חדו"א 1
סמסטר א' תשפד
עבודת בית 9

שאלה 1

א) מהי הפונקציה הקדומה ל- $f(x)$? תן את ההגדרה ואת הדוגמאות המתאימות.

ב) בדקו שהפונקציה $F(x)$ הנתונה היא קדומה ל- $f(x)$:

1)

$$F(x) = \arctan\left(\frac{x}{3}\right),\ f(x) = \frac{3}{9+x^2}$$

2)

$$F(x) = \ln\left(\frac{x-1}{x+1}\right),\ f(x) = \frac{2}{x^2-1}$$

ג) רשום פונקציה קדומה לפונקציה הנתונה:

1)

$$f(x) = e^{3x}$$

2)

$$f(x) = \cos(5x+6)$$

3)

$$f(x) = (mx+n)^{1/3},\ m \neq 0$$

4)

$$f(x) = (ax+b)^n \quad (a \neq 0,\ n \neq -1)$$

שאלה 2 חשב את האינטגרלים הבאים:

א) $$\int \frac{(x+2)^2}{\sqrt[3]{x}}\,dx$$

ב) $$\int e^{-3x+2}\,dx$$

ג) $$\int \frac{3^{2x}+5^{x+1}}{4^x}\,dx$$

ד) $$\int \frac{x}{x+1}\,dx$$

ה) $$\int \sin^2 x\,dx$$

ו) $$\int \frac{1}{x^2+2x+3}\,dx$$

ז) $$\int \frac{1}{\sqrt{2x^2-12x}}\,dx$$

ח) $$\int \frac{1}{\sqrt{12x-2x^2}}dx$$

ט) $$\int \sqrt[3]{2-5x}\,dx$$

י) $$\int \frac{x-1}{\sqrt{x}+1}dx$$

יא) $$\int (\sin x+\cos x)^2\,dx$$

יב) $$\int \sin(3x)\cos(5x)\,dx$$

יג) $$\int \cot^2(x)\,dx$$

יד) $$\int \frac{(2x+1)}{x^2+x+1}\,dx$$

שאלה 3 חשב את האינטגרלים הבאים:

א) $$\int \ln x\,dx$$

ב) $$\int 2x\ln x\,dx$$

ג) $$\int x^2\sin(2x)\,dx$$

ד) $$\int x^2\arctan(x)\,dx$$

ה) $$\int 9x^2e^{3x}\,dx$$

שאלה 4 חשב את האינטגרלים הבאים:

א) $$\int \frac{(x+1)}{x(x-3)}\,dx$$

ב) $$\int \frac{2x^2+x+3}{(x+2)(x^2+x+1)}\,dx$$

ג) $$\int \frac{5x^3-17x^2+18x-5}{(x-1)^3(x-2)}\,dx$$

ד) $$\int \frac{x^2}{x^2-4x+3}\,dx$$

ה) $$\int \frac{x^3+x^2}{x^2-6x+5}\,dx$$

ו) $$\int \frac{3x^3+x^2+5x+1}{x^3+x}\,dx$$

שאלה 5 חשב את האינטגרלים הבאים:

א) $$\int \frac{1}{3+\sin x}\,dx$$

ב) $$\int \frac{1}{1+5\cos x}\,dx$$

ג) $$\int \frac{1}{\sin x+2\cos x+6}\,dx$$

שאלה 6 בצעו את ההצבה המביאה את חישוב האינטגרל לאינטגרציה של שבר אלגברי וחשבו את האינטגרל:

א) $$\int \frac{\sqrt{x}}{1+x}\,dx$$

ב) $$\int \frac{\sqrt{x+2}}{x}\,dx$$

ג) $$\int \frac{1+e^x}{(1-e^{2x})e^x}\,dx$$

שאלה 7 חשב את האינטגרלים הבאים:

א) $$\int \sin^4 x\cos^5 x\,dx$$

ב) $$\int \sin^2 x \cos^2 x \, dx$$

ג) $$\int \cos^6 x \, dx$$

ד) $$\int \sin^2 x \cos^4 x \, dx$$

תשובות

שאלה 1

א) <u>הגדרה</u>: $F(x)$ פונקציה קדומה של פונקציה $f(x)$ אם $F'(x)=f(x)$.

<u>דוגמאות</u>:

$(x^2)'=2x \Leftarrow F(x)=x^2$ פונקציה קדומה ל $f(x)=2x$.

$(a^x)'=a^x\ln a \Leftarrow F(x)=a^x$ פונקציה קדומה ל $f(x)=a^x\ln a$.

