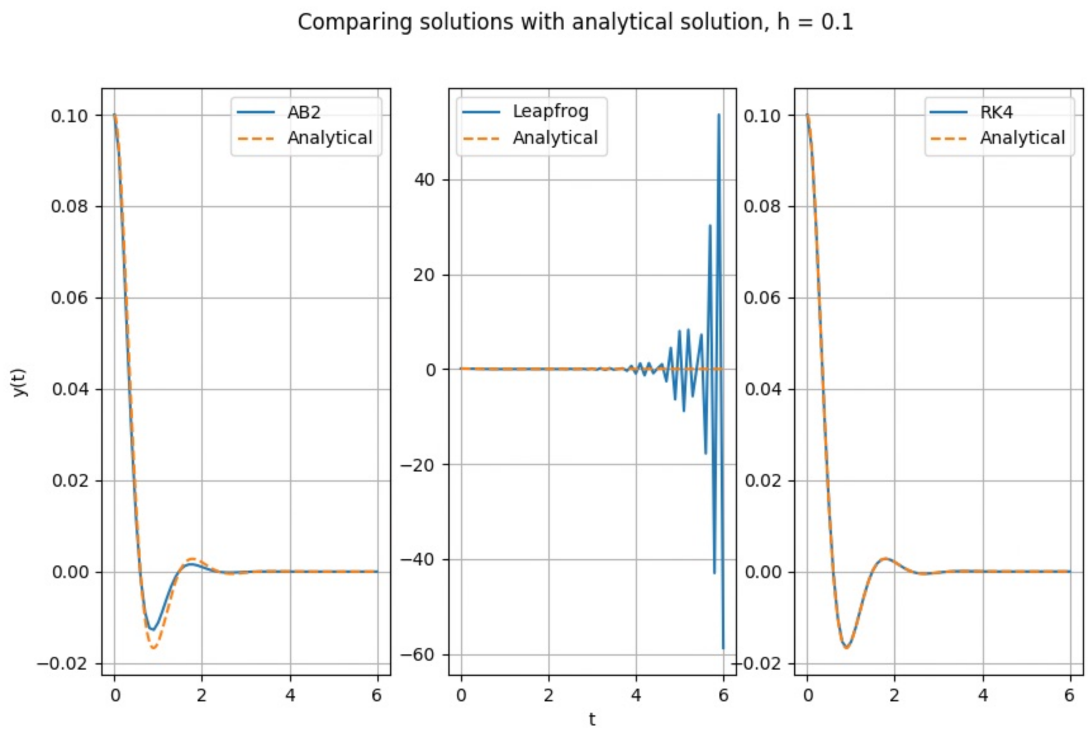
# TAM 470 HW #6 Solutions

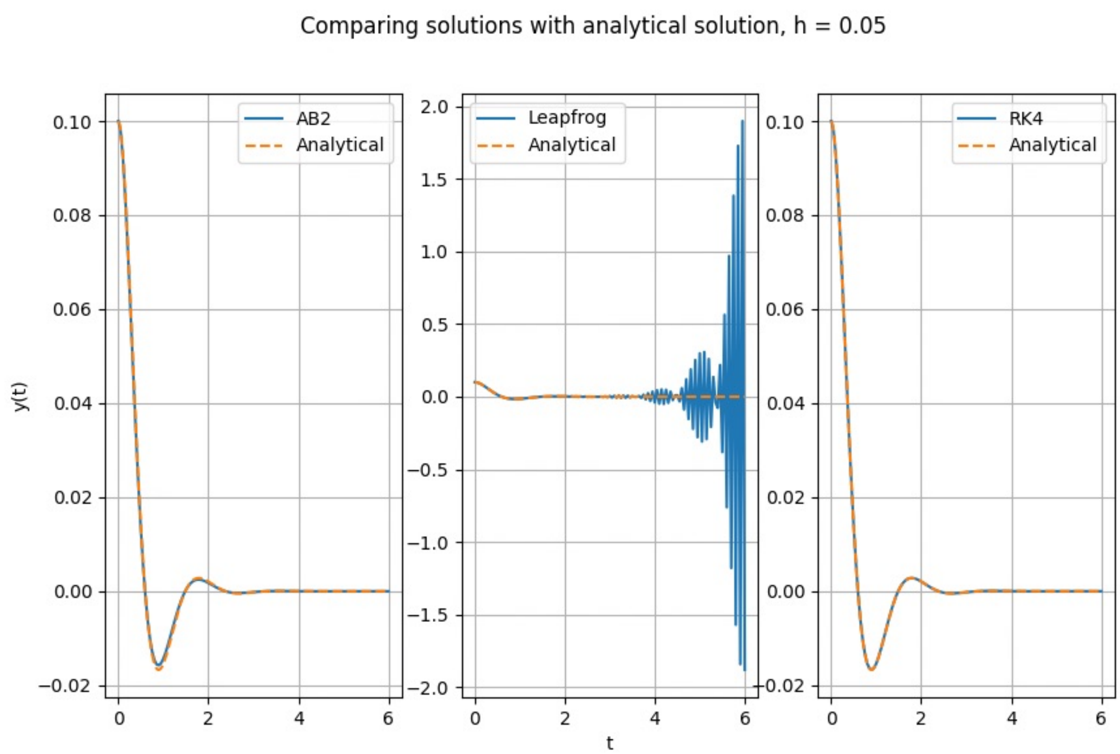
## Problem 2




$h = 0.1$, AB2 $\rightarrow$ Slightly inaccurate at the minima but stable

Leapfrog $\rightarrow$ Unstable

RK4 $\rightarrow$ Stable and accurate

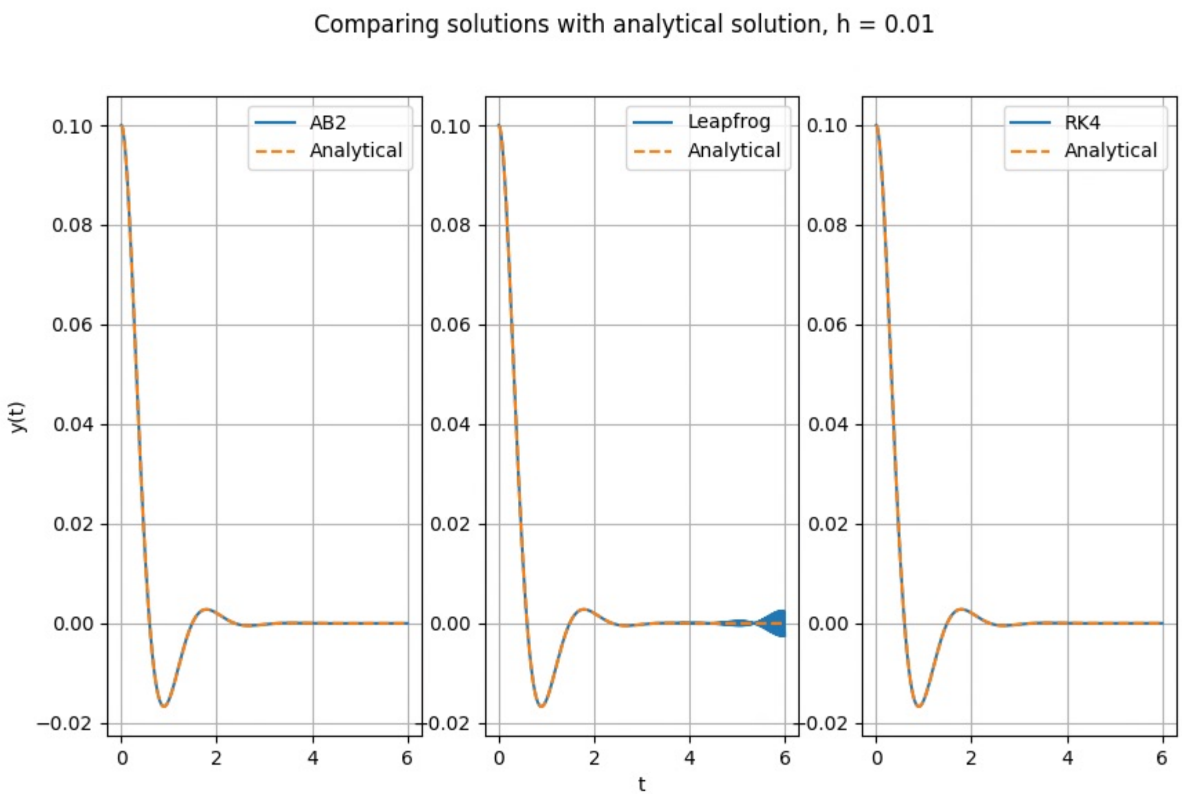


h = 0.05

AB2 → Stable and better accuracy than previous.

Leapfrog → Still unstable but the errors are lesser than previous.

RK4 → Stable and accurate




h = 0.01, AB2 → Stable and accurate

Leapfrog → Good approximation upto around t = 4, starts oscillating after that and starts becoming unstable.

RK4 → Stable and accurate.

