TAM 470 - HW #4 solutions

Problem 2

$$\frac{dy}{dt} = -0.2y - 2\cos(2t)\cdot y^2 \; ; \; y(0) = 1$$

(a) $f(y,t) = -0.2y - 2y^2\cos(2t)$

Following the form of (4.4) in Moin,

$$\lambda = \left.\frac{\partial f}{\partial y}\right|_{(y_0,t_0)}$$

$$\Rightarrow \lambda = -0.2 - 4y\cos(2t)\Big|_{(1,0)}$$

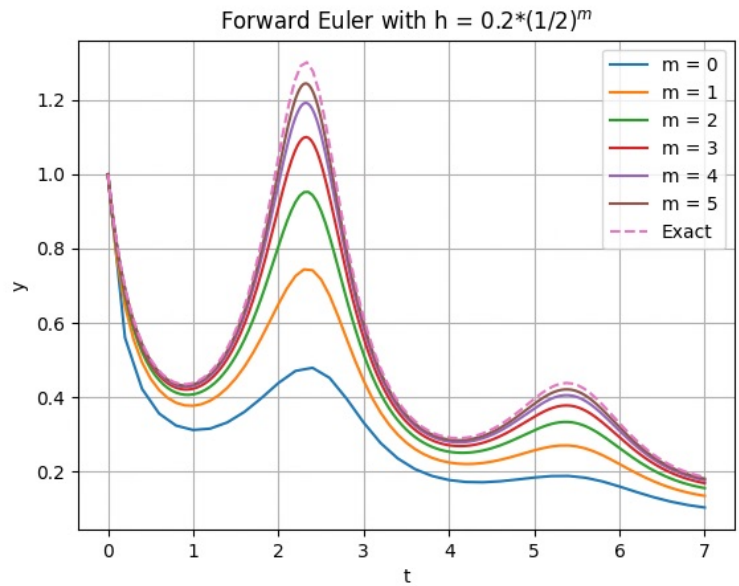
$$= -0.2 - 4 = -4.2$$

The maximum allowable time step (4.9) in Moin,

$$h \le \frac{2}{|\lambda|}$$

$$\rightarrow \boxed{h_{max} = \frac{2}{4.2} = 0.4762}$$

(b)



The exact solution is given by

$$y_{exact} = \frac{101}{100\sin(2t) - 10\cos 2t + 111 e^{0.2t}}$$

(c) One convergence criteria can be

$$\max \left| \frac{y_{m+1} - y_m}{y_m} \right| < rel\_tol \qquad (*)$$

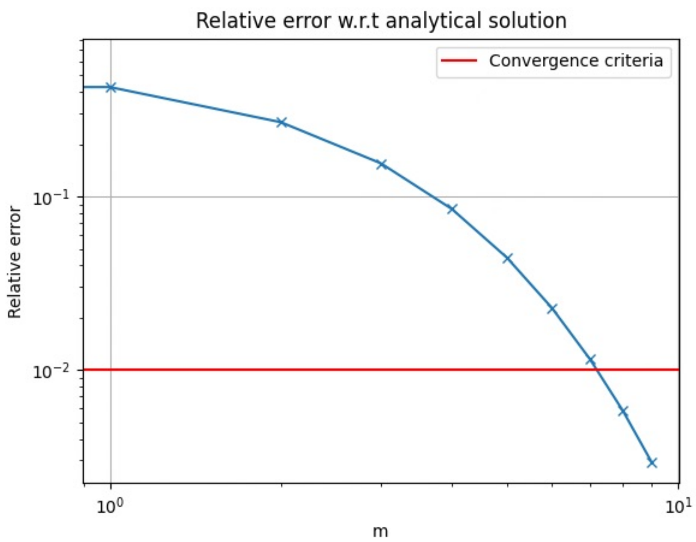
where rel_tol can be the tolerance for relative error. If we take rel_tol = 1%, then we need to find m such that

(*) holds.

Since we have exact solution as well, we can use a criteria

$$\max \left| \frac{y_{exact} - y_m}{y_{exact}} \right| < rel\_tol \quad (**)$$

The figure below is a demonstration of (**) with $m = 0, 1 \dots 9$ and $rel\_tol = 0.01$.



We see that $m = 8$ is the first value of m where the calculated error goes below 0.01.

