



Linear Algebra Primer

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Another, very in-depth linear algebra review from CS229 is available here:

<http://cs229.stanford.edu/section/cs229-linalg.pdf>

And a video discussion of linear algebra from EE263 is here (lectures 3 and 4):

<https://see.stanford.edu/Course/EE263>



Outline

- Vectors and matrices
 - Basic Matrix Operations
 - Determinants, norms, trace
 - Special Matrices
- Transformation Matrices
 - Homogeneous coordinates
 - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculus



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Vectors and matrices are just collections of ordered numbers that represent something: location in space, speed, pixel brightness, etc. We'll define some common uses and standard operations on them.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \mathbf{v}^T = [v_1 \quad v_2 \quad \dots \quad v_n]$$
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$



Vector

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where

$$\mathbf{v}^T = [v_1 \quad v_2 \quad \dots \quad v_n]$$

T denotes the transpose operation



Vector

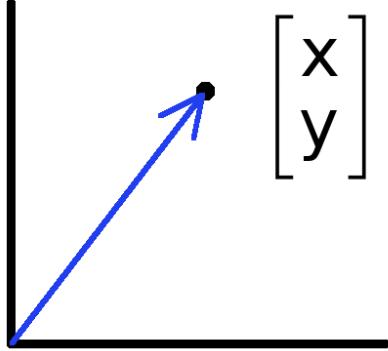
- We'll default to column vectors in this class

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

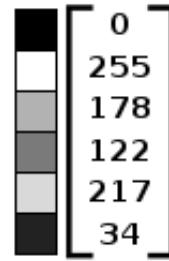
- You'll want to keep track of the orientation of your vectors when programming in python



Some vectors have a geometric interpretation, others don't...



- Some vectors have a geometric interpretation:
 - Points are just vectors from the origin.
 - We can make calculations like “distance” between 2 vectors



- Other vectors don't have a geometric interpretation:
 - Vectors can represent any kind of data (pixels, gradients at an image keypoint, etc)
 - Such vectors don't have a geometric interpretation
 - We can still make calculations like “distance” between 2 vectors



Matrix

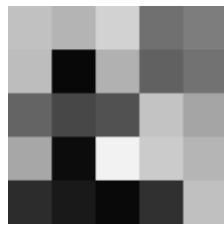
- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size m by n, i.e. m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that \mathbf{A} is square.



Images



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$$\begin{bmatrix} 193 & 180 & 210 & 112 & 125 \\ 189 & 8 & 177 & 97 & 114 \\ 100 & 71 & 81 & 195 & 165 \\ 167 & 12 & 242 & 203 & 181 \\ 44 & 25 & 9 & 48 & 192 \end{bmatrix}$$

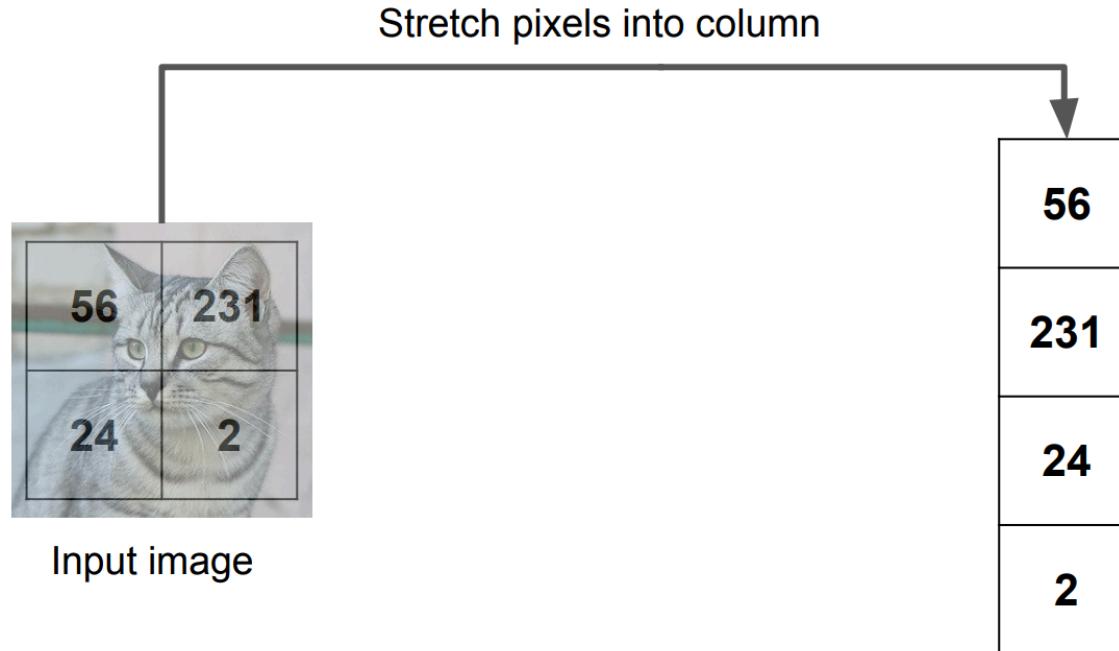
- Python represents an image as a matrix of pixel brightnesses
- Note that the upper left corner is $[x, y] = (0,0)$

row column Python indices
start at 0



Images can be represented as a **matrix** of pixels.

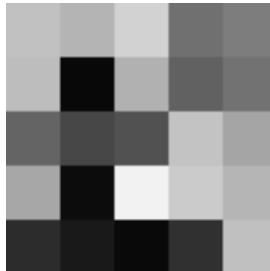
Images can also be represented as a **vector** of pixels



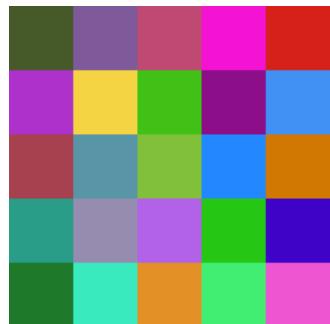


Color Images

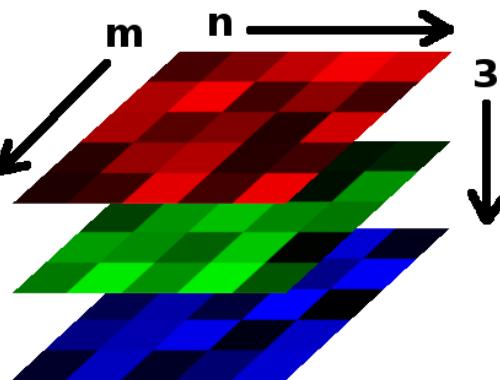
- **Grayscale images** have **1** number per pixel, and are stored as an **$m \times n$ matrix**.



- **Color images** have **3** numbers per pixel – red, green, and blue brightnesses (RGB) - and are stored as an **$m \times n \times 3$ matrix**



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Basic Matrix Operations

- We will discuss:
 - Addition
 - Scaling
 - Dot product
 - Multiplication
 - Transpose
 - Inverse / pseudoinverse
 - Determinant / trace



Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

– We can only add a matrix with matching dimensions, or a scalar.
Good to know for Python assignments ☺

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$



Vector Norms

- Examples of vector norms

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

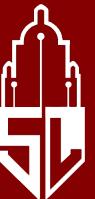
$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}.$$

$$\|x\|_\infty = \max_i |x_i|$$



Vector Norms

- More formally, a norm is any function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies the following 4 properties:
- **Non-negativity:** For all $x \in \mathbb{R}^n$, $f(x) \geq 0$
- **Definiteness:** $f(x) = 0$ if and only if $x = [0,0\dots 0]$.
- **Homogeneity:** For all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t|f(x)$
- **Triangle inequality:** For all $x, y \in \mathbb{R}^n$, $f(x + y) \leq f(x) + f(y)$



Vector Operation: inner product

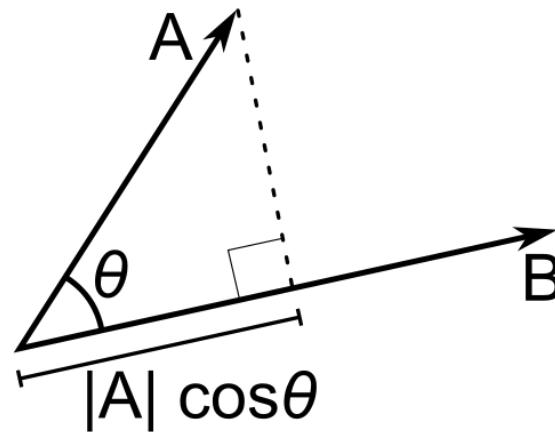
- **Inner product (dot product) of two vectors**
 - Multiply corresponding entries of two vectors and add up the result
 - $x \cdot y$ is also $|x| |y| \cos(\text{the angle between } x \text{ and } y)$

$$\mathbf{x}^T \mathbf{y} = [x_1 \quad \dots \quad x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n x_i y_i \quad (\text{scalar})$$



Vector Operation: inner product

- **Inner product (dot product) of two vectors**
 - If B is a **unit** vector:
 - Then $A \cdot B = |A| |B| \cos(\Theta) = |A| \times 1 \times \cos(\Theta) = |A| \cos(\Theta)$
 - $A \cdot B$ gives the length of A which lies in the direction of B





Matrix Operations

- The **product of two matrices**

$$A \in \mathbb{R}^{m \times n}$$

$$B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

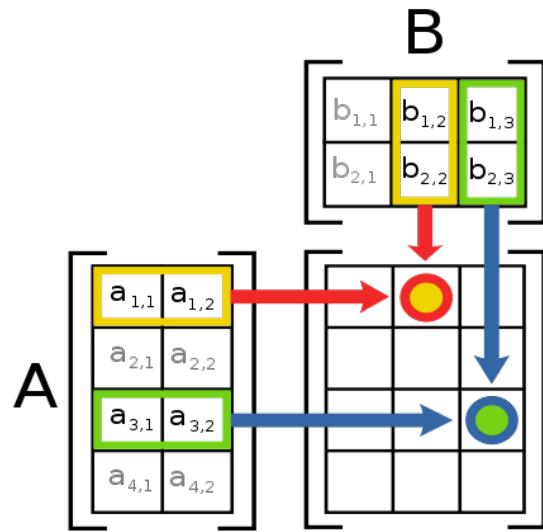
$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} & a_1^T & \\ & a_2^T & \\ \vdots & & \\ & a_m^T & \end{bmatrix} \begin{bmatrix} | & & & | \\ b_1 & b_2 & \cdots & b_p \\ | & & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}.$$



Matrix Operations

- Multiplication
- The product AB is:



- Each entry in the result is:
(that row of A) dot product with (that column of B)
- Many uses, which will be covered later



Matrix Operations

- Multiplication example:

$$\begin{array}{c} A \times B \\ \downarrow \quad \nearrow \\ \begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \end{array}$$

– Each entry of the matrix product is made by taking: the dot product of the corresponding row in the left matrix, with the corresponding column in the right one.

$$0 \cdot 3 + 2 \cdot 7 = 14$$



Matrix Operations

- The product of two matrices

Matrix multiplication is associative: $(AB)C = A(BC)$.