## Problem 3

$$y_{n+1} = y_n + \frac{3h}{2} f(y_n, t_n) - \frac{h}{2} f(y_{n-1}, t_{n-1})$$

We know that $y'_n = f(y_n, t_n)$, $y'_{n-1} = f(y_{n-1}, t_{n-1})$

Let $\tau = y_{n+1} - y_n - \frac{3h}{2} y'_n + \frac{h}{2} y'_{n-1} + O(?)$

Taylor expansion,

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \ldots \quad \text{---} \quad (3.1)$$

$$y_{n-1} = y_n - h y'_n + \frac{h^2}{2} y''_n - \frac{h^3}{6} y'''_n + \ldots$$

$$y'_{n-1} = y'_n - h y''_n + \frac{h^2}{2} y'''_n - \frac{h^3}{6} y^{iv}_n + \ldots \quad \text{---} \quad (3.2)$$

Using (3.1) and (3.2) in original equation for $\tau$,

$$\tau = \left(\cancel{y_n} + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \ldots\right) - \cancel{y_n} - \frac{3h}{2} y'_n$$

$$+ \frac{h}{2}\left(y'_n - h y''_n + \frac{h^2}{2} y'''_n + \ldots\right)$$

$$= \cancel{\left(h - \frac{3h}{2} + \frac{h}{2}\right)} y'_n + \cancel{\left(\frac{h^2}{2} - \frac{h^2}{2}\right)} y''_n + \left(\frac{h^3}{6} + \frac{h^3}{4}\right) y'''_n + \ldots$$

$$\therefore \tau = \frac{5h^3}{12} y'''_n + H.O.T$$

The leading error term is $\frac{5h^3}{12} y_n'''$ which is $O(h^3)$.

$\Longrightarrow$ AB2 is $O(h^3)$ locally and $O(h^2)$ globally.

## Problem 4

BDF 2:- $y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f(y_{n+1}, t_{n+1})$

Assume the model problem with $y' = f = \lambda y$.

$$\therefore \quad y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} \lambda y_{n+1}$$

$$y_{n+1} = \frac{1}{\left(1 - \frac{2h\lambda}{3}\right)} \cdot \left(\frac{4}{3} y_n - \frac{1}{3} y_{n-1}\right)$$

Let $y_n = \sigma^n y_0$

$$\therefore \quad \sigma^{n+1} = \frac{1}{1 - \frac{2h\lambda}{3}} \left(\frac{4}{3} \sigma^n - \frac{1}{3} \sigma^{n-1}\right)$$

$$\Longrightarrow \quad \sigma = \frac{1}{1 - \frac{2h\lambda}{3}} \left(\frac{4}{3} - \frac{1}{3\sigma}\right)$$

Rearranging and grouping,

$$(3 - 2h\lambda)\sigma^2 - 4\sigma + 1 = 0$$

$$\sigma = \frac{4 \pm \sqrt{16 - 4(3 - 2h\lambda)}}{2(3 - 2h\lambda)}$$

$$= \frac{4 \pm 2\sqrt{4-3+2h\lambda}}{2(3-2h\lambda)}$$

$$= \frac{2 \pm \sqrt{1+2h\lambda}}{(3-2h\lambda)}$$

$$\sigma_1 = \frac{2+\sqrt{1+2h\lambda}}{(3-2h\lambda)} \quad , \quad \sigma_2 = \frac{2-\sqrt{1+2h\lambda}}{(3-2h\lambda)}$$

$$\lim_{h\to 0} \sigma_1 = 1 \quad , \quad \underbrace{\lim_{h\to 0} \sigma_2 = \frac{1}{3}}_{\text{Spurious root!}}$$

$\therefore$ The Spurious root is $\sigma_2 = \dfrac{2-\sqrt{1+2h\lambda}}{(3-2\lambda h)}$

