Uniform convergence (Section 24)

Definition (24.1-2)

Suppose $f, f_1, f_2, \ldots$ are functions from $S \subset \mathbb{R}$ to $\mathbb{R}$.
$f_n \to f$ pointwise on S if $f_n(x) \to f(x)$, $\forall x \in S$: $\forall \varepsilon > 0, x \in S$ $\exists N = N(\varepsilon, x) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \geqslant N$.
$f_n \to f$ uniformly on S if $\forall \varepsilon > 0$ $\exists N = N(\varepsilon) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \geqslant N$, $\forall x \in S$. Equivalently, $\lim_n \sup_{x \in S} |f_n(x) - f(x)| = 0$.

Uniform convergence $\Rightarrow$ pointwise convergence.
The converse is false.

Uniform convergence preserves continuity

Theorem (24.3)

Suppose $f_n \to f$ uniformly on S, and $\forall n$, f_n is continuous at $x_0 \in S$. Then f is continuous at x_0.

Proof by "$\frac{\varepsilon}{3}$ argument". Fix $\varepsilon > 0$. Need to find $\delta > 0$ s.t. $|f(x_0) - f(x)| < \varepsilon$ whenever $x \in S$, $|x - x_0| < \delta$.
Find n s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$, $\forall x \in S$. f_n is continuous at x_0, hence $\exists \delta > 0$ s.t. $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$ whenever $x \in S$, $|x - x_0| < \delta$.
This δ works for us: if $|x - x_0| < \delta$, then
$|f(x_0) - f(x)| \leqslant |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)| < 3\frac{\varepsilon}{3} = \varepsilon$.
■

More examples of convergence

$f_n(x) = \frac{x}{1+nx^2}$. Does the sequence (f_n) converge pointwise on $\mathbb{R}$? If yes, find the limit, and determine whether the convergence is uniform. (Example 7 from p. 198)

$f_n(0) = 0$ for any n. If $x \neq 0$, then $f_n(x) = \frac{x/n}{1/n+x^2}$, so $\lim_n f_n(x) = 0$.
$f_n \to f$ pointwise, where $f(x) = 0$.

$f_n \to f$ uniformly iff $\lim_n \sup_{x\in\mathbb{R}} \left|f_n(x) - f(x)\right| = 0$.

$\sup_{x\in\mathbb{R}} \left|f_n(x) - f(x)\right| = \sup_{x\in\mathbb{R}} \frac{|x|}{1+nx^2}$.
AGM Inequality: for $a, b \geqslant 0, \sqrt{ab} \leqslant \frac{a+b}{2}$.
Take $a = 1$, $b = nx^2$: $\sqrt{n}|x| \leqslant \frac{1+nx^2}{2}$, hence $\frac{|x|}{1+nx^2} \leqslant \frac{1}{2\sqrt{n}}$.
$\lim_n \frac{1}{2\sqrt{n}} = 0$, hence $\lim_n \sup_{x\in\mathbb{R}} \left|f_n(x) - f(x)\right| = 0$.

Conclusion: $f_n \to f$ uniformly ($f(x) = 0$).

More examples of convergence

$f_n(x) = n^2 x^n (1-x)$. Does the sequence (f_n) converge pointwise on $[0,1]$? If yes, find the limit, and determine whether the convergence is uniform. (Example 8 from p. 198)

$f_n(0) = f_n(1) = 0$. Recall Exercise 9.12: if $\lim_n \left|\frac{s_{n+1}}{s_n}\right| < 1$, then $\lim s_n = 0$. Take $s_n = f_n(x) = n^2 x^n (1-x)$ $(0 < x < 1)$, then $\lim_n \left|\frac{s_{n+1}}{s_n}\right| = x$, hence $\lim f_n(x) = 0$.

$f_n \to f$ pointwise, where $f(x) = 0$.
$f_n \to f$ uniformly iff $\lim_n \sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = 0$.

To find $\sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = \max_{0\leqslant x\leqslant 1} f_n(x)$, differentiate:
$f_n'(x) = n^2\left(nx^{n-1} - (n+1)x^n\right) = n^2(n+1)x^{n-1}\left(x - \frac{n}{n+1}\right)$.
$f_n\left(\frac{n}{n+1}\right) = n^2\left(\frac{n}{n+1}\right)\frac{1}{n+1} = \frac{n^{n+1}}{(n+1)^n} = \frac{n^2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n$.
$\lim_n \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, hence $\lim_n \sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = +\infty$.

Conclusion: $f_n \to f$ pointwise, but not uniformly ($f(x) = 0$).

Uniformly Cauchy sequences (Section 25)

Definition (25.3)

A sequence (f_n) of functions $S \to \mathbb{R}$ is called uniformly Cauchy if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $|f_i(x) - f_j(x)| < \varepsilon \; \forall x \in S$ whenever $i, j \geqslant N$.

Equivalently: $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. for $i, j \geqslant N$, $\sup_{x \in S} |f_i(x) - f_j(x)| < \varepsilon$.

