Robot Motion Planning
Mathmatics Preliminaries

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1 Vector Space

In [1, Ch.8, Sec.8.3.1], there is a definition of a vector space. Here we will review its more rigorous definition.

Consider a nonempty set $\mathcal{X}$ on which the operations of addition "+" of elements and scalar multiplication "$\cdot$" with elements are defined (i.e., closure). Addition is closed: $\forall x, y \in \mathcal{X} \to x + y \in \mathcal{X}$. Also scalar multiplication is closed: for example, $\forall x \in \mathcal{X}, \alpha \in \mathbb{C} \to \alpha \cdot x \in \mathcal{X}$. The triplet $(\mathcal{X}, +, \cdot)$ is said to be a vector space (or linear vector space, in short LVS) if the following properties are satisfied with $\forall x, y, z \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{C}$:

1. $x + y = y + x$, commutativity of vector addition
2. $(x + y) + z = x + (y + z)$, associativity of vector addition
3. There is a null vector, $\mathbf{0}$, in $\mathcal{X}$ such that $\mathbf{0} + x = x$
4. There exists $-x$ in $\mathcal{X}$ for each $x \in \mathcal{X}$ such that $x + (-x) = \mathbf{0}$.
5. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$, distributivity over vector addition
6. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$, distributivity over scalar addition
7. $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$, associativity of scalar-vector multiplication.
8. $1 \cdot x = x$

The above example is called a complex vector space. If $\mathbb{R}$ is used in place of $\mathbb{C}$, then we call it a real vector space. If $\mathbb{C}$ is replaced by a general field $\mathbb{F}$ (e.g., $\mathbb{R}$., $\mathbb{C}$, or $\mathbb{Q}$), then we say a vector space over the field $\mathbb{F}$. Note that here we do not denote vectors as bold-faced letters. In general, vector spaces are a general mathematical space that even includes a space that consists of functions (e.g., $C^k(\mathbb{R})$: the set of k-differentiable functions $f : \mathbb{R} \to \mathbb{R}$).

(Note: For the definition of the field, refer to [1, Ch.4, Sec.4.4.1]).

We denote n-dimensional real vector space as $\mathbb{R}^n$. Likewise n-dimensional complex vector space is as $\mathbb{C}^n$. We will extensively use $\mathbb{R}^n$ (in particular, $n = 2, 3$).

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be arbitrary vectors in a vector space. The set of all linear combinations is called the *span* of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$:

$$\text{span}\left\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\right\} \doteq \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \;\middle|\; \alpha_1, \cdots, \alpha_n \in \mathbb{R} \right\}.$$

Note that if $\mathbf{y}$ can be expressed as a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, then

$$\text{span}\left\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n, \mathbf{y}\right\} = \text{span}\left\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\right\}.$$

A set of vectors $\{\mathbf{a}_1, \cdots \mathbf{a}_n\}$ is linearly independent if the equality containing a linear combination form $\sum_{i=1}^{n} \alpha_i \mathbf{a}_i = \mathbf{0}$ implies $\alpha_i = 0$ $(\forall i = 1, \cdots, n)$. With this set, the *span* of these vectors is defined as

$$\text{span}\left\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\right\} = \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{a}_i \;\middle|\; \alpha_1, \cdots, \alpha_n \in \mathbb{R} \right\}$$

and these vectors become the *basis vectors* of the spanned space. For $\mathbb{R}^n$, the *standard basis vectors* (or natural basis) are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

2D and 3D Cartesian space are respectively denoted as $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, let $\mathbf{p} \in \mathbb{R}^2$. For a vector, we will denote it as a bold-faced letter $\mathbf{p}$. Then we can write it as a column vector form as

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = [p_1 \; p_2]^T$$

where the superscript T means the transpose of the matrix/vector. The above component form is obtained by using the standard basis vectors $\mathbf{e}_1 = [1 \; 0]^T$ and $\mathbf{e}_2 = [0 \; 1]^T$, i.e., $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2$. 3D space $\mathbb{R}^3$ can be expressed similarly. In general, for $\mathbf{p} \in \mathbb{R}^n$, it can be expressed as $\mathbf{p} = \sum_{i=1}^{n} p_i \mathbf{e}_i$ or $\mathbf{p} = [p_1 \; p_2 \; \cdots \; p_n]^T$.

