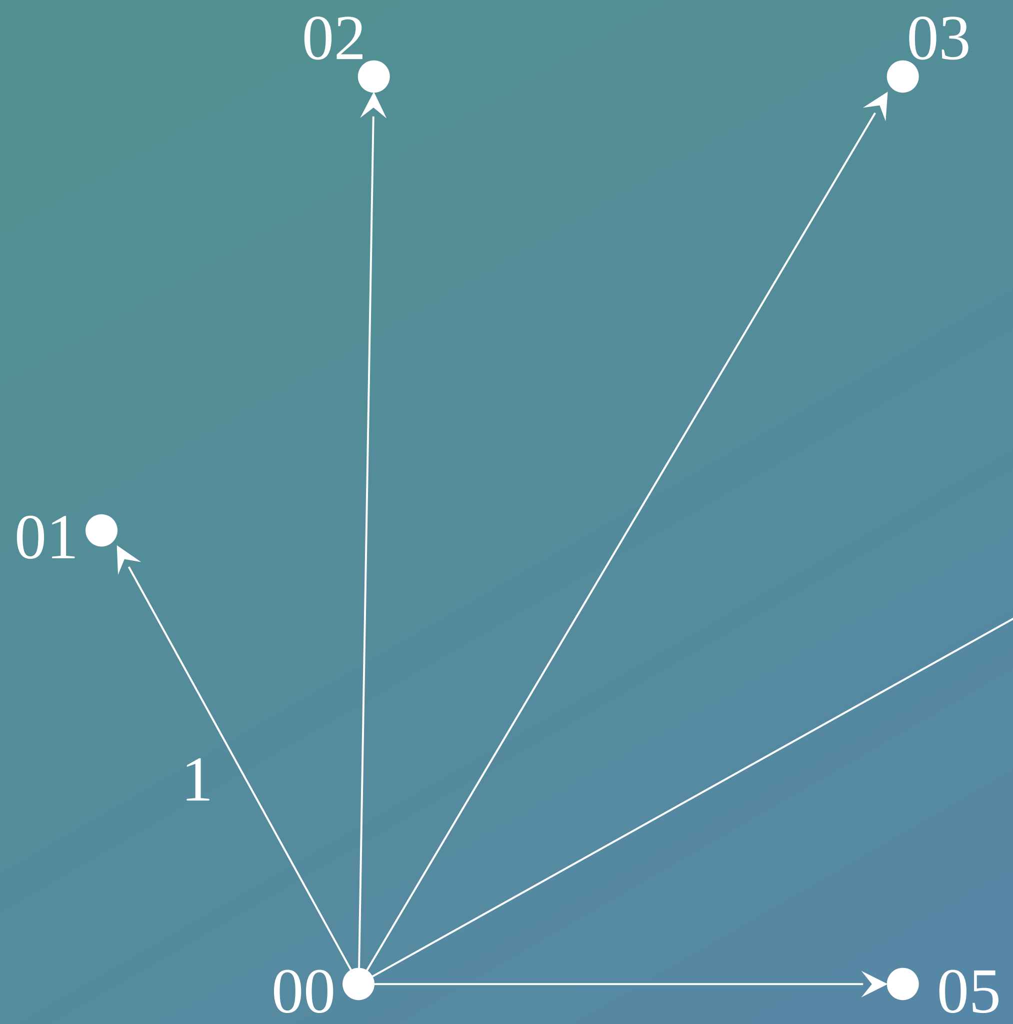
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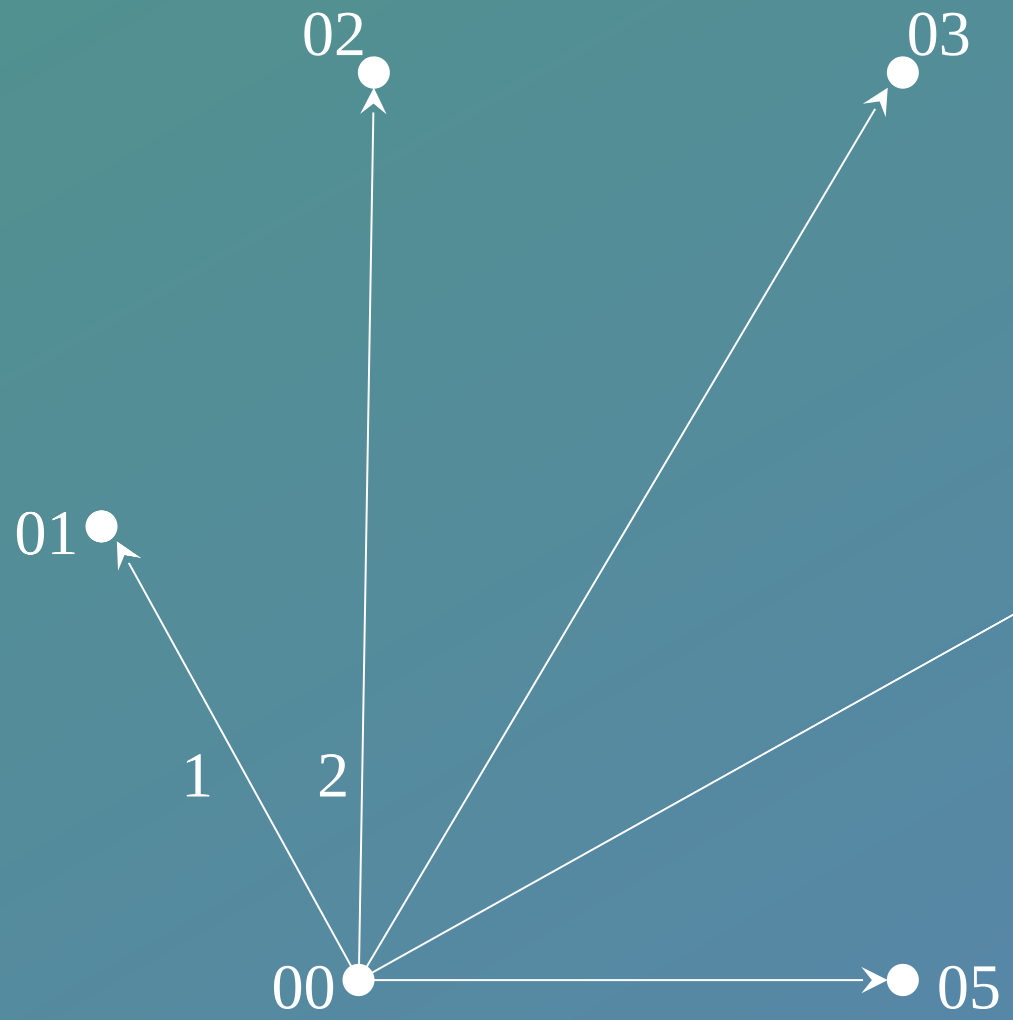
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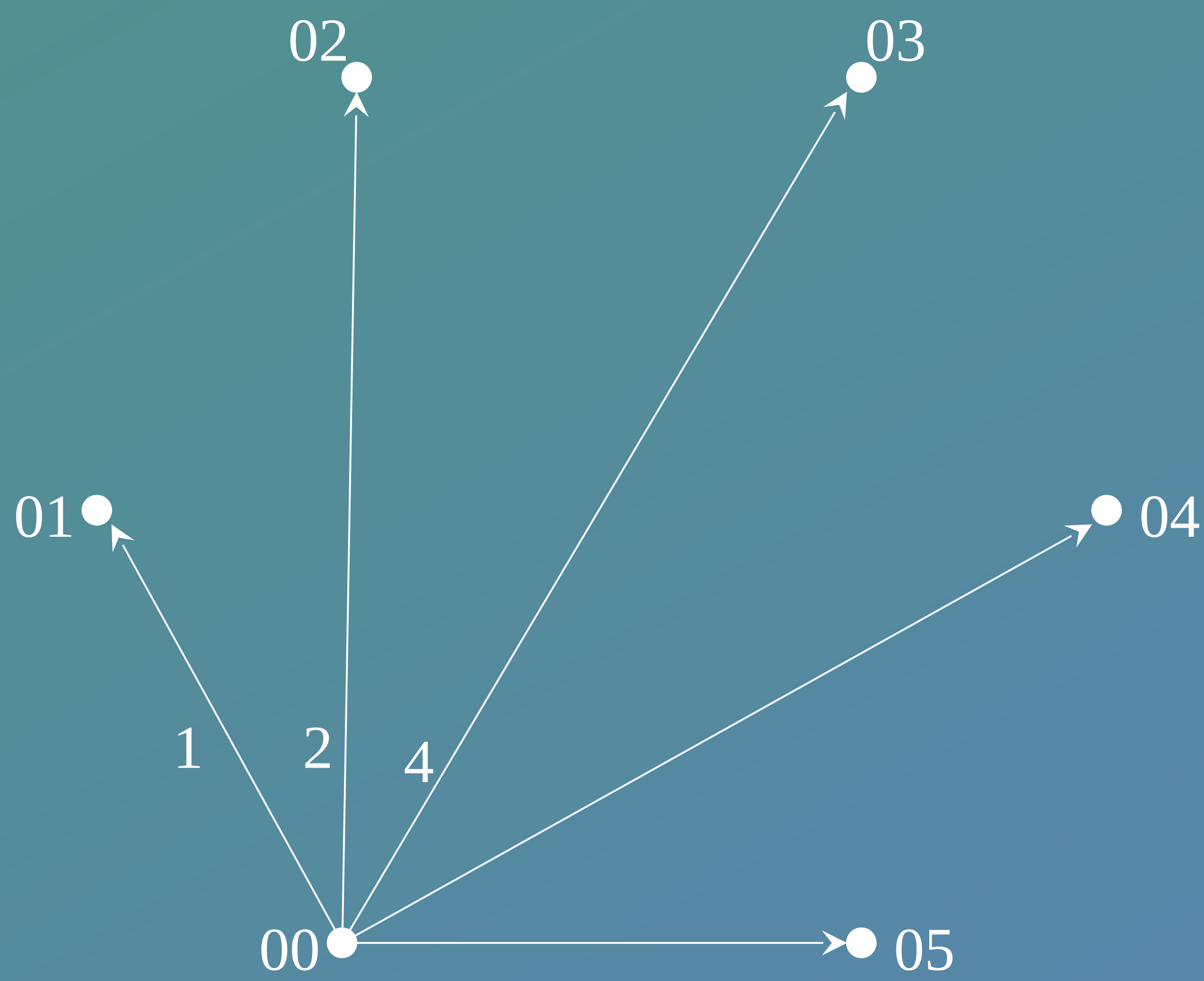
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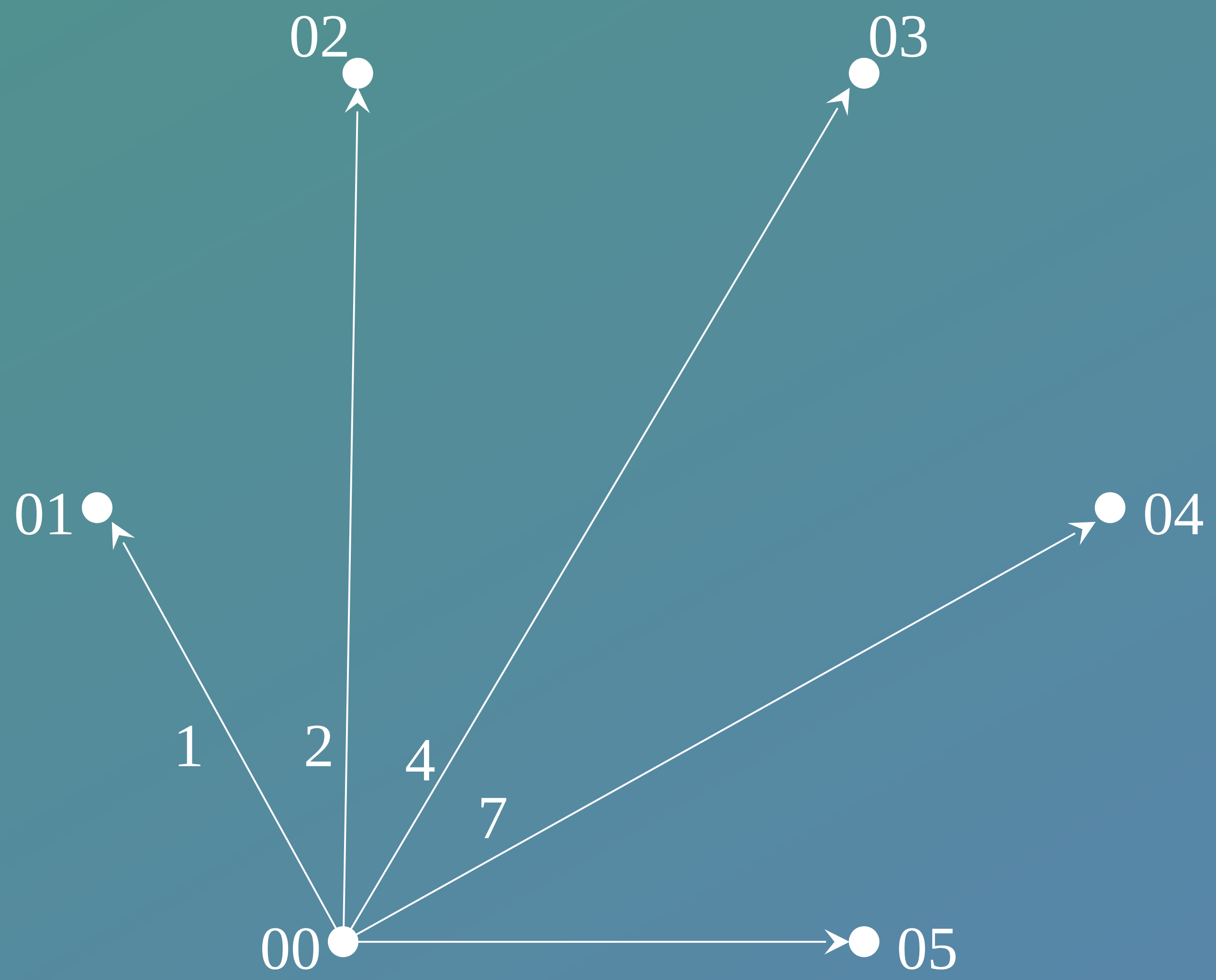
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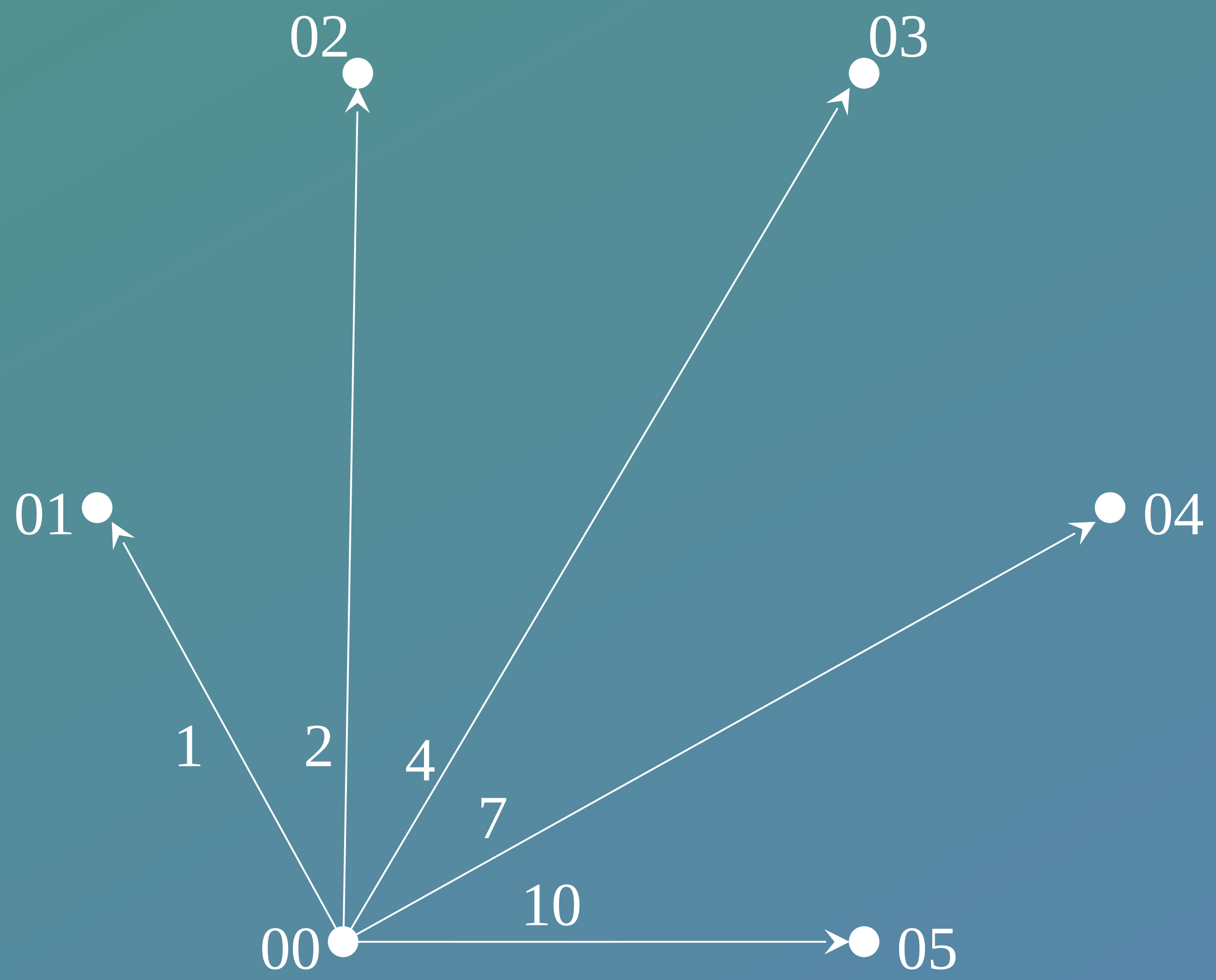

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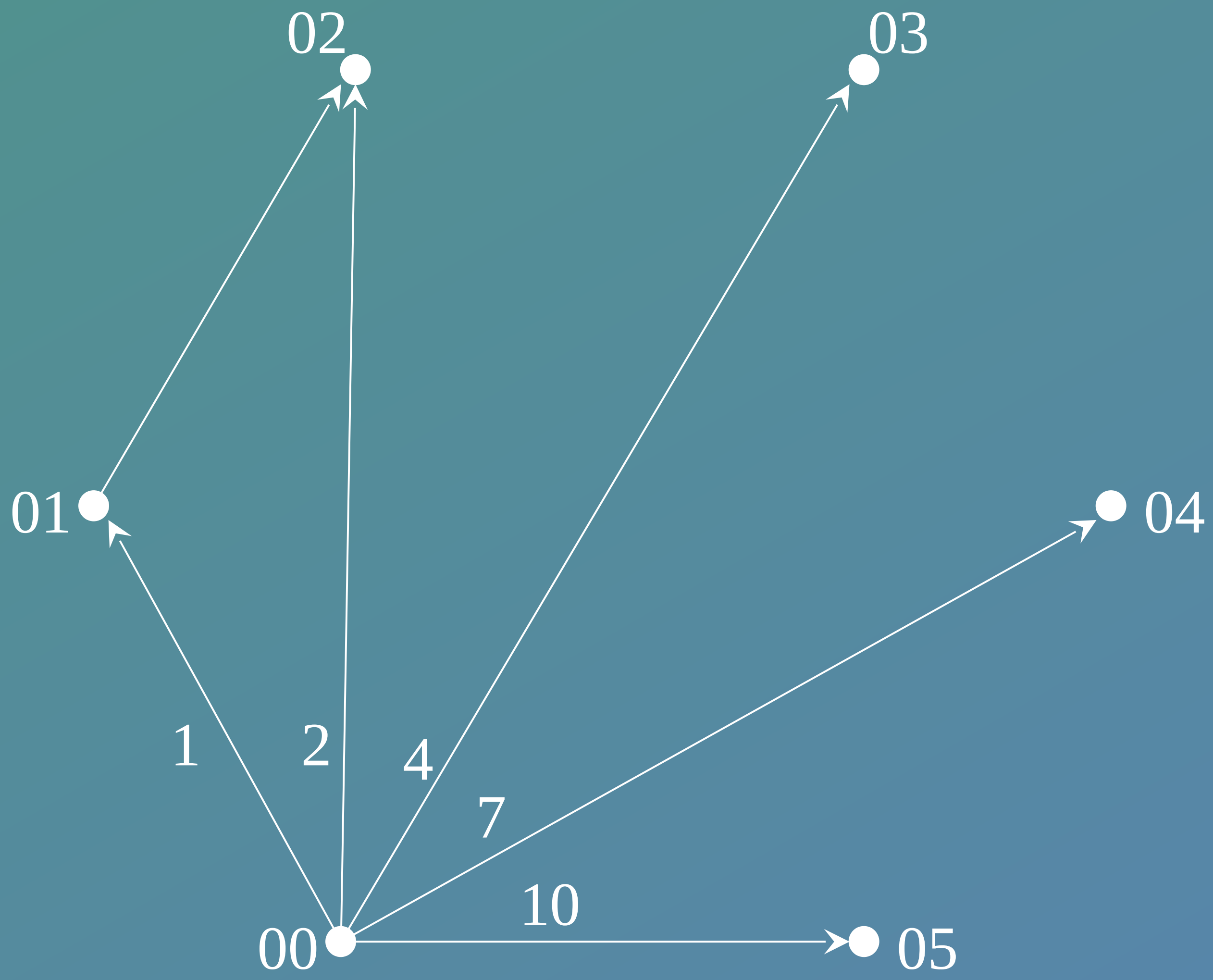
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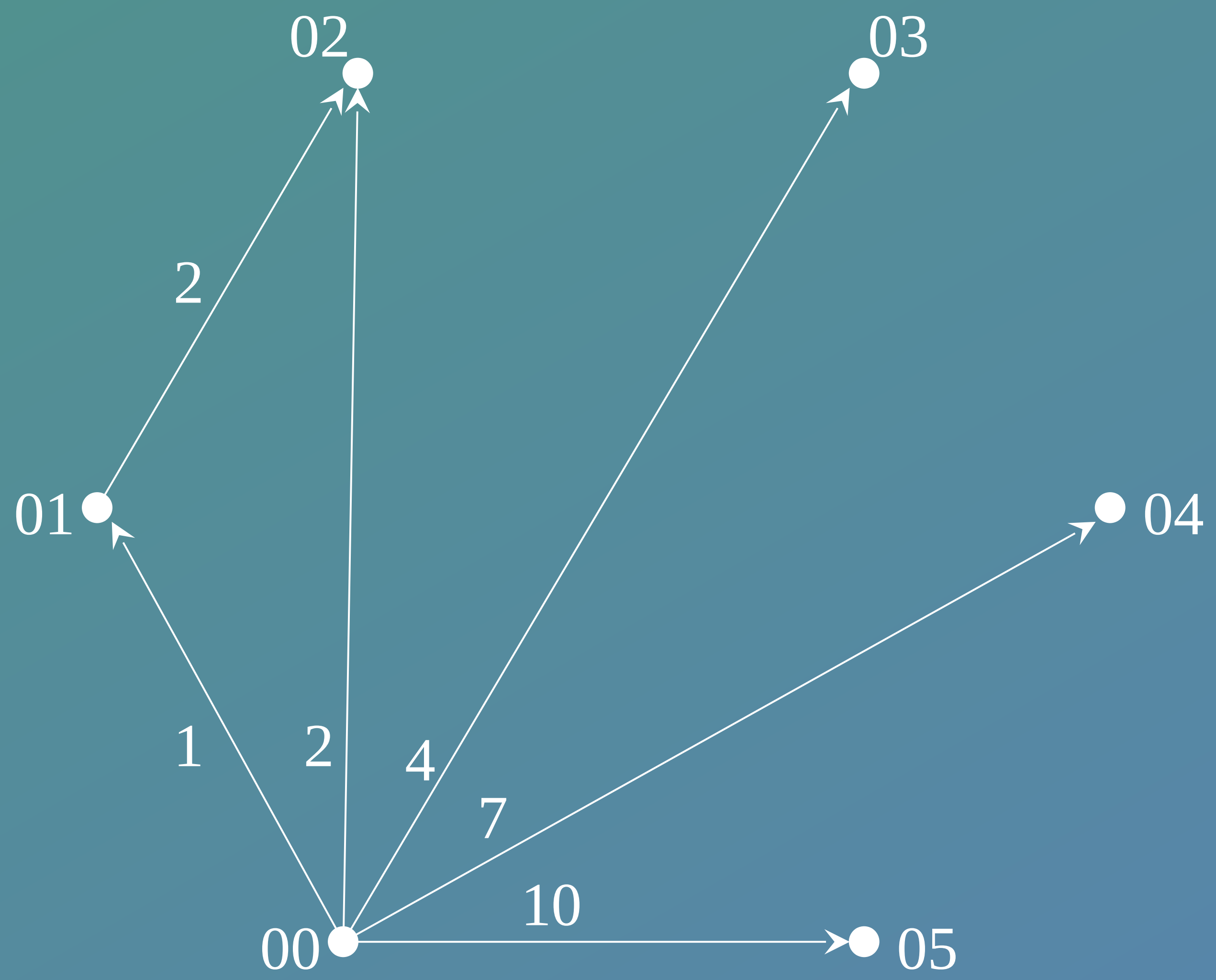
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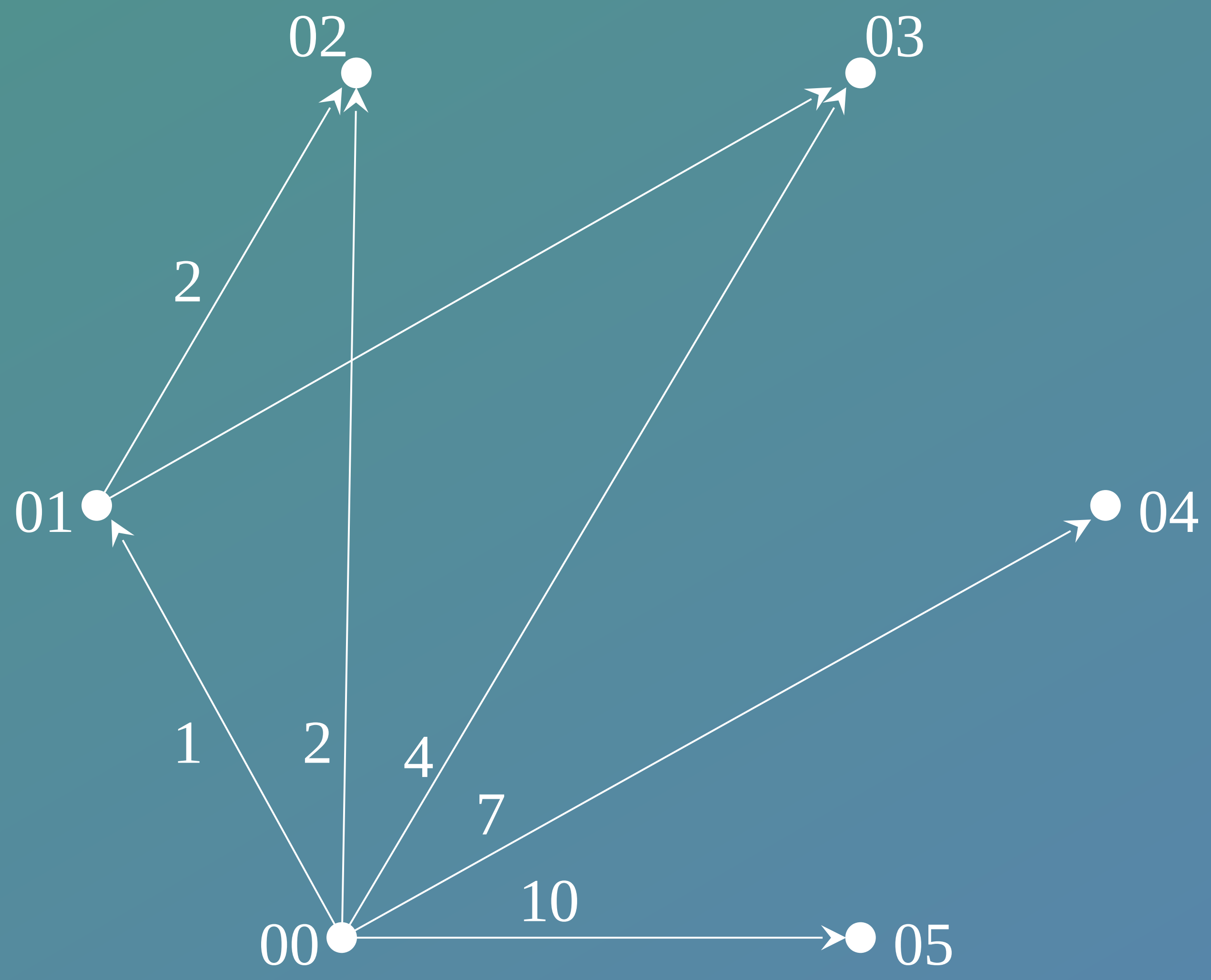
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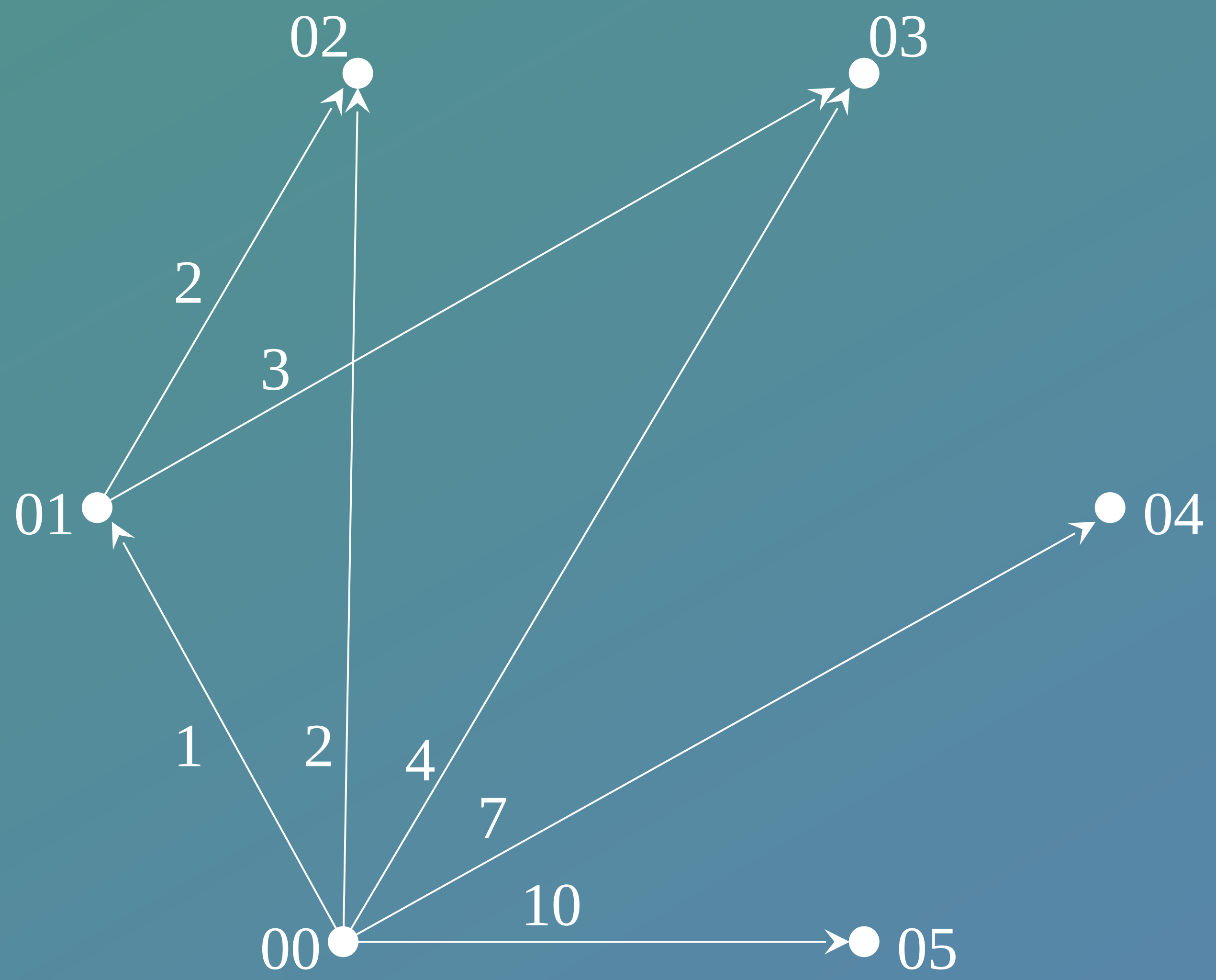

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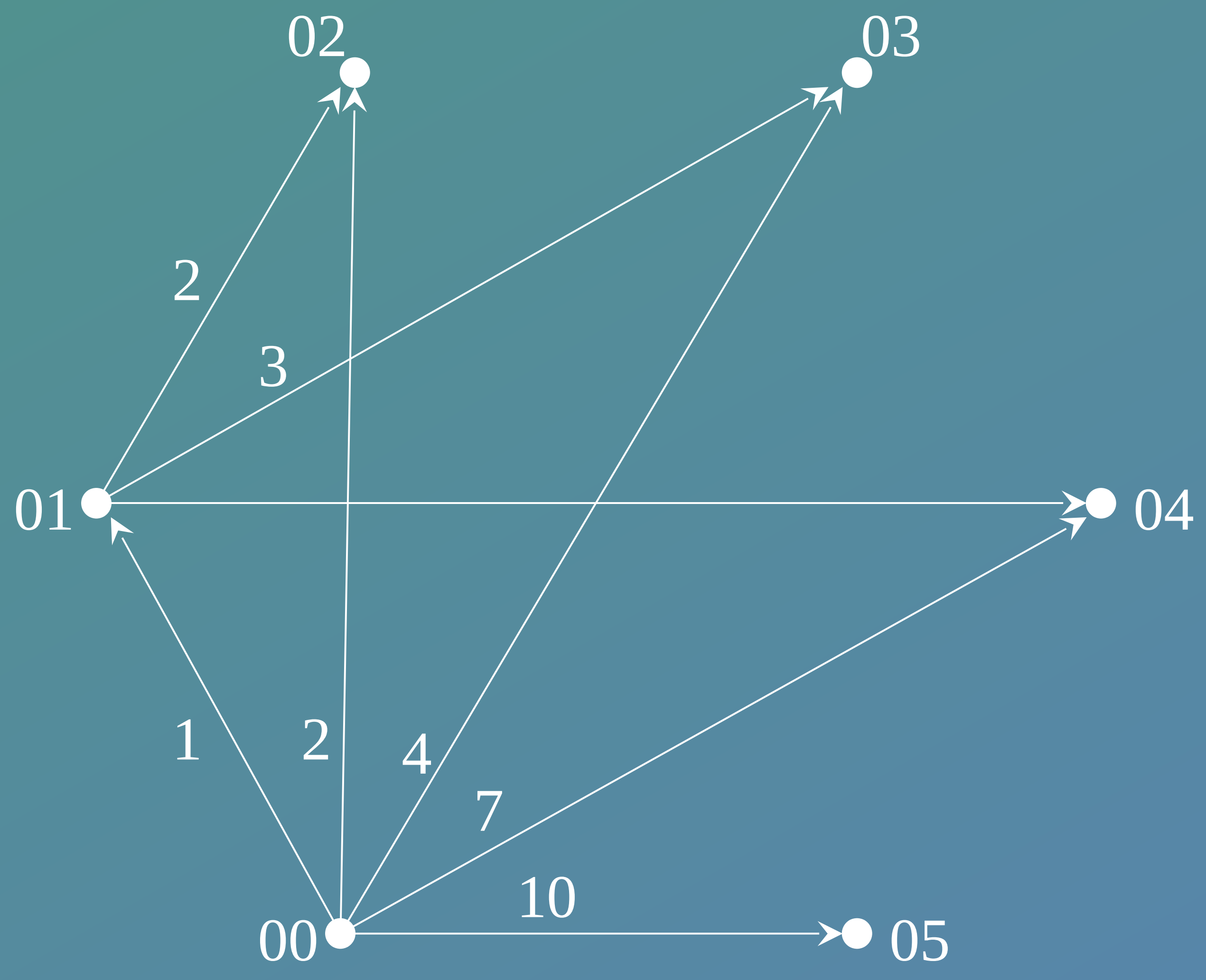

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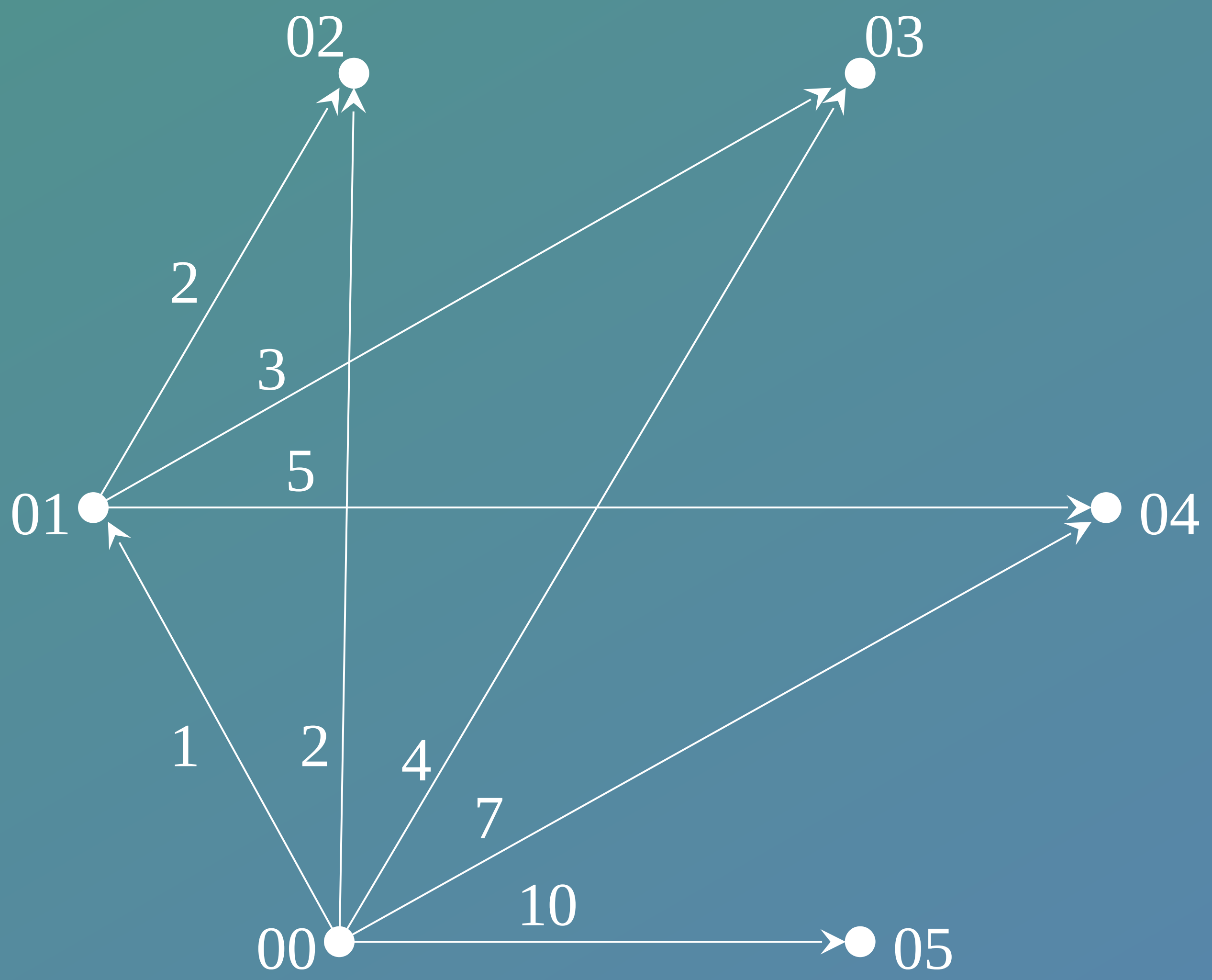
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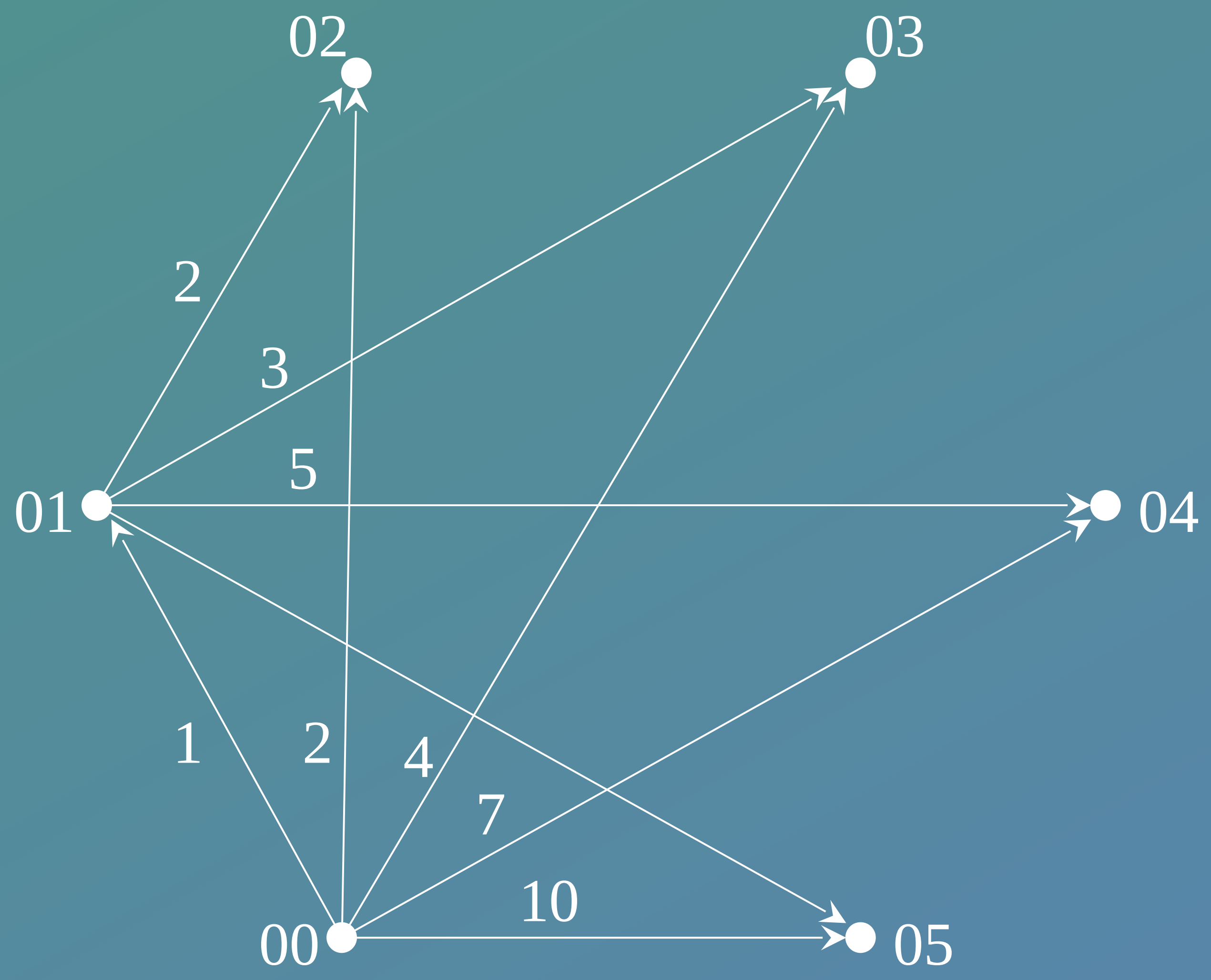
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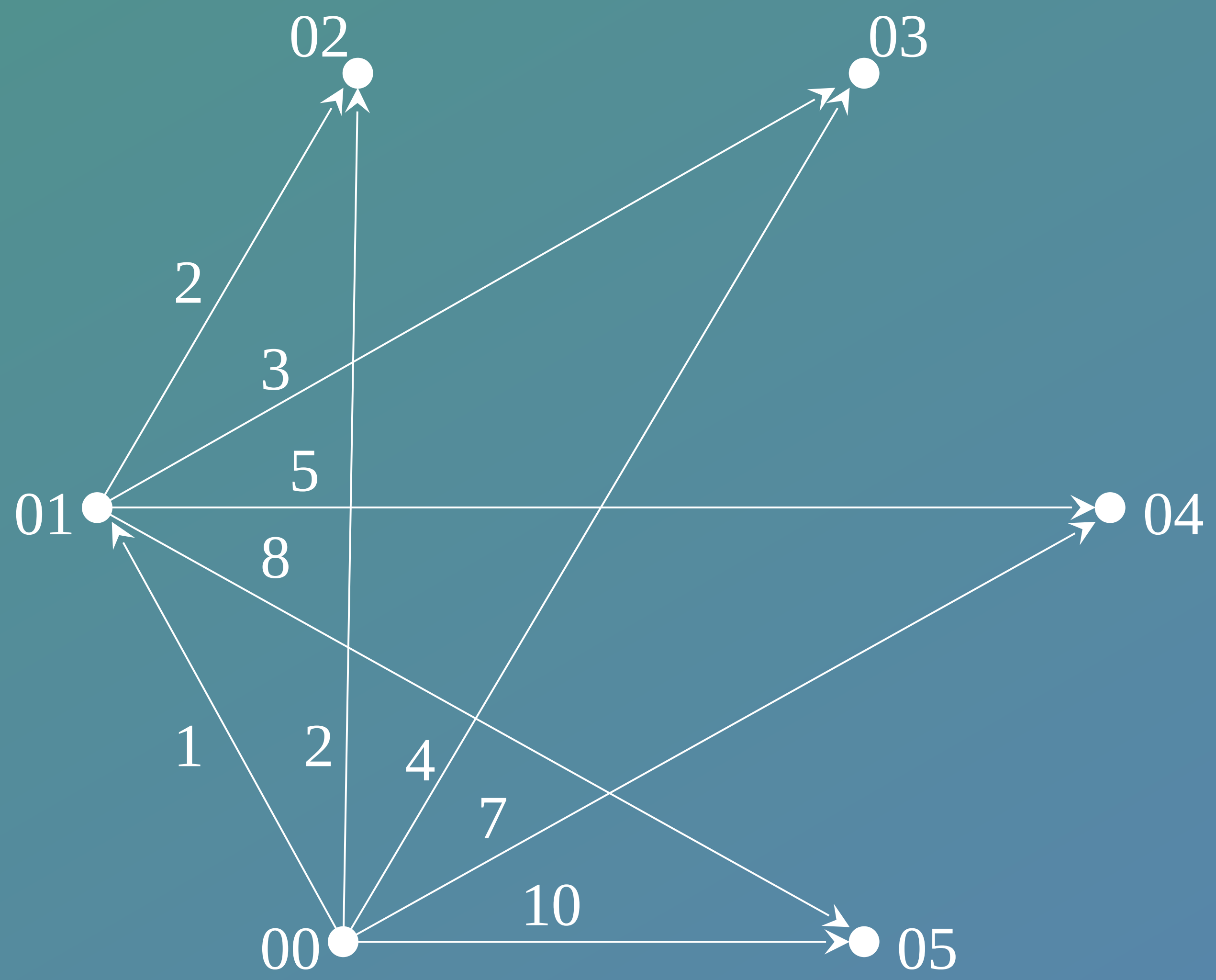
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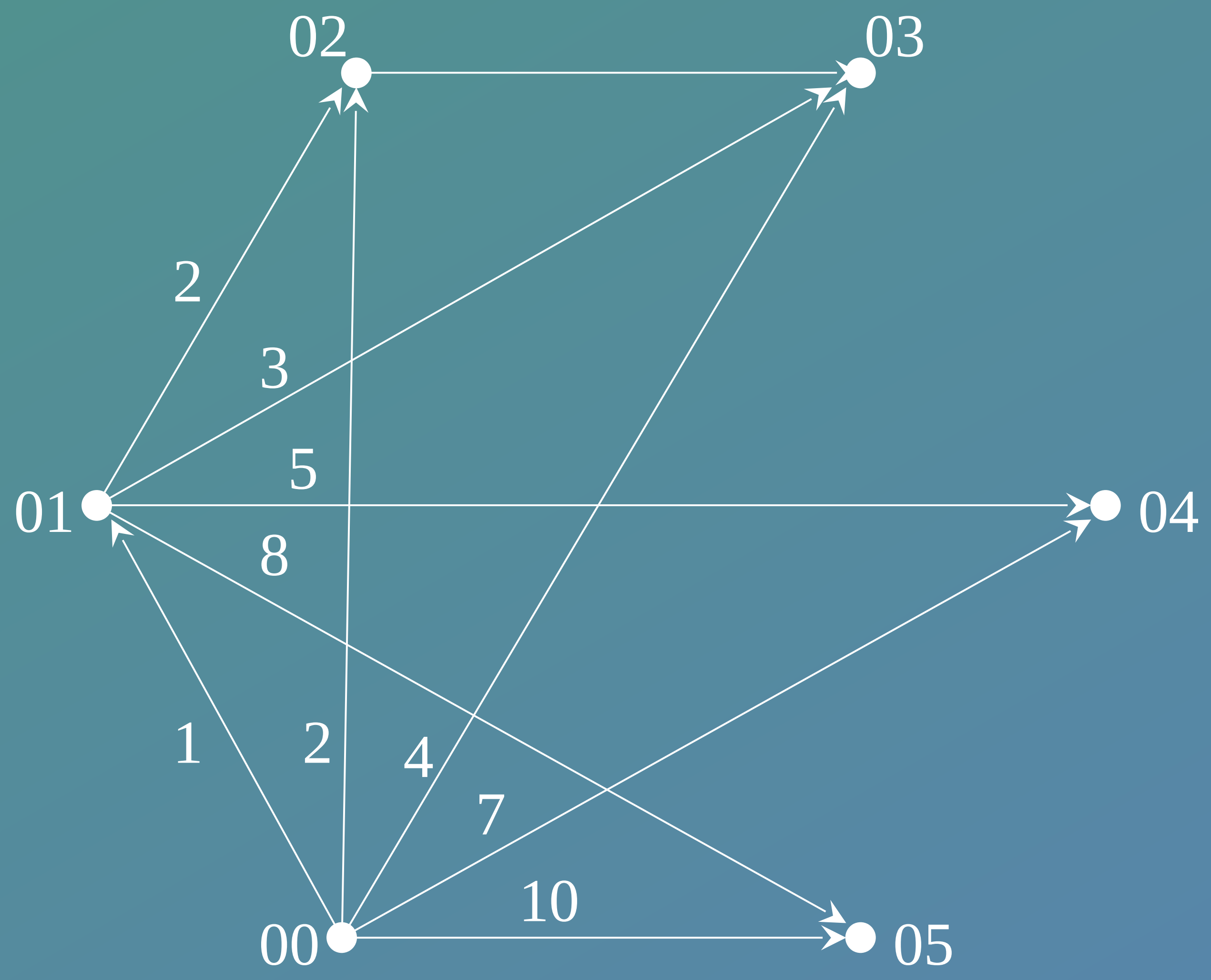
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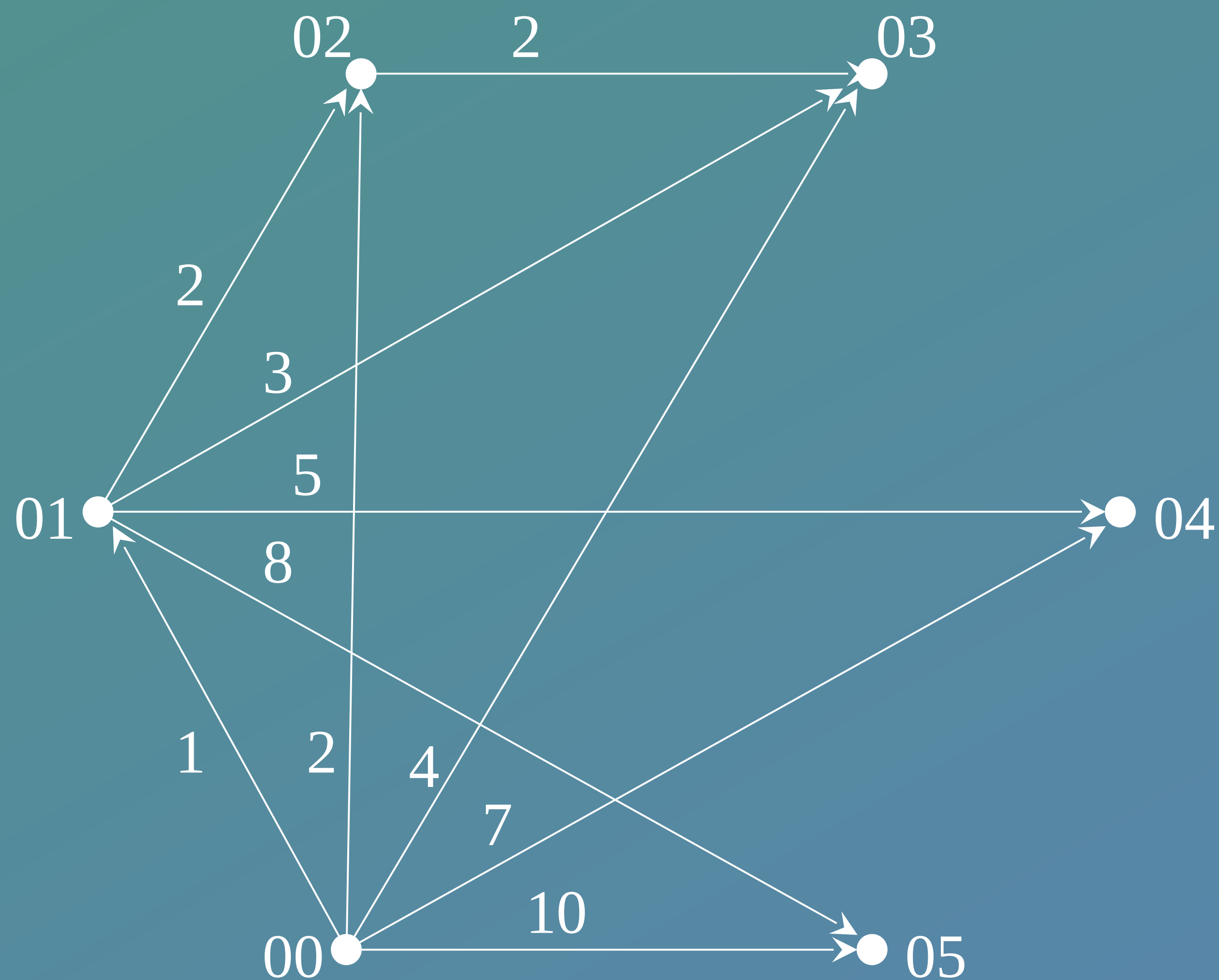

