Example

Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Does (s_n) converge?

(s_n) is increasing: for any n, $s_{n+1} = s_n + \frac{1}{n+1}$. Is (s_n) bounded? **No!**

$s_{2^m} = \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{m-1}+1} + \ldots + \frac{1}{2^m}\right)$
$= \frac{1}{1} + \frac{1}{2} + \sum_{j=1}^{m-1} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \geqslant \frac{3}{2} + \frac{1}{2}(m-1) = \frac{m+2}{2}$.

The sequence (s_n) is unbounded, hence it **diverges to** $+\infty$.

Optional: s_n "behaves like" $\ln n$. $\lim \frac{s_n}{\ln n} = 1$, and $\lim(s_n - \ln n) = \gamma$ – the Euler-Mascheroni constant. $\gamma = 0.577\ldots$.

Open question: is γ rational?

Definition of $\liminf$ and $\limsup$

Convention: if a non-void set $A \subset \mathbb{R}$ is not bounded above (below), set $\sup A = +\infty$ (resp. $\inf A = -\infty$).

For a sequence (s_n) and $N \in \mathbb{N}$, define $u_N = \sup\{s_n : n > N\}$ (also denoted by $\sup_{n>N} s_n$).
If (s_n) is not bounded above, then $u_N = +\infty$ for any N.
If (s_n) is bounded above, then (u_N) is a decreasing sequence of real numbers, hence it has a limit (in $\mathbb{R} \cup \{-\infty\}$).

Definition (10.6 – $\limsup$ of the sequence (s_n))

$\limsup s_n = \lim u_N$ (this equals $+\infty$ if $\forall N$, $u_N = +\infty$).

Similarly, let $v_N = \inf\{s_n : n > N\}$ (also denoted by $\inf_{n>N} s_n$).
Either $v_N = -\infty$ for any N (if (s_n) is not bounded below), or (v_N) is an increasing sequence of real numbers.
Define $\liminf s_n = \lim v_N$.

More on lim inf and lim sup

Observation: $\boxed{\limsup s_n \geqslant \liminf s_n}$.
Indeed, $u_N = \sup\{s_n : n > N\} \geqslant \inf\{s_n : n > N\} = v_N$, hence
$\limsup s_n = \lim u_N \geqslant \lim v_N = \liminf s_n$.

Example: $s_n = \begin{cases} 1/n & n \textbf{ even} \\ -n & n \textbf{ odd} \end{cases}$. **Find** $\limsup s_n$, $\liminf s_n$.

(1) $\limsup s_n = \lim_N u_N$, where $u_N = \sup_{n>N} s_n$.
$u_1 = \sup_{n>1} s_n = \sup\{\frac{1}{2}, -3, \frac{1}{4}, -5, \ldots\} = \frac{1}{2}$,
$u_2 = \sup\{-3, \frac{1}{4}, -5, \ldots\} = \frac{1}{4}$, $u_3 = \sup\{\frac{1}{4}, -5, \ldots\} = \frac{1}{4}$, ...
In general, for $k \in \mathbb{N}$, $u_{2k} = u_{2k+1} = \frac{1}{2k+2}$. $0 < u_n < \frac{1}{n}$, hence, by Squeeze Theorem, $\lim u_N = 0$. Thus, $\boldsymbol{\limsup s_n = 0}$.

(2) $\liminf s_n = \lim_N v_N$, where $v_N = \inf_{n>N} s_n$.
(s_n) is not bounded below, hence $v_N = -\infty$ for any N, and so,
$\boldsymbol{\liminf s_n = -\infty}$.

$\liminf$ and $\limsup$ vs. $\lim$

Theorem (Theorem 10.7)

(1) If $\lim s_n$ *is defined, then* $\liminf s_n = \lim s_n = \limsup s_n$.
(2) If $\liminf s_n = s = \limsup s_n$, *then* $\lim s_n$ *is defined, and* $= s$.

Proof of (1). Recall: $u_N = \sup_{n>N} s_n$, $v_N = \inf_{n>N} s_n$, $\limsup s_n = \lim u_N$, $\liminf s_n = \lim v_N$.

• $\lim s_n = +\infty$. Need: $\lim u_N = +\infty = \lim v_N$ – that is, $\forall A \in \mathbb{R}\ \exists K \in \mathbb{N}$ s.t. $u_N, v_N \geqslant A$ for $N > K$.

$\forall A \in \mathbb{R}\ \exists K \in \mathbb{N}$ s.t. $s_n > A$ for $n > K$. For $N > K$, $u_N = \sup_{n>N} s_n > A$, and $v_N = \inf_{n>N} s_n \geqslant A$ (indeed, A is a lower bound for $\{s_n : n > N\}$). $\square$

• $\lim s_n = -\infty$: handled similarly. $\square$

$\liminf$ and $\limsup$ vs. $\lim$, page 2

Theorem (Theorem 10.7)

(1) If $\lim s_n$ is defined, then $\liminf s_n = \lim s_n = \limsup s_n$.
(2) If $\liminf s_n = s = \limsup s_n$, then $\lim s_n$ is defined, and $= s$.

Proof of (1), continued.

- $\lim s_n = s \in \mathbb{R}$. Show that $\lim u_N = s = \lim v_N$ – that is, for $\varepsilon > 0$ $\exists K \in \mathbb{N}$ s.t. $|u_N - s|, |v_N - s| \leqslant \varepsilon$ for $N > K$.