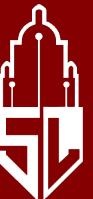
Matrix multiplication is distributive: $A(B + C) = AB + AC$.

Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$. (For example, if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times q}$, the matrix product BA does not even exist if m and q are not equal!)



Matrix Operations

- Powers
 - By convention, we can refer to the matrix product AA as A^2 , and AAA as A^3 , etc.
 - **Important: only square matrices can be multiplied that way!**
(make sure you understand why)



Matrix Operations

- Transpose a matrix: flip the matrix, so row 1 becomes column 1

$$\begin{bmatrix} 0 & 1 & \dots \\ \downarrow & \curvearrowright & \\ \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$



Matrix Operations

- **Determinant**

- $\det(\mathbf{A})$ returns a scalar
- Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix

- For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$

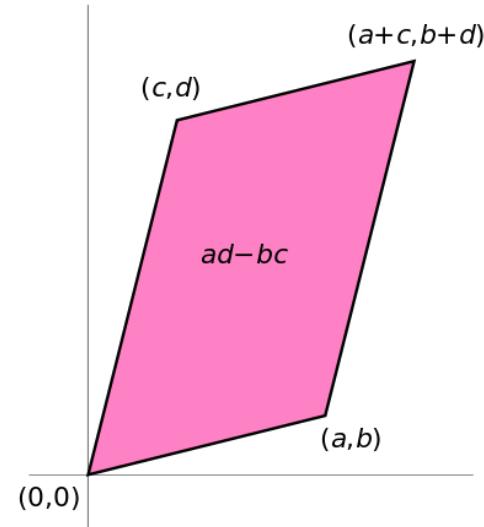
- Properties:

$$\det(\mathbf{AB}) = \det(\mathbf{BA})$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \text{ is singular}$$





Matrix Operations

- **Trace**

$\text{tr}(\mathbf{A})$ = sum of diagonal elements

$$\text{tr}\left(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}\right) = 1 + 7 = 8$$

- Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely in this class though.)
- Properties:

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$$

$$\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$$



Matrix Operations

- **Vector Norms (we've talked about them earlier)**

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \|x\|_\infty = \max_i |x_i|.$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}. \quad \|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

- **Matrix norms:** Norms can also be defined for matrices, such as

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n A_{ij}^2} = \sqrt{\text{tr}(A^T A)}.$$



Special Matrices

- **Identity matrix \mathbf{I}**

Square matrix, 1's along diagonal, 0's elsewhere

$$\mathbf{I} \cdot [\text{a matrix } A] = [\text{that matrix } A]$$
$$[\text{a matrix } A] \cdot \mathbf{I} = [\text{that matrix } A]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Diagonal matrix**

Square matrix with numbers along diagonal, 0's elsewhere

[diagonal matrix A] · [another matrix B] scales the rows of matrix B

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$



Special Matrices

- Symmetric matrix

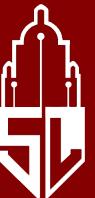
$$\mathbf{A}^T = \mathbf{A}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix

$$\mathbf{A}^T = -\mathbf{A}$$

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$



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Matrix multiplication can be used to transform vectors.
A matrix used in this way is called a **transformation matrix**.



Transformation: scaling

- **Matrices can be used to transform vectors** in useful ways, through multiplication: $Ax = x'$
- Simplest transformation is **scaling**:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

(Verify to yourself that the matrix multiplication works out this way)