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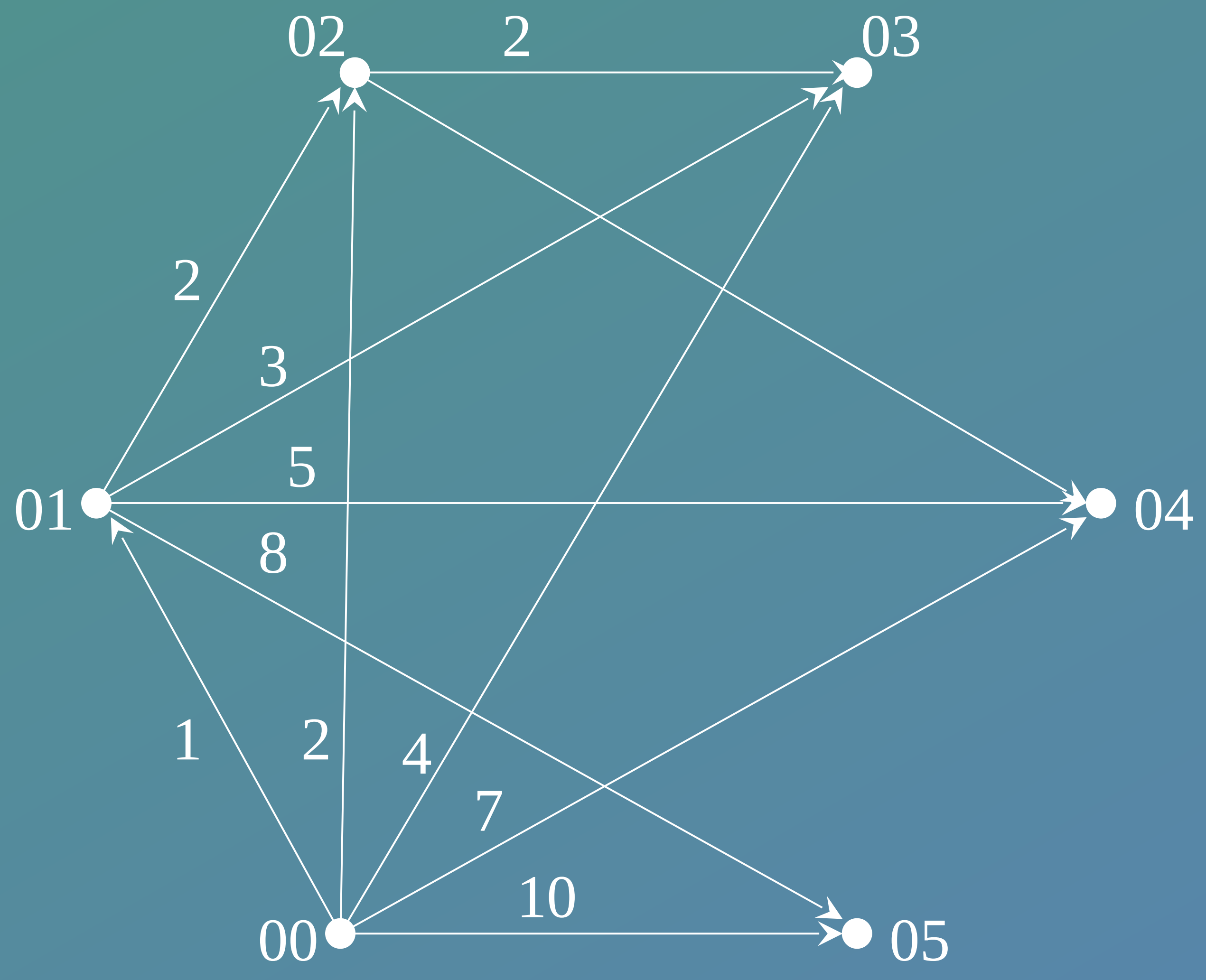

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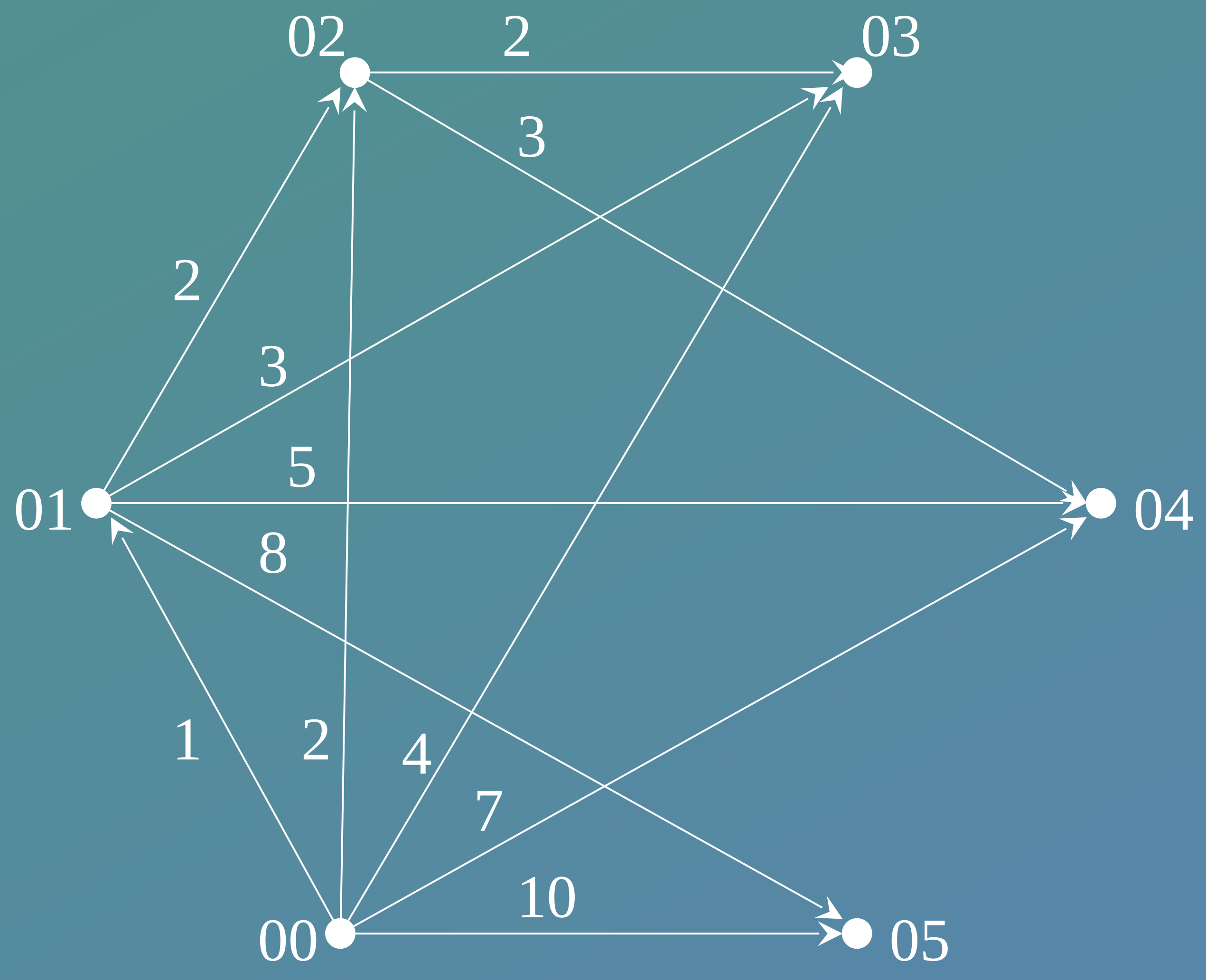
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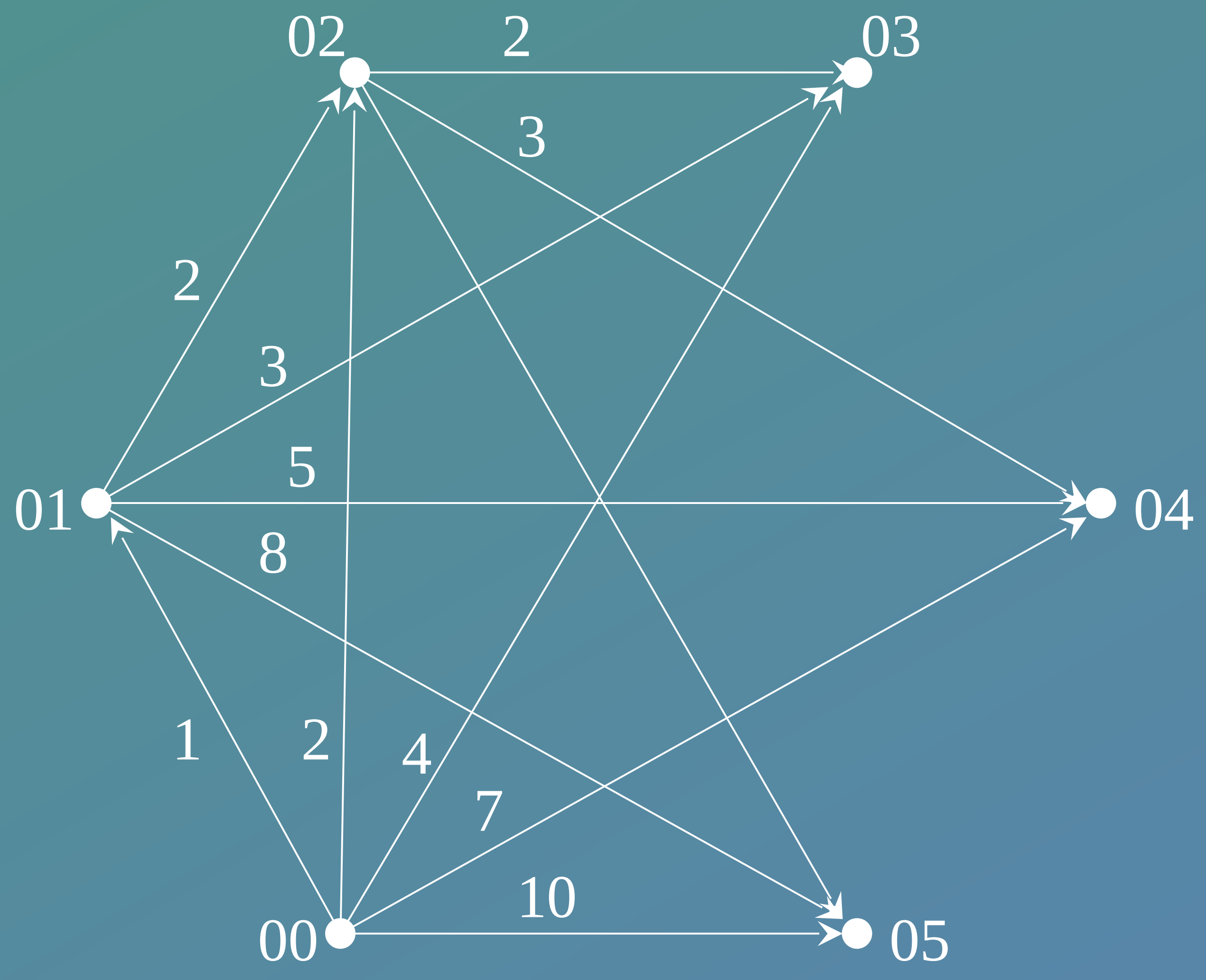
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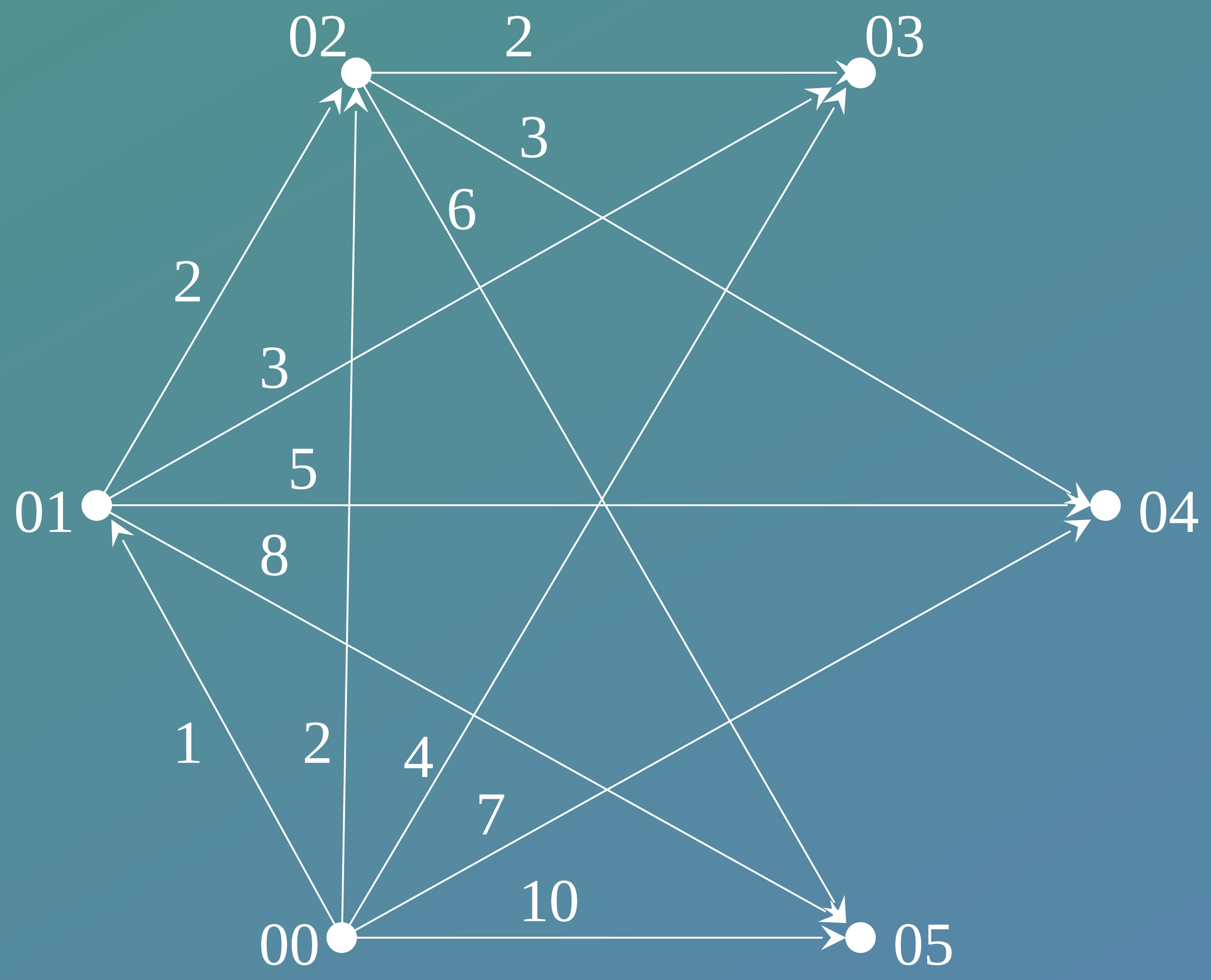
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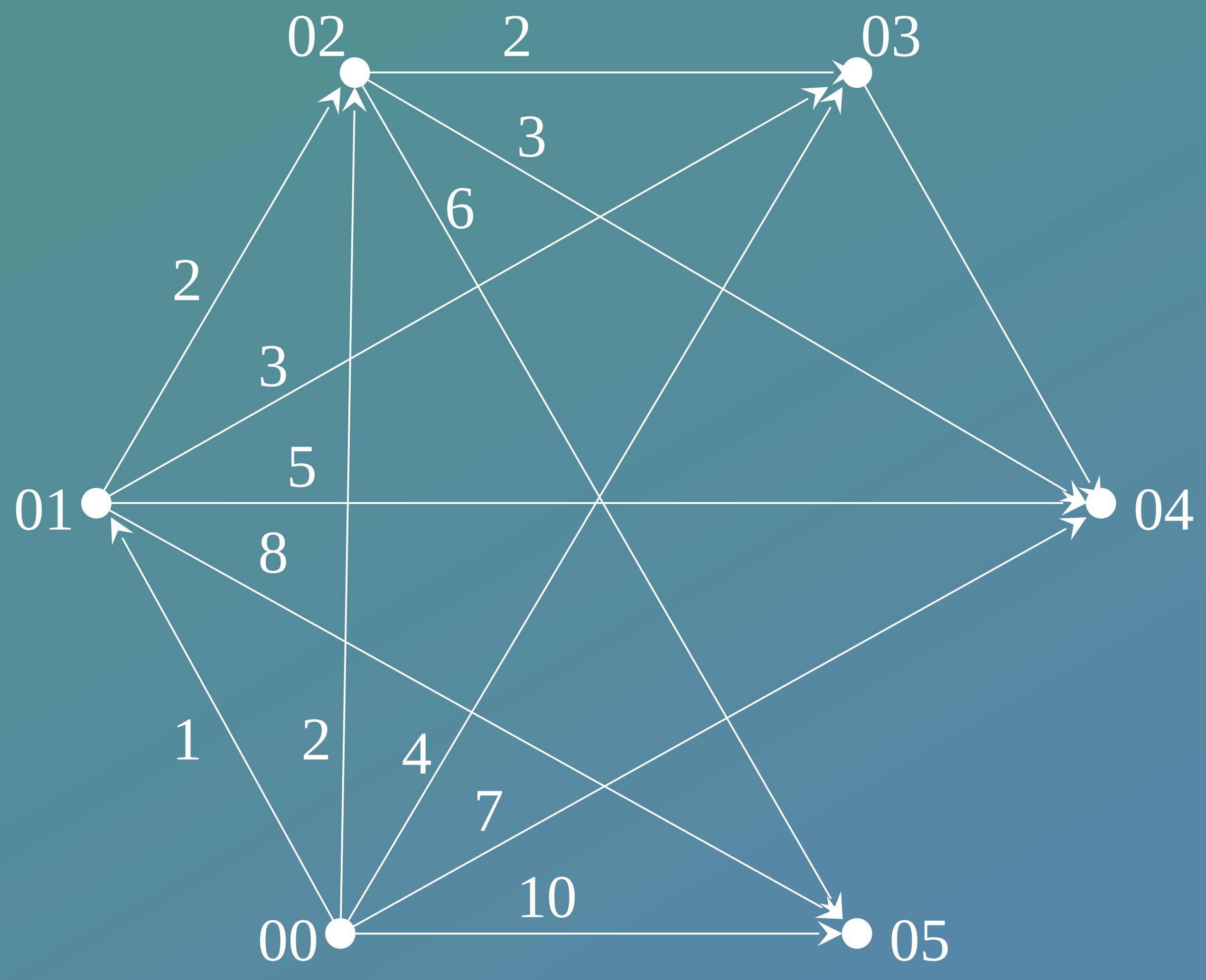
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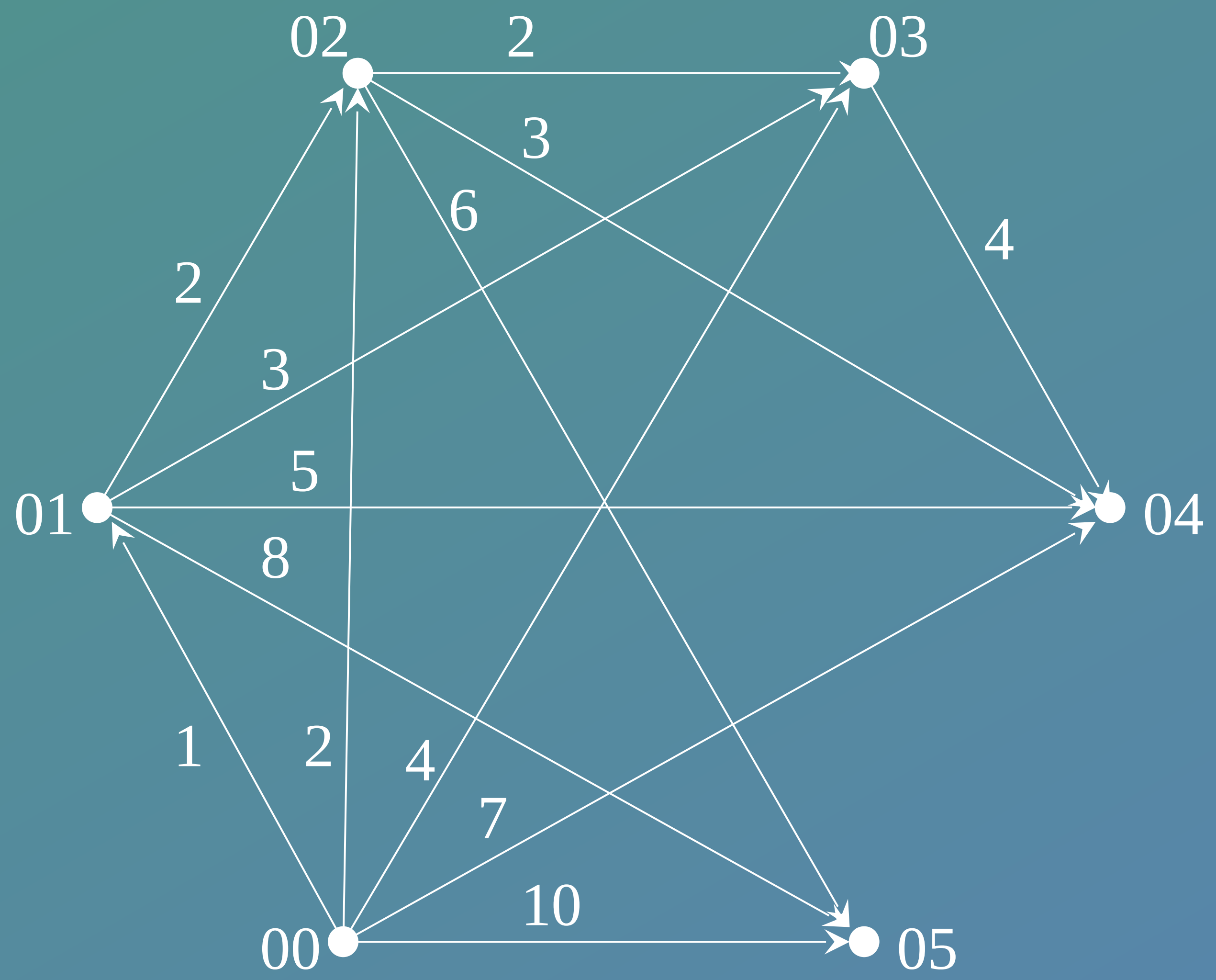
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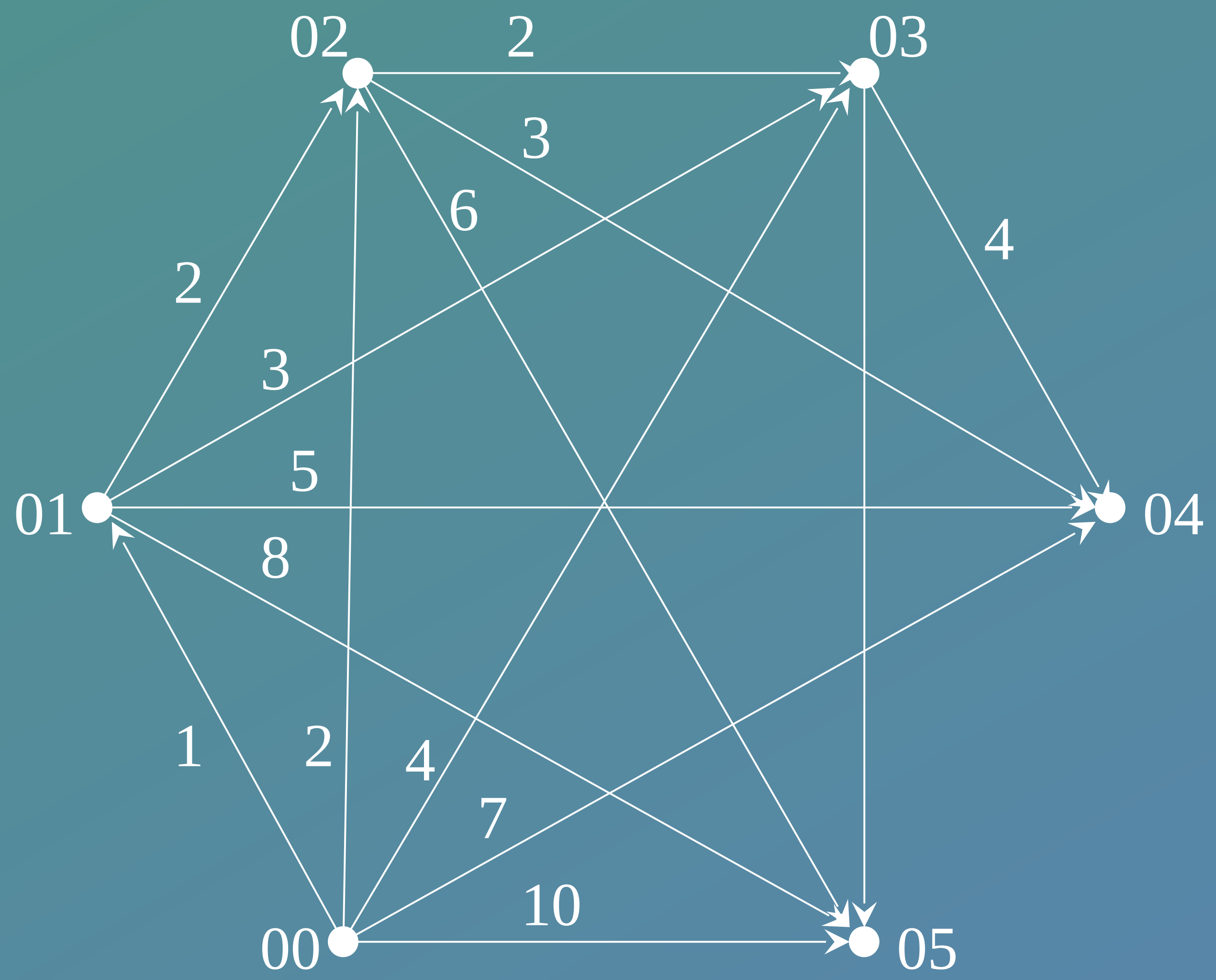
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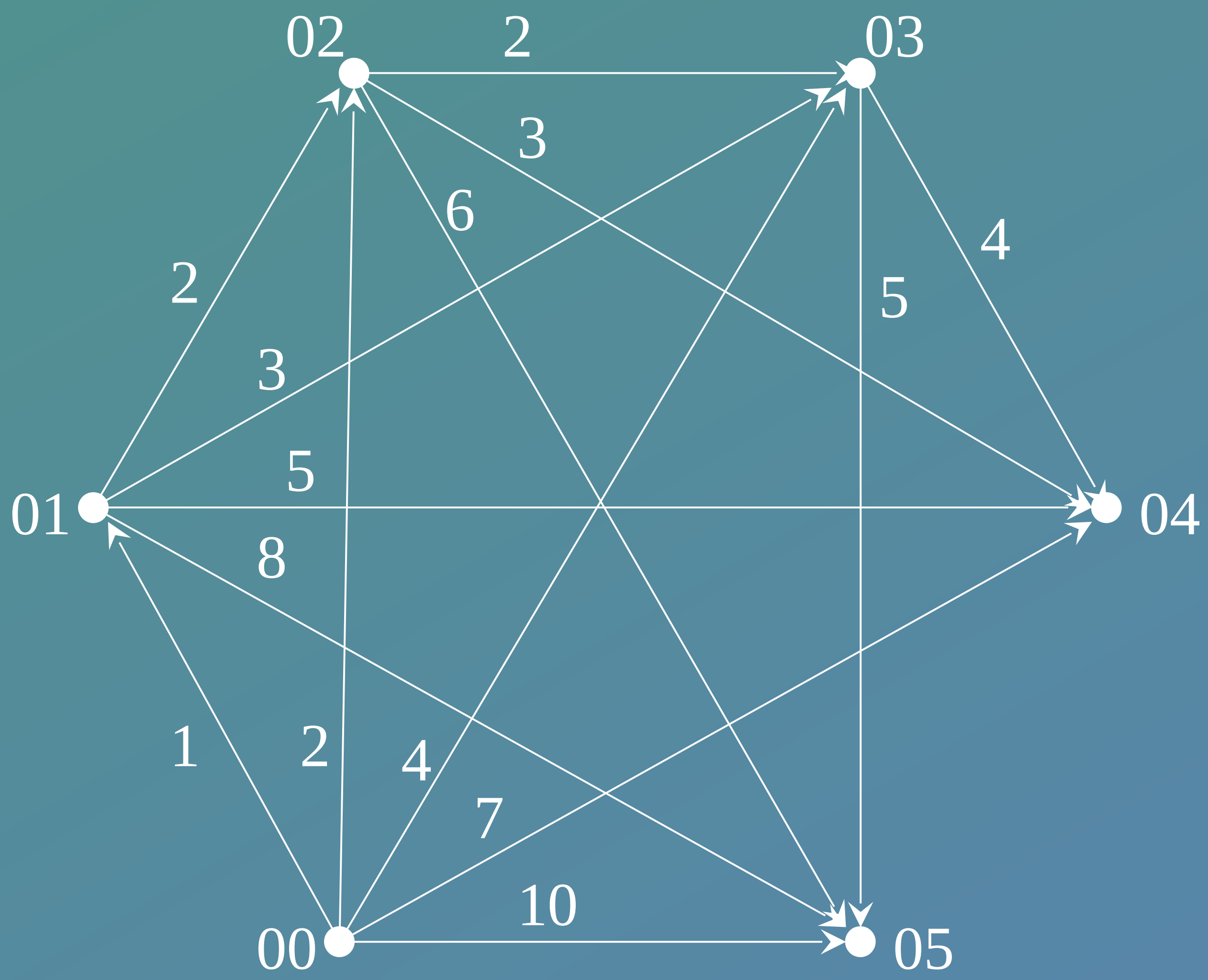
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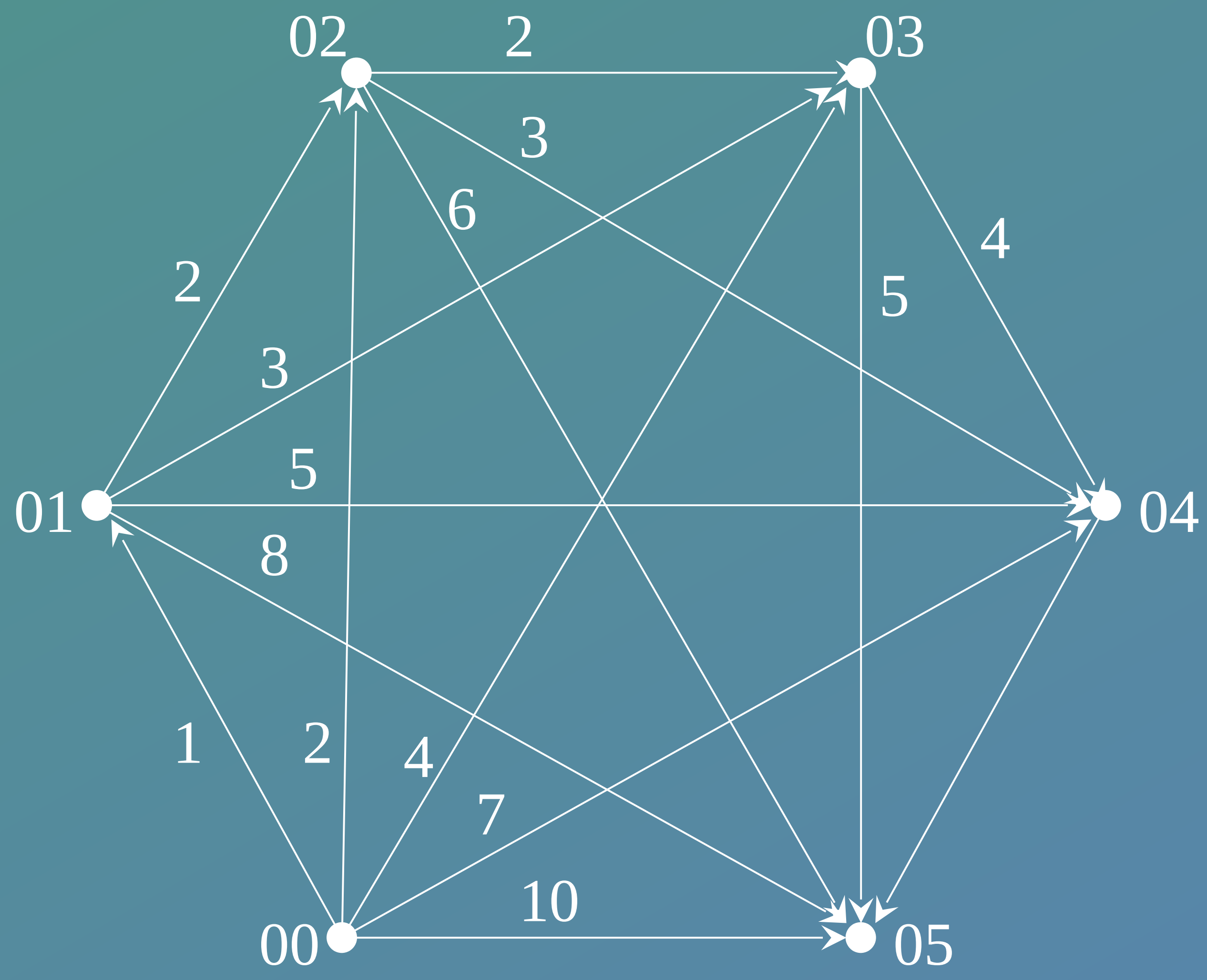
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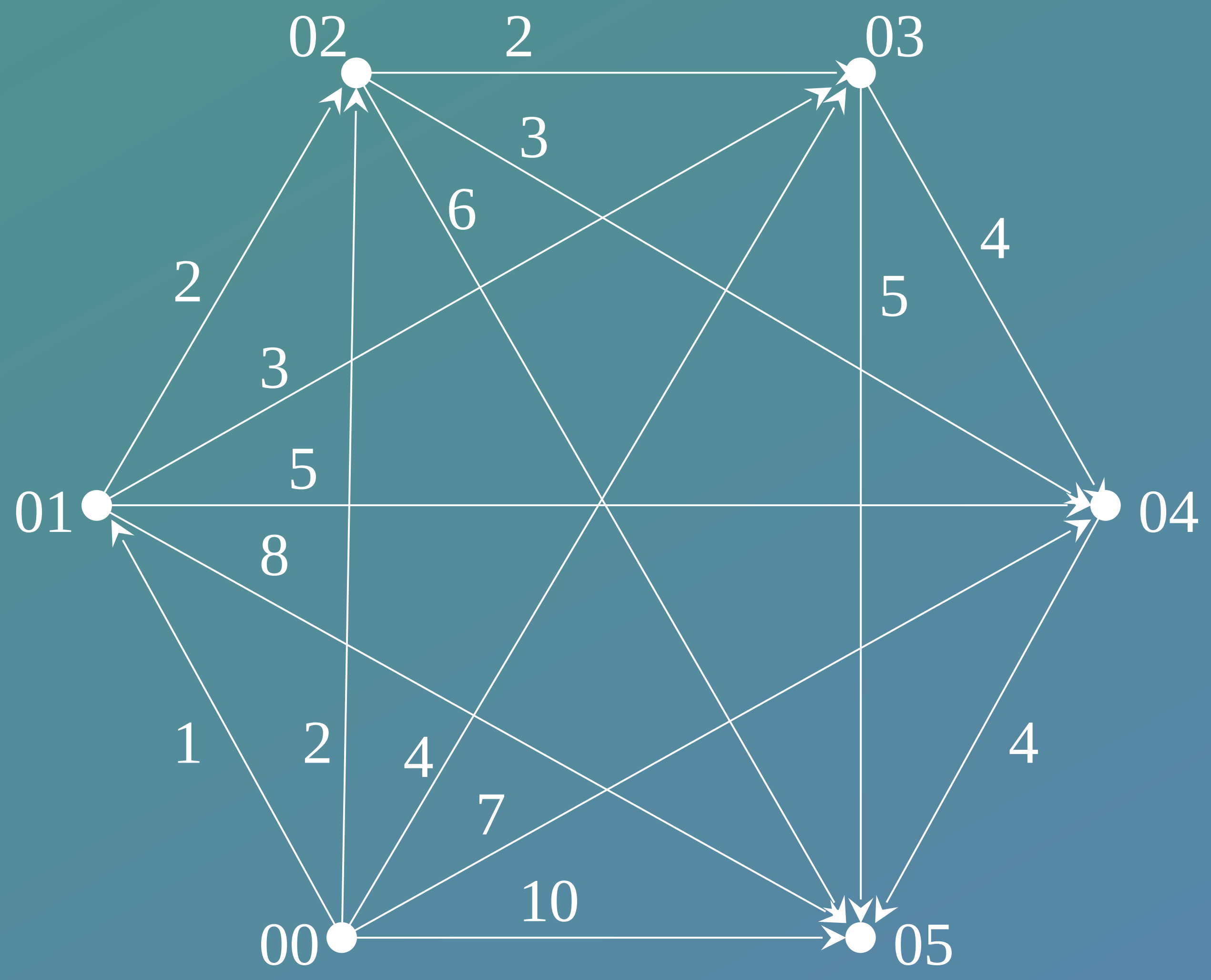
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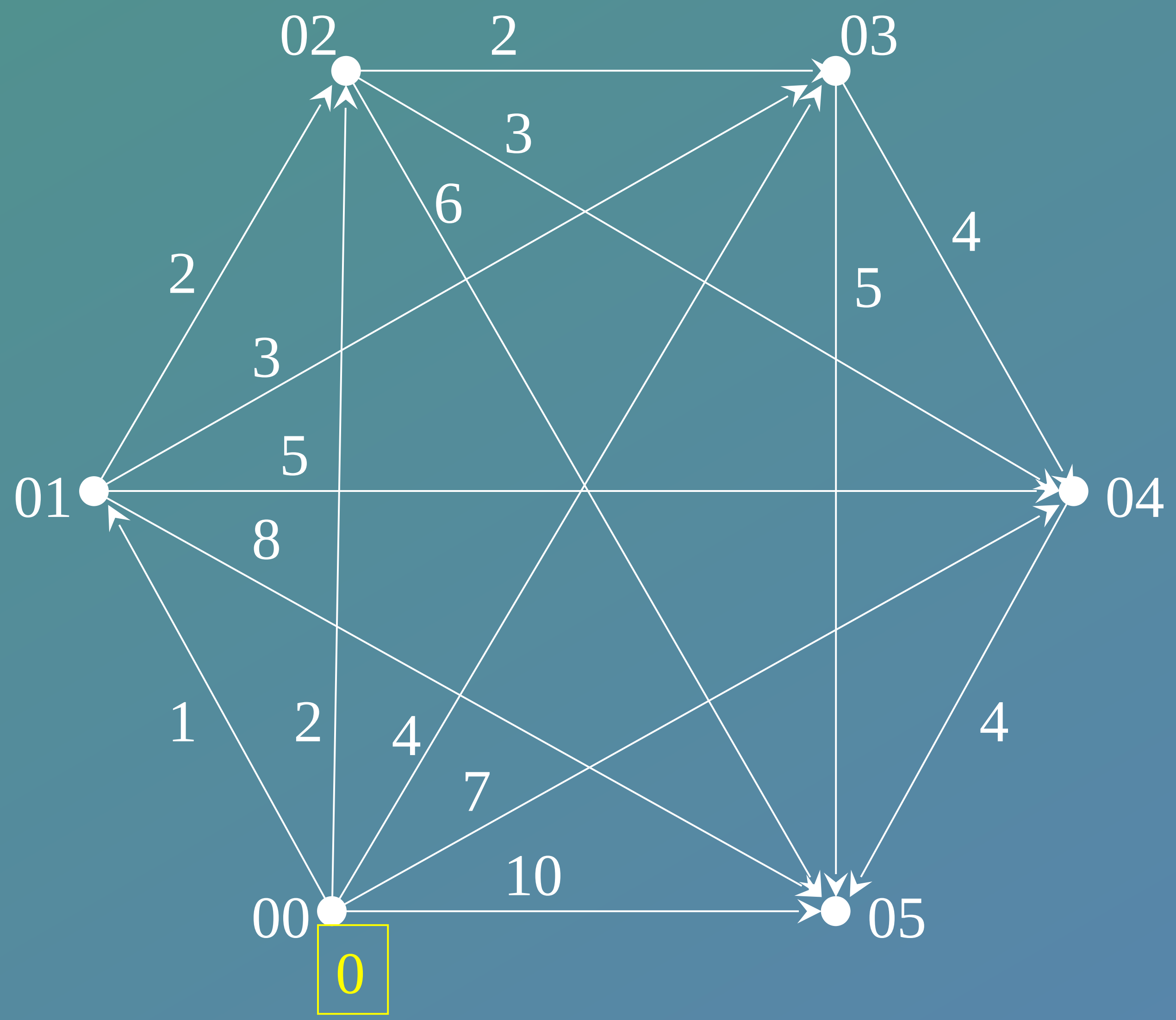
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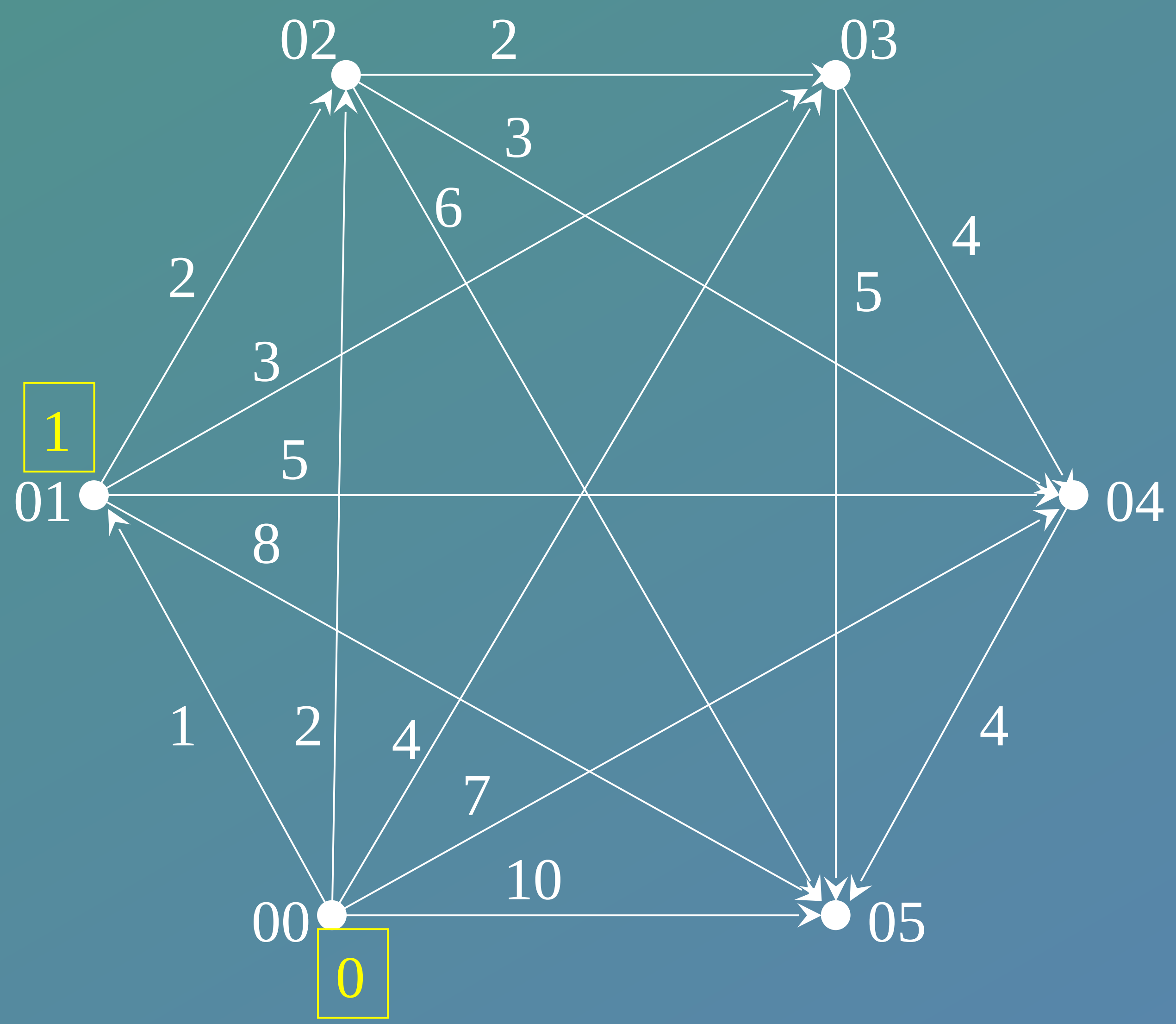