ב)

ג) **1)**

$$f(x)=e^{3x}$$
$$F(x)=\frac{1}{3}e^{3x}$$

2)

$$f(x)=\cos(5x+6)$$
$$F(x)=\frac{1}{5}\sin(5x+6)$$

3)

$$f(x)=(mx+n)^{1/3},\quad m\neq 0$$
$$F(x)=\frac{3}{4}\cdot\frac{1}{m}\cdot(mx+n)^{4/3}$$

4)

$$f(x)=(ax+b)^n\quad (a\neq 0,\ n\neq -1)$$
$$F(x)=\frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}$$

■

שאלה 2

א)

$$\begin{aligned}\int\frac{(x+2)^2}{\sqrt[3]{x}}\,dx&=\int\frac{x^2+4x+4}{\sqrt[3]{x}}\,dx\\&=\int\left(x^{5/3}+4x^{2/3}+4x^{-1/3}\right)\,dx\\&=\frac{3}{8}x^{8/3}+\frac{12}{5}x^{5/3}+6x^{2/3}+C\end{aligned}$$

ב)

$$\int e^{-3x+2}\,dx=-\frac{1}{3}e^{-3x+2}+C$$

ג)

$$\begin{aligned}\int\frac{3^{2x}+5^{x+1}}{4^x}\,dx&=\int\left(\left(\frac{9}{4}\right)^x+5\cdot\left(\frac{5}{4}\right)^x\right)\,dx\\&=\frac{\left(\frac{9}{4}\right)^x}{\ln\left(\frac{9}{4}\right)}+5\cdot\frac{\left(\frac{5}{4}\right)^x}{\ln\left(\frac{5}{4}\right)}+C\end{aligned}$$

ד)

$$\int \frac{x}{x+1}\,dx = \int \frac{x+1-1}{x+1}\,dx$$
$$= x - \ln|x+1| + C$$

ה)

$$\int \sin^2 x\,dx = \int \frac{(1-\cos(2x))}{2}\,dx$$
$$= \frac{1}{2}(x - \frac{1}{2}\sin(2x)) + C$$

ו)

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$$
$$= \int \frac{1}{t^2+2}\,dt$$
$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right) + C$$
$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C$$

ז)

$$\int \frac{1}{\sqrt{2x^2-12x}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2-6x}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x-3)^2-9}}\,dx$$

$$t = x-3$$
$$t' = 1$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{t^2-9}}\,dt$$
$$\overset{\text{נוסחה 11 בדף הנוסחאות}}{=} \frac{1}{\sqrt{2}}\ln\left|t + \sqrt{t^2-9}\right| + C$$
$$= \frac{1}{\sqrt{2}}\ln\left|x - 3 + \sqrt{x^2-6x}\right| + C$$

ח)

$$\int \frac{1}{\sqrt{12x-2x^2}}dx = \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{6x-x^2}}dx$$
$$= \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{-(x^2-6x)}}dx$$
$$= \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{-((x-3)^2-9)}}dx$$
$$= \int \frac{1}{\sqrt{2}}\frac{1}{\sqrt{9-(x-3)^2}}dx$$

$$t = x-3$$
$$t' = 1$$

$$= \int \frac{1}{\sqrt{2}} \frac{1}{\sqrt{9-t^2}} dt$$

$$\overset{\text{נוסחה 10 בדף הנוסחאות}}{=} \frac{1}{\sqrt{2}} \arcsin\left(\frac{t}{3}\right) + C$$

$$= \frac{1}{\sqrt{2}} \arcsin\left(\frac{x-3}{3}\right) + C$$

ט)

$$\int \sqrt[3]{2-5x}\, dx$$

$$t = 2-5x$$
$$t' = -5$$

$$-\frac{1}{5} \int t^{1/3}\, dx = -\frac{1}{5} \frac{t^{4/3}}{\frac{4}{3}} + C$$

$$= -\frac{3}{20} (2-5x)^{4/3} + C$$

י)

$$\int \frac{x-1}{\sqrt{x}+1} dx$$

$$t = \sqrt{x}$$
$$t' = \frac{1}{2\sqrt{x}}$$

$$\int \frac{t^2-1}{t+1} \cdot 2t \cdot dt = 2\left(\frac{t^3}{3} - \frac{t^2}{2}\right) + C$$

$$= 2\left(\frac{x^{3/2}}{3} - \frac{x}{2}\right) + C$$

יא)

$$\int (\sin x + \cos x)^2 \; dx = \int \left(\sin^2 x + \cos^2 x + 2 \sin x \cos x\right) \; dx$$

$$= \int (1 + 2 \sin x \cos x) \; dx$$

$$= \int (1 + \sin(2x)) \; dx$$

$$= x - \frac{1}{2} \cos(2x) + C \; .$$

יב)

$$\int \sin(3x) \cos(5x)\, dx = \int \frac{\sin(8x) - \sin(2x)}{2}$$

$$= \frac{1}{2}\left(-\frac{1}{8} \cos(8x) + \frac{1}{2} \cos(2x)\right) + C$$

יג)

$$\int \cot^2(x)\,dx = \int \left(\frac{1}{\sin^2(x)} - 1\right)\,dx$$
$$= -\cot(x) - x + C$$

