Reminder: interior points in metric space (S, d)

Definition (Open ball – not in textbook)

Suppose $s_0 \in S$, $r > 0$. The open ball with center s_0 and radius r is $\mathbf{B}_r^o(s_0) = \{s \in S : d(s, s_0) < r\}$.

Definition (13.6 – interior of $E \subset S$)

$s_0 \in S$ is called interior to E if $\exists\, r > 0$ s.t. $\mathbf{B}_r^o(s_0) \subset E$. The set of interior points is denoted by E^o, and called the interior of E.

Definition (13.6 – open sets in S)

$E \subset S$ is called open if $E = E^o$.

Reminder: open sets in metric space (S, d)

Fact (13.7 – (iii) and (iv) proved in Homework 4)

1. (i) *S is open.*
2. (ii) *$\emptyset$ is open.*
3. (iii) *A union of any collection of open sets is open.*
4. (iv) *An intersection of finitely many open sets is open.*

Proposition (not in textbook)

1. *Any open ball is open.*
2. *For any E, E^o is open – that is, $(E^o)^o = E^o$.*

Proposition (not in textbook)

A set is open iff it is a union of open balls.

Closed sets in metric space (S, d)

Definition (13.6 – closed sets in S)

$E \subset S$ is called closed if $S \backslash E$ is open.

Fact (obtained from 13.7 using de Morgan's laws)

1. *S is closed.*
2. *$\emptyset$ is closed.*
3. *An intersection of any collection of closed sets is closed.*
4. *A union of finitely many closed sets is closed (does not generalize to infinite unions).*

Fact (some de Morgan's laws, proved in Homework 4)

Suppose $(A_i)_{i \in I}$ are subsets of S. Then
$S \backslash (\cup_i A_i) = \cap_i (S \backslash A_i)$, $S \backslash (\cap_i A_i) = \cup_i (S \backslash A_i)$.

Examples of open and closed sets

Example: finite sets are closed.
Conisder $E = \{x_1, \ldots, x_n\} \subset S$.
For $1 \leqslant i \leqslant n$, $S \backslash \{x_i\}$ is open, hence $\{x_i\}$ is closed.
$E = \{x_1\} \cup \ldots \cup \{x_n\}$ is closed, as a finite union of closed sets.

Example: intervals in $\mathbb{R}$. Suppose $a < b$.
(a, b) is open, not closed. $[a, b]$ is closed, not open.
$(a, b]$, $[a, b)$ are neither.

Example: discrete metric. For $x, y \in S$, $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$.

Describe open and closed sets.
Hint. Suppose $s \in S$. Is $\{s\}$ closed? open?
Every set is both open and closed!
Moral of the story: sometimes a set can be both open and closed.

Being open or closed depends on the ambient space

Suppose (S, d) is a metric space, and $E \subset S$.
Being open or closed is a property of the position of E inside of S, and not of E itself.
Example: Suppose d is the usual metric on $\mathbb{R}$, which is inherited by $E = \mathbb{Q}$.
If $S = \mathbb{Q}$ itself: E is both open and closed in S.
If $S = \mathbb{R}$: E is neither open nor closed in S.
Indeed, if E were closed, then $\mathbb{R} \backslash \mathbb{Q}$ would have to be open.
That is, for any $x_0 \in \mathbb{R} \backslash \mathbb{Q}$, there would exist $r > 0$ s.t. $(x_0 - r, x_0 + r) = \mathbf{B}_r^o(x_0) \subset \mathbb{R} \backslash \mathbb{Q}$. This is impossible, due to the denseness of rationals.
The possibility of E being open is ruled out similarly.

Closure and boundary of a set

Definition (13.6 – closure)

The closure of $E \subset S$ (denote by by E^-) is the intersection of all closed sets containing E.

Observations. (1) $E \subset E^-$.
(2) E^- is closed, as an intersection of closed sets; this is the smallest closed set containing E.

Definition (13.6 – boundary)

The boundary of $E \subset S$ is $\partial E = E^- \setminus E^o$.

Descriptions of closure and boundary

Proposition (13.9)

a. *$E = E^-$ iff E is closed.*
b. *E is closed iff the limit of any sequence of points in E is in E.*
c. *$s \in E^-$ iff it is a limit of a sequence of points in E.*
d. $\partial E = E^- \cap (S \backslash E)^-$.

Proof. (a) is clear.
(c) $\Rightarrow$ (b): E^- is the set of limits of sequences from E. (b) implies that $E^- = E$, hence by (a), E is closed. ∎

More about E^-

Lemma (not in textbook)

$x_0 \notin E^-$ *iff* $\exists\, r > 0$ *s.t.* $\mathbf{B}_r^o(x_0) \cap E = \emptyset$.

Proof. $\Leftarrow$: $S \backslash \mathbf{B}_r^o(x_0)$ is closed, contains E. E^- is the smallest closed set containing E, hence $E^- \subset S \backslash \mathbf{B}_r^o(x_0)$.

$\Rightarrow$: If $x_0 \notin E^-$, then x_0 belongs to open set $F = S \backslash E^-$. $F = F^o$, hence $\mathbf{B}_r^o(x_0) \subset F$ for some F. ■

Proof: $s \in E^-$ iff it is a limit of a sequence of points in E.
If $s \notin E^-$, then $\exists r$ s.t. $\mathbf{B}_r^o(s) \cap E = \emptyset$. No sequence $(s_n) \subset E$ can converge to s, since $d(s_n, s) \geqslant r$.
If $s \in E^-$, then for any $n \in \mathbb{N}$ we can find $s_n \in \mathbf{B}_{1/n}^o(s) \cap E$; then $\lim s_n = s$. ■

Further examples

Example. Find the closure of $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$.
$E^- = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

Recall: $s \in E^-$ iff $\mathbf{B}_r^o(s) \cap E \neq \emptyset$, $\forall r > 0$. Clearly all points of E has this property, as does 0 (due to the Archimedean Property of reals).
If $s < 0$, then $\mathbf{B}_{|s|}^o(s) \cap E = \emptyset$, so $s \notin E^-$.
If $s > 1$, then $\mathbf{B}_{s-1}^o(s) \cap E = \emptyset$, so $s \notin E^-$.
If $s \in (0,1)\setminus E$, let $n = \lfloor \frac{1}{s} \rfloor$, then $\frac{1}{n+1} < s < \frac{1}{n}$. Thus, $\mathbf{B}_r^o(s) \cap E = \emptyset$, for $r = \min\left\{\frac{1}{n} - s, s - \frac{1}{n+1}\right\}$.
Conclude: $E^- = E \cup \{0\}$.