MATH 447: Real Variables - Homework #4

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Problem 1 (13.3(a))**.** Let B be the set of all bounded sequences $\mathbf{x} = (x_1, x_2, \ldots)$ and define $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$.

1. Show d is a metric for B.

Solution 1. We will show that d is a metric for B by showing that it satisfies the symmetry, non-degeneracy, and triangle inequality with respect to B, i.e. the set of all bounded sequences.

Proof.

1. Symmetry: We wish to show that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.

$$d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\} \tag{1}$$

$$d(\mathbf{x}, \mathbf{y}) = \sup\{|y_j - x_j| : j = 1, 2, \ldots\} \tag{2}$$

$$\tag{3}$$

From the properties of absolute value, we know that $|a - b| = |b - a|$. Setting $a = x_j$ and $b = y_j$, we can state:

$$|x_j - y_j| = |y_j - x_j| \tag{4}$$

2. Non-degeneracy We wish to show that $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\} = 0$ iff $\mathbf{x} = \mathbf{y}$.

$$|x_j - y_j| = 0 \iff x_j = y_j \tag{5}$$

$$x_j - y_j = 0 \tag{6}$$

3. Triangle Inequality We wish to show that $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

$$\text{RHS} = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{zy}) \tag{7}$$

$$= \sup\{|x_j - y_j| : j = 1, 2, \ldots\} + \sup\{|y_j - z_j| : j = 1, 2, \ldots\} \tag{8}$$

$$\geq \sup\{|x_j - y_j| + |y_j - z_j| : j = 1, 2, \ldots\} \tag{9}$$

$$\geq \sup\{(x_j - y_j) + (y_j - z_j)\} \tag{10}$$

$$= \sup\{|x_j - z_j|\} = \text{LHS} \tag{11}$$

Lines 8 and 9 can be shown in Homework 3 Problem 4. $\square$

Problem 2 (13.4)**.** Prove (iii) and (iv) in Discussion 13.7

1. The union of *any* collection of open sets is open.
2. The intersection of *finitely many* open sets is again an open set.

Solution 2.

Proof. Suppose we have a collection of open sets, $\{U_k\}$. Then if $x \in \{U_k\}$, then $x \in \cup_k \{U_k\}$. Because $\{U_k\}$ is a collection of open sets, $\cup_k \{U_k\}$ is also itself an open set. □

Proof. Suppose we have a collection of finitely open sets, $\{U_k\}$. Then, for each element x in the intersection $\cap_k U_k$, there exists a neighborhood N_k about the element x for each open set $U_{k=1,2,\ldots k}$ such that $N_k \subset U_k, k = 1, 2, \ldots k$. For each N_k, there exists a radius r_k. We can choose the smallest radius out of all neighborhoods N_k such that $r_{\text{min}} = \min\{N_1, N_2, \ldots N_k\}$. Then, this neighborhood is a subset of all open sets U_k, $N_{k_{\text{min}(r)}} \subset \{U_k\}$. Therefore, the intersection $\cap_k U_k$ is open for a finite collection of open sets. □

Problem 3 (13.10(a))**.** Show that the interior of each of the following sets is the empty set.

1. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Solution 3.

Proof. The definition of the interior point of a set E is a point that contains at least one neighborhood such that $N \subset E$. The interior of a set E is a set that contains only interior points. A set E is called *open* if the set E is equal to its interior. Therefore, to show that the interior of the set $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is the empty set, we must show that there exists a point in E such that its neighborhood contains elements that are included in the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. We will show that for any point in E, there is no neighborhood of s_o such that $B_r^o(s_o) \subset E$ for all r. We will first show that between any two rational numbers, there exists an irrational number.

$$0 < \frac{1}{\sqrt{2}} < 1 \tag{12}$$

$$r_1 + 0 < r_1 + \frac{1}{\sqrt{2}}(r_1 - r_2) < r_1 + (r_2 - r_1) \tag{13}$$

$$r_1 < r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1) < r_2 \tag{14}$$

$$\tag{15}$$

Suppose that the radius of any neighborhood about any point in E is a rational number. Then, by above, there exists members of $B_r^o(s_o)$ that are irrational, and $\notin E$, as E contains only rational numbers. Now suppose that we set our radius r to be an irrational number. We know that between any rational and irrational number exists an irrational number. Let A be a rational number, and C to be an irrational number. Then, $B = \frac{A+C}{2}$ is irrational, and $A < B < C$. Therefore, we have shown that there is no value of radius r that satisfies $B_r^o(s_o) \subset E$ for any rational point s_o in E. e □

Problem 4 (13.12(b))**.** Let (S, d) be any metric space.