יד)

$$\int \frac{(2x+1)}{x^2+x+1}\,dx$$

$$t = x^2 + x + 1$$
$$t' = 2x + 1$$

$$\int \frac{t'}{t}\,dx = \int \frac{1}{t}\,dt$$
$$= \ln|t| + C$$
$$= \ln|x^2 + x + 1| + C \ .$$

■

שאלה 3

א) $\int \ln x\,dx$

$u = \ln x$

$\mathrm{v}' = 1$

$u' = \frac{1}{x}$

$\mathrm{v}' = x$

$$\int \ln x\,dx = x\ln x - \int 1\,dx = x\ln x - x + C$$

■

ב) $\int 2x\ln x\,dx$ הנוסחה של אינטגרציה בחלקים:

$$\int u\cdot \mathrm{v}'\,dx = u\cdot \mathrm{v} - \int u'\cdot \mathrm{v}\,dx$$

עבור u ו v' נבחר:

$$u = \ln x\ , \qquad \mathrm{v}' = 2x\ , \qquad u' = \frac{1}{x}\ , \qquad \mathrm{v} = x^2$$

כך ש

$$\int u\cdot \mathrm{v}'\,dx = u\cdot \mathrm{v} - \int u'\cdot \mathrm{v}\,dx\ ,$$

$$\int 2x\ln x\,dx = x^2\ln x - \int x\,dx = x^2\ln x - \frac{x^2}{2} + C$$

■

ג)
$$\int x^2 \sin(2x)\,dx$$
$$u = x^2$$
$$\text{v}' = \sin(2x)$$
$$u' = 2x$$
$$\text{v} = -\tfrac{1}{2}\cos(2x)$$

$$\begin{aligned}\int x^2 \sin(2x)\,dx =& x^2\left(-\frac{1}{2}\cos(2x)\right) - \int \left(-\frac{1}{2}\cos(2x)\right)\cdot 2x\,dx \\ =& -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\,dx\end{aligned}$$

$$u = x$$
$$\text{v}' = \cos(2x)$$
$$u' = 1$$
$$\text{v} = \tfrac{1}{2}\sin(2x)$$

$$\begin{aligned}&= -\frac{x^2}{2}\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx \\ &= -\frac{x^2}{2}\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C\ .\end{aligned}$$

■

ד)
$$\int x^2 \arctan(x)\,dx$$
$$u = \arctan(x)$$
$$\text{v}' = x^2$$
$$u' = \tfrac{1}{1+x^2}$$
$$\text{v} = \tfrac{x^3}{3}$$

$$\begin{aligned}\int x^2 \arctan(x) =& \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx \\ =& \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int \left(x - \frac{x}{x^2+1}\right)\,dx \\ =& \frac{x^3}{3}\arctan(x) - \frac{1}{3}\int x\,dx + \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}\,dx \\ =& \frac{x^3}{3}\arctan(x) - \frac{x^2}{6} + \frac{1}{6}\ln|x^2+1| + C\ .\end{aligned}$$

■

ה)
$$\int 9x^2 e^{3x}\,dx$$
$$u = 9x^2$$
$$\text{v}' = e^{3x}$$
$$u' = 18x$$
$$\text{v} = \tfrac{e^{3x}}{3}$$

$$\int 9x^2e^{3x}\,dx = 3x^2e^{3x} - \int 6xe^{3x}\,dx$$

$$u = 6x$$
$$\mathrm{v}' = e^{3x}$$
$$u' = 6$$
$$\mathrm{v} = \tfrac{e^{3x}}{3}$$

$$\begin{aligned}\int 9x^2e^{3x}\,dx &= 3x^2e^{3x} - 2xe^{3x} + \int 2e^{3x}\,dx \\ &= 3x^2e^{3x} - 2xe^{3x} + \frac{2}{3}e^{3x} + C \ .\end{aligned}$$

■

שאלה 4

א) $\displaystyle\int \frac{(x+1)}{x(x-3)}\,dx$

$$\frac{(x+1)}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} \ , \qquad A(x-3) + Bx = x+1$$

