חדו"א 1
סמסטר א' תשפד
עבודת בית 9

שאלה 1

א) מהי הפונקציה הקדומה ל- $f(x)$? תן את ההגדרה ואת הדוגמאות המתאימות.

ב) בדקו שהפונקציה $F(x)$ הנתונה היא קדומה ל- $f(x)$:

1)

$$F(x) = \arctan\left(\frac{x}{3}\right), \quad f(x) = \frac{3}{9+x^2}$$

2)

$$F(x) = \ln\left(\frac{x-1}{x+1}\right), \quad f(x) = \frac{2}{x^2-1}$$

ג) רשום פונקציה קדומה לפונקציה הנתונה:

1)

$$f(x) = e^{3x}$$

2)

$$f(x) = \cos(5x+6)$$

3)

$$f(x) = (mx+n)^{1/3}, \quad m \neq 0$$

4)

$$f(x) = (ax+b)^n \quad (a \neq 0, \ n \neq -1)$$

שאלה 2 חשב את האינטגרלים הבאים:

א) $$\int \frac{(x+2)^2}{\sqrt[3]{x}}\,dx$$

ב) $$\int e^{-3x+2}\,dx$$

ג) $$\int \frac{3^{2x}+5^{x+1}}{4^x}\,dx$$

ד) $$\int \frac{x}{x+1}\,dx$$

ה) $$\int \sin^2 x\,dx$$

ו) $$\int \frac{1}{x^2+2x+3}\,dx$$

ז) $$\int \frac{1}{\sqrt{2x^2-12x}}\,dx$$

ח) $$\int \frac{1}{\sqrt{12x-2x^2}}dx$$

ט) $$\int \sqrt[3]{2-5x}\,dx$$

י) $$\int \frac{x-1}{\sqrt{x}+1}dx$$

יא) $$\int (\sin x+\cos x)^2\,dx$$

יב) $$\int \sin(3x)\cos(5x)\,dx$$

יג) $$\int \cot^2(x)\,dx$$

יד) $$\int \frac{(2x+1)}{x^2+x+1}\,dx$$

שאלה 3 חשב את האינטגרלים הבאים:

א) $$\int \ln x\,dx$$

ב) $$\int 2x\ln x\,dx$$

ג) $$\int x^2\sin(2x)\,dx$$

ד) $$\int x^2\arctan(x)\,dx$$

ה) $$\int 9x^2e^{3x}\,dx$$

שאלה 4 חשב את האינטגרלים הבאים:

א) $$\int \frac{(x+1)}{x(x-3)}\,dx$$

ב) $$\int \frac{2x^2+x+3}{(x+2)(x^2+x+1)}\,dx$$

ג) $$\int \frac{5x^3-17x^2+18x-5}{(x-1)^3(x-2)}\,dx$$

ד) $$\int \frac{x^2}{x^2-4x+3}\,dx$$

ה) $$\int \frac{x^3+x^2}{x^2-6x+5}\,dx$$

ו) $$\int \frac{3x^3+x^2+5x+1}{x^3+x}\,dx$$

שאלה 5 חשב את האינטגרלים הבאים:

א) $$\int \frac{1}{3+\sin x}\,dx$$

ב) $$\int \frac{1}{1+5\cos x}\,dx$$

ג) $$\int \frac{1}{\sin x+2\cos x+6}\,dx$$

שאלה 6 בצעו את ההצבה המביאה את חישוב האינטגרל לאינטגרציה של שבר אלגברי וחשבו את האינטגרל:

א) $$\int \frac{\sqrt{x}}{1+x}\,dx$$

ב) $$\int \frac{\sqrt{x+2}}{x}\,dx$$

ג) $$\int \frac{1+e^x}{(1-e^{2x})e^x}\,dx$$

שאלה 7 חשב את האינטגרלים הבאים:

א) $$\int \sin^4 x\cos^5 x\,dx$$

ב) $\int \sin^2 x \cos^2 x \, dx$

ג) $\int \cos^6 x \, dx$

ד) $\int \sin^2 x \cos^4 x \, dx$

תשובות

שאלה 1

א) הגדרה: $F(x)$ פונקציה קדומה של פונקציה $f(x)$ אם $F'(x)=f(x)$.
דוגמאות:
$(x^2)'=2x \Leftarrow F(x)=x^2$ פונקציה קדומה ל $f(x)=2x$.
$(a^x)'=a^x\ln a \Leftarrow F(x)=a^x$ פונקציה קדומה ל $f(x)=a^x\ln a$.