1. Show that the finite union of compact sets in S is compact.

Solution 4. *Proof.* To show that the finite union of compact sets in S is compact, we will use the property of compactness of each set U_k in the finite union $\cup_k \{U_k\}$. If a set U_k is compact, then there exists a finite subcover for every open cover of U_k. Then, there exists $\cup_k U_k$ for finitely many k. □

Problem 5 (A)**.** Show that a sequence in a metric space (S, d) cannot have more than one limit.

Solution 5. We will show that if a sequence s_n converges to L_1 and L_2, $\forall \varepsilon > 0, \exists N_1 \ s.t. \ m, n > N \implies |s_n - L_1| < \frac{\varepsilon}{2}$ and $\exists N_2 \ s.t. \ n > N_2 \implies |s_n - L_2| < \frac{\varepsilon}{2}$. Choose $N_{\max} = \max(N_1, N_2)$. Then, $n > N_{\max} \implies$

$$d\left(s_n - L_1 + (L_2 - s_n)\right) < d\left(\frac{\varepsilon}{2}\right) + d\left(\frac{\varepsilon}{2}\right) < \varepsilon \tag{16}$$

$$= d\left(\frac{\varepsilon}{2}\right) < \varepsilon \tag{17}$$

This implies that $L_1 = L_2$, so there cannot be more than one limit in (S, d).

Problem 6 (B)**.**
1. Suppose a sequence s_n in a metric space (S, d) converges to $s \in S$. Prove that any subsequence of s_n converges to s as well.

Solution 6. Let s_n be a sequence in S. By the given, $\forall \varepsilon > 0, \exists N \ s.t. \ n > N \implies d(s_n - s)$. Let s_{n_k} be a subsequence of s_n. We know that $n_k > k$ for all k. Now let $s = \lim_{n\to\infty} s_n$ and let $\varepsilon > 0$. There exists N such that $n > N$ implies $d(s_n - s) < \varepsilon$. Now $k > N$ implies $n_k > N$, implying $d(s_{n_k} - s) < \varepsilon$. Therefore,

$$\lim_{k\to\infty} s_{n_k} = s \tag{18}$$

Problem 7 (C)**.** Suppose (S, d) is a complete metric space, and $E \subset S$. We can view E as a metric space, equipped with the metric inherited from S. Prove that E is complete iff it is a closed subset of S.

Solution 7. $\implies$: Suppose that E is not complete. Then, E does not contain its limit points. By definition, a closed set is a set that contains all of its limit points. Therefore, E cannot be closed. $\impliedby$: Suppose that E is not a closed subset of S. Then, E does not contain all of its limit points. Suppose that E is complete. Then, every Cauchy sequence in E converges to some limit point in E.

Problem 8 (D)**.** Suppose s_n is a Cauchy sequence in a metric space (S, d) which has a convergent subsequence. Is it true that the sequence s_n itself converges?

Solution 8. We know that the following holds:

$$\forall \varepsilon > 0, \exists N \ s.t. \ m, n > N \implies d(s_m - s_n) < \varepsilon \tag{19}$$

Want: If s_n has a convergent subsequence, then $L \in S$, $d(s_{n_k} - L) < \varepsilon \implies d(s_n - L) < \varepsilon$. Choose N such that $n > N \implies d(s_{n_k} - L) < \frac{\varepsilon}{2}$. Choose N_1 such that $d(s_{n_{k+1}} - s_{n_k}) < \frac{\varepsilon}{2}$ Then we get:

$$d(s_{n_{k+1}} - s_{n_k} + s_{n_k} - L) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \tag{20}$$

$$d(s_{n_{k+1}} - L) < \varepsilon \tag{21}$$

Problem 9 (Bonus Problem)**.** Is the metric space (B, d) defined in Problem 13.3(a) complete?

Solution 9.

