TAM 470 / CSE 450

Homework 7

Problem 1 (40 points)

The steady 1D heat equation for a long, thin structure with insulated lateral surfaces and a cross-sectional area $A(x)$ that varies along its length is

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A}\frac{dA}{dx}\right)\frac{dT}{dx} + \frac{Q}{k} = 0 \tag{1}$$

where Q represents internal volumetric heating and k is the thermal conductivity.

A key assumption in equation 1 is that the temperature variation in the cross-section for a given position x does not significantly vary, such that we can assume the temperature distribution in any cross-sectional segment is uniform. Hence temperature $T(x)$ is a function of x only.

Suppose we wish to model the temperature distribution in a structure on the domain $x \in [0, L]$ that has a solid circular cross-section where the circular radius r varies according to

$$r(x) = (r_0 - r_L)\left(\frac{x}{L} - 1\right)^2 + r_L \quad x \in [0, L] \tag{2}$$

where r_0 is the radius at $x = 0$ and r_L is the radius at $x = L$.

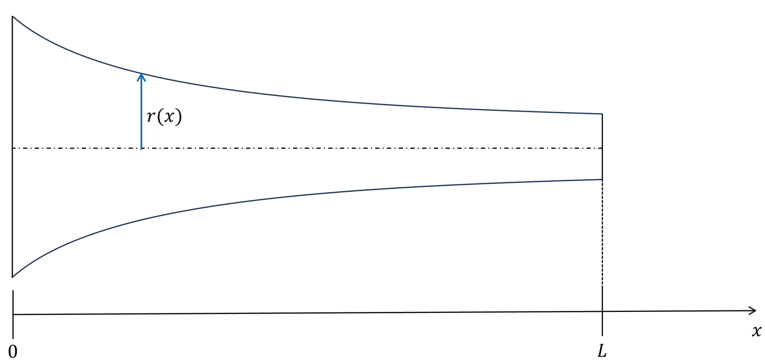

Figure 1: Domain for the 1D steady heat transfer problem.

Further suppose that the structure has a fixed temperature T_0 at the left end:

$$T(0) = T_0 \tag{3}$$

At the right end, there is an applied heat flux q_L into the domain, which corresponds to the boundary condition

$$kT'(L) = q_L \tag{4}$$

Part 1: (10 points)

Using the formulation discussed in lecture for the discretization of a second order linear ODE using centered differences, write down the expressions for α_j, β_j, γ_j, and f_j for this specific problem.

Additionally, indicate any modifications to the standard system of equations needed to implement the boundary conditions given in equations 3 and 4. Use a one-sided difference scheme for equation 4.

Part 2: PrairieLearn (20 points):

Go to PrairieLearn to write a function that numerically solves the ODE in Equation 1 using the boundary conditions given in Equations 3–4 using first-order one-sided difference schemes to approximate the Neumann boundary condition.

Part 3 (10 points):

Use the code you wrote in Part 2 to answer the following questions for $L = 10$, $r_0 = L/10$, $r_L = L/20$, $T_0 = 100$, and a grid of $N = 40$ equally spaced intervals ($h = L/N$).

(**a**) (5 pts) Using parameters $k = 5$, $Q = 8$, plot the temperature distributions for the cases of $q_L = +10$ (heat flux enters the domain) and $q_L = -10$ (heat flux exists the domain).

(**b**) (5 pts) Taking $k = 5$ and $q_L = 0$ (insulated boundary) estimate (to the nearest whole number) the largest value of Q for which the temperature in the body does not exceed $T = 200$. Plot the associated temperature distribution for this value of Q on the domain.

Problem 2: 4 credit-hour students only (10 points)

Derive the ODE given in Equation 1 by performing an energy balance on a control volume. List all assumptions and show your work to obtain the final result.