ב)

ג) **1)**
$$f(x)=e^{3x}$$
$$F(x)=\frac{1}{3}e^{3x}$$
2)
$$f(x)=\cos(5x+6)$$
$$F(x)=\frac{1}{5}\sin(5x+6)$$
3)
$$f(x)=(mx+n)^{1/3},\ m\neq 0$$
$$F(x)=\frac{3}{4}\cdot\frac{1}{m}\cdot(mx+n)^{4/3}$$
4)
$$f(x)=(ax+b)^n \quad (a\neq 0,\ n\neq -1)$$
$$F(x)=\frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}$$
■

שאלה 2

א)
$$\begin{aligned}\int\frac{(x+2)^2}{\sqrt[3]{x}}\,dx&=\int\frac{x^2+4x+4}{\sqrt[3]{x}}\,dx\\&=\int\left(x^{5/3}+4x^{2/3}+4x^{-1/3}\right)\,dx\\&=\frac{3}{8}x^{8/3}+\frac{12}{5}x^{5/3}+6x^{2/3}+C\end{aligned}$$

ב)
$$\int e^{-3x+2}\,dx=-\frac{1}{3}e^{-3x+2}+C$$

ג)
$$\begin{aligned}\int\frac{3^{2x}+5^{x+1}}{4^x}\,dx&=\int\left(\left(\frac{9}{4}\right)^x+5\cdot\left(\frac{5}{4}\right)^x\right)\,dx\\&=\frac{\left(\frac{9}{4}\right)^x}{\ln\left(\frac{9}{4}\right)}+5\cdot\frac{\left(\frac{5}{4}\right)^x}{\ln\left(\frac{5}{4}\right)}+C\end{aligned}$$

ד)

$$\begin{aligned}\int \frac{x}{x+1}\,dx =& \int \frac{x+1-1}{x+1}\,dx \\ =& x - \ln|x+1| + C\end{aligned}$$

ה)

$$\begin{aligned}\int \sin^2 x\,dx =& \int \frac{(1-\cos(2x))}{2}\,dx \\ =& \frac{1}{2}(x - \frac{1}{2}\sin(2x)) + C\end{aligned}$$

ו)

$$\begin{aligned}\int \frac{1}{x^2+2x+3}\,dx =& \int \frac{1}{(x+1)^2+2}\,dx \\ =& \int \frac{1}{t^2+2}\,dt \\ =& \frac{1}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right) + C \\ =& \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C\end{aligned}$$

ז)

$$\begin{aligned}\int \frac{1}{\sqrt{2x^2-12x}}\,dx =& \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2-6x}}\,dx \\ =& \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x-3)^2-9}}\,dx\end{aligned}$$

$$\begin{aligned}t &= x-3 \\ t' &= 1\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{t^2-9}}\,dt \\ &\overset{\text{נוסחה 11 בדף הנוסחאות}}{=} \frac{1}{\sqrt{2}}\ln\left|t+\sqrt{t^2-9}\right| + C \\ &= \frac{1}{\sqrt{2}}\ln\left|x-3+\sqrt{x^2-6x}\right| + C\end{aligned}$$

ח)

$$\begin{aligned}\int \frac{1}{\sqrt{12x-2x^2}}dx =& \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{6x-x^2}}dx \\ =& \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{-(x^2-6x)}}dx \\ =& \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{-((x-3)^2-9)}}dx \\ =& \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{9-(x-3)^2}}dx\end{aligned}$$

$$t = x - 3$$
$$t' = 1$$

$$= \int \frac{1}{\sqrt{2}} \frac{1}{\sqrt{9-t^2}} dt$$
$$\overset{\text{נוסחה 10 בדף הנוסחאות}}{=} \frac{1}{\sqrt{2}} \arcsin\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{\sqrt{2}} \arcsin\left(\frac{x-3}{3}\right) + C$$

ט)

$$\int \sqrt[3]{2-5x}\, dx$$

$$t = 2 - 5x$$
$$t' = -5$$

$$-\frac{1}{5} \int t^{1/3}\, dx = -\frac{1}{5} \frac{t^{4/3}}{\frac{4}{3}} + C$$
$$= -\frac{3}{20}(2-5x)^{4/3} + C$$

י)

$$\int \frac{x-1}{\sqrt{x}+1} dx$$

$$t = \sqrt{x}$$
$$t' = \tfrac{1}{2\sqrt{x}}$$

$$\int \frac{t^2-1}{t+1} \cdot 2t \cdot dt = 2\left(\frac{t^3}{3} - \frac{t^2}{2}\right) + C$$
$$= 2\left(\frac{x^{3/2}}{3} - \frac{x}{2}\right) + C$$

