Lunar perturbation of the metric associated to the averaged orbital transfer

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Abstract

In a series of previous article we introduced a Riemannian metric associated to the energy minimizing orbital transfer with low propulsion. The aim of this article is to study the deformation of this metric due to the perturbation caused by the lunar attraction. Using Hamiltonian formalism, we describe the effects of the perturbations on the orbital transfers and the deformation of the conjugate and cut loci of the original metric.

Introduction

Recent space missions like lunar Smart-1 mission, Boeing orbital transfer, using electric propulsion are innovative design feature to reduce launch costs and lead to the analyse of the low thrust controlled Kepler equation using averaging techniques in optimal control. Pioneering work in this direction associated to the energy minimization problem are due to Edelbaum [4], Epenoy-Geffroy [5], and more recently to Bonnard-Caillau [1]. Under some simplifying assumption they lead to the definition of a Riemannian distance between Keplerian orbits, and this is a preliminary step in computing the time minimal or find mass maximizing solutions using numerical continuation techniques [3].

Main Objectives

1. Compute a double averaged on the lunar perturbation potential.
2. From the averaged Hamiltonian associated to the energy minimization, define a perturbed Hamiltonian related to a Zermelo navigation problem.
3. Using the Pontryagin Maximum Principle (PMP), solve numerically a boundary-value problem via a simple shooting method.
4. Study the second order optimality conditions by computing the conjugate points via an algorithm based on the Jacobi fields.
5. Compare the trajectories and cut loci of the perturbed and unperturbed problem

The Riemannian metric

From the controlled planar Kepler equation, the ellipse of motion of a satellite can be described by the system

$$\frac{dx}{dt} = \sum_{i=1}^{2} u_i F_i(x,l)$$
$$\frac{dl}{dt} = w_0(x,l) + g(x,l,u). \tag{1}$$

where $x = (n, \rho, \theta)$ are the slow orbital elements of the satellite (n is the mean motion, ρ the eccentricity and θ the argument of periapsis), and l is the fast angle pointing out the position of the satellite on its orbit.

The optimal control problem associated to (1) is to find, given two boundary values (x_0, x_f) and an interval $[0, t_f]$, a control $u = (u_1, u_2)$ such that $|u| \leq 1$ and minimizing the cost is $\int_{[0,t_f]} |u(s)|^2 ds$.

Applying the Pontryagin Maximum Principle, the averaged Hamiltonian with respect to the true longitude l is given by

$$\overline{H} = \frac{1}{4n^{\frac{5}{3}}}\left[18n^2p_n^2 + 5(1-\rho^2)p_\rho^2 + (5-4\rho^2)\frac{p_\theta^2}{\rho^2}\right].$$

The Riemannian metric associated to $\overline{H}$ is

$$g = \frac{1}{9n^{\frac{1}{3}}}dn^2 + \frac{2n^{\frac{5}{3}}}{5(1-\rho^2)}d\rho^2 + \frac{2n^{\frac{5}{3}}}{5-4\rho^2}d\theta^2.$$

Model for the lunar perturbation

The perturbed dynamic can be written as

$$\frac{dx}{dt} = F_0(x,l,l') + \sum_{i=1}^{2} u_i F_i(x,l)$$
$$\frac{dl}{dt} = \tilde{w}_0(x,l) + g(x,l,u)$$
$$\frac{dl'}{dt} = w_1(l')$$

where l' is the fast angle pointing out the position of the Moon on his orbit.

The drift F_0 corresponds to the perturbed vector field induced by the lunar attraction. This perturbation is modelled as the effect of the Moon on the two body system {Earth + satellite} where these three mass points belong to the orbital plane of the Moon. This perturbation acceleration derives from a potential $R(x,l,l')$ whose double averaged is given by

$$\overline{R}(x) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} R(x,l,l')dMdM' = \frac{n'^2}{4n^{\frac{4}{3}}}(1+\frac{3}{2}\rho^2)$$

where n' is the mean motion of the Moon supposed constant and M, M' are the mean anomaly respectively of the satellite and the Moon. The double averaged drift vector field is deduced from the Lagrange equations

$$\overline{F_0}(n,p,\theta) = \frac{3n'^2}{4\;n}\sqrt{1-\rho^2}\frac{\partial}{\partial\theta}(n,p,\theta).$$

Hamiltonian from a Zermelo navigation problem

A Zermelo navigation problem on a n-dimensional Riemannian manifold $(\mathcal{X}, g)$ is a time minimal problem associated to the system

$$\frac{dx}{dl} = X_0(x) + \sum_{i=1}^{n} u_i X_i(x)$$

where F_i form an orthonormal frame for the metric g and $|u| \leq 1$. F_0 represents the current of magnitude $|F_0|_g$. Applying the PMP, the associated Hamiltonian has the form

$$G(x,p) = G_0(x,p) + \epsilon\sqrt{G(x,p)}$$

where $G_i(x,p) = \langle p, X_i(x)\rangle$.