The dot product of $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_n]^T \in \mathbb{R}^n$ and $\mathbf{y} = [y_1 \; y_2 \; \cdots \; y_n]^T \in \mathbb{R}^n$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = \mathbf{x}^T \mathbf{y}.$$

The Euclidean norm of a vector $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_n]^T \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

By using the Euclidean norm, the distance between two point vectors $\mathbf{x}$ and $\mathbf{y}$ is computed as

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

2 Matrices

Let $A \in \mathbb{R}^{m\times n}$, i.e., the set of $m \times n$ real matrices. We can write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \;\; (i = 1, \cdots, m, \;\; j = 1, \cdots, n).$$

The transpose of $A = (a_{ij})$ is defined as: $A^T = (a_{ji})$.

The column rank of $A \in \mathbb{R}^{m\times n}$ is the dimension of the column space. Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ where $\mathbf{a}_i \in \mathbb{R}^m$ $(i = 1, 2, \cdots, n)$. Then the column space of A is

$$\mathcal{R}(A) = \text{span}\{\mathbf{a}_1, \cdots ., \mathbf{a}_n\}.$$

It is also called as the *r*ange of A. Likewise row rank is the dimension of the row space of A, or equivalently, the column space of A^T. Fundamental fact is that row rank and column rank of A are the same, which uniquely defines the *rank* of A. For $A \in \mathbb{R}^{m\times n}$, it follows that

$$\text{rank}(A) \doteq \dim\left(\text{span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}\right) \leq \min(m, n).$$

If $\text{rank}(A) = \min(m, n)$ then A is called full-rank, and if $\text{rank}(A) < \min(m, n)$ then A is called rank-deficient.

For square matrices, that $A \in \mathbb{R}^{n\times n}$ is full-rank is equivalent to saying that A is non-singular (i.e., A is invertible, or $\det(A) \neq 0$).

The null space of a matrix $A \in \mathbb{R}^{m\times n}$ is defined as

$$\mathcal{N}(A) \doteq \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Important matrix decompositions include eigenvalue problem/decomposition and singular value decomposition.

3 Mapping

A map relates elements of one set (called a *domain*) to elements of another set (called a *codomain*). Mappings can be described by a function or a function of functions. For example:

$$\begin{aligned} &f : x \mapsto \sin(x) \;:\; [0, \pi) \to [-1, 1] \\ &f(x) = \sin(x) \;:\; [0, \pi) \to [-1, 1] \;:\; x \mapsto \sin(x) \end{aligned}$$

For example, let $f : \mathbb{R}^n \to \mathbb{R}$. Then $\mathbb{R}^n$ is the domain, and $\mathbb{R}$ is the codomain. For any $\mathbf{x} \in \mathbb{R}^n$, the set of $f(\mathbf{x})$ is called the *image*.

- *injective*: The function $f : A \to B$ is injective iff $\forall a, b \in A$, we have $f(a) = f(b) \to a = b$.
- *surjective*: The function $f : A \to B$ is surjective iff $\forall b \in B$, there is an $a \in A$ s.t. $f(a) = b$.

- *bijective*: The function $f : A \to B$ is bijective iff $\forall b \in B$, there is a unique $a \in A$ s.t. $f(a) = b$.

Examples of injective functions include: $\exp : x \mapsto e^x : \mathbb{R} \to \mathbb{R}$.

Examples of surjective functions include: $f : x \mapsto \sin(x) : \mathbb{R} \to [-1, 1]$.

Examples of bijective functions include:

$$\begin{aligned} f : x \mapsto x &: \mathbb{R} \to \mathbb{R} \\ g : x \mapsto x^2 &: \mathbb{R}^+ \to \mathbb{R}^+ \\ \exp : x \mapsto e^x &: \mathbb{R} \to \mathbb{R}^+ \end{aligned}$$

4 Jacobian of a Mapping

Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. In other words, for $\mathbf{x} = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_m(x_1, x_2, \cdots, x_n) \end{pmatrix}.$$

By the chain rule, an infinitesimal motion of each component in $\mathbf{f}$ is computed as

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

for $i = 1, 2, \cdots, m$. Collecting all the terms lead to the expression as

$$d\mathbf{f} = J(\mathbf{x}) d\mathbf{x}$$

where the matrix J is called the Jacobian matrix (or simply Jacobian) of the mapping, of which the i, j-th element is defined as

$$(J(\mathbf{x}))_{ij} = J_{ij}(\mathbf{x}) \doteq \frac{\partial f_i}{\partial x_j}.$$