יא)

$$\int (\sin x + \cos x)^2\ dx = \int \left(\sin^2 x + \cos^2 x + 2 \sin x \cos x\right)\ dx$$
$$= \int (1 + 2\sin x \cos x)\ dx$$
$$= \int (1 + \sin(2x))\ dx$$
$$= x - \frac{1}{2}\cos(2x) + C\ .$$

יב)

$$\int \sin(3x)\cos(5x)\, dx = \int \frac{\sin(8x) - \sin(2x)}{2}$$
$$= \frac{1}{2}\left(-\frac{1}{8}\cos(8x) + \frac{1}{2}\cos(2x)\right) + C$$

יג)

$$\int \cot^2(x)\,dx = \int \left(\frac{1}{\sin^2(x)} - 1\right)\,dx$$
$$= -\cot(x) - x + C$$

יד)

$$\int \frac{(2x+1)}{x^2+x+1}\,dx$$

$$t = x^2 + x + 1$$
$$t' = 2x + 1$$

$$\int \frac{t'}{t}\,dx = \int \frac{1}{t}\,dt$$
$$= \ln|t| + C$$
$$= \ln|x^2+x+1| + C\ .$$

■

שאלה 3

א)

$$\int \ln x\,dx$$
$$u = \ln x$$
$$\mathrm{v}' = 1$$
$$u' = \tfrac{1}{x}$$
$$\mathrm{v}' = x$$

$$\int \ln x\,dx = x\ln x - \int 1\,dx = x\ln x - x + C$$

■

ב) $\int 2x\ln x\,dx$ הנוסחה של אינטגרציה בחלקים:

$$\int u\cdot \mathrm{v}'\,dx = u\cdot \mathrm{v} - \int u'\cdot \mathrm{v}\,dx$$

עבור u ו v' נבחר:

$$u = \ln x\ ,\qquad \mathrm{v}' = 2x\ ,\qquad u' = \frac{1}{x}\ ,\qquad \mathrm{v} = x^2$$

כך ש

$$\int u\cdot \mathrm{v}'\,dx = u\cdot \mathrm{v} - \int u'\cdot \mathrm{v}\,dx\ ,$$

$$\int 2x\ln x\,dx = x^2\ln x - \int x\,dx = x^2\ln x - \frac{x^2}{2} + C$$

■

ג) $$\int x^2 \sin(2x)\, dx$$

$$u = x^2$$
$$\mathrm{v}' = \sin(2x)$$
$$u' = 2x$$
$$\mathrm{v} = -\tfrac{1}{2}\cos(2x)$$

$$\begin{aligned}\int x^2 \sin(2x)\, dx =& x^2\left(-\frac{1}{2}\cos(2x)\right) - \int\left(-\frac{1}{2}\cos(2x)\right)\cdot 2x\, dx \\ =& -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\, dx\end{aligned}$$

$$u = x$$
$$\mathrm{v}' = \cos(2x)$$
$$u' = 1$$
$$\mathrm{v} = \tfrac{1}{2}\sin(2x)$$

$$\begin{aligned}&= -\frac{x^2}{2}\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\, dx \\ &= -\frac{x^2}{2}\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C\ .\end{aligned}$$

■

ד) $$\int x^2 \arctan(x)\, dx$$

$$u = \arctan(x)$$
$$\mathrm{v}' = x^2$$
$$u' = \tfrac{1}{1+x^2}$$
$$\mathrm{v} = \tfrac{x^3}{3}$$

$$\begin{aligned}\int x^2 \arctan(x) =& \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int \frac{x^3}{x^2+1}\, dx \\ =& \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)\, dx \\ =& \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int x\, dx + \frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{x^2+1}\, dx \\ =& \frac{x^3}{3}\arctan(x) - \frac{x^2}{6} + \frac{1}{6}\ln|x^2+1| + C\ .\end{aligned}$$

■

ה) $$\int 9x^2 e^{3x}\, dx$$

$$u = 9x^2$$
$$\mathrm{v}' = e^{3x}$$
$$u' = 18x$$
$$\mathrm{v} = \tfrac{e^{3x}}{3}$$

$$\int 9x^2e^{3x}\,dx = 3x^2e^{3x} - \int 6xe^{3x}\,dx$$

$$u = 6x$$
$$\mathrm{v}' = e^{3x}$$
$$u' = 6$$
$$\mathrm{v} = \tfrac{e^{3x}}{3}$$

$$\begin{aligned}\int 9x^2e^{3x}\,dx =& 3x^2e^{3x} - 2xe^{3x} + \int 2e^{3x}\,dx\\ =& 3x^2e^{3x} - 2xe^{3x} + \frac{2}{3}e^{3x} + C\ .\end{aligned}$$