# Problem 5

(a) Real axis $\lambda$. $\lambda = -10$

$$|\lambda| h \leq \frac{6}{11}$$

$$\Rightarrow h \leq \frac{6}{110} \quad \text{or} \quad h \leq 0.05454$$

(b) Imaginary $\lambda$, $\lambda = 4i$

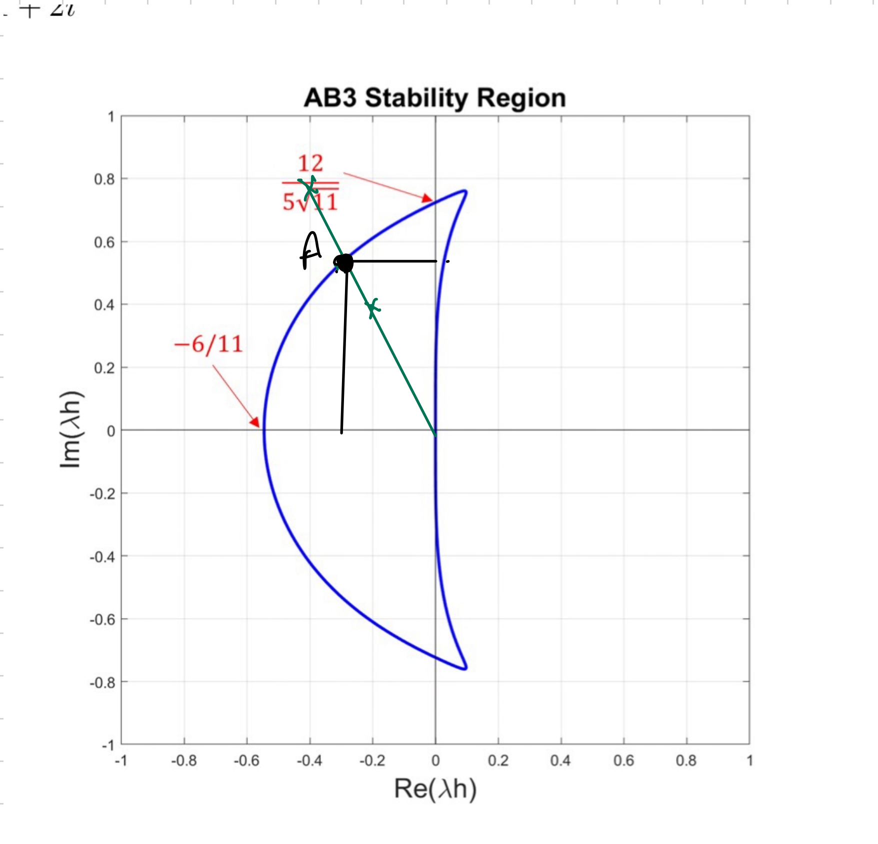
$$|\lambda| h \leq \frac{12}{5\sqrt{11}}$$

$$\Rightarrow h \leq \frac{3}{5\sqrt{11}} \quad \text{or} \quad h \leq 0.1809$$

(c) For a complex $\lambda$ with non-zero real and imaginary parts, $\lambda = -1 + 2i$

$|\lambda| = \sqrt{5}$. This can be done in many way (since we don't know the exact equation of blue curve).

(i) Reading out the values from the given graph. We can trace the value of $\lambda$ on the graph and see where it intersect the blue line.




— is the line $-1+2i$ on complex plane

Point A is intersection of line and curve. Since it is on the line, it should be a multiple of $-1+2i$. From the figure, we can approximately say that it is around $-0.27 + 0.54i$. We need h such that

$$h|\lambda| \leq |-0.27 + 0.54i|$$

$$\Rightarrow h \leq \sqrt{\frac{0.27^2 + 0.54^2}{5}}$$

$$\Rightarrow \underline{\underline{h \leq 0.27}}$$

(ii) A conservative approach without reading from graph, Since we know the real and imaginary intercepts, we can use them as

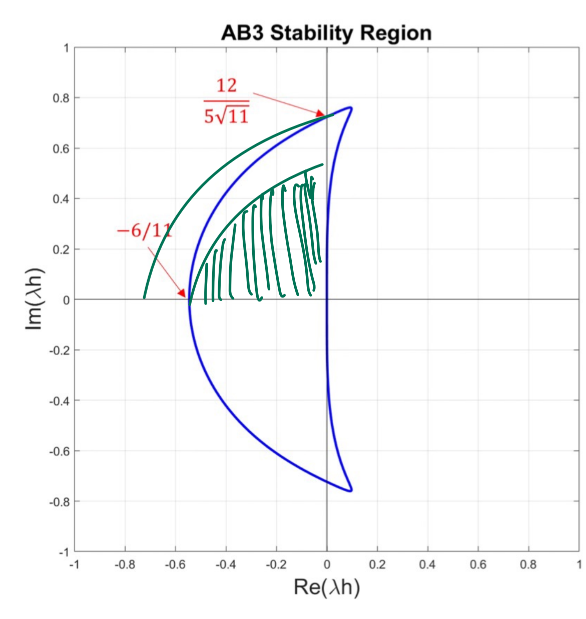
$$|\lambda| h \leq \frac{6}{11} \quad \text{and} \quad |\lambda| h \leq \frac{12}{5\sqrt{11}}$$

$$h \leq \frac{6}{11\sqrt{5}} \quad \text{and} \quad h \leq \frac{12}{5\sqrt{55}}$$

$$h \leq 0.2439 \quad \text{and} \quad h \leq 0.3236$$

Picking least $\Longrightarrow$ $\underline{\underline{h \leq 0.2439}}$.