We consider the following Hamiltonian

$$H_{pert}(x,p) = \overline{H}_0(x,p) + \epsilon\sqrt{\overline{H}(x,p)} \tag{2}$$

where $\overline{H}_0 = \langle p, \overline{F}_0\rangle$. To such an Hamiltonian, it corresponds a Zermelo navigation problem.

The geometric concept of conjugate point

Let $\vec{H}_{pert}$ the Hamiltonian vector field associated to H_{pert}.

- *Let $z = (x,p)$ be a reference extremal solution of $\vec{H}_{pert}$ on $[0, t_f]$. The variational equation*

$$\dot{\delta z}(t) = d\vec{H}_{pert}(z(t))\delta z(t) \tag{3}$$

is called the Jacobi equation. A Jacobi field is a non trivial solution $\delta z = (\delta x, \delta p)$ of (3) and it is said to be vertical at time t if $\delta x(t) = 0$.

- *The first conjugate time t_c is a time from which the exponential map $p_0 \to exp_{x_0}(t, p_0) = x(t, x_0, p_0)$ associated to the flow of $\vec{H}_{pert}$ is not an immersion at $t = t_c$. $x(t_c)$ is said to be conjugate to x_0.*

This singularity can be characterised by the independence of the Jacobi fields $(\delta x_1(t), \delta x_2(t))$ vertical at $t = 0$ and such that δp_i is orthogonal to p_0 $(i = 1, 2)$. In particular, at the first conjugate time t_c, the family $(\delta x_1(t), \delta x_2(t))_{|t=t_c}$ is not of rank 2.

The following test condition allow us to compute t_c for a given trajectory $x(t, x_0, p_0)$ [2]

$$det[\delta x_1(t_c), \delta x_2(t_c), \dot{x}(t_c)] = 0. \tag{4}$$

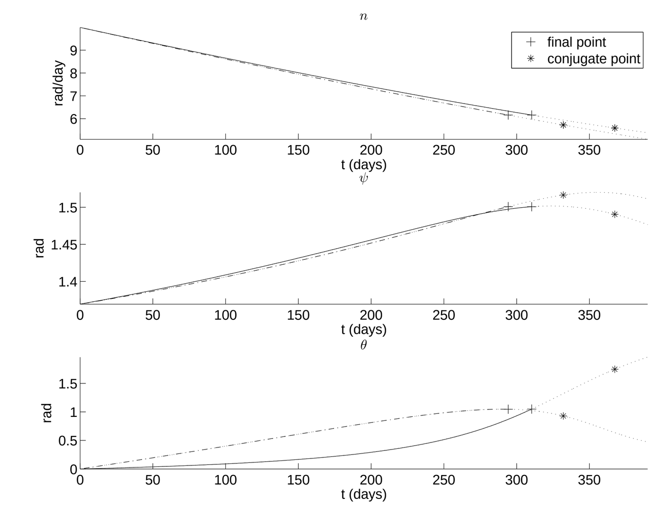
Computational results

Simulations provided below represent BC-extremals from the Hamiltonian system associated to (2). The boundary value problem is given by $x_0 = (10, 2e-1, 0)$, $x_f = (6.16, 7e-2, \frac{\pi}{3})$ and t_f is free. These extremals correspond to zeros of the shooting mapping

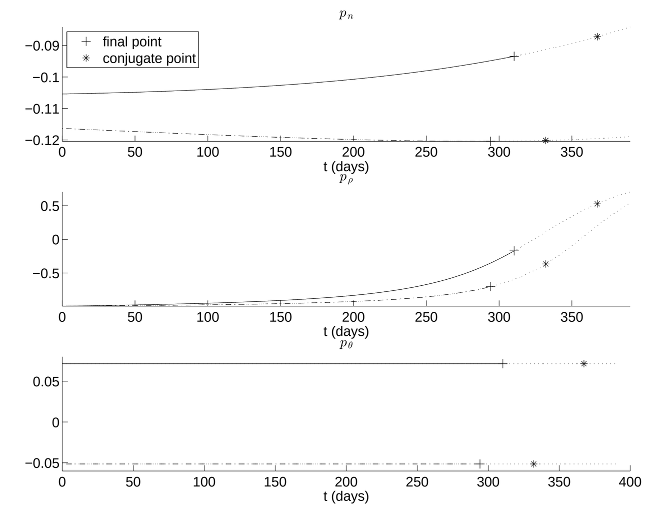
$$S : (t_f, p_0) \to (x_{t_f}(x_0, p_0)) - x_f, |p(0)|^2 - 1).$$

In order to check the optimality of our trajectories, conjugate points are computed via the verticality test (4). Simulations were performed via the package HAMPATH [3] which implements a indirect method (Newton) to find a zero of S.

