SOLUTIONS FOR HOMEWORK 4

13.3. **(a)** The properties of symmetry and non-degeneracy clearly hold. It remains to establish the triangle inequality: for sequences $x = (x_i)$, $y = (y_i)$, and $z = (z_i)$, we need to show that

$$\sup_i |x_i - z_i| \le \sup_i |x_i - y_i| + \sup_i |y_i - z_i|. \tag{1}$$

We clearly have $\sup_i |x_i - y_i| + \sup_i |y_i - z_i| \ge \sup_i \big(|x_i - y_i| + |y_i - z_i|\big)$. By the triangle inequality, $|x_i - y_i| + |y_i - z_i| \ge |x_i - z_i|$, hence

$$\sup_i |x_i - y_i| + \sup_i |y_i - z_i| \ge \sup_i \big(|x_i - y_i| + |y_i - z_i|\big) \ge \sup_i |x_i - z_i|,$$

establishing (1).

Bonus problem – very little partial credit given: is the metric space (B, d) defined in (a) complete?

Yes, (B, d) is complete. Indeed, suppose $(x^{(k)})_k$ is a Cauchy sequence; show it converges. Clearly, for every i, $|x_i^{(k)} - x_i^{(m)}| \le d(x^{(k)}, x^{(m)})$, hence the sequence $(x_i^{(k)})_k$ is Cauchy. By the completeness of $\mathbb{R}$, the latter sequence converges to some $x_i \in \mathbb{R}$. We have to show that $x = (x_i)$ belongs to B, and is the limit of $(x^{(k)})_k$.

(a) $x \in B$. Find N s.t. $d(x^{(k)}, x^{(m)}) < 1$ for $k, m > N$. Then for any $k > N$ and $i \in \mathbb{N}$, $|x_i^{(k)} - x_i^{(N+1)}| < 1$, hence $|x_i^{(k)}| < |x_i^{(N+1)}| + 1$. Passing to the limit, obtain that $|x_i| \le \sup_i |x_i^{(N+1)}| + 1$, hence $x \in B$.

(b) $x = \lim_k x^{(k)}$. Fix $\varepsilon > 0$, and show that that there exists K s.t. $d(x, x^{(k)}) \le \varepsilon$ for $k > K$ – that is, for $k > K$ and $i \in \mathbb{N}$, $|x_i - x_i^{(k)}| \le \varepsilon$. To this end, find K s.t. $d(x^{(k)}, x^{(m)}) < \varepsilon$ for $k, m > K$. For such k and m, $|x_i^{(k)} - x_i^{(m)}| < \varepsilon$ for any i. Passing to the limit as $m \to \infty$, we obtain $|x_i - x_i^{(k)}| \le \varepsilon$ for any $i \in \mathbb{N}$, and $k > K$. Consequently, $d(x, x^{(k)}) \le \varepsilon$, as we wanted.

Note: in contrast to $\mathbb{R}^n$, the convergence of $(x_i^{(k)})_k$ for every i **does not** imply the convergence of $(x^{(k)})_k$. As an example consider $x_i^{(k)} = \begin{cases} 1 & i = k \\ 0 & i \ne k \end{cases}$.

13.4. We prove two statements from Discussion 13.7.

(iii): A UNION OF ANY COLLECTION OF OPEN SETS IS OPEN. Suppose $\{U : U \in \mathcal{U}\}$ is a collection of open sets. Let $E = \cup_{U \in \mathcal{U}} U$, and show that E is open – that is, for any $x \in E$ there exists $r > 0$ s.t. $\mathbf{B}_r^o(x) \subset E$. As x belongs to the union of U's, it must belong to some U. This set U is open, hence $\mathbf{B}_r^o(x) \subset U$, for some $r > 0$. As $U \subset E$, we conclude that $\mathbf{B}_r^o(x) \subset E$.

(iv): AN INTERSECTION OF FINITELY MANY OPEN SETS IS OPEN. Let $E = U_1 \cap \ldots \cap U_n$, where $U_1, \ldots, U_n$ are open. If $x \in E$, then $x \in U_k$ for every k. Consequently, for $1 \le k \le n$ there exists $r_k > 0$ with $\mathbf{B}_{r_k}^o(x) \subset U_k$. Let $r = \min\{r_1, \ldots, r_n\}$, then $\mathbf{B}_r^o(x) \subset E$.