$B = \frac{4}{3}$, $A = \frac{-1}{3}$

$$\int \left(-\frac{1}{3x} + \frac{4}{3(x-3)}\right) dx = -\frac{1}{3}\ln|x| + \frac{4}{3}\ln|x-3| + C$$

■

ב) $\displaystyle\int \frac{2x^2+x+3}{(x+2)(x^2+x+1)}\,dx$

$$\frac{2x^2+x+3}{(x+2)(x^2+x+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1} \ , \qquad A(x^2+x+1) + (Bx+C)(x+2) = 2x^2+x+3$$

$$x^2 : , A+B = 2 \ , \qquad x : \ A+2B+C = 1 \ , \qquad x^0 : \ A+2C = 3$$

$B = -1$, $A = 3$
$C = 0$

$$\int \left(\frac{3}{x+2} - \frac{x}{x^2+x+1}\right) dx = 3\ln|x+2| - \int \frac{x}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx$$

$$t = x + \tfrac{1}{2}$$
$$t' = 1$$

$$
\begin{aligned}
&=3\ln|x+2|-\int\frac{t-\frac{1}{2}}{t^2+\frac{3}{4}}\,dx\\
&=3\ln|x+2|-\frac{1}{2}\int\frac{2t}{t^2+\frac{3}{4}}\,dt+\frac{1}{2}\int\frac{1}{t^2+\frac{3}{4}}\,dt\\
&=3\ln|x+2|-\frac{1}{2}\ln|t^2+\frac{3}{4}|+\frac{1}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}t\right)+C\\
&=3\ln|x+2|-\frac{1}{2}\ln|x^2+x+1|+\frac{1}{2}\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C
\end{aligned}
$$

■

ג)

$$\int\frac{5x^3-17x^2+18x-5}{(x-1)^3(x-2)}\,dx$$

$$\frac{5x^3-17x^2+18x-5}{(x-1)^3(x-2)}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{D}{x-2}$$

$$A(x-2)+B(x-1)(x-2)+C(x-1)^2(x-2)+D(x-1)^3=5x^3-17x^2+18x-5$$

$$
\begin{aligned}
A&=-1\\
B&=0\\
C&=2
\end{aligned}
$$

$$
\begin{aligned}
&=\int\frac{-1}{(x-1)^3}\,dx+2\int\frac{1}{x-1}\,dx+3\int\frac{1}{x-2}\,dx\\
&=\frac{1}{2(x-1)^2}+2\ln|x-1|+3\ln|x-2|+C
\end{aligned}
$$

■

ד)

$$\int\frac{x^2}{x^2-4x+3}\,dx=\int\left(1+\frac{4x-3}{x^2-4x+3}\right)dx=x+\int\frac{4x-3}{x^2-4x+3}\,dx=x+\int\frac{4x-3}{(x-1)(x-3)}\,dx$$

$$\frac{4x-3}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}\ ,\qquad A(x-3)+B(x-1)=4x-3\ .$$

$$
\begin{aligned}
A&=-\tfrac{1}{2}\\
B&=\tfrac{9}{2}
\end{aligned}
$$

$$x+\int\left(-\frac{1}{2(x-1)}+\frac{9}{2(x-3)}\right)dx=x-\frac{1}{2}\ln|x-1|+\frac{9}{2}\ln|x-3|+C$$

■

ה)

$$\int\frac{x^3+x^2}{x^2-6x+5}\,dx$$

$$x^2-6x+5\overline{\big)\, x^3+x^2 \quad}$$

$$\begin{array}{r|l} & x \\ \hline x^2-6x+5 & x^3+\ x^2 \\ & \underline{x^3-6x^2}+5x \\ & \quad 7x^2-5x \end{array}$$

$$\begin{array}{r|l} & x\ +7 \\ \hline x^2-6x+5 & x^3+\ x^2 \\ & \underline{x^3-6x^2}+\ 5x \\ & \quad 7x^2-\ 5x \\ & \quad \underline{7x^2-42x}+35 \\ & \qquad\quad 37x-35 \end{array}$$

$$\frac{x^3+x^2}{x^2-6x+5}=x+7+\frac{37x-35}{x^2-6x+5}=x+7+\frac{37x-35}{(x-5)(x-1)}$$

$$\frac{37x-35}{(x-5)(x-1)}=\frac{A}{x-5}+\frac{B}{x-1}=\frac{A(x-1)+B(x-5)}{(x-5)(x-1)}$$

$$A(x-1)+B(x-5)=37x-35$$

נציב $x=1$:

$$-4B=2 \qquad \Rightarrow \qquad B=-\frac{1}{2}\ .$$

נציב $x=5$:

$$4A=150 \qquad \Rightarrow \qquad A=\frac{75}{2}\ .$$

לכן

$$\frac{37x-35}{(x-5)(x-1)}=\frac{75}{2(x-5)}-\frac{1}{2(x-1)}$$

ובסה"כ

$$\frac{x^3+x^2}{x^2-6x+5}=x+7+\frac{75}{2(x-5)}-\frac{1}{2(x-1)}\ .$$

לכן

$$\int\frac{x^3+x^2}{x^2-6x+5}\,dx=\int\left(x+7+\frac{75}{2(x-5)}-\frac{1}{2(x-1)}\right)dx=\frac{x^2}{2}+7x+\frac{75}{2}\ln|x-5|-\frac{1}{2}\ln|x-1|\ .$$

■

ו) $$\int\frac{3x^3+x^2+5x+1}{x^3+x}\,dx$$

$$x^3+x\overline{\big)\, 3x^3+x^2+5x+1}$$

$$
\begin{array}{r}
3 \qquad\qquad\qquad \\
x^3+x \overline{\big)\, 3x^3+x^2+5x+1} \\
\underline{3x^3+ \qquad\; 3x} \qquad\;\; \\
x^2+2x+1
\end{array}
$$

$$\frac{3x^3+x^2+5x+1}{x^3+x} = 3 + \frac{x^2+2x+1}{x^3+x}$$

$$\frac{x^2+2x+1}{x^3+x} = \frac{x^2+2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$A(x^2+1)+(Bx+C)x = x^2+2x+1$$

x^2:	$A+B=1$
x:	$C=2$
x^0:	$A=1$

לכן $A=1$, $B=0$, $C=2$ ונקבל

$$\frac{x^2+2x+1}{x^3+x} = \frac{1}{x} + \frac{2}{x^2+1}$$

ובסה"כ

$$\frac{3x^3+x^2+5x+1}{x^3+x} = 3 + \frac{1}{x} + \frac{2}{x^2+1}$$

לכן

$$\int \frac{3x^3+x^2+5x+1}{x^3+x}\,dx = \int \left(3 + \frac{1}{x} + \frac{2}{x^2+1}\right) dx = 3x + \ln|x| + 2\arctan(x) + C$$

$\blacksquare$

שאלה 5

א) $$\int \frac{1}{3+\sin x}\,dx$$

$$t = \tan\left(\tfrac{x}{2}\right)$$

$$t' = \tfrac{1}{2}\cdot\tfrac{1}{\cos^2\left(\frac{x}{2}\right)} = \tfrac{1}{2}\left(1+\tan^2\left(\tfrac{x}{2}\right)\right) = \tfrac{1}{2}(1+t^2)$$

$\sin x = \dfrac{2t}{1+t^2}$ (זהות)

$$
\begin{aligned}
\int \frac{1}{3+\sin x}\,dx &= \int \frac{1}{3+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,t' dx \\
&= \int \frac{2}{3(1+t^2)+2t}\,dt \\
&= \int \frac{2}{3+3t^2+2t}\,dt \\
&= \frac{2}{3}\cdot\int \frac{1}{t^2+\frac{2}{3}t+1}\,dt \\
&= \frac{2}{3}\cdot\int \frac{1}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}}\,dt \ .
\end{aligned}
$$

$z = t + \frac{1}{3}$

$z'_t = 1$

$$
\begin{aligned}
\frac{2}{3}\cdot\int \frac{1}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}}\,dt &= \frac{2}{3}\cdot\int \frac{1}{z^2+\frac{8}{9}}\,dz \\
&= \frac{2}{3}\cdot\frac{3}{\sqrt{8}}\arctan\left(\frac{3}{\sqrt{8}}z\right)+C \\
&= \frac{2}{3}\cdot\frac{3}{\sqrt{8}}\arctan\left(\frac{3\left(t+\frac{1}{3}\right)}{\sqrt{8}}\right)+C \\
&= \frac{2}{3}\cdot\frac{3}{\sqrt{8}}\arctan\left(\frac{3\left(\tan\frac{x}{2}+\frac{1}{3}\right)}{\sqrt{8}}\right)+C \ .
\end{aligned}
$$

