TAM 470 HW 7 Solutions

Problem 1

Part 1:

$$T''(x) + \left(\frac{1}{A}\frac{dA}{dx}\right)T'(x) + \frac{Q}{k} = 0$$

Using central differences,

$$\frac{T_{j+1} - 2T_j + T_{j-1}}{h^2} + \left[\frac{1}{A_j}\left(\frac{dA}{dx}\right)_j\right]\left(\frac{T_{j+1} - T_{j-1}}{2h}\right) + \frac{Q_j}{k_j} = 0$$

$$\Rightarrow \left[\frac{1}{h^2} + \frac{1}{2hA_j}\left(\frac{dA}{dx}\right)_j\right]T_{j+1} - \frac{2}{h^2}T_j + \left[\frac{1}{h^2} - \frac{1}{2hA_j}\left(\frac{dA}{dx}\right)_j\right]T_{j-1} = -\frac{Q_j}{k_j} \rightarrow (1)$$

$$\Rightarrow \left.\begin{aligned} \alpha_j &= \left[\frac{1}{h^2} + \frac{1}{2hA_j}\left(\frac{dA}{dx}\right)_j\right] \\ \beta_j &= -\frac{2}{h^2} \\ \gamma_j &= \left[\frac{1}{h^2} - \frac{1}{2hA_j}\left(\frac{dA}{dx}\right)_j\right] \end{aligned}\right\} \rightarrow (2)$$

$$f_j = -\frac{Q_j}{k_j}$$

When T_0 is prescribed, we look at equation with $j=1$,

$$\alpha_1 T_2 + \beta_1 T_1 + \gamma_1 T_0 = -\frac{Q}{k}$$

$$\Rightarrow \quad \alpha_1 T_2 + \beta_1 T_1 = \underbrace{-\frac{Q_1}{k_1} - \gamma_1 T_0}_{\text{All quantities known}} \quad - (3)$$

For Neuman condition at the right end,

$$k T' = q_r$$

$$\Rightarrow k(T_N - T_{N-1}) = h q_r$$

$$\Rightarrow T_N = T_{N-1} + \frac{h q_r}{k} \quad - (4)$$

Consider (1) with $j = N-1$,

$$\alpha_{N-1} T_N + \beta_{N-1} T_{N-1} + \gamma_{N-1} T_{N-2} = -\frac{Q}{k}$$

Using T_N from (4),

$$\alpha_{N-1}\left(T_{N-1} + \frac{h q_r}{k}\right) + \beta_{N-1} T_{N-1} + \gamma_{N-1} T_{N-2} = -\frac{Q}{k}$$

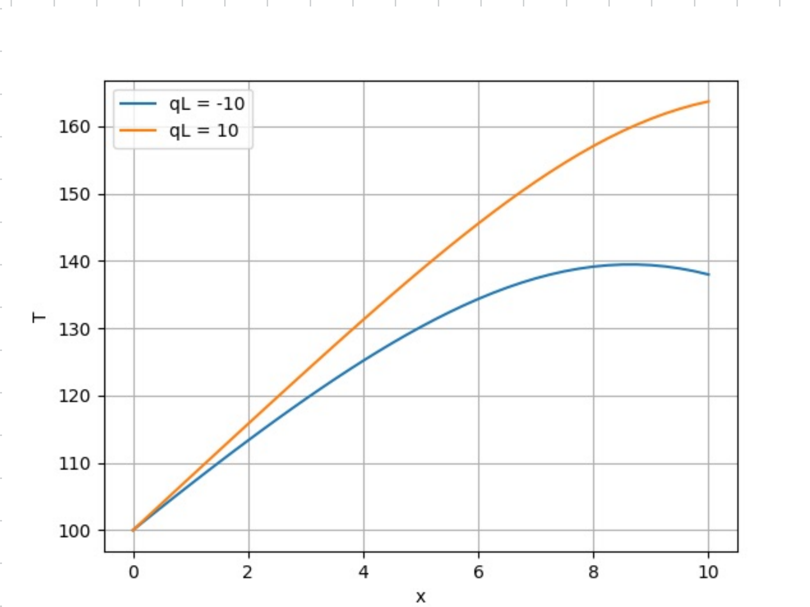
$$\Rightarrow (\alpha_{N-1} + \beta_{N-1}) T_{N-1} + \gamma_{N-1} T_{N-2} = -\frac{Q}{k} - \alpha_{N-1}\frac{h q_r}{k} \quad - (5)$$

When building the system to solve, we (1) for $j=2$ to $j=N-2$, (3) for $j=1$ and (5) for $j=N-1$