Scaling
matrix

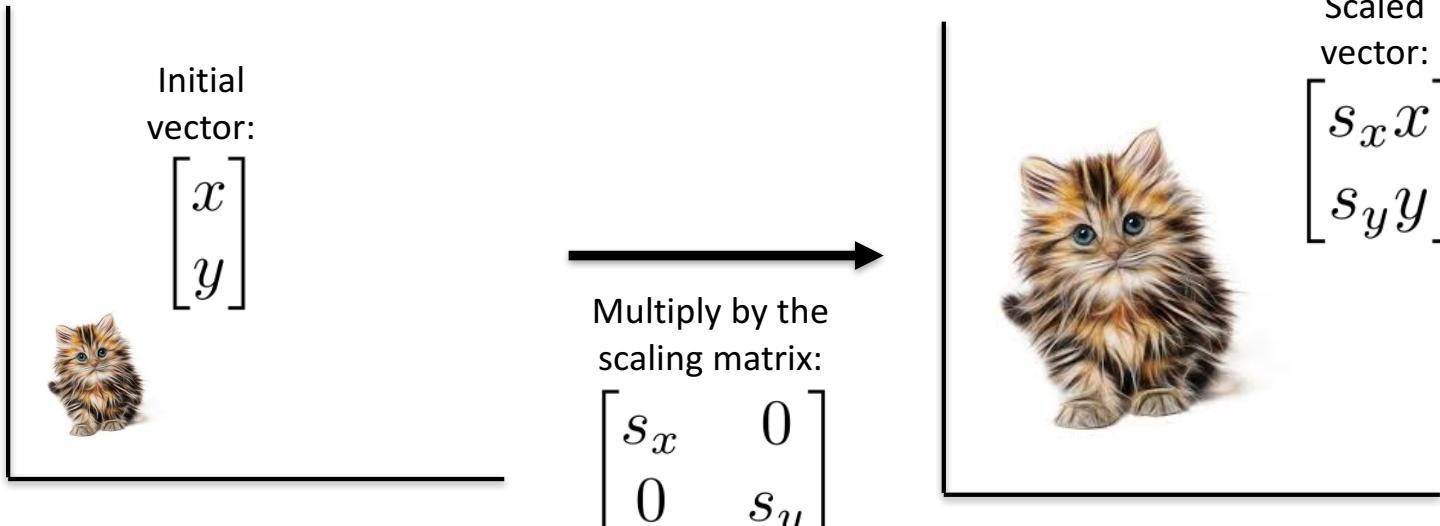
Initial
vector

Scaled
vector



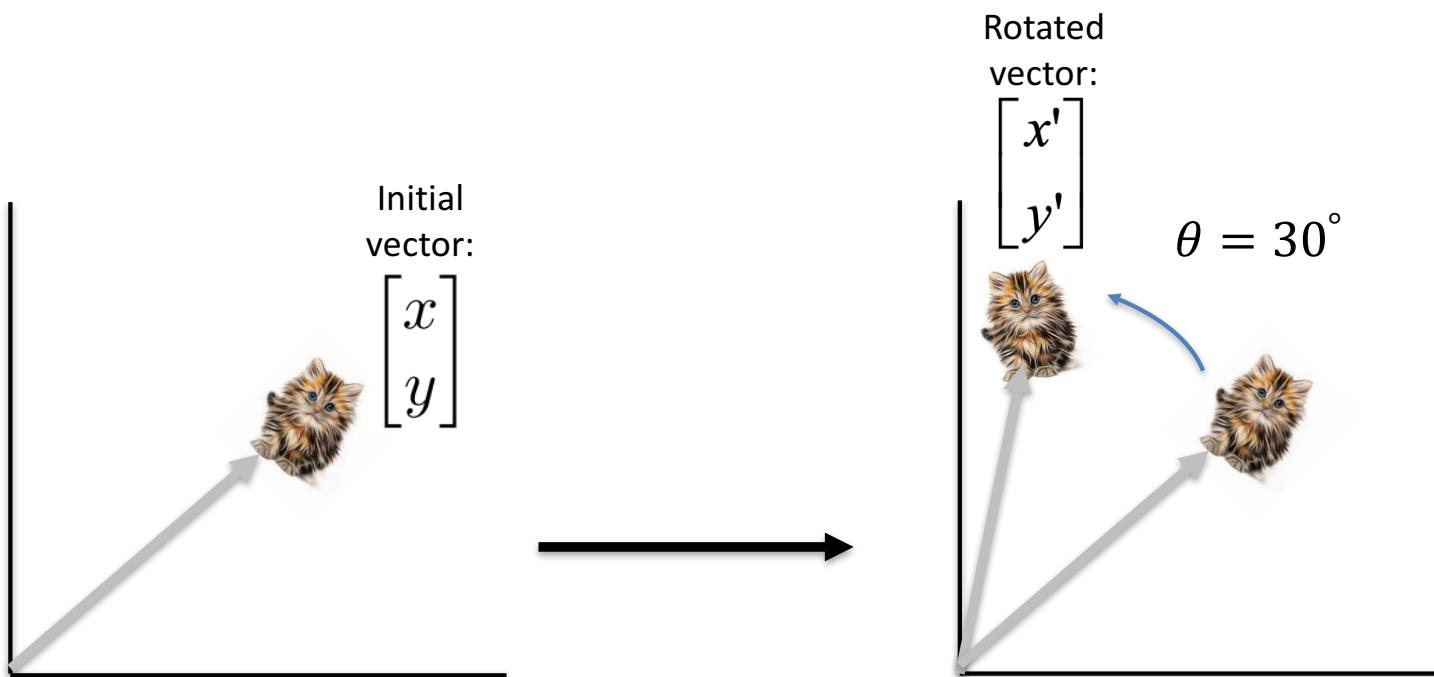
Transformation: scaling

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$





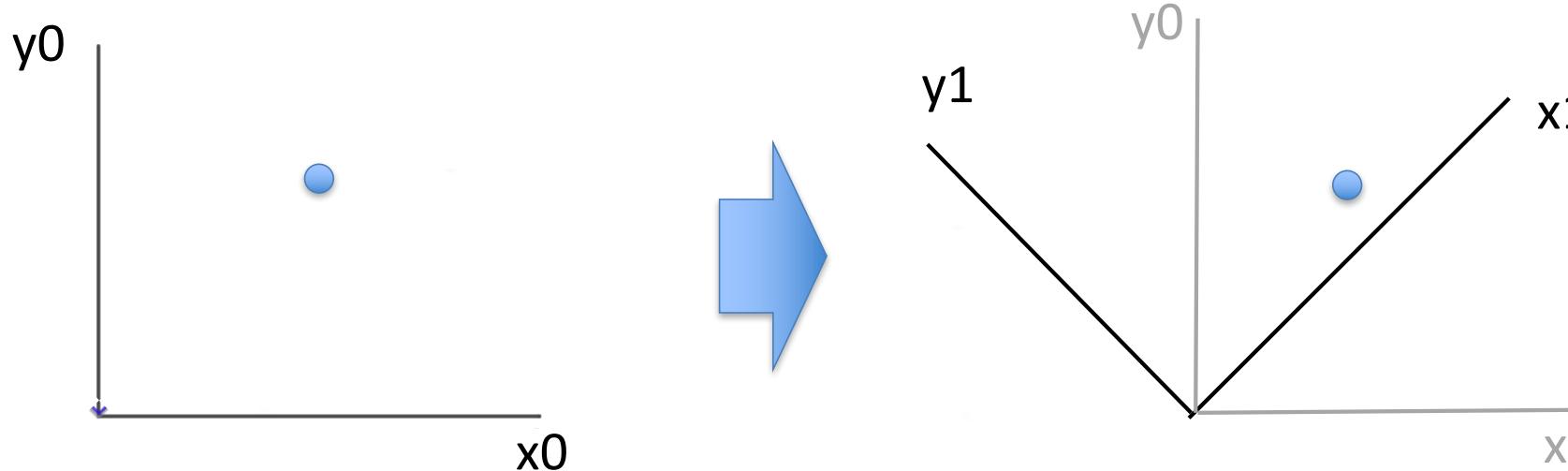
Transformation: rotation





Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame "0" to a new, rotated coordinate frame "1"?

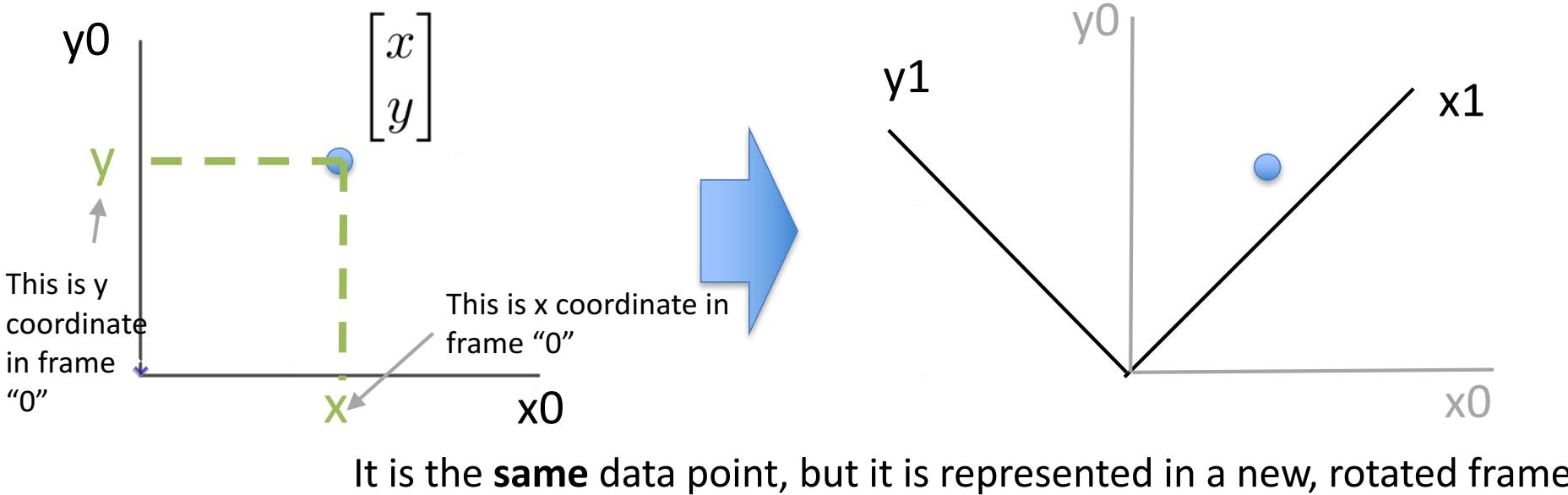


It is the **same** data point, but it is represented in a new, rotated frame.



Before learning how to rotate a vector, let's understand how to do something slightly different

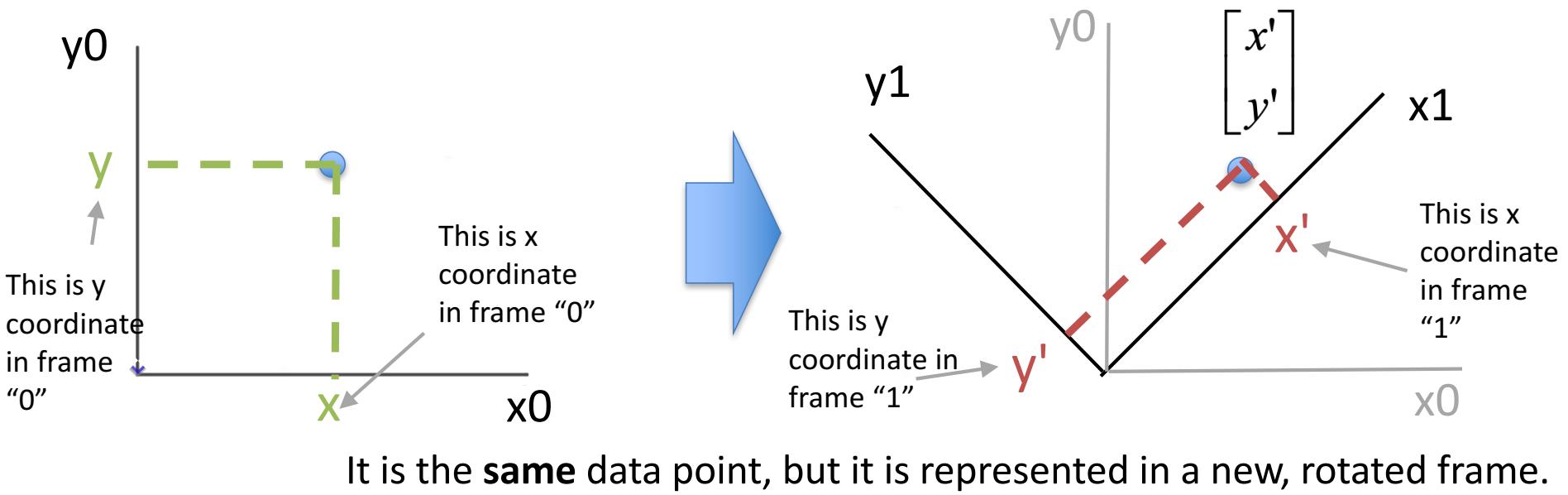
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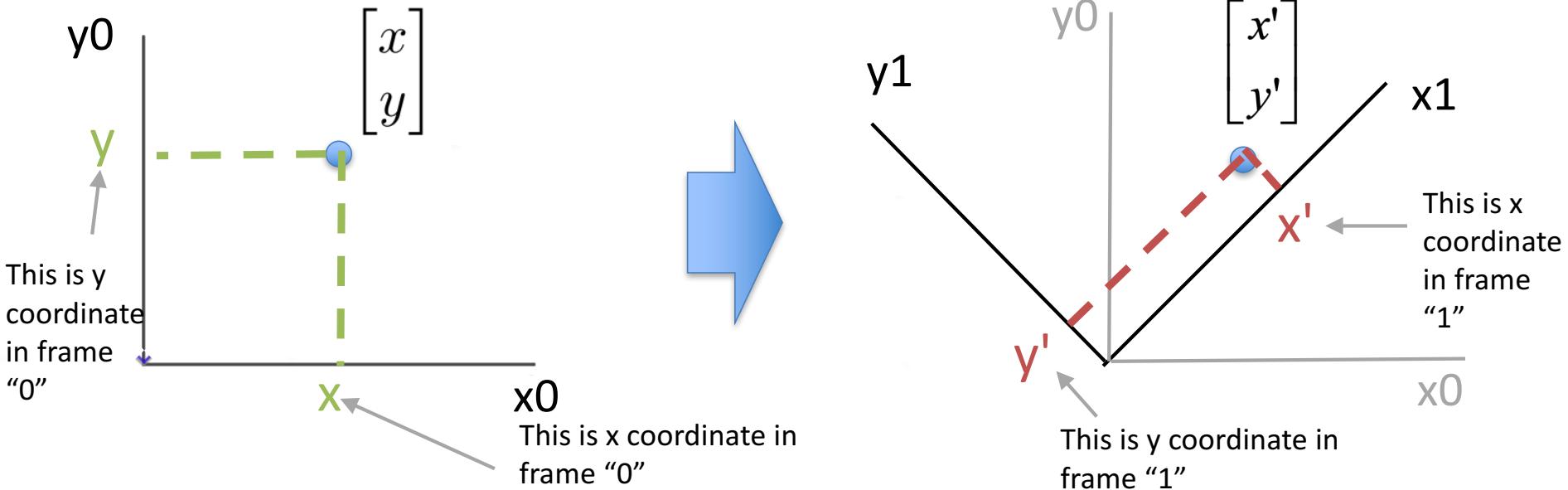




Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame "0" to a new, rotated coordinate frame "1"?
- Remember what a vector is:

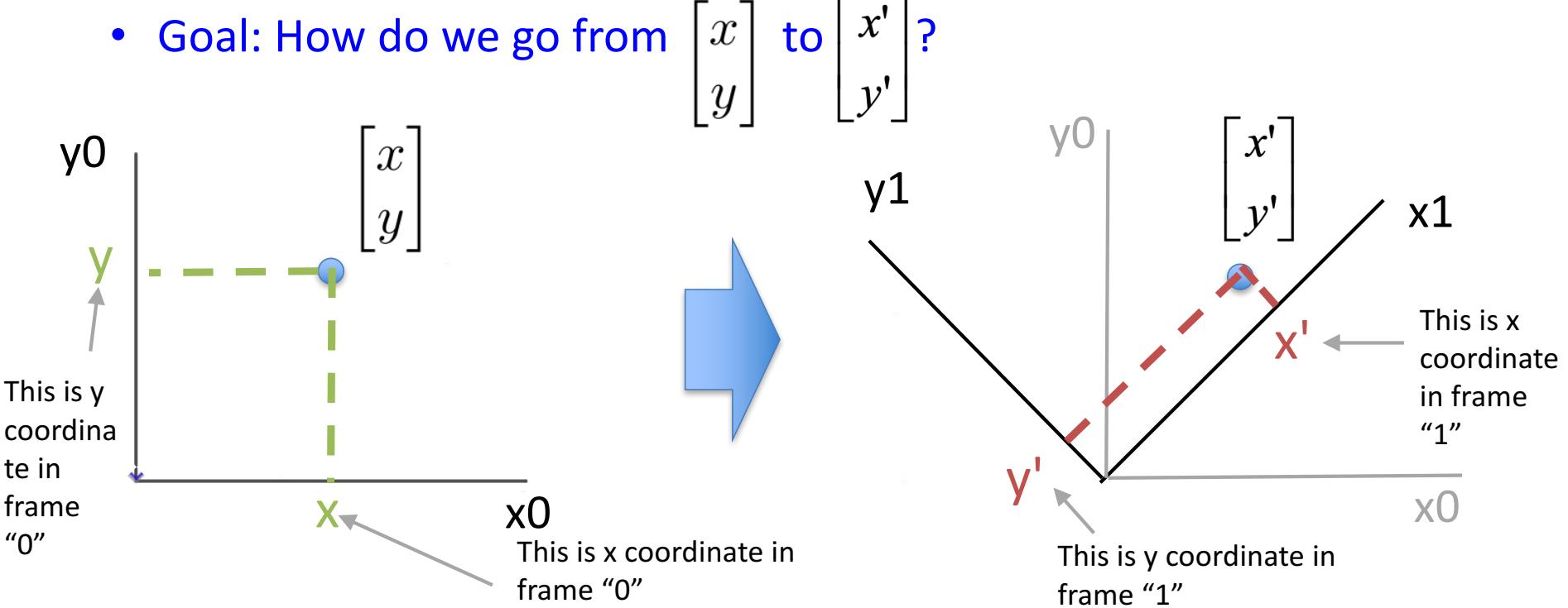
[component in direction of the frame's x axis, component in direction of y axis]





Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame "0" to a new, rotated coordinate frame "1"?
- Goal: How do we go from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$?





Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?
- Goal: How do we go from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$?
- **Answer: we use dot products!!**
 - x' (the new x coordinate) is the length of the original vector which lies in the direction of the new x axis
 - y' (the new y coordinate) is the length of the original vector which lies in the direction of the new y axis
- So:
 - x' is [original vector] **dot** [the new x axis]
 - y' is [original vector] **dot** [the new y axis]



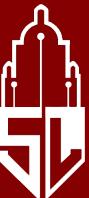
Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?
- Goal: How do we go from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$?
 - x' (the new x coordinate) is [original vector] **dot** [the new x axis]
 - y' (the new y coordinate) is [original vector] **dot** [the new y axis]
- So:
 - $x' = [x, y]$ **dot** [the new x axis]
 - $y' = [x, y]$ **dot** [the new y axis]



Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?
- Goal: How do we go from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$?
 - $x' = [x, y]$ **dot** [the new x axis]
 - $y' = [x, y]$ **dot** [the new y axis]
- So:
 - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} [\text{the new x axis}] \text{ dot } [x, y] \\ [\text{the new y axis}] \text{ dot } [x, y] \end{bmatrix}$



Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?
- Goal: How do we go from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$?

- $\begin{bmatrix} x' \\ y' \end{bmatrix}$ $\left[\begin{array}{l} \text{[the new x axis] dot [x, y]} \\ \text{[the new y axis] dot [x, y]} \end{array} \right]$

- So:

- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \left[\begin{array}{l} \text{[the new x axis]} \\ \text{[the new y axis]} \end{array} \right] \times \begin{bmatrix} x \\ y \end{bmatrix}$

- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \left[\begin{array}{l} \text{2x2 Rotation Matrix} \end{array} \right] \times \begin{bmatrix} x \\ y \end{bmatrix}$

→ it is a matrix-vector multiplication!!



Before learning how to rotate a vector, let's understand how to do something slightly different

- How can you convert a vector represented in frame “0” to a new, rotated coordinate frame “1”?
- Goal: How do we go from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$?

→ answer: it is a matrix-vector multiplication!!

$$\circ \begin{bmatrix} x' \\ y' \end{bmatrix} = \left[\text{2x2 Rotation Matrix} \right] \times \begin{bmatrix} x \\ y \end{bmatrix}$$

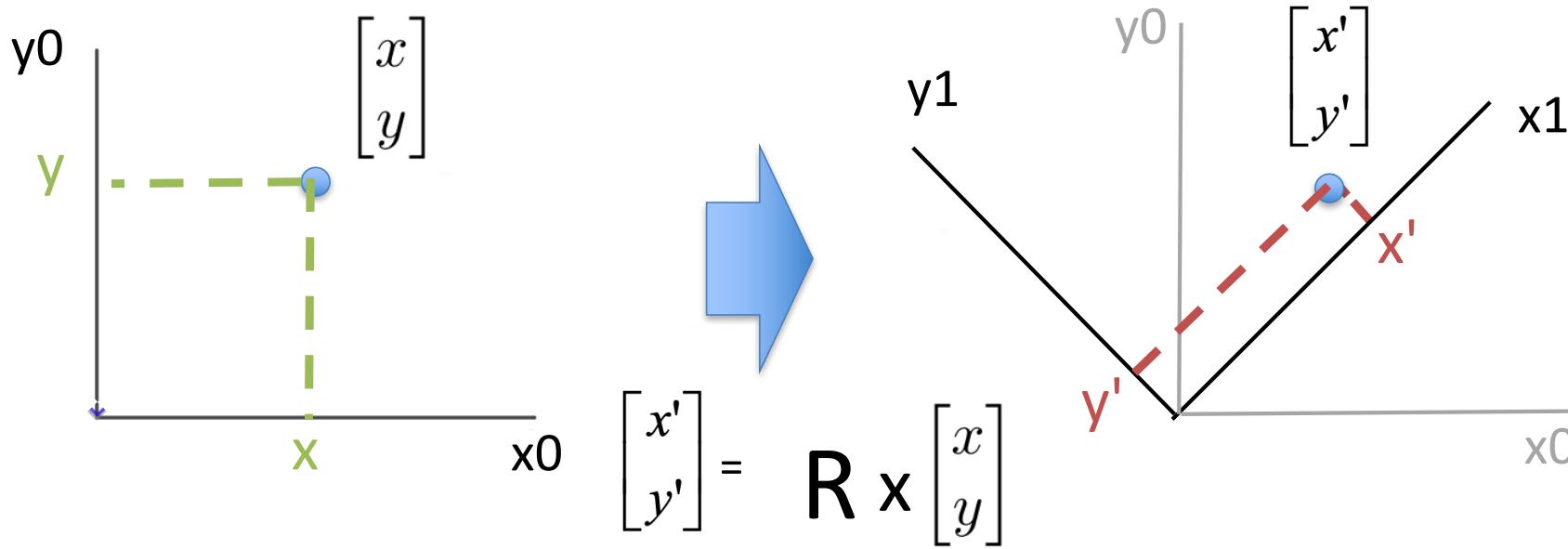
$$\circ \begin{bmatrix} x' \\ y' \end{bmatrix} = R \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \mathbf{P}$$



Before learning how to rotate a vector, let's understand how to do something slightly different

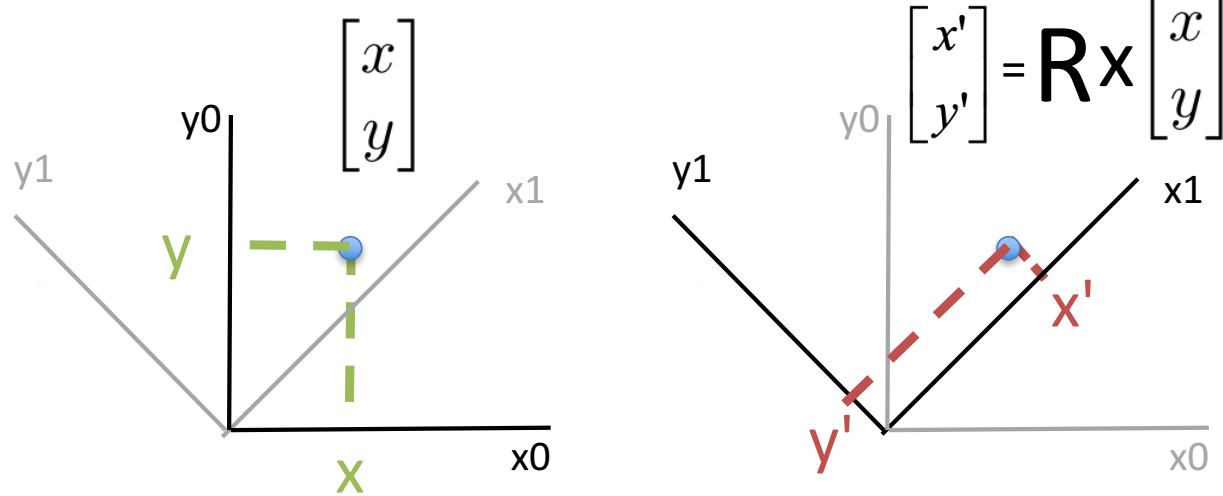
- How can you convert a vector represented in frame "0" to a new, rotated coordinate frame "1"?
- Answer: using a matrix-vector multiplication!!





Now, here's how we do a rotation!!