■

שאלה 4

א) $$\int \frac{(x+1)}{x(x-3)}\,dx$$

$$\frac{(x+1)}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}\ , \qquad A(x-3) + Bx = x+1$$

$$B = \tfrac{4}{3}\ ,\ A = \tfrac{-1}{3}$$

$$\int \left(-\frac{1}{3x} + \frac{4}{3(x-3)}\right)dx = -\frac{1}{3}\ln|x| + \frac{4}{3}\ln|x-3| + C$$

■

ב) $$\int \frac{2x^2+x+3}{(x+2)(x^2+x+1)}\,dx$$

$$\frac{2x^2+x+3}{(x+2)(x^2+x+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1}\ , \qquad A(x^2+x+1) + (Bx+C)(x+2) = 2x^2+x+3$$

$$x^2 :\ , A+B = 2\ , \qquad x :\ A+2B+C = 1\ , \qquad x^0 :\ A+2C = 3$$

$$B = -1\ ,\ A = 3$$
$$C = 0$$

$$\int \left(\frac{3}{x+2} - \frac{x}{x^2+x+1}\right)dx = 3\ln|x+2| - \int \frac{x}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx$$

$$t = x + \tfrac{1}{2}$$
$$t' = 1$$

$$
\begin{aligned}
=&3\ln|x+2| - \int \frac{t-\frac{1}{2}}{t^2+\frac{3}{4}}\,dx \\
=&3\ln|x+2| - \frac{1}{2}\int \frac{2t}{t^2+\frac{3}{4}}\,dt + \frac{1}{2}\int \frac{1}{t^2+\frac{3}{4}}\,dt \\
=&3\ln|x+2| - \frac{1}{2}\ln|t^2+\frac{3}{4}| + \frac{1}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}t\right) + C \\
=&3\ln|x+2| - \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}
$$

■

ג) $\displaystyle\int \frac{5x^3-17x^2+18x-5}{(x-1)^3(x-2)}\,dx$

$$\frac{5x^3-17x^2+18x-5}{(x-1)^3(x-2)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x-2}$$

$$A(x-2) + B(x-1)(x-2) + C(x-1)^2(x-2) + D(x-1)^3 = 5x^3-17x^2+18x-5$$

$$
\begin{aligned}
A &= -1 \\
B &= 0 \\
C &= 2
\end{aligned}
$$

$$
\begin{aligned}
=&\int \frac{-1}{(x-1)^3}\,dx + 2\int \frac{1}{x-1}\,dx + 3\int \frac{1}{x-2}\,dx \\
=&\frac{1}{2(x-1)^2} + 2\ln|x-1| + 3\ln|x-2| + C
\end{aligned}
$$

■

ד)

$$\int \frac{x^2}{x^2-4x+3}\,dx = \int \left(1 + \frac{4x-3}{x^2-4x+3}\right)dx = x + \int \frac{4x-3}{x^2-4x+3}\,dx = x + \int \frac{4x-3}{(x-1)(x-3)}\,dx$$

$$\frac{4x-3}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}\ , \qquad A(x-3) + B(x-1) = 4x-3\ .$$

$$
\begin{aligned}
A &= -\tfrac{1}{2} \\
B &= \tfrac{9}{2}
\end{aligned}
$$

$$x + \int \left(-\frac{1}{2(x-1)} + \frac{9}{2(x-3)}\right)dx = x - \frac{1}{2}\ln|x-1| + \frac{9}{2}\ln|x-3| + C$$

■

ה) $\displaystyle\int \frac{x^3+x^2}{x^2-6x+5}\,dx$

$$x^2-6x+5 \overline{\big) \, x^3+x^2}$$

$$\begin{array}{r} x \\ x^2-6x+5 \overline{\big) \, x^3+x^2} \\ \underline{x^3-6x^2+5x} \\ 7x^2-5x \end{array}$$

$$\begin{array}{r} x+7 \\ x^2-6x+5 \overline{\big) \, x^3+x^2} \\ \underline{x^3-6x^2+5x} \\ 7x^2-5x \\ \underline{7x^2-42x+35} \\ 37x-35 \end{array}$$

$$\frac{x^3+x^2}{x^2-6x+5} = x+7+\frac{37x-35}{x^2-6x+5} = x+7+\frac{37x-35}{(x-5)(x-1)}$$

