TAM 470 / CSE 450
Homework 3

Problem 1 (10 points)

Please go to the Homework 3 set on PrairieLearn to complete this question:

Write a function that implements a fourth-order Padé scheme with a third-order scheme for boundary nodes (i.e. the matrix equation (2.18) from the Moin text) to numerically calculate the derivative of a function $f(x)$ on a grid of equally-spaced points.

Problem 2 (10 points)

(a) Use your function from Problem 1 to compute the Padé approximated derivative of

$$f(x) = \cos(2x^2)$$

on the interval $[0, \pi]$ for $N = 6, 11$, and 21 grid points on this interval. Submit 3 plots, each containing the exact derivative (smoothly plotted, using e.g. 100 grid points on the interval $[0, \pi]$) and the Padeé derivative approximations at the N uniformly distributed gird points on the interval $[0, \pi]$.

(b) Generate a log-log plot of truncation error vs grid spacing h for the points $x = 0$ and $x = \frac{\pi}{2}$, using the same function and interval as discussed in part (a). Be sure compute the truncation error for enough grid spacings h to produce the expected linear relationship on a log-log plot. Estimate the slopes of the log-log plots by using `numpy.polyfit` or similar (see Jupyter notebook example from class) and comment on whether the error behavior in the log-log plots matches your expectations for this Padé scheme at both interior and boundary grid points.

Problem 3 (10 points)

Compute

$$I = \int_0^{\frac{\pi}{2}} x^{1.5} \cos(x^2)\, dx$$

using the methods requested below. Parts (a) through (d) must be done by hand; you can use Python to help with calculations if you do not use any of the numerical integration libraries:

(a) (2 pts) Trapezoid rule using 4 equal-sized panels of width $h = \frac{\pi}{8}$ on the interval $[0, \frac{\pi}{2}]$

(b) (2 pts) Midpoint rule using the midpoints of the panels described in part (a)

(c) (2 pts) Trapezoid rule with end correction using the same 4 panels

(d) (2 pts) Simpson's rule using the same 4 panels (use 2 Simpson panels of size $\Delta_{Simpson} = 2h$, where h is the grid spacing of the panels from part (a)).

(e) (2 pts) Use `scipy.integrate.quad` with default error tolerances to compute the integral. Which of your answers in parts (a) through (d) most closely matches the value computed here?

Problem 4 (10 points)

Moin Chapter 3 Problem 9. The Simpson rule including error terms has the form

$$I = \tilde{I} + c_1 h^4 + c_2 h^6 + c_3 h^8 + ...,$$

where I is the exact integral, $\tilde{I}$ is the estimate using the Simpson rule with grid spacing h, and c_1, c_2, c_3, etc are constants.

<u>Hint</u>: Use knowledge of the form of the truncation error for Simpson's method to apply Romberg integration to improve on the approximated integral value.