Compactness

Definition (13.11)

Suppose $E \subset S$. A family $\mathcal{U}$ of open sets is an open cover for E is $E \subset \cup_{U \in \mathcal{U}} U$. A subcover is a subfamily of $\mathcal{U}$ which is also an open cover. E is called compact if any open cover has finite subcover.

Note. A cover $\mathcal{U}$ is a collection of sets, not their union. In other words, a cover is a subset not of S, but of $\mathcal{P}(S)$ (the power set of S).

Compact sets are complete

Proposition (not in textbook)

Suppose $E \subset S$. If E is compact, then E is complete.

Recall: (E, d) is complete if any Cauchy sequence in E converges.

Completeness is intrinsic. Suppose E is a subset of a metric space (S, d); suppose also that E is a subset of another metric space (S', d'), so that $d|_E = d'|_E$. Then E is either complete in both S and S', or in none.

Is any complete metric space compact? **No!** $\mathbb{N}$ with discrete metric ($d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y$) is complete and bounded, but not compact.

Proof. Suppose E is not complete – that is, $\exists$ a Cauchy sequence $(s_n)_{n \in \mathbb{N}} \subset E$ which has no limit in E. For $k \in \mathbb{N}$ find n_k s.t. $d(s_m, s_\ell) < 2^{-k}$ for $m, \ell \geqslant n_k$.
Let $U_k = \left\{ s \in S : d(s, s_{n_k}) > 2^{-k} \right\}$. Note that U_k is an open set: if $s \in U_k$, then $\mathbf{B}_r^o(s) \subset U_k$, with $r = d(s, s_{n_k}) - 2^{-k}$.

Continues on the next slide

Compactness, completeness, etc.

Any compact set is complete (proof continues).

Claim 1. The family $(U_k)_{k\in\mathbb{N}}$ is an open cover for E.
Suppose, for the sake of contradiction, that $x \in E\backslash(\cup_k U_k)$. Thus, $d(s_{n_k}, x) \leqslant 2^{-k}$ for any k, hence (s_{n_k}) converges to x.
Homework 4: as (s_n) is Cauchy, and has a subsequence converging to x, then (s_n) itself converges to x, contradiction!

Claim 2. The family $(U_k)_{k\in\mathbb{N}}$ has no finite subcover.
$s_{n_m} \notin \cup_{j<m} U_j$, since $d(s_{n_m}, s_{n_j}) \leqslant 2^{-j}$. ■

Corollary (not in textbook)

Any compact set is closed.

Fact: Any complete set is closed (can assign this as homework).

Criterion for compactness

Definition (not in textbook)

$E \subset S$ is called totally bounded if $\forall \varepsilon > 0\ \exists s_1, \ldots, s_n \in S$ s.t. $E \subset \cup_{i=1}^n \mathbf{B}^o_\varepsilon(s_i)$.

Observation. Total boundedness is intrinsic: we can assume $s_i \in E$. Indeed, find $x_1, \ldots, x_n \in S$ s.t. $E \subset \cup_{i=1}^n \mathbf{B}^o_{\varepsilon/2}(x_i)$ (assume $E \cap \mathbf{B}^o_{\varepsilon/2}(x_i) \neq \emptyset$, for any i). Pick $s_i \in E \cap \mathbf{B}^o_{\varepsilon/2}(x_i)$, then $\mathbf{B}^o_{\varepsilon/2}(x_i) \subset \mathbf{B}^o_\varepsilon(s_i)$, hence $E \subset \cup_{i=1}^n \mathbf{B}^o_\varepsilon(s_i)$.

Observation. Any totally bounded set is bounded. Indeed, suppose $E \subset \cup_{i=1}^n \mathbf{B}^o_1(s_i)$. Fix $s_0 \in S$, and let $r = \max_{1 \leqslant i \leqslant n} d(s_0, s_i)$. Then $E \subset \mathbf{B}^o_{r+1}(s_0)$. The converse is false (consider $\mathbb{N}$ with the metric $d(x, y) = 1$ whenever $x \neq y$).

Compactness versus convergence of sequences

Theorem (Characterization of compactness; not in textbook)

For a subset E of a metric space, the following are equivalent:

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Both completeness and total boundedness are intrinsic, hence

Compactness is intrinsic.

Proof of (1) $\Rightarrow$ (2). Suppose E is compact. Know: E is complete. To prove total boundedness, fix $\varepsilon > 0$. $E \subset \cup_{s \in E} \mathbf{B}^o_\varepsilon(s)$ (open cover). Find a finite subcover $E \subset \cup_{i=1}^n \mathbf{B}^o_\varepsilon(s_i)$. ∎

Total boundedness and Cauchy sequences

Proposition (not in textbook)

E is totally bounded iff any sequence in E has a Cauchy subsequence.

Proof: any sequence in E has a Cauchy subsequence $\Rightarrow$ E is totally bounded. We prove: if E is not totally bounded, then $\exists$ sequence $(x_i) \subset E$ with no Cauchy subsequence.
$\exists\, \varepsilon > 0$ s.t. no finite collection of open balls of radius ε covers E.
Need to find a sequence $(x_i)_{i \in \mathbb{N}} \subset E$ s.t. $d(x_i, x_j) \geqslant \varepsilon$ when $i \neq j$. Such a sequence has no Cauchy subsequences.