$$\frac{37x-35}{(x-5)(x-1)} = \frac{A}{x-5}+\frac{B}{x-1} = \frac{A(x-1)+B(x-5)}{(x-5)(x-1)}$$

$$A(x-1)+B(x-5) = 37x-35$$

נציב $x=1$:

$$-4B=2 \qquad \Rightarrow \qquad B=-\frac{1}{2} \ .$$

נציב $x=5$:

$$4A=150 \qquad \Rightarrow \qquad A=\frac{75}{2} \ .$$

לכן

$$\frac{37x-35}{(x-5)(x-1)} = \frac{75}{2(x-5)}-\frac{1}{2(x-1)}$$

ובסה"כ

$$\frac{x^3+x^2}{x^2-6x+5} = x+7+\frac{75}{2(x-5)}-\frac{1}{2(x-1)} \ .$$

לכן

$$\int \frac{x^3+x^2}{x^2-6x+5}\,dx = \int \left(x+7+\frac{75}{2(x-5)}-\frac{1}{2(x-1)}\right)dx = \frac{x^2}{2}+7x+\frac{75}{2}\ln|x-5|-\frac{1}{2}\ln|x-1| \ .$$

■

ו) $$\int \frac{3x^3+x^2+5x+1}{x^3+x}\,dx$$

$$x^3+x \overline{\big) \, 3x^3+x^2+5x+1}$$

$$\begin{array}{r} 3 \\ x^3+x\overline{\big)\ 3x^3+x^2+5x+1} \\ \underline{3x^3+\qquad 3x} \\ x^2+2x+1 \end{array}$$

$$\frac{3x^3+x^2+5x+1}{x^3+x}=3+\frac{x^2+2x+1}{x^3+x}$$

$$\frac{x^2+2x+1}{x^3+x}=\frac{x^2+2x+1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$A(x^2+1)+(Bx+C)x=x^2+2x+1$$

x^2:	$A+B=1$
x:	$C=2$
x^0:	$A=1$

לכן $A=1$, $B=0$, $C=2$ ונקבל

$$\frac{x^2+2x+1}{x^3+x}=\frac{1}{x}+\frac{2}{x^2+1}$$

ובסה"כ

$$\frac{3x^3+x^2+5x+1}{x^3+x}=3+\frac{1}{x}+\frac{2}{x^2+1}$$

לכן

$$\int\frac{3x^3+x^2+5x+1}{x^3+x}\,dx=\int\left(3+\frac{1}{x}+\frac{2}{x^2+1}\right)dx=3x+\ln|x|+2\arctan(x)+C$$

■

שאלה 5

א) $$\int\frac{1}{3+\sin x}\,dx$$

$$t=\tan\left(\tfrac{x}{2}\right)$$

$$t'=\tfrac{1}{2}\cdot\tfrac{1}{\cos^2\left(\frac{x}{2}\right)}=\tfrac{1}{2}\left(1+\tan^2\left(\tfrac{x}{2}\right)\right)=\tfrac{1}{2}(1+t^2)$$

$$\sin x=\frac{2t}{1+t^2}$$ (זהות)

$$\begin{aligned}\int \frac{1}{3+\sin x}\,dx =& \int \frac{1}{3+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,t' dx\\ =& \int \frac{2}{3(1+t^2)+2t}\,dt\\ =& \int \frac{2}{3+3t^2+2t}\,dt\\ =& \frac{2}{3}\cdot \int \frac{1}{t^2+\frac{2}{3}t+1}\,dt\\ =& \frac{2}{3}\cdot \int \frac{1}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}}\,dt\ .\end{aligned}$$

$z = t+\frac{1}{3}$

$z'_t = 1$

$$\begin{aligned}\frac{2}{3}\cdot \int \frac{1}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}}\,dt =& \frac{2}{3}\cdot \int \frac{1}{z^2+\frac{8}{9}}\,dz\\ =& \frac{2}{3}\cdot \frac{3}{\sqrt{8}}\arctan\left(\frac{3}{\sqrt{8}}z\right)+C\\ =& \frac{2}{3}\cdot \frac{3}{\sqrt{8}}\arctan\left(\frac{3\left(t+\frac{1}{3}\right)}{\sqrt{8}}\right)+C\\ =& \frac{2}{3}\cdot \frac{3}{\sqrt{8}}\arctan\left(\frac{3\left(\tan\frac{x}{2}+\frac{1}{3}\right)}{\sqrt{8}}\right)+C\ .\end{aligned}$$