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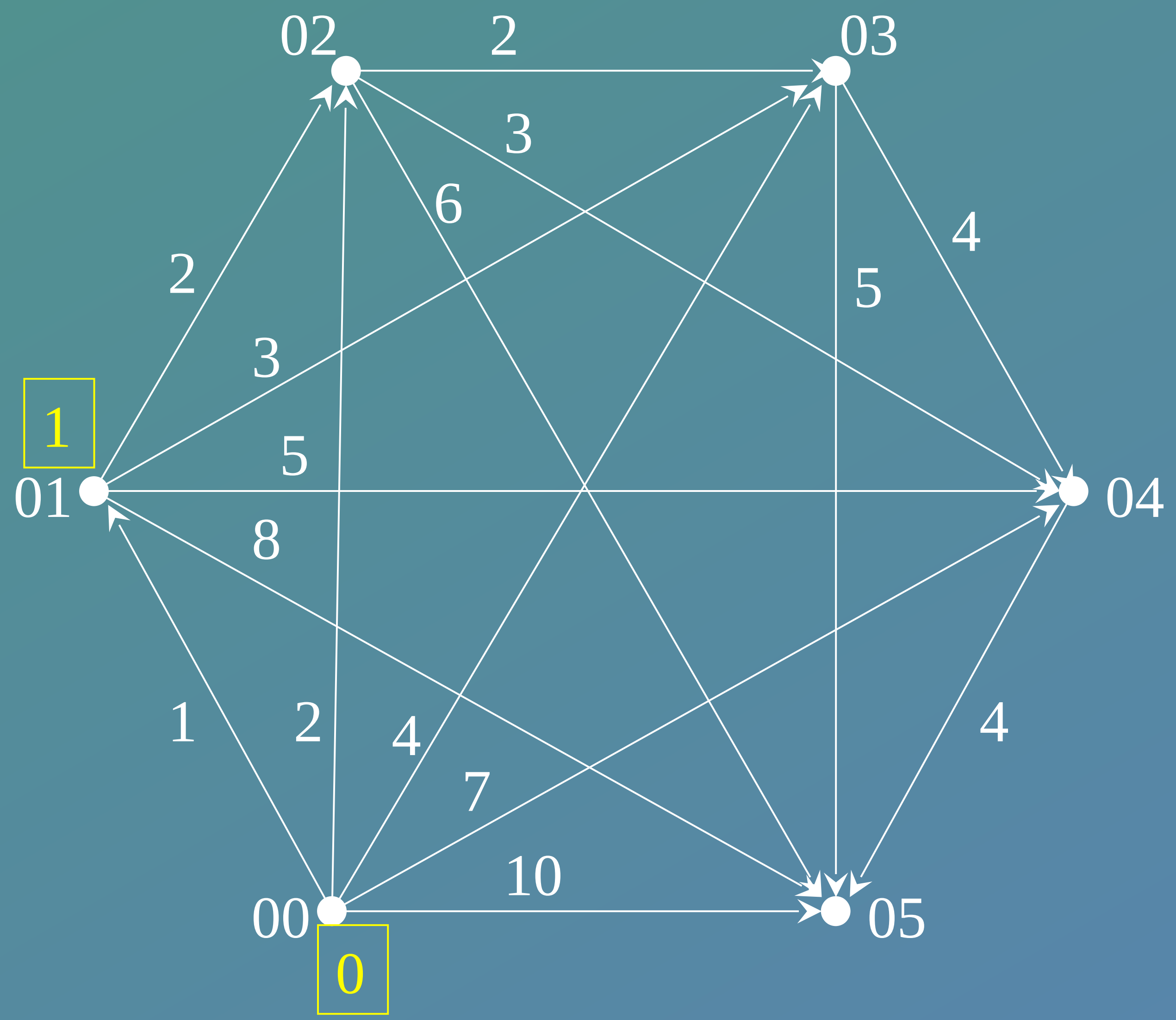
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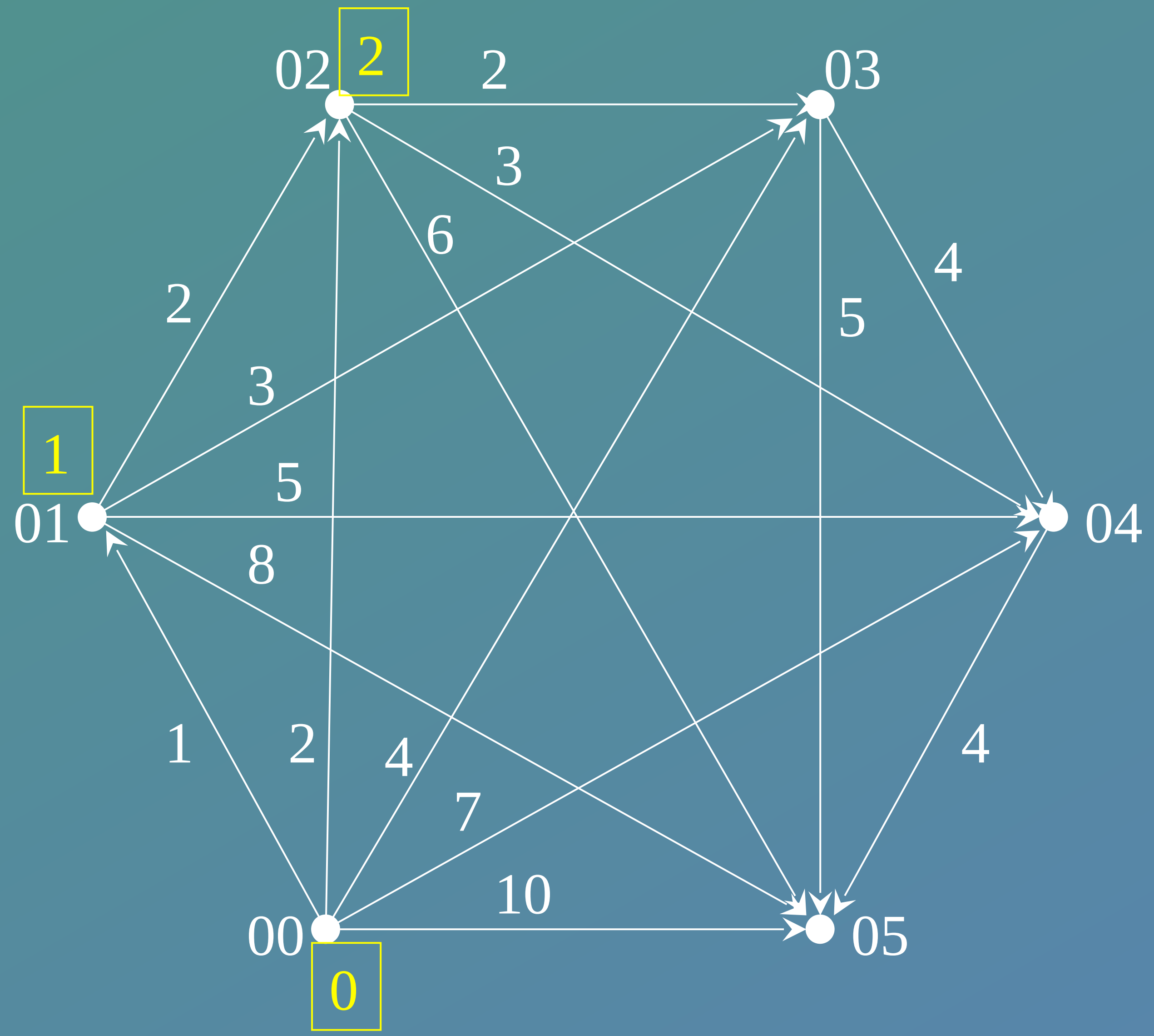
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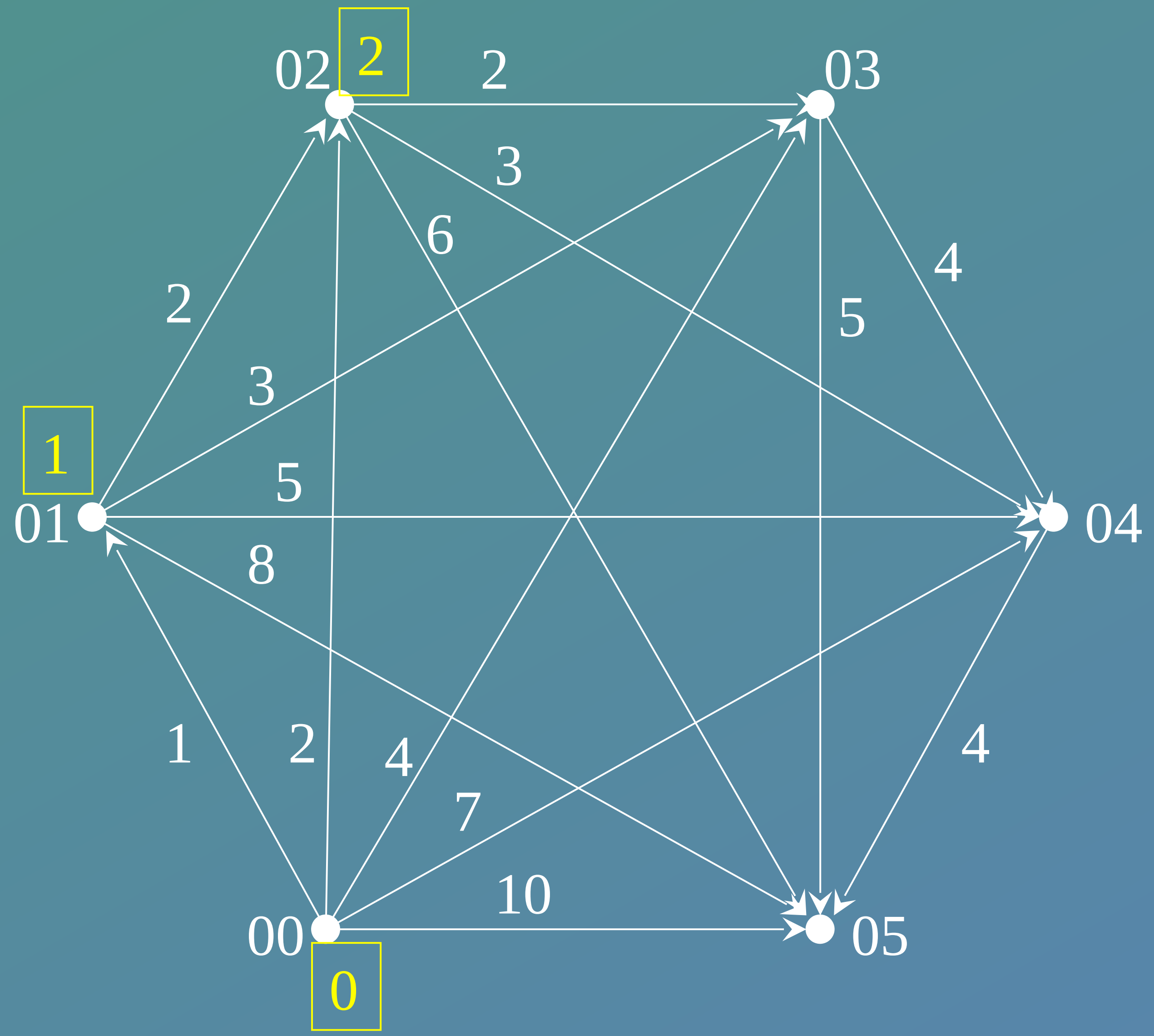
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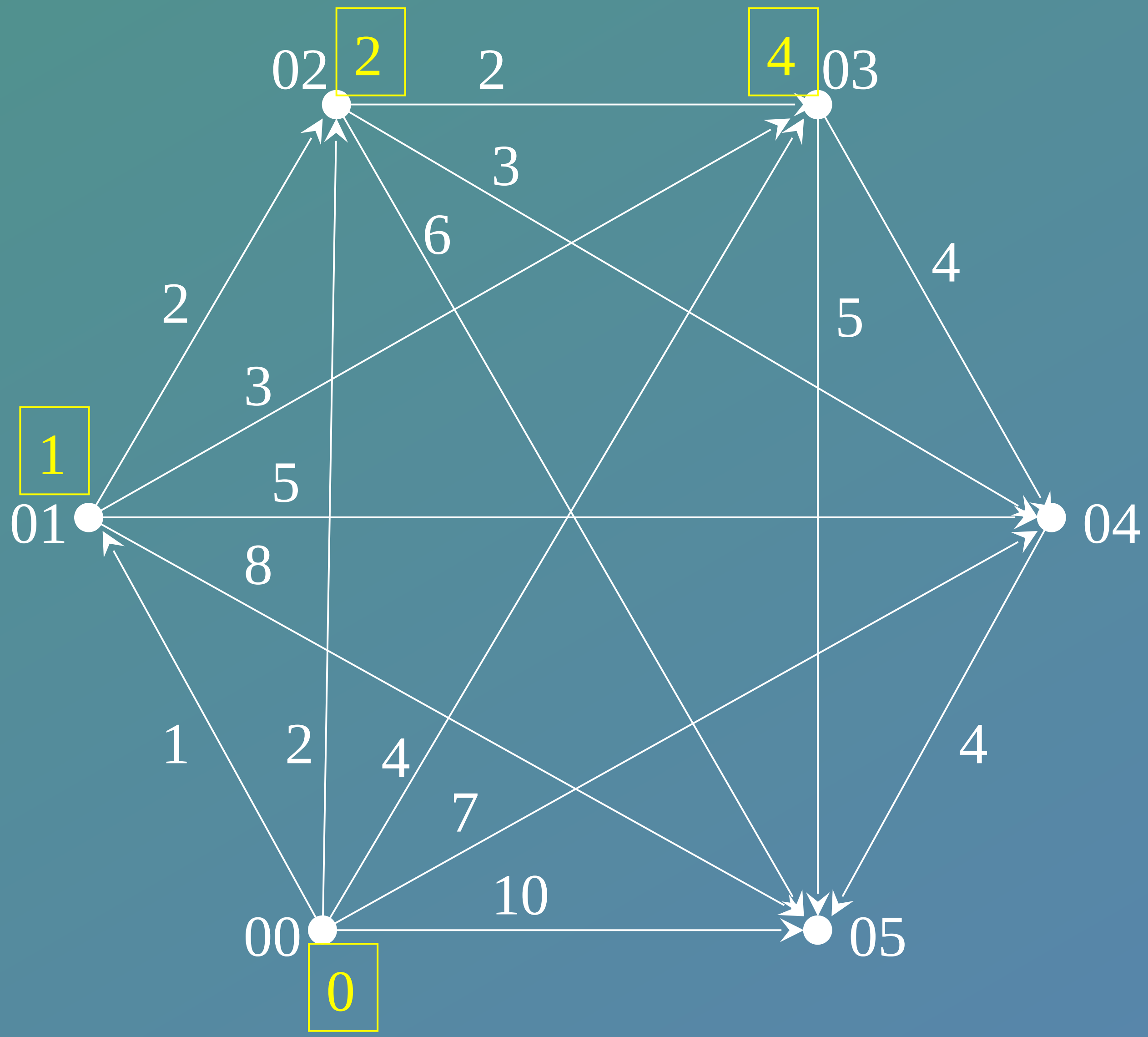
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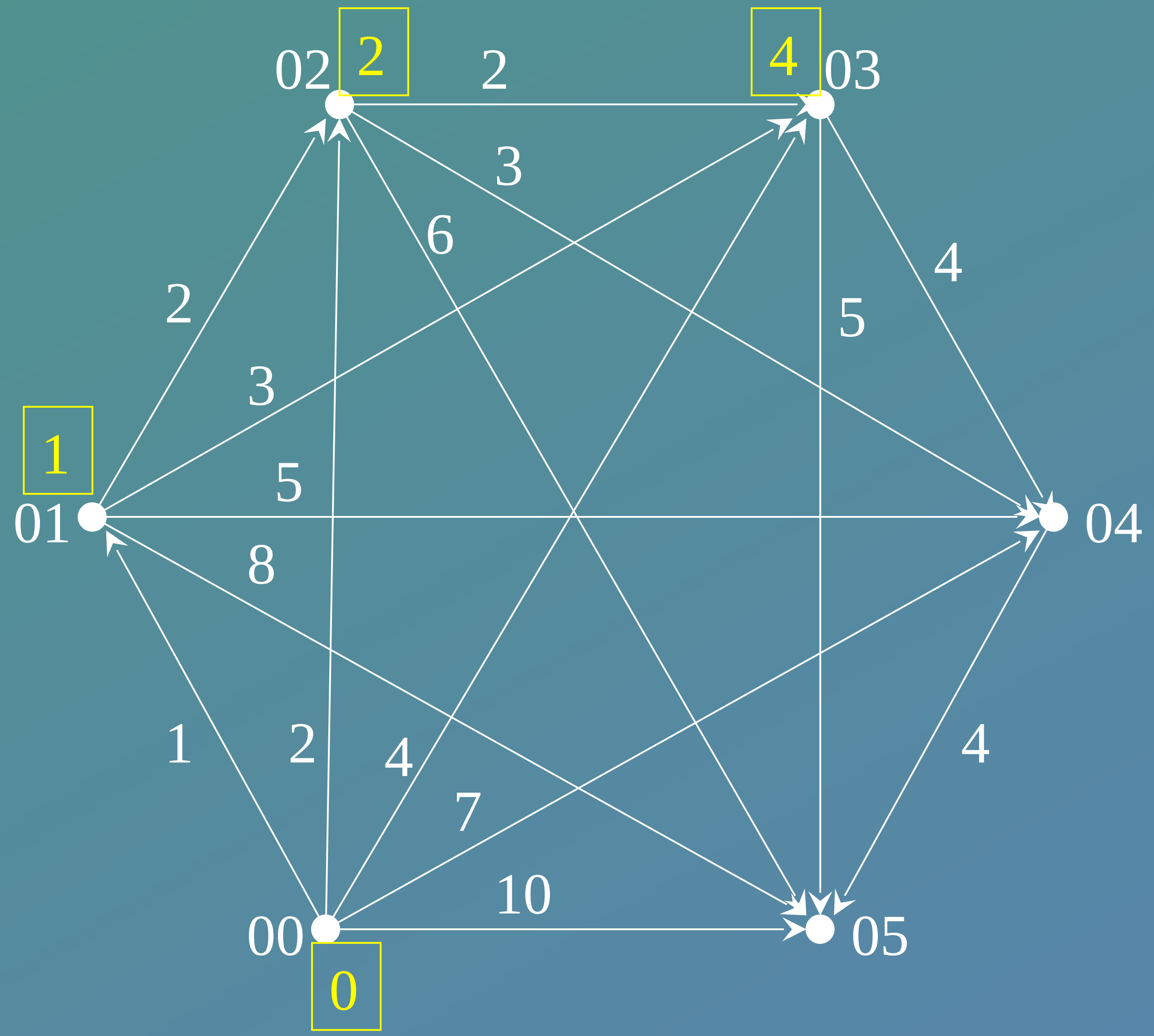
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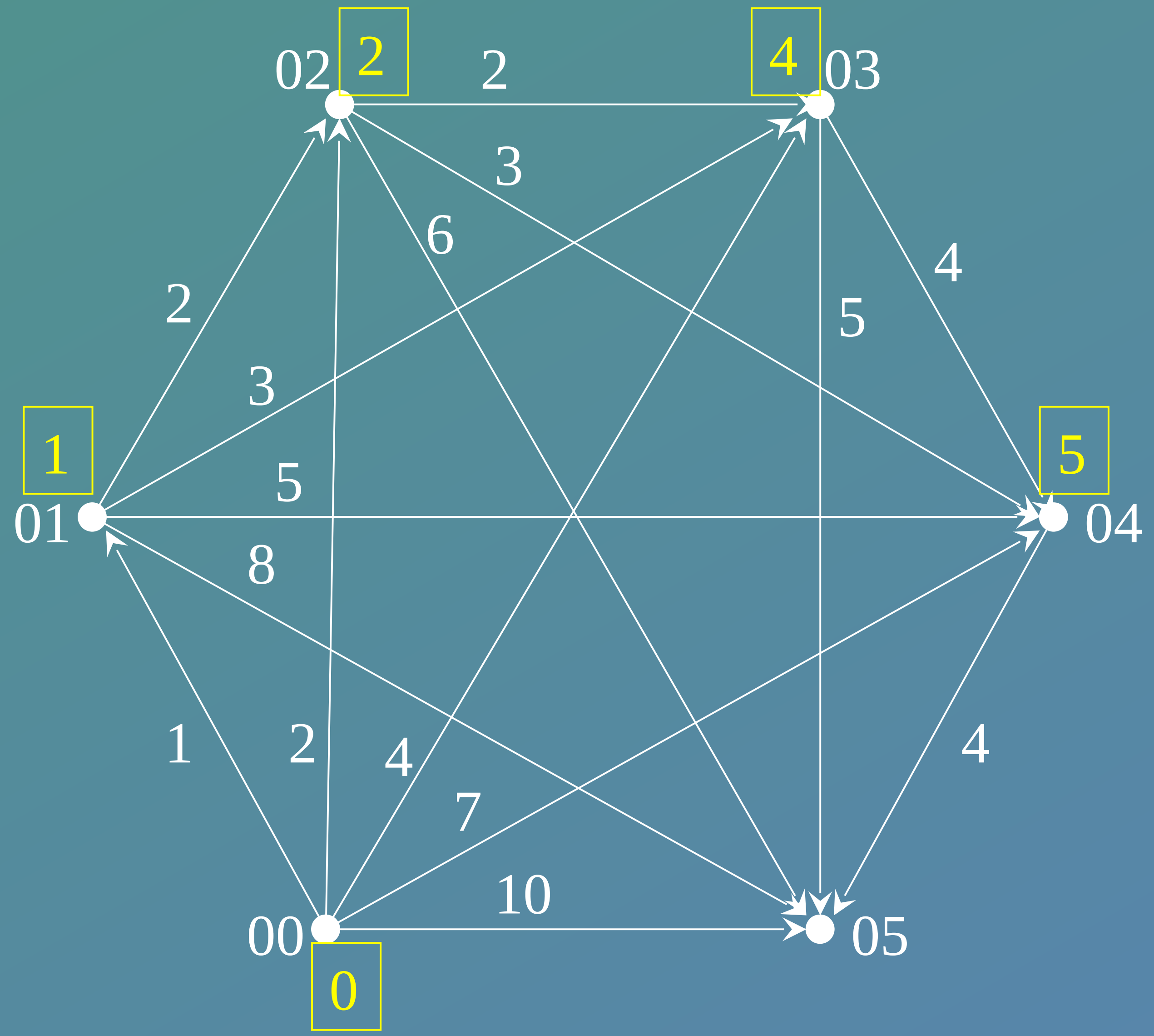

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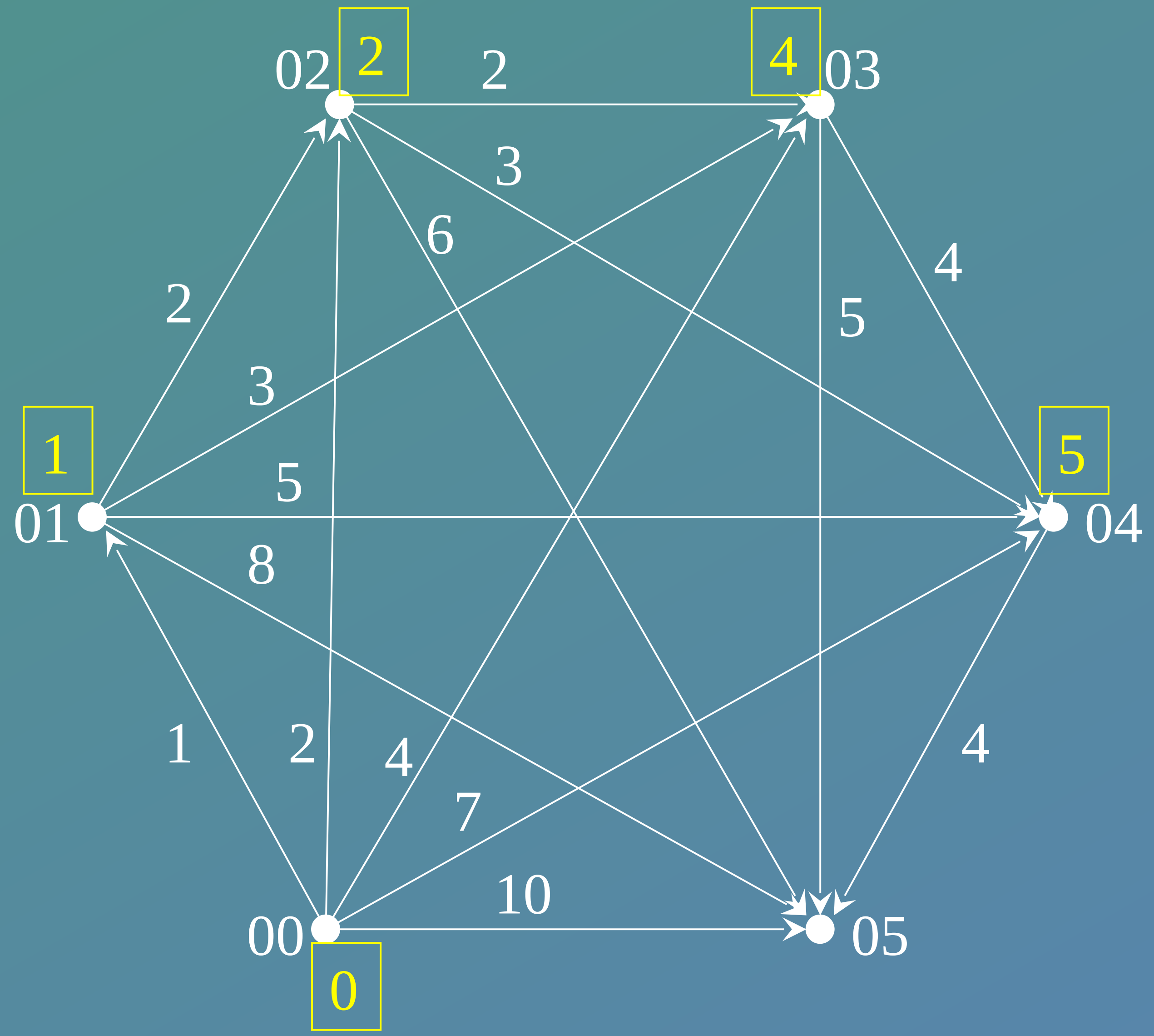

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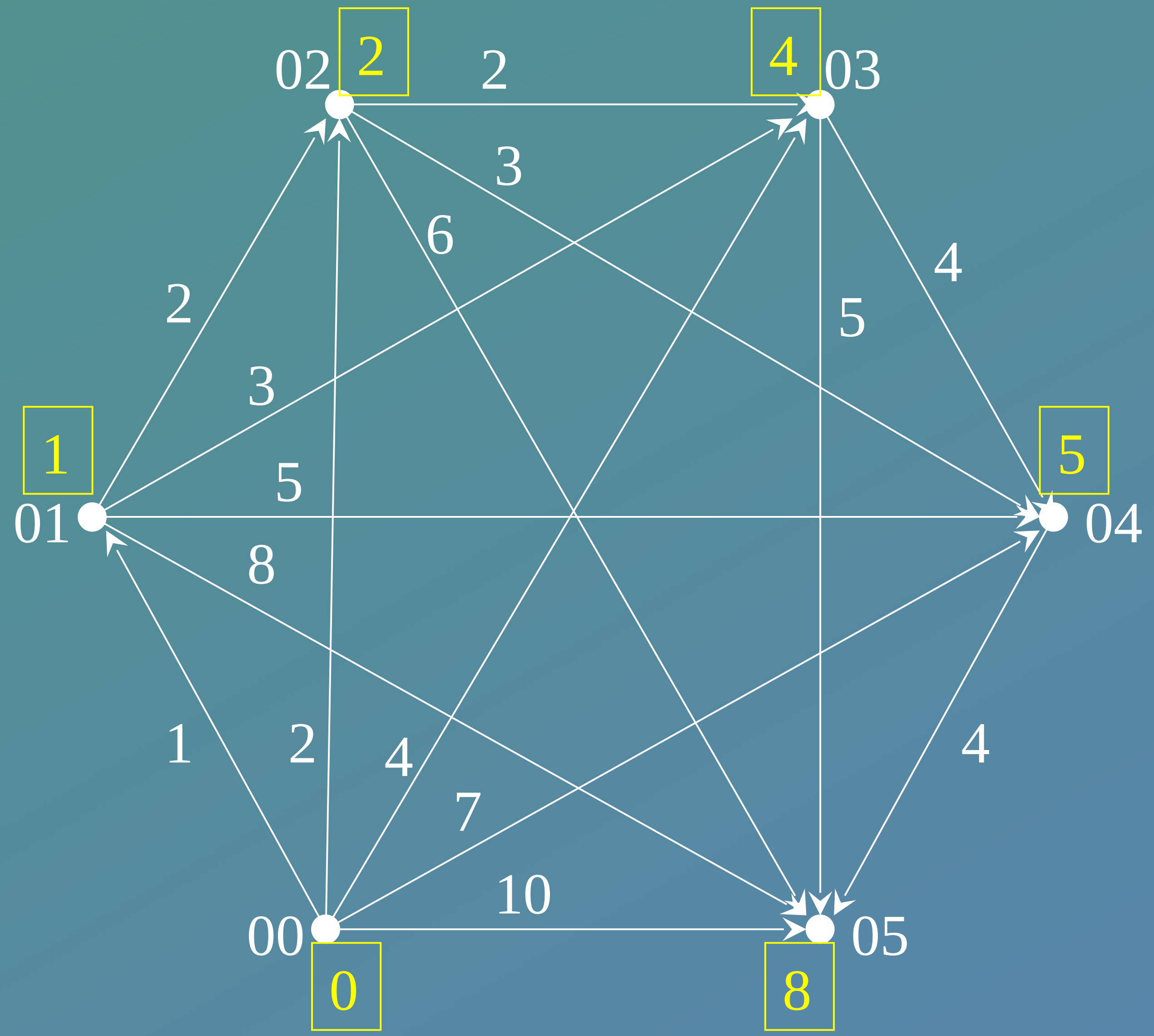
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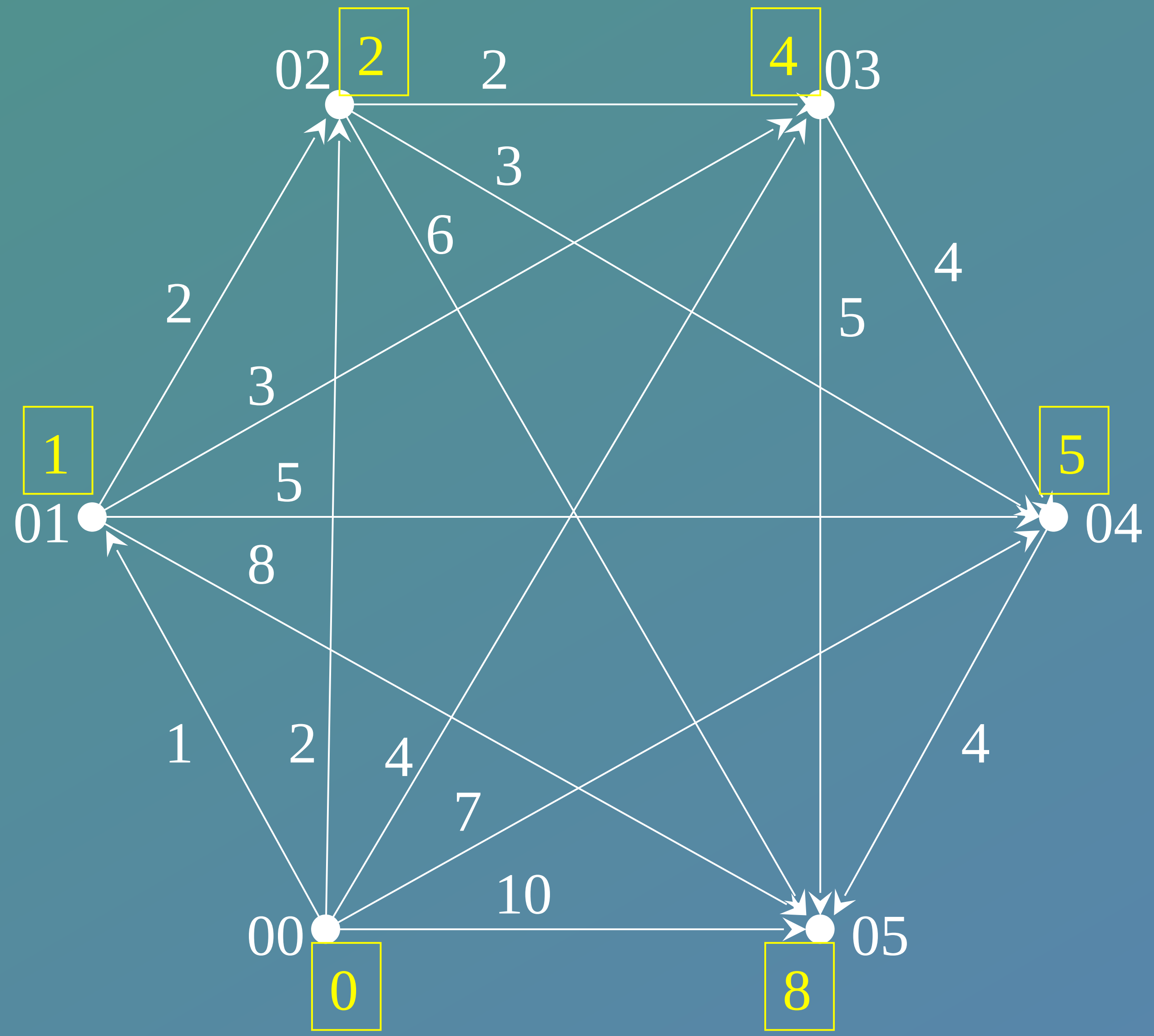
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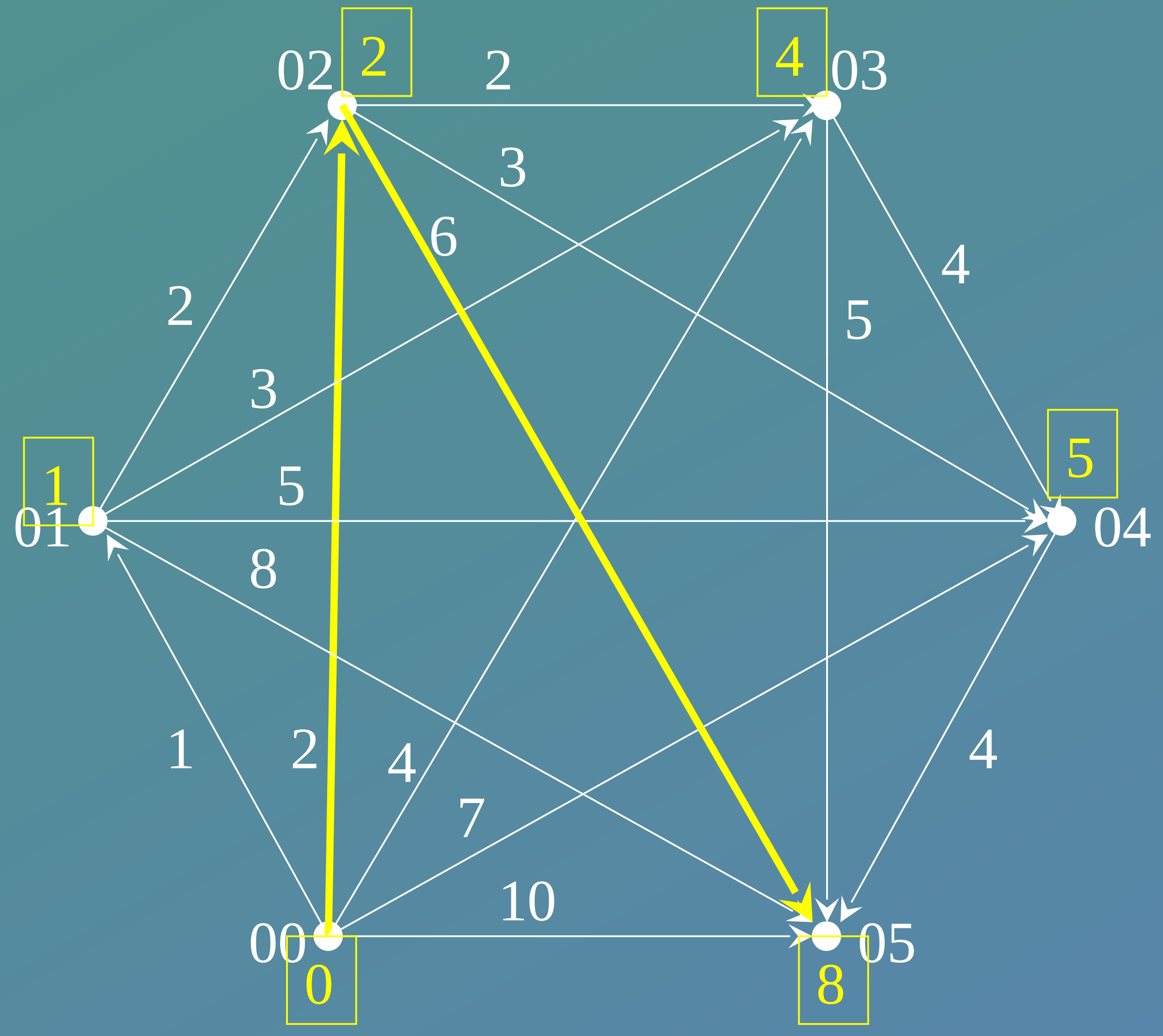
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Algorithme de Dijkstra.

Algorithme de Dijkstra.

Soit G=(X, E) un graphe orienté admettant une arborescence couvrante enracinée en **r**; d: E→R une fonction-distance non-négative. L'algorithme suivant fournit l'arborescence des plus courts chemins enracinée en r, ainsi que les distances $\pi(x)$ de r à x:

Algorithme de Dijkstra.

Soit G=(X, E) un graphe orienté admettant une arborescence couvrante enracinée en **r**; d: $E \rightarrow R$ une fonction-distance non-négative. L'algorithme suivant fournit l'arborescence des plus courts chemins enracinée en r, ainsi que les distances $\pi(x)$ de r à x:

initialisation: $\boldsymbol{\pi}(\mathbf{r}) := \mathbf{0}$; $\pi(x) := \infty$ pour $x \neq \mathbf{r}$;

S:= {**r**}; A := Ø; p = **r**;

itération 1: corrections des marque temporaires:

si S=X ***alors*** **FIN**, ***sinon:***

pour $x \in X \backslash S$ tels que $e=(p;x) \in E$; si $\pi(x) > \pi(p) + d(e)$, alors $\pi(x) := \pi(p) + d(e)$;

itération 2: acceptation définitive d'une marque minimale:

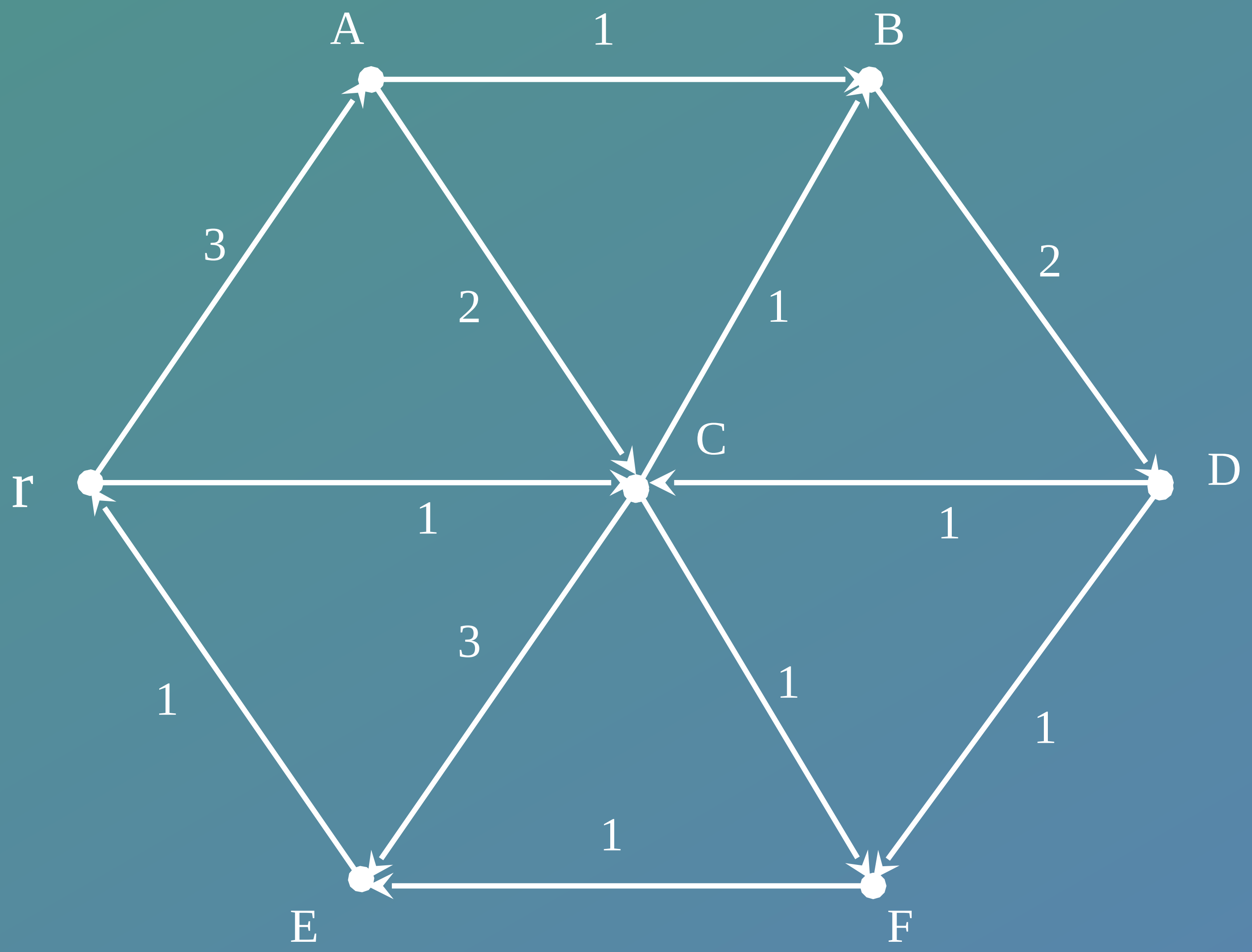
soit $\mathbf{x} \in X \backslash S$; tel que $\pi(x) = \min\{ \pi(y) ;\ y \in X \backslash S\}$;

soit $\mathbf{e} \in E$, $e=(z;x)$, tel que $z \in S$ et $\pi(x) = \pi(z) + d(e)$;

$S := S \cup \{\mathbf{x}\}$; $A := A \cup \{\mathbf{e}\}$; p:= **x** ;

revenir à l'itération 1;

FIN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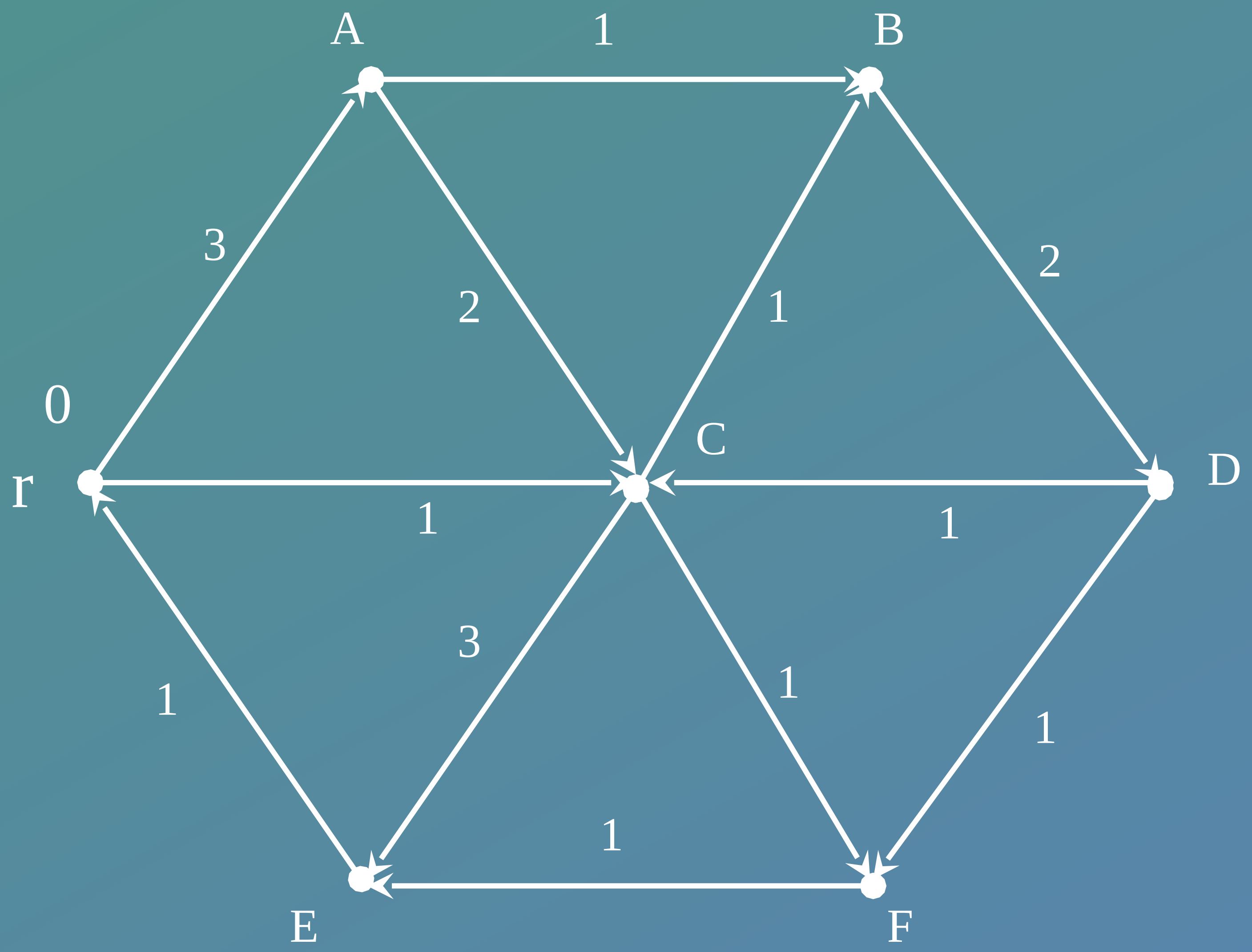

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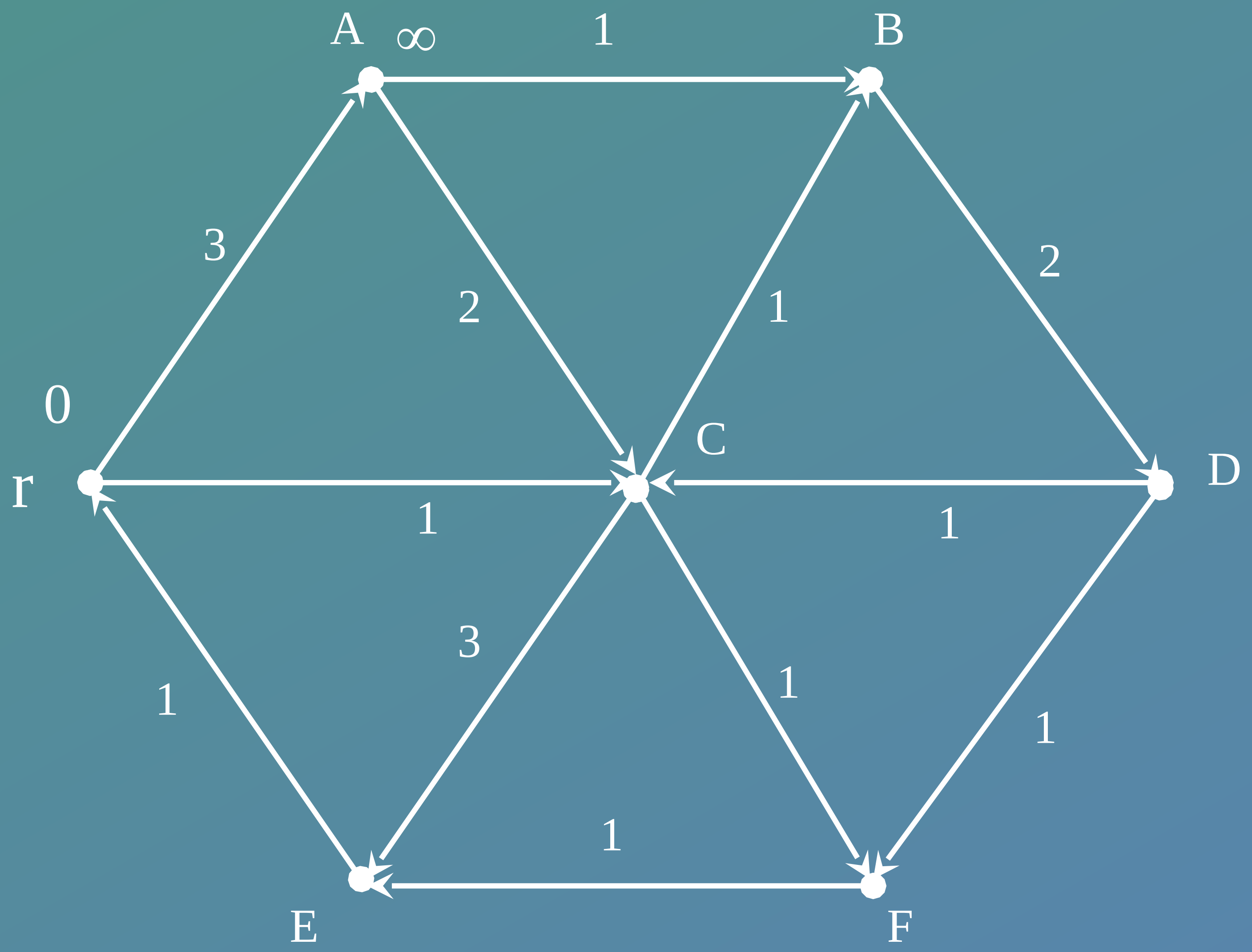
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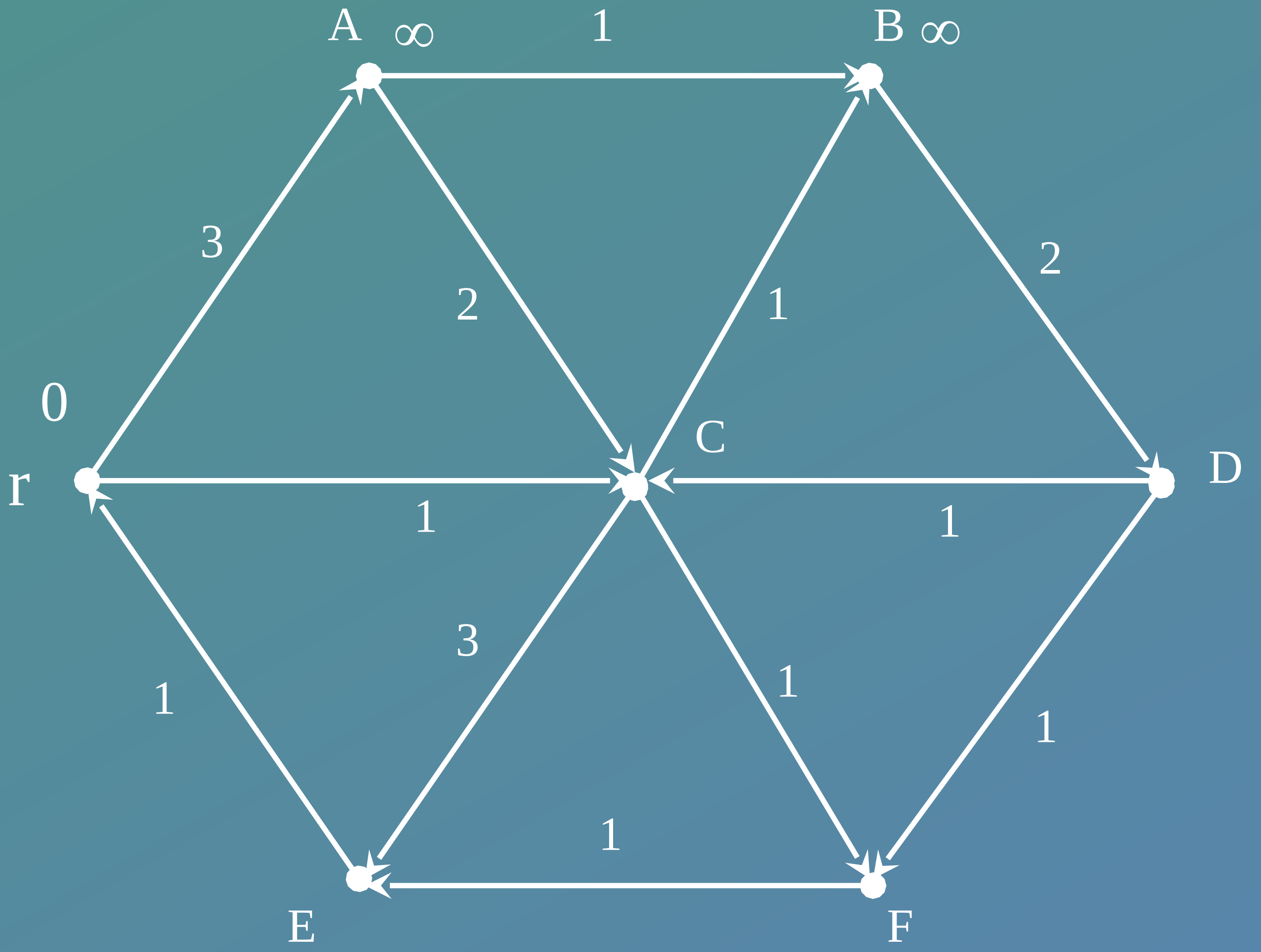
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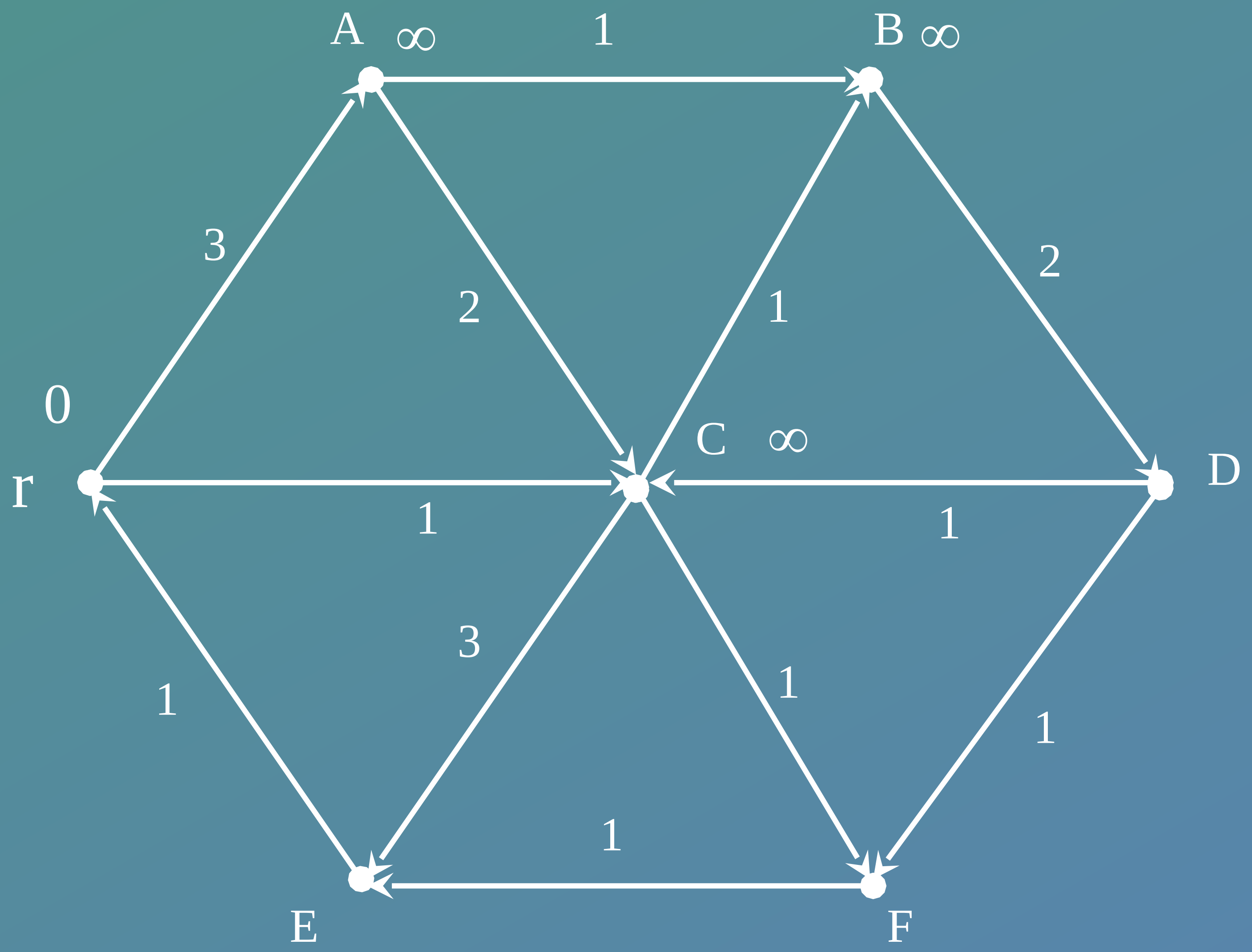

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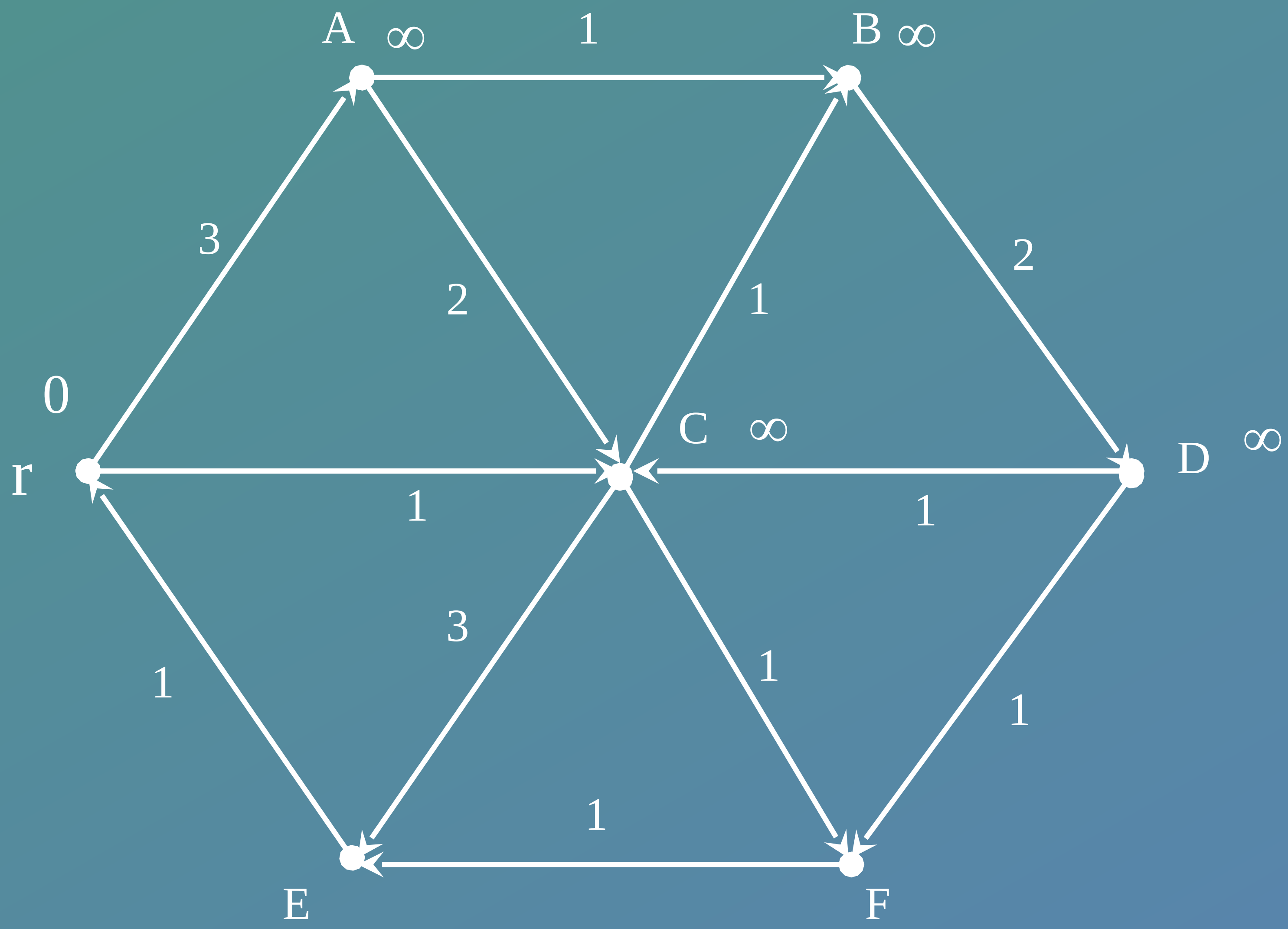

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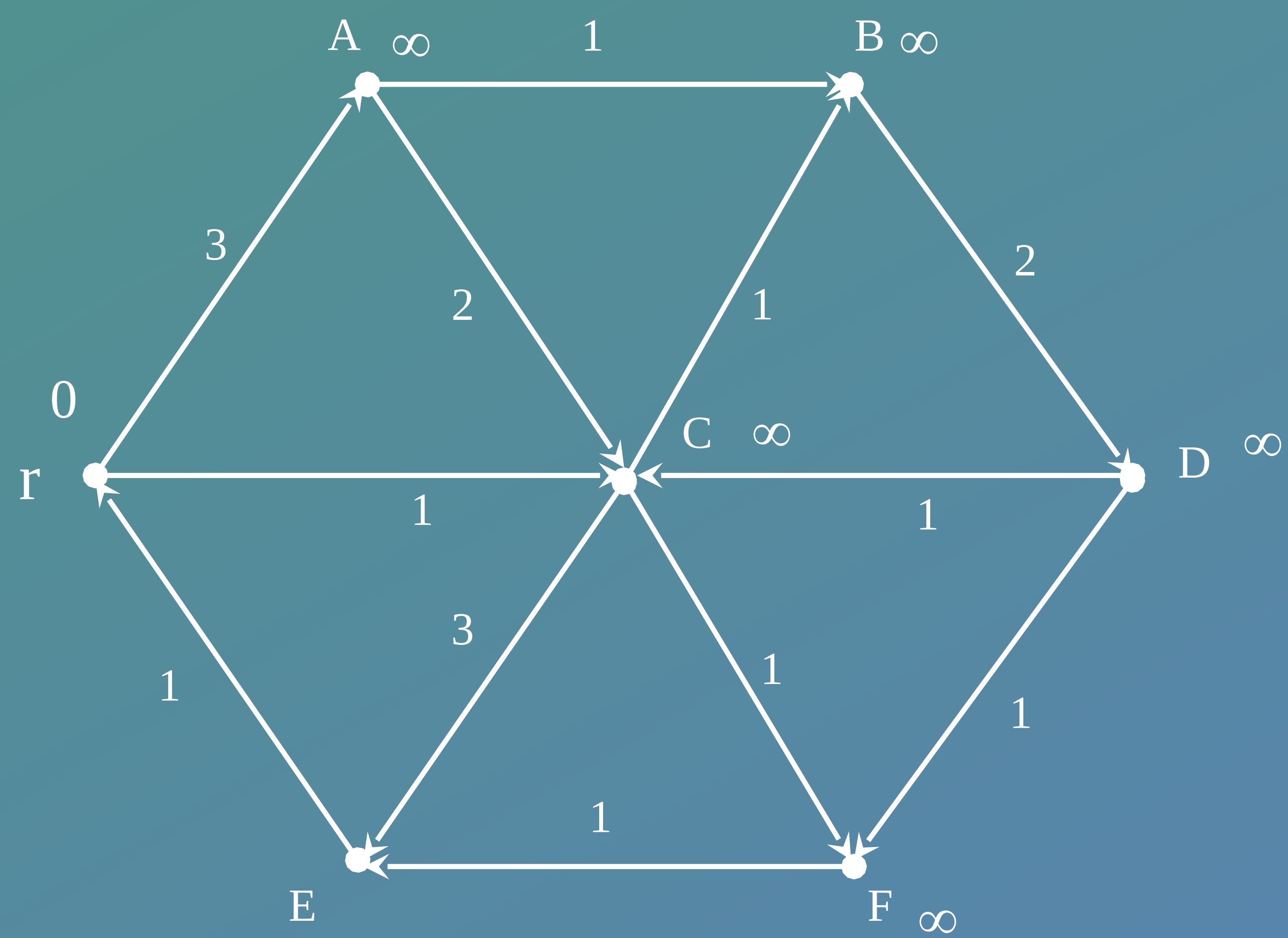