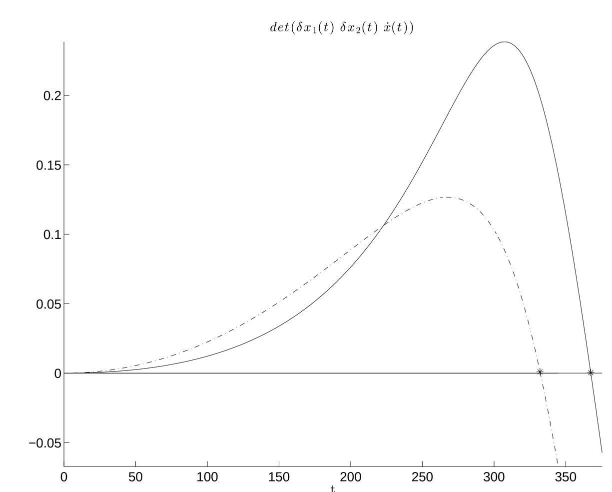
Time evolutions of the extremal components in the perturbed case (dash-dot line) and the unperturbed case (solid line) are illustrated.



Evolution of the state vector $(n(t), \psi(t) = \frac{\pi}{2} - asin(\rho(t)), \theta(t))$.

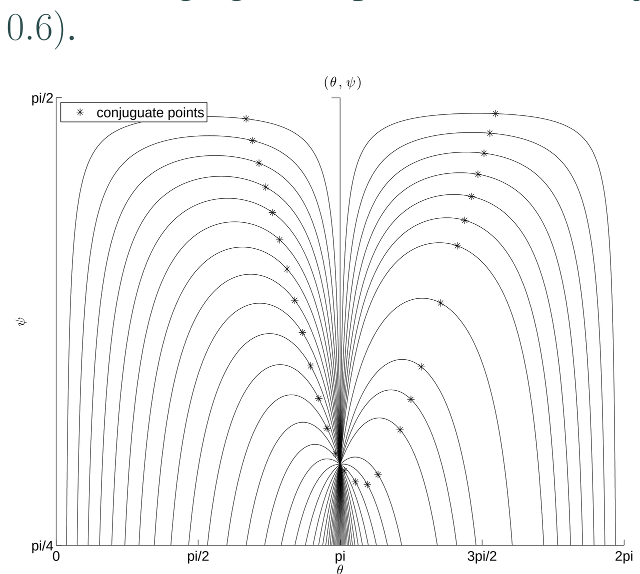


Evolution of the adjoint vector $(p_n(t), p_\rho(t), p_\theta(t))$.

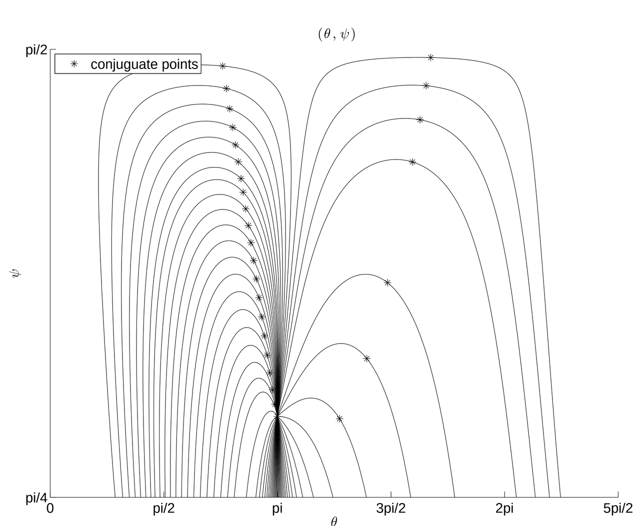


Rank test condition (4) *for the perturbed and unperturbed problem.*

The following figures represent several trajectories in the coordinates (θ, ψ) starting from $(\theta_0, \rho_0) = (\pi, 0.6)$.



Trajectories from the unperturbed case.



Trajectories from the perturbed case.

References

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Acknowledgements

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