Continuous image of compact set is compact

Theorem (21.3)

$f : S \to S^*$ is continuous iff $f^{-1}(U)$ is open $\forall$ open $U \subset S^*$.

Theorem (21.4(i))

Suppose $f : S \to S^*$ is continuous ((S, d) and (S^*, d^*) are metric spaces), and $E \subset S$ is compact. Then $f(E) \subset S^*$ is compact.

Proof: f continuous, E compact $\Rightarrow$ $f(E)$ compact.

Suppose $(U_i)_{i \in I}$ is an open cover for $f(E)$. We prove that a finite subcover exists.

For $i \in I$, $V_i = f^{-1}(U_i)$ is open. Note that V_i's form a cover for E: if $s \in E$, then $f(s) \in U_i$ for some i, hence $s \in V_i$.

By compactness, we can find $i_1, \ldots, i_n \in I$ s.t. $E \subset \cup_{k=1}^n V_{i_k}$. Then $f(E) \subset \cup_{k=1}^n f\left(V_{i_k}\right) = \cup_{k=1}^n U_{i_k}$. ■

Continuous function attains its max and min

Corollary (similar to 18.1)

If $f : S \to \mathbb{R}$ is continuous, and $E \subset S$ is compact, then $f(E)$ is bounded. Moreover, f attains its maximum and minimum – that is, there exist $x, y \in E$ s.t. $f(x) = \sup_{e \in E} f(e) = \max_{e \in E} f(e)$, and $f(y) = \inf_{e \in E} f(e) = \min_{e \in E} f(e)$.

Proof. $\mathbb{R} \supset f(E)$ is compact $\Leftrightarrow$ closed and bounded. Let $a = \sup f(E)$. Need to show: $a \in f(E)$.

Suppose, for the sake of contradiction, that $a \notin f(E)$. Then $(a - \varepsilon, a) \cap f(E) \neq \emptyset$ for any $\varepsilon > 0$, hence $\exists\ a_1, a_2, \ldots \in f(E)$ s.t. $a_i \to a$. This contradicts $f(E)$ being closed. ■

Intermediate Value Property

Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is a function. f has the Intermediate Value Property (IVP) on I if, whenever $a, b \in I$, $a < b$, and y lies between $f(a)$ and $f(b)$, then $\exists\ x \in (a, b)$ s.t. $f(x) = y$.

Theorem (18.2)

Any continuous function has the IVP.

Proof for the case of $f(a) < y < f(b)$. Let $S = \{s \in [a, b] : f(s) < y\}$; $a \in S$ (so $S \neq \emptyset$), $b \notin S$. We show that $x = \sup S$ works, by proving that (i) $f(x) \leqslant y$; (ii) $f(x) \geqslant y$.

(i) $\forall n\ \exists s_n \in (x - \frac{1}{n}, x] \cap S$. $s_n \to x \Rightarrow f(s_n) \to f(x)$. $f(s_n) < y$. $\Rightarrow$ $f(x) \leqslant y$. $x < b$, since $f(b) > y$.
(ii) Let $t_n = \min\left\{x + \frac{1}{n}, b\right\}$. $t_n \notin S \Rightarrow f(t_n) \geqslant y$. $t_n \to x \Rightarrow$ $f(t_n) \to f(x)$, so $f(x) \geqslant y$. ∎

More about IVP

Corollary (18.3 – continuous images of intervals.)

If I is an interval, and $f : I \to \mathbb{R}$ has the IVP, then $f(I)$ is either an interval, or a single point.

Proof. Let $J = f(I)$. If $\inf J = \sup J$, then $J = \{\sup J\}$. Otherwise, if $\inf J < y < \sup J$, then $y \in J$. Indeed, pick $u, v \in J$ s.t. $u \leqslant y \leqslant v$. $u = f(a), v = f(b)$. By IVP, $y = f(x)$, for some $x \in I$. Thus, J contains $(\inf J, \sup J)$. So, J is either $(\inf J, \sup J)$, $[\inf J, \sup J)$, $(\inf J, \sup J]$, or $[\inf J, \sup J]$. ■

Examples (1) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ fails IVP (on any interval).

(2) $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has IVP (on any interval).

Roots of polynomial of odd degree

Proposition (Exercise 18.9)

Any polynomial of odd degree has at least one real root.

Is the same true for polynomials of even degree? **No!** For instance, $p(x) = x^2 + 2x + 2 = (x+1)^2 + 1 > 0$ for any x.

Proof. Write $p(x) = c_0 + c_1 x + \ldots + c_n x^n$, with n odd. The polynomials p and $q(x) = \frac{1}{c_n} p(x) = x^n + \sum_{k=0}^{n-1} \gamma_k x^k$ (with $\gamma_k = \frac{c_k}{c_n}$) have the same roots; it suffices to show that q has a root.