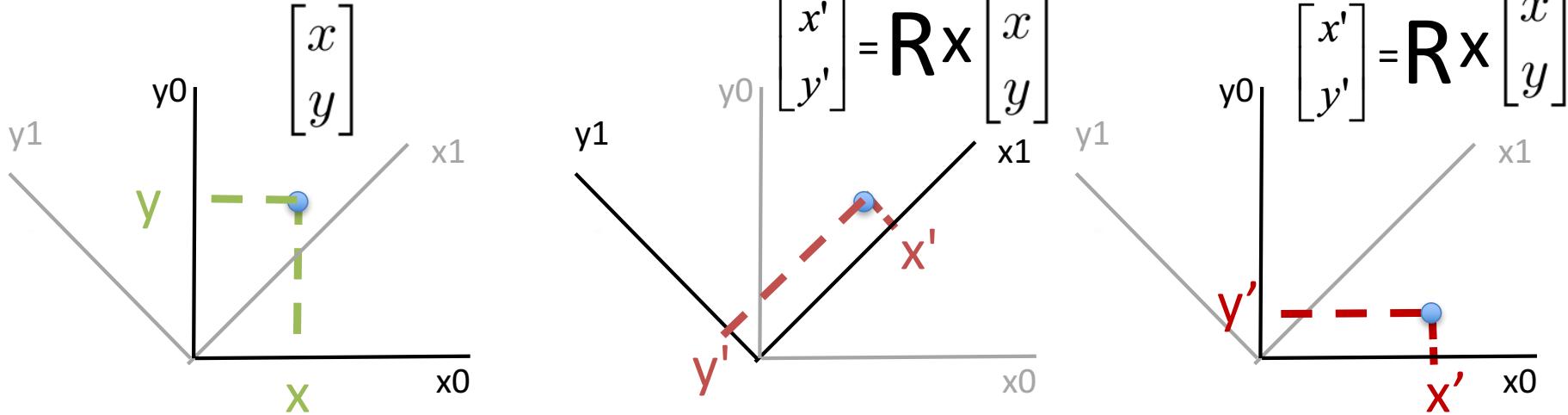
- Until now, we've seen how to convert a data point represented in frame "0" to a new, rotated coordinate frame "1". We rotated the frame to the left.





Now, here's how we do a rotation!!

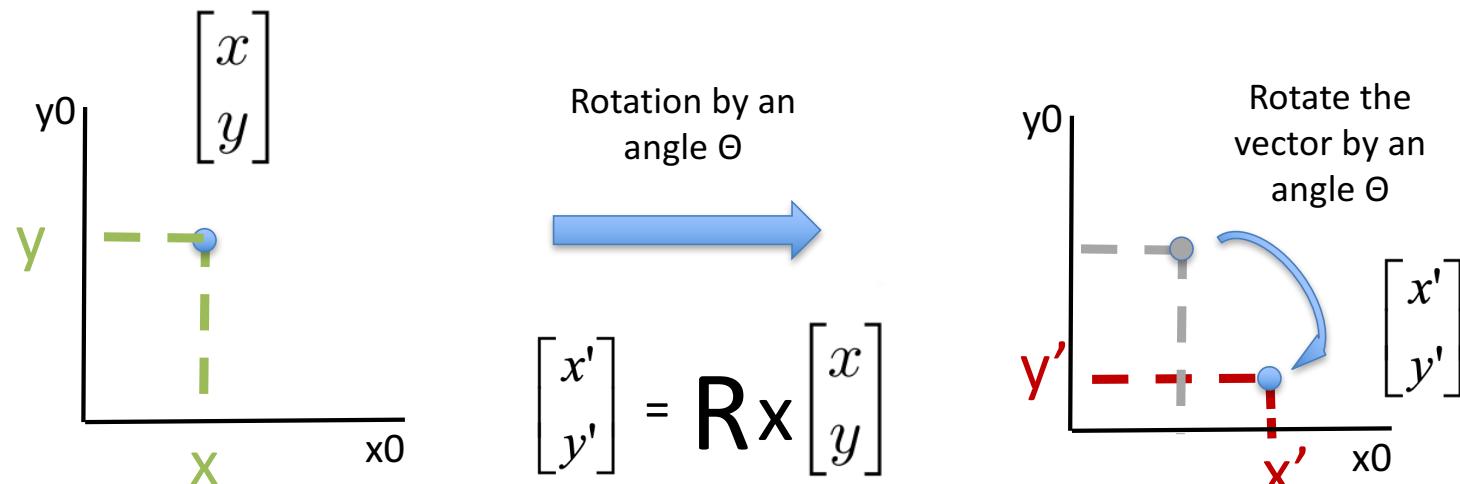
- Until now, we've seen how to convert a data point represented in frame "0" to a new, rotated coordinate frame "1". We rotated the frame to the left.
- Insight: rotating the frame to the left == rotating a data point to the right**
 - Suppose we express a point in the new coordinate system "1"
 - If we plot the result in the **original** coordinate system, we have rotated the point right





Now, here's how we do a rotation!!

- Until now, we've seen how to convert a data point represented in frame "0" to a new, rotated coordinate frame "1".
- Insight: rotating the frame to the left == rotating a data point to the right

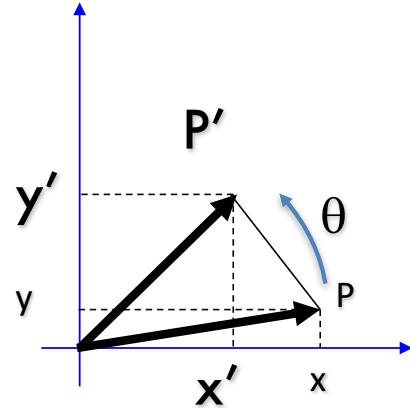


- Thus, rotation matrices can be used to rotate vectors. We'll usually think of them in that sense-- as operators to rotate vectors



2D Rotation Matrix Formula: what is R?

Counter-clockwise rotation by an angle θ



$$x' = \cos \theta x - \sin \theta y$$

$$y' = \cos \theta y + \sin \theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \mathbf{P}$$



Transformation Matrices

- Multiple transformation matrices can be used to transform a point:
 $p' = R_2 R_1 S p$
- The effect of this is to apply their transformations one after the other, from **right to left**.
- In the example above, the result is
 $(R_2 (R_1 (S p)))$
- The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:
 $p' = (R_2 R_1 S) p$



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Homogeneous system

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- This is sufficient for scale, rotate, skew transformations.
- But notice, we can't add a constant! 😞

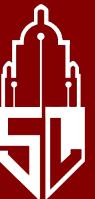


Homogeneous system

- The (somewhat hacky) solution? Stick a “1” at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale, and skew like before, **AND** translate (note how the multiplication works out, above)
- This is called “homogeneous coordinates”



Homogeneous system

- In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Generally, a homogeneous transformation matrix will have a bottom row of [0 0 1], so that the result has a “1” at the bottom too.



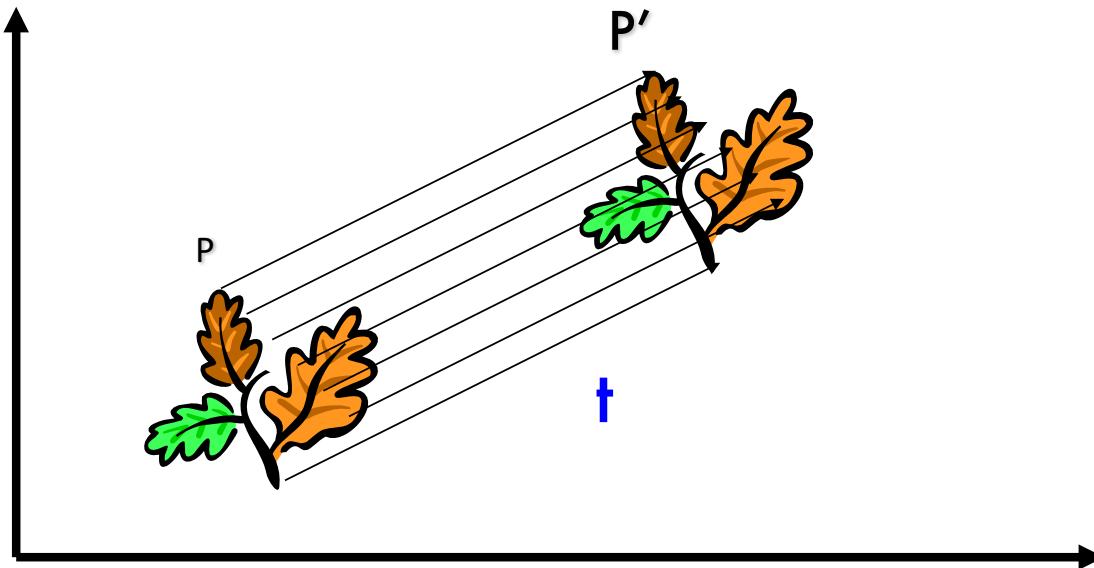
Homogeneous system

- One more thing we might want: to divide the result by something
 - For example, we may want to divide by a coordinate, to make things scale down as they get farther away in a camera image
 - Matrix multiplication can't actually divide
 - So, **by convention**, in homogeneous coordinates, we'll divide the result by its last coordinate after doing a matrix multiplication

$$\begin{bmatrix} x \\ y \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} x/7 \\ y/7 \\ 1 \end{bmatrix}$$

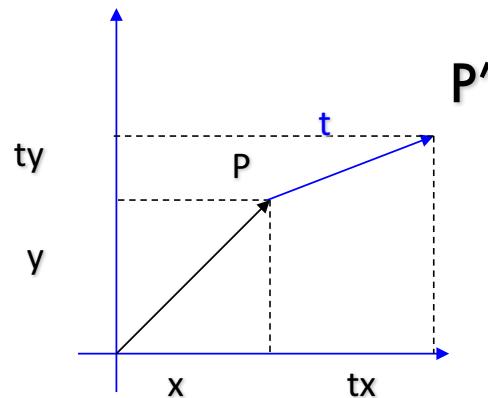


2D Translation using Homogeneous Coordinates





2D Translation using Homogeneous Coordinates



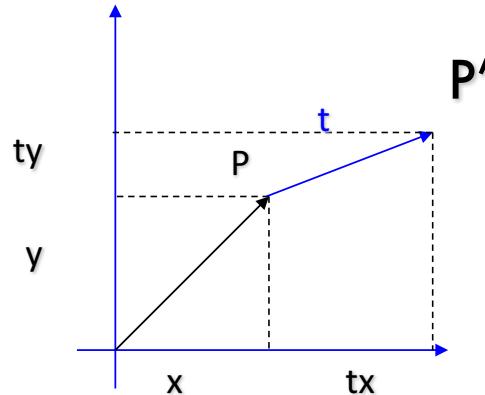
$$P = (x, y) \rightarrow (x, y, 1)$$

$$t = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$P' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & t_x & t_y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



2D Translation using Homogeneous Coordinates



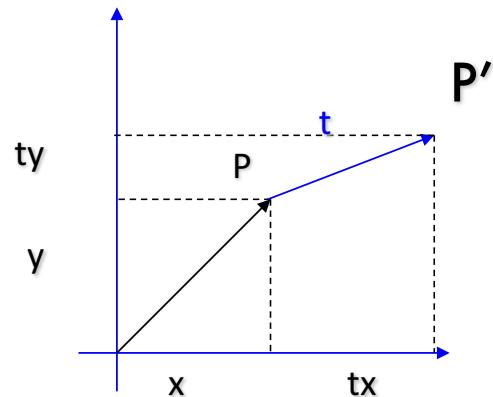
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