Proposition. If B is the set of all bounded sequences $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, endowed with the distance function $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i| : i = 1, 2, \ldots n\}$, then the metric space (B, d) is complete.

Proof. We wish to find a proof that (B, d) such that B is the set of all bounded sequences, and d is the distance defined as:

$$d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i| : i = 1, 2, \ldots n\} \tag{22}$$

The idea is to treat the set B of sequences as a set of infinite dimensional vectors in $\mathbb{R}$. We will utilize and prove the following lemma:

Lemma 1. Consider a sequence $(\mathbf{x}^k)_k$ in $\mathbb{R}^n$, with $\mathbf{x}^k = (x_i^k)_{i=1}^n$

1. $(\mathbf{x}^{(k)})_k$ is Cauchy iff $(\mathbf{x}^k)_k$ is Cauchy for $1 \leq i \leq n$
2. $(\mathbf{x}^{(k)})_k$ converges to $\mathbf{x} = (x_i)_{i=1}^n$ iff $\lim_{k\to\infty} x_i^{(k)}$ for $1 \leq i \leq n$

$\implies$: To prove part one of the lemma above, we will suppose that $(\mathbf{x}^k)_k$ is Cauchy. Fix i. Want: $(x_i^{(k)})_k$ is Cauchy. For $\varepsilon > 0$ we need to find N *s.t.* $\left|x_i^{(k)} - x_i^{(m)}\right| < \varepsilon$ for $k, m > N$. We need to find an N *s.t.* $d(\mathbf{x}^k, \mathbf{x}^m) < \varepsilon$ for all $k, m > N$. $\varepsilon > d(\mathbf{x}^{(k)}, \mathbf{x}^{(m)}) = \sup\{|x_i - y_i| : i = 1, 2, \ldots, n\} \geq |x_i - y_i|$. Therefore, this N works in the forward direction. $\impliedby$: Suppose $(x_i^{(k)})_k$ is Cauchy $\forall\ i$. We want to find: $(\mathbf{x}^{(k)})_k$ is Cauchy. We fix $\varepsilon > 0$. We need to find N *s.t.* $d(\mathbf{x}^k, \mathbf{x}^m) < \varepsilon$ for $k, m > N$. For $1 \leq i \leq n$ find $N_i \in \mathbb{N}$ *s.t.* $\left|x_i^{(k)} - x_i^{(m)}\right| < \varepsilon$ for $k, m > N_i$. Then, we know that:

$$x_i^{(k)} < \varepsilon + x_i^{(m)} \tag{23}$$

$$x_i^{(m)} < \varepsilon + x_i^{(k)} \tag{24}$$

Therefore, by the properties of the least upper bound, we have the following:

$$\sup x_i^{(k)} \leq \varepsilon + x_i^{(m)} \tag{25}$$

$$\sup x_i^{(m)} \leq \varepsilon + x_i^{(k)} \tag{26}$$

$$\tag{27}$$

We can prove a quick fact proposition about differences of supremums of sequences:

Proposition.

$$\sup(s_n - t_n) \le \sup s_n - t_n$$

Proof.

$$s_n \le \sup s_n \tag{28}$$

$$t_n \le \sup t_n \tag{29}$$

$$s_n - t_n \le \sup s_n - \sup t_n \tag{30}$$

$$\sup(s_n - t_n) \le \sup s_n - \sup t_n \tag{31}$$

$\square$

Therefore, we can state the following:

$$\sup(x_i^{(k)} - x_i^{(m)}) \le \sup x_i^{(k)} - \sup x_i^{(m)} \tag{32}$$

$$\le (x_i^{(m)} + \varepsilon) - (x_i^{(k)} + \varepsilon) \tag{33}$$

We know from the implication that the following is true:

$$\left| x_i^{(k)} - x_i^{(m)} \right| < \varepsilon \tag{34}$$

$$\sup(x_i^{(k)} - x_i^{(m)}) < \varepsilon \tag{35}$$

Therefore, we can say the following:

$$\sup\left(\left| x_i^{(k)} - x_i^{(m)} \right|\right) < \varepsilon \tag{36}$$

To show the second part of Lemma 1, we can use similar reasoning as the first part of Lemma 1 to show that the limit exists B. $\square$