Find K s.t. $|s_n - s| < \varepsilon$ for $n > K$. In other words, $s - \varepsilon < s_n < s + \varepsilon$ for such n.
If $N > K$, then $s + \varepsilon$ is an upper bound for $\{s_n : n > N\}$, hence $u_N = \sup_{n>N} s_n \leqslant s + \varepsilon$.
Similarly, $v_N = \inf_{n>N} s_n \geqslant s - \varepsilon$ for $N > K$.
Also, $u_N = \sup_{n>N} \geqslant v_N = \inf_{n>N} s_n$.
Conclude: for $N > K$, $s + \varepsilon \geqslant u_N \geqslant v_N \geqslant s - \varepsilon$. Thus, $|u_N - s|, |v_N - s| \leqslant \varepsilon$ for such N. ■

lim inf and lim sup vs. lim, page 3

Theorem (Theorem 10.7)

(1) If $\lim s_n$ *is defined, then* $\liminf s_n = \lim s_n = \limsup s_n$.
(2) If $\liminf s_n = s = \limsup s_n$, *then* $\lim s_n$ *is defined, and* $= s$.

Proof of (2) – case of $\limsup s_n = \liminf s_n = s \in \mathbb{R}$.
For $\varepsilon > 0$, find $N \in \mathbb{N}$ s.t. $|s_n - s| < \varepsilon$ for $n > N$.
Note: for $n > N$, $u_N = \sup_{k>N} s_k \geqslant s_n \geqslant \inf_{k>N} s_k = v_N$.
Find K_1 (K_2) s.t. $|u_N - s| < \varepsilon$ ($|v_N - s| < \varepsilon$) for $N > K_1$ (resp. $N > K_2$).
Pick $N > \max\{K_1, K_2\}$. If $n > N$, then $u_N < s + \varepsilon$ and $v_N > s - \varepsilon$.
If $n > N$, then $s - \varepsilon < v_N \leqslant s_n \leqslant u_N < s + \varepsilon$, hence $|s - s_n| < \varepsilon$.

Cases of $\limsup s_n = \liminf s_n = \pm\infty$: exercise! ∎

Cauchy sequences

Definition (Cauchy sequences – 10.8)

A sequence (s_n) is called Cauchy if $\forall \varepsilon > 0 \; \exists N$ s.t. $|s_n - s_m| < \varepsilon$ for $n, m > N$.

Theorem (Theorem 10.11 mostly)

A sequence (s_n) converges iff it is Cauchy.

Being Cauchy gives an intrinsic criterion for convergence of a sequence (we do not need to guess the limit s).

Example from homework. If $|s_{n+1} - s_n| < 2^{-n}$ for any n, then (s_n) is Cauchy, hence convergent.

Lemma (10.9)

Any convergent sequence is Cauchy.

Cauchy sequences

Proof: if (s_n) converges, then it is Cauchy. For $\varepsilon > 0$ find N s.t. $|s_n - s_m| < \varepsilon$ for $n, m > N$. Let $s = \lim s_n$. Find N s.t. $|s_k - s| < \frac{\varepsilon}{2}$ for $k > N$. If $n, m > N$, then
$|s_n - s_m| = |(s_n - s) - (s_m - s)| \leqslant |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ∎

Lemma (Lemma 10.10)

Every Cauchy sequence is bounded.

Proof. If (s_n) is Cauchy, find N s.t. $|s_k - s_m| < 1$ for $k, m > N$.
Pick $m > N$, then $|s_k| < |s_m| + 1$ for $k > N$ ($\triangle$ Ineq.).
For any n, $|s_n| \leqslant \max\{\max_{k \leqslant N} |s_k|, |s_m| + 1\}$. ∎

Proof of Theorem 10.11: Cauchy $\Leftrightarrow$ convergent

Cauchy $\Leftarrow$ convergent: done before (Lemma 10.9).

Proof: (s_n) Cauchy $\Rightarrow$ convergent. (s_n) is bounded, hence $\limsup s_n, \liminf s_n \in \mathbb{R}$. We need: $\limsup s_n = \liminf s_n$.
$\limsup s_n \geqslant \liminf s_n$, hence it suffices to show that, if $\varepsilon > 0$, then $\limsup s_n \leqslant \liminf s_n + \varepsilon$.
Find N s.t. $|s_k - s_m| < \frac{\varepsilon}{2}$ for $k, m > N$.
Pick $m > N$, then $s_k < s_m + \frac{\varepsilon}{2}$ for $k > N$.
For $j \geqslant N$, $u_j = \sup_{k>j} s_k \leqslant s_m + \frac{\varepsilon}{2}$, hence $\limsup s_n = \lim u_j \leqslant s_m + \frac{\varepsilon}{2}$.
Similarly, $\liminf s_n \geqslant s_m - \frac{\varepsilon}{2}$.
Conclude $\limsup s_n \leqslant \liminf s_n + \varepsilon$. ∎

Subsequences (Section 11)

Definition (Definition 11.1)

A sequence (t_k) is a subsequence of (s_n) if there exists a strictly increasing sequence $n_1 < n_2 < \dots$ so that $t_k = s_{n_k}$ for any k.

Example. Suppose $s_n = \frac{1}{n}$.
Subsequences: $t_k = \frac{1}{k^2}$ $(n_k = k^2)$
$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \dots$ (reciprocals of prime numbers). $n_k = k$-th prime number.
Not subsequences: $\frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{1}{8}, \dots$ (one term of (s_n) is repeated),
$\frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \dots$ (order is changed).

Example. $s_n = (-1)^n + \frac{1}{n}$. Then (s_n) diverges. $s_{2k} = 1 + \frac{1}{2k}$, so (s_{2k}) converges.

To be proved later: every sequence has a subsequence with limit.