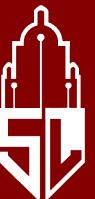
2D Translation using Homogeneous Coordinates



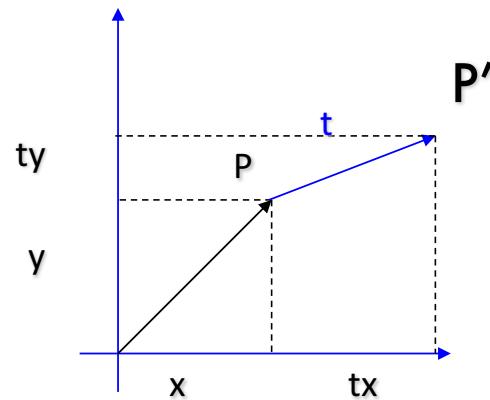
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots \\ \vdots & \ddots & \vdots \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



2D Translation using Homogeneous Coordinates



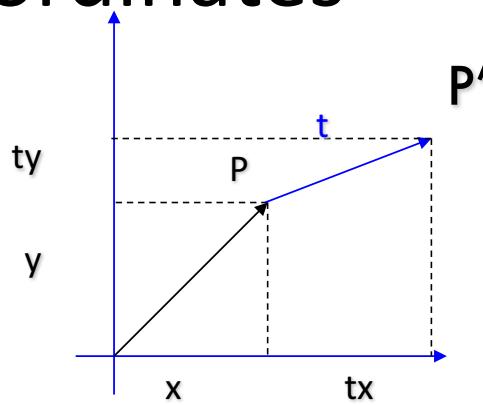
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



2D Translation using Homogeneous Coordinates

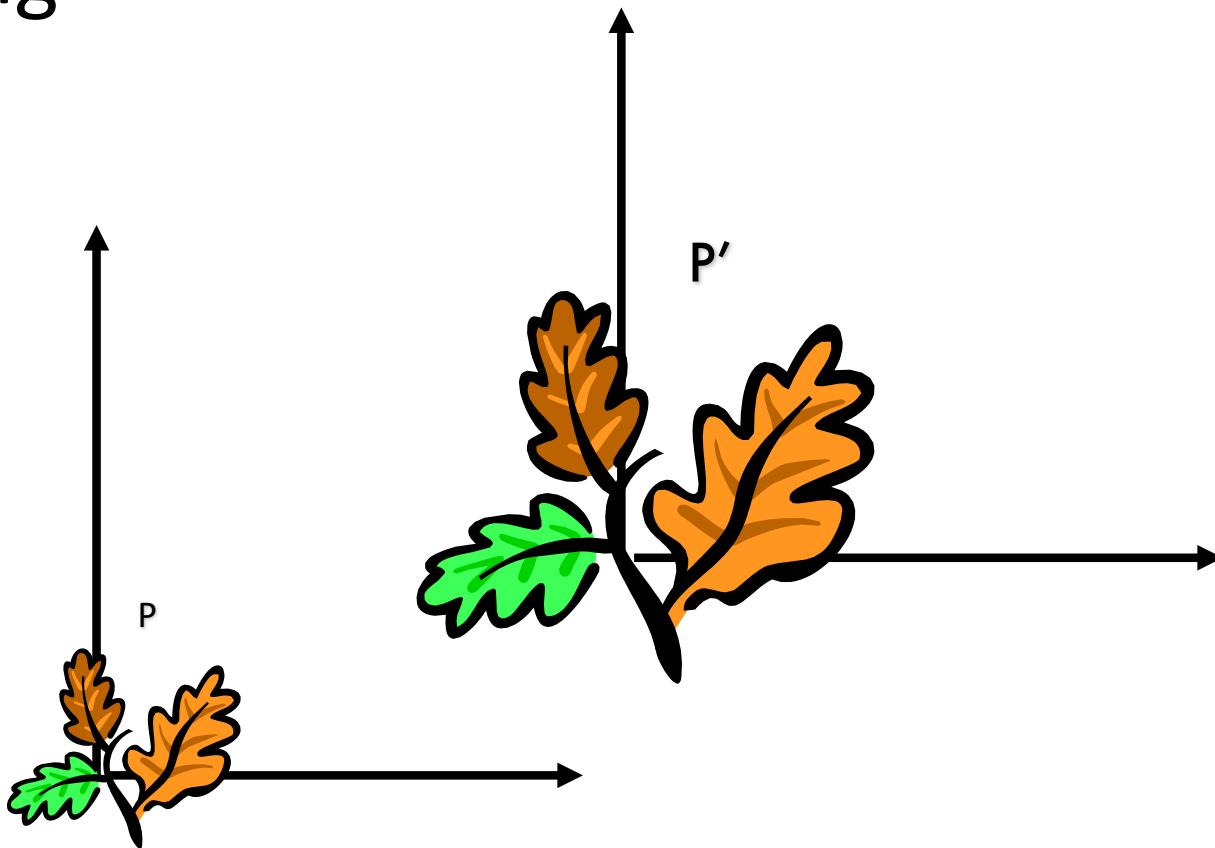


$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

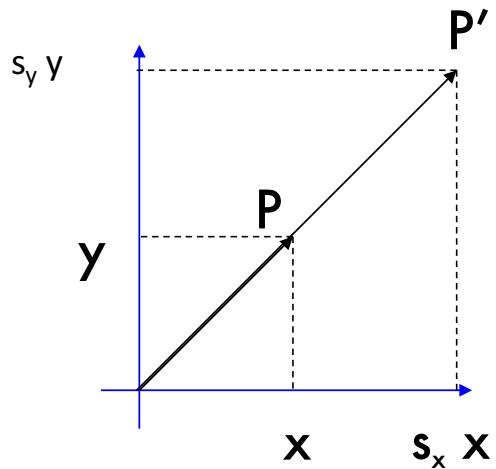
$$\begin{aligned}\mathbf{P}' &\rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}\end{aligned}$$

Scaling





Scaling Equation



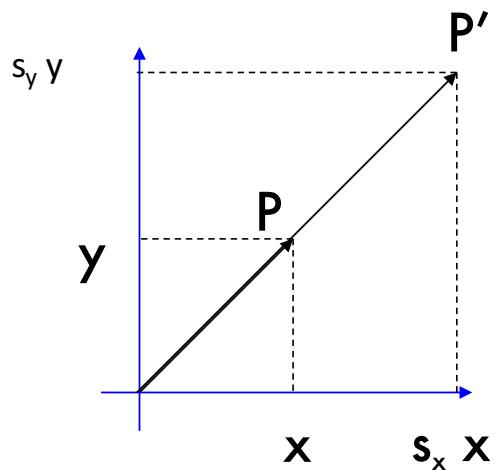
$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$



Scaling Equation



$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

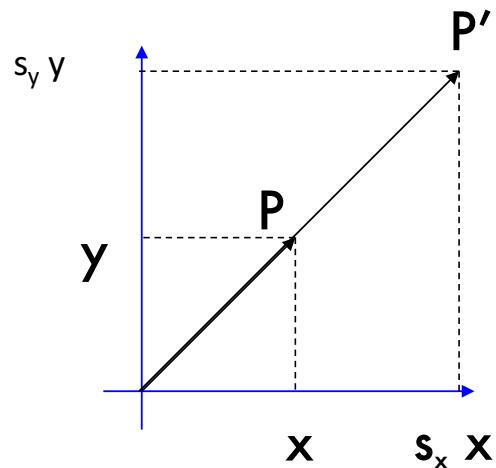
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Scaling Equation



$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

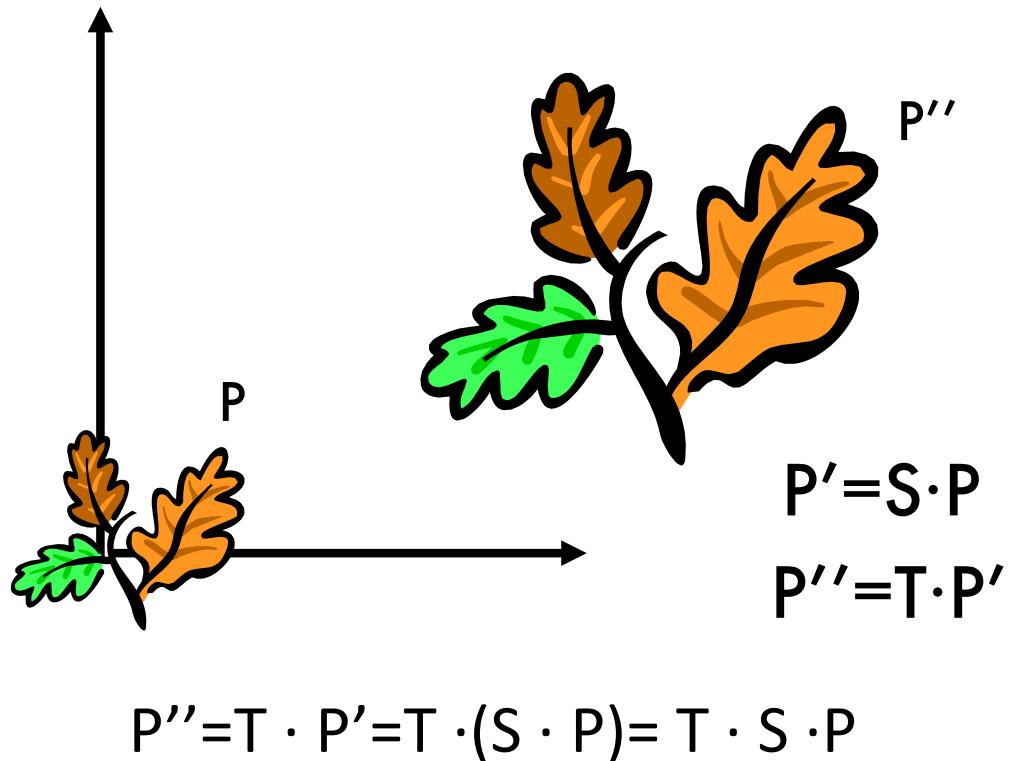
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$



Scaling & Translating





Scaling & Translating

$$\mathbf{P}'' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Scaling & Translating

$$\begin{aligned} \mathbf{P}'' &= \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\ &= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{aligned}$$



