# TAM 470 / CSE 450

# Homework 5

## Problem 1 (25 points)

Consider a simple pendulum (point mass $m$ connected to a string of negligible mass) of length $l$ forming an angle $\theta(t)$ with the vertical axis (see Moin Ch. 4 Problem 8 for a figure). Assuming small amplitude swings ($\sin\theta \approx \theta$) and no air resistance or other sources of damping, the ODE governing the motion of the pendulum is

$$\theta''(t) + \frac{g}{l}\theta(t) = 0 \ , \tag{1}$$

where $g$ is the gravitational constant. The initial conditions specify the position $\theta_0$ and angular velocity $\omega_0$ at $t = 0$:

$$\theta(0) = \theta_0, \quad \theta'(0) = \omega_0 \ . \tag{2}$$

Note that $\theta(t)$ is specified in radians and $\theta'(t)$ is in radians per second.

- **(a)** (2 pts) Rewrite (1)-(2) as a linear system of first-order ODEs: $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}$, where $\mathbf{A}$ is a matrix of constant coefficients and $\mathbf{y}(t) = (\theta(t), \theta'(t))$ with initial condition $\mathbf{y}(0) = (\theta_0, \omega_0)$.
- **(b)** (2 pts) Find an expression for the eigenvalues of the matrix $\mathbf{A}$ in terms of the constants $g$ and $l$.
- **(c)** (2 pts) State whether the **forward Euler** scheme will be unconditionally stable, conditionally stable, or unstable for this ODE. If conditionally stable, state the time step restriction required for stability. Be sure to justify your answer with principles from linear stability analysis.
- **(d)** (2 pts) Repeat part (c) for the **backward Euler** scheme.
- **(e)** (2 pts) Repeat part (c) for the **trapezoidal** scheme.
- **(f)** (10 pts, PL) Go to PrairieLearn to write functions program that numerically solve (1)–(2) with the (i) forward Euler scheme, (ii) backward Euler scheme, and (iii) trapezoidal scheme.
- **(g)** (5 pts) Using your code from Part (f), take $g = 9.81$ m/sec$^2$, $l = 1.0$ m, with initial conditions $(\theta_0, \omega_0) = (0.1, 0)$, and solve the ODE using all three schemes up to a time of $T = 10$ using a time step of $h = 0.005$. Plot all three solutions for $\theta(t)$ on the same figure, for $0 \leq t \leq T$. Discuss whether the solution behaviors are consistent with your expectations with respect to stability.

## Problem 2 (20 points)

Consider the differential equation and initial condition

$$y' = e^{\sin(y)} - ty, \qquad y(0) = 1 \tag{3}$$

**(a)** (2 pts) Write down the trapezoidal integration scheme for this particular ODE in the form

$$F(y_{n+1}) = 0$$

**(b)** (2 pts) Write down the Jacobian $\dfrac{\partial F}{\partial y_{n+1}}$ of the equation from part (a).

**(c)** (2 pts) Write down the associated Newton-Raphson update scheme that you could use to solve $F(y_{n+1}) = 0$ for $y_{n+1}$ at each time step, i.e. write $y_{n+1}^{k+1}$ in terms of $y_{n+1}^{k}$, $y_n$, $t_n$ and $t_{n+1}$, where $k$ represents the Newton-Raphson iteration count on a given time step.

**(d)** (10 pts, PL) Go to PrairieLearn to write a function that implements the trapezoidal scheme to solve Equation 3. Use Newton iterations to solve the nonlinear equation at each time step, with a stopping criteria of $|F(y_{n+1}^{k+1})| < 10^{-12}$, where superscripts refer to the iteration count and subscripts refer to the time value.

**(e)** (4 pts) Using your code from part (d), plot the solution for $h = 0.5$ and $h = 0.1$ for $t \in [0, 8]$.

## Problem 3: 4 credit-hour students only (10 points)

Recall our discussion of the following ODE from lecture:

$$y' = \lambda(y - \cos t) - \sin t, \quad y(0) = y_0 \tag{4}$$

which has exact solution

$$y(t) = e^{\lambda t}(y_0 - 1) + \cos t \tag{5}$$

Recall the oscillating behavior shown in class for large negative $\lambda$; for example, Figure 1 shows the solution using backward Euler and trapezoidal methods for $h = 0.2$ and $\lambda = -10^6$:

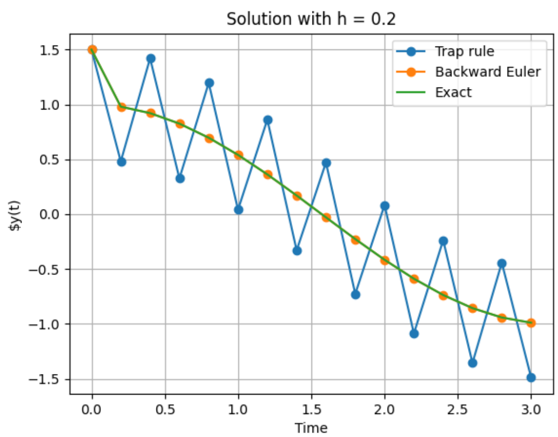


Figure 1: Numerical solutions of Equation 4 using a time step of $h = 0.2$

In the steps below, you'll perform some analysis to explain why exactly the trapezoidal scheme exhibits the behavior shown in Figure 1 when applied to the ODE in Equation 4.

**(a)** (2 pts) Write down the trapezoidal integration scheme for this specific ODE, that is, express $y_{n+1}$ in terms of $y_n$, $t_n$, $t_{n+1}$, time step size $h$ and the parameter $\lambda$.

**(b)** (2 pts) Write code that implements the trapezoidal scheme to solve this problem for $\lambda = -10^6$ and $y_0 = 1.5$ with a time step of $h = 0.1$ on the interval $t \in [0, 3]$. Plot the results.

**(c)** (2 pts) Show that if both $|h\lambda| \gg 1$ and $|\lambda| \gg 1$, the trapezoidal update scheme for $y_{n+1}$ can be approximated as

$$y_{n+1} \approx -y_n + \cos t_n + \cos t_{n+1} \tag{6}$$

**(d)** (2 pts) Show that the approximation from part (c) implies that the numerical solution at time $t_n$ can be written in terms of initial condition $y_0$ as

$$y_n \approx (-1)^n(y_0 - 1) + \cos t_n \tag{7}$$

**(e)** (2 pts) Argue using words and/or a few calculations why the result from part (d) explains the oscillating behavior of the trapezoidal scheme shown in Figure 1. Additionally, identify the initial condition for which oscillations in the numerical solution will not occur.