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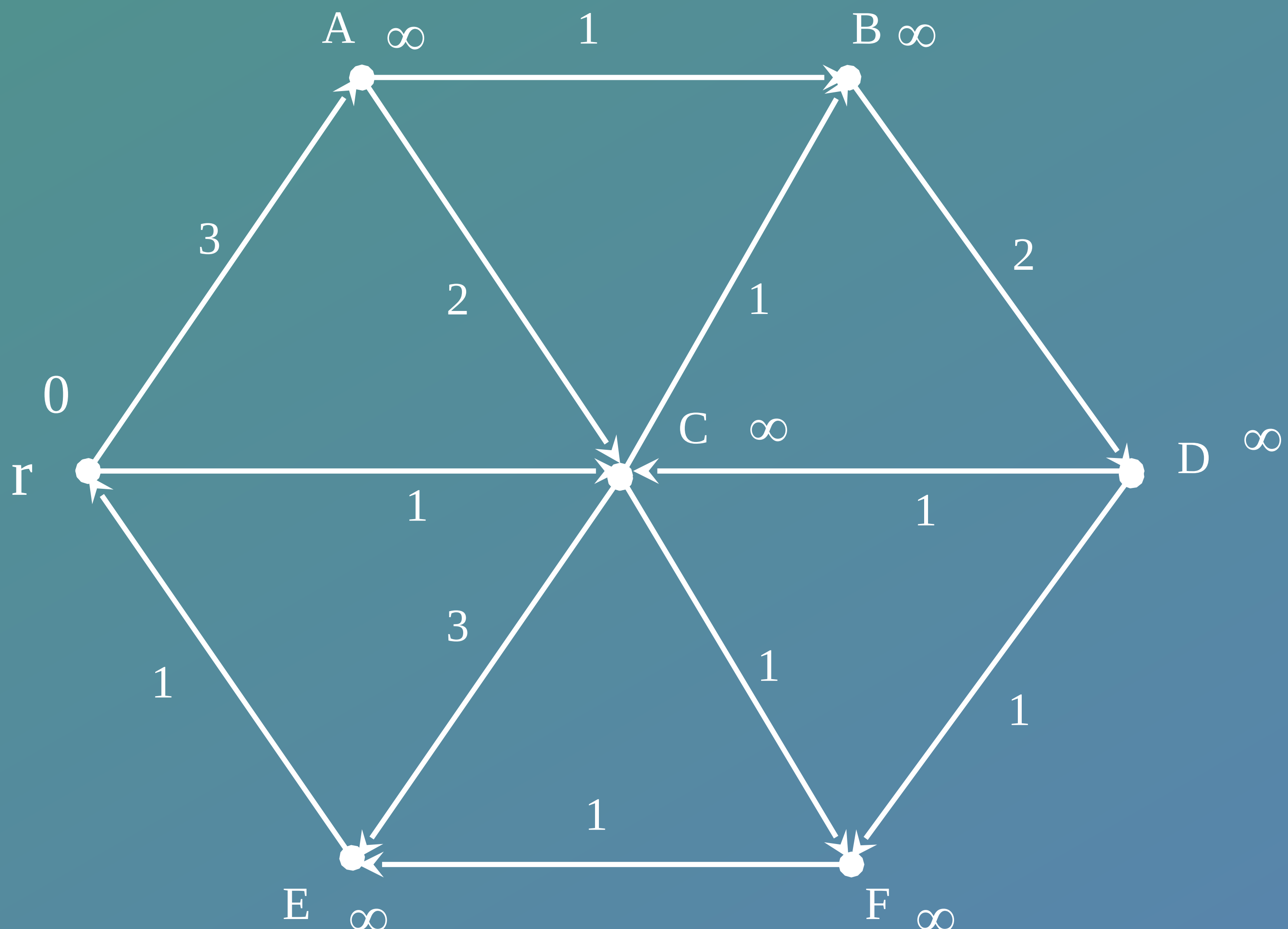

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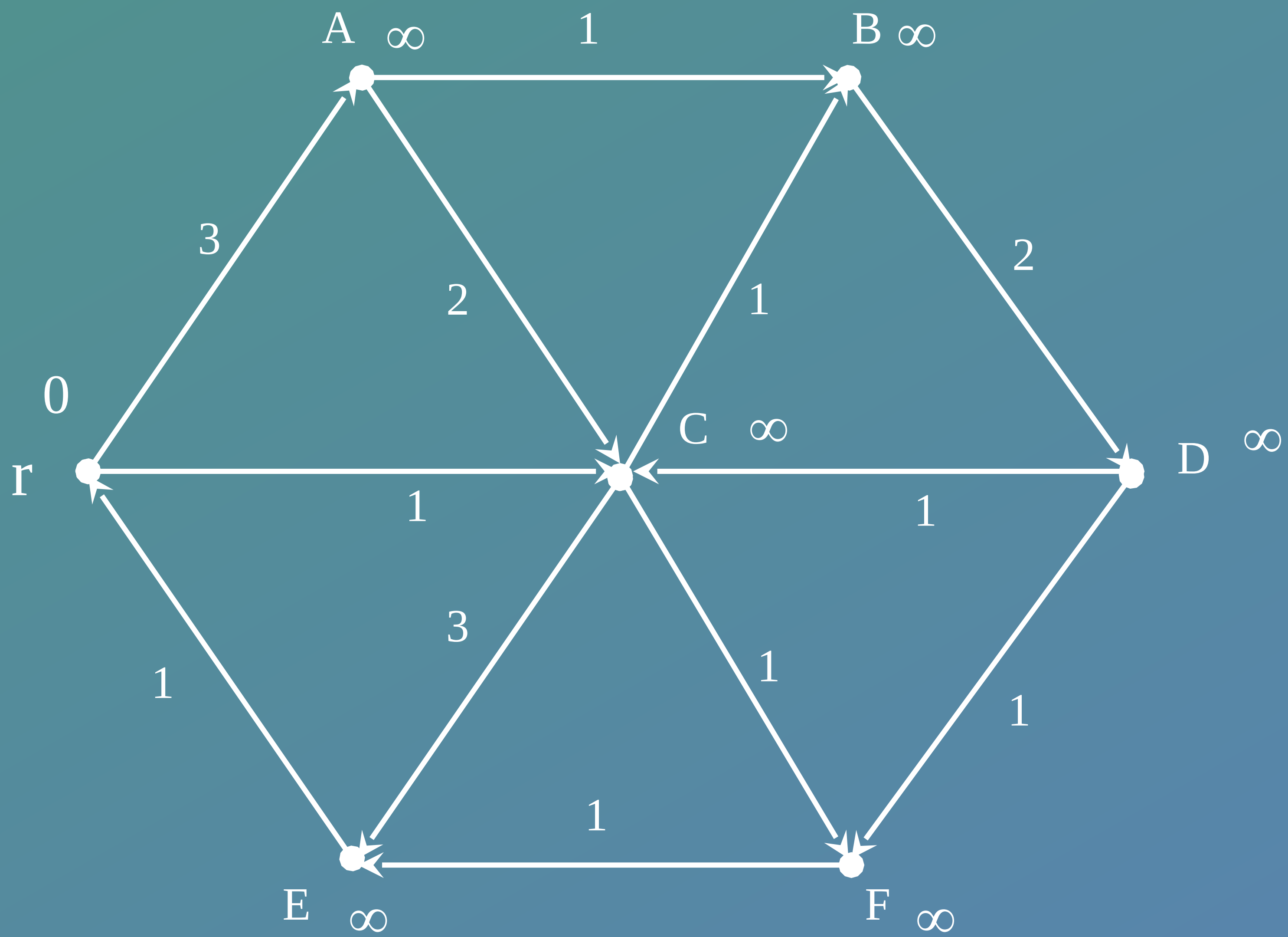
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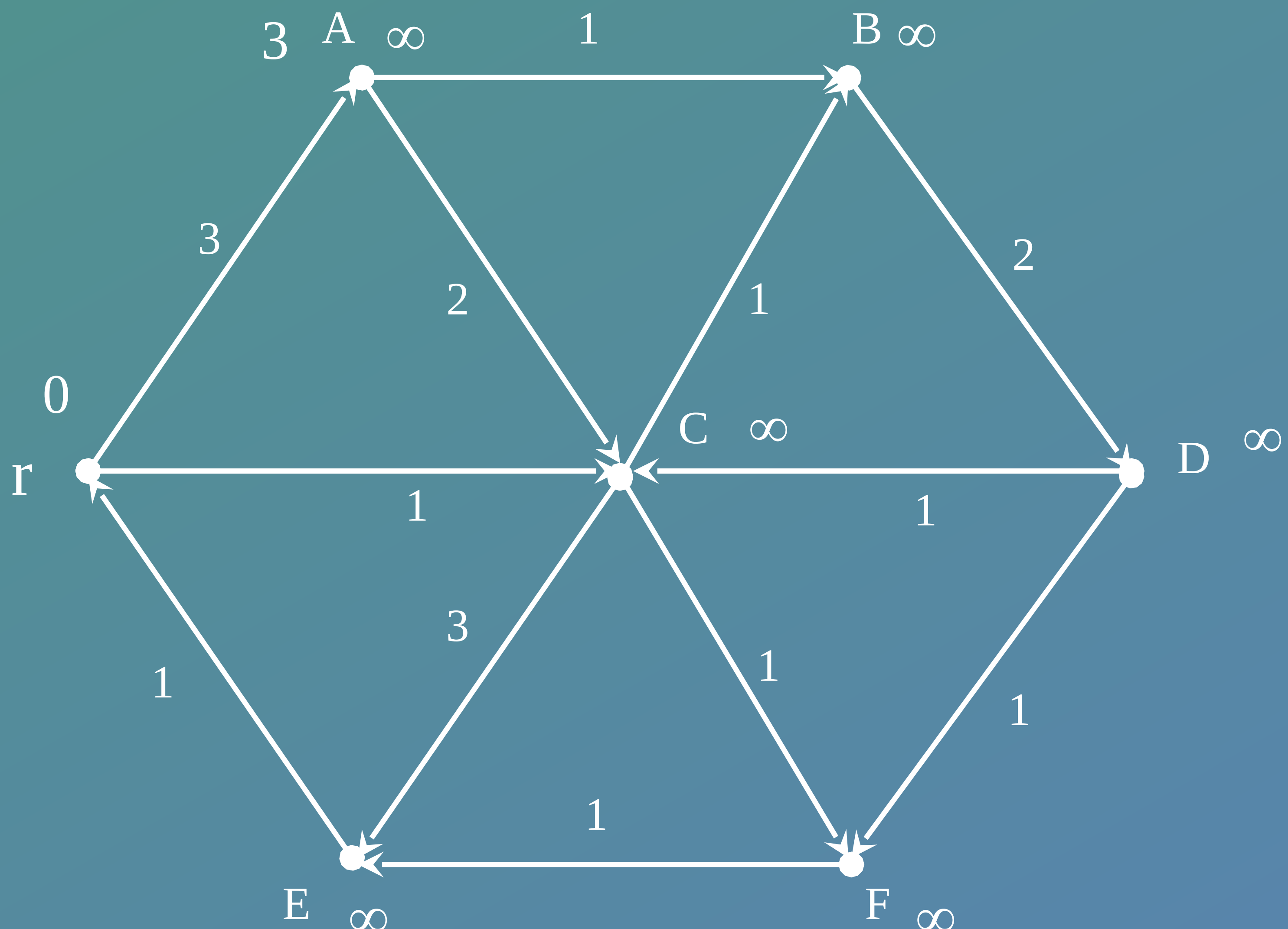

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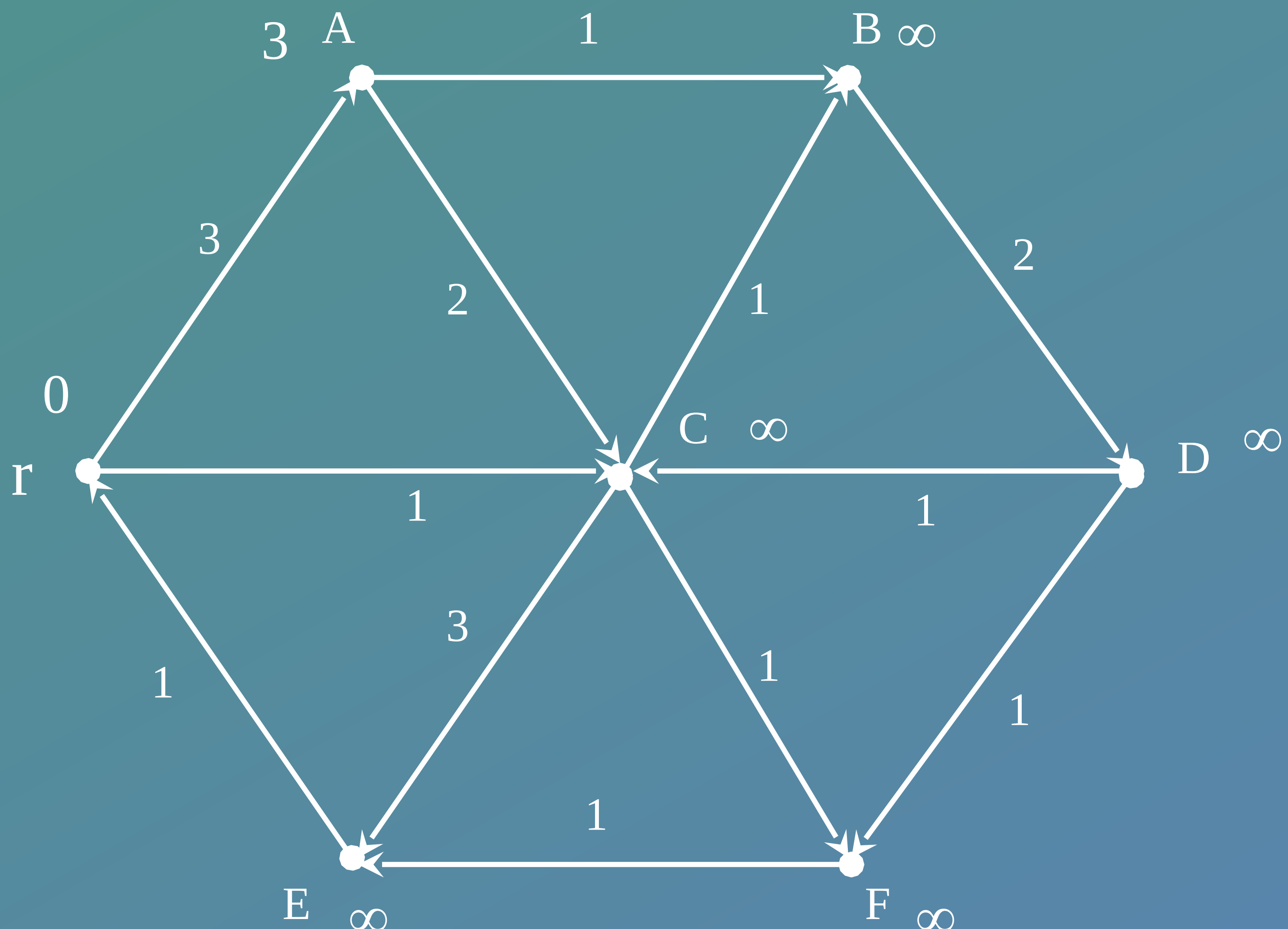
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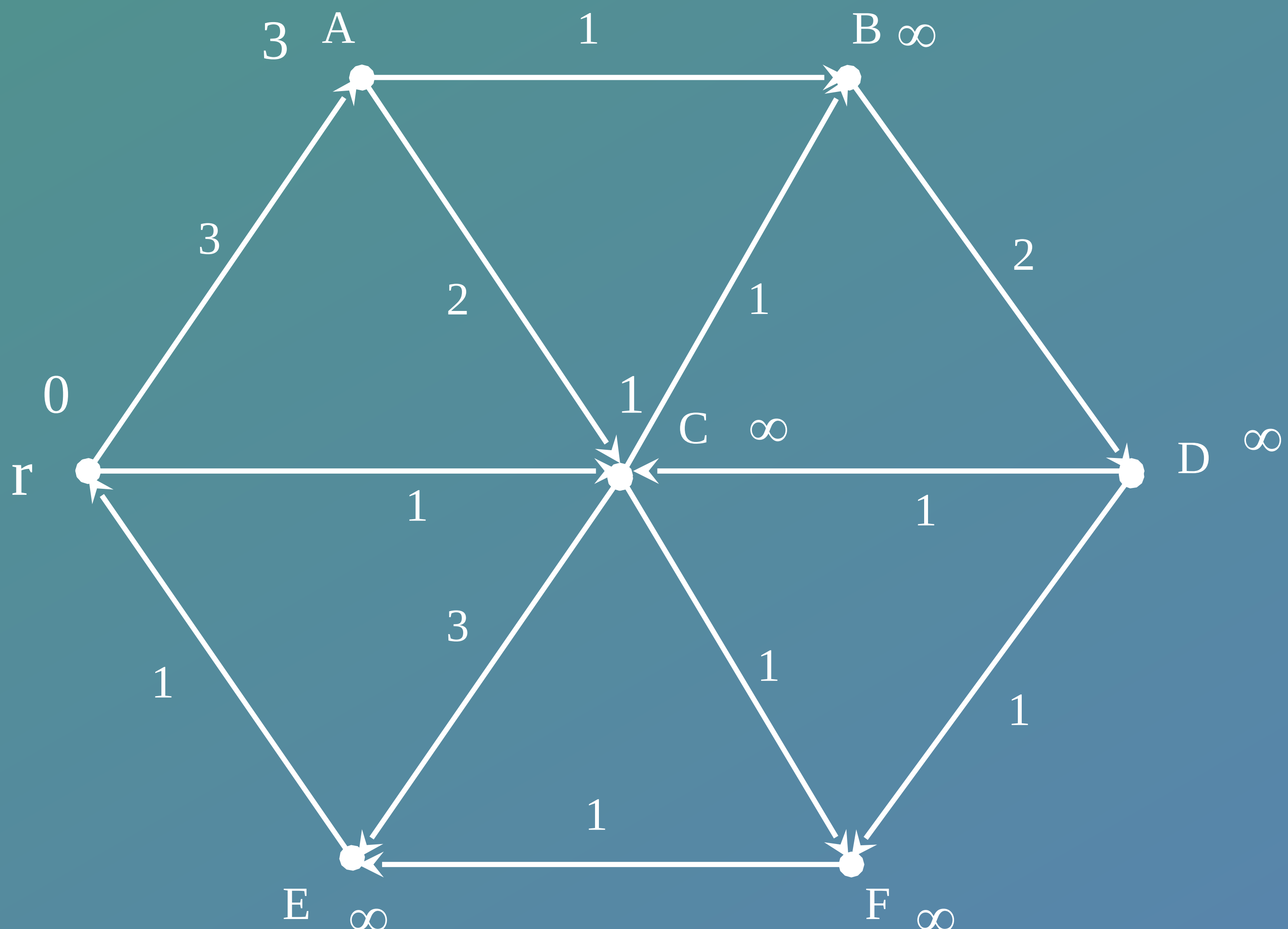
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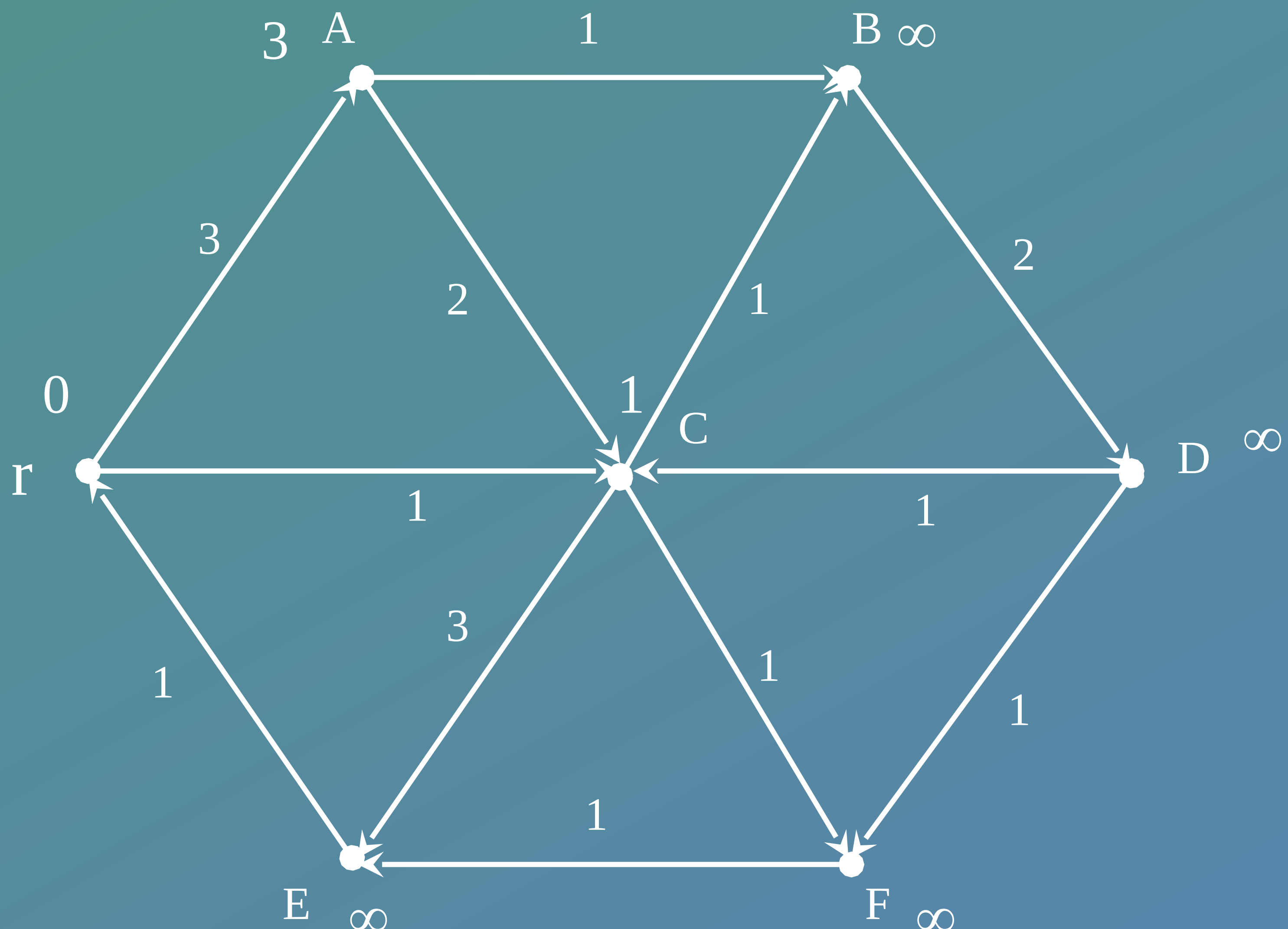
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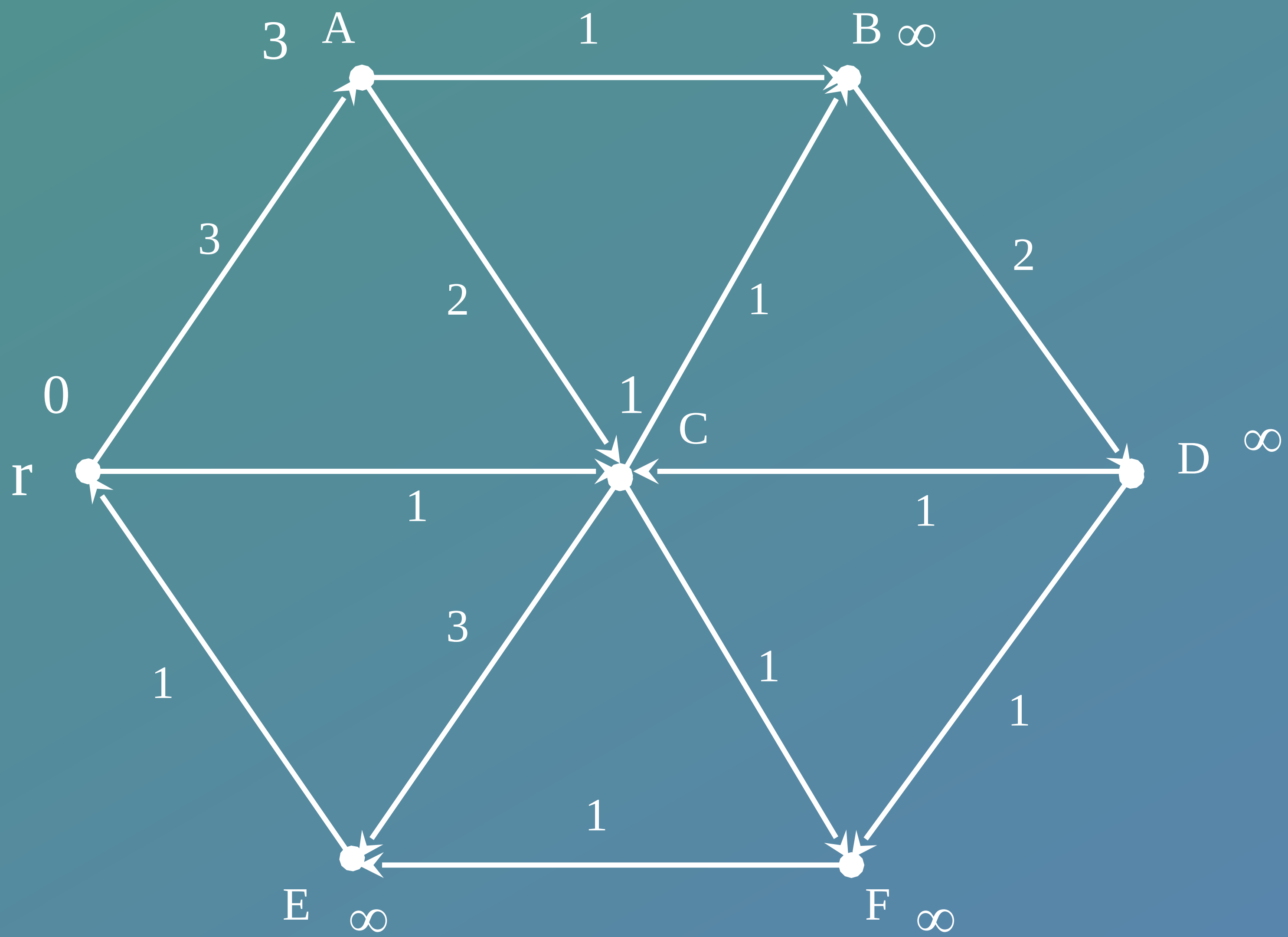

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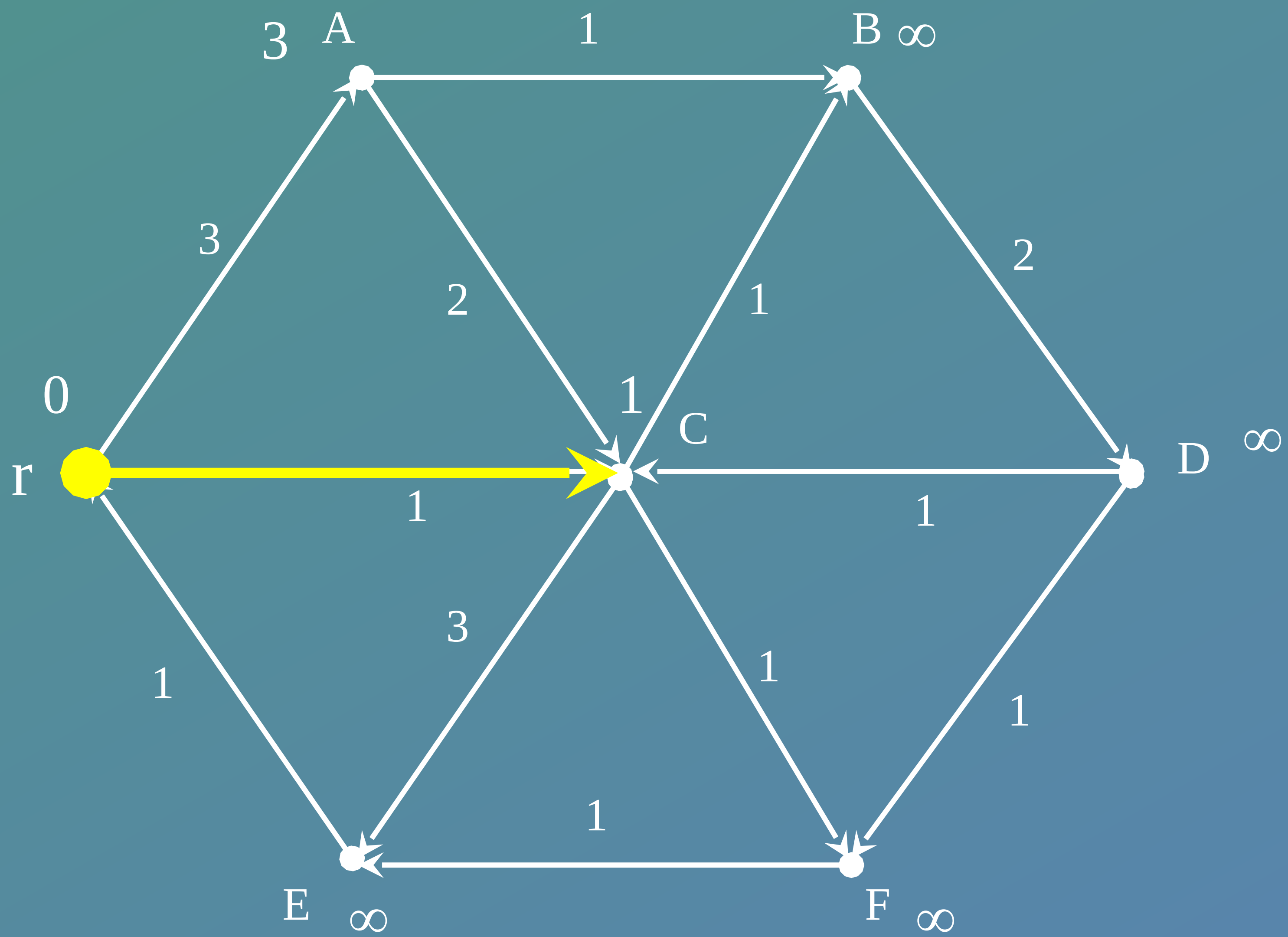

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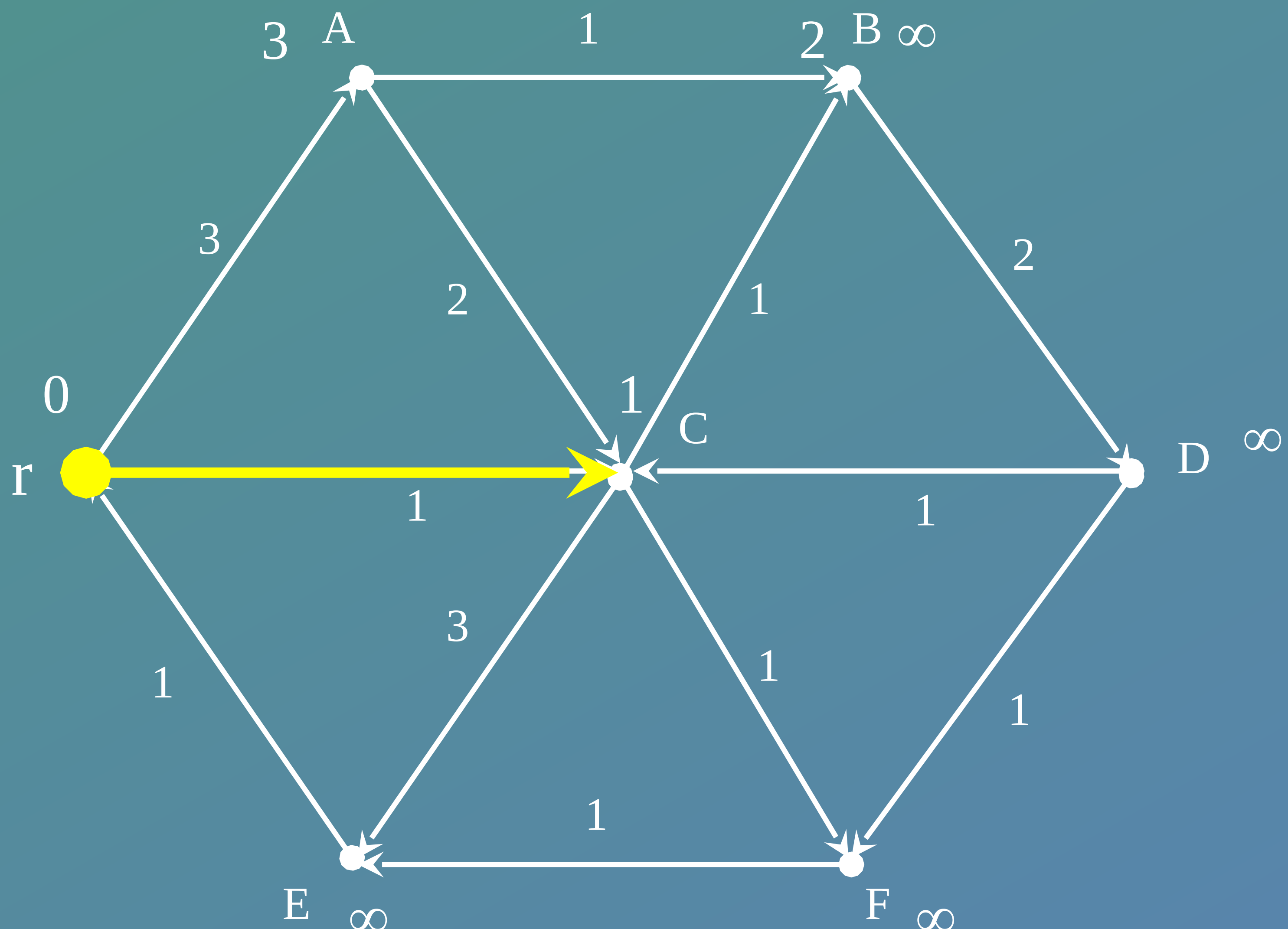

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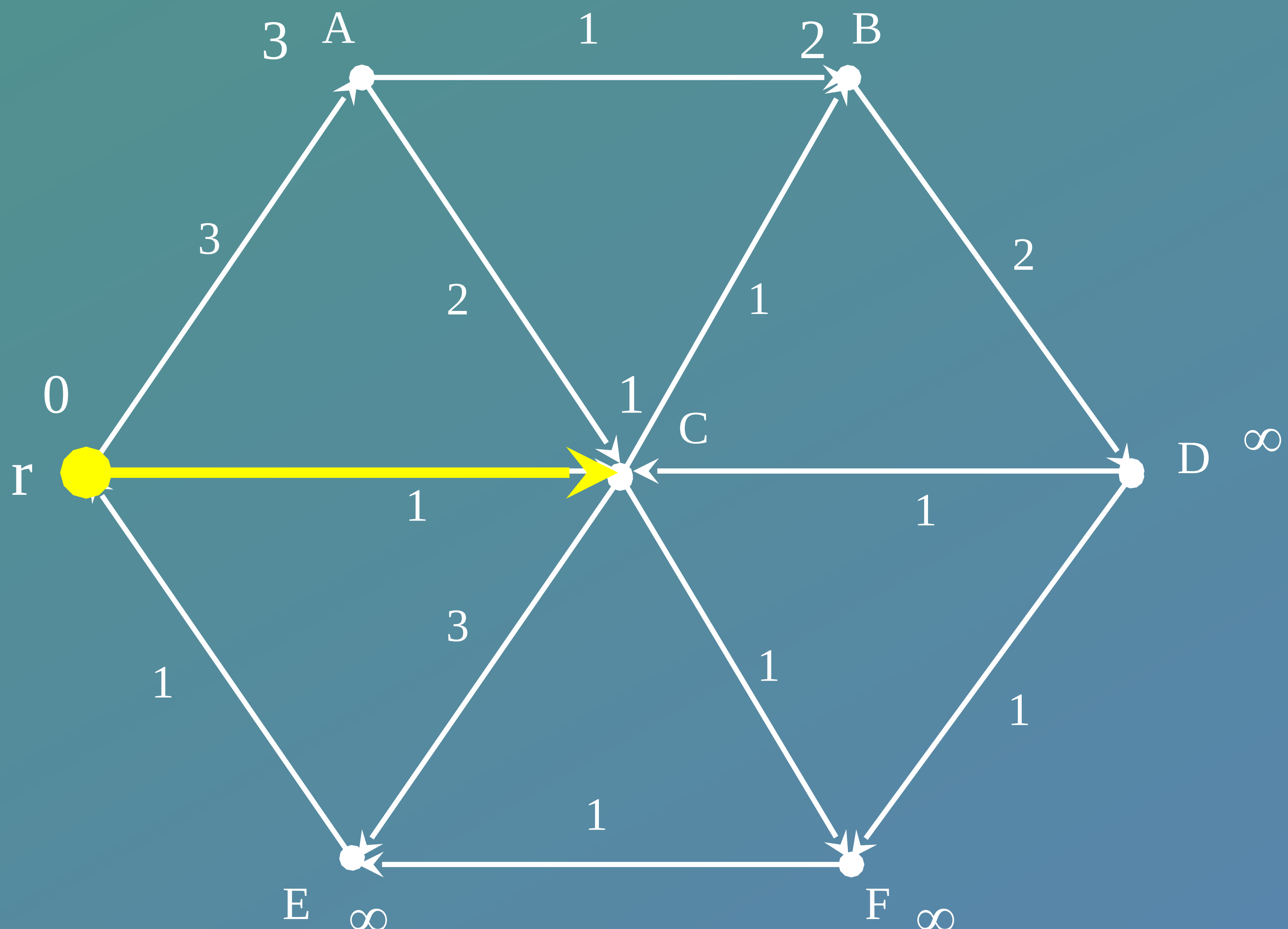

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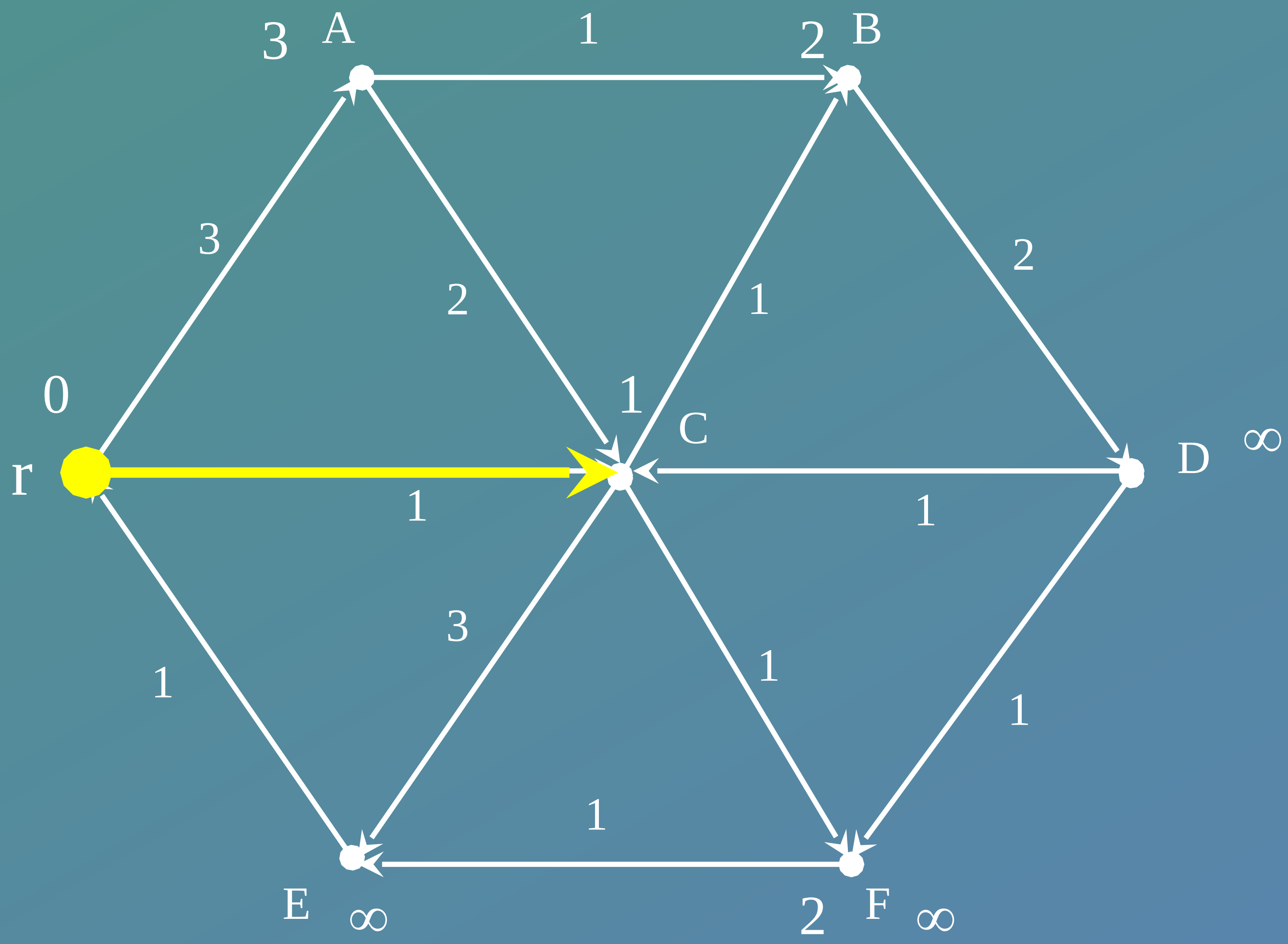
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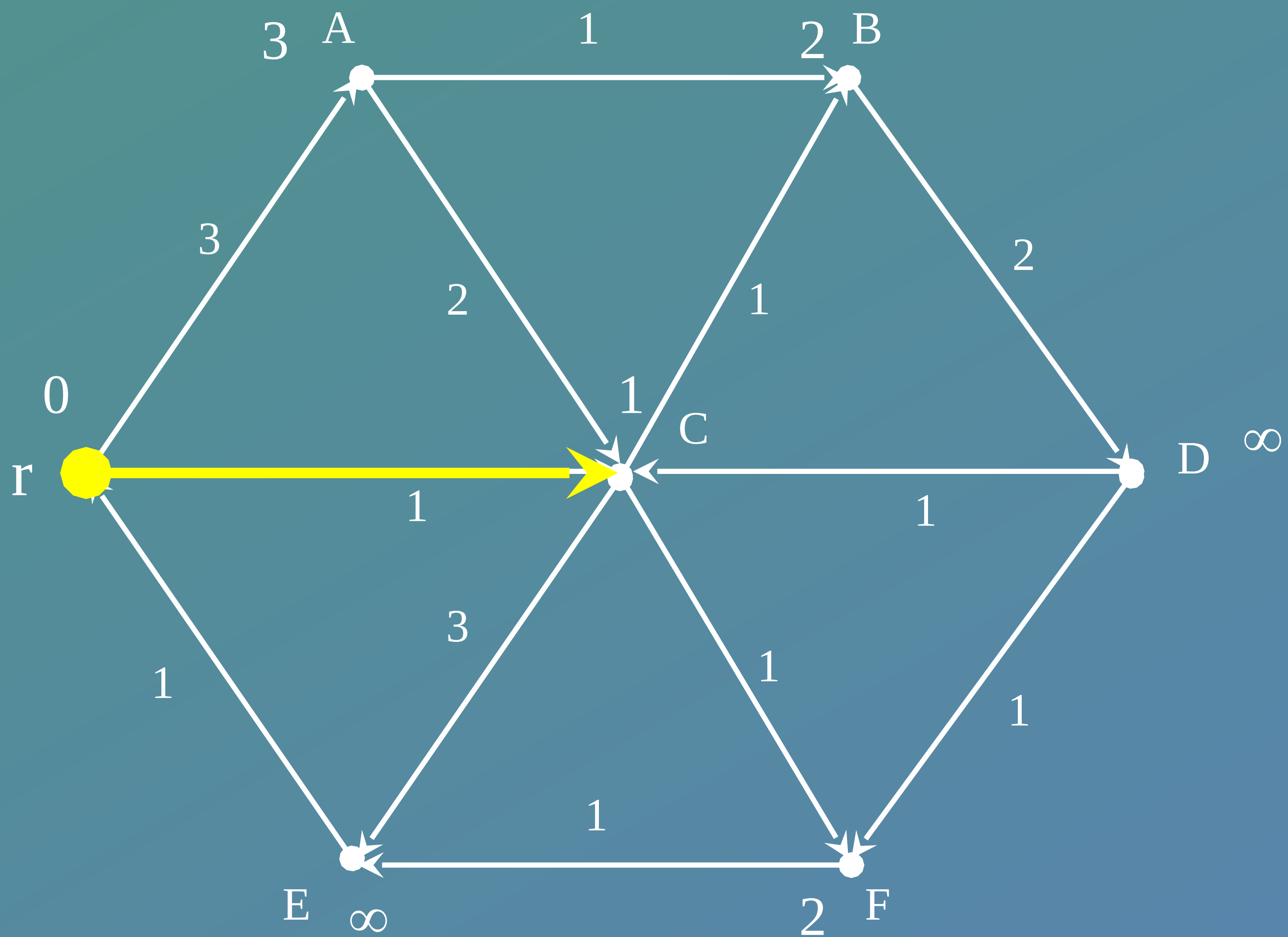
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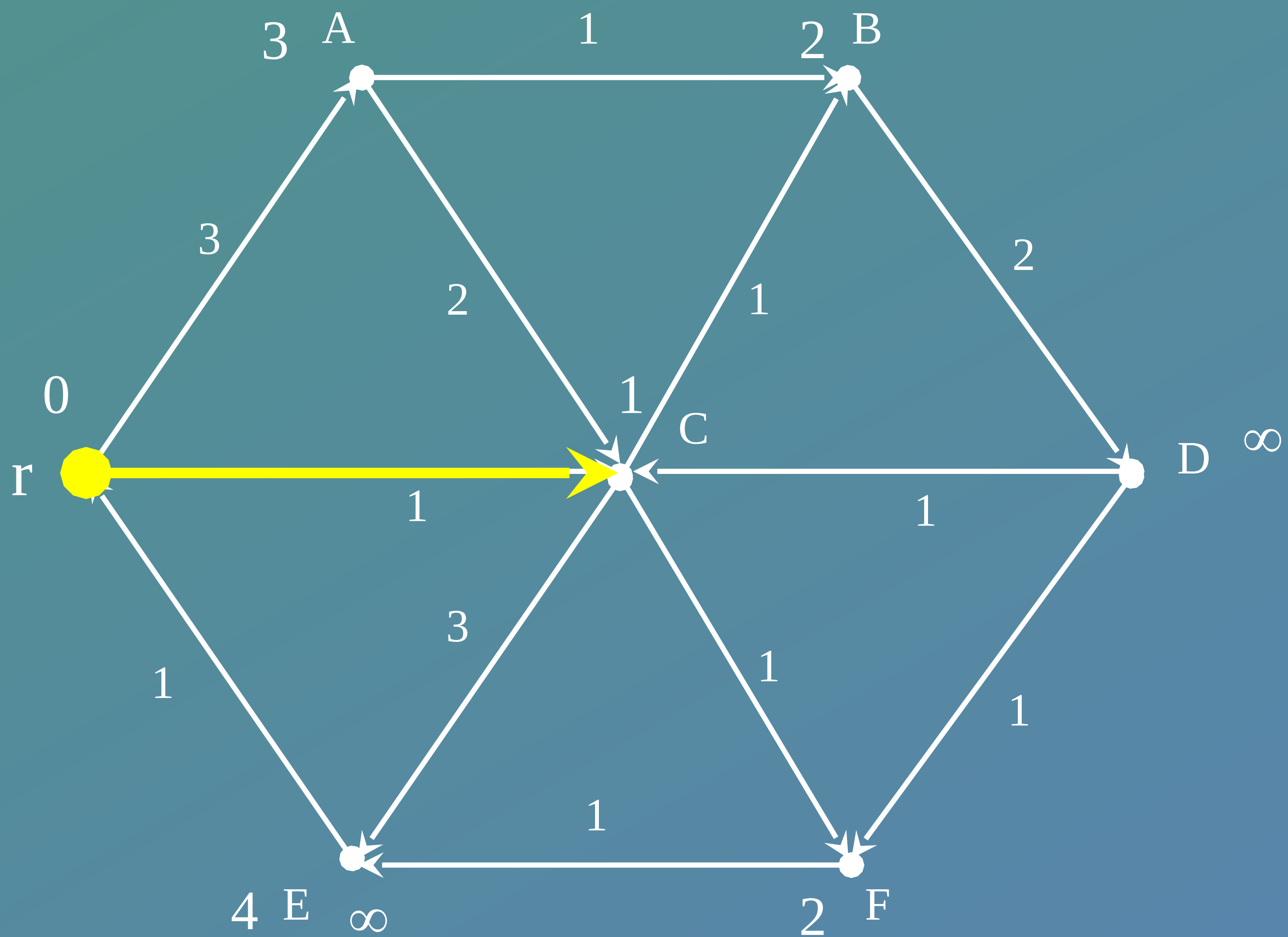