Translating & Scaling != Scaling & Translating

$$\mathbf{P}''' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$



Translating & Scaling != Scaling & Translating

$$\mathbf{P}''' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$\mathbf{P}''' = \mathbf{S} \cdot \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$



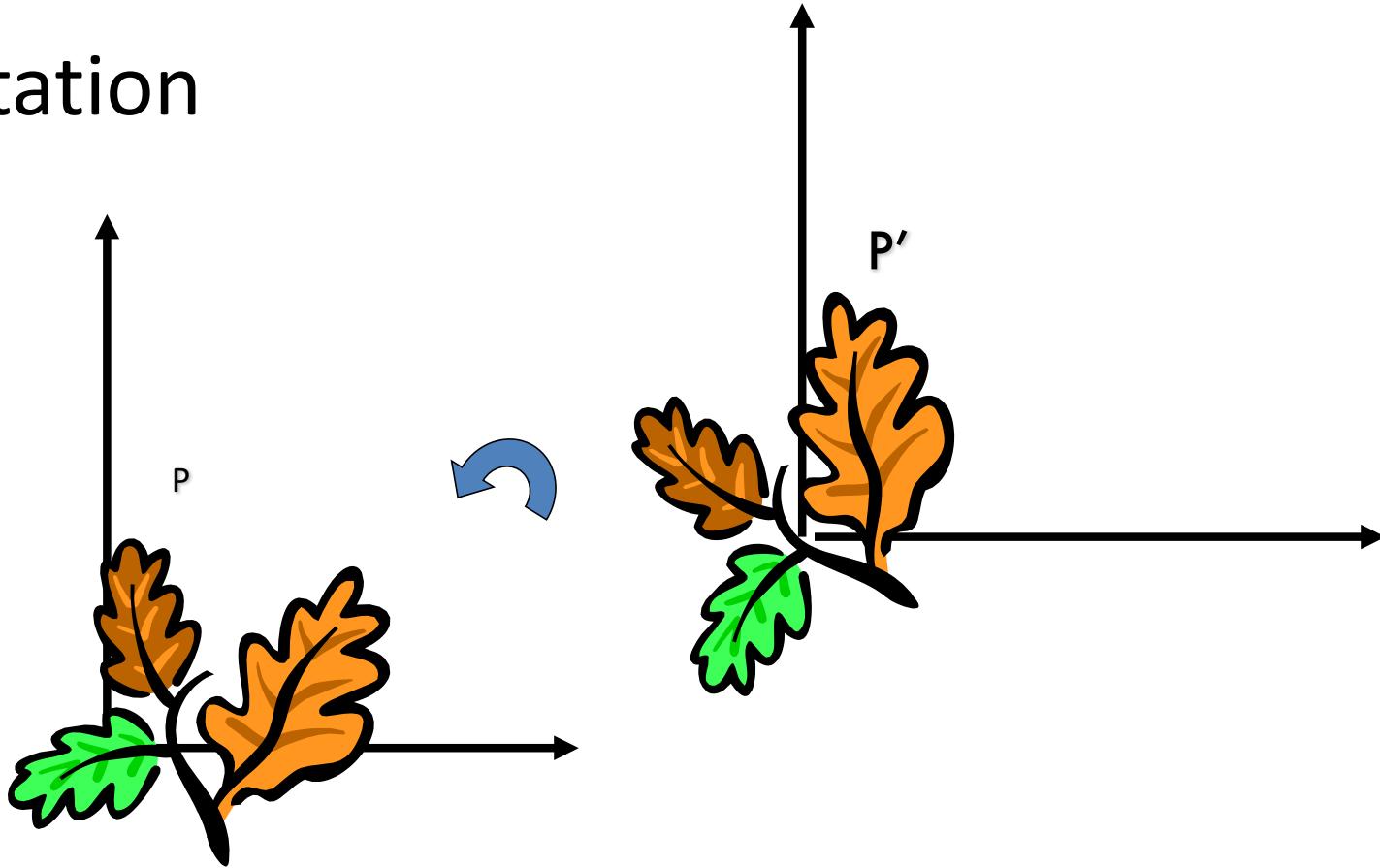
Translating & Scaling != Scaling & Translating

$$\mathbf{P}''' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$\begin{aligned}\mathbf{P}''' &= \mathbf{S} \cdot \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\ &= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}\end{aligned}$$



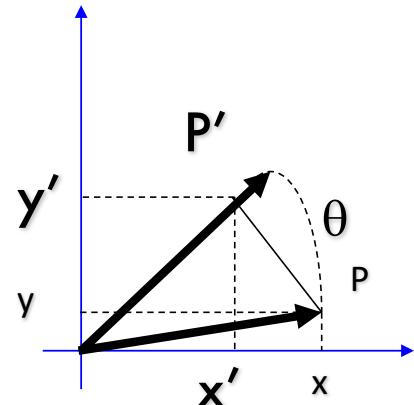
Rotation





Rotation Equations

Counter-clockwise rotation by an angle θ



$$x' = \cos \theta x - \sin \theta y$$

$$y' = \cos \theta y + \sin \theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \mathbf{P}$$



Rotation Matrix Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix is 2x2

$$x' = \cos \theta x - \sin \theta y$$
$$y' = \cos \theta y + \sin \theta x$$

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$



Rotation Matrix Properties

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
 - (and so are its columns)



Rotation Equation

$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (\cos\theta \cdot x - \sin\theta \cdot y, \quad \cos\theta \cdot y + \sin\theta \cdot x)$$



$$(x, y, 1)$$

$$(\cos\theta \cdot x - \sin\theta \cdot y, \cos\theta \cdot y + \sin\theta \cdot x, 1)$$



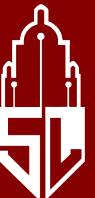
Rotation Equation

$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (\cos\theta \cdot x - \sin\theta \cdot y, \cos\theta \cdot y + \sin\theta \cdot x)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\begin{aligned}\mathbf{P}' &= (\cos\theta \cdot x - \sin\theta \cdot y, \cos\theta \cdot y + \sin\theta \cdot x) \\ &\rightarrow (\cos\theta \cdot x - \sin\theta \cdot y, \cos\theta \cdot y + \sin\theta \cdot x, 1)\end{aligned}$$

$$\begin{aligned}\mathbf{P}' &\rightarrow \begin{bmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \cos\theta \cdot y + \sin\theta \cdot x \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{R} \cdot \mathbf{P}\end{aligned}$$



Scaling + Rotation + Translation

$$\mathbf{P}' = (\mathbf{T} \mathbf{R} \mathbf{S}) \mathbf{P}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

This is the form of the general-purpose transformation matrix

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$



Outline

- Vectors and matrices
 - Basic Matrix Operations
 - Determinants, norms, trace
 - Special Matrices
- Transformation Matrices
 - Homogeneous coordinates
 - Translation
- **Matrix inverse**
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculate



The inverse of a transformation matrix reverses its effect



Inverse

- Given a matrix \mathbf{A} , its inverse \mathbf{A}^{-1} is a matrix such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- E.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$
- Inverse does not always exist. If \mathbf{A}^{-1} exists, \mathbf{A} is *invertible* or *non-singular*. Otherwise, it's *singular*.
- Useful identities, for matrices that are invertible:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-T} \triangleq (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$



Matrix Operations

- Pseudoinverse
 - Fortunately, there are workarounds to solve $AX=B$ in these situations. And python can do them!
 - Instead of taking an inverse, directly ask python to solve for X in $AX=B$, by typing **np.linalg.solve(A, B)**
 - Python will try several appropriate numerical methods (including the pseudoinverse if the inverse doesn't exist)
 - Python will return the value of X which solves the equation
 - If there is no exact solution, it will return the closest one
 - If there are many solutions, it will return the smallest one



Matrix Operations

- Python example:

$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>> import numpy as np
>> x = np.linalg.solve(A,B)
x =
    1.0000
   -0.5000
```



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- The rank of a transformation matrix tells you how many dimensions it transforms a vector to.



Linear independence

- Suppose we have a set of vectors v_1, \dots, v_n
- If we can express v_1 as a linear combination of the other vectors $v_2 \dots v_n$, then v_1 is linearly *dependent* on the other vectors.
 - The direction v_1 can be expressed as a combination of the directions $v_2 \dots v_n$. (E.g. $v_1 = .7 v_2 - .7 v_4$)



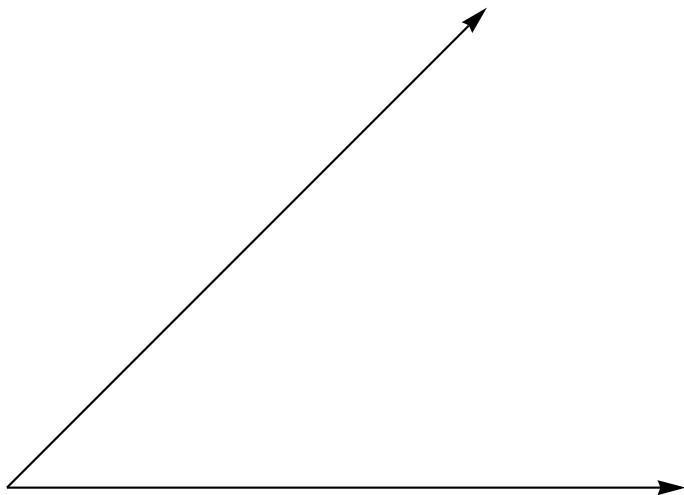
Linear independence

- Suppose we have a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$
- If we can express \mathbf{v}_1 as a linear combination of the other vectors $\mathbf{v}_2 \dots \mathbf{v}_n$, then \mathbf{v}_1 is linearly *dependent* on the other vectors.
 - The direction \mathbf{v}_1 can be expressed as a combination of the directions $\mathbf{v}_2 \dots \mathbf{v}_n$. (E.g. $\mathbf{v}_1 = .7 \mathbf{v}_2 - .7 \mathbf{v}_4$)
- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
 - Common case: a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero)

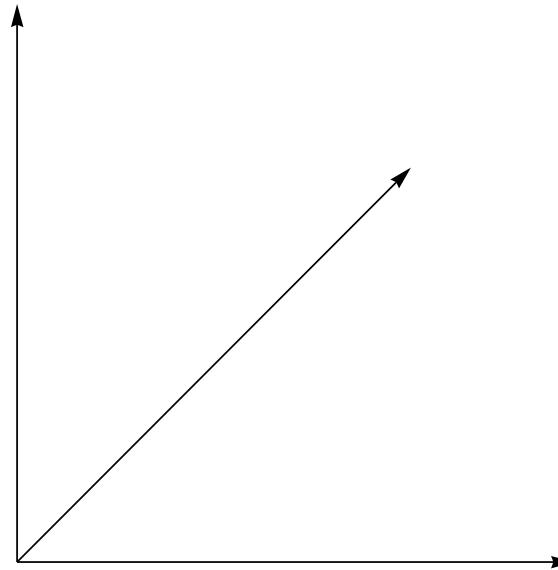


Linear independence

Linearly independent set



Not linearly independent





Matrix rank

- **Column/row rank**

$\text{col-rank}(\mathbf{A})$ = the maximum number of linearly independent column vectors of \mathbf{A}

$\text{row-rank}(\mathbf{A})$ = the maximum number of linearly independent row vectors of \mathbf{A}

- Column rank always equals row rank

- **Matrix rank**

$$\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$$



Matrix rank

- For transformation matrices, the rank tells you the dimensions of the output
- E.g. if rank of \mathbf{A} is 1, then the transformation

$$\mathbf{p}' = \mathbf{Ap}$$

maps points onto a line.

- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 2y \end{bmatrix} \leftarrow \text{All points get mapped to the line } y=2x$$



A matrix has “full rank” \Leftrightarrow it is invertible

- If an $m \times m$ matrix is rank m , we say it's “full rank”
 - Maps an $m \times 1$ vector uniquely to another $m \times 1$ vector
 - An inverse matrix can be found
- If rank $< m$, we say it's “singular”
 - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
 - Inverse does not exist
- Inverse also doesn't exist for non-square matrices



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Eigenvector and Eigenvalue

- An eigenvector x of a linear transformation A is a non-zero vector that, when A is applied to it, does not change direction.

$$Ax = \lambda x, \quad x \neq 0.$$



Eigenvector and Eigenvalue

- An eigenvector x of a linear transformation A is a non-zero vector that, when A is applied to it, does not change direction.
- Applying A to the eigenvector only scales the eigenvector by the scalar value λ , called an eigenvalue.

$$Ax = \lambda x, \quad x \neq 0.$$



Eigenvector and Eigenvalue

- We want to find all the eigenvalues of A:

$$Ax = \lambda x, \quad x \neq 0.$$

- Which can we written as:

$$Ax = (\lambda I)x \quad x \neq 0.$$

- Therefore:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$



Eigenvector and Eigenvalue

- We can solve for eigenvalues by solving:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

- Since we are looking for non-zero x , we can instead solve the above equation as:

$$|(\lambda I - A)| = 0.$$



Properties

- The trace of a A is equal to the sum of its eigenvalues:

$$\text{tr}A = \sum_{i=1}^n \lambda_i.$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^n \lambda_i.$$

- The rank of A is equal to the number of non-zero eigenvalues of A .
- The eigenvalues of a diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ are just the diagonal entries d_1, \dots, d_n



Spectral theory

- We call an eigenvalue λ and an associated eigenvector an **eigenpair**.
- The space of vectors where $(A - \lambda I) = 0$ is often called the **eigenspace** of A associated with the eigenvalue λ .
- The set of all eigenvalues of A is called its **spectrum**:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is singular}\}.$$



Spectral theory

- The magnitude of the largest eigenvalue (in magnitude) is called the spectral radius

$$\rho(A) = \max \{|\lambda_1|, \dots, |\lambda_n|\}$$

- Where C is the space of all eigenvalues of A



Spectral theory

- The spectral radius is bounded by infinity norm of a matrix:
$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}$$
- Proof: Turn to a partner and prove this!



Spectral theory

- The spectral radius is bounded by infinity norm of a matrix:
$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}$$
- Proof: Let λ and \mathbf{v} be an eigenpair of A :

$$|\lambda|^k \|\mathbf{v}\| = \|\lambda^k \mathbf{v}\| = \|A^k \mathbf{v}\| \leq \|A^k\| \cdot \|\mathbf{v}\|$$

and since $\mathbf{v} \neq 0$ we have

$$|\lambda|^k \leq \|A^k\|$$

and therefore

$$\rho(A) \leq \|A^k\|^{\frac{1}{k}}.$$



Diagonalization

- An $n \times n$ matrix A is diagonalizable if it has n linearly independent eigenvectors.
- Most square matrices (in a sense that can be made mathematically rigorous) are diagonalizable:
 - Normal matrices are diagonalizable
 - Matrices with n distinct eigenvalues are diagonalizable

Lemma: Eigenvectors associated with distinct eigenvalues are linearly independent.



Diagonalization

- Eigenvalue equation:

$$AV = VD$$

$$A = VDV^{-1}$$

- Where D is a diagonal matrix of the eigenvalues

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$



Diagonalization

- Eigenvalue equation:

$$\begin{aligned}AV &= VD \\A &= VDV^{-1}\end{aligned}$$

- Assuming all λ_i 's are unique:

$$A = VDV^T$$

- Remember that the inverse of an orthogonal matrix is just its transpose and the eigenvectors are orthogonal



Symmetric matrices

- Properties:
 - For a symmetric matrix A , all the eigenvalues are real.
 - The eigenvectors of A are orthonormal.

$$A = VDV^T$$



Symmetric matrices

- Therefore:

$$x^T A x = x^T V D V^T x = y^T D y = \sum_{i=1}^n \lambda_i y_i^2$$

– where $y = V^T x$

- So, what can you say about the vector x that satisfies the following optimization? $\max_{x \in \mathbb{R}^n} x^T A x$ subject to $\|x\|_2^2 = 1$



Symmetric matrices

- Therefore:

$$x^T A x = x^T V D V^T x = y^T D y = \sum_{i=1}^n \lambda_i y_i^2$$

– where $y = V^T x$

- So, what can you say about the vector x that satisfies the following optimization? $\max_{x \in \mathbb{R}^n} x^T A x$ subject to $\|x\|_2^2 = 1$

– Is the same as finding the eigenvector that corresponds to the largest eigenvalue of A .



Some applications of Eigenvalues

- PageRank
- Schrodinger's equation
- PCA
- We are going to use it to compress images in future classes



Outline

- Vectors and matrices
 - Basic Matrix Operations
 - Determinants, norms, trace
 - Special Matrices
- Transformation Matrices
 - Homogeneous coordinates
 - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculus



Matrix Calculus – The Gradient

- Let a function $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ take as input a matrix A of size $m \times n$ and return a real value.
- Then the **gradient** of f :

$$\nabla_A f(A) \in \mathbb{R}^{m \times n} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \frac{\partial f(A)}{\partial A_{21}} & \frac{\partial f(A)}{\partial A_{22}} & \cdots & \frac{\partial f(A)}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{m1}} & \frac{\partial f(A)}{\partial A_{m2}} & \cdots & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$



Matrix Calculus – The Gradient

- Every entry in the matrix is: $\nabla_A f(A))_{ij} = \frac{\partial f(A)}{\partial A_{ij}}$.
- the size of $\nabla_A f(A)$ is always the same as the size of A. So if A is just a vector x:

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$



Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = [b_1 \quad b_2 \quad \dots \quad b_n]^T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Find:
 $\frac{\partial f(x)}{\partial x_k} = ?$

$$\nabla_x f(x) = ?$$



Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \sum_{i=1}^n b_i x_i$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^n b_i x_i = b_k.$$

- From this we can conclude that: $\nabla_x b^T x = b$.



Matrix Calculus – The Gradient

- Properties

- $\nabla_x(f(x) + g(x)) = \nabla_x f(x) + \nabla_x g(x).$
- For $t \in \mathbb{R}$, $\nabla_x(t f(x)) = t \nabla_x f(x).$



Matrix Calculus – The Hessian

- The Hessian matrix with respect to x , written $\nabla_x^2 f(x)$ or simply as H :

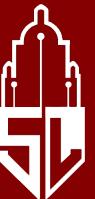
$$\nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

- The Hessian of n -dimensional vector is the $n \times n$ matrix.



Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.
- Exercise: Why is the Hessian always symmetric?



Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.
- The Hessian is always symmetric, because
$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}.$$
- This is known as Schwarz's theorem: The order of partial derivatives don't matter as long as the second derivative exists and is continuous.



Matrix Calculus – The Hessian

- Note that the hessian is not the gradient of whole gradient of a vector (this is not defined). It is actually the gradient of **every entry** of the gradient of the vector.

$$\nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$



Matrix Calculus – The Hessian

- Eg, the first column is the gradient of $\frac{\partial f(x)}{\partial x_1}$

$$\nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$



Exercise

- Example:

consider the quadratic function $f(x) = x^T Ax$

$$f(x) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

Exercise

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$





Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right]\end{aligned}$$

Divide the summation into 3 parts depending on whether:

- $i == k$ or
- $j == k$



Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\ &= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2A_{kk} x_k\end{aligned}$$



Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \boxed{\sum_{i \neq k} A_{ik} x_i x_k} + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\ &= \boxed{\sum_{i \neq k} A_{ik} x_i} + \sum_{j \neq k} A_{kj} x_j + 2A_{kk} x_k\end{aligned}$$



Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \boxed{\sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2} \right] \\ &= \sum_{i \neq k} A_{ik} x_i + \boxed{\sum_{j \neq k} A_{kj} x_j} + 2A_{kk} x_k\end{aligned}$$



Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + \boxed{A_{kk} x_k^2} \right] \\ &= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + \boxed{2A_{kk} x_k}\end{aligned}$$



Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\ &= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2A_{kk} x_k\end{aligned}$$



Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j \\ &= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\ &= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2A_{kk} x_k \\ &= \sum_{i=1}^n A_{ik} x_i + \sum_{j=1}^n A_{kj} x_j = 2 \sum_{i=1}^n A_{ki} x_i,\end{aligned}$$



Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^n A_{\ell i} x_i \right]$$



Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} &= \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^n A_{\ell i} x_i \right] \\ &= 2A_{\ell k} = 2A_{k\ell}.\end{aligned}$$



Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$$

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} &= \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^n 2A_{\ell i} x_i \right] \\ &= 2A_{\ell k} = 2A_{k\ell}.\end{aligned}$$

$$\nabla_x^2 f(x) = 2A$$



What we have learned

- Vectors and matrices
 - Basic Matrix Operations
 - Special Matrices
- Transformation Matrices
 - Homogeneous coordinates
 - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculate