# Monotone sequences

**Definition (Increasing, decreasing, monotone sequences)**

A sequence $(s_n)$ is called increasing (decreasing) if $s_n \leqslant s_{n+1}$ (resp. $s_n \geqslant s_{n+1}$) for any $n$.
A sequence is monotone if it is either increasing or decreasing.

Our "increasing" sequences are sometimes called "nondecreasing." We do not require $s_n < s_{n+1}$. Similarly, our "decreasing" sequences are also referred to as "nonincreasing."

**Example 1.** $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. This sequence is increasing: $x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n$ for any $n$.

**Example 2.** $y_n = \frac{(-1)^n}{n^2}$. This sequence is not monotone: $y_{n+1} > y_n$ if $n$ is odd, $y_{n+1} < y_n$ if $n$ is even.

# Bounded monotone sequences converge

**Theorem (Theorem 10.2)**

*Any monotone bounded sequence converges.*

Any convergent sequence is **bounded** (proved before).

A convergent sequence need not be **monotone** – e.g. $y_n = \frac{(-1)^n}{n^2}$.

**Example.** Let $s_n = \sum_{k=0}^{n-1} \frac{1}{k!}$. Does $(s_n)$ converge? Recall: $0! = 1$.

$s_{n+1} = s_n + \frac{1}{n!}$, hence $(s_n)$ is increasing.

Is $(s_n)$ bounded? Does there exists $A > 0$ s.t. $s_n \leqslant A$ for any $n$?

Let $s_n = \sum_{k=0}^{n-1} \frac{1}{k!}$. Does there exists $A > 0$ s.t. $s_n \leqslant A$ for any $n$?

For $k \geqslant 1$, $k! \geqslant 2^{k-1}$ (prove this using induction!).

$s_n = \frac{1}{0!} + \frac{1}{1!} + \ldots + \frac{1}{(n-1)!} \leqslant 1 + \frac{1}{2^0} + \frac{1}{2^1} + \ldots + \frac{1}{2^{n-2}}$

$1 + \frac{1-1/2^{n-1}}{1-1/2} < 1 + \frac{1}{1-1/2} = 1 + 2 = 3.$

We already know: $(s_n)$ is increasing. Thus, $(s_n)$ **converges**.

In fact: $\lim s_n = e = 2.71828\ldots$.

# Bounded monotone sequences converge: proof

**Theorem (Theorem 10.2)**

*Any monotone bounded sequence converges.*

**Proof: Any bounded increasing sequence converges.**
Let $u = \sup\{s_1, s_2, \ldots\}$. Show that $\lim s_n = u$.
Want: $\forall \varepsilon > 0\ \exists N \in \mathbb{R}$ s.t. $|s_n - u| < \varepsilon$ for $n > N$.
As $s_n \leqslant u$, it suffices to verify $s_n > u - \varepsilon$.
$u - \varepsilon$ is not an upper bound for $\{s_1, s_2, \ldots\}$, hence $s_N > u - \varepsilon$ for some $N$.
For $n > N$, $s_n \geqslant s_N > u - \varepsilon$. ∎

Decreasing sequences are handled similarly.

# Example (recursively defined sequence: pp. 57-58)

**Let $s_0 = 5$, $s_{n+1} = \frac{s_n^2+5}{2s_n}$ for $n \geqslant 0$.**
**Does $\lim s_n$ exist, and if yes, what is it?**

**(1) If lim exists, what is it?** *Suppose* $s = \lim s_n$. Pass to the limit in $s_{n+1} = \frac{s_n^2+5}{2s_n}$ to obtain $s = \frac{s^2+5}{2s} \Rightarrow 2s^2 = s^2 + 5 \Rightarrow s^2 = 5 \Rightarrow s = \pm\sqrt{5}$.
$s_n > 0$ for any $n$, hence $s \geqslant 0$, so $s = \sqrt{5}$.

**(2) Does $(s_n)$ converge?** $s_0 = 5$, $s_1 = 3$, $s_2 = \frac{7}{3}$, ...
*We guess:* $(s_n)$ is decreasing to $\sqrt{5}$, hence bounded. This will imply convergence.
Show: for $n \geqslant 0$, $\sqrt{5} \leqslant s_{n+1} \leqslant s_n$.
We show: if $n \geqslant 0$ and $s_n \geqslant \sqrt{5}$, then $\sqrt{5} \leqslant s_{n+1} \leqslant s_n$.

## Example continued. $s_0 = 5$, $s_{n+1} = \frac{s_n^2+5}{2s_n}$ for $n \geqslant 0$

We show: if $n \geqslant 0$ and $s_n \geqslant \sqrt{5}$, then $\sqrt{5} \leqslant s_{n+1} = \frac{s_n^2+5}{2s_n} \leqslant s_n$.