Pick $x_1 \in E$. $E \not\subset \mathbf{B}^o_\varepsilon(x_1)$, hence $\exists\, x_2 \in E$ s.t. $d(x_1, x_2) \geqslant \varepsilon$.
$E \not\subset \mathbf{B}^o_\varepsilon(x_1) \cup \mathbf{B}^o_\varepsilon(x_2)$, hence $\exists\, x_3 \in E$ s.t. $d(x_1, x_3), d(x_2, x_3) \geqslant \varepsilon$.
Continue in the same manner. ∎

Proof of Proposition p. 2

Proof: E totally bounded $\Rightarrow$ any sequence in E has a Cauchy subsequence. Suppose (s_i) is a sequence in a totally bounded set E; we look for a Cauchy subsequence.

Find $x_{11}, \ldots x_{1N_1} \in S$ so that $E \subset \cup_{j=1}^{N_1} \mathbf{B}^o_{2^{-2}}(x_{1j})$. Find $j_1 \in \{1, \ldots, N_1\}$ so that $I_1 = \{i : s_i \in \mathbf{B}^o_{2^{-2}}(x_{1j_1})\}$ is infinite. Pick $i_1 := \min I_1$.

Find $x_{21}, \ldots x_{2N_2} \in S$ so that $E \cap \mathbf{B}^o_{2^{-2}}(x_{1j_1}) \subset \cup_{j=1}^{N_2} \mathbf{B}^o_{2^{-3}}(x_{2j})$. Find $j_2 \in \{1, \ldots, N_2\}$ so that $I_2 = \{i \in I_1 : s_i \in \mathbf{B}^o_{2^{-3}}(x_{2j_2})\}$ is infinite. Pick $i_2 := \min(I_2 \backslash \{i_1\})$.

Note: $s_{i_1} \in \mathbf{B}^o_{2^{-2}}(x_{1j_1})$, $s_{i_2} \in \mathbf{B}^o_{2^{-2}}(x_{1j_1}) \cap \mathbf{B}^0_{2^{-3}}(x_{2j_2})$.

Proceed to obtain $i_1 < i_2 < \ldots$ s.t. $s_{i_k} \in \mathbf{B}^o_{2^{-1-\ell}}(x_{\ell j_\ell})$ for $k \geqslant \ell$. For $k \geqslant \ell$, $d(s_{i_k}, s_{i_\ell}) \leqslant d(s_{i_k}, x_{\ell j_\ell}) + d(x_{\ell j_\ell}, s_{i_\ell}) < 2 \cdot 2^{-1-\ell} = 2^{-\ell}$. Thus, (s_{i_k}) is Cauchy. $\blacksquare$

Compactness versus convergence of sequences

Theorem (Characterization of compactness; not in textbook)

For a subset E of a metric space, the following are equivalent:

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Proof of (2) $\Rightarrow$ (3). Suppose E is totally bounded and complete. Any sequence has a Cauchy subsequence (total boundedness), which has limit in E (completeness). ■

Proof of (3) $\Rightarrow$ (2). If E is not totally bounded, then it contains a sequence with no Cauchy subsequences, hence no convergent subsequences. If E is not complete, then it has a Cauchy sequence with no limit in E, hence with no convergent subsequences (Homework 4). ■

Proof of (3) $\Rightarrow$ (1) (or (2) $\Rightarrow$ (1)): omitted for lack of time.

Compactness and convergence of sequences

Lemma (Established for $\mathbb{R}$, true for any metric space)

A sequence (s_n) in a metric space (S, d) has a subsequence converging to s iff, for any $\varepsilon > 0$, $|\{n \in \mathbb{N} : d(s_n, s) < \varepsilon\}| = \infty$.

Proof of Characterization of Compactness, $(1) \Rightarrow (3)$. Shall show the contrapositive: $\neg(3) \Rightarrow \neg(1)$.
Suppose a sequence $(s_n) \subset E$ has no subsequence converging to a limit in E. Construct an open cover for E without a finite subcover.

For each $x \in E$, find $r(x) > 0$ s.t. $|\mathbf{B}^o_{r(x)}(x) \cap \{s_n : n \in \mathbb{N}\}| < \infty$. Then $\{\mathbf{B}^o_{r(x)}(x) : x \in E\}$ is an open cover for E. There is no finite subcover: for any finite collection $(x_i)_{i=1}^m$, $\cup_{i=1}^m \mathbf{B}^o_{r(x_i)}(x_i)$ contains only finitely many s_n's.
■

Compact subsets of $\mathbb{R}^n$

Theorem (13.13 – Heine-Borel)

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof of HB: closed and bounded $\Rightarrow$ compact. Recall: compact $\Leftrightarrow$ complete and totally bounded.
Suppose $E \subset \mathbb{R}^n$ is closed and bounded.
$\mathbb{R}^n$ is complete, E is closed $\Rightarrow$ E is complete.
Remains to show: E is totally bounded – that is,
$\forall\ \varepsilon > 0\ \exists\ \vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ s.t. $E \subset \cup_{j=1}^{n} \mathbf{B}_{\varepsilon}^{o}(\vec{x}^{(j)})$.
Pick $A > 0$ s.t. $\forall \vec{x} = (x_i)_{i=1}^{n} \in E$ we have $\|\vec{x}\| < A$. Write $\vec{x} = (x_i)_{i=1}^{n} \in E$, then $|x_i| < A$. Find $N \in \mathbb{N}$ s.t. $\frac{A}{N} < \frac{\varepsilon}{\sqrt{n}}$. Let $\vec{x}^{(j)}$ be vectors with coordinates $p\frac{A}{N}$, with $p \in \{-N, \ldots, N\}$.
For $\vec{x} \in E$, find j s.t. $|x_i^{(j)} - x_i| < \frac{\varepsilon}{\sqrt{n}}$, for $1 \leqslant i \leqslant n$. Then
$d(\vec{x}, \vec{x}^{(j)}) = \left(\sum_i (x_i^{(j)} - x_i)^2\right)^{1/2} < \varepsilon$. ∎