A continuous function on a compact set is unif. cont.

Theorem (21.4(ii))

Suppose $(S,d),(S^*,d^*)$ are metric spaces, $f:S\to S^*$ is continuous, $E\subset S$ is compact. Then $f|_E$ is uniformly continuous.

Proof. For $\varepsilon>0$, find $\delta>0$ s.t. $d^*\big(f(s),f(t)\big)<\varepsilon$ when $d(s,t)<\delta$. For $s\in S$ find $\delta_s>0$ s.t. $d^*\big(f(s),f(t)\big)<\frac{\varepsilon}{2}$ when $d(s,t)<\delta_s$.
$E\subset\cup_{s\in E}\mathbf{B}^o_{\delta_s/2}(s)$, hence $\exists s_1,\ldots,s_n$ s.t. $E\subset\cup_{k=1}^n\mathbf{B}^o_{\delta_{s_k}/2}(s_k)$.
We claim that $\delta=\frac{1}{2}\min_{1\leqslant k\leqslant n}\delta_{s_k}$ works. Suppose $d(t,s)<\delta$. Find k s.t. $s\in\mathbf{B}^o_{\delta_{s_k}/2}(s_k)\Leftrightarrow d(s,s_k)<\frac{\delta_{s_k}}{2}$.
$d(t,s_k)\leqslant d(t,s)+d(s,s_k)<\delta+\frac{\delta_{s_k}}{2}\leqslant\delta_{s_k}$.
$d^*\big(f(s),f(s_k)\big),d^*\big(f(t),f(s_k)\big)<\frac{\varepsilon}{2}$.
Thus, $d^*\big(f(s),f(t)\big)\leqslant d^*\big(f(s),f(s_k)\big)+d^*\big(f(t),f(s_k)\big)<\varepsilon$ ∎

Extensions of uniformly continuous functions

Theorem (similar to 19.5)

Suppose $E \subset S$, E^- compact, $f : E \to S^*$, S^* complete. Then f is uniformly continuous iff it extends to a continuous $\tilde{f} : E^- \to S^*$.

Proof: a unif. cont. f has a continuous extension $\tilde{f}$.

Suppose $x \in E^- \backslash E$. $\exists (x_n) \subset E$ s.t. $x_n \to x$. $\tilde{f}(x) := \lim_n f(x_n)$.
Want: $\tilde{f}(x)$ doesn't depend on the choice of (x_n). Suppose $(y_n) \subset E$, $y_n \to x$. Want: $\lim f(x_n) = \lim f(y_n)$. The sequence $(z_k) = (x_1, y_1, x_2, y_2, \ldots)$ is Cauchy, hence $\big(f(x_1), f(y_1), \ldots\big)$ must be Cauchy, hence $\lim f(x_n) = \lim f(z_k) = \lim f(y_n)$.
Continuity of $\tilde{f}$: know that $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d^*\big(f(x), f(y)\big) < \varepsilon$ if $x, y \in E$, $d(x, y) < \delta$. Show: if $x, y \in E^-$, $d(x, y) < \delta$, then $d^*\big(\tilde{f}(x), \tilde{f}(y)\big) \leqslant \varepsilon$. Find $(x_n), (y_n) \subset E$, $x_n \to x$, $y_n \to y$. For large n, $d(x_n, y_n) < \delta$. $\tilde{f}(x) = \lim f(x_n)$, $\tilde{f}(x) = \lim f(x_n)$, hence $d^*\big(\tilde{f}(x), \tilde{f}(y)\big) = \lim_n d^*\big(f(x_n), f(y_n)\big) \leqslant \varepsilon$. ∎

Connectedness (Section 22)

Definition (22.1)

Suppose (S, d) is a metric space. $E \subset S$ is called disconnected if $\exists$ open $U_1, U_2 \subset S$ s.t.: (i) $E \subset U_1 \cup U_2$; (ii) $(E \cap U_1) \cap (E \cap U_2) = \emptyset$; (iii) $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$.
E is connected if it is not disconnected.

Observation. $E \subset \mathbb{R}$ is connected iff it is empty, a singleton, or an interval.

Intervals are connected: will be proved later.

Suppose E has $\geqslant 2$ points, is not an interval.
Show that E is disconnected.

Find $a < b < c$, $a, c \in E$, $b \notin E$. Take $U_1 = (-\infty, b)$, $U_2 = (b, \infty)$. In Definition, (i), (ii), (iii) are satisfied! ■

Connectedness (Section 22)

Definition (22.1)

Suppose (S, d) is a metric space. $E \subset S$ is called disconnected if $\exists$ open $U_1, U_2 \subset S$ s.t.: (i) $E \subset U_1 \cup U_2$; (ii) $(E \cap U_1) \cap (E \cap U_2) = \emptyset$; (iii) $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$.

Proposition (Definition 22.1 + remark following it)

E is disconnected iff $\exists A, B \subset E$ s.t. $E = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B^- = \emptyset = A^- \cap B$.

Conclusion: $\boxed{\text{Connectedness is an intrinsic property of } E}$

$x \in A^- \cap B$ iff $x \in B$, and $\exists (a_n) \subset A$, $a_n \to x$.
This does not depend on the ambient space S.

Connectedness (Section 22)

Proof: E disconnected $\Rightarrow$ exist A and B...

Claim: $A = E \cap U_1$, $B = E \cap U_2$ work. Suppose, for contradiction, that $b \in A^- \cap B$. $b \in U_2 \Rightarrow \exists r > 0$ s.t. $\mathbf{B}_r^o(b) \subset U_2$.
$\mathbf{B}_r^o(b) \cap A \subset U_2 \cap (E \cap U_1) = (E \cap U_2) \cap (E \cap U_1) = \emptyset$, hence $b \notin A^-$; contradiction! ■

Proof: E disconnected $\Leftarrow$ exist A and B...

Claim: $U_1 = S \backslash B^-$, $U_2 = S \backslash A^-$ work.
Note: $E \cap U_1 = A$. Indeed, if $x \in E \cap U_1$, then $x \notin B$, hence $x \in A$. If $x \in A$, then $x \notin B^-$, hence $x \in E \cap U_1$. Similarly, $E \cap U_2 = B$. ■

Intervals are connected

Proposition (Example 1, p. 179)

Any interval $I \subset \mathbb{R}$ is connected.

Proof. Suppose, for contradiction, an interval I is disconnected – that is, $I = A \cup B$, with A, B non-empty, $A^- \cap B = \emptyset = A \cap B^-$.

Find $a \in A$, $b \in B$, say $a < b$. Let $c = \sup\{x \in A : x < b\}$. Note that $b \notin A^-$, hence $\exists \delta > 0$ s.t. $(b - \delta, b + \delta) \cap I \subset B$. So, $c < b$. Similarly, $c > a$.

• If $c \in A$, then $c \notin B^-$, hence $\exists \sigma > 0$ s.t. $(c - \sigma, c + \sigma) \subset A$. c is not an upper bound for $\{x \in A : x < b\}$: contradiction!

• If $c \in B$, then $\exists \sigma > 0$ s.t. $(c - \sigma, c + \sigma) \subset B$. c is not the least upper bound for $\{x \in A : x < b\}$: contradiction! ■