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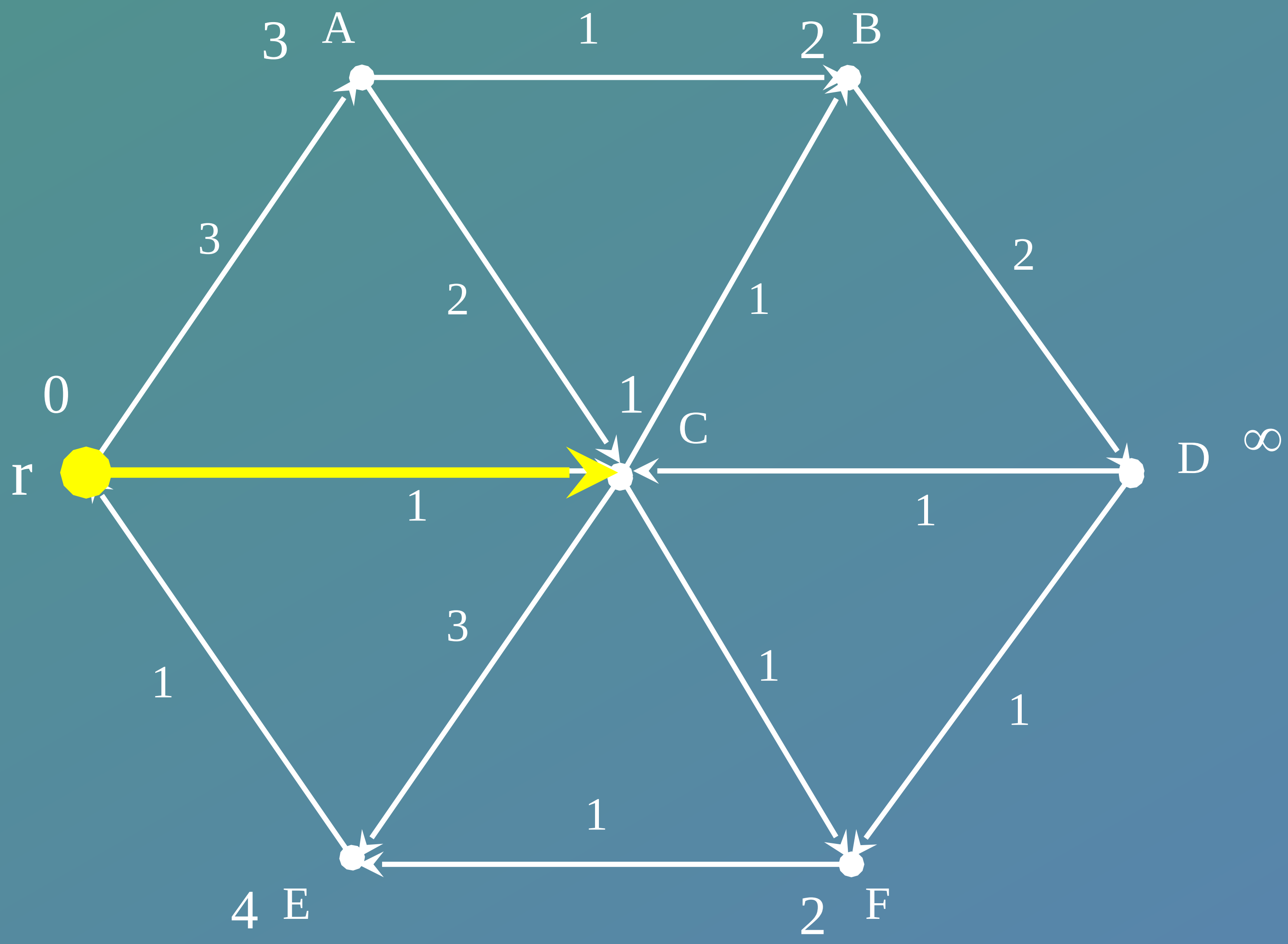
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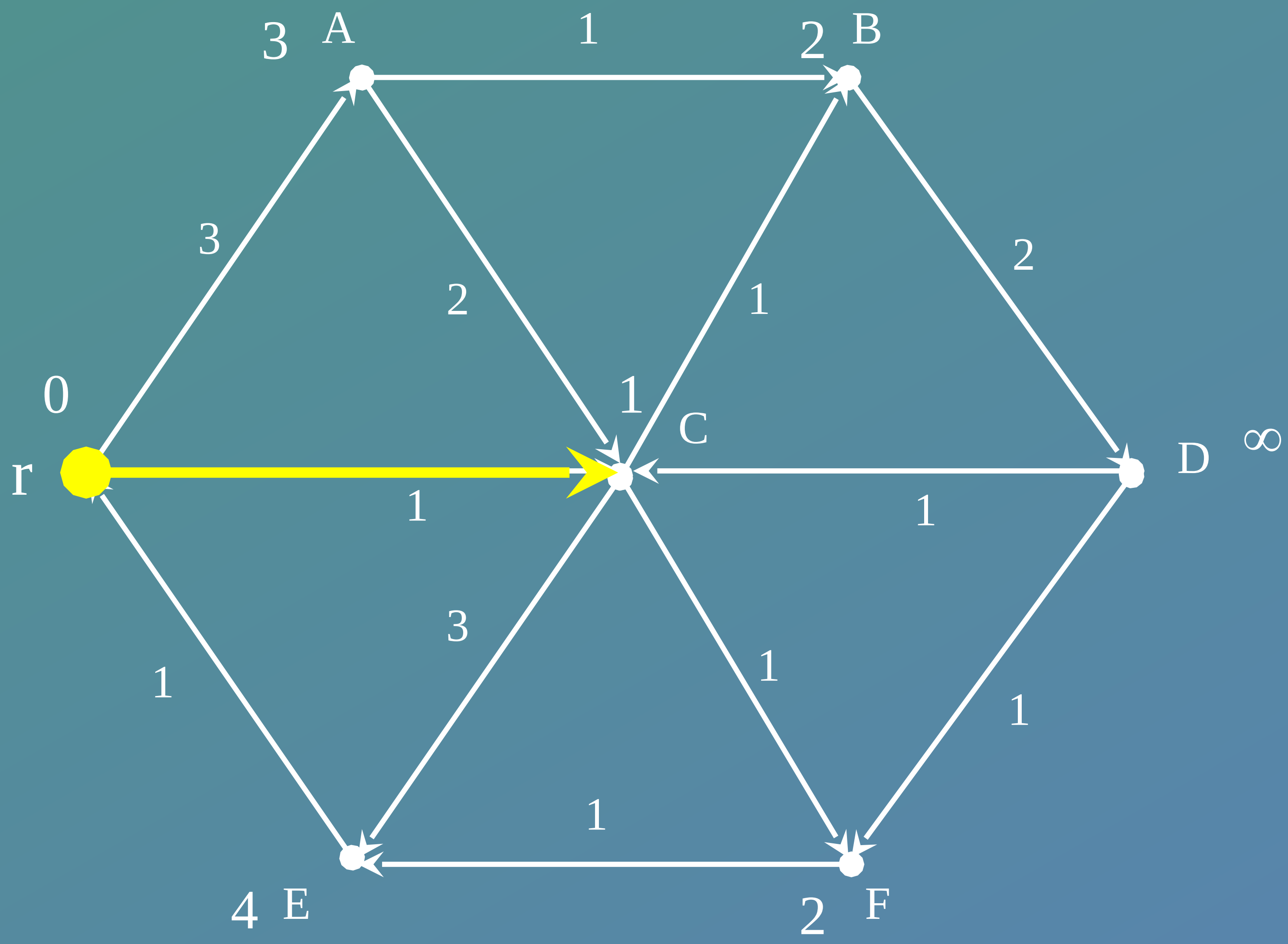
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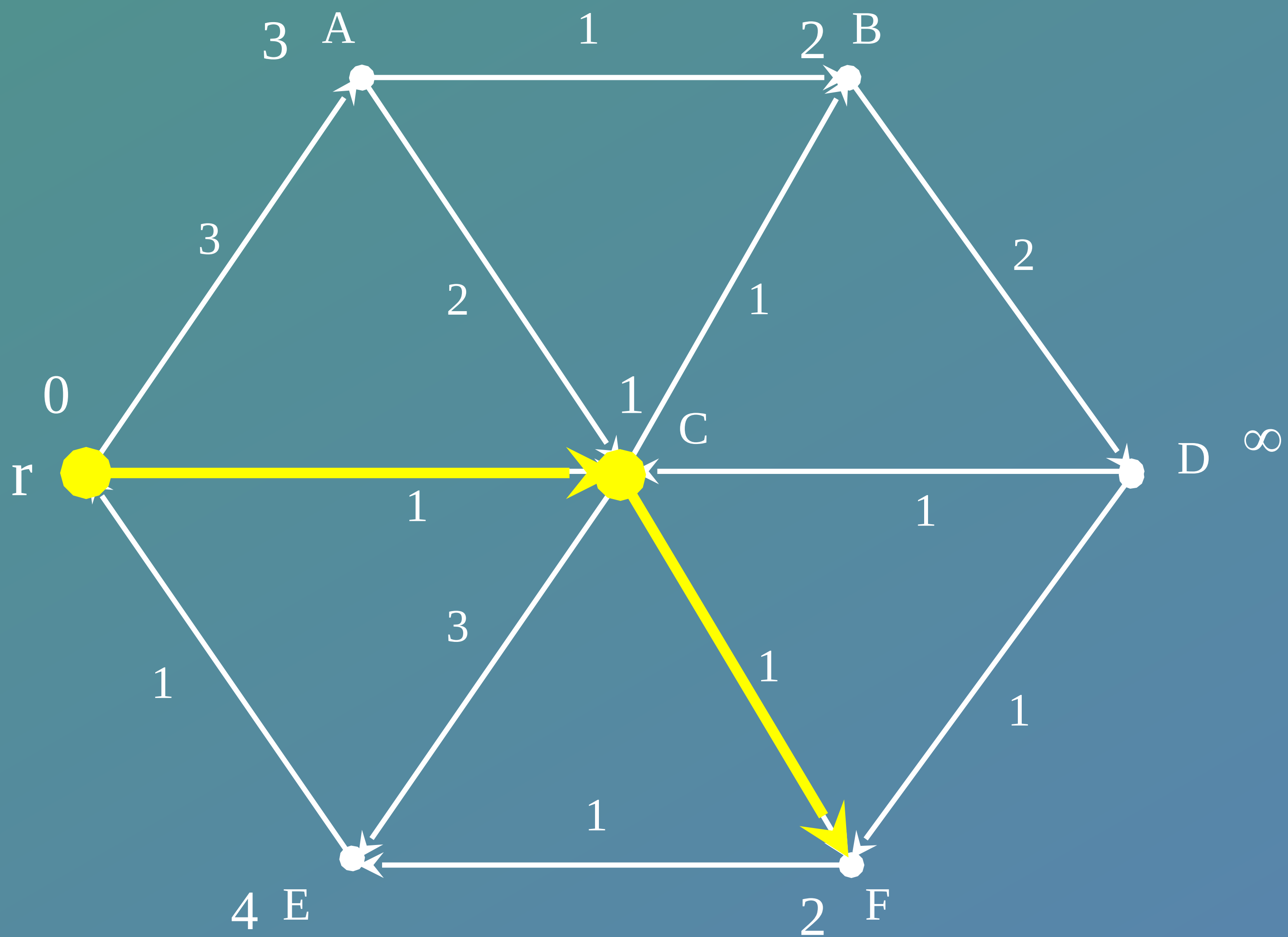
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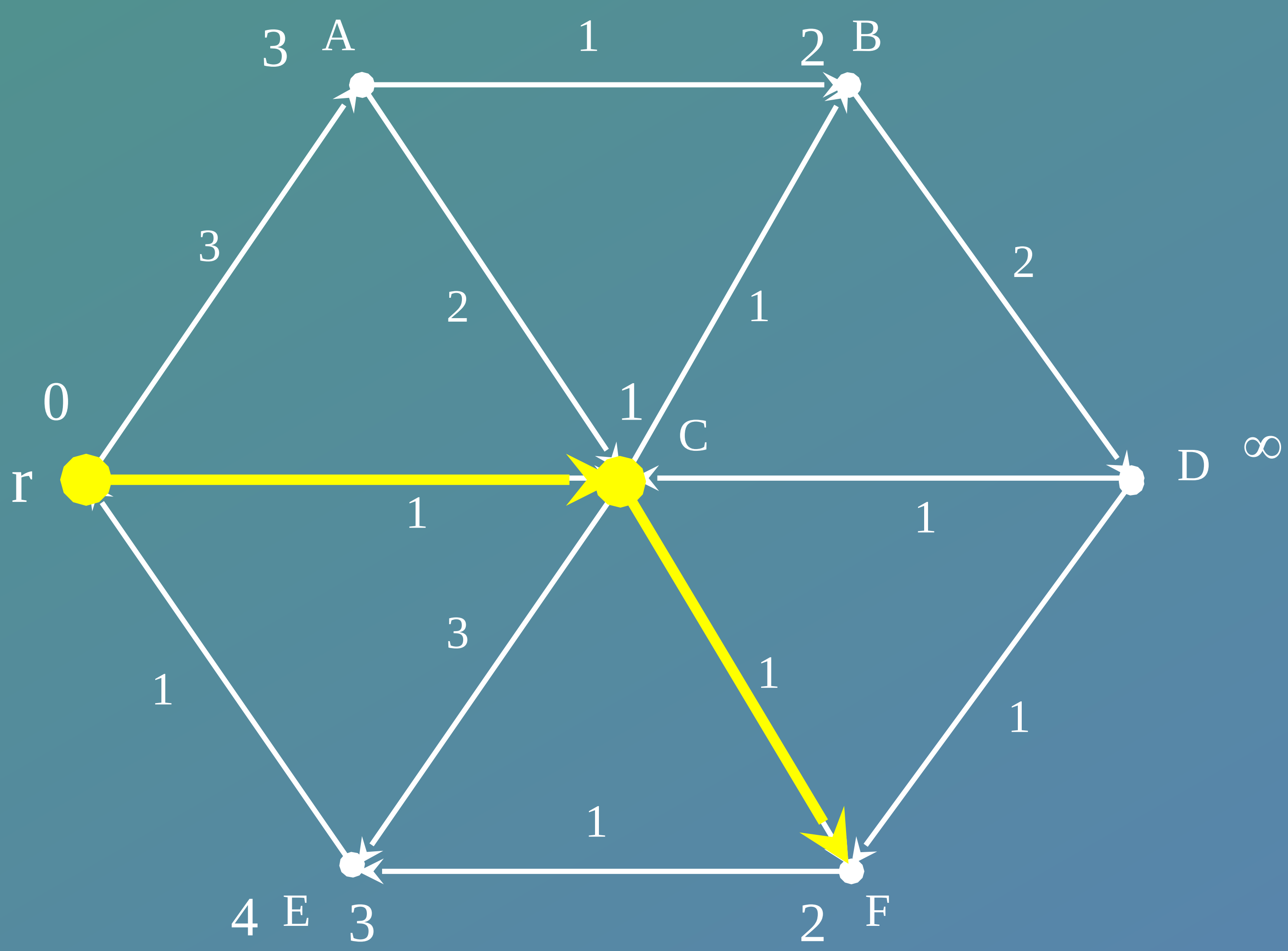

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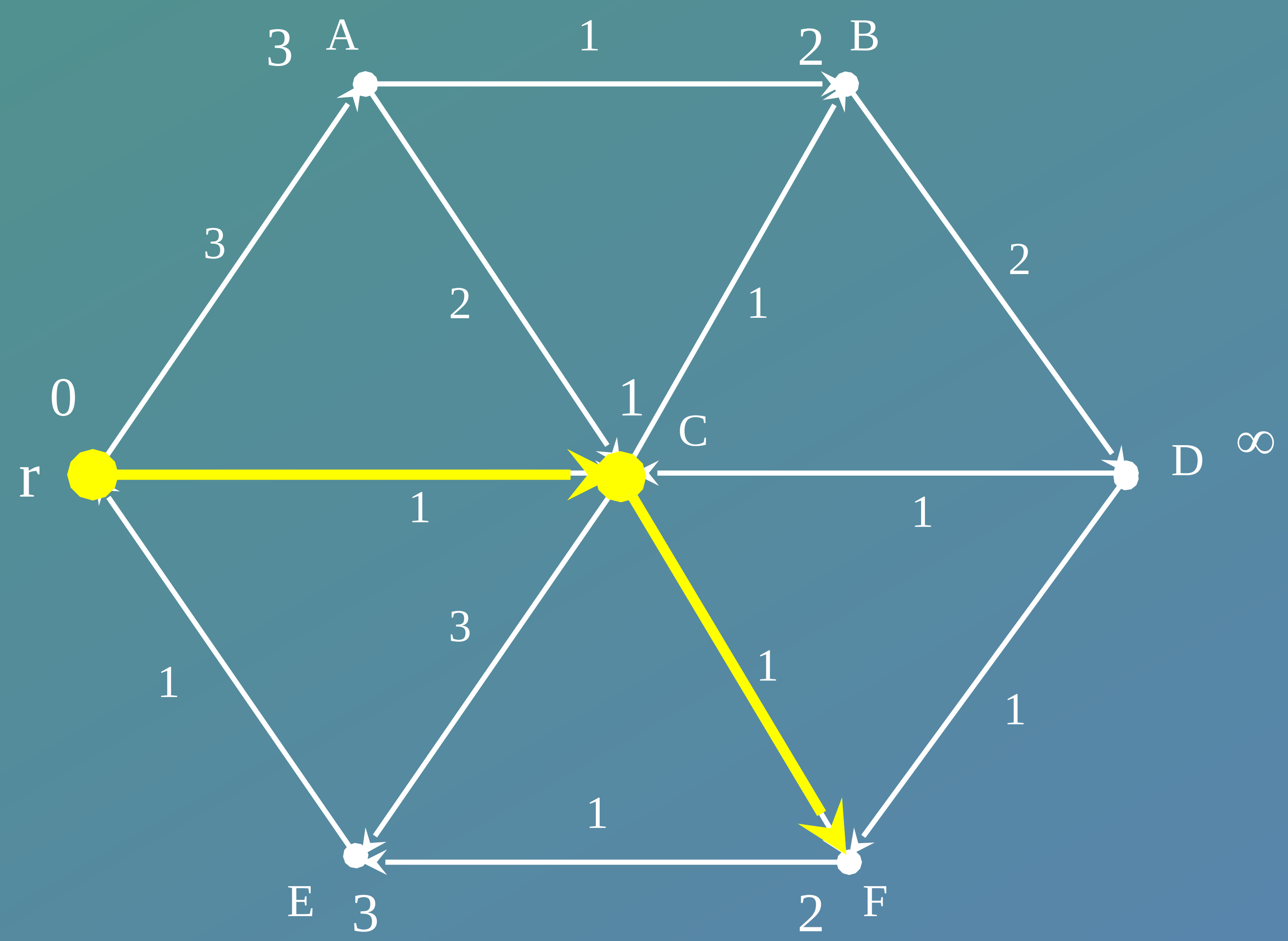

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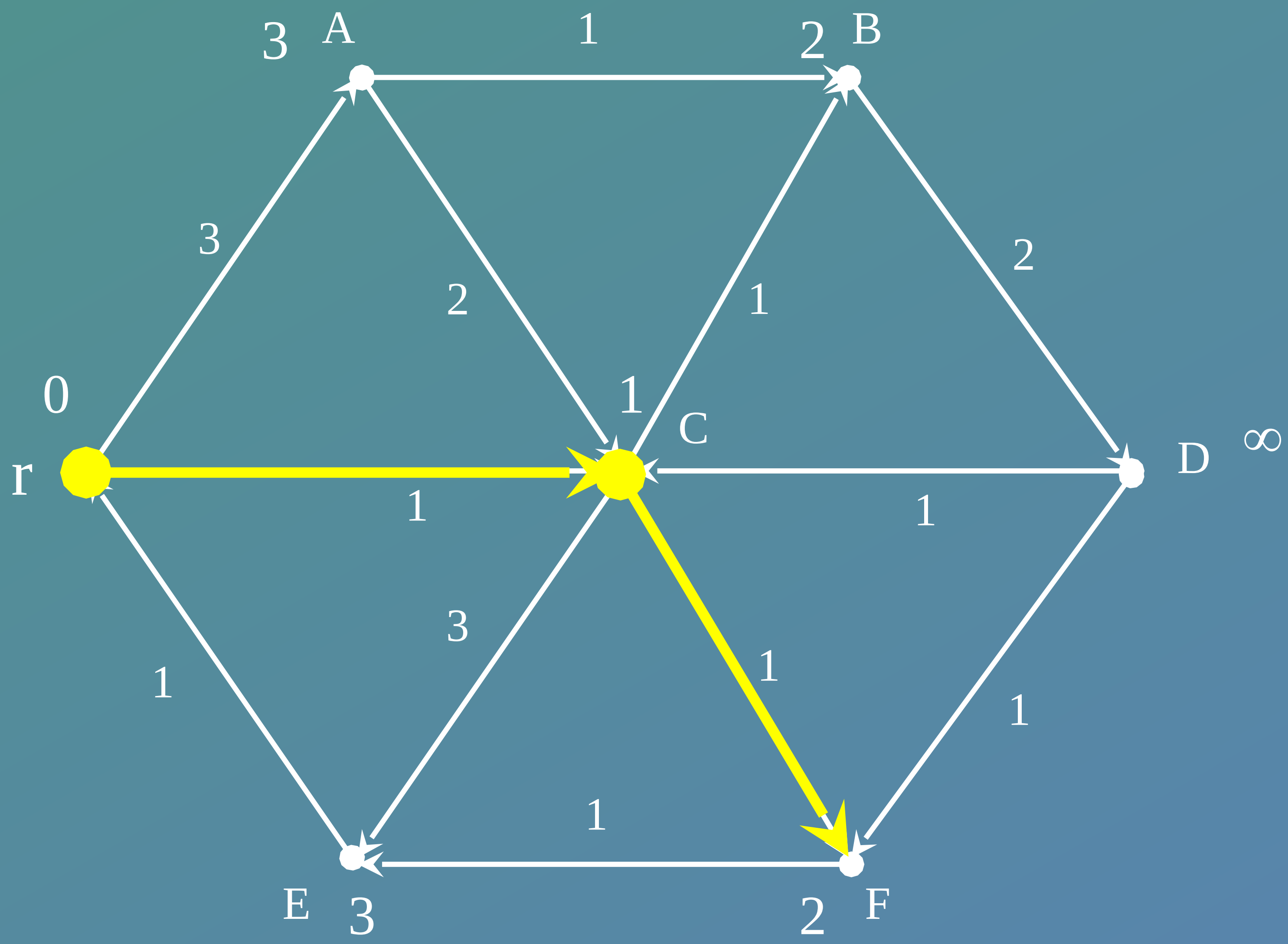

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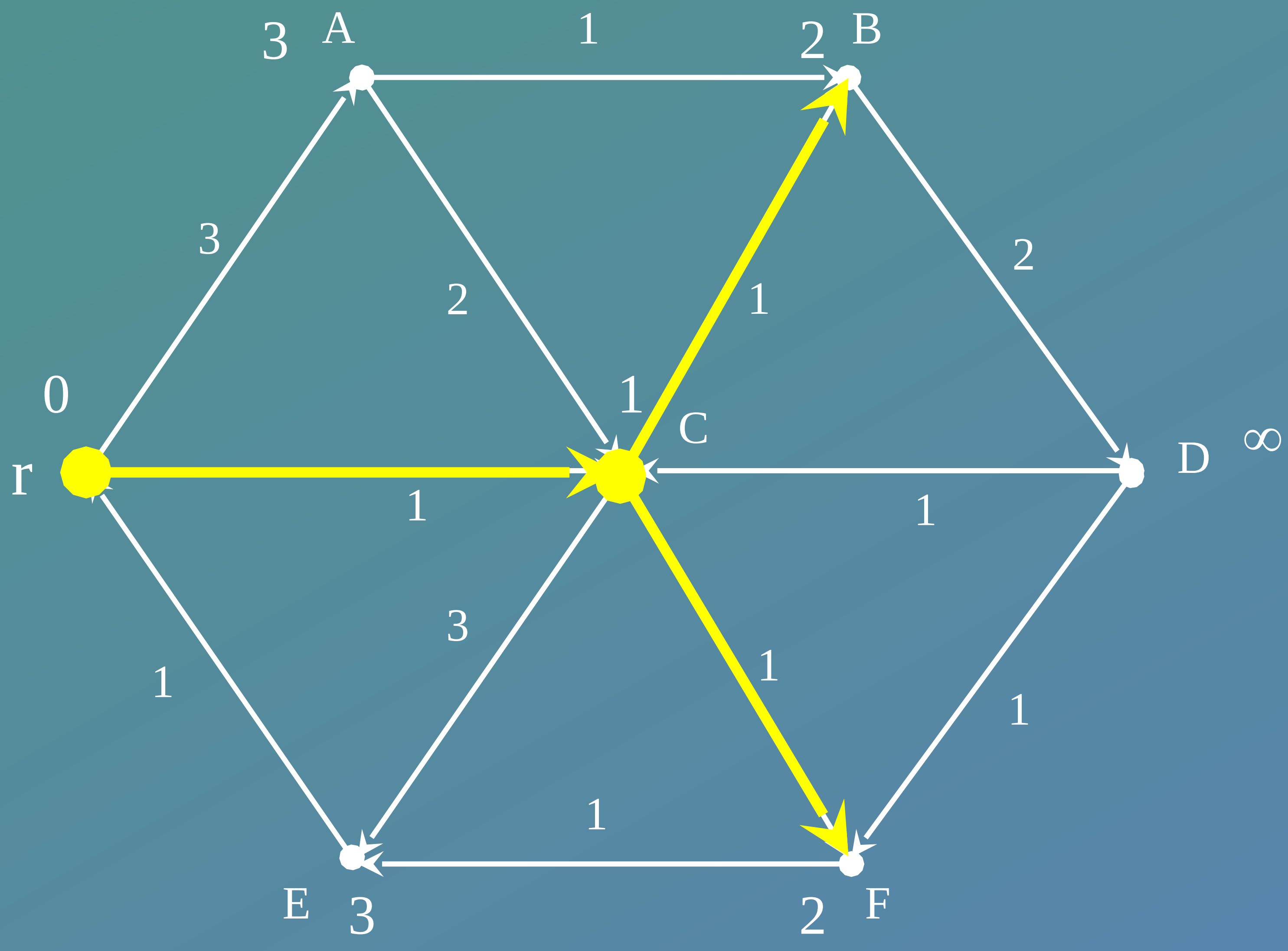
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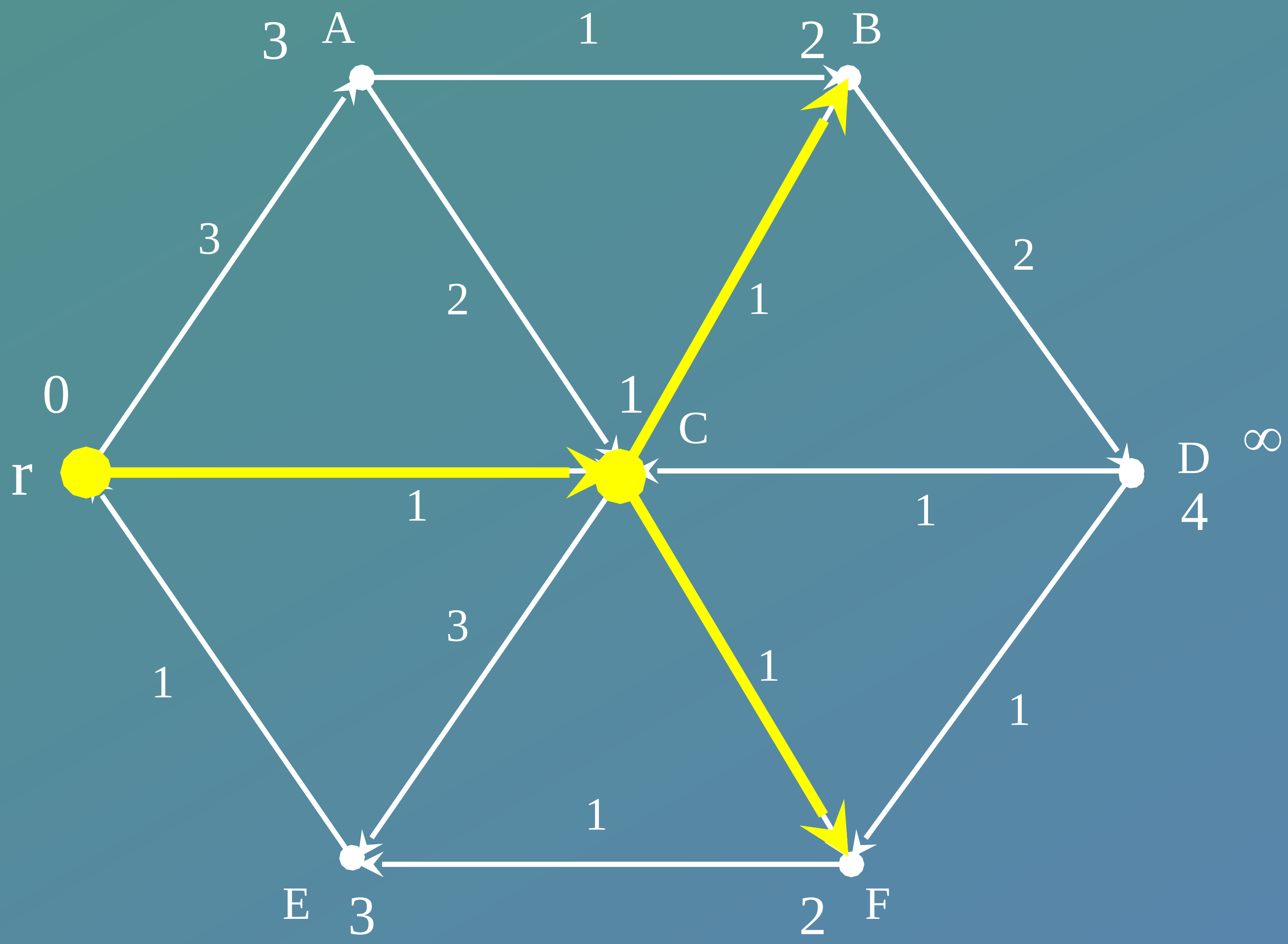
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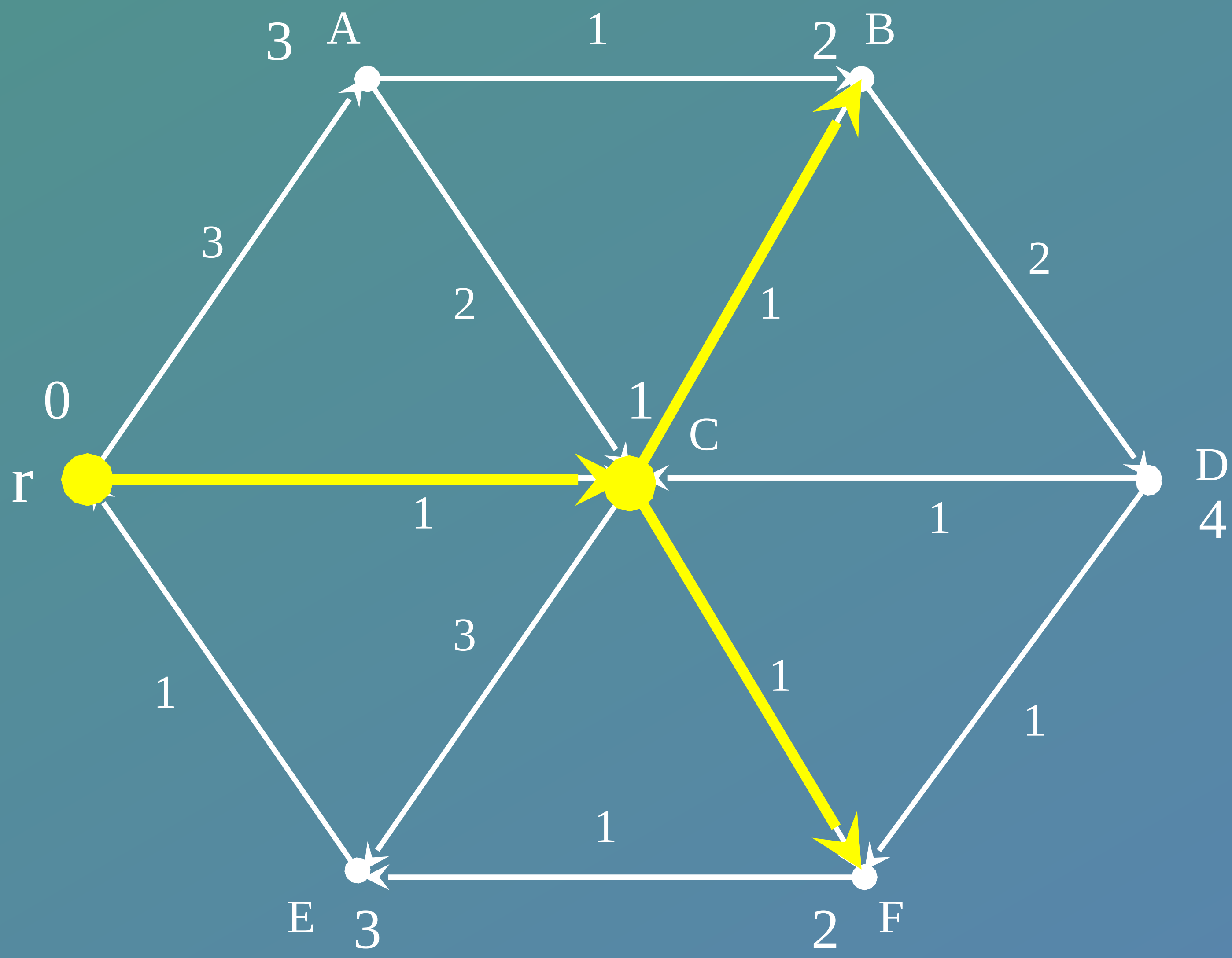
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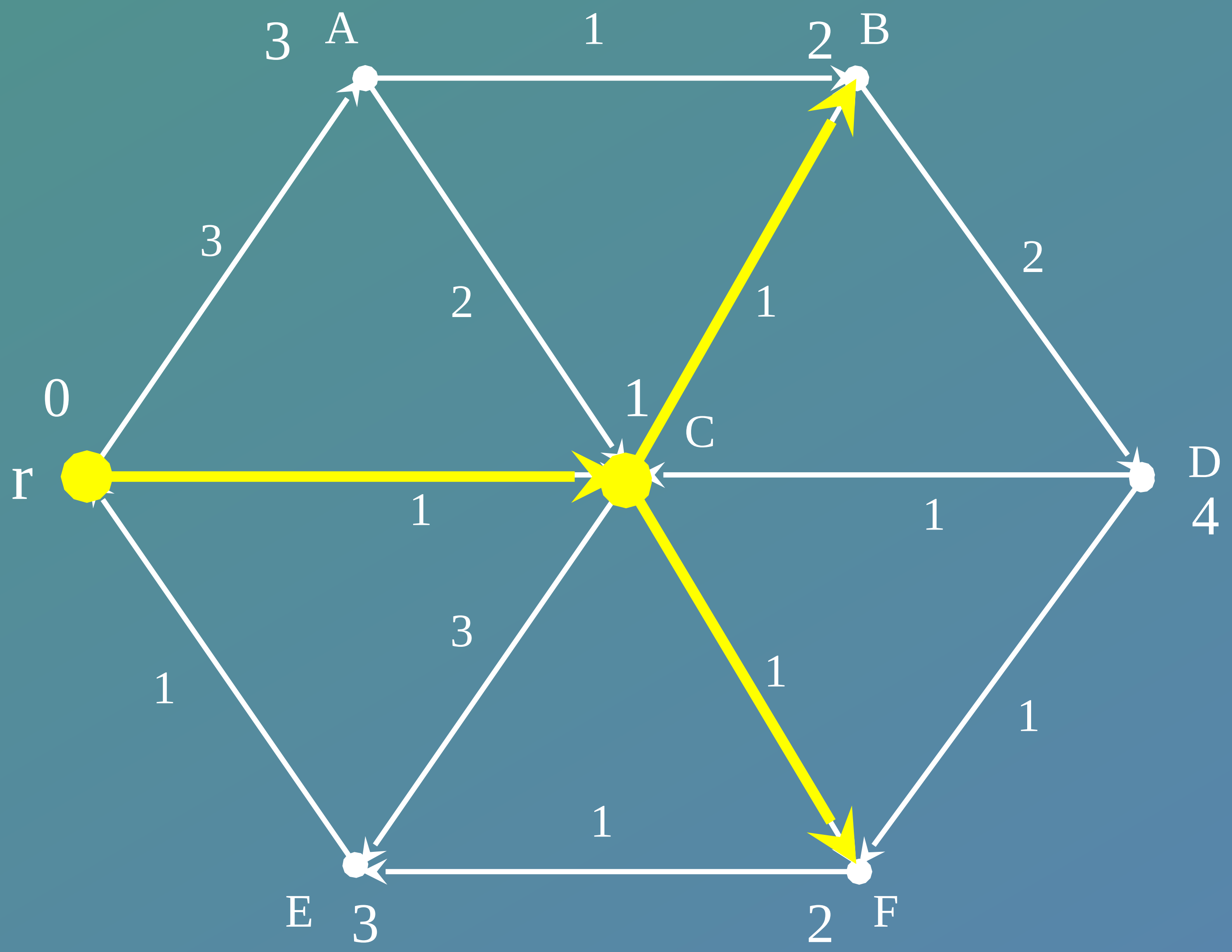

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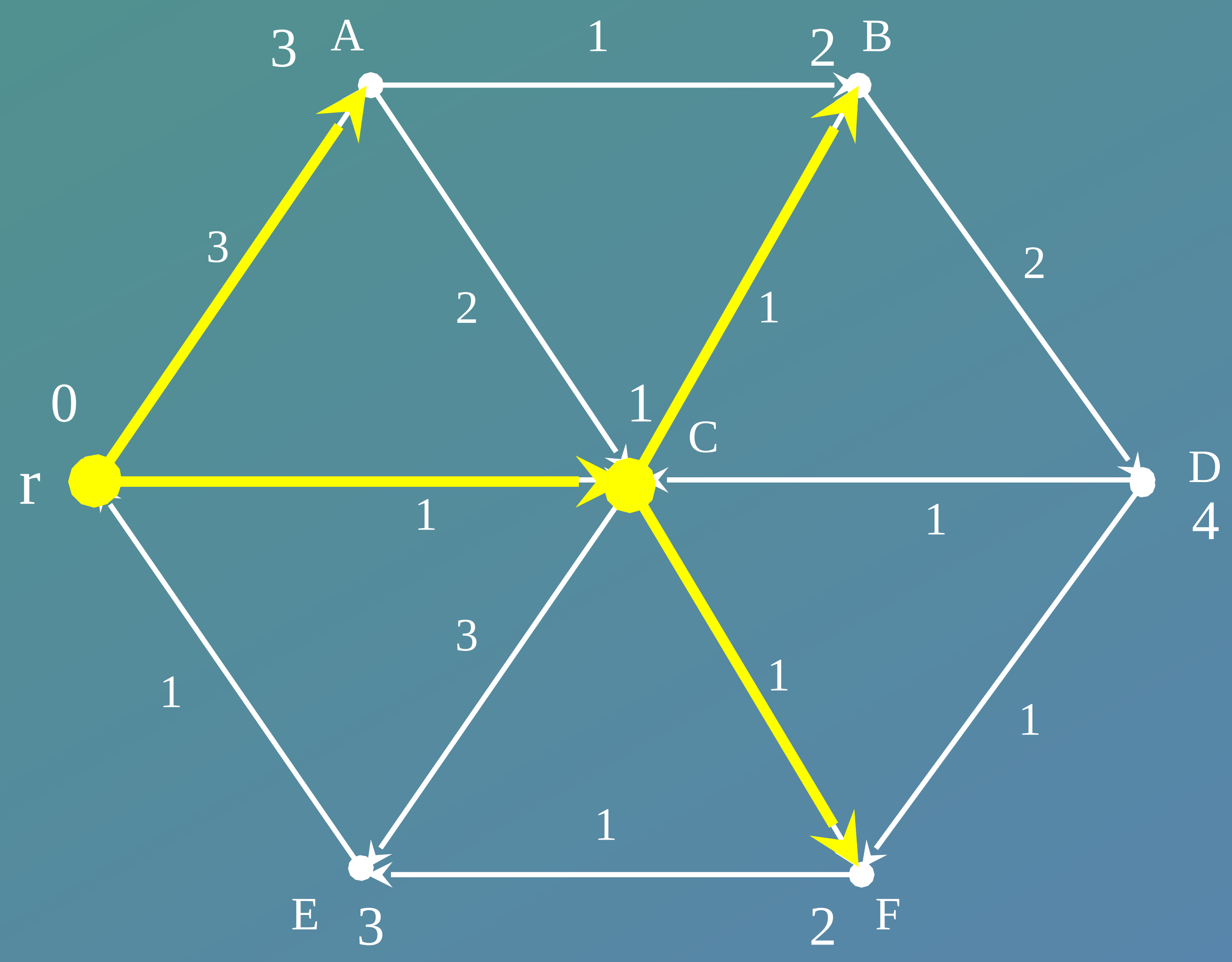

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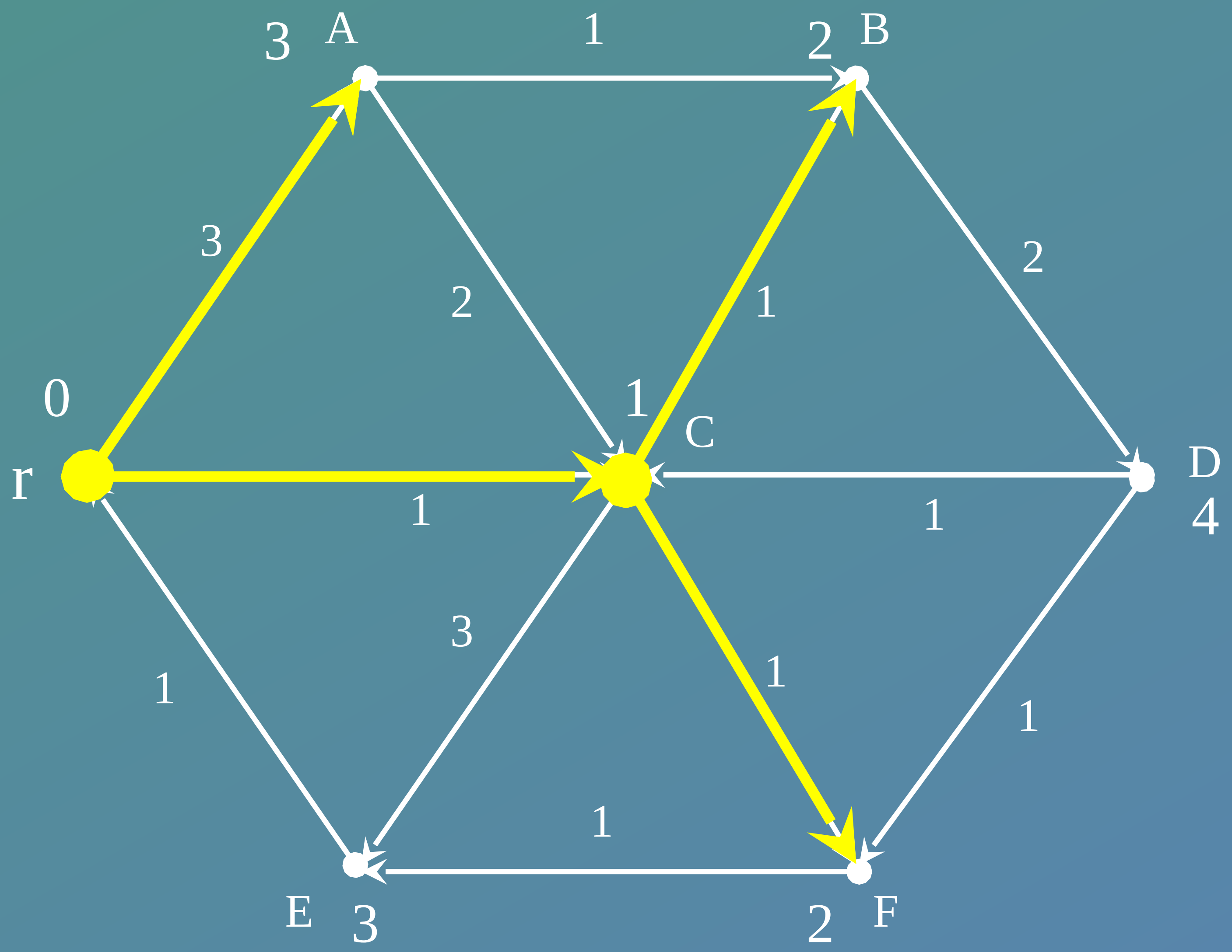
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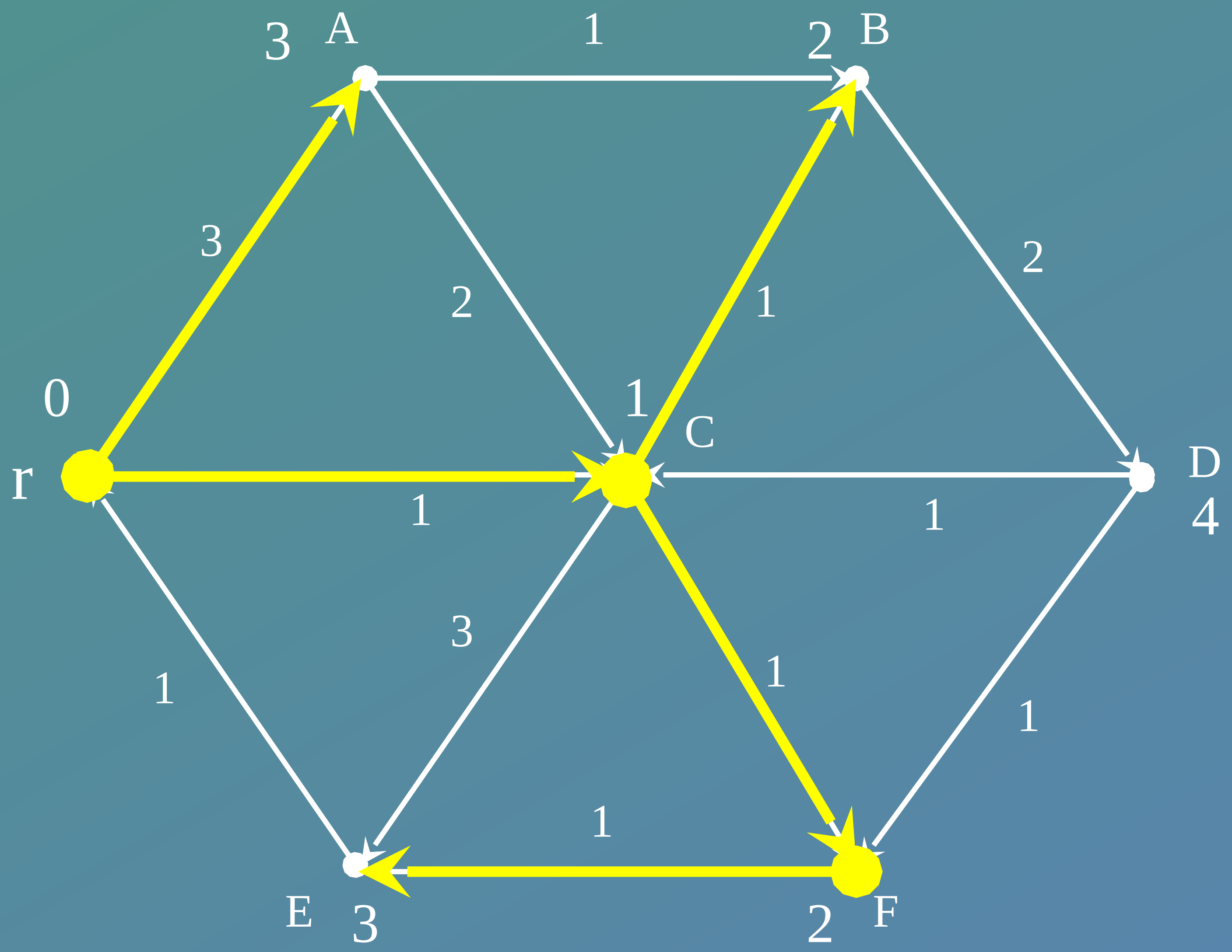
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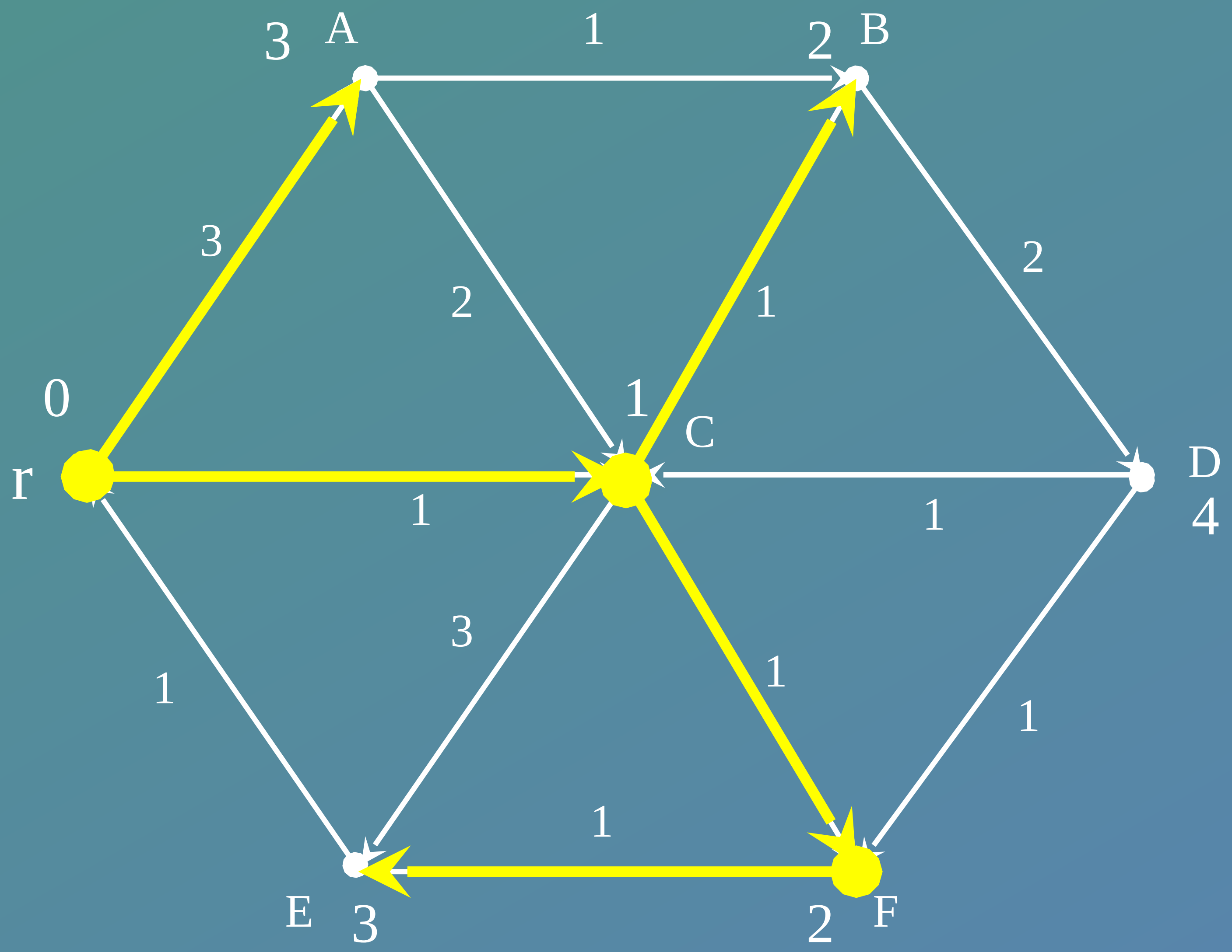
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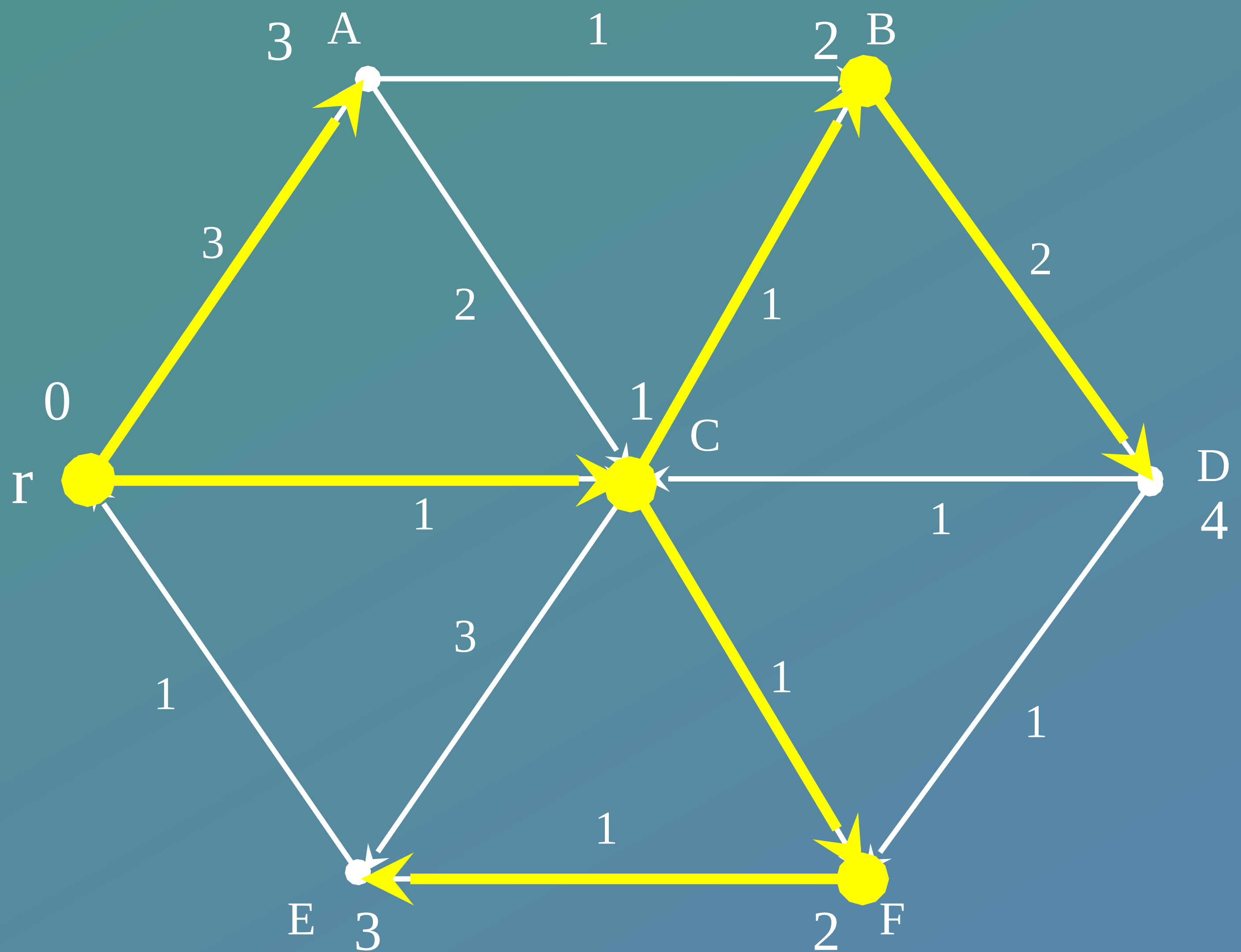

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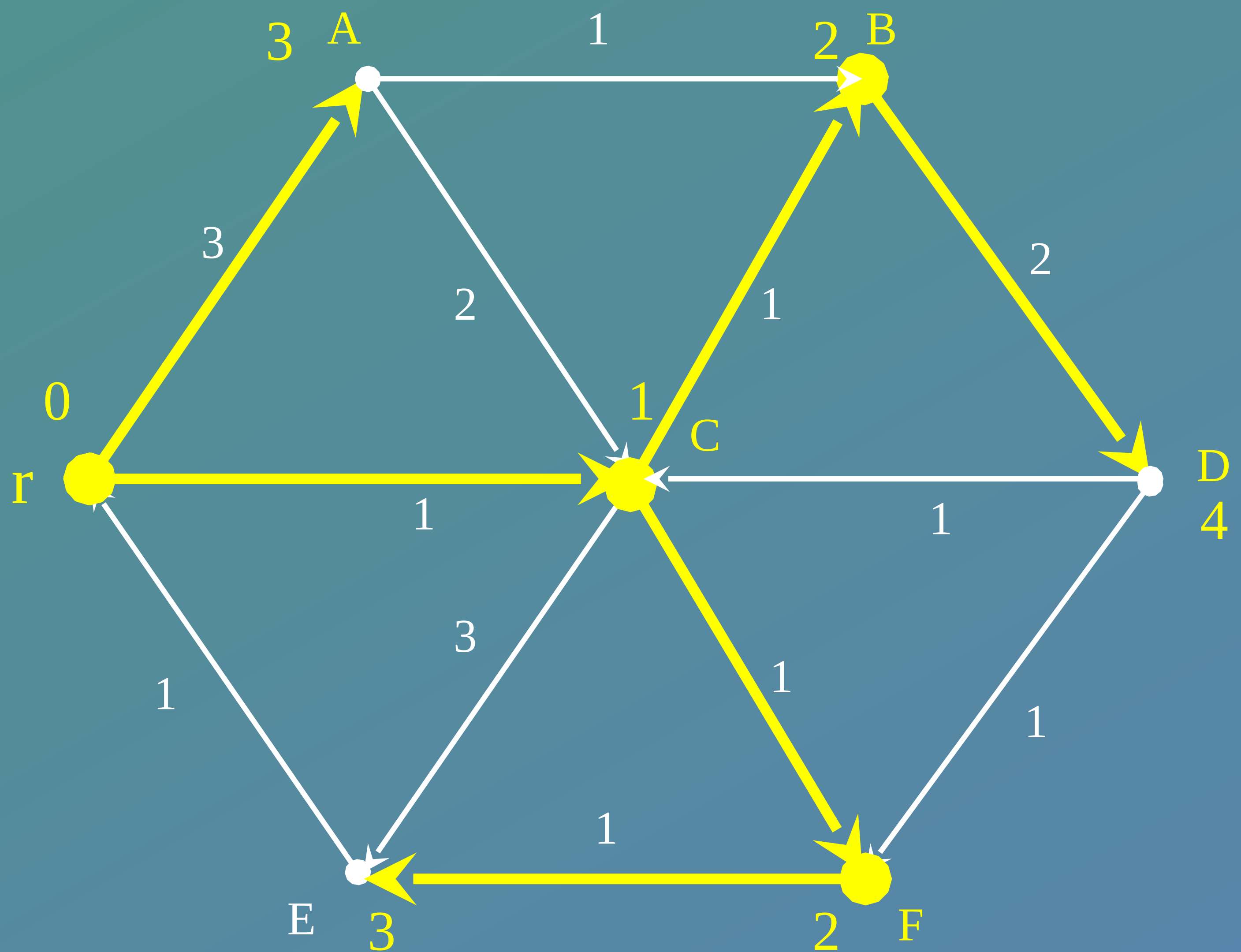