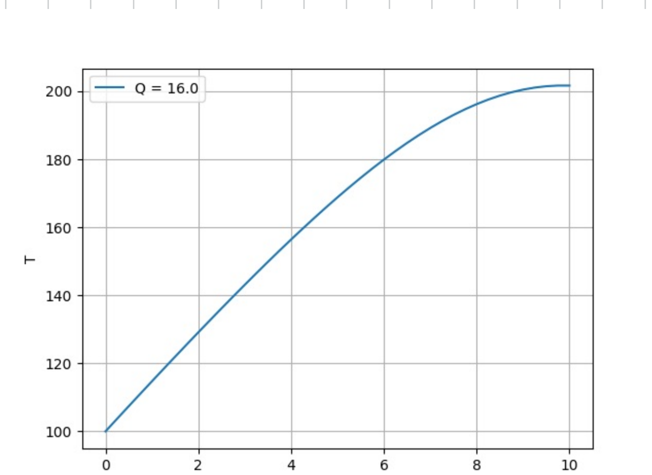
$$
\begin{bmatrix}
\beta_1 & \alpha_1 & 0 & \cdots & \cdots & 0 \\
\gamma_2 & \beta_2 & \alpha_2 & 0 & \cdots & 0 \\
0 & \gamma_3 & \beta_3 & \alpha_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & & & 0 & \gamma_{N-1} & (\alpha_{N-1}+\beta_{N-1})
\end{bmatrix}
\begin{bmatrix}
T_1 \\ T_2 \\ \vdots \\ T_{N-1}
\end{bmatrix}
=
\begin{bmatrix}
f_1 - \gamma_1 T_0 \\ f_2 \\ \vdots \\ f_{N-1} - \alpha_{N-1}\dfrac{h q_L}{k}
\end{bmatrix}
$$

Part 2 :- Implement on PL

Part 3 :- (a)

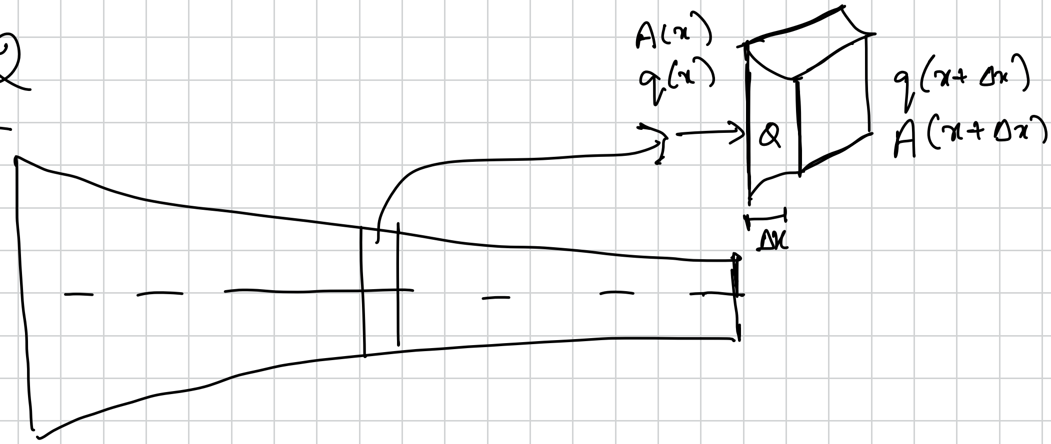


(b)



At $Q = 16$, $T_{max} \approx 201^\circ C$ just crosses $200^\circ c$.

Problem 2



$$\dot{E} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = 0$$

Fourier's law, $q = -kT' \Rightarrow q' = (-kT')'$

$$\Rightarrow q(x) \cdot A(x) - q(x+\Delta x) A(x+\Delta x) + Q \Delta x A(x) = 0$$

$$q(x) A(x) - \left(q(x) + \Delta x\, q'(x) + O(\Delta x^2)\right)\left(A(x) + \Delta x A'(x) + O(\Delta x^2)\right) + Q A \Delta x = 0$$

$$\Rightarrow \cancel{q(x) A(x)} - \cancel{q(x) A(x)} + \Delta x \left[-q'(x) A(x) - q(x) A'(x) + Q A(x)\right] = 0$$

$\Rightarrow$ Assuming constant k and using Fourier law,

$$kT'' A + kT' A' + Q A = 0$$

$$\longrightarrow \boxed{T'' + \left(\frac{A'}{A}\right) T' + \frac{Q}{k} = 0}$$