Definition of continuity

Definition (21.1)

Suppose (S, d) and (S^*, d^*) are metric spaces. The funciton $f : \mathrm{dom}(f) \to S^*$ (with $\mathrm{dom}(f) \subset S$) is called continuous at $x \in \mathrm{dom}(f)$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d^*(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.
f is called continuous on E ($E \subset S$) if it is continuous $\forall x \in E$ – that is, $\forall x \in E$, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d^*(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

Theorem (17.1 + 17.2, more or less)

$f : S \to S^*$ is continuous at $x \in S$ iff $f(x_n) \to f(x)$ whenever a sequence (x_n) converges to x.

Example: a function discontinuous on $\mathbb{R}$

Dirichlet function (Exercise 17.13(a)). $f : \mathbb{R} \to \mathbb{R}$: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

f is discontinuous everywhere.

Fix $x \in \mathbb{R}$, and show that f is discontinuous there. To this end, find a sequence $x_n \to x$ s.t. $f(x_n) \not\to f(x)$.

Case 1: $x \notin \mathbb{Q}$. By denseness of $\mathbb{Q}$, find $x_1, x_2, \ldots \in \mathbb{Q}$ s.t. $x_n \to x$. $f(x_n) = 1$ for any n, yet $f(x) = 0$.

Case 2: $x \in \mathbb{Q}$. By denseness of $\mathbb{R}\backslash\mathbb{Q}$, find $x_1, x_2, \ldots \notin \mathbb{Q}$ s.t. $x_n \to x$. $f(x_n) = 0$ for any n, yet $f(x) = 1$. ■

Example: a function continuous at exactly one point

Dirichlet function, modified (Exercise 17.13(b)). $g : \mathbb{R} \to \mathbb{R}$:
$g(x) = xf(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. **$g$ is continuous only at 0.**

First show g is continuous at 0. If $x_n \to 0$, then $|x_n| \to 0$.
$|g(x_n) - g(0)| \leqslant |x_n|$, hence $g(x_n) \to g(0)$ by Comparison Test.

Fix $x \in \mathbb{R} \backslash \{0\}$, and show that f is discontinuous there. To this end, find a sequence $x_n \to x$ s.t. $g(x_n) \not\to g(x)$.

Case 1: $x \notin \mathbb{Q}$. By denseness of $\mathbb{Q}$, find $x_1, x_2, \ldots \in \mathbb{Q}$ s.t. $x_n \to x$.
$g(x_n) = x_n \to x$, yet $g(x) = 0 \neq x$.

Case 2: $x \in \mathbb{Q} \backslash \{0\}$. By denseness of $\mathbb{R} \backslash \mathbb{Q}$, find $x_1, x_2, \ldots \notin \mathbb{Q}$ s.t.
$x_n \to x$. $g(x_n) = 0$ for any n, yet $g(x) = x$. ∎

Example: a function continuous on $\mathbb{R} \backslash \mathbb{Q}$ only

Thomae function (Exercise 17.14). If $x \notin \mathbb{Q}$, let $h(x) = 0$. If $x \in \mathbb{Q} \backslash \{0\}$, write $x = \frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\gcd(a, b) = 1$. Set $h(x) = \frac{1}{b}$. Let $h(0) = 1$. **h is continuous at x iff $x \notin \mathbb{Q}$.**

(1) If $x \in \mathbb{Q}$, then h is discontinuous at x. Find $(x_n) \subset \mathbb{R} \backslash \mathbb{Q}$, $x_n \to x$. Then $h(x_n) \not\to h(x)$.

(2) $x \notin \mathbb{Q}$. For $\varepsilon > 0$ find $\delta > 0$ s.t. $h(y) = |h(x) - h(y)| < \varepsilon$ when $|x - y| < \delta$. Only worry about $y \in \mathbb{Q}$.

Find $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. Need to find $\delta > 0$ s.t. if $|\frac{a}{b} - x| < \delta$, then $b > N$. Indeed, then $f\left(\frac{a}{b}\right) = \frac{1}{b} < \frac{1}{N} < \varepsilon$.