Geometrically, we draw two circles of radius $\frac{6}{11}$ and $\frac{12}{5\sqrt{11}}$ and say that $h$ should make the quantity $|\lambda| h$ stay within inner circle.



## Problem 6

(a) Backward Euler,

$$y_{n+1} - y_n = h f(y_{n+1}, t_{n+1}) + O(h^2)$$

Taylor expand $f(y_{n+1}, t_{n+1})$ about $(y_n, t_{n+1})$

$$f(y_{n+1}, t_{n+1}) = f(y_n, t_{n+1}) + (y_{n+1} - y_n) f_y(y_n, t_{n+1}) + O(h^2)$$

Substitute back in Backward Euler equation,

$$y_{n+1} - y_n = h \left( f(y_n, t_{n+1}) + (y_{n+1} - y_n) f_y(y_n, t_{n+1}) \right) + O(h^2)$$

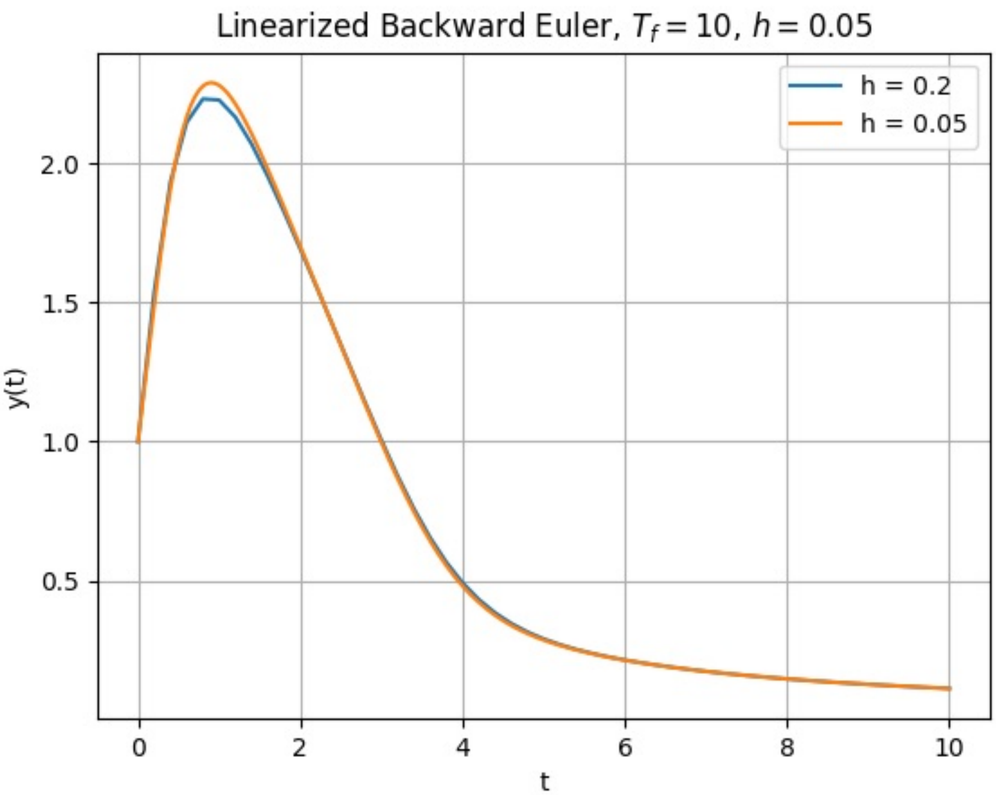
$$= h f(y_n, t_{n+1}) + h y_{n+1} f_y(y_n, t_{n+1}) - h y_n f_y(y_n, t_{n+1}) + O(h^2)$$

$$\Rightarrow \left(1 - h f_y(y_n, t_{n+1})\right) y_{n+1} = \left(1 - h f_y(y_n, t_{n+1})\right) y_n + h f(y_n, t_{n+1})$$

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{h f(y_n, t_{n+1})}{\left(1 - h f_y(y_n, t_{n+1})\right)}}$$

(b) Implement on PL

(c)



Linearized Backward Euler, $T_f = 10$, $h = 0.05$
h = 0.2
h = 0.05
y(t)
t
2.0
1.5
1.0
0.5
0
2
4
6
8
10