■

ב) $\displaystyle\int \frac{1}{1+5\cos x}\,dx$

$t = \tan\left(\frac{x}{2}\right)$

$t' = \frac{1}{2}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1+t^2)$

$\cos x = \dfrac{1-t^2}{1+t^2}$ (זהות)

$$\int \frac{1}{1+5\cos x}\,dx = \int \frac{1}{1+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \left(\frac{2}{1+t^2}\right) \cdot t'dx$$
$$= \int \frac{2}{1+t^2+5(1-t^2)}\,dt$$
$$= \int \frac{2}{6-4t^2}\,dt$$
$$= \frac{-1}{2}\int \frac{1}{t^2-\frac{3}{2}}\,dt\ .$$

$$\frac{1}{t^2-\frac{3}{2}} = \frac{1}{\left(t+\sqrt{\frac{3}{2}}\right)\left(t-\sqrt{\frac{3}{2}}\right)} = \frac{A}{t+\sqrt{\frac{3}{2}}} + \frac{B}{t-\sqrt{\frac{3}{2}}}$$

$$A\left(t-\sqrt{\frac{3}{2}}\right) + B\left(t+\sqrt{\frac{3}{2}}\right) = 1$$

$$A = \frac{1}{\sqrt{6}}\ , \qquad B = -\frac{1}{\sqrt{6}}$$

$$\frac{-1}{2}\int\left(\frac{1}{\sqrt{6}}\frac{1}{\left(t-\sqrt{\frac{3}{2}}\right)} - \frac{1}{\sqrt{6}}\frac{1}{\left(t+\sqrt{\frac{3}{2}}\right)}\right)dt = -\frac{1}{2\sqrt{6}}\ln\left|t-\sqrt{\frac{3}{2}}\right| + \frac{1}{2\sqrt{6}}\ln\left|t+\sqrt{\frac{3}{2}}\right|$$
$$= \frac{1}{2\sqrt{6}}\ln\left|\frac{t+\sqrt{\frac{3}{2}}}{t-\sqrt{\frac{3}{2}}}\right|$$
$$= \frac{1}{2\sqrt{6}}\ln\left|\frac{\tan\left(\frac{x}{2}\right)+\sqrt{\frac{3}{2}}}{\tan\left(\frac{x}{2}\right)-\sqrt{\frac{3}{2}}}\right| + C$$

■

ג) $\int \frac{1}{\sin x + 2\cos x + 6}\,dx$. זהויות של הצבה אוניברסלית:

$$t = \tan\left(\frac{x}{2}\right)\ , \qquad t' = \frac{1}{2}(1+t^2)\ , \cos x = \frac{1-t^2}{1+t^2}\ , \qquad \sin x = \frac{2t}{1+t^2}$$

$$
\begin{aligned}
\int \frac{1}{\sin x + 2\cos x + 6}\,dx =& \int \frac{1}{\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) + 6} \cdot \left(\frac{2}{1+t^2}\right) \cdot t' dx \\
=& \int \frac{2}{2t + 2 - 2t^2 + 6 + 6t^2}\,dt \\
=& \int \frac{2}{8 + 2t + 4t^2}\,dt \\
=& \frac{1}{2}\int \frac{1}{t^2 + \frac{t}{2} + 2}\,dt \\
=& \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{4}\right)^2 + \frac{31}{16}}\,dt
\end{aligned}
$$

$$
z = t + \frac{1}{4}\,, \qquad z'_t = 1
$$

$$
\begin{aligned}
\frac{1}{2}\int \frac{1}{z^2 + \frac{31}{16}}\,dz =& \frac{1}{2}\frac{4}{\sqrt{31}} \arctan\left(\frac{4z}{\sqrt{31}}\right) + C \\
=& \frac{1}{2}\frac{4}{\sqrt{31}} \arctan\left(\frac{4t+1}{\sqrt{31}}\right) + C \\
=& \frac{2}{\sqrt{31}} \arctan\left(\frac{4\tan\left(\frac{x}{2}\right)+1}{\sqrt{31}}\right) + C
\end{aligned}
$$

■

שאלה 6

א) $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

$$t = \sqrt{x}$$

$$t' = \tfrac{1}{2\sqrt{x}} = \tfrac{1}{2t}$$

$$
\begin{aligned}
\int \frac{\sqrt{x}}{1+x}\,dx &= \int \frac{t}{1+t^2}\,dx \\
&= \int \frac{t}{1+t^2}\cdot\frac{1}{t'}\cdot t' dx \\
&= \int \frac{t}{1+t^2}\cdot\frac{1}{t'}\ dt \\
&= \int \frac{t}{1+t^2}\cdot 2t\,dt \\
&= \int \frac{2t^2}{1+t^2}\,dt \\
&= \int \frac{2(1+t^2-1)}{1+t^2}\,dt \\
&= \int \frac{2(1+t^2)-2}{1+t^2}\,dt \\
&= \int \left(\frac{2(1+t^2)}{1+t^2}-\frac{2}{1+t^2}\right)\,dt \\
&= \int \left(2-\frac{2}{1+t^2}\right)\,dt \\
&= 2t-2\arctan t+C \\
&= 2\sqrt{x}-2\arctan\sqrt{x}+C\ .
\end{aligned}
$$