■

ב) $\displaystyle\int \frac{1}{1+5\cos x}\,dx$

$t=\tan\left(\frac{x}{2}\right)$

$t' = \frac{1}{2}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1+t^2)$

$\cos x = \dfrac{1-t^2}{1+t^2}$ (זהות)

$$
\begin{aligned}
\int \frac{1}{1+5\cos x}\,dx &= \int \frac{1}{1+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \left(\frac{2}{1+t^2}\right) \cdot t' dx \\
&= \int \frac{2}{1+t^2+5(1-t^2)}\,dt \\
&= \int \frac{2}{6-4t^2}\,dt \\
&= \frac{-1}{2} \int \frac{1}{t^2-\frac{3}{2}}\,dt \ .
\end{aligned}
$$

$$
\frac{1}{t^2-\frac{3}{2}} = \frac{1}{\left(t+\sqrt{\frac{3}{2}}\right)\left(t-\sqrt{\frac{3}{2}}\right)} = \frac{A}{t+\sqrt{\frac{3}{2}}} + \frac{B}{t-\sqrt{\frac{3}{2}}}
$$

$$
A\left(t-\sqrt{\frac{3}{2}}\right) + B\left(t+\sqrt{\frac{3}{2}}\right) = 1
$$

$$
A = \frac{1}{\sqrt{6}} \ , \qquad B = -\frac{1}{\sqrt{6}}
$$

$$
\begin{aligned}
\frac{-1}{2} \int \left(\frac{1}{\sqrt{6}} \frac{1}{\left(t-\sqrt{\frac{3}{2}}\right)} - \frac{1}{\sqrt{6}} \frac{1}{\left(t+\sqrt{\frac{3}{2}}\right)} \right) dt &= -\frac{1}{2\sqrt{6}} \ln\left|t-\sqrt{\frac{3}{2}}\right| + \frac{1}{2\sqrt{6}} \ln\left|t+\sqrt{\frac{3}{2}}\right| \\
&= \frac{1}{2\sqrt{6}} \ln\left|\frac{t+\sqrt{\frac{3}{2}}}{t-\sqrt{\frac{3}{2}}}\right| \\
&= \frac{1}{2\sqrt{6}} \ln\left|\frac{\tan\left(\frac{x}{2}\right)+\sqrt{\frac{3}{2}}}{\tan\left(\frac{x}{2}\right)-\sqrt{\frac{3}{2}}}\right| + C
\end{aligned}
$$

■

ג) $\int \frac{1}{\sin x + 2\cos x + 6}\,dx$. זהויות של הצבה אוניברסלית:

$$
t = \tan\left(\frac{x}{2}\right) \ , \qquad t' = \frac{1}{2}(1+t^2) \ , \cos x = \frac{1-t^2}{1+t^2} \ , \qquad \sin x = \frac{2t}{1+t^2}
$$

$$\int \frac{1}{\sin x + 2\cos x + 6}\,dx = \int \frac{1}{\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) + 6} \cdot \left(\frac{2}{1+t^2}\right) \cdot t'dx$$

$$= \int \frac{2}{2t + 2 - 2t^2 + 6 + 6t^2}\,dt$$

$$= \int \frac{2}{8 + 2t + 4t^2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{t^2 + \frac{t}{2} + 2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{4}\right)^2 + \frac{31}{16}}\,dt$$

$$z = t + \frac{1}{4}\,, \qquad z'_t = 1$$

$$\frac{1}{2}\int \frac{1}{z^2 + \frac{31}{16}}\,dz = \frac{1}{2}\frac{4}{\sqrt{31}}\arctan\left(\frac{4z}{\sqrt{31}}\right) + C$$

$$= \frac{1}{2}\frac{4}{\sqrt{31}}\arctan\left(\frac{4t+1}{\sqrt{31}}\right) + C$$

$$= \frac{2}{\sqrt{31}}\arctan\left(\frac{4\tan\left(\frac{x}{2}\right)+1}{\sqrt{31}}\right) + C$$

■

שאלה 6

א) $\int \frac{\sqrt{x}}{1+x}\,dx$

$t = \sqrt{x}$

$t' = \frac{1}{2\sqrt{x}} = \frac{1}{2t}$

$$\begin{aligned}
\int \frac{\sqrt{x}}{1+x}\,dx &= \int \frac{t}{1+t^2}\,dx\\
&= \int \frac{t}{1+t^2}\cdot\frac{1}{t'}\cdot t' dx\\
&= \int \frac{t}{1+t^2}\cdot\frac{1}{t'}\;dt\\
&= \int \frac{t}{1+t^2}\cdot 2t\,dt\\
&= \int \frac{2t^2}{1+t^2}\,dt\\
&= \int \frac{2(1+t^2-1)}{1+t^2}\,dt\\
&= \int \frac{2(1+t^2)-2}{1+t^2}\,dt\\
&= \int \left(\frac{2(1+t^2)}{1+t^2}-\frac{2}{1+t^2}\right)\,dt\\
&= \int \left(2-\frac{2}{1+t^2}\right)\,dt\\
&=2t-2\arctan t+C\\
&=2\sqrt{x}-2\arctan\sqrt{x}+C\ .
\end{aligned}$$