Let $z = xN!$, $\gamma = \min\{z - \lfloor z \rfloor, \lfloor z \rfloor + 1 - z\}$ (distance from z to the nearest integer), $\delta = \frac{\gamma}{N!}$. Suppose, for contradiction, that $|\frac{a}{b} - x| < \delta$, and $b \leqslant N$. Then $|\frac{a}{b}N! - xN!| < N!\delta = \gamma$. This is impossible, since $\frac{a}{b}N! \in \mathbb{Z}$. ∎

Operations on continuous functions

Theorem (17.3, 17.4)

Suppose f and g are continuous at $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \subset S$, where (S, d) is a metric space. Then $|f|$, kf $(k \in \mathbb{R})$, $f + g$, and fg are continuous at x_0. $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.

Proof: fg is continuous at x_0.
Need to show: if $x_n \to x_0$, then $f(x_n)g(x_n) \to f(x_0)g(x_0)$.
Know: $\lim f(x_n) = f(x_0)$, $\lim g(x_n) = g(x_0)$.
$\lim f(x_n)g(x_n) = \big[\lim f(x_n)\big] \cdot \big[\lim g(x_n)\big] = f(x_0)g(x_0)$. ∎

Proposition (Sec. 17, Example 5)

If f and g are functions into $\mathbb{R}$, continuous at x_0, then $\max\{f, g\}$ and $\min\{f, g\}$ are continuous at x_0.

Proof: continuity of $\max\{f, g\}$. For $a, b \in \mathbb{R}$, $\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$.
$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$. $f + g, f - g$ are cont. at $x_0 \Rightarrow$ so is $|f - g|$. ∎

Further examples

Proposition (Exercises 17.5, 17.6)

1. *Any polynomial is continuous.*
2. *Any rational function is continuous.*

Polynomial: $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ (assume $a_n \neq 0$). $\mathrm{dom}(p) = \mathbb{R}$.
Rational function: $f = \frac{p}{q}$, where p, q are polynomials.
$\mathrm{dom}(f) = \mathbb{R} \setminus \{x : q(x) = 0\}$.

Proof. **(1)** Prove that $x \mapsto x^m$ is continuous, for $m \in \{0, 1, 2, \ldots\}$ (induction; products of continuous functions are continuous).
(2) p is continuous (sums of continuous functions are continuous).
(3) $\frac{p}{q}$ is continuous (ratios of continuous functions are continuous). ∎

Compositions of continuous functions

Theorem (17.5 more or less)

Suppose (S_1, d_1), (S_2, d_2), (S_3, d_3) *are metric spaces, and we have functions* $f : \mathrm{dom}(f) \to S_2$ *and* $g : \mathrm{dom}(g) \to S_3$ $(\mathrm{dom}(f) \subset S_1$, $\mathrm{dom}(g) \subset S_2)$. *Suppose* $x_0 \in \mathrm{dom}(f)$, $f(x_0) \in \mathrm{dom}(g)$, f *is continuous at* x_0, g *is continuous at* $f(x_0)$. *Then* $g \circ f$ *is continuous at* x_0.

Proposition (Homework)

The function $[0, \infty) \to [0, \infty) : x \mapsto \sqrt{x}$ *is continuous.*

Corollary

Suppose f *is continuous at* x_0. *If* $f(x_0) \geqslant 0$, *then* $\sqrt{f}$ *(defined on* $\{x \in \mathrm{dom}(f) : f(x) \geqslant 0\}$*) is continuous at* x_0.

Another characterization of continuity

Theorem (21.3)

Suppose (S, d) and (S^*, d^*) are metric spaces. $f : S \to S^*$ is continuous iff $f^{-1}(U)$ is open for every open $U \subset S^*$. Here,
$f^{-1}(U) = \{s \in S : f(s) \in U\}$.

Proof: next time.

Corollary (Exercise 21.4 – Homework)

Suppose (S, d) is a metric space. Then a function $f : S \to \mathbb{R}$ is continuous iff $f^{-1}((a, b))$ is open whenever $a < b$.

Lemma (Exercise 21.2 – Homework)

f is continuous at $s_0 \in S$ iff for any open set $U \ni f(s_0)$ $\exists$ open set $V \ni s_0$ s.t. $f(V) \subset U$.

Note: $V \subset f^{-1}(U) \Leftrightarrow f(V) \subset U$.