9.49 / 9.490 Neural Circuits for Cognition
Problem Set 1
Due: Friday Oct 4, by midnight

1) **f-I curve for the leaky (LIF) integrate-and-fire neuron.** As we discussed in class, a highly simplified description of neuron dynamics is given by the Leaky Integrate-and-Fire (LIF) model. The subthreshold neural voltage is given by:

$$C_m \frac{dV(t)}{dt} = -g_m(V - V_m) + I_{inj} \tag{1}$$

where C_m is the membrane capacitance, g_m a fixed (leak) conductance that maintains the cell at a resting potential V_m in the absence of inputs, and I_{inj} is an injected current. This equation is augmented by a threshold condition: when V reaches a threshold V_θ, the voltage is immediately reset to $V_{reset} < V_\theta$. At that point, the neuron is considered to have spiked.

a. Derive analytically the critical current (I^*_{inj}) below which the neuron will never fire and above which it will as a function of the parameters g_m, V_m. Explain in words the dependence of I^*_{inj} on g_m: why does it vary the way it does?

b. Derive the the f-I curve of the neuron: that is, derive the relationship between I_{inj} and the firing rate ν of the LIF neuron. (Hint: Solve for $V(t)$ at a constant I_{inj} and solve for how long it takes to go from $V = V_{reset}$ to V_θ). Plot ν versus I_{inj}, using parameters $V_m = -60$ mV, $g_m = 0.1$ mS/cm^2, $C_m = 1$ μF/cm^2, $V_E = 0$ mV, $V_\theta = -50$ mV, $V_{reset} = -55$ mV.

c. Use the expansion $\log(1 + \epsilon) \approx \epsilon$ (which is valid when ϵ is small) on your expressions from b., to show that the f-I curve is approximately linear. Add this approximate linear expression for the f-I curve to your plot in b. to show that the linear approximation is good for large values of I_{inj}.

2) **Numerical simulation of the LIF neuron and assessment of the method-of-averaging to obtain rate-based expressions.** Consider Equation 1 (and the reset condition) together with the following equation for synaptic activation:

$$\tau \frac{ds}{dt} = -s + \sum_a \delta(t - t_a), \tag{2}$$

where the sum over a in the synaptic equation is over all the times t_a at which the neuron spikes.

a. Numerically integrate Equations (1)-(2) to obtain the time-evolution of the voltage and synaptic activation of a LIF neuron, over a 1 second interval. (To numerically integrate: replace the derivative by a finite-time difference over an interval Δt according to the "Euler method", as we saw in class, and iterate. Use $\Delta t = 0.1$ ms.) Use parameters from Problem 1. Initial conditions: s(0)=0 and $V(0) = V_{reset}$. At $t = 1$ s, step I_{inj} from 0 to a value that produces spiking at approximately 60 Hz; hold this step until $t = 2$, then return I_{inj} to 0. Plot the simulated V(t) and s(t) versus time in ms, with s(t) simulated under three conditions: $\tau_s = 10$ ms, $\tau_s = 50$ ms, and $\tau_s = 100$ ms.

b. As in class, averaging over the fast spiking variable – specifically over one inter-spike interval – reduces this pair of equations to a single differential equation for s in terms of the firing rate $\nu(I_{inj})$ of the neuron. Assume that τ_s is large compared to the interval between spikes in the neuron. Show the rate-based equation results for $s(t)$ on top of $s(t)$ obtained from the spiking simulations in a. above. When is the rate-based equation more accurate?

3) **Single neuron with two time scales.** Consider a single neuron with two self-synapses, one excitatory and one inhibitory, with different characteristic time scales. The rate of the neuron is given by:

$$r = \alpha(s_1 - s_2), \quad \alpha > 0$$

and the synaptic activities:

$$\tau_1 \dot{s}_1 = -s_1 + r$$
$$\tau_2 \dot{s}_2 = -s_2 + r$$

Here $\tau_1 > 0$, $\tau_2 > 0$ are the characteristic time scales.

a. Write the system dynamics in a two dimensional matrix form, $\dot{s} = As$. Calculate the eigenvalues of the matrix A as a function of α, τ_1, τ_2. What are the possible fixed points of the system?

b. For $\alpha = 1$, what will be the nature of the dynamics (exponential decay/growth, oscillations...) in case of $\tau_1 > \tau_2$? and for $\tau_2 > \tau_1$?. Give an intuitive explanation for the difference between these cases. Draw qualitatively (by hand or using a computer) a representing solution trajectory in the $s_1 s_2$ plane for each of the cases.

c. For $\alpha \gg 1$, what is the condition for the system to be stable? Explain.

d. Now, assume that the synaptic activities have different amplitudes in the rate equation:

$$r = \alpha_1 s_1 - \alpha_2 s_2, \; \alpha_1, \alpha_2 > 0$$

What is the condition on α_1, α_2 in order for the system to have a continuum of fixed points? Write an equation (s_1 as a function of s_2) describing the continuum of fixed points when the condition you found is realized, and draw it qualitatively.

e. Under the condition found in the last sub question, find a condition for the continuum of fixed points to be attractive.

Acknowledgement

Question 3 is adapted from Prof. Haim Sompolinsky's course materials.