Specifically,

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

We also write the Jacobian as

$$J = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}.$$

5 Introductory Graph Theory

5.1 Graphs

Let us consider the following story as an example. An airline is trying to construct its air service map. In this case, each city that has the air service of the company can be marked as a point, and the air service between cities can be represented as a line. Then for the purpose of the air service network identification, we do not need an actual map: just collection of cities and the connections between the cities. This is an example of a *Graph*. Now let us review the graph theory (see [2, 3] for more detail, from which we extract the summary as below). A short introduction can be found in [4].

A *graph* G is defined with a set V of *nodes* (or *vertices*), and a set of *edges* E where edges mean unordered pairs of vertices. Symbolically, we write $G = (V, E)$. Both V and E can be either infinite or finite.

- The *order* of a graph is the number of its nodes.
- The *size* is the number of its edges.
- If u and v are two vertices of a graph and if the *unordered pair* $\{u, v\}$ is an edge denoted as e, we say that e joins u and v, or that it is an edge between u and v (equivalently, u and v are incident on e, or e is incident to both u and v). Or we say that u and v are *adjacent*.
- *parallel edges*: two or more edges that join the same pair of distinct nodes.
- *loop*: an edge represented by an unordered pair in which the two elements are not distinct.
- *multigraph*: a graph with no loops.
- *simple graph*: a graph with no parallel edges and loops.

An *undirected graph* is a graph that has no direction associated with its edges.

A *directed graph* (or *digraph*) is a graph that has directions in its edges. In other words, the graph has a set A of *ordered* pairs of distinct nodes (called arcs). If the ordered pair $\{u, v\}$ is an arc a, we say that a is *directed* from u to v. In this case, arc a is *adjacent from* vertex u and is *adjacent to* vertex v (i.e., u and v are *adjacent*).

A *path* (or *walk*) in a graph is defined as a sequence of nodes (or vertices) such that for adjacent nodes v_i and v_{i+1}, an edge $e_{i,i+1}$ that connects v_i and v_{i+1} exists. This is applied to both undirected and directed graphs. Specifically, for a directed graph, a path is called a *directed path* (or *dipath*).

A graph is said to be *connected* if for all nodes in the graph, there exists a path connecting v_i and v_j. Also applied both to undirected and directed graphs.

5.2 Graph isomorphism

Two graphs $G = (V, E)$ and $G' = (V', E')$ are identical if $V = V'$ and $E = E'$. However this is too restrictive and rigid. Even though two graphs are not identical, they can share the same structure. For example, a simple graph with $\{a, b, c\}$ and another simple graph with $\{p, q, r\}$ can be structurally the same, i.e., edges will be of the same structure. Hence we define graph isomorphism.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if there exists a one-to-one correspondence f, called *isomorphism*, from V to V' such that there is an edge between $f(v)$ and $f(w)$ in G' iff there is an edge between v and w in G. For the practical purpose, two isomorphic graphs are considered as one same graph.

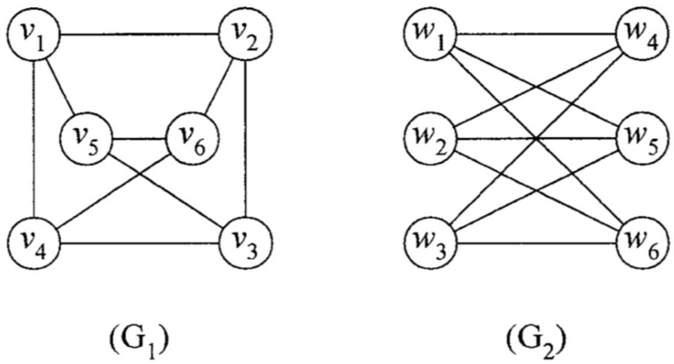


Figure 1: Example of graph isomorphism, from [3].

5.3 Adjacency matrices

Let $G = (V, E)$ be a graph (undirected graph) where the set $V = \{1, 2, \cdots, n\}$. The *adjacency matrix* of the graph is the $n \times n$ matrix, denoted here as $A = (a_{ij})$, where the non-diagonal entry is the number of edges joining nodes i and j, and the diagonal entry is twice the number of loops at node i.