1) Right hand inequality. Want $\frac{s_n^2+5}{2s_n} \leqslant s_n$
$\Leftrightarrow s_n^2 + 5 \leqslant 2s_n^2 \Leftrightarrow 5 \leqslant s_n^2$ , which is true.

2) Left hand inequality. Want $\frac{s_n^2+5}{2s_n} \geqslant \sqrt{5} \Leftrightarrow s_n^2 + 5 \geqslant 2s_n\sqrt{5}$.

> Arithmetic-Geometric Means (AGM) Inequality: for $a, b \in [0, \infty)$, $\frac{a+b}{2} \geqslant \sqrt{ab}$.

Proof of AGM: $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a + b - 2\sqrt{a}\sqrt{b}\right)^2 = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 \geqslant 0$.
Apply AGM with $a = s_n^2, b = 5$ to get $s_n^2 + 5 \geqslant 2s_n\sqrt{5}$. $\blacksquare$

# Optional: Newtons' method for finding zeros (p. 259)

Need to find (numerically) $x_*$ s.t. $f(x_*) = 0$. Use iterative process.
Find $(x_n)$ converging to $x_*$ (starting from a certain $x_0$).
Approximate $f$ by its tangent line at $(x_n, f(x_n))$:
$y = f'(x_n)(x - x_n) + f(x_n)$. $x_{n+1}$ is the $x$-intercept of this tangent line.

$0 = f'(x_n)(x_{n+1} - x_n) + f(x_n)$, hence $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$.

Do we have $\lim_n x_n = x_*$? Can we estimate $|x_n - x_*|$ (so that we would know when to stop)? Most often, yes!

If $C > 0$, then $\sqrt{C}$ is the positive root of $f(x) = x^2 - C$. Then

$x_{n+1} = x_n - \frac{x_n^2 - C}{2x_n} = \frac{2x_n^2 - (x_n^2 - C)}{2x_n} = \frac{x_n^2 + C}{2x_n}$. $\boxed{x_{n+1} = \dfrac{x_n^2 + C}{2x_n}}$.

# Decimal expansions of positive numbers (10.3)

A decimal expansion $K.d_1d_2d_3\dots$ ($K \in \{0, 1, 2, \dots\}$, $d_1, d_2, \dots \in \{0, \dots, 9\}$) determines a non-negative real number.
Let $s_n = K.d_1d_2d_3\dots d_n = K + \sum_{k=1}^{n} \frac{d_k}{10^k}$.
We claim that $\lim s_n$ is a real number.
The sequence $(s_n)$ is increasing. Need to show boundedness:
$s_n \leqslant K + \sum_{k=1}^{n} \frac{9}{10^k} = K + \frac{9}{10} \cdot \frac{1-1/10^n}{1-1/10} = K + 1 - \frac{1}{10^n} < K + 1.$

**Note**. §16 shows that any real number has a decimal expansion.

**However**, some real numbers have two decimal expansions.
$1 = 1.000\dots = 0.999\dots$, since $\lim_n \sum_{k=1}^{n} \frac{9}{10^k} = 1$.

# Unbounded monotone sequences

**Theorem (Theorem 10.4)**

*If $(s_n)_{n \geqslant m}$ is an unbounded increasing (decreasing) sequence, then $\lim s_n = +\infty$ (resp.$= -\infty$).*

**Proof when $(s_n)_{n \geqslant m}$ is unbounded increasing.**

Need to show: $\forall\, A \in \mathbb{R}\ \exists\, N \in \mathbb{R}$ s.t. $s_n > A$ whenever $n > N$.
The sequence $(s_n)$ is bounded below by $s_m$, hence it is not bounded above.
In particular, $A$ is not an upper bound.
$\exists\, N \in \{m, m+1., \ldots\}$ s.t. $s_N > A$. We claim that this $N$ works:
if $n > N$, then $s_n \geqslant s_N > A$. ■

**Corollary (Corollary 10.5)**

*If $(s_n)_{n \geqslant m}$ is a monotone sequence, then $\lim s_n$ exists, and is equal to a real number, or to $\pm\infty$.*

# Example

**Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Does $(s_n)$ converge?**

$(s_n)$ is increasing: for any $n$, $s_{n+1} = s_n + \frac{1}{n+1}$. Is $(s_n)$ bounded? **No!**

$$s_{2^m} = \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{m-1}+1} + \ldots + \frac{1}{2^m}\right)$$
$$= \frac{1}{1} + \frac{1}{2} + \sum_{j=1}^{m-1} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \geqslant \frac{3}{2} + \frac{1}{2}(m-1) = \frac{m+2}{2}.$$

The sequence $(s_n)$ is unbounded, hence it **diverges to** $+\infty$.