■

ב) $$\int \frac{\sqrt{x+2}}{x}\,dx$$

$$t=\sqrt{x+2}\ ,\qquad t'=\frac{1}{2\sqrt{x+2}}=\frac{1}{2t}$$

$$\Leftarrow\ x+2=t^2 \quad \Leftarrow \quad x=t^2-2$$

$$\begin{aligned}
\int \frac{t}{t^2-2}\,dx &= \int \frac{t}{t^2-2}\cdot\frac{1}{t'}\cdot t'\,dx\\
&= \int \frac{t}{t^2-2}\cdot\frac{1}{t'}\;dt\\
&= \int \frac{t}{t^2-2}\cdot 2t\,dt\\
&= \int \frac{2t^2}{t^2-2}\,dt
\end{aligned}$$

השבר בתוך האינטגרל ניתן לרשום כשברים חלקיים:

$$\begin{aligned}\frac{2t^2}{t^2-2} &= \frac{2(t^2-2+2)}{t^2-2}\\ &= \frac{2(t^2-2)+2)}{t^2-2}\\ &= \frac{2(t^2-2)}{t^2-2}+\frac{4}{t^2-2}\\ &= 2+\frac{4}{t^2-2}\\ &= 2+\frac{4}{(t-\sqrt{2})(t+\sqrt{2})}\end{aligned}$$

$$\frac{4}{(t-\sqrt{2})(t+\sqrt{2})} = \frac{A}{t+\sqrt{2}}+\frac{B}{t-\sqrt{2}} = \frac{A(t-\sqrt{2})+B(t+\sqrt{2})}{(t-\sqrt{2})(t+\sqrt{2})}$$

$t=\sqrt{2}$: $\Leftarrow 2\sqrt{2}B=4 \Leftarrow B=\sqrt{2}$

$t=-\sqrt{2}$: $\Leftarrow -2\sqrt{2}A=4 \Leftarrow A=-\sqrt{2}$

לכן

$$\frac{2t^2}{t^2-2} = 2+\frac{\sqrt{2}}{t-\sqrt{2}}-\frac{\sqrt{2}}{t+\sqrt{2}}$$

והאינטגרל הופך ל

$$\begin{aligned}\int \frac{2t^2}{t^2-2}\,dt &= \int \left(2+\frac{\sqrt{2}}{t-\sqrt{2}}-\frac{1}{t+\sqrt{2}}\right)dt\\ &= 2t+\sqrt{2}\ln|t-\sqrt{2}|-\sqrt{2}\ln|t+\sqrt{2}|+C\\ &= 2\sqrt{x+2}+\sqrt{2}\ln|\sqrt{x+2}-\sqrt{2}|-\sqrt{2}\ln|\sqrt{x+2}+\sqrt{2}|+C\end{aligned}$$

■

ג) $$\int \frac{1+e^x}{(1-e^{2x})e^x}\,dx$$

$$t=e^x$$

$$t'=e^x=t$$

$$\int \frac{1+t}{(1-t^2)t}dx = \int \frac{1+t}{(1-t^2)t^2}\cdot t'\,dx = \int \frac{1+t}{(1-t^2)t^2}dt = \int \frac{1}{(1-t)t^2}dt$$

$$\frac{1}{(1-t)t^2} = \frac{A}{1-t}+\frac{B}{t^2}+\frac{C}{t} = \frac{At^2+B(1-t)+Ct(1-t)}{(1-t)t^2}$$

$$At^2+B(1-t)+Ct(1-t)=1$$

$$t=0 \ \rightsquigarrow B=1 \ , \qquad t=1 \ \rightsquigarrow A=1 \ , \qquad t=2 \ \rightsquigarrow 4A-B-2C=1 \ \Rightarrow C=1 \ .$$

$$\int \left(\frac{1}{1-t}+\frac{1}{t^2}+\frac{1}{t}\right) dt = -\ln|1-t| - \frac{1}{t} + \ln|t| + C = -\ln|1-e^x| - \frac{1}{e^x} + x + C$$

