1. (a) $f''(x) + (2\pi n)^2 f = 0$

its correspoding characteristic polynomial is:

$$r^2 + (2\pi n)^2 = 0$$

$$\Rightarrow r^2 = -(2\pi n)^2$$

$$\Rightarrow r_1 = 2\pi n i \quad r_2 = -2\pi n i$$

hence the general solution comes as:

$$f = C_1 \cos(2\pi n x) + C_2 \sin(2\pi n x)$$

Since $n \in \mathbb{Z}^+$

$$\begin{cases} f(0) = 1 & \Rightarrow C_1 = 1 \\ f'(1) = 0 & \Rightarrow C_2 = 0 \end{cases}$$

$$\Rightarrow f \in F \quad , F = \{ \cos(2\pi n x) \mid n \in 0, 1, 2, \cdots \}$$

let the F^+ denote linear combinetion set.

(b) Let X, Y, Z be the members of set F^+

let $a, b, c \in \mathbb{R}$

We can see linear combination of members in F^+ is still member of F^+, hence it's closure

test 8 properties:

① $x+y = a\cos(2\pi n x) + b\cos(2\pi n y)$

$= b\cos(2\pi n y) + a\cos(2\pi n x)$

$= y + x$

② $(x+y)+z = [a\cos(2\pi n x) + b\cos(2\pi n y)] + c\cos(2\pi n z)$

$= a\cos(2\pi n x) + [b\cos(2\pi n y) + c\cos(2\pi n z)]$

$= x + (y+z)$

③ $0 + X = 0 + a\cos(2\pi n X) = X + 0$

④ $-X + X = -a\cos(2\pi n x) + a\cos(2\pi n x) = 0$

⑤ $0x = 0 \cdot a\cos(2\pi n X) = 0$

⑥ $1 \cdot x = 1 \cdot a\cos(2\pi n X) = a\cos(2\pi n X) = x$

⑦ $(mn)x = mn\, a\cos(2\pi n x) = m(nx)$

⑧ $(m+n)X = (m+n)\, a\cos(2\pi n X)$

$= ma\cos(2\pi n X) + na\cos(2\pi n X)$

$= mX + nX$

(c) The function space should be infinite-dimensional

2. (a) $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 2 & 4 \\ 3 & 9 & 4 & 6 \end{pmatrix}$

$$Ax = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 2 & 4 \\ 3 & 9 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Simplify A to echlon form:

$$ef(A) = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

still. $ef(A)x = 0$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$rref(A) \Rightarrow \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

x_2 and x_4 are free variables

if we assign $x_2 = 1$ and $x_4 = 0$

then $\vec{x} = (-3, 1, 0, 0)^T$

if we assign $x_2 = 0$ and $x_4 = 1$

then $\vec{x} = (-2, 0, 0, 1)^T$

hence the null space of A is:

$$\mathcal{N}(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(b) $\text{rank}(A) = 4 - 2 = 2$

(C)

```
>> A = [ 1 3 0 2;2 6 2 4;3 9 4 6]

A =

     1     3     0     2
     2     6     2     4
     3     9     4     6

>> null(A,'r')

ans =

    -3    -2
     1     0
     0     0
     0     1

>> rank(A)

ans =

     2
```

3. ① eigenvalue of B

λ_i is the eigenvalue of A

$$A - \lambda_i I = PBP^{-1} - \lambda_i I$$

$$= PP^{-1}(PBP^{-1} - \lambda_i I)PP^{-1}$$

$$= P(BP^{-1}P - \lambda_i P^{-1} I P)P^{-1}$$

$$\Rightarrow A - \lambda I = P(B - \lambda_i I)P^{-1}$$

$\therefore \det(A - \lambda_i I) = \det(P)\det(B - \lambda_i I)\det(P^{-1}) = 0$

Since P is nonsingular. $\det(P) \neq 0$

$\therefore \det(B - \lambda_i I = 0$

$\therefore$ the eigenvalue of B is λ_i

③ eigenvectors of B

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$PBP^{-1}\vec{v}_i = \lambda_i \vec{v}_i$$

$$BP^{-1}\vec{v}_i = \lambda_i P^{-1}\vec{v}_i$$

$$\therefore B(P^{-1}\vec{v}_i) = \lambda_i (P^{-1}\vec{v}_i)$$

Since λ_i are eigenvalues of B

$P^{-1}\vec{v}_i$ are eigenvector of B

4. (a) $A = \|\vec{x} \times \vec{y}\|$

$$dA = \|d\vec{x} \times d\vec{y}\| =$$

$$= \left\| \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \times \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \right\|$$

$$= \left\| \frac{\partial x \partial y}{(\partial u)^2} du^2 + \frac{\partial x \partial y}{\partial u \partial v} du dv + \frac{\partial x \partial y}{\partial v \partial u} dv du + \frac{\partial x \partial y}{(\partial v)^2} (dv)^2 \right\|$$

omit 2^{nd} order term

$$= \left\| \frac{\partial x \partial y}{\partial u \partial v} du dv - \frac{\partial x \partial y}{\partial v \partial u} du dv \right\|$$

$$= |\det J(u, v)| \, du dv$$

(b) $\begin{cases} x = f(r, \theta) = r\cos\theta \\ y = g(r, \theta) = r\sin\theta \end{cases}$

$$dA = \| r\cos^2\theta + r\sin^2\theta \| dr d\theta = r dr d\theta$$

(C)

5. (a) every vertex can have up to $(n-1)$ edges with other vertices

hence there are $n(n-1)$ edges in total

Since it's a simple graph, all the edges are counted twice, so the maximum value should be divided by two.

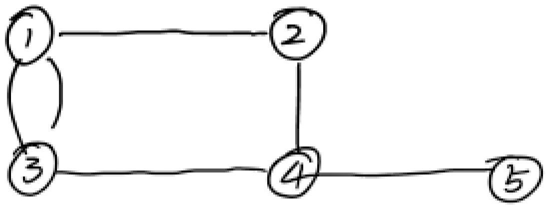
hence $|E| \leq n(n-1)/2$

since $|V| = n$

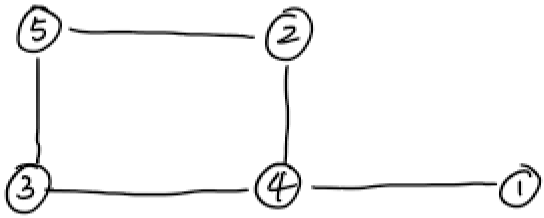
$$\Rightarrow \quad |E| \leq |V|(|V|-1)/2 = \binom{|V|}{2}$$

(b)

G:



G'



(c) G' is a simple graph, G is not hence they are no isomorphic

(d) Adjacent Matrix of G:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Adjacent Matrix of G':

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$