TAM 470 / CSE 450
Homework 6

Problem 1 (10 points, PL)

Recall the linearized damped pendulum equation:

$$\theta''(t) + c\theta'(t) + \frac{g}{l}\theta(t) = 0 \tag{1}$$

With specified initial conditions $(\theta(0), \theta'(0)) = (\theta_0, \omega_0)$.

Go to PrairieLearn to write functions that solve Equation 1 using:

(**a**) RK4 (fourth-order Runge-Kutta explicit)

(**b**) Leapfrog (use the RK2 explicit scheme with $\alpha = \frac{1}{2}$ for the first time step)

(**c**) AB2 (second-order Adams-Bashforth explicit; use the RK2 explicit scheme with $\alpha = \frac{1}{2}$ for the first time step).

Problem 2 (10 points)

Using the functions you wrote in Problem 1, use all three numerical schemes to solve Equation 1 for $t \in [0, 6]$ seconds, with parameters $c = 4\ \text{sec}^{-1}$, $g = 9.81\ \text{m/sec}^2$, and $l = 0.6$ m, and initial conditions $(\theta_0, \omega_0) = (0.1, 0)$. Note that θ is in radians. For each scheme, try $h = 0.1$, $h = 0.05$, $h = 0.01$. For each of these 9 total simulations, plot the numerical solution for $\theta(t)$ against the exact solution (valid for $c < 2\sqrt{\frac{g}{l}}$):

$$\theta(t) = e^{-\xi t}\left[\theta_0\left(\cos\omega t + \frac{\xi}{\omega}\sin\omega t\right) + \frac{\omega_0}{\omega}\sin\omega t\right] \tag{2}$$

where $\xi = \frac{c}{2}$ and $\omega = \sqrt{\left|\frac{c^2}{4} - \frac{g}{l}\right|}$.

Discuss the results in terms of what you know about the stability and accuracy of these schemes (in the discussion of stability, you should be sure to demonstrate whether the scheme is stable for the time steps under consideration).

Problem 3 (10 points)

Derive the leading error term of the local (one-step) error for the AB2 method:

$$y_{n+1} = y_n + \frac{3h}{2} f(y_n, t_n) - \frac{h}{2} f(y_{n-1}, t_{n-1}) \tag{3}$$

The error term should be expressed in terms of h and derivatives of $y(t)$. State the order of the local and global error for this method.

Problem 4 (10 points)

Consider the BDF2 method below:

$$y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2}{3} h f(y_{n+1}, t_{n+1}) \tag{4}$$

Find the amplification factors $\sigma_1(z)$ and $\sigma_2(z)$ where $z = h\lambda$ (i.e. the roots of the equation associated with linear stability analysis) associated with the BDF2 method. Identify the spurious root by computing the values of each amplification factor in the limit $h \to 0$ (the spurious root is the one that does <u>not</u> have an amplification factor of 1 as $h \to 0$).

Problem 5 (10 points)

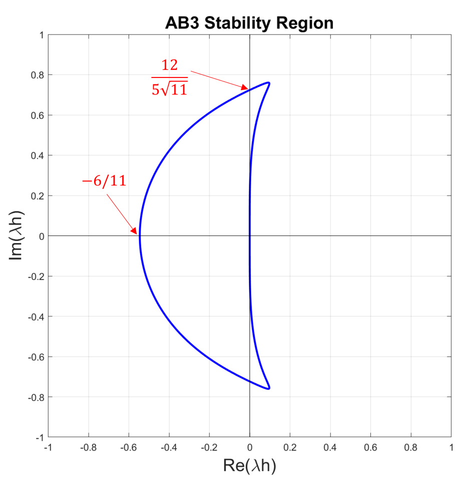
The diagram below is the linear stability diagram for the AB3 method. Stable values of λh lie inside (or on the boundary of) the region enclosed by the blue curve. The arrows and accompanying numbers indicate the locations where the blue curve intercepts the real and imaginary axes.

For the values of λ indicated below, use the figure to calculate (estimate if needed) the maximum allowable time step h for stability of the AB3 scheme.

(**a**) (4 pts) $\lambda = -10$

(**b**) (4 pts) $\lambda = 4i$

(**c**) (2 pts) $\lambda = -1 + 2i$



Problem 6: 4 credit-hour students only (10 points)

(a) (3 pts) Derive the linearized backward Euler scheme, using the same methodology as the linearized trapezoid scheme discussed in lecture and the Moin textbook. Write the update rule assuming a scalar ODE, i.e. write $y_{n+1} = ...$

(b) (5 pts, PL) Go to PrairieLearn to implement a function that uses the linearized backwards Euler scheme to solve

$$y' = e^{\sin(y)} - ty, \qquad y(0) = y_0 \tag{5}$$

(c) (2 pts) Use your code with initial condition $y(0) = 1$ to create a plot of the solution for $h = 0.2$ and $h = 0.05$ for $t \in [0, 10]$.