- Obviously $A = A^T$ since $a_{ij} = a_{ji}$.
- The adjacency matrix of a simple graph is a binary matrix (all elements are either 0 or 1) where each diagonal element is zero.

The adjacency matrix of a directed graph with $V = \{1, 2, \cdots, n\}$ is the $n \times n$ matrix $A = (a_{ij})$ where $a_{ij} = 1$ iff there is an arc from node i to j.

- Each diagonal element is zero, in this case.
- A does not need to be symmetric.

5.4 Weighted graphs

As often as not, it can be useful to define *weights* that are associated with each edge in a graph. Recall that previously, when two nodes are connected, then the corresponding element in the adjacency matrix is 1, and 0 otherwise. However this is replaced by a *weight* in a weighted graph. Examples of weights include:

- Distance between connected (or adjacent) nodes
- Travel time for adjacent nodes
- Fuel consumption during travel between adjacent nodes
- and so on ...

For disconnected nodes, the weights are considered *infinite*.

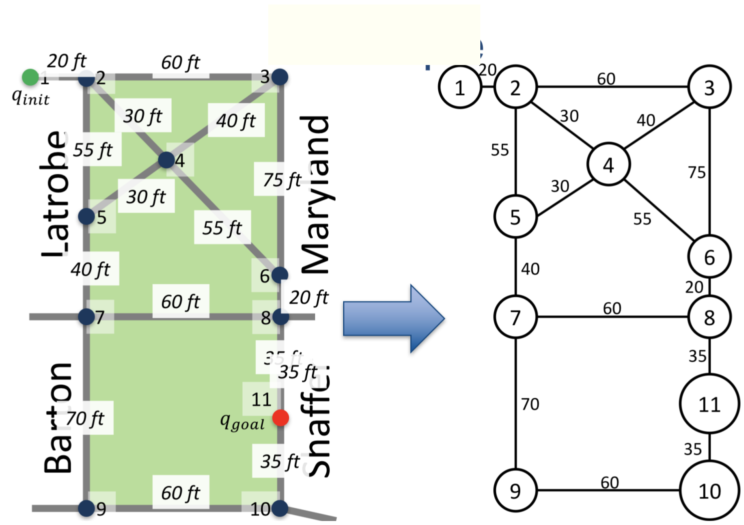
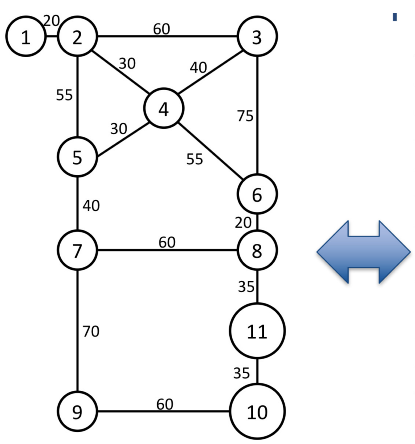


Figure 2: Example of weighted graph.



	1	2	3	4	5	6	7	8	9	10	11
1	0	40	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	40	0	60	30	55	∞	∞	∞	∞	∞	∞
3	∞	60	0	40	∞	75	∞	∞	∞	∞	∞
4	∞	30	40	0	30	55	∞	∞	∞	∞	∞
5	∞	55	∞	30	0	∞	40	∞	∞	∞	∞
6	∞	∞	75	55	∞	0	∞	20	∞	∞	∞
7	∞	∞	∞	∞	40	∞	0	60	70	∞	∞
8	∞	∞	∞	∞	∞	20	60	0	∞	∞	35
9	∞	∞	∞	∞	∞	∞	70	∞	0	60	∞
10	∞	∞	∞	∞	∞	∞	∞	∞	60	0	35
11	∞	∞	∞	∞	∞	∞	∞	35	∞	35	0

Figure 3: Example of weighted graph.

References

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[3] V. K. Balakrishnan. *Schaum's Outline of Theory and Problems of Graph Theory*. McGraw-Hill, 1997.

[4] Howie Choset, Kevin Lynch, Seth Hutchinson, George Kantor, Wolfram Burgard, Lydia Kavraki, and Sebastian Thrun. *Princinples of Robot Motion: Theory, Algorithms, and Implementations*. MIT Press, 2005.