13.10. (a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}^o = \emptyset$. We need to verify that, for any $n \in \mathbb{N}$, $\frac{1}{n}$ is not an interior point – in other words, that for no $r > 0$ is $\mathbf{B}^o_r(\frac{1}{n}) = \left(\frac{1}{n} - r, \frac{1}{n} + r\right)$ contained in $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. This however, is trivially true.

13.12. (b) Suppose $E = E_1 \cup \ldots \cup E_n$, where $E_1, \ldots, E_n$ are compact sets. To prove that E is compact, we need to show that any open cover $\{U : U \in \mathcal{U}\}$ for E has a finite subcover. Note that $\{U : U \in \mathcal{U}\}$ is an open cover for E_k, for any k; thus, it has a finite subcover $\{U : U \in \mathcal{U}_k\}$ (with finite $\mathcal{U}_k \subset \mathcal{U}$). Then $\{U : U \in \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n\}$ is a finite open subcover for E.

A. Show that a sequence in a metric space (S, d) cannot have more than one limit.

Suppose, for the sake of contradiction, that a sequence (s_n) has two distinct limits, say $a, b \in S$. Let $\varepsilon = d(a, b)/3$. Find N_1, N_2 s.t. $d(s_n, a) < \varepsilon$ for $n > N_1$, and $d(s_n, s) < \varepsilon$ for $n > N_2$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then $3\varepsilon = d(a, b) \leq d(a, s_n) + d(s_n, b) < 2\varepsilon$, a contradiction!

B. (a) Suppose a sequence (s_n) in a metric space (S, d) converges to $s \in S$. Prove that any subsequence of (s_n) converges to s as well.

(b) Is a subsequence of a Cauchy sequence necessarily Cauchy?

(a) Consider a subsequence (s_{n_k}), with $n_1 < n_2 < \ldots$. We have to show that for any $\varepsilon > 0$ there exists K s.t. $d(s, s_{n_k}) < \varepsilon$ for $k > K$. To this end, find N s.t. $d(s, s_n) < \varepsilon$ for $n > N$. Any K with $n_K > N$ will work for us.

(b) Yes, a subsequence of a Cauchy sequence must be Cauchy. The proof is similar to (a). Suppose $n_1 < n_2 < \ldots$. We have to show that for any $\varepsilon > 0$ there exists K s.t. $d(s_{n_k}, s_{n_\ell}) < \varepsilon$ for $k, \ell > K$. To this end, find N s.t. $d(s_m, s_n) < \varepsilon$ for $n, m > N$. Any K with $n_K > N$ will work for us.

C. Suppose (S, d) is a complete metric space, and $E \subset S$. We can view E as a metric space, equipped with the metric inherited from S. Prove that E is complete iff it is a closed subset of S.

(i) Suppose E is a closed subset of S, and show that any Cauchy sequence $(s_n) \subset E$ has a limit in E. Indeed, by the completeness of S, (s_n) converges to some $s \in S$. By Proposition 13.9, $s \in E$.

(ii) Conversely, suppose (E, d) is complete. According to Proposition 13.9, we have to show that, whenever $(s_n) \subset E$ has a limit $s \in S$, then actually $s \in E$. As (s_n) has a limit in S, it is Cauchy, hence, by the completeness of (E, d), it must converge to a limit in E. By the uniqueness of the limit, s must belong to E.

Note that (i) uses the completeness of S, but (ii) does not.

D. Suppose (s_n) is a Cauchy sequence in a metric space (S, d), which has a convergent subsequence. Is it true that the sequence (s_n) itself converges?

Yes, the sequence (s_n) does converge. Indeed, find $n_1 < n_2 < \ldots$ so that $\lim_k s_{n_k} = s$, and show that $\lim_n s_n = s$ as well. Fix $\varepsilon > 0$. We need to find N s.t. $d(s_n s) < \varepsilon$ for $n > N$.

To this end, find N s.t. $d(s_n, s_m) < \varepsilon/2$ for $n, m > N$. To show that this N works for us, find K s.t. $d(s_{n_k}, s), \varepsilon/2$ for $k > K$. Now pick $k > K$ with $n_k > N$. Then, for $n > N$, $d(s_n, s) \leq d(s_n, s_{n_k}) + d(s_{n_k}, s) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.