**Optional:** $s_n$ "behaves like" $\ln n$. $\lim \frac{s_n}{\ln n} = 1$, and $\lim(s_n - \ln n) = \gamma$ – the Euler-Mascheroni constant. $\gamma = 0.577\ldots$.

**Open question: is $\gamma$ rational?**

# Definition of $\liminf$ and $\limsup$

**Convention:** if a non-void set $A \subset \mathbb{R}$ is not bounded above (below), set $\sup A = +\infty$ (resp. $\inf A = -\infty$).

For a sequence $(s_n)$ and $N \in \mathbb{N}$, define $u_N = \sup\{s_n : n > N\}$ (also denoted by $\sup_{n>N} s_n$).
If $(s_n)$ is not bounded above, then $u_N = +\infty$ for any $N$.
If $(s_n)$ is bounded above, then $(u_N)$ is a decreasing sequence of real numbers, hence it has a limit (in $\mathbb{R} \cup \{-\infty\}$).

**Definition (10.6 – $\limsup$ of the sequence $(s_n)$)**

$\limsup s_n = \lim u_N$ (this equals $+\infty$ if $\forall N$, $u_N = +\infty$).

Similarly, let $v_N = \inf\{s_n : n > N\}$ (also denoted by $\inf_{n>N} s_n$).
Either $v_N = -\infty$ for any $N$ (if $(s_n)$ is not bounded below), or $(v_N)$ is an increasing sequence of real numbers.
Define $\liminf s_n = \lim v_N$.

# More on $\liminf$ and $\limsup$

**Observation:** $\boxed{\limsup s_n \geqslant \liminf s_n}$.
Indeed, $u_N = \sup\{s_n : n > N\} \geqslant \inf\{s_n : n > N\} = v_N$, hence $\limsup s_n = \lim u_N \geqslant \lim v_N = \liminf s_n$.

**Example:** $s_n = \begin{cases} 1/n & n \textbf{ even} \\ -n & n \textbf{ odd} \end{cases}$. **Find** $\limsup s_n$, $\liminf s_n$.

**(1)** $\limsup s_n = \lim_N u_N$, where $u_N = \sup_{n>N} s_n$.
$u_1 = \sup_{n>1} s_n = \sup\{\frac{1}{2}, -3, \frac{1}{4}, -5, \ldots\} = \frac{1}{2}$,
$u_2 = \sup\{-3, \frac{1}{4}, -5, \ldots\} = \frac{1}{4}$, $u_3 = \sup\{\frac{1}{4}, -5, \ldots\} = \frac{1}{4}$, ...
In general, for $k \in \mathbb{N}$, $u_{2k} = u_{2k+1} = \frac{1}{2k+2}$. $0 < u_n < \frac{1}{n}$, hence, by Squeeze Theorem, $\lim u_N = 0$. Thus, $\boldsymbol{\limsup s_n = 0}$.

**(2)** $\liminf s_n = \lim_N v_N$, where $v_N = \inf_{n>N} s_n$.
$(s_n)$ is not bounded below, hence $v_N = -\infty$ for any $N$, and so, $\boldsymbol{\liminf s_n = -\infty}$.

# lim inf and lim sup vs. lim

**Theorem (Theorem 10.7)**

*(1) If* $\lim s_n$ *is defined, then* $\liminf s_n = \lim s_n = \limsup s_n$.
*(2) If* $\liminf s_n = s = \limsup s_n$*, then* $\lim s_n$ *is defined, and* $= s$.

**Proof of (1).** Recall: $u_N = \sup_{n>N} s_n$, $v_N = \inf_{n>N} s_n$, $\limsup s_n = \lim u_N$, $\liminf s_n = \lim v_N$.

• $\lim s_n = +\infty$. Need: $\lim u_N = +\infty = \lim v_N$ – that is, $\forall A \in \mathbb{R}\ \exists K \in \mathbb{N}$ s.t. $u_N, v_N \geqslant A$ for $N > K$.

$\forall A \in \mathbb{R}\ \exists K \in \mathbb{N}$ s.t. $s_n > A$ for $n > K$. For $N > K$, $u_N = \sup_{n>N} s_n > A$, and $v_N = \inf_{n>N} s_n \geqslant A$ (indeed, $A$ is a lower bound for $\{s_n : n > N\}$). $\square$

• $\lim s_n = -\infty$: handled similarly. $\square$