Problem 3

$$\theta''(t) + c\theta'(t) + \frac{g}{l}\theta(t) = 0 \qquad + \text{ICs.}$$

(a) Let $\theta'(t) = \beta(t)$

$\therefore$ The above equation becomes,

$$\beta'(t) = -c\beta(t) - \frac{g}{l}\theta(t) \qquad + \text{ICs}$$

In matrix form

$$\boxed{\begin{bmatrix} \theta'(t) \\ \beta'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/l & -c \end{bmatrix} \begin{bmatrix} \theta(t) \\ \beta(t) \end{bmatrix} \quad + \text{ ICs}}$$

(b) $A = \begin{bmatrix} 0 & 1 \\ -g/l & -c \end{bmatrix}$

Let λ be eigen values

$$\det |A - I\lambda| = 0 \Longrightarrow \begin{vmatrix} -\lambda & 1 \\ -g/l & -\lambda - c \end{vmatrix} = 0$$

$$\Longrightarrow \lambda(\lambda + c) + g/l = 0$$

$$\lambda^2 + c\lambda + g/l = 0$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - \frac{4g}{l}}}{2}$$

$$\boxed{\lambda = \frac{c}{2}\left[-1 \pm \sqrt{1 - \frac{4}{c^2}\left(\frac{g}{l}\right)}\right]}$$

(c) With $c = 4$, $\frac{g}{l} = 2$,

$$\lambda = 2\left[-1 \pm \sqrt{1 - \frac{4}{4^2} \cdot 2}\right] = 2\left[-1 \pm \frac{1}{\sqrt{2}}\right]$$

$$= -2 \pm \sqrt{2}$$

$$|1+\lambda h| \leq 1$$

$$\Rightarrow -1 \leq 1+\lambda h \leq 1, \text{ when } \lambda \in \mathbb{R}^-$$

$$\Rightarrow -2 \leq \lambda h \leq 0$$

$$\therefore \quad h \leq \frac{2}{|\lambda|}$$

Using the above eigen values,

$$h \leq \frac{2}{0.5857} \quad \text{and} \quad h \leq \frac{2}{3.4142}$$

$$\Rightarrow h \leq 3.414 \quad \text{and} \quad h \leq 0.5857$$

$$\therefore \quad \underline{\underline{h_{max} = 0.5857}}$$

(d) When $c=4$, $g/l = 5$

$$\lambda = 2\left[-1 \pm \sqrt{1 - \frac{4 \cdot 5}{4^2}}\right]$$

$$= 2\left[-1 \pm \sqrt{-\frac{1}{4}}\right] = 2\left[-1 \pm \frac{i}{2}\right] = -2 \pm i$$

$$|1+h\lambda| \leq 1$$

$$\Rightarrow |1+h(-2 \pm i)| \leq 1$$

$$\Rightarrow \left| (1-2h) \pm ih \right| \leq 1$$

$$\sqrt{(1-2h)^2 + h^2} \leq 1$$

$$\Rightarrow 5h^2 + 1 - 4h - 1 \leq 0$$

$$\Rightarrow h(5h-4) \leq 0$$

$$\Rightarrow h \leq 4/5 \quad \text{and} \quad h > 0$$

$$\therefore \underline{h_{max} = 4/5}$$

Problem 4

$$u''(t) + [u(t)]^2 u'(t) + t^2 u(t) = 0$$

(a) Let $u'(t) = w(t)$ and substitute above.

$$\therefore \; w'(t) + [u(t)]^2 w(t) + t^2 u(t) = 0$$

$$\begin{bmatrix} u'(t) \\ w'(t) \end{bmatrix} = \begin{bmatrix} w(t) \\ -[u(t)]^2 w(t) - t^2 u(t) \end{bmatrix} = f(\underline{y}, t)$$

with $u(0) = U_0$ and $w(0) = W_0$ are ICs.

(b) $$f(u, w, t) = \begin{bmatrix} w(t) \\ -t^2 u(t) - [u(t)]^2 w(t) \end{bmatrix}$$

$$J = \nabla_{u,w} f(u, w, t) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial w} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -t^2 - 2uw & -u^2 \end{bmatrix}$$

Applying ICs, $u(0) = 1$, $w(0) = 0$.

$$J = \nabla_{u,w} f(u, w, t)\Big|_{t=0} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Eigen values of J be λ

$$\lambda(\lambda + 1) = 0 \Rightarrow \lambda = 0 \text{ and } \lambda = -1$$

As $\lambda \in \mathbb{R}$, we take the non-zero eigen value for stability.

$$h \leq \frac{2}{|\lambda|} \Rightarrow \underline{\underline{h_{max} = 2}}$$