$\blacksquare$

ב) $\int \frac{\sqrt{x+2}}{x}\,dx$

$$t=\sqrt{x+2}\ , \qquad t'=\frac{1}{2\sqrt{x+2}}=\frac{1}{2t}$$

$$\Leftarrow\ x+2=t^2 \quad \Leftarrow \quad x=t^2-2$$

$$
\begin{aligned}
\int \frac{t}{t^2-2}\,dx &= \int \frac{t}{t^2-2}\cdot\frac{1}{t'}\cdot t'\,dx \\
&= \int \frac{t}{t^2-2}\cdot\frac{1}{t'}\ dt \\
&= \int \frac{t}{t^2-2}\cdot 2t\,dt \\
&= \int \frac{2t^2}{t^2-2}\,dt
\end{aligned}
$$

השבר בתוך האינטגרל ניתן לרשום כשברים חלקיים:

$$\begin{aligned}\frac{2t^2}{t^2-2} =& \frac{2(t^2-2+2)}{t^2-2}\\ =& \frac{2(t^2-2)+2)}{t^2-2}\\ =& \frac{2(t^2-2)}{t^2-2}+\frac{4}{t^2-2}\\ =& 2+\frac{4}{t^2-2}\\ =& 2+\frac{4}{(t-\sqrt{2})(t+\sqrt{2})}\end{aligned}$$

$$\frac{4}{(t-\sqrt{2})(t+\sqrt{2})} = \frac{A}{t+\sqrt{2}}+\frac{B}{t-\sqrt{2}} = \frac{A(t-\sqrt{2})+B(t+\sqrt{2})}{(t-\sqrt{2})(t+\sqrt{2})}$$

$t=\sqrt{2}$: $\Leftarrow 2\sqrt{2}B=4 \Leftarrow B=\sqrt{2}$

$t=-\sqrt{2}$: $\Leftarrow -2\sqrt{2}A=4 \Leftarrow A=-\sqrt{2}$

לכן

$$\frac{2t^2}{t^2-2} = 2+\frac{\sqrt{2}}{t-\sqrt{2}}-\frac{\sqrt{2}}{t+\sqrt{2}}$$

והאינטגרל הופך ל

$$\begin{aligned}\int\frac{2t^2}{t^2-2}\,dt =& \int\left(2+\frac{\sqrt{2}}{t-\sqrt{2}}-\frac{1}{t+\sqrt{2}}\right)dt\\ =& 2t+\sqrt{2}\ln|t-\sqrt{2}|-\sqrt{2}\ln|t+\sqrt{2}|+C\\ =& 2\sqrt{x+2}+\sqrt{2}\ln|\sqrt{x+2}-\sqrt{2}|-\sqrt{2}\ln|\sqrt{x+2}+\sqrt{2}|+C\end{aligned}$$

■

ג)

$$\int\frac{1+e^x}{(1-e^{2x})e^x}\,dx$$

$$t=e^x$$

$$t'=e^x=t$$

$$\int\frac{1+t}{(1-t^2)t}dx = \int\frac{1+t}{(1-t^2)t^2}\cdot t'\,dx = \int\frac{1+t}{(1-t^2)t^2}dt = \int\frac{1}{(1-t)t^2}dt$$

$$\frac{1}{(1-t)t^2} = \frac{A}{1-t}+\frac{B}{t^2}+\frac{C}{t} = \frac{At^2+B(1-t)+Ct(1-t)}{(1-t)t^2}$$

$$At^2+B(1-t)+Ct(1-t)=1$$

$$t=0 \ \rightsquigarrow B=1\ , \qquad t=1 \ \rightsquigarrow A=1\ , \qquad t=2 \ \rightsquigarrow 4A-B-2C=1 \ \Rightarrow C=1\ .$$

$$\int\left(\frac{1}{1-t}+\frac{1}{t^2}+\frac{1}{t}\right)dt=-\ln|1-t|-\frac{1}{t}+\ln|t|+C=-\ln|1-e^x|-\frac{1}{e^x}+x+C$$

$\blacksquare$

שאלה 7

א) $\displaystyle\int \sin^4 x\cos^5 x\,dx$

$$t=\sin x\ , \qquad t'=\cos x$$

$$\cos^2 x=1-\sin^2 x=1-t^2\ , \qquad \cos^4 x=(1-t^2)^2\ .$$

$$\begin{aligned}\int \sin^4 x\cos^5 x\,dx&=\int t^4(1-t^2)^2\,t'\,dx\\&=\int t^4(1-t^2)^2\,dt\\&=\int(t^8-2t^6+t^4)\,dt\\&=\frac{t^9}{9}-\frac{2t^7}{7}+\frac{t^5}{5}+C\\&=\frac{\sin^9 x}{9}-\frac{2\sin^7 x}{7}+\frac{\sin^5 x}{5}+C\end{aligned}$$