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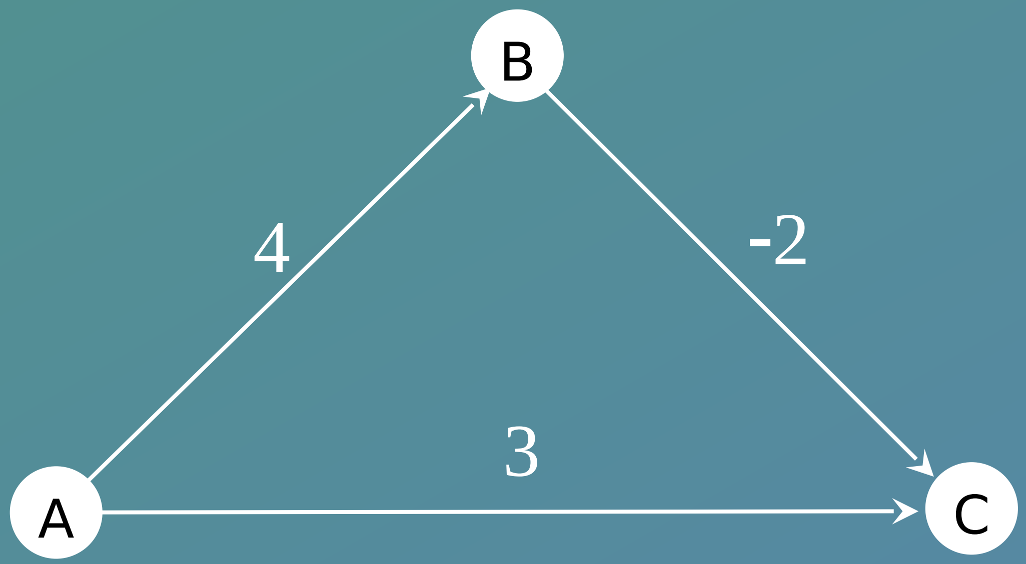
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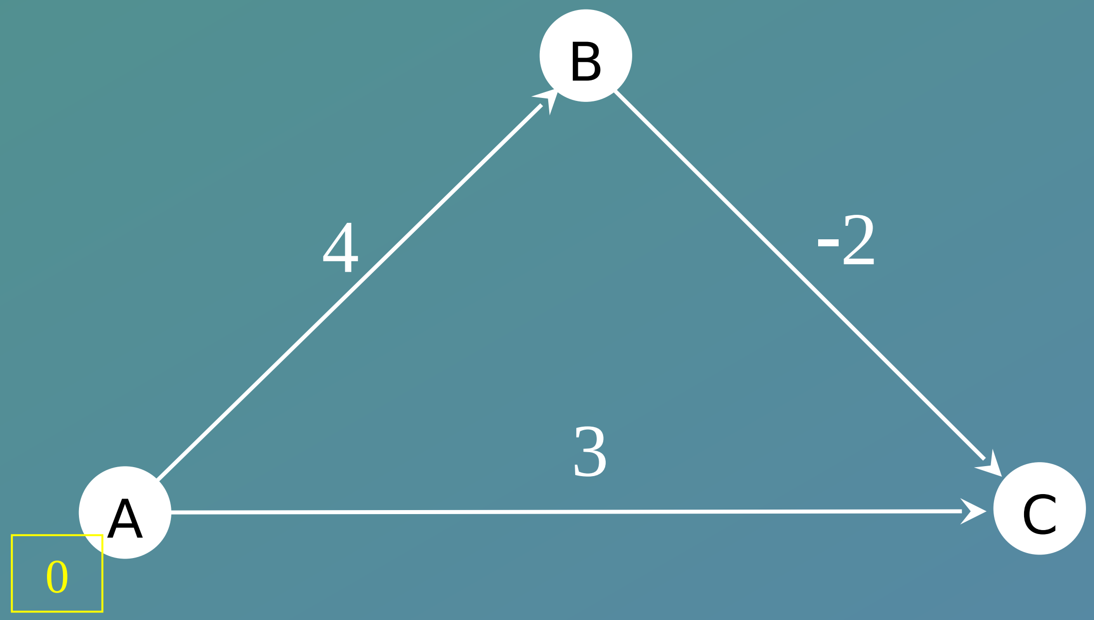
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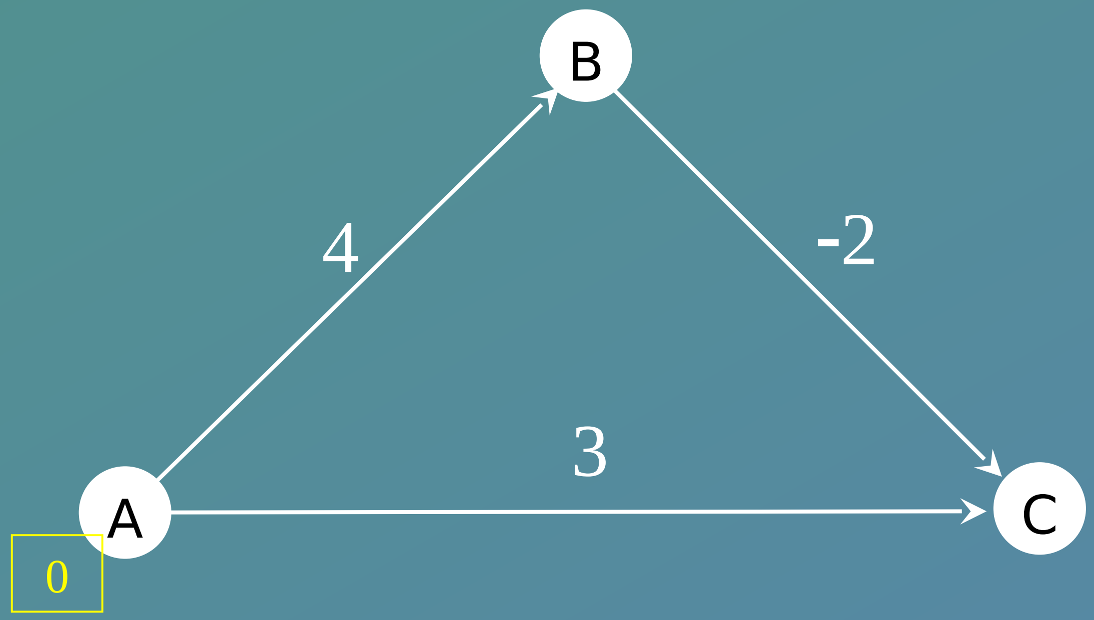
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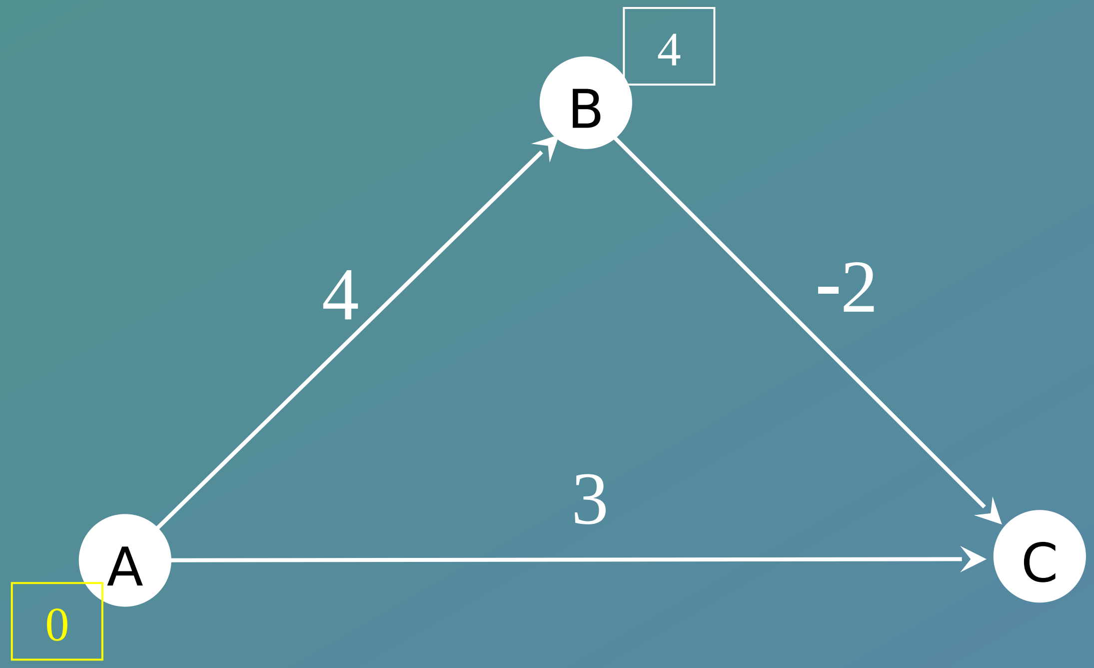
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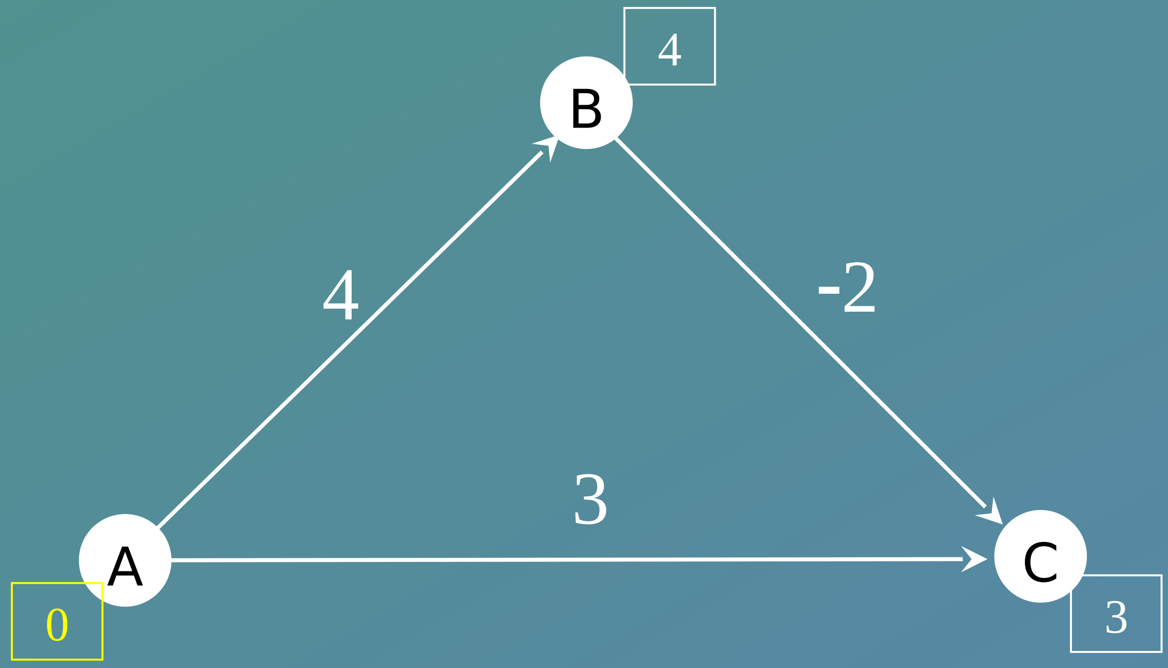
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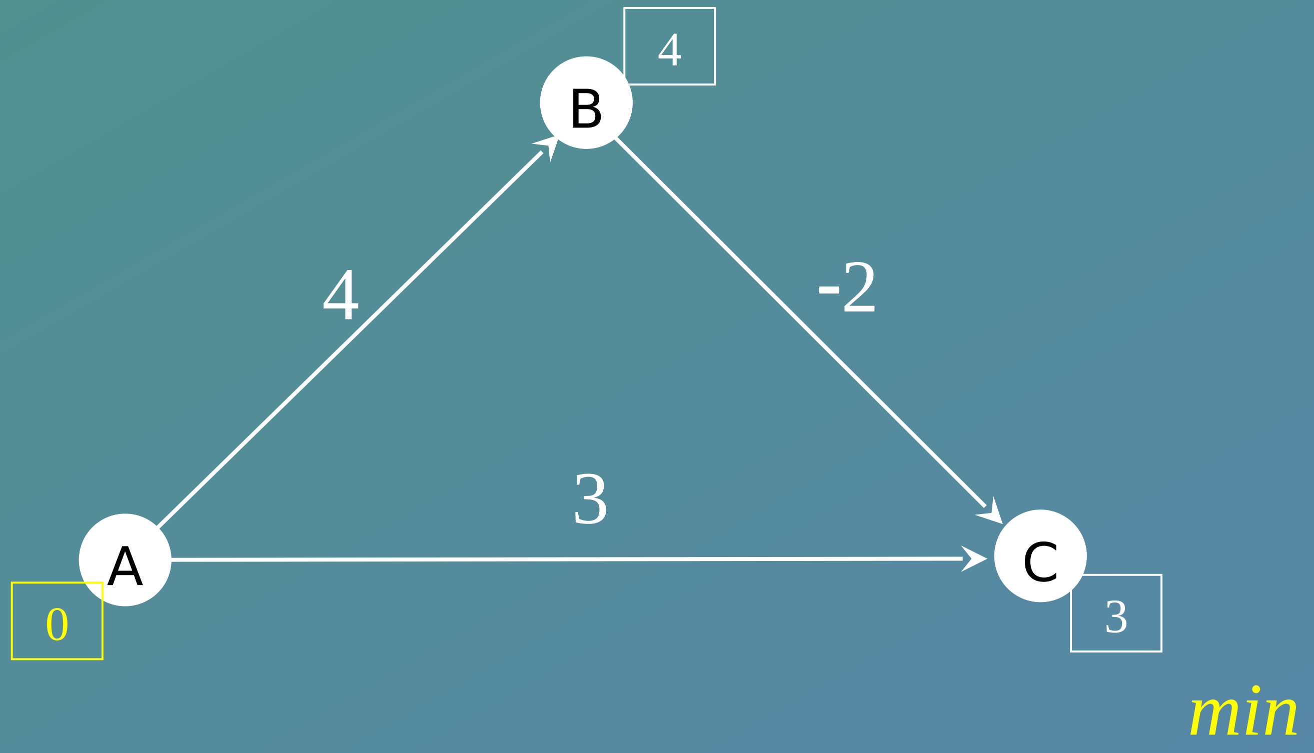
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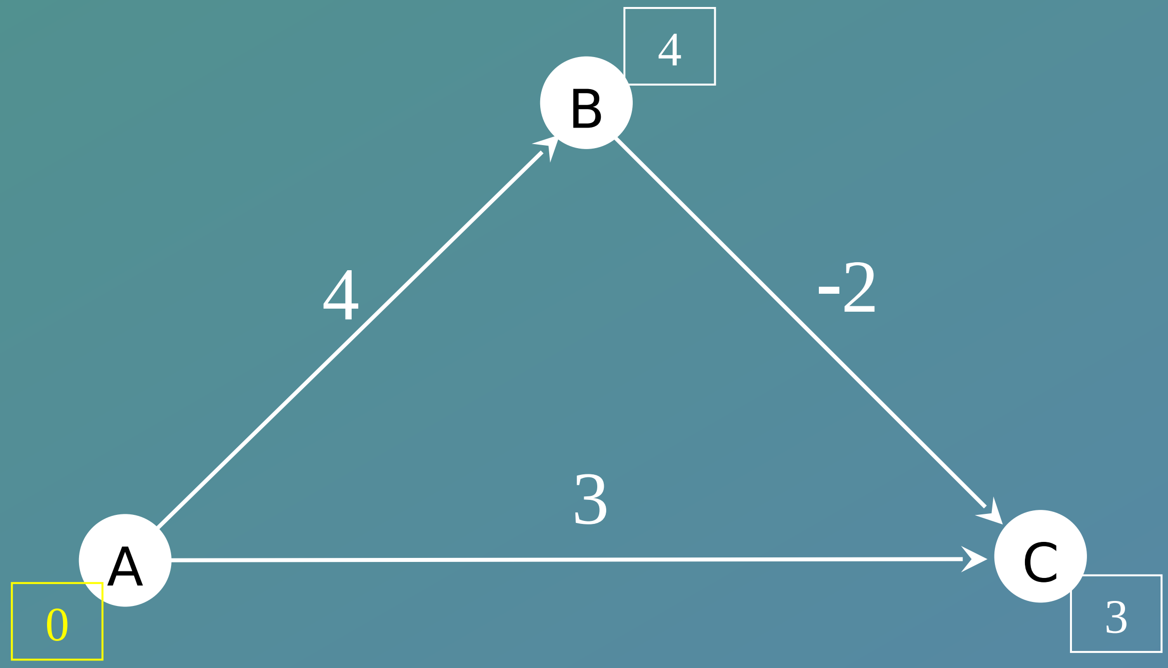

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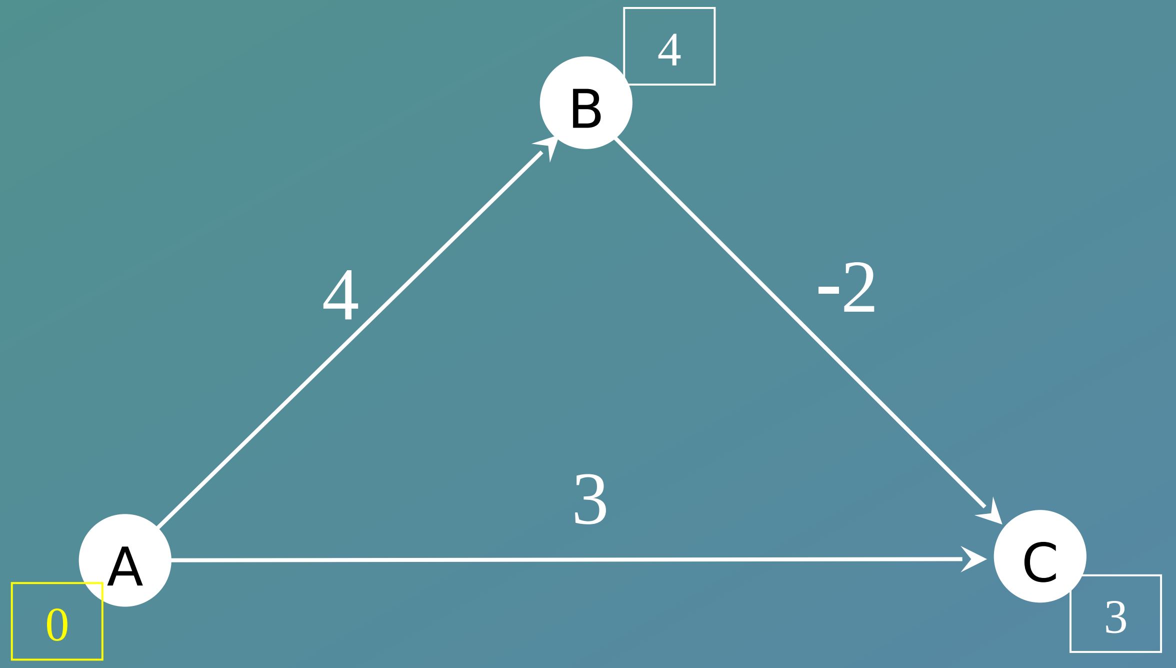

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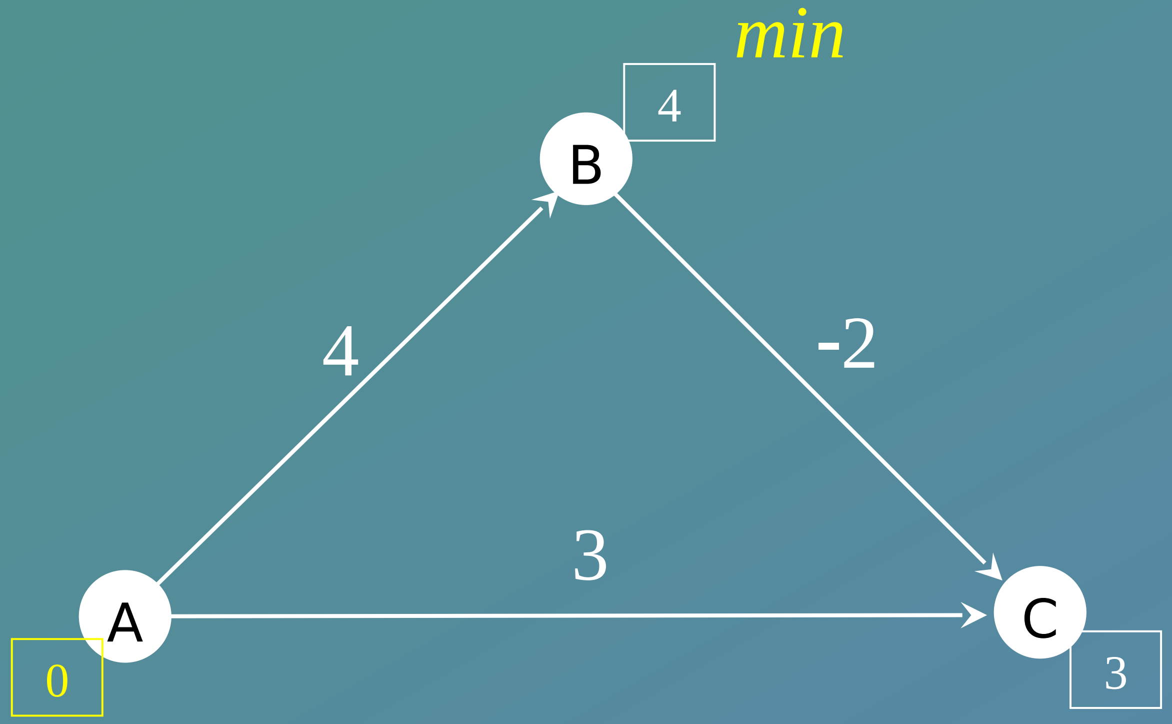
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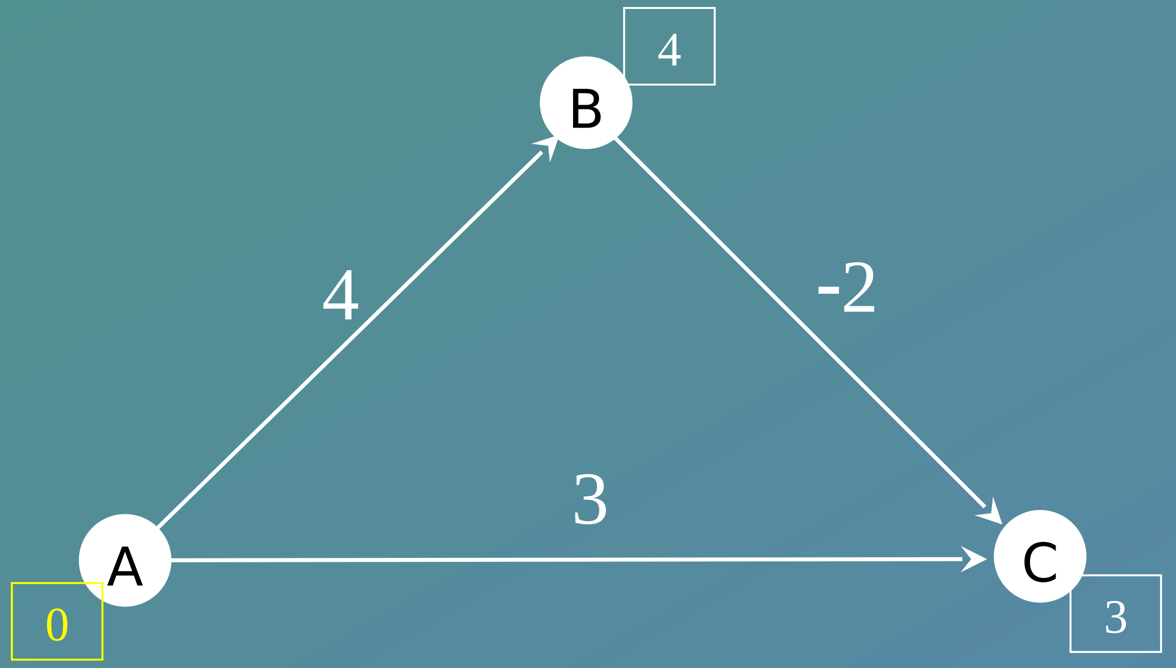
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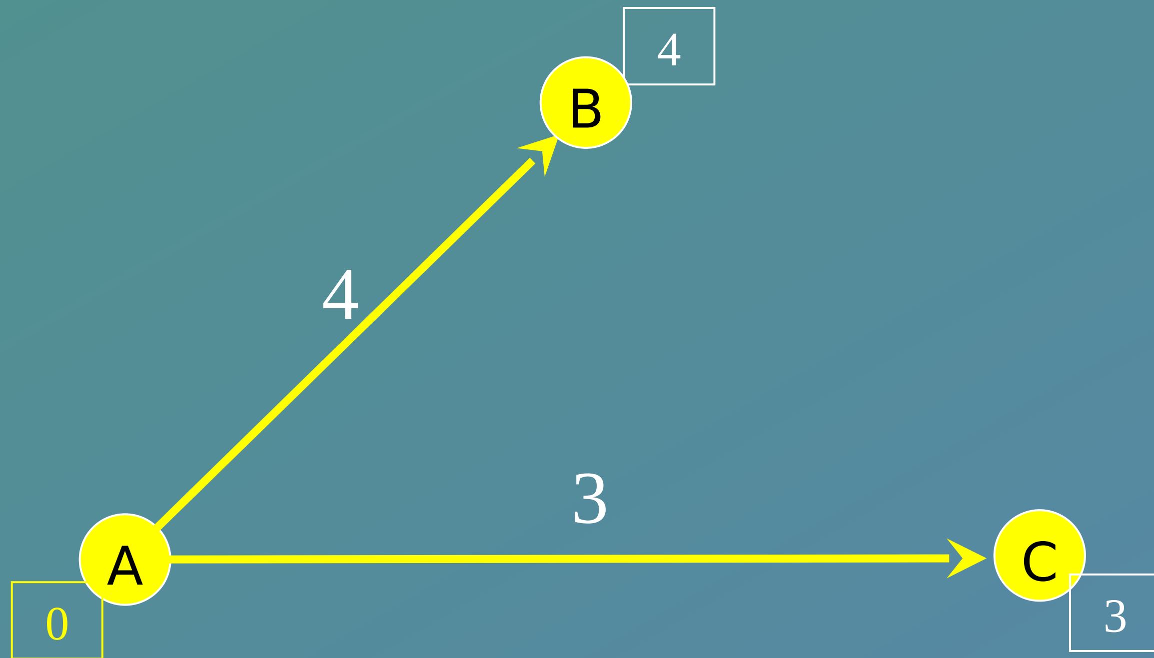
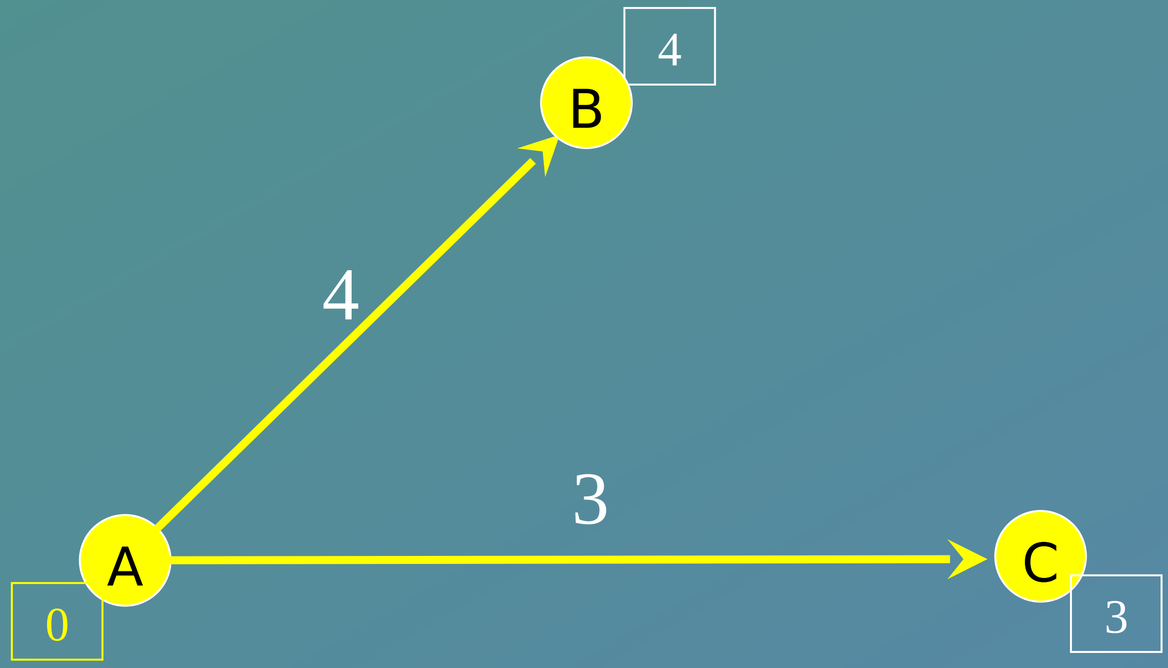


-2 = 2

une arborescence - mais pas celle des plus courts chemins



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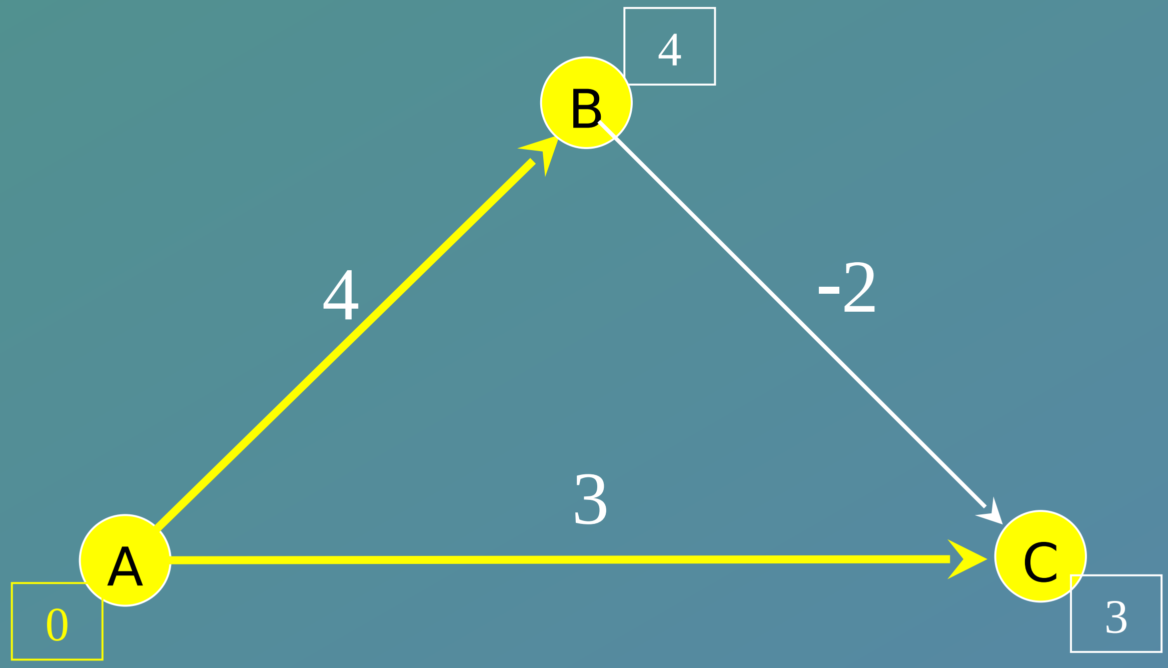

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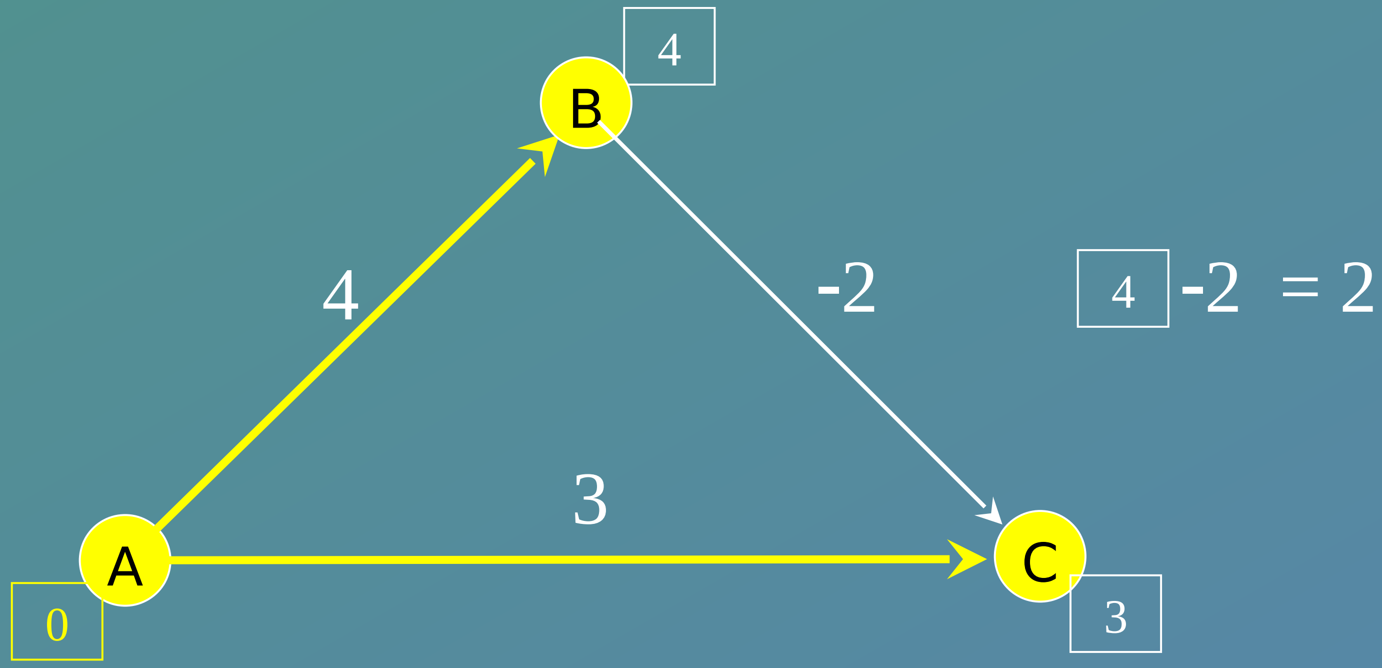

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4 -2 = 2
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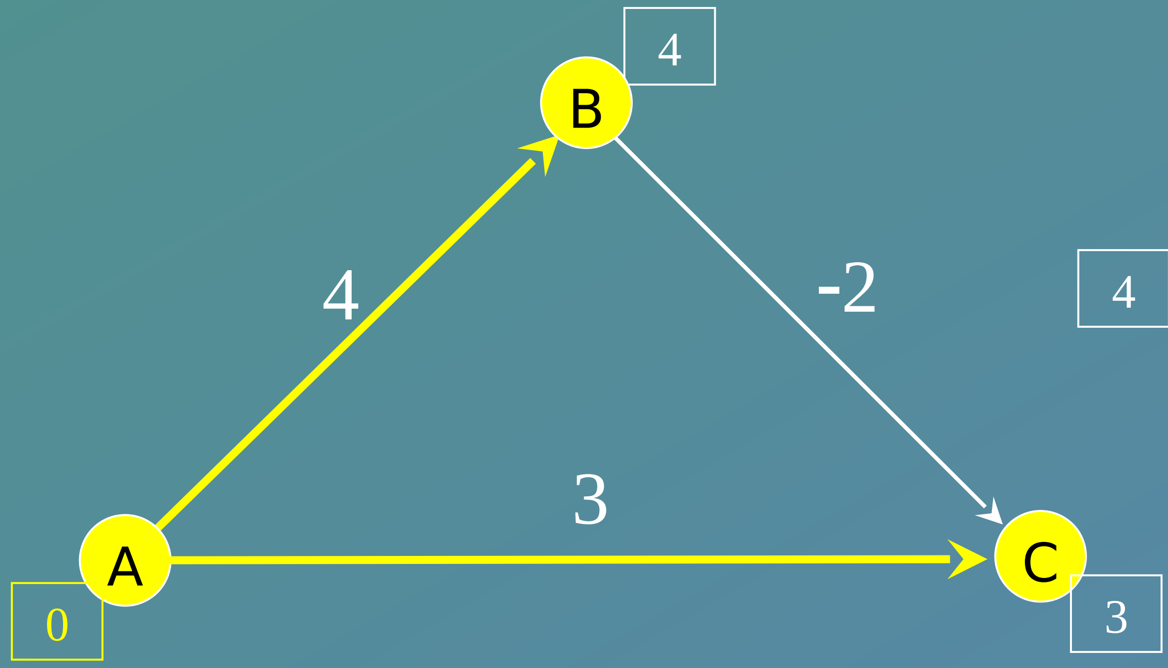
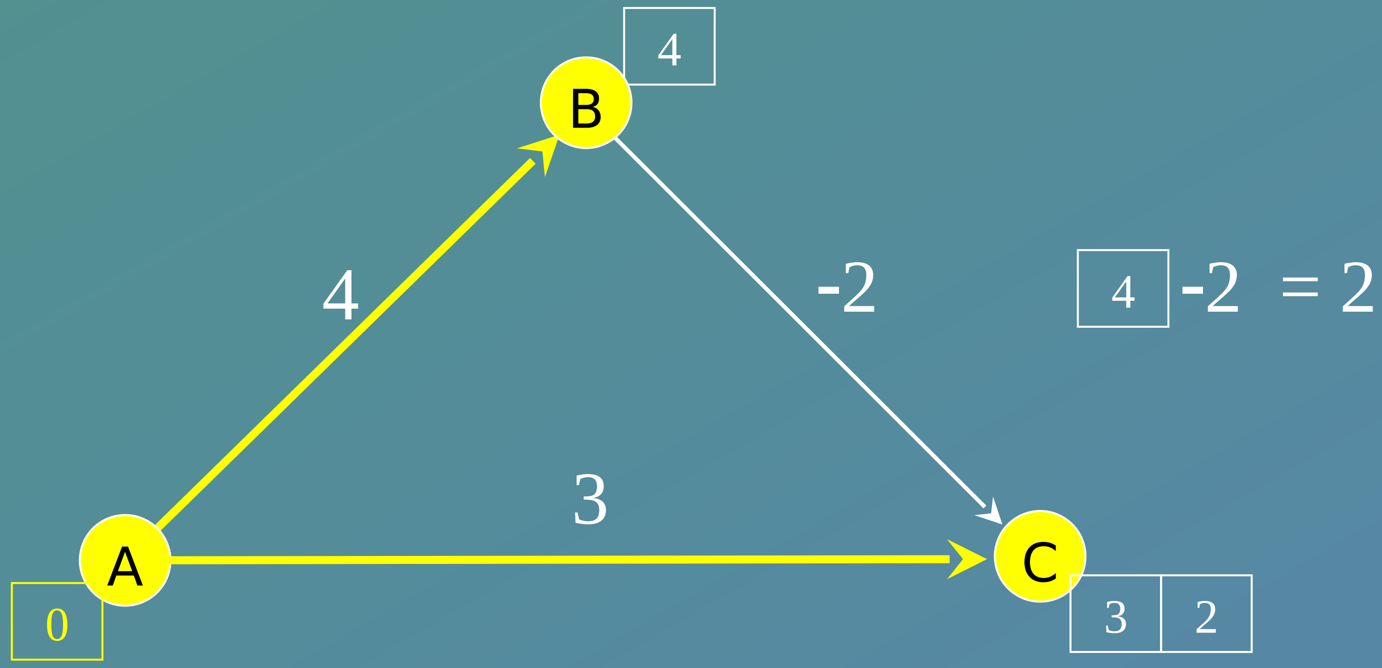


$\boxed{4} - 2 = 2$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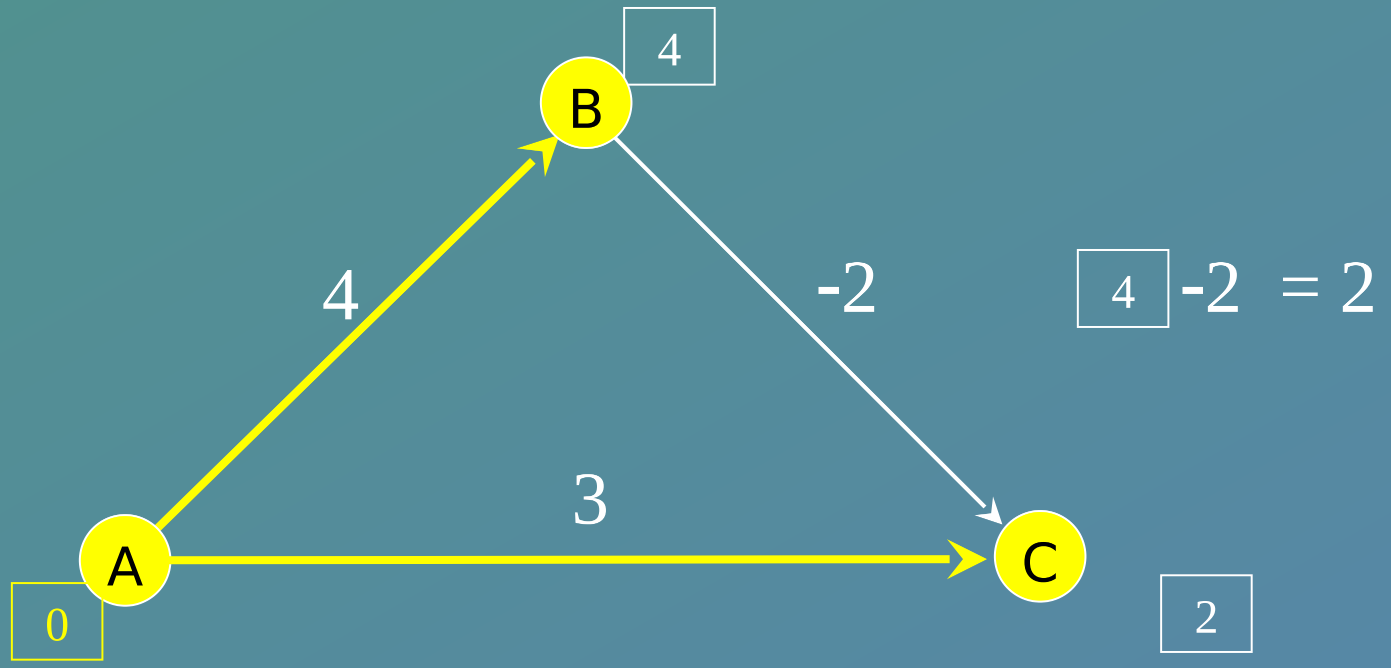
-2 = 2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4 -2 = 2
3
A
C
0
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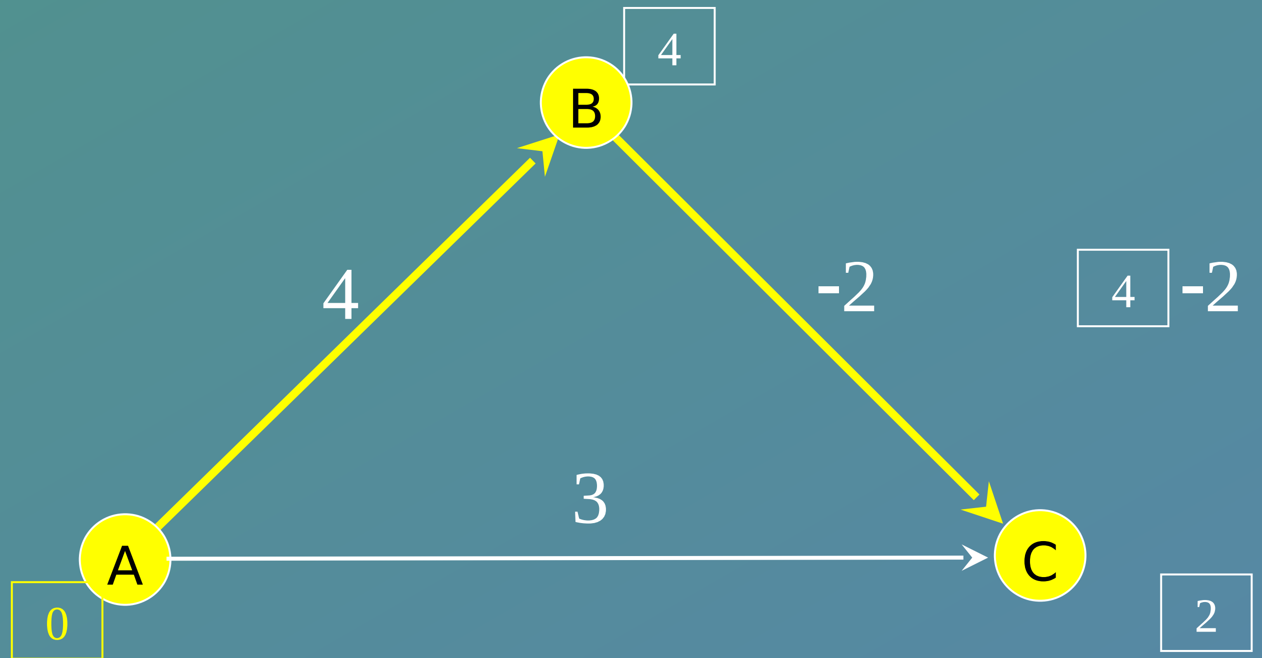


$\boxed{4} - 2 = 2$

Exercice 1:

Dans chacun des cinq pays A, B, C, D et E, le beurre a un prix différent: certains de ces pays surproduisent, d’autres doivent importer. Le gouvernement de chaque pays décide de créer un organisme dont le rôle est d’aider les échanges. Pour cela, dans un premier temps, chaque pays passe un accord commercial avec certains de ses voisins fixant des aides pour l’exportation vers des pays déficitaires et des taxes pour l’exportation vers des pays surproducteurs. Le premier tableau est celui des taxes : t_{ij} représente la taxe payée pour passer du pays i au pays j. Le deuxième tableau est celui des aides : le producteur reçoit a_{ij} s’il exporte de i vers j. S’il n’y a ni aide ni taxe, on supposera qu’il n’y a pas d’échange direct possible. Par ailleurs le transport du beurre coûte lui-même un certain prix précisé dans le troisième tableau (taxes, aides et coûts étant donnés en centaines d’euros par tonne).

	TAXES				
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

TAXES

i j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

AIDES

i j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

COÛTS

i j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

a) Modéliser le problème de transfère du beurre du pays C vers le pays B sous forme d'un problème de plus court chemin dans un graphe orienté valué.

	TAXES				
i j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

	TAXES				
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

	TAXES				
i j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

TAXES

i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

AIDES

i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

COÛTS

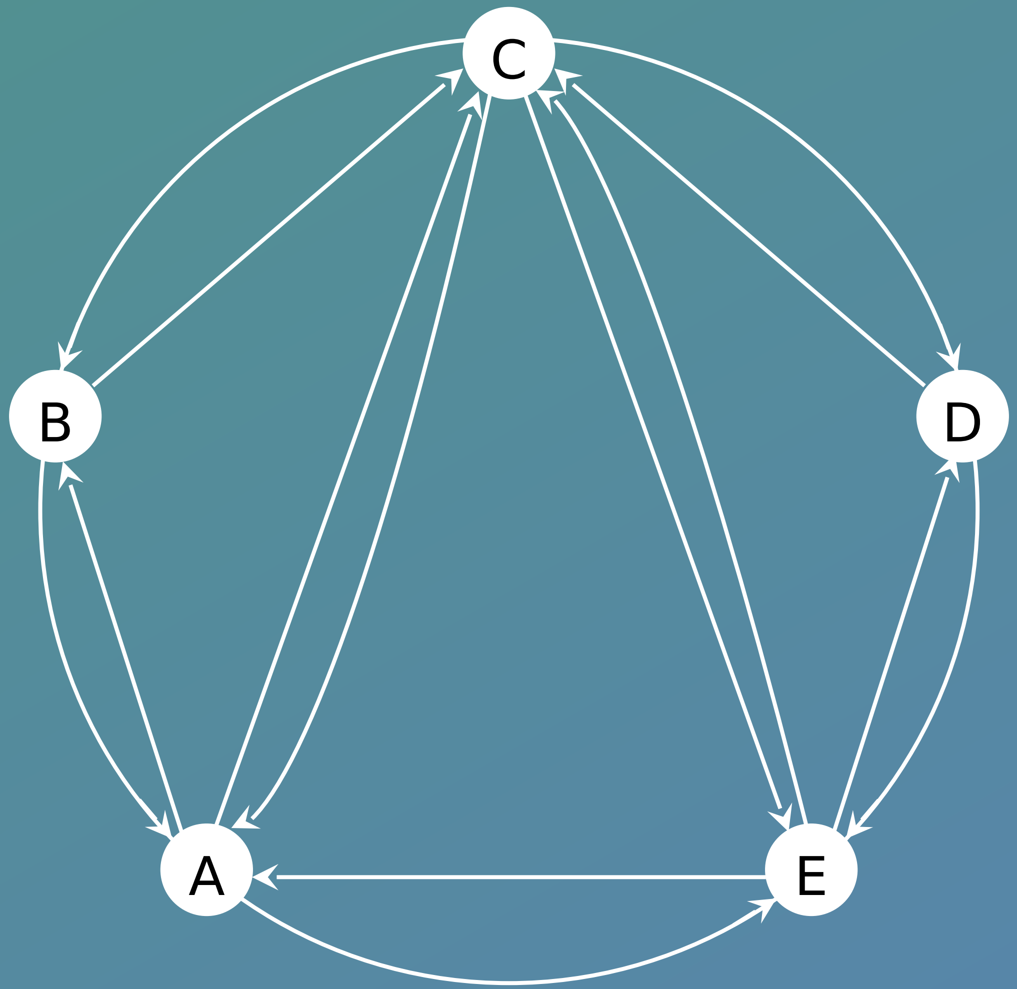
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

			TAXES		
i j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

			AIDES		
i j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

			COÛTS		
i j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

i j	A	B	C	D	E
A	0	1	1	0	1
B	1	0	1	0	0
C	1	1	0	1	1
D	0	0	1	0	1
E	1	0	1	1	0


C
B
D
A
E

	TAXES				
i j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

	TAXES				
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

+

TAXES					
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

AIDES					
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

COÛTS					
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

+

+

		TAXES			
i j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

		AIDES			
i j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

		COÛTS			
i j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

+

-

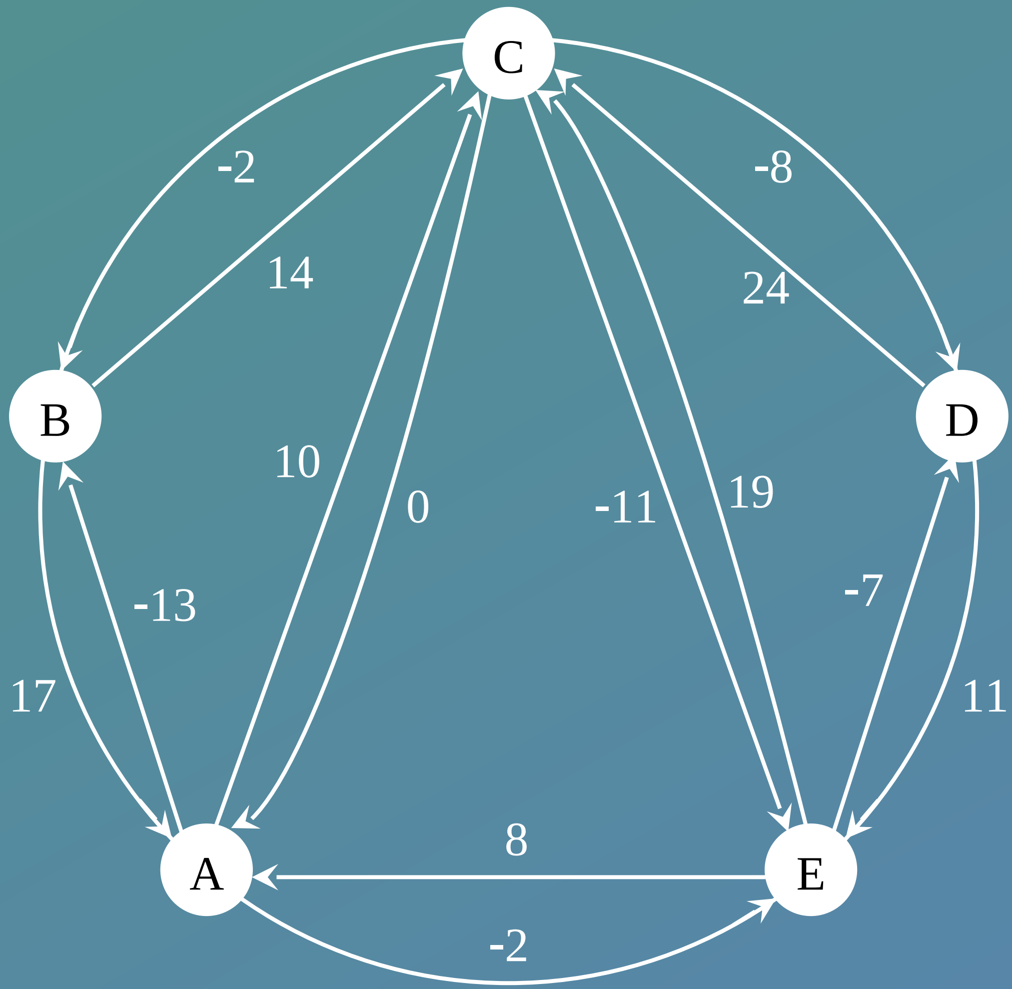
+

	TAXES				
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

i \ j	A	B	C	D	E
A		-13	10		-2
B	17		14		
C	0	-2		-8	-11
D			24		11
E	8		19	-7	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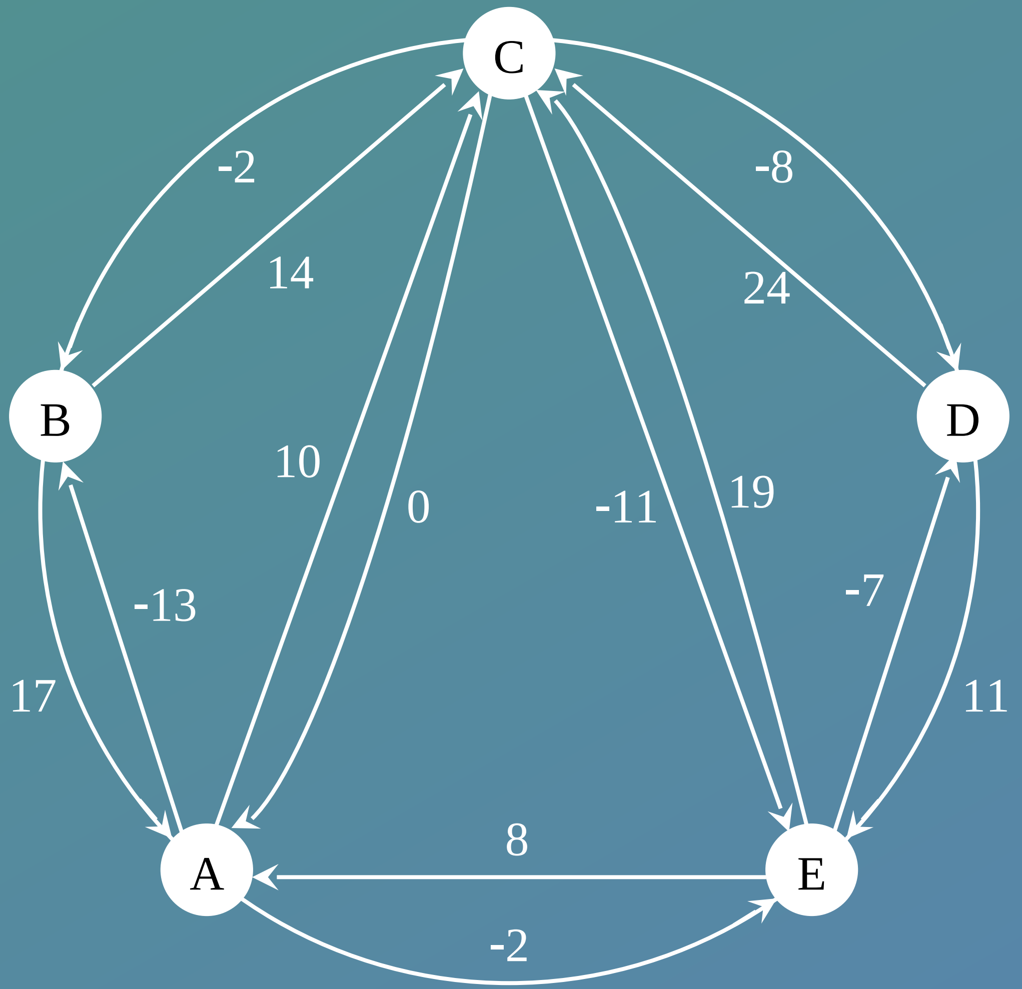
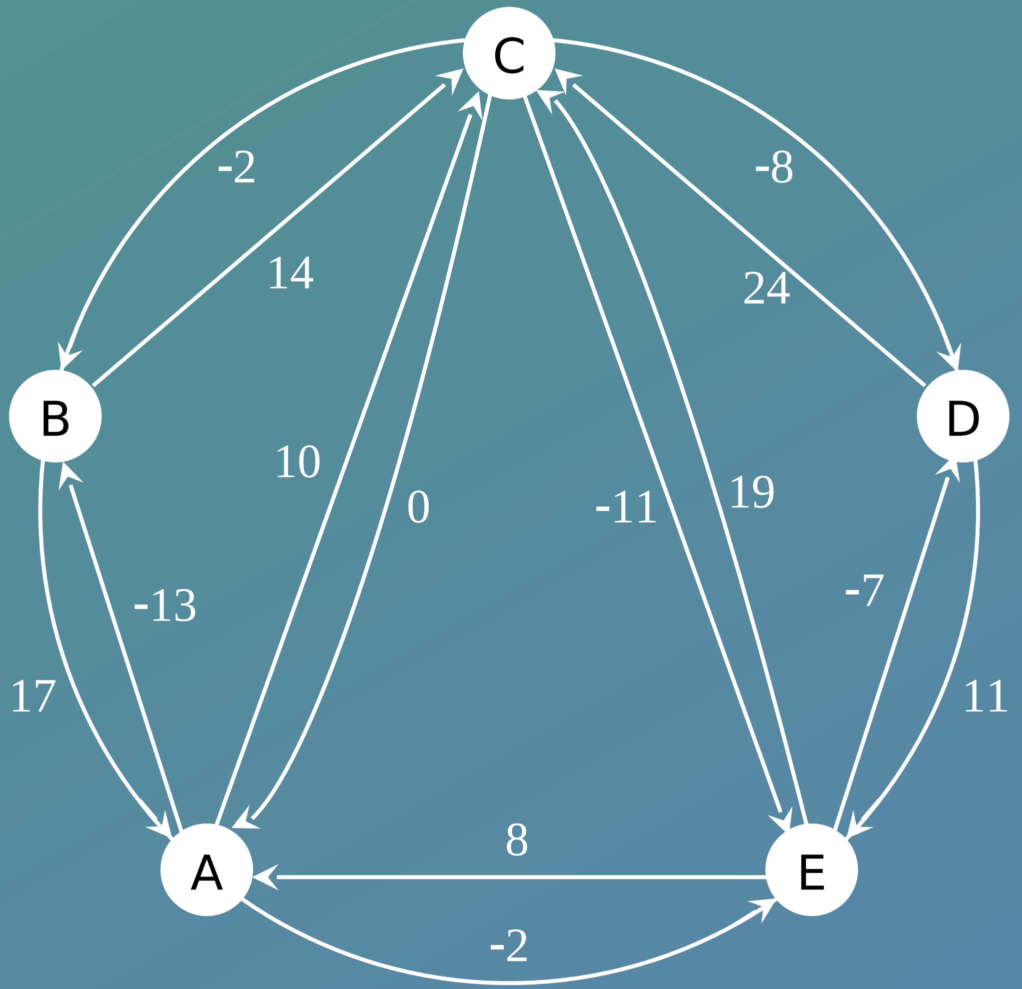
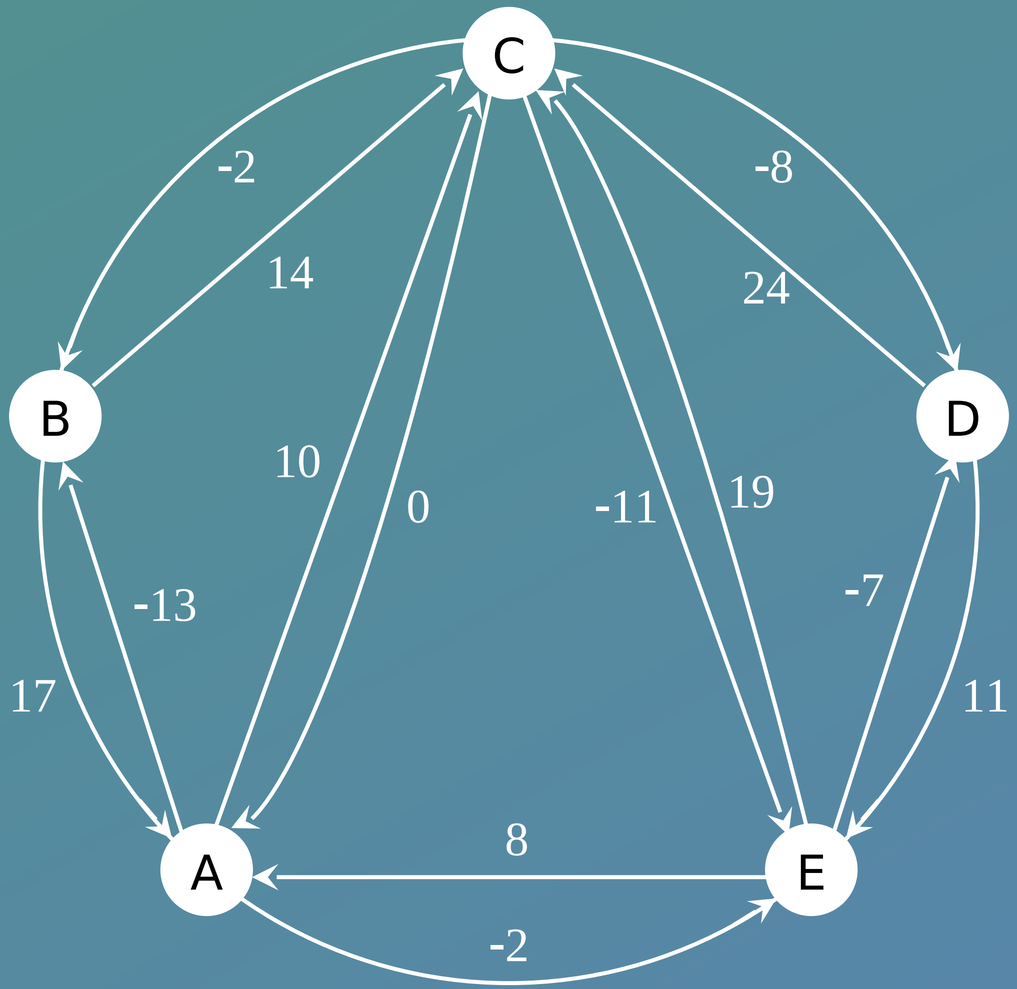


b) Peut-on appliquer un algorithme vu en cours pour résoudre ce problème? (Justifier)

c) Appliquer formellement l'algorithme de DIJKSTRA à partir du sommet C.