Let $A = \sum_{k=0}^{n-1} |\gamma_k| + 1$. Once we show that $q(-A) < 0 < q(A)$, we will be done (q is continuous, hence we can apply IVT to q on $[-A, A]$). Show that $q(A) > 0$; $q(-A) < 0$ is handled similarly. $A \geqslant 1$, hence $A^k \leqslant \frac{1}{A} \cdot A^n$ for $0 \leqslant k \leqslant n-1$, hence $q(A) = A^n + \sum_{k=0}^{n-1} \gamma_k A^k \geqslant$ $A^n \big(1 - \frac{1}{A} \sum_{k=0}^{n-1} |\gamma_k|\big) = A^{n-1} \big(A - \sum_{k=0}^{n-1} |\gamma_k|\big) = A^{n-1} > 0$. ∎

More consequences of Intermediate Value Theorem

Proposition (Existence of fixed point – pp. 135-136)

Any continuous function $f : [0,1] \to [0,1]$ has a fixed point – that is, a point $x \in [0,1]$ s.t. $f(x) = x$.

Proof. $g(x) = f(x) - x$ is continuous on $[0,1]$, with $g(0) = f(0) \geqslant 0$, and $g(1) = f(1) - 1 \leqslant 0$. By IVT, $\exists x$ s.t. $g(x) = 0$; then $f(x) = x$. ■

Proposition (Existence of m-th root – p. 136)

For any $m \in \mathbb{N}$ and $y > 0$ there is $x > 0$ s.t. $x^m = y$.

Sketch of a proof. Fix m. Let $b = \max\{1, y\}$, then $b^m \geqslant y$. Apply IVT to the continuous function $f(x) = x^m$ on $[0, b]$. ■

Continuity of inverse functions

A function $f : I \to \mathbb{R}$ ($I \subset \mathbb{R}$) is strictly increasing if $f(x) < f(y)$ whenever $x < y$.

Theorem (18.4)

Suppose $I \subset \mathbb{R}$ is an interval, $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $J = f(I)$ is an interval; $f^{-1} : J \to I$ is strictly increasing and continuous.

Proof: next time.

The function $f : I = [0, \infty) \to \mathbb{R} : x \mapsto x^m$ is strictly increasing, with $J = f(I) = [0, \infty)$ ($\forall y \geqslant 0\ \exists x \geqslant 0$ s.t. $x^m = y$). The inverse function $f^{-1}(y)$ is denoted by $y^{1/m}$ (the m-th root).

Corollary

The function $x \mapsto x^{1/m}$ (taking $[0, \infty)$ to itself) is continuous.

Proof of continuity of f^{-1}

Theorem 18.4. Suppose $I \subset \mathbb{R}$ is an interval, $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $J = f(I)$ is an interval; $f^{-1} : J \to I$ is strictly increasing and continuous.

Proof (from 18.5, essentially). Clearly $g = f^{-1}$ is strictly increasing. Need to show continuity.

Pick $y_0 = f(x_0) \in J$ (so $x_0 = g(y_0)$), show that g is cont. at y_0. Assume $x_0 \notin \partial I$ – that is, $\exists \varepsilon_0 > 0$ s.t. $(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset I$.
Need: if $\varepsilon \in (0, \varepsilon_0)$, then $\exists \delta > 0$ s.t. $g(y) \in (x_0 - \varepsilon, x_0 + \varepsilon)$ whenever $|y - y_0| < \delta$.

Let $y_1 = f(x_0 - \varepsilon)$, $y_2 = f(x_0 + \varepsilon)$. Let $\delta = \min\{y_2 - y_0, y_0 - y_1\}$. If $|y - y_0| < \delta$, then $y_1 < y < y_2$, hence
$x_0 - \varepsilon = g(y_1) < g(y) < g(y_2) = x_0 + \varepsilon$. ∎

Monotonicity of injective functions on intervals

Theorem (18.6)

Suppose f is a continuous $1-1$ function on an interval I. Then f is strictly monotone.

Proof. Pick $a, b \in I$, with $a < b$. Suppose $f(a) < f(b)$. Prove that f is sitrictly increasing.

(1) Suppose $c \in (a, b)$, show that $f(c) \in \big(f(a), f(b)\big)$.
If $f(c) > f(b)$, then, by IVT, $\exists x \in (a, c)$ s.t. $f(x) = f(b)$. But f is $1-1$, so no such x can exist. $f(c) < f(a)$ is ruled out similarly.
(2) Similarly, $c < a$ ($c > b$) $\Rightarrow f(c) < f(a)$ (resp. $f(c) > f(b)$).
(3) Conclusion: $f(c) < f(a)$ ($f(c) > f(a)$) if $c < a$ (resp. $c > a$).

(4) Suppose $x, y \in I$, $x < y$. Want: $f(x) < f(y)$.
- If $x < a$, then $f(x) < f(a)$, hence $f(y) > f(x)$.
- If $x > a$, then $f(x) > f(a)$, hence $f(y) > f(x)$. ■