■

שאלה 7

א) $\displaystyle\int \sin^4 x \cos^5 x \, dx$

$$t=\sin x \ , \qquad t'=\cos x$$

$$\cos^2 x = 1-\sin^2 x = 1-t^2 \ , \qquad \cos^4 x = (1-t^2)^2 \ .$$

$$\begin{aligned}\int \sin^4 x \cos^5 x \, dx &= \int t^4(1-t^2)^2 \, t' \, dx \\ &= \int t^4(1-t^2)^2 \, dt \\ &= \int (t^8-2t^6+t^4) \, dt \\ &= \frac{t^9}{9} - \frac{2t^7}{7} + \frac{t^5}{5} + C \\ &= \frac{\sin^9 x}{9} - \frac{2\sin^7 x}{7} + \frac{\sin^5 x}{5} + C\end{aligned}$$

■

ב) $\displaystyle\int \sin^2 x \cos^2 x \, dx$

$$\sin^2 x \cos^2 = (\sin x \cos x)^2 = \left(\frac{\sin 2x}{2}\right)^2 = \frac{1}{4}\sin^2 2x \ , = \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right) = \frac{1}{8}(1-\cos 4x)$$

$$\begin{aligned}\int \frac{1}{8}(1-\cos 4x) \ dx =& \frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C \\ \frac{x}{8} - \frac{\sin 4x}{32} + C&\end{aligned}$$

■

ג) $\int \cos^6 x\, dx$

$$\begin{aligned}\cos^6 x &= \left(\cos^2 x\right)^3 \\ &= \left(\frac{1+\cos 2x}{2}\right)^3 \\ &= \frac{1}{8}\left(1+3\cos 2x + 3\cos^2 2x + \cos^3 2x\right) \\ &= \frac{1}{8}\left(1+3\cos 2x + 3\left(\frac{1+\cos 4x}{2}\right) + \cos^3 2x\right)\end{aligned}$$

$$\int \frac{1}{8}\left(1+3\cos 2x + 3\left(\frac{1+\cos 4x}{2}\right) + \cos^3 2x\right) dx = \frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x + \int \cos^3 2x dx$$

$$t = \sin 2x \ , \qquad t' = 2\cos 2x$$

$$\begin{aligned}&\frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x + \frac{1}{8}\int \left(1-\sin^2 2x\right)\cos 2x\, dx \\ =&\frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x + \frac{1}{8}\int \left(1-t^2\right)\frac{t'}{2} dx \\ =&\frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x + \frac{1}{16}\int \left(1-t^2\right) dt \\ =&\frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x + \frac{1}{16}\int \left(1-t^2\right) dt \\ =&\frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x + \frac{1}{16}\left(t - \frac{t^3}{3}\right) \\ =&\frac{5x}{16} + \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) \\ =&\frac{5x}{16} + \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{16}\sin 2x - \frac{\sin^3 2x}{48} \\ =&\frac{5x}{16} + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C\end{aligned}$$

■

ד)

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\sin x \cos x\right)^2 \cos^2 x\, dx$$

$$\sin x \cos x = \frac{\sin 2x}{2} \ , \qquad \cos^2 x = \frac{1+\cos 2x}{2} \ .$$

$$\begin{aligned}\int \sin^2 x \cos^4 x \, dx &= \int (\sin x \cos x)^2 \cos^2 x \, dx \\ &= \int \frac{\sin^2 x}{4} \cdot \frac{1+\cos 2x}{2} \, dx \\ &= \frac{1}{8} \int \left(\sin^2 x + \sin^2 2x \cos 2x\right) \, dx \\ &= \frac{1}{8} \int \left(\frac{1-\cos 4x}{2} + \sin^2 2x \cos 2x\right) \, dx \\ &= \frac{1}{8} \int \left(\frac{1-\cos 4x}{2} + \sin^2 2x \cos 2x\right) \, dx \\ &= \frac{1}{16} \int (1-\cos 4x) \, dx + \frac{1}{8} \int \sin^2 2x \cos 2x \, dx \\ &= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{8} \int \sin^2 2x \cos 2x \end{aligned}$$

$$t = \sin 2x \ , \qquad t' = 2\cos 2x$$

$$\begin{aligned}\frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{8} \int \sin^2 2x \cos 2x &= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{8} \int t^2 \frac{t'}{2} \, dx \\ &= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{16} \int t^2 \, dt \\ &= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{16}\frac{t^3}{3} + C \\ &= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{16}\frac{\sin^3 2x}{3} + C \\ &= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C\end{aligned}$$