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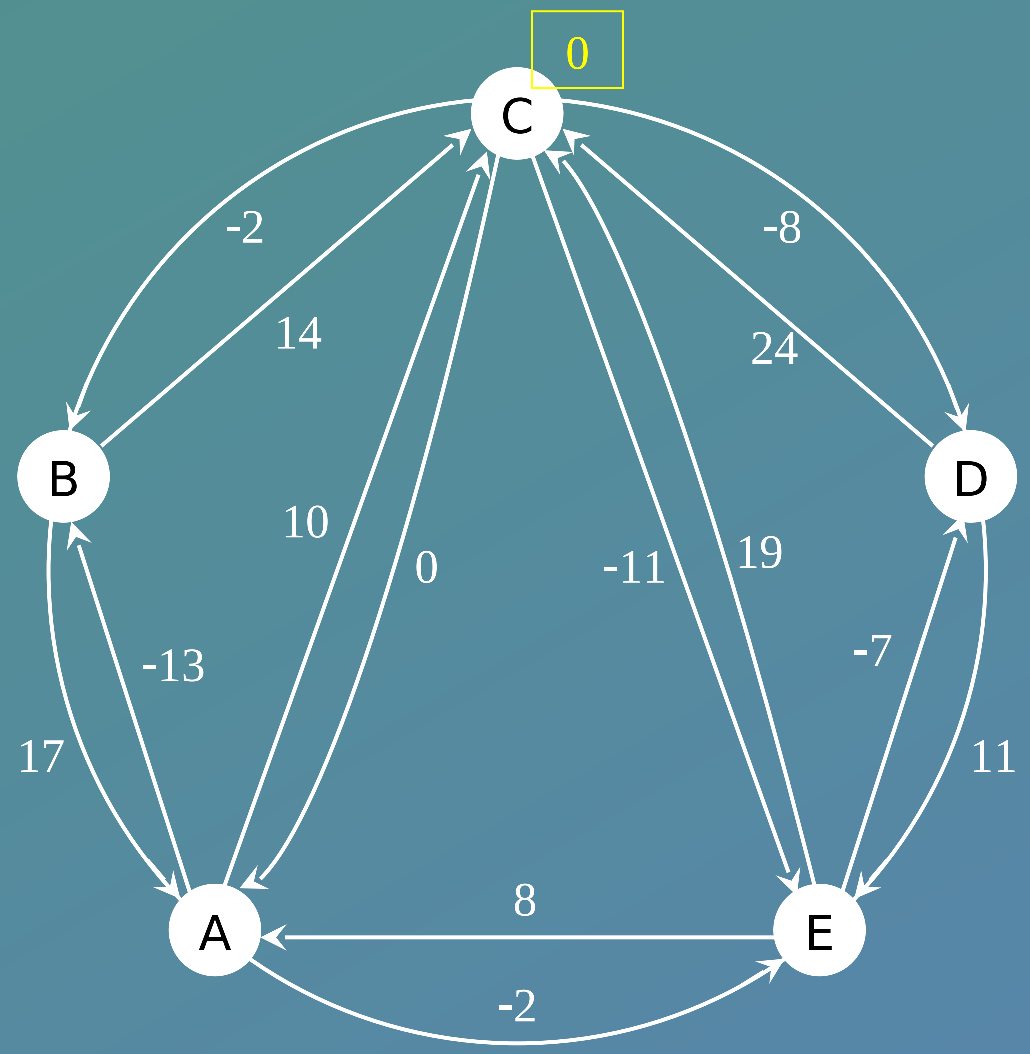

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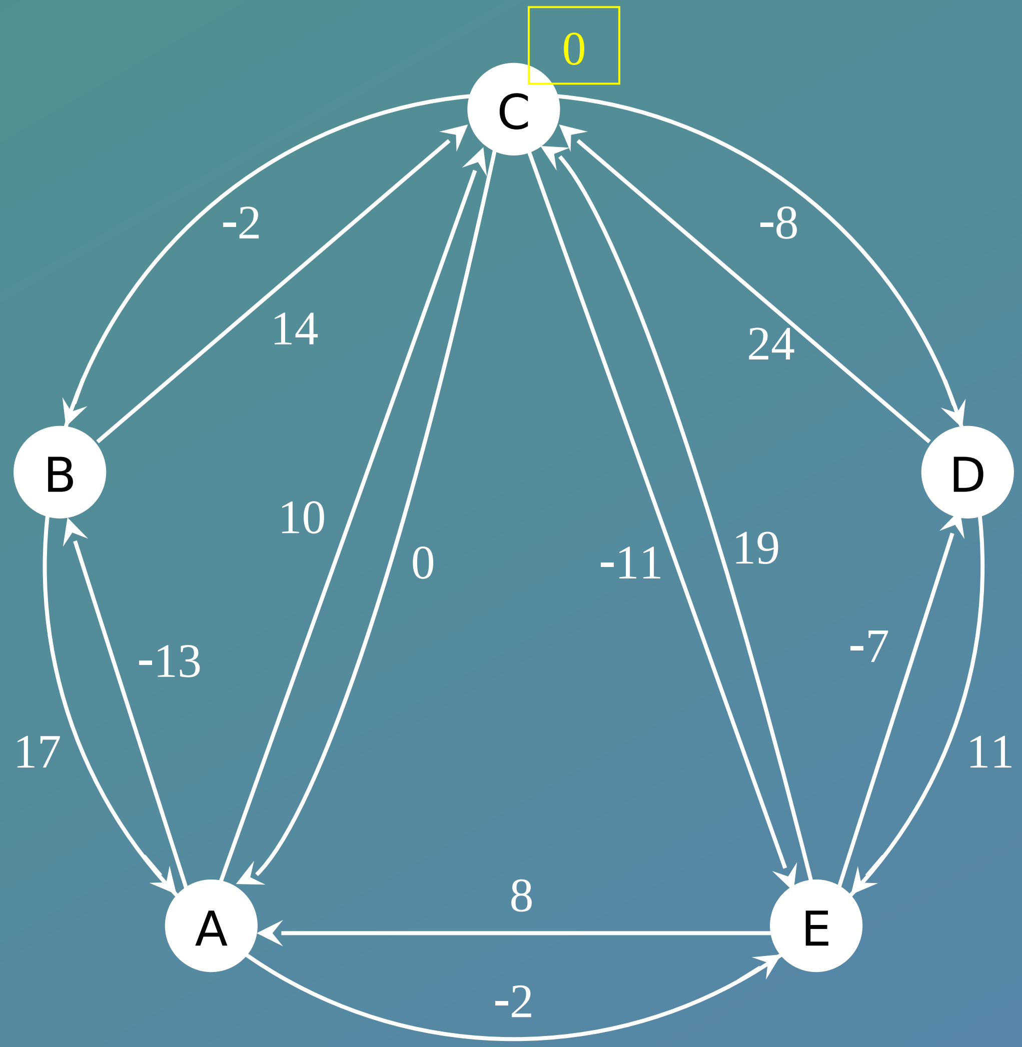
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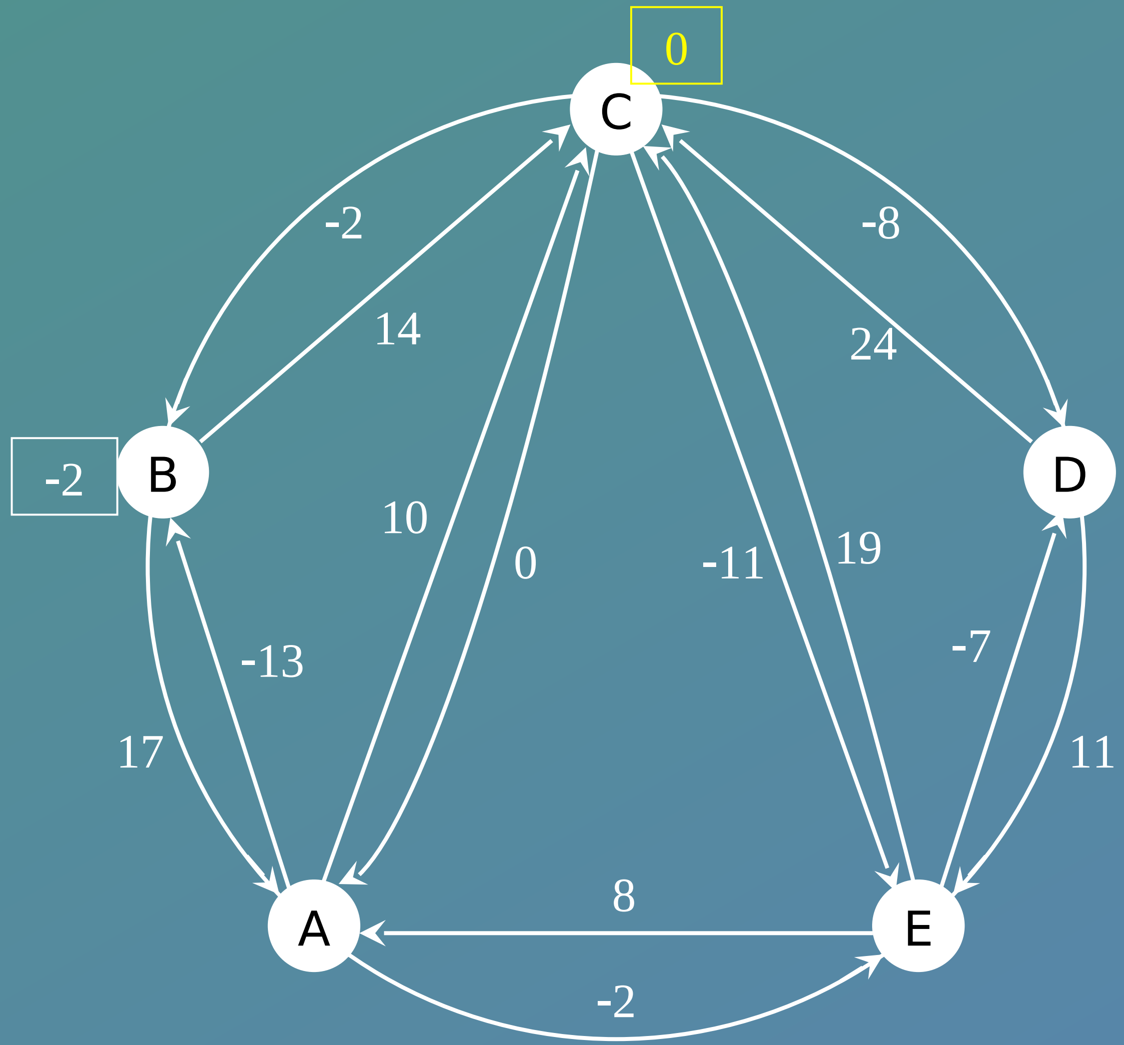
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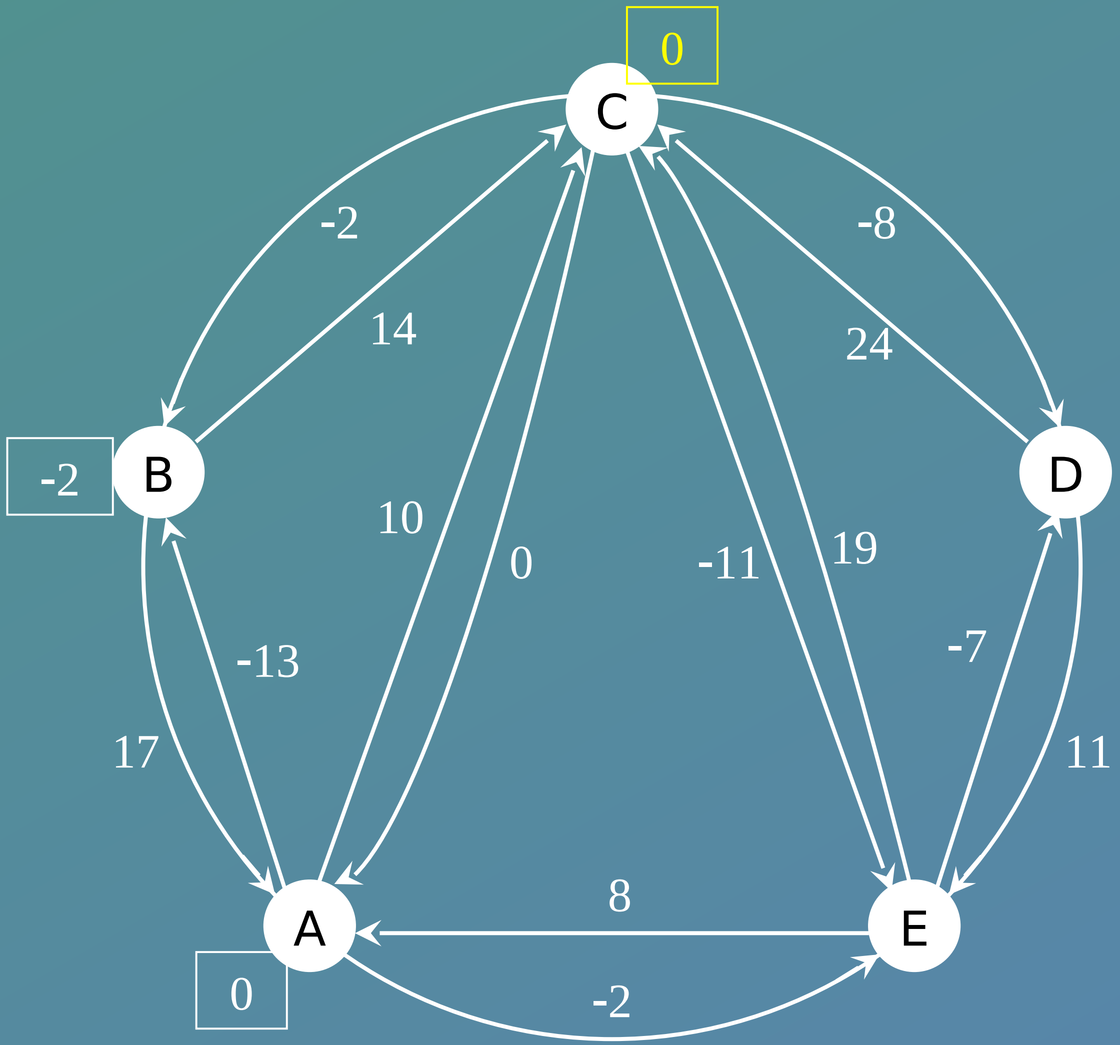
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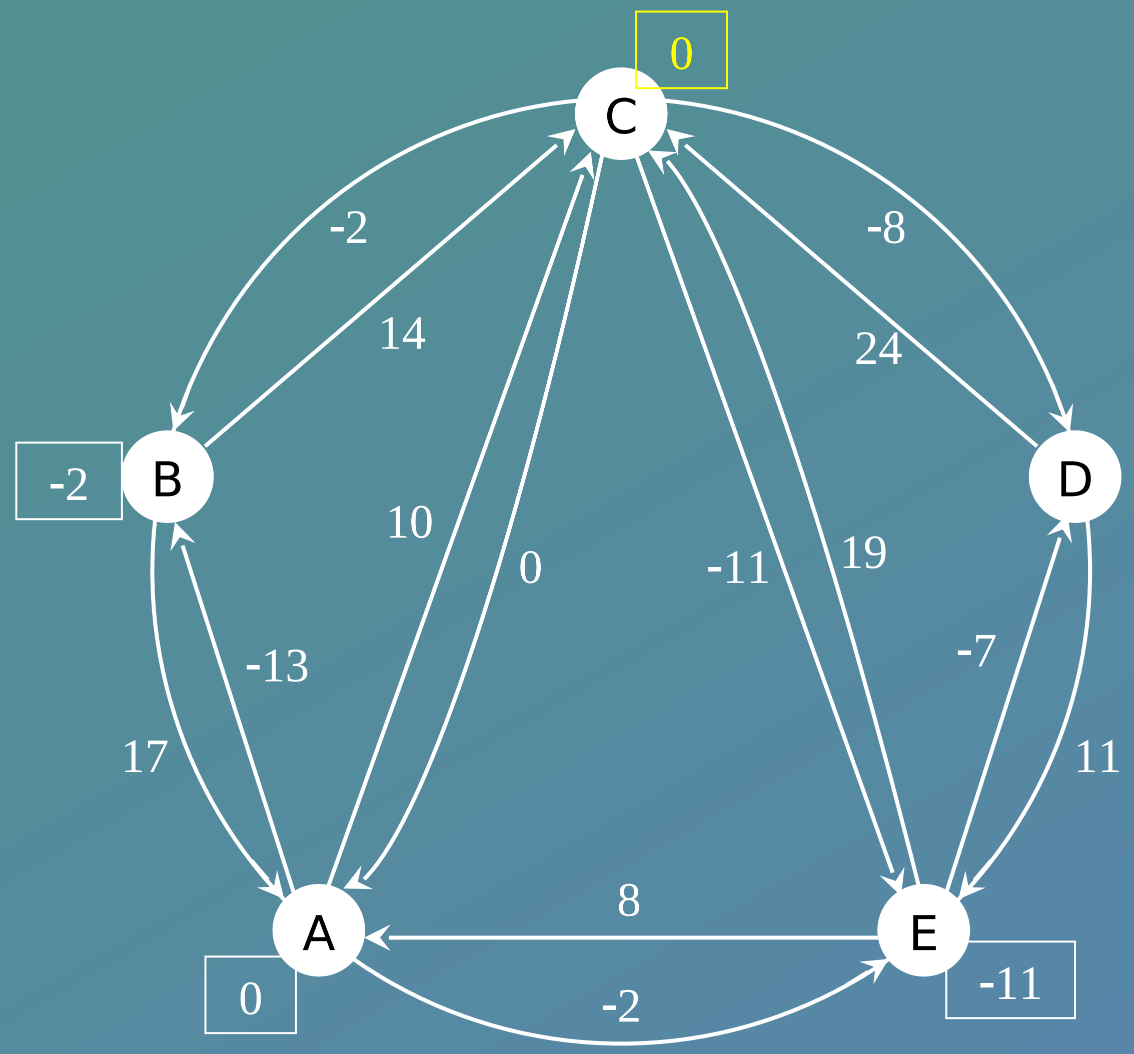
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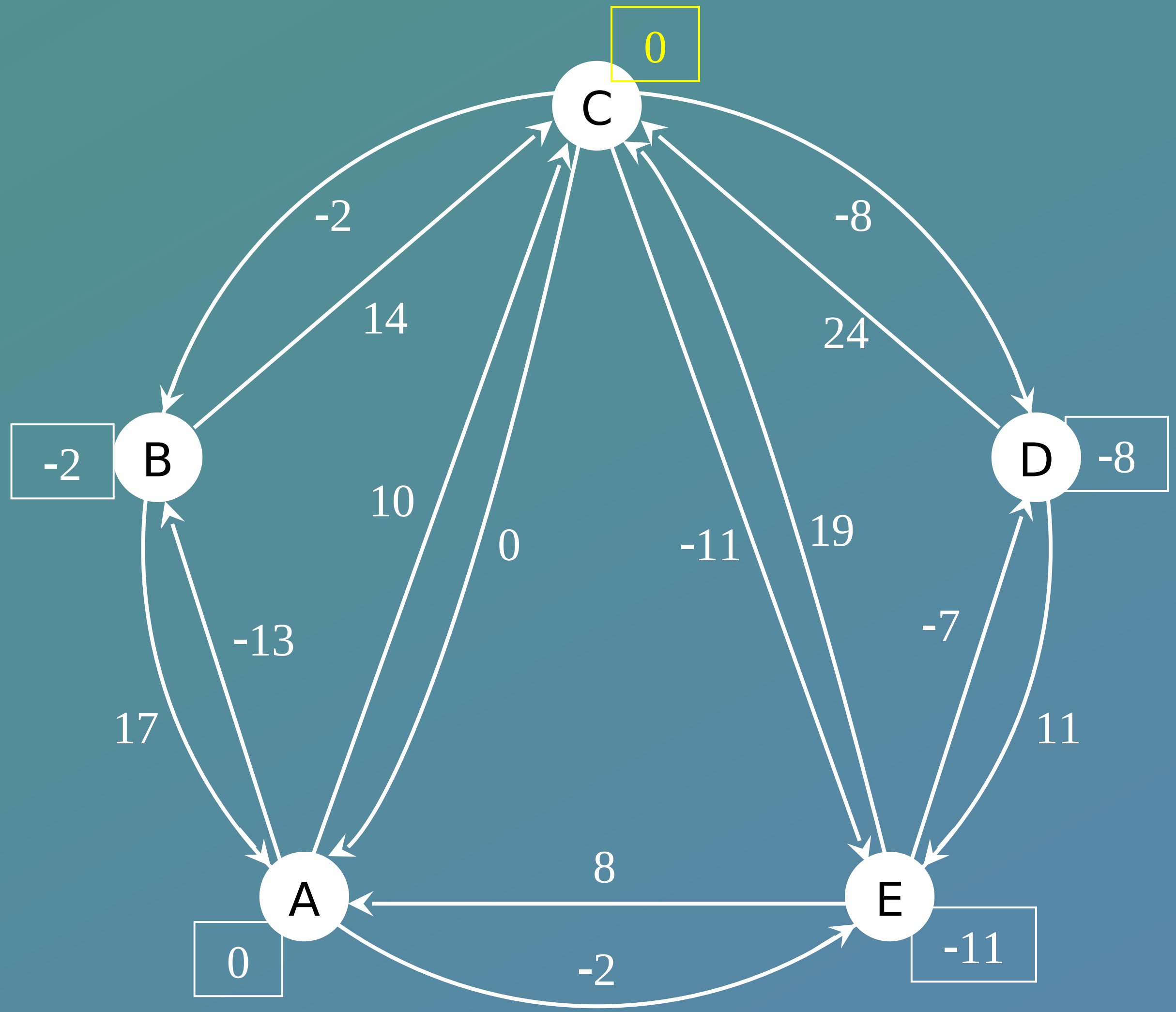

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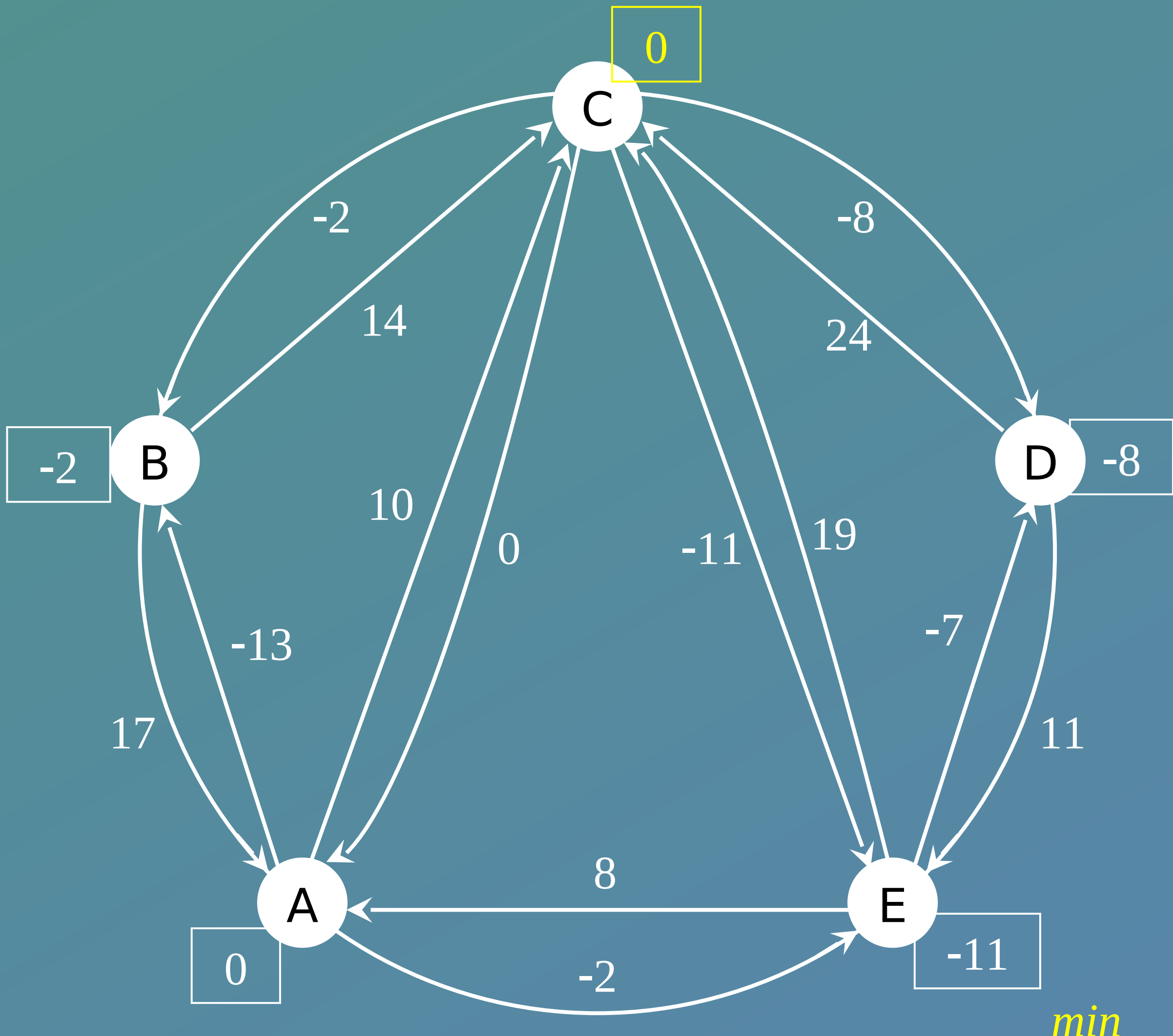
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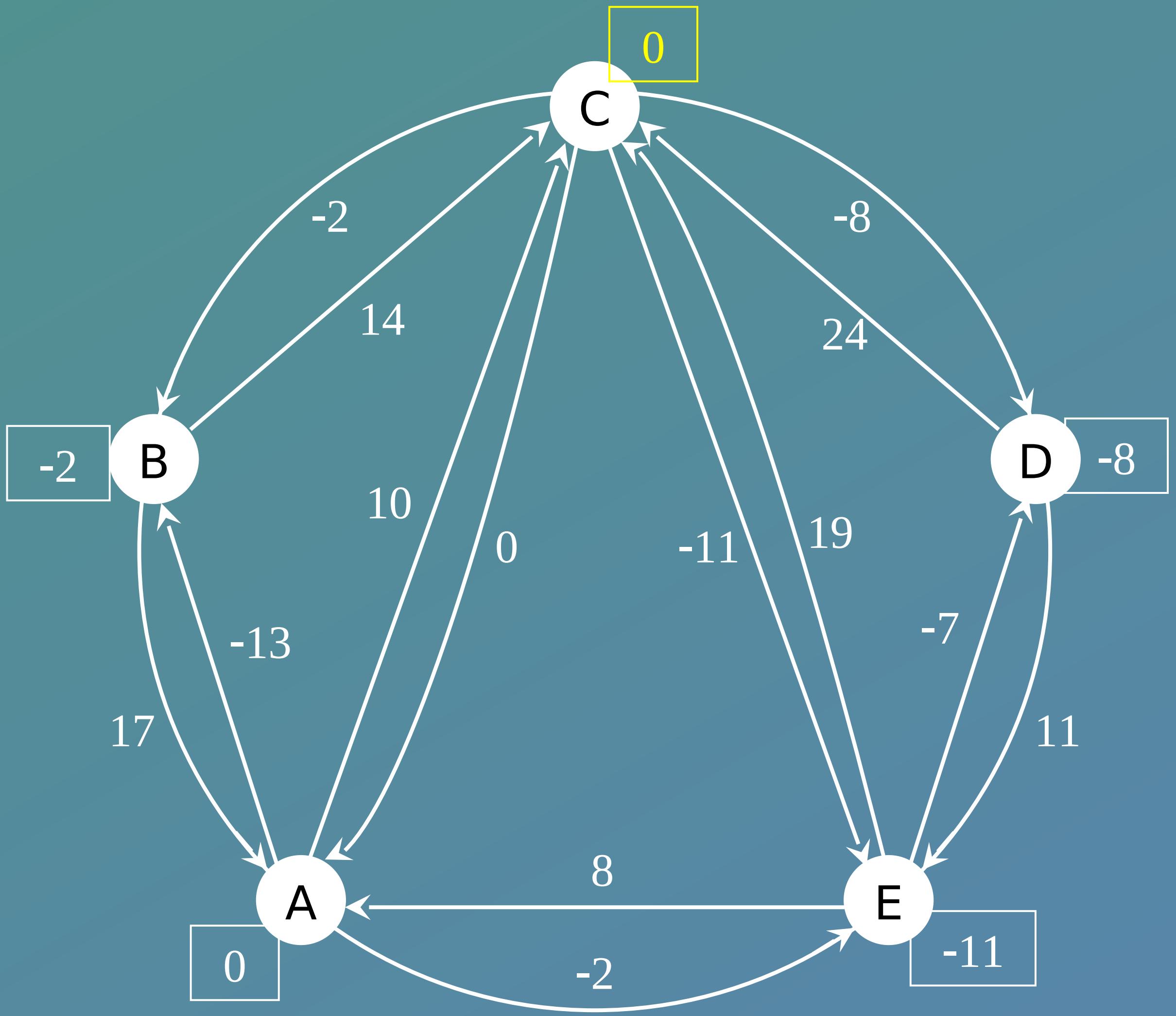
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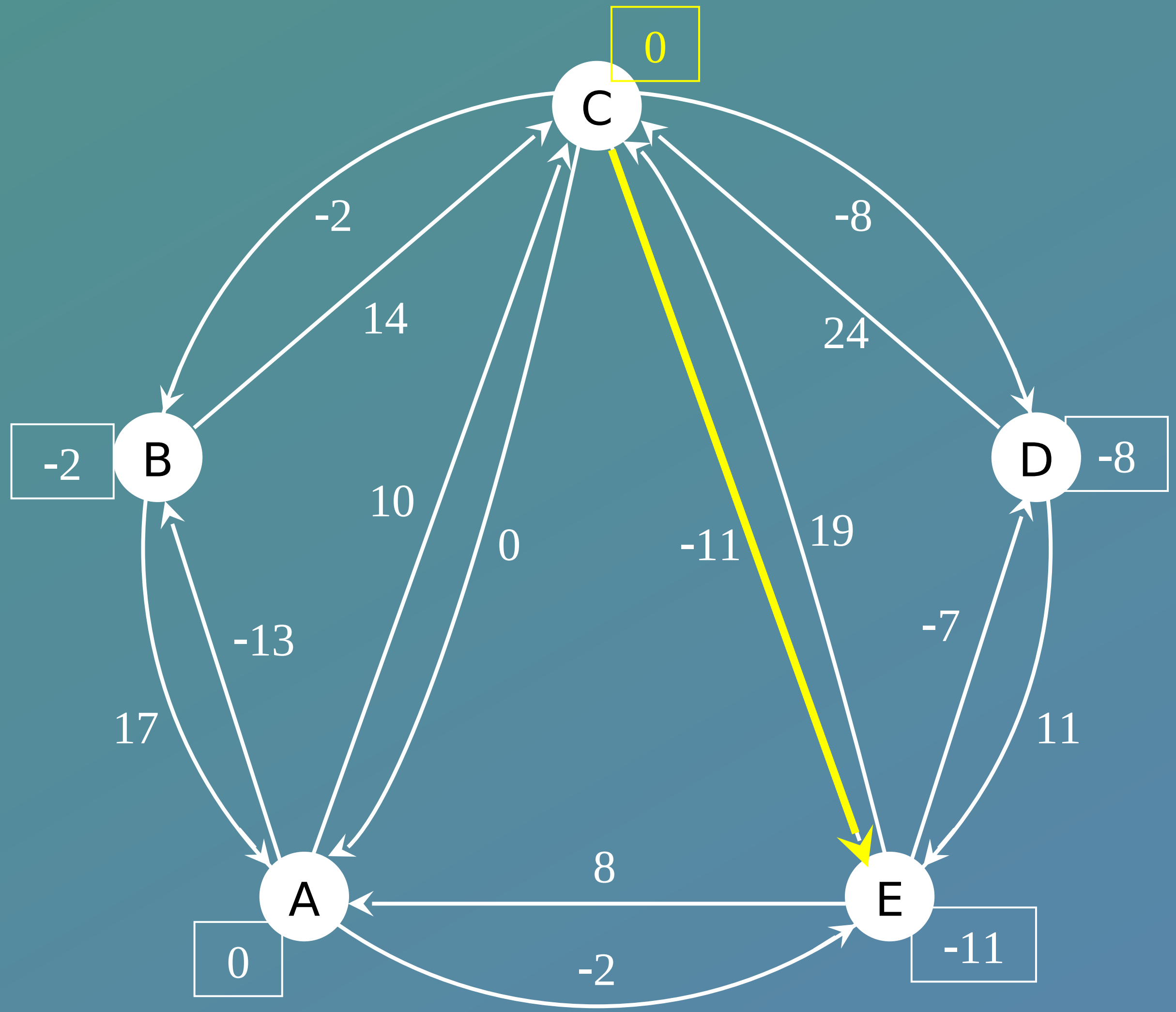

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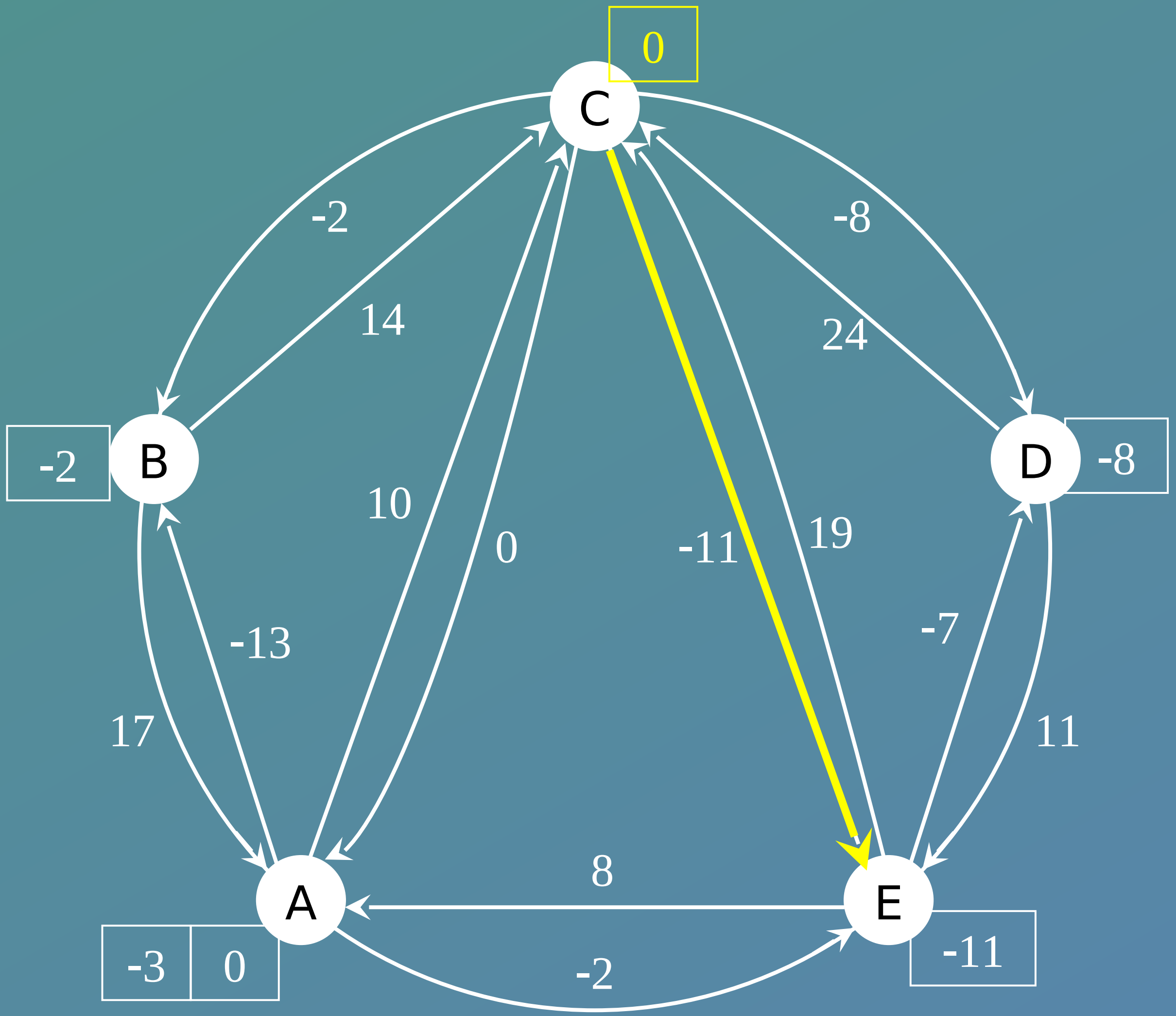

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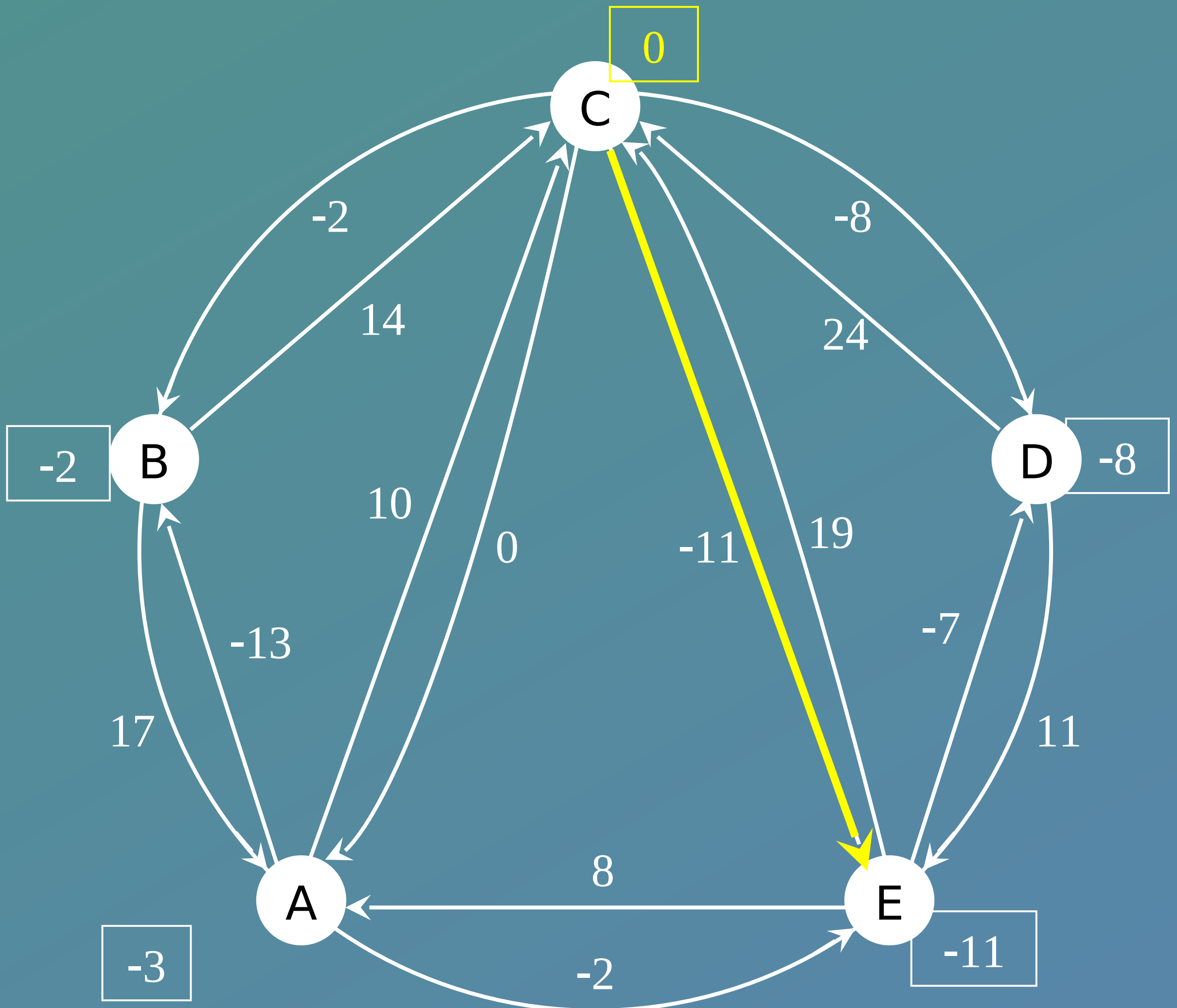
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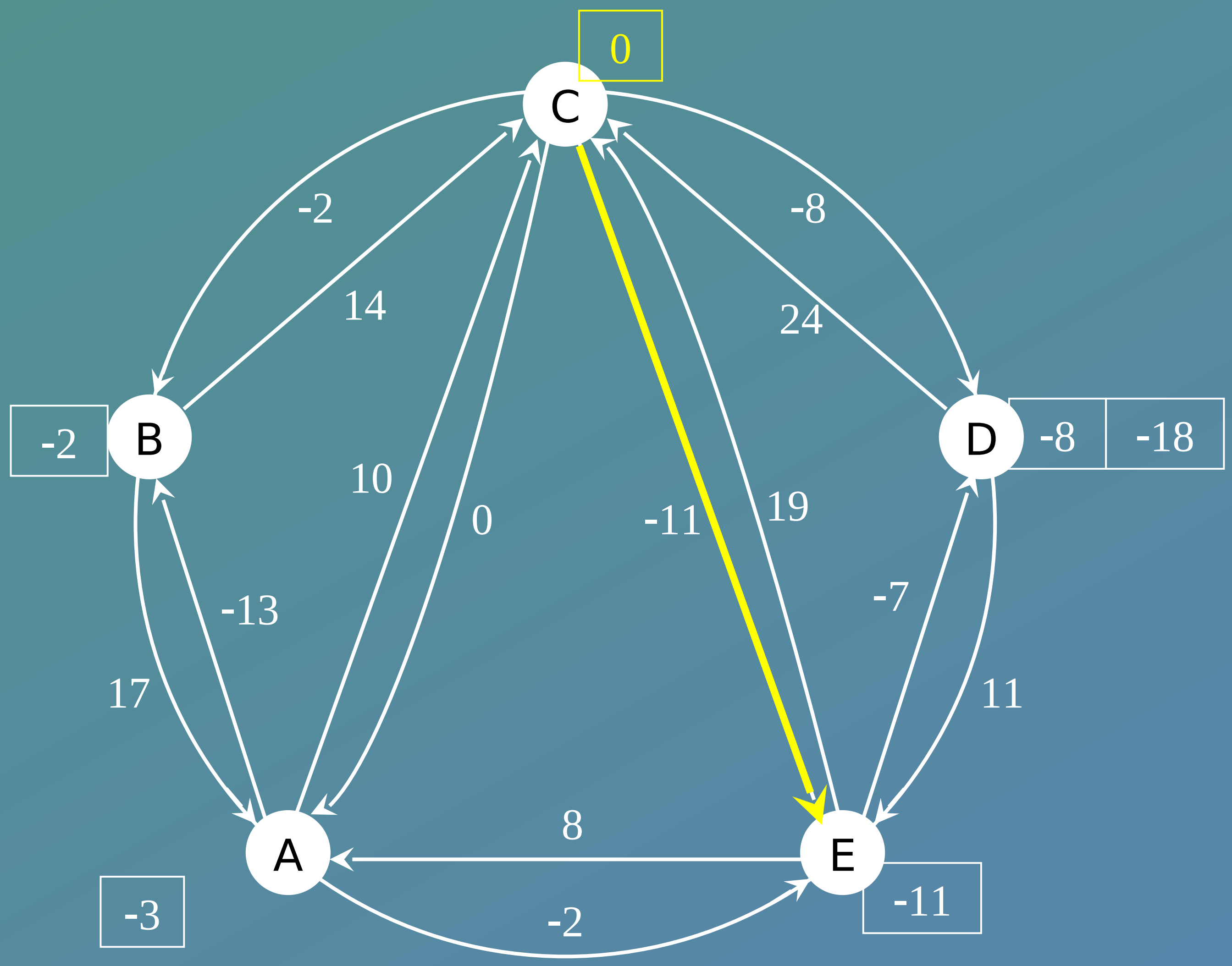

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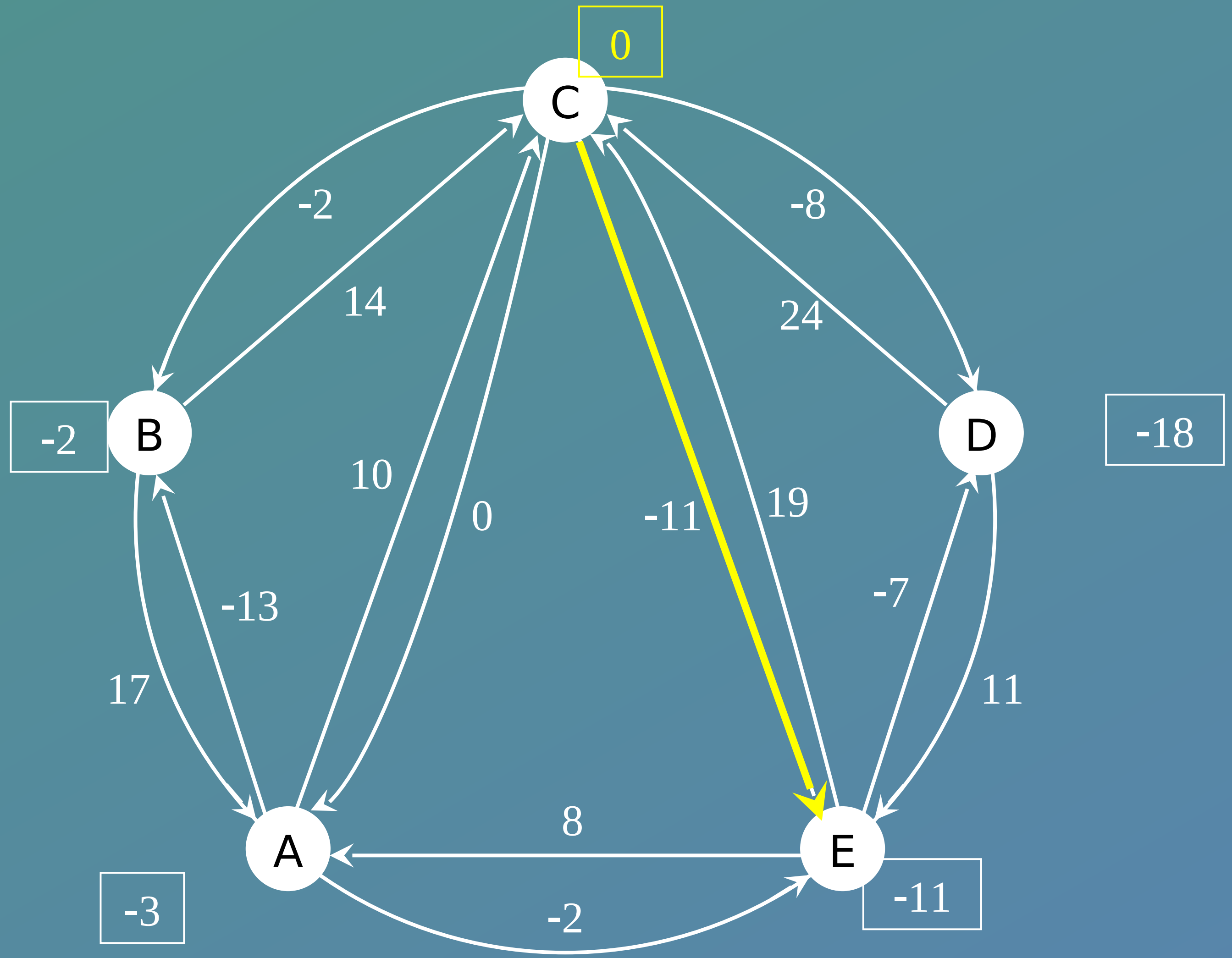