$\blacksquare$

ב) $\displaystyle\int \sin^2 x\cos^2 x\,dx$

$$\sin^2 x\cos^2=(\sin x\cos x)^2=\left(\frac{\sin 2x}{2}\right)^2=\frac{1}{4}\sin^2 2x\ ,=\frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)=\frac{1}{8}(1-\cos 4x)$$

$$\begin{aligned}\int\frac{1}{8}(1-\cos 4x)\ dx=&\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C\\&\frac{x}{8}-\frac{\sin 4x}{32}+C\end{aligned}$$

$\blacksquare$

ج) $\int \cos^6 x\, dx$

$$\begin{aligned}\cos^6 x &= \left(\cos^2 x\right)^3 \\ &= \left(\frac{1+\cos 2x}{2}\right)^3 \\ &= \frac{1}{8}\left(1+3\cos 2x+3\cos^2 2x+\cos^3 2x\right) \\ &= \frac{1}{8}\left(1+3\cos 2x+3\left(\frac{1+\cos 4x}{2}\right)+\cos^3 2x\right)\end{aligned}$$

$$\int \frac{1}{8}\left(1+3\cos 2x+3\left(\frac{1+\cos 4x}{2}\right)+\cos^3 2x\right)dx = \frac{x}{8}+\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x+\int \cos^3 2x dx$$

$$t=\sin 2x \ , \qquad t' = 2\cos 2x$$

$$\begin{aligned}&\frac{x}{8}+\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x+\frac{1}{8}\int\left(1-\sin^2 2x\right)\cos 2x\, dx \\ =&\frac{x}{8}+\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x+\frac{1}{8}\int\left(1-t^2\right)\frac{t'}{2}dx \\ =&\frac{x}{8}+\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x+\frac{1}{16}\int\left(1-t^2\right)dt \\ =&\frac{x}{8}+\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x+\frac{1}{16}\int\left(1-t^2\right)dt \\ =&\frac{x}{8}+\frac{3}{16}\sin 2x+\frac{3}{16}x+\frac{3}{64}\sin 4x+\frac{1}{16}\left(t-\frac{t^3}{3}\right) \\ =&\frac{5x}{16}+\frac{3}{16}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{16}\left(\sin 2x-\frac{\sin^3 2x}{3}\right) \\ =&\frac{5x}{16}+\frac{3}{16}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{16}\sin 2x-\frac{\sin^3 2x}{48} \\ =&\frac{5x}{16}+\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x-\frac{1}{48}\sin^3 2x+C\end{aligned}$$

■

د)

$$\int \sin^2 x\cos^4 x\, dx = \int (\sin x\cos x)^2\cos^2 x\, dx$$

$$\sin x\cos x = \frac{\sin 2x}{2}\ , \qquad \cos^2 x = \frac{1+\cos 2x}{2}\ .$$

$$
\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int (\sin x \cos x)^2 \cos^2 x \, dx \\
&= \int \frac{\sin^2 x}{4} \cdot \frac{1+\cos 2x}{2} \, dx \\
&= \frac{1}{8} \int \left(\sin^2 x + \sin^2 2x \cos 2x \right) \, dx \\
&= \frac{1}{8} \int \left(\frac{1-\cos 4x}{2} + \sin^2 2x \cos 2x \right) \, dx \\
&= \frac{1}{8} \int \left(\frac{1-\cos 4x}{2} + \sin^2 2x \cos 2x \right) \, dx \\
&= \frac{1}{16} \int (1-\cos 4x) \, dx + \frac{1}{8} \int \sin^2 2x \cos 2x \, dx \\
&= \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \frac{1}{8} \int \sin^2 2x \cos 2x
\end{aligned}
$$

$$
t = \sin 2x \ , \qquad t' = 2\cos 2x
$$

$$
\begin{aligned}
\frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \frac{1}{8} \int \sin^2 2x \cos 2x &= \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \frac{1}{8} \int t^2 \, \frac{t'}{2} \, dx \\
&= \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \frac{1}{16} \int t^2 \, dt \\
&= \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \frac{1}{16} \frac{t^3}{3} + C \\
&= \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \frac{1}{16} \frac{\sin^3 2x}{3} + C \\
&= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C
\end{aligned}
$$