If the functions (f_i) are bounded, then (f_i) is uniformly Cauchy iff it is Cauchy in the metric space $(B(S), d)$.
Recall $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$.

Theorem (25.4)

(f_n) is uniformly Cauchy iff it converges uniformly to some f.

Corollary. $B(S)$ is a complete metric space.

Proof: uniformly Cauchy $\Leftrightarrow$ uniformly converges

Theorem 25.4. (f_n) is uniformly Cauchy iff it converges uniformly to some f.

Proof. (1) If $f_n \to f$ uniformly, then (f_n) is uniformly Cauchy. For $\varepsilon > 0$, find $N \in \mathbb{N}$ s.t., for $i \geqslant N$, $|f_i(x) - f(x)| < \frac{\varepsilon}{2}$, $\forall x \in S$. If $i, j \geqslant N$, $x \in S$, then $|f_i(x) - f_j(x)| \leqslant |f_i(x) - f(x)| + |f_j(x) - f(x)| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.

(2) Suppose (f_n) is uniformly Cauchy. Find uniform limit f.
(i) $\forall x \in S$, $(f_n(x))$ is Cauchy, hence convergent. Let $f(x) = \lim_n f_n(x)$, then $f_n \to f$ pointwise.

(ii) Show: $f_n \to f$ uniformly. Fix $\varepsilon > 0$; we need to find $N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| \leqslant \varepsilon \ \forall x \in S, \forall n \geqslant N$. Find N s.t. $|f_n(x) - f_m(x)| < \varepsilon$ $\forall x \in S$ when $n, m \geqslant N$. If $n \geqslant N$, then
$|f_n(x) - f(x)| = \lim_m |f_n(x) - f_m(x)| \leqslant \varepsilon \ \forall x \in S$. ∎

Uniform convergence for series of functions

We are given functions $g_n : S \to \mathbb{R}$. A series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges (uniformly) if the sequence $s_k(x) = \sum_{n=1}^{k} g_n(x)$ of partial sums converges (uniformly).

Theorem (25.5)

If the functions (g_n) are continuous on S, and $\sum_n g_n$ converges uniformly on S, then $\sum_n g_n$ is continuous.

Proof. If the functions (g_n) are continuous on S, then so are the partial sums s_k. $\sum_{n=1}^{\infty} g_n(x) = \lim_k s_k(x)$, and the convergence is uniform; uniform limits of continuous functions are continuous. ■

The Cauchy criterion for series

Definition (p. 205)

$\sum_n g_n(x)$ satisfies the Cauchy criterion uniformly on S if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\left|\sum_{n=i}^{j} g_n(x)\right| < \varepsilon\ \forall x \in S$ whenever $j \geqslant i > N$.

Theorem (25.6)

If $\sum_n g_n$ satisfies the Cauchy criterion uniformly on S, then the series converges uniformly.

Remark. The converse is also true.

Proof. We need to verify that the sequence of partial sums $s_k(x) = \sum_{n=1}^{k} g_n(x)$ is uniformly Cauchy: $\forall \varepsilon > 0\ \exists N$ s.t. $|s_j(x) - s_i(x)| < \varepsilon$ for any $x \in S$, whenever $j > i \geqslant N$.
N is in the definition works: if $j > i \geqslant N$, then, for $x \in S$,
$\left|s_j(x) - s_i(x)\right| = \left|\sum_{n=1}^{j} g_n(x) - \sum_{n=1}^{i} g_n(x)\right| = \left|\sum_{n=i+1}^{j} g_n(x)\right| < \varepsilon.$ ∎

Weierstrass M-test for uniform convergence

Theorem (25.7)

Suppose $M_1, M_2, \ldots \in [0, \infty)$, and $\sum_k M_k < \infty$. If $|g_k| \leqslant M_k$ on S for any k, then $\sum g_k$ converges uniformly on S.

Proof. Fix $\varepsilon > 0$. Find N s.t. $\sum_{k=i}^{j} M_k < \varepsilon$ for $j \geqslant i \geqslant N$. Then for $x \in S$, $\left|\sum_{k=i}^{j} g_k(x)\right| \leqslant \sum_{k=i}^{j} |g_k(x)| \leqslant \sum_{k=i}^{j} M_k < \varepsilon$. ∎

Corollary

A power series $\sum_k a_k x^k$ converges uniformly (to a continuous function) on $[-b, b]$, if $b < R$ ($R = \left(\limsup |a_n|^{1/n}\right)^{-1}$).

Proof. Let $M_k = |a_k| b^k$; $\limsup_k M_k^{1/k} = b \limsup |a_k|^{1/k} = b/R < 1$, so $\sum_k M_k < \infty$. Apply Weierstrass M-Test. ∎

Remark. Convergence need not be uniform on $(-R, R)$. Indeed, $\sum_{k=0}^{\infty} x^k = f(x) = \frac{1}{1-x}$ for $x \in (-1, 1)$; the convergence is not uniform, since partial sums are bounded, but f is not.