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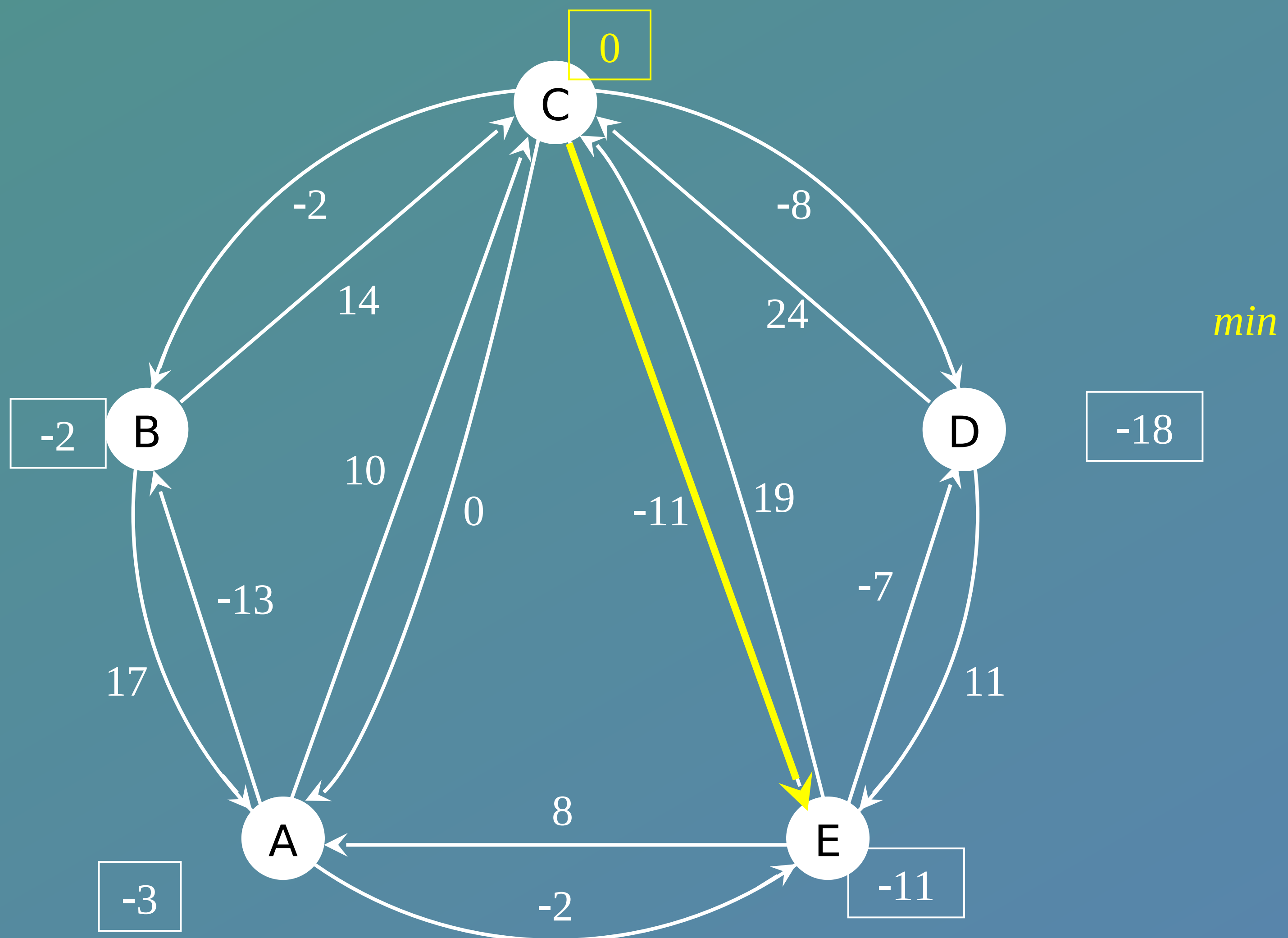

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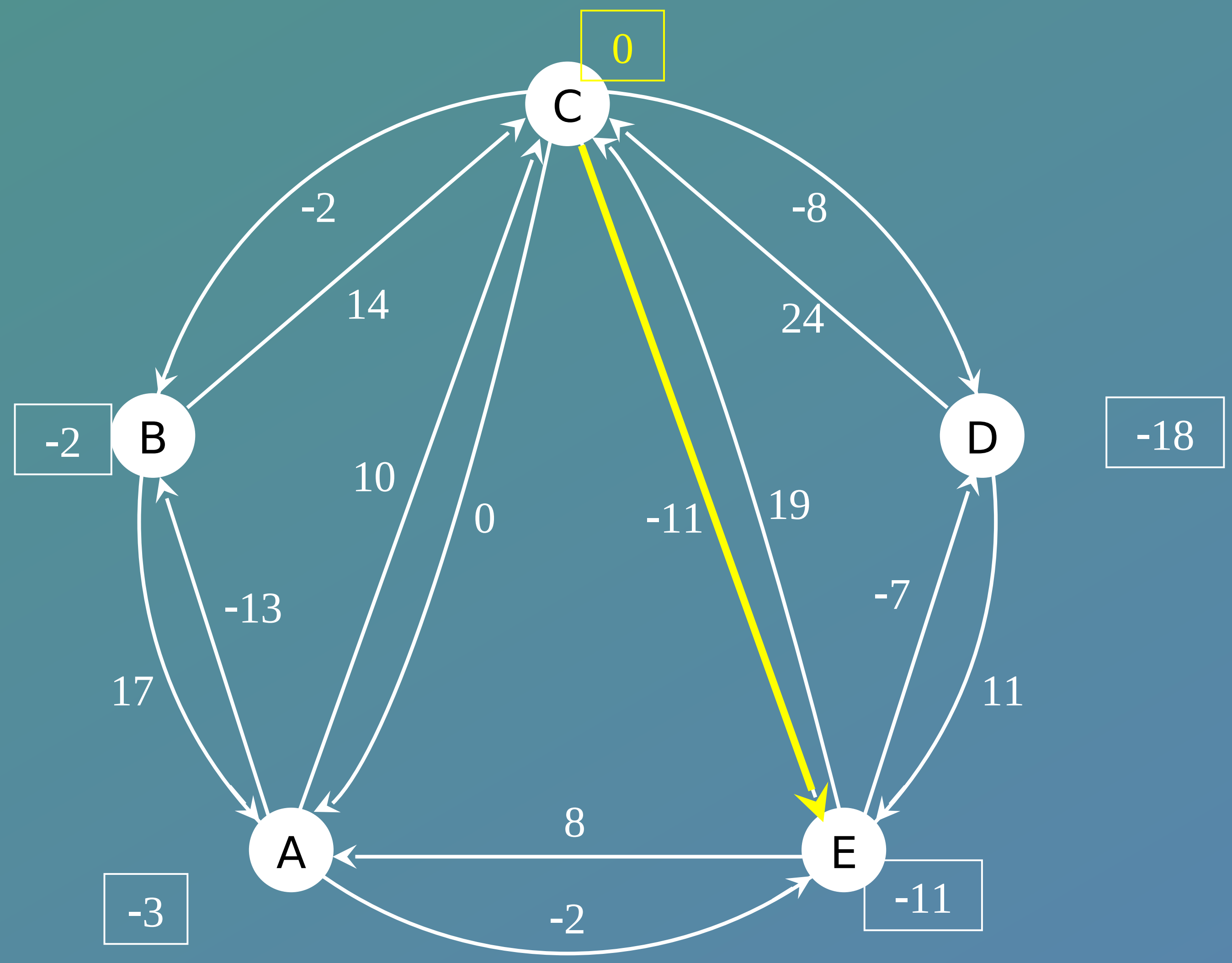
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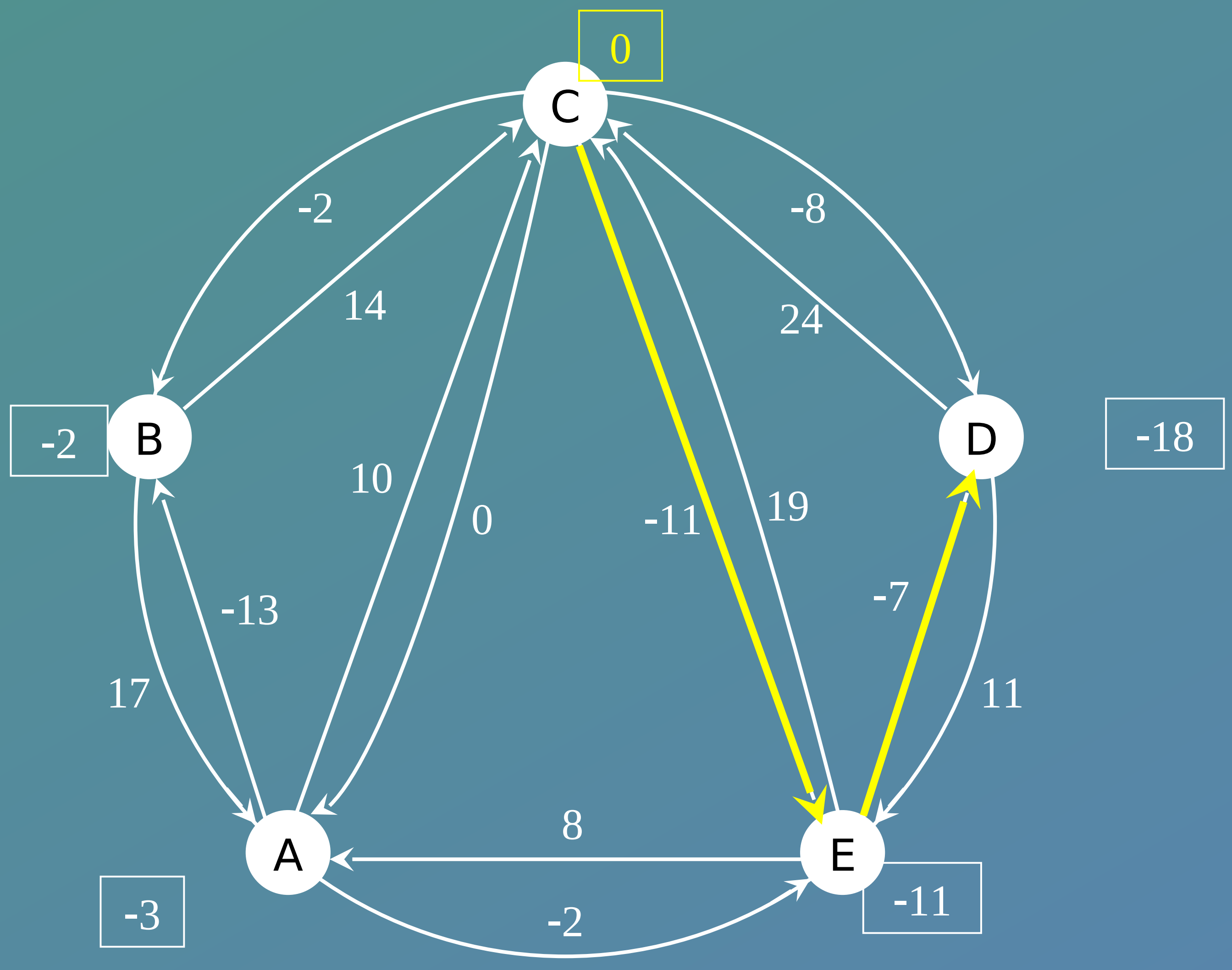
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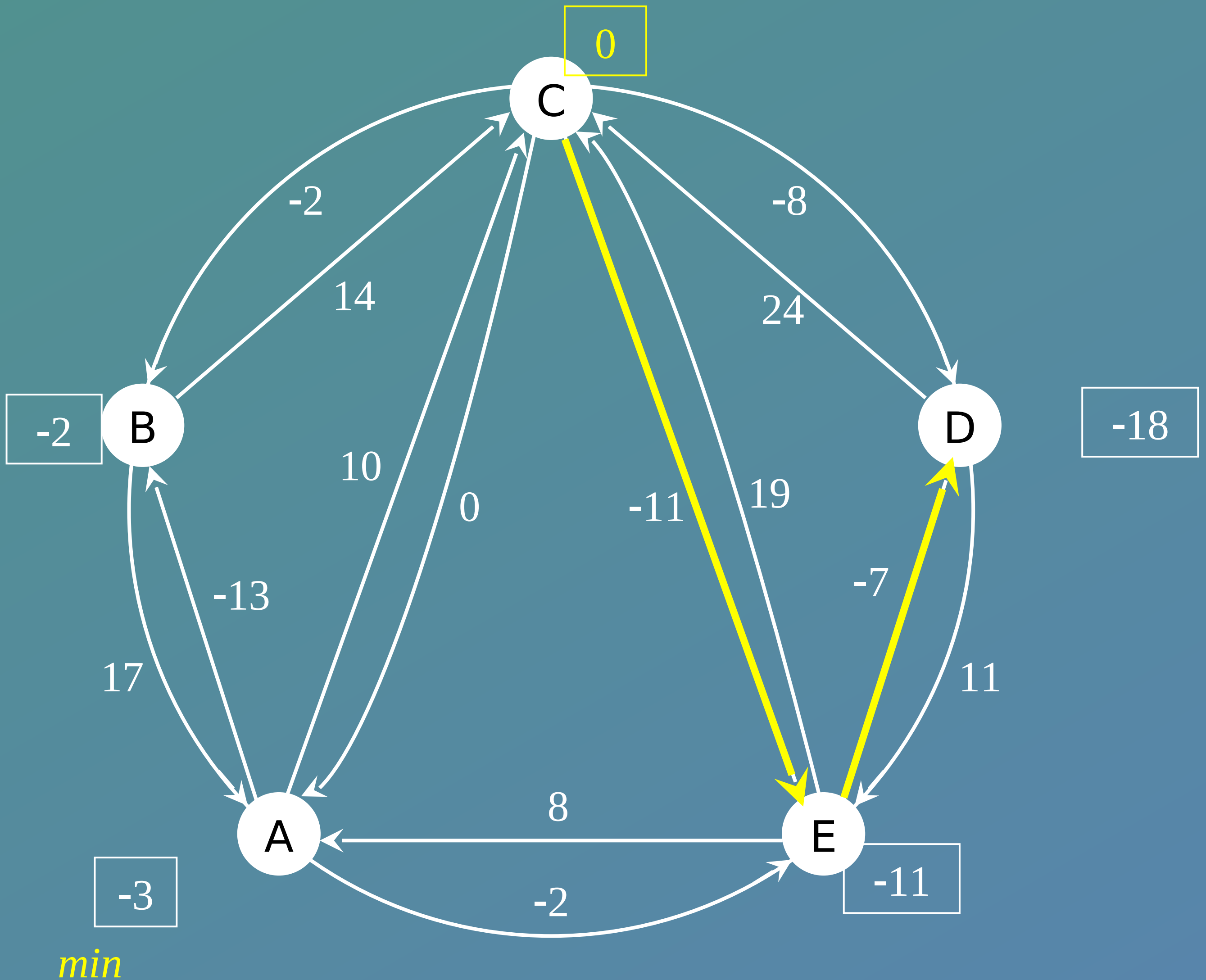

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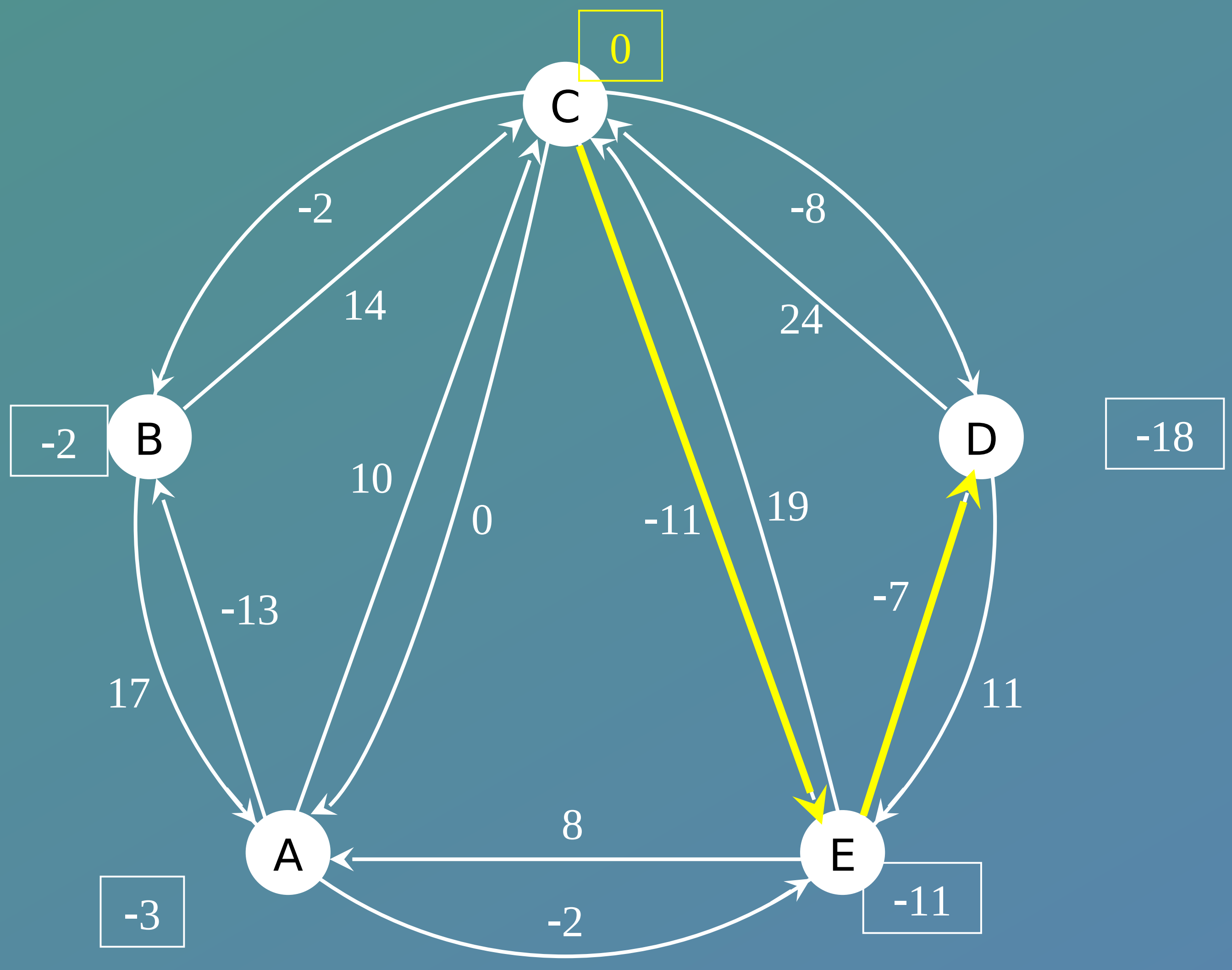
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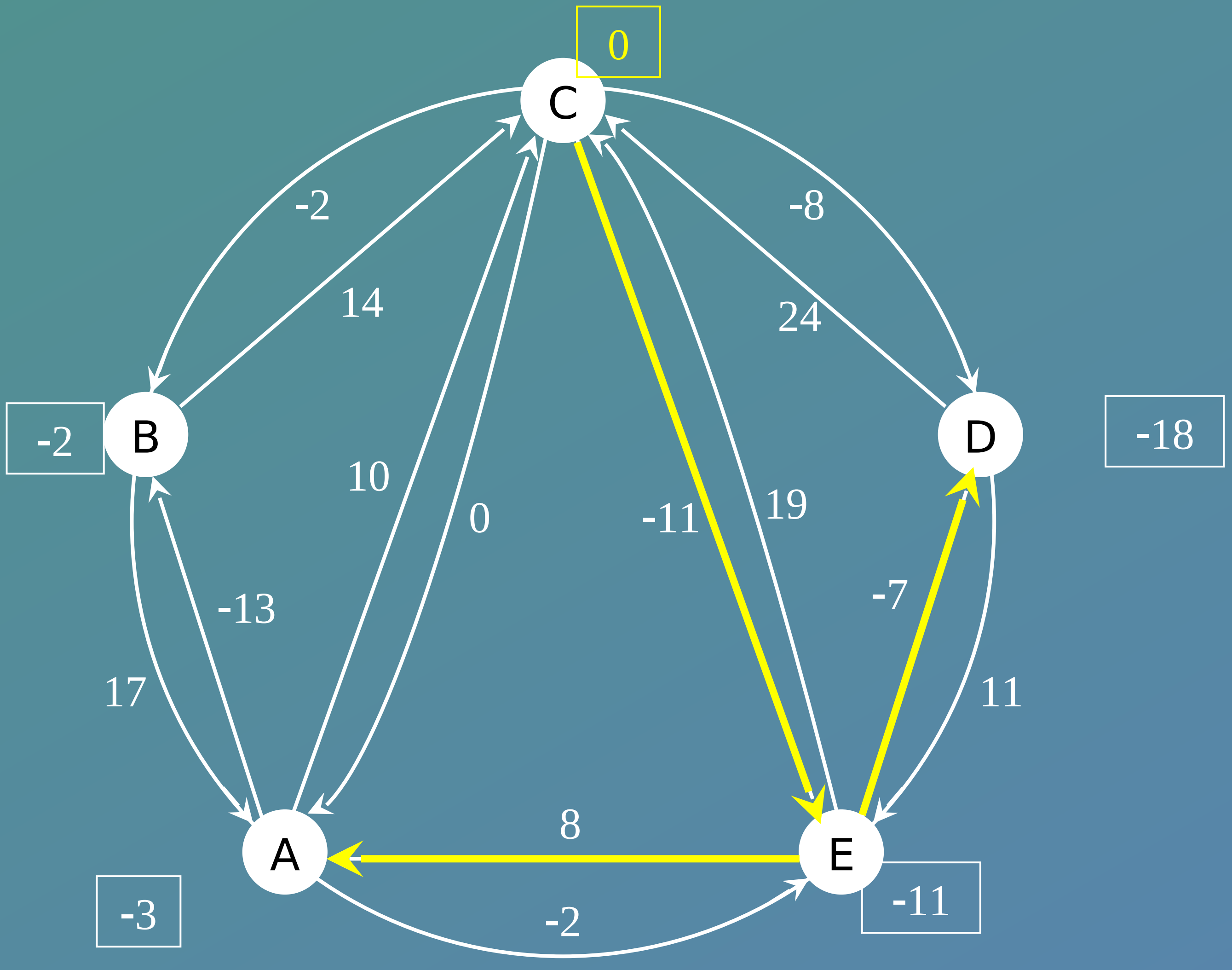
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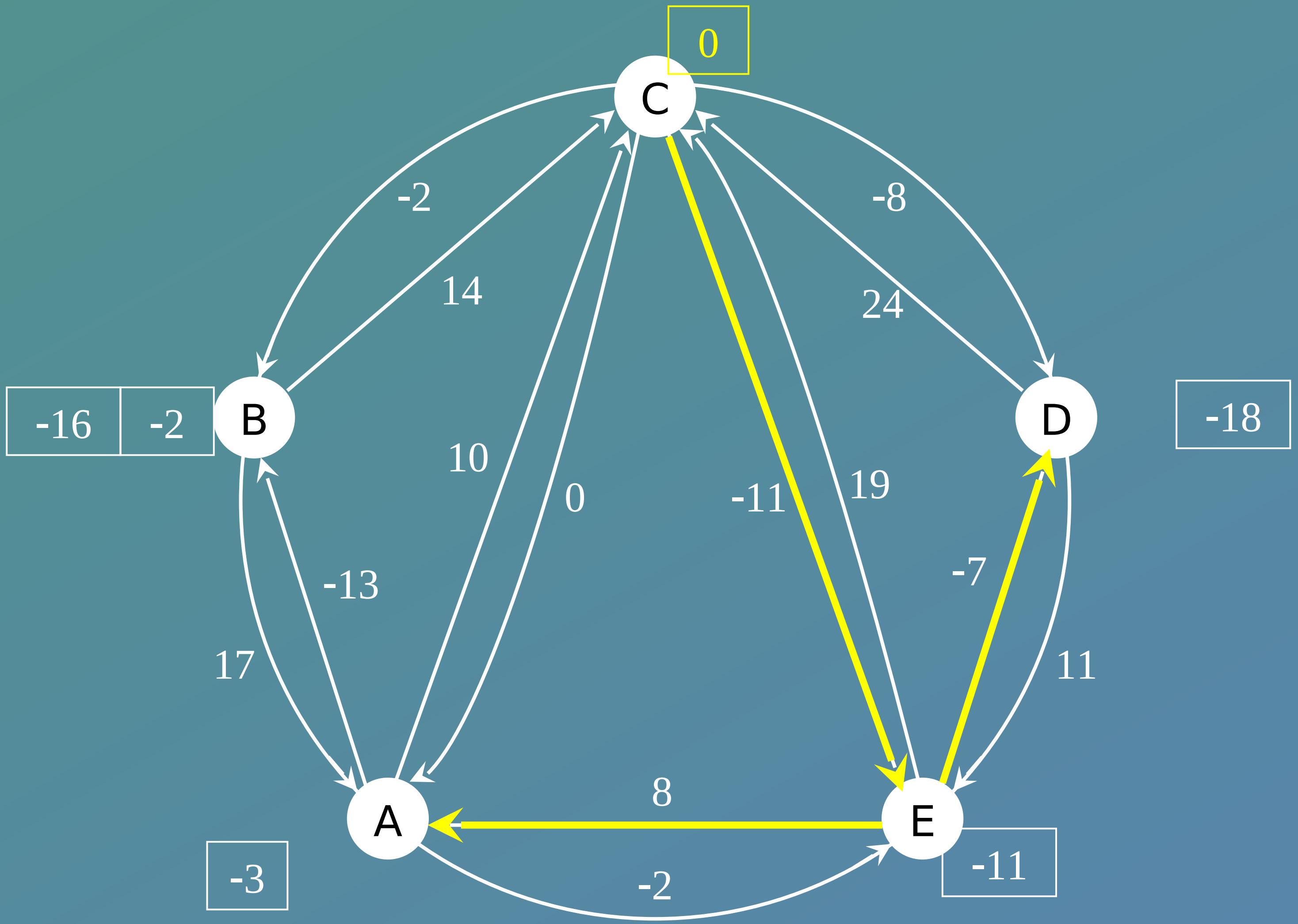

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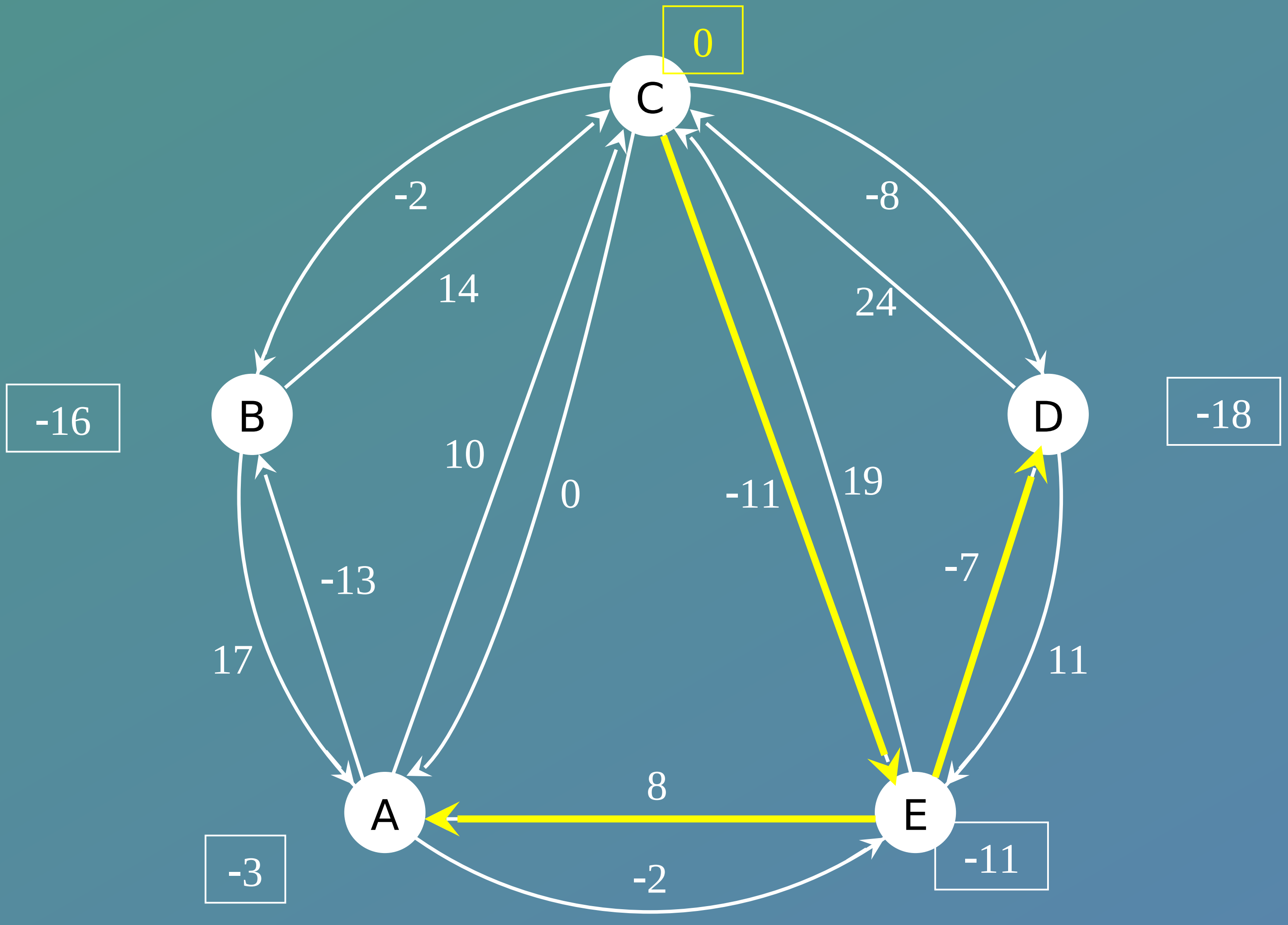
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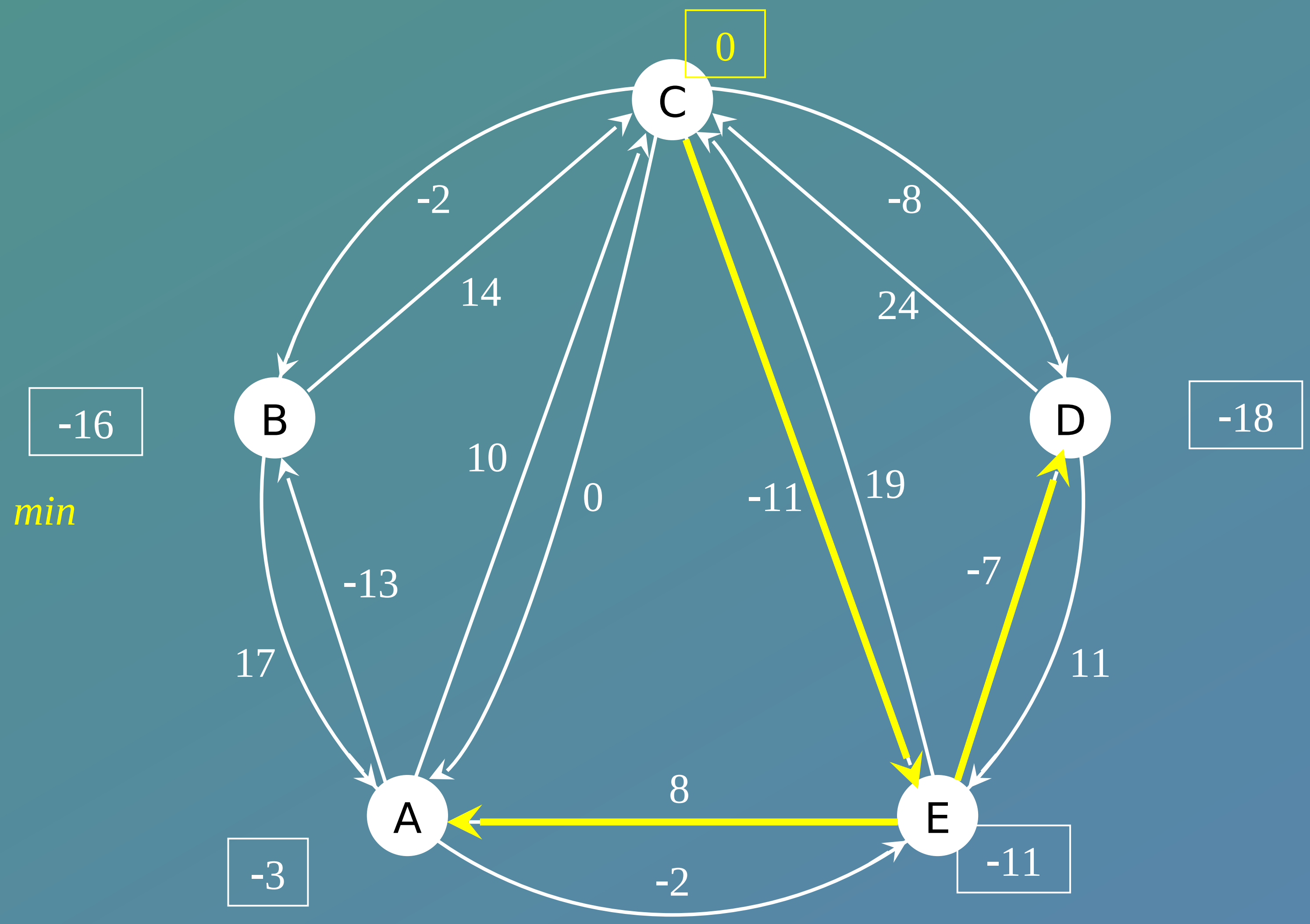
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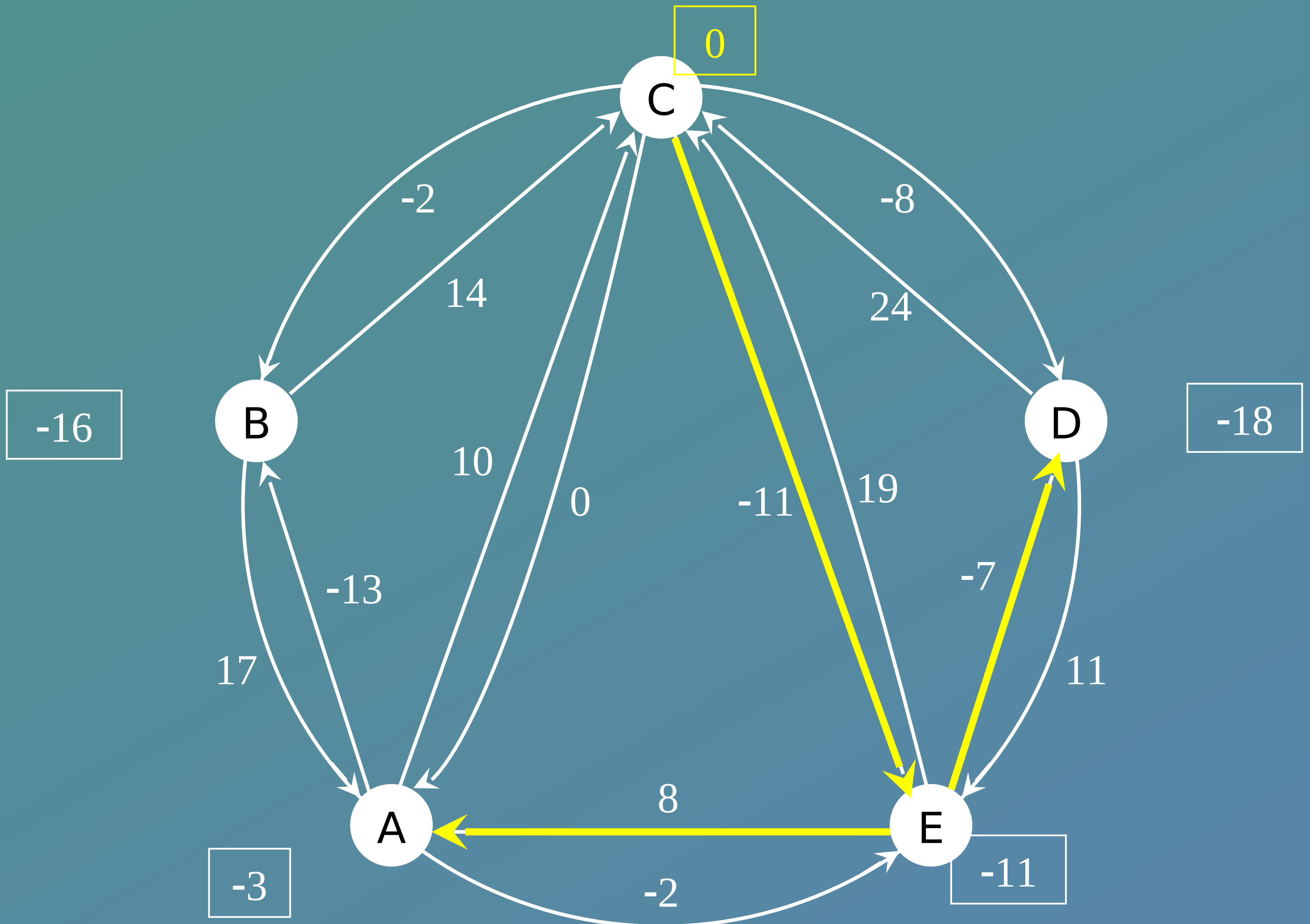
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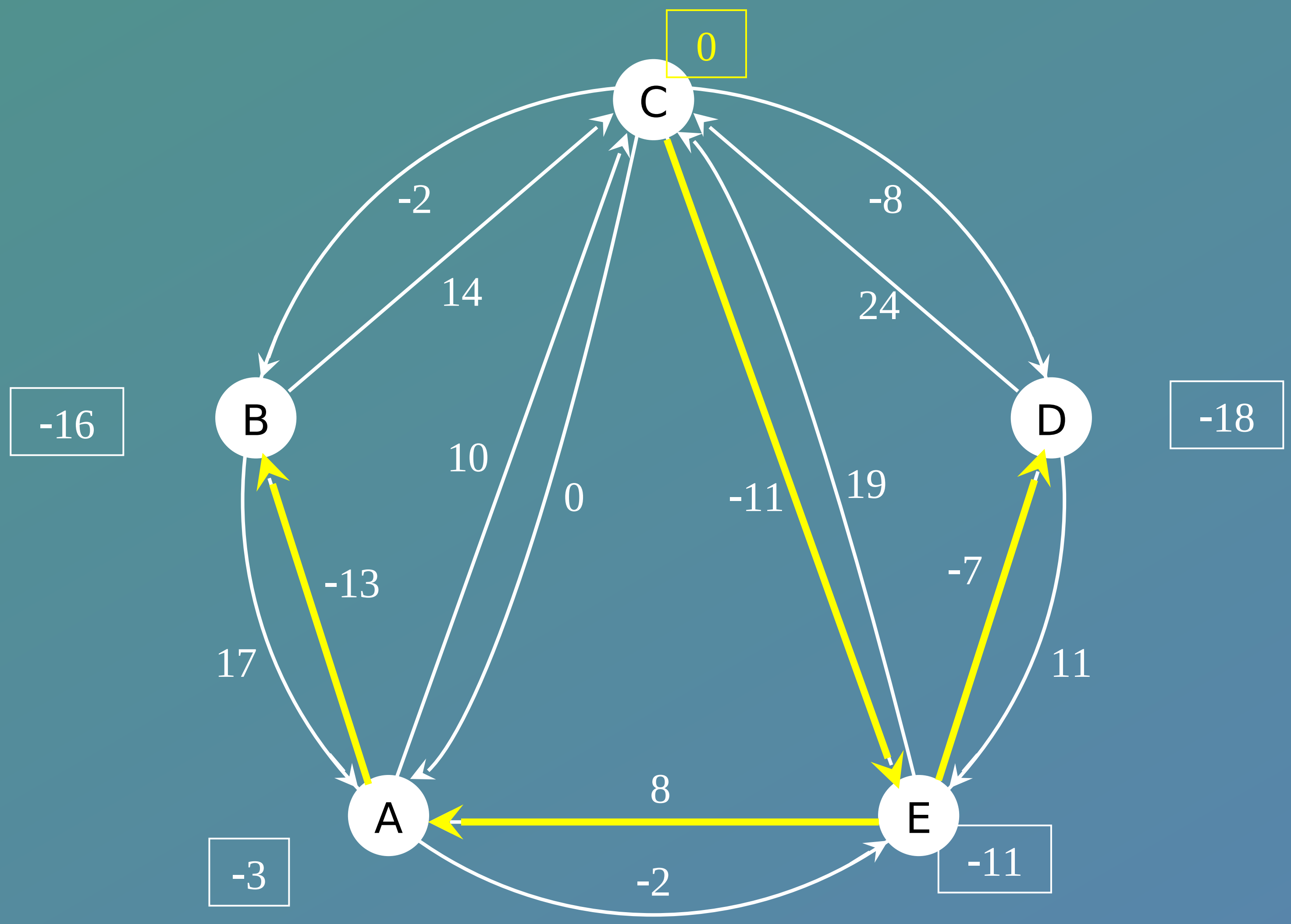
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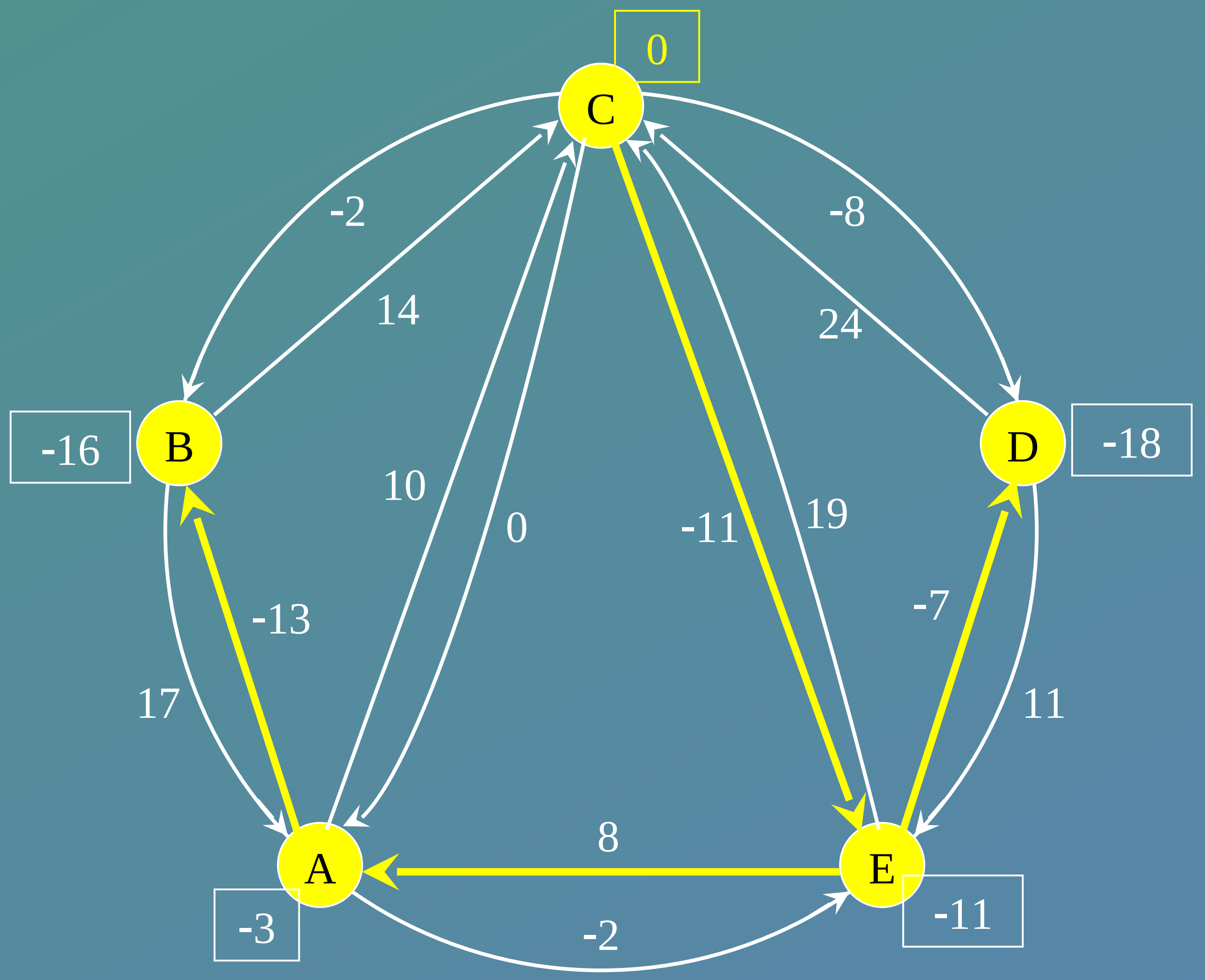
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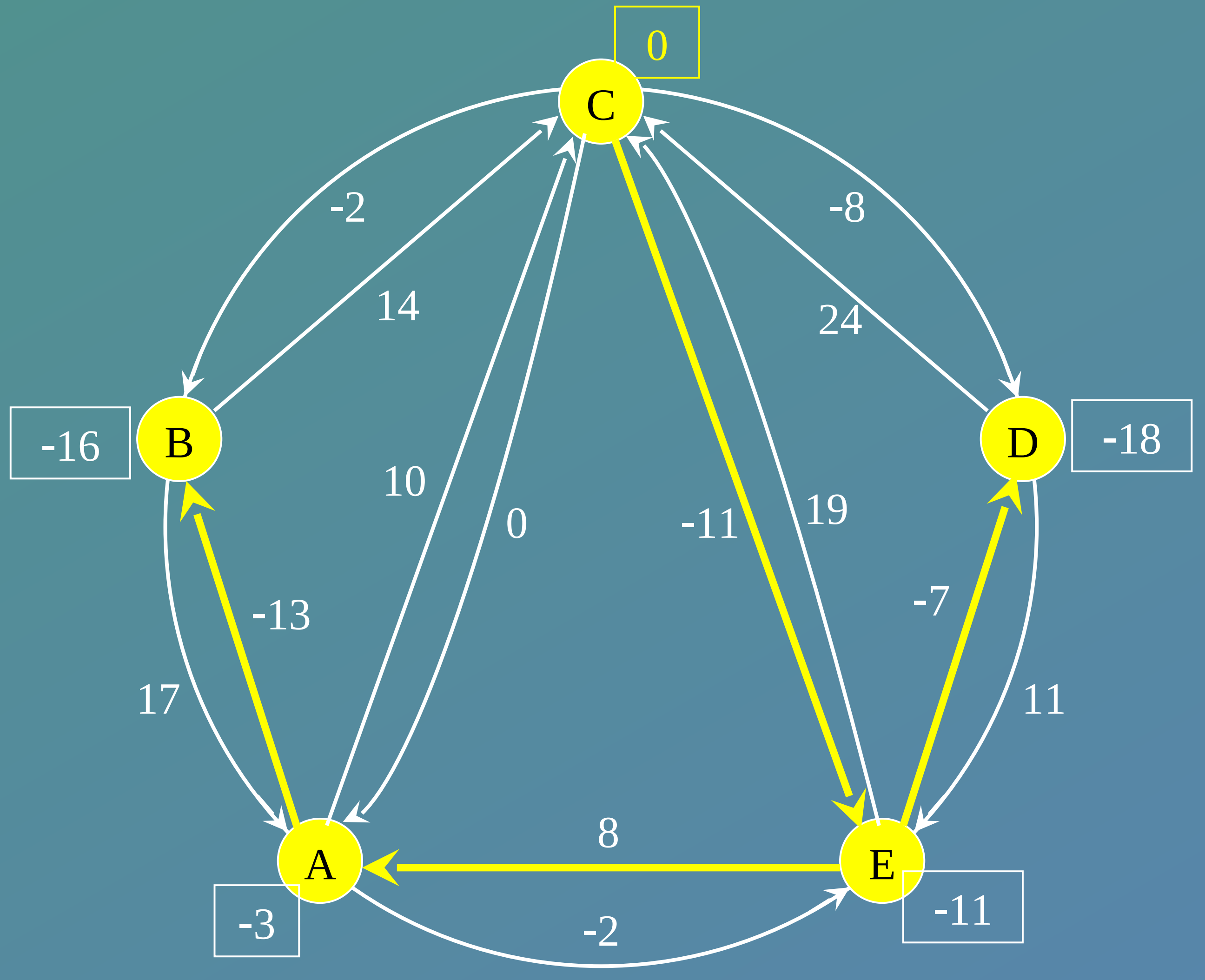
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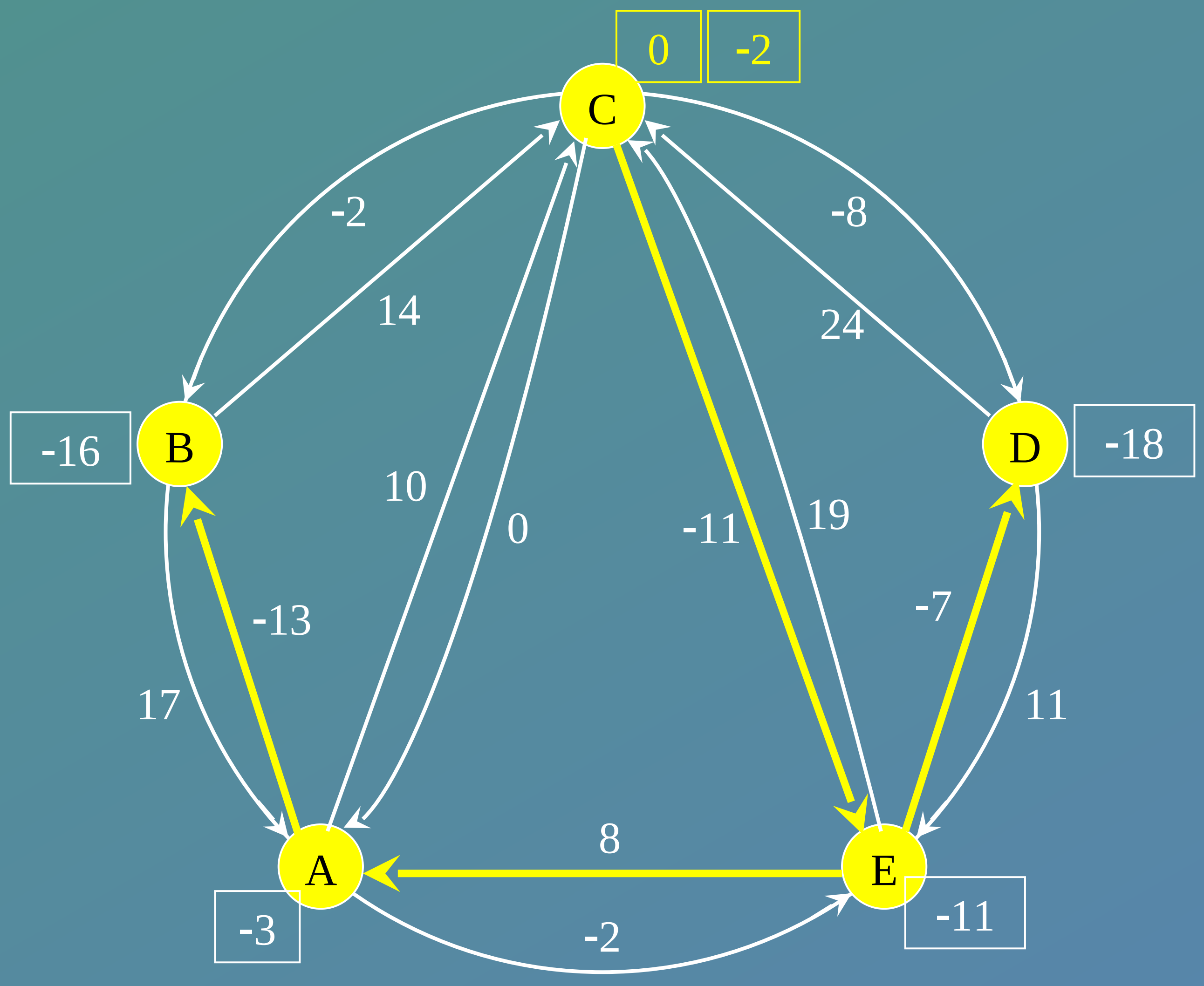

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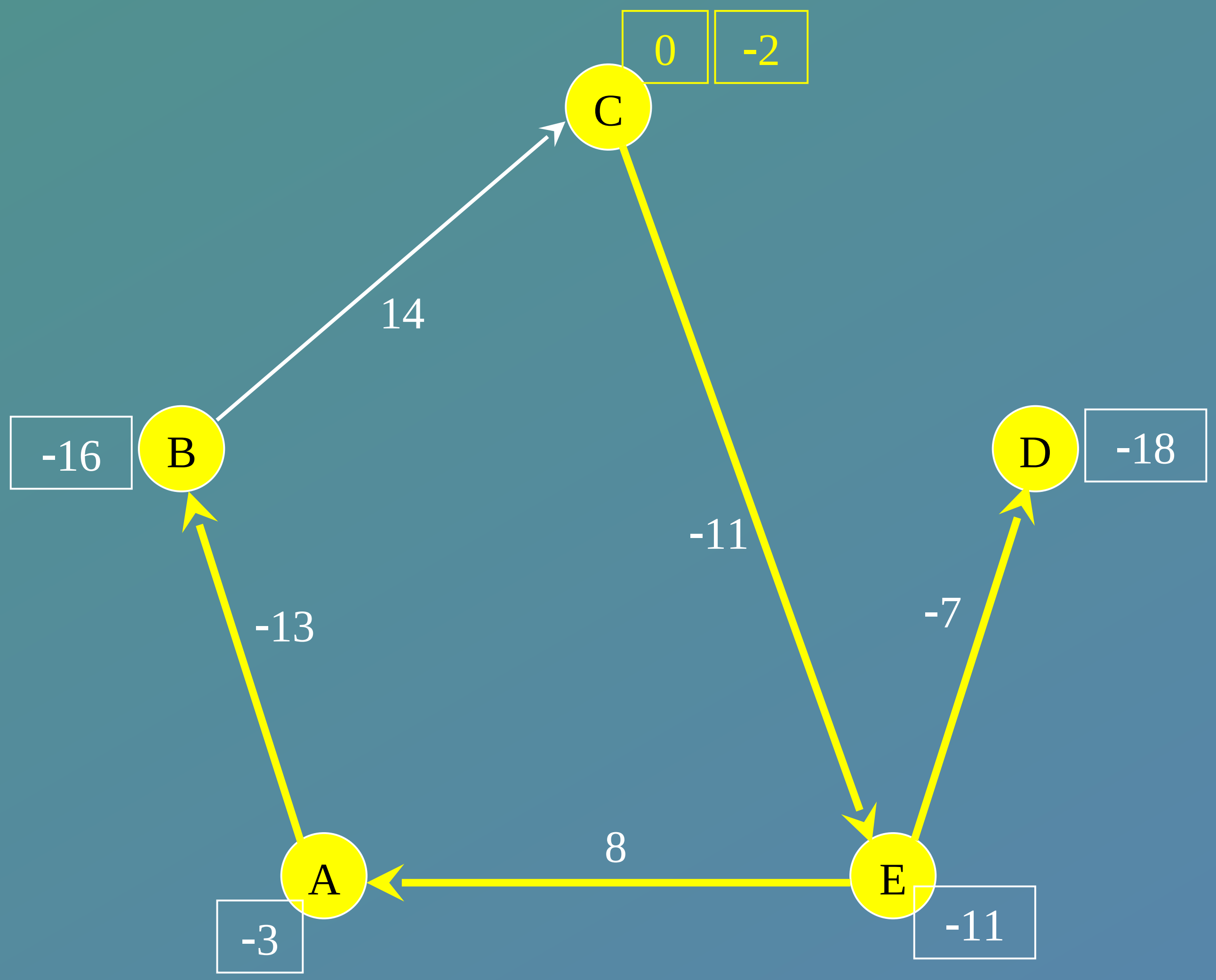

C-E-A-B-C - le circuit absorbant de longueur -2

		TAXES			
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

		AIDES			
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

		COÛTS			
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

C-E-A-B-C - le circuit absorbant de longueur -2

	TAXES				
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

	AIDES				
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

	COÛTS				
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

C-E-A-B-C - le circuit absorbant de longueur -2

TAXES

i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

AIDES

i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

COÛTS

i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

C-E-A-B-C - le circuit absorbant de longueur -2

TAXES					
i \ j	A	B	C	D	E
A			5		
B	15		8		
C					
D			22		8
E	5		15		

AIDES					
i \ j	A	B	C	D	E
A		15			5
B					
C	5	8		10	15
D					
E				10	

COÛTS					
i \ j	A	B	C	D	E
A		2	5		3
B	2		6		
C	5	6		2	4
D			2		3
E	3		4	3	

C-E-A-B-C - le circuit absorbant de longueur -2