TAM 470 - Homework 2 Solutions

Problem 1

Consider Taylor expansions of f_{j-1} and f_{j-2}.

$$f_{j-1} = f_j - h f_j' + \frac{h^2}{2} f_j'' - \frac{h^3}{6} f_j''' + \frac{h^4}{24} f_j^{iv} + \ldots\ldots$$

$$f_{j-2} = f_j - 2h f_j' + \frac{(2h)^2}{2} f_j'' - \frac{(2h)^3}{6} f_j''' + \frac{(2h)^4}{24} f_j^{iv} + \ldots\ldots$$

Lets evaluate RHS of given equation using above expressions.

$$\frac{f_j - 2f_{j-1} + f_{j-2}}{h^2} = \frac{1}{h^2}\Big[\cancel{f_j} - 2\Big(\cancel{f_j} - \cancel{hf_j'} + \frac{h^2}{2} f_j'' - \frac{h^3}{6} f_j''' + \ldots\Big) + \cancel{f_j} - \cancel{2hf_j'} + 2h^2 f_j'' - \frac{4h^3}{3} f_j''' \ldots\Big]$$

$$= \frac{1}{h^2}\left[h^2 f_j'' + \left(\frac{h^3}{3} - \frac{4h^3}{3}\right) f_j''' + \ldots\right]$$

$$= f_j'' - h f_j''' + \text{H.O.T}$$

Rearranging,

$$f_j'' = \frac{f_j - 2f_{j-1} + f_{j-2}}{h^2} + h f_j''' + H.O.T$$

$\Longrightarrow \tau = h f_j'''$

$\Longrightarrow$ This scheme is 1^{st} order accurate.

NOTE :- H.O.T means higher order terms.

Problem 2

Start with an expression similar to Moin 2.10.

For our case, we look at $x_{i-1}, x_i, x_{i+1}, x_{i+2}$.

$$\therefore\ f_i' + \sum_{k=-1}^{2} a_k f_{i+k} = O(?) \qquad (*)$$

a_{-1}, a_0, a_1, a_2 are to be determined.

Construct the Taylor table as shown below.

	f_i	f_i'	f_i''	f_i'''	f_i^{iv}	f_i^{v}
f_i'	0	1	0	0	0	0
$a_{-1} f_{i-1}$	a_{-1}	$-a_{-1}h$	$+\frac{a_{-1}h^2}{2}$	$-\frac{a_{-1}h^3}{6}$	$+\frac{a_{-1}h^4}{24}$	$-\frac{a_{-1}h^5}{120}$
$a_0 f_i$	a_0	0	0	0	0	0
$a_1 f_{i+1}$	a_1	$a_1 h$	$\frac{a_1 h^2}{2}$	$\frac{a_1 h^3}{6}$	$\frac{a_1 h^4}{24}$	$\frac{a_1 h^5}{120}$
$a_2 f_{i+2}$	a_2	$a_2(2h)$	$\frac{a_2(2h)^2}{2}$	$\frac{a_2(2h)^3}{6}$	$\frac{a_2(2h)^4}{24}$	$\frac{a_2(2h)^5}{120}$

Set as many lower derivative columns to 0

We have four unknowns $\Rightarrow$ we look to solve using first 4 columns.

$$\therefore \quad a_{-1} + a_0 + a_1 + a_2 = 0 \quad - (2.1)$$

$$1 + h(-a_{-1} + a_1 + 2a_2) = 0 \quad - (2.2)$$

$$h^2\left(\frac{a_{-1}}{2} + \frac{a_1}{2} + 2a_2\right) = 0 \quad - (2.3)$$

$$h^3\left(-\frac{a_{-1}}{6} + \frac{a_1}{6} + \frac{8a_2}{6}\right) = 0 \quad - (2.4)$$

Solving.

$$\boxed{\begin{aligned} a_{-1} &= \frac{1}{3h} \\ a_0 &= \frac{1}{2h} \\ a_1 &= -\frac{1}{h} \\ a_2 &= \frac{1}{6h} \end{aligned}}$$

Leading term will have the 5^{th} column i.e f^{IV} term. The leading term is,

$$\frac{h^4}{24} \cdot \frac{1}{h}\left[\frac{1}{3} - 1 + \frac{16}{6}\right] f_i^{IV} = \frac{h^3}{12} f_i^{IV}$$

Using a_{-1}, a_0, a_1, a_2 in (*), we get,

$$\therefore f_i' = \frac{-2f_{i-1} - 3f_i + 6f_{i+1} - f_{i+2}}{6h} + \frac{h^3}{12} f_j^{iv} + H.O.T$$

$\therefore$ The method is 3^{rd} order accurate.

Problem 3 (Moin exercise 4)

(a) $f_0' + \alpha f_1' - \frac{1}{h}\left(af_0 + bf_1 + cf_2 + df_3\right) = O(?)$

Taylor table

	f_0	f_0'	f_0''	f_0'''	f_0^{IV}	f_0^{V}	f_0^{VI}
f_0'	0	1	0	0	0	0	0
$\alpha f_1'$	0	α	αh	$\frac{\alpha h^2}{2}$	$\frac{\alpha h^3}{6}$	$\frac{\alpha h^4}{24}$	$\frac{\alpha h^5}{120}$
$-\frac{a}{h} f_0$	$-\frac{a}{h}$	0	0	0	0	0	0
$-\frac{b}{h} f_1$	$-\frac{b}{h}$	$-b$	$-\frac{bh^2}{h2}$	$-\frac{bh^2}{6}$	$-\frac{bh^3}{24}$	$-\frac{bh^4}{120}$	$-\frac{bh^5}{720}$
$-\frac{c}{h} f_2$	$-\frac{c}{h}$	$-2c$	$-\frac{c}{h}\frac{(2h)^2}{2}$	$-\frac{c}{h}\frac{(2h)^3}{6}$	$-\frac{c}{h}\frac{(2h)^4}{24}$	$-\frac{c}{h}\frac{(2h)^5}{120}$	$-\frac{c}{h}\frac{(2h)^6}{720}$
$-\frac{d}{h} f_3$	$-\frac{d}{h}$	$-3d$	$-\frac{d}{h}\frac{(3h)^2}{2}$	$-\frac{d}{h}\frac{(3h)^3}{6}$	$-\frac{d}{h}\frac{(3h)^4}{24}$	$-\frac{d}{h}\frac{(3h)^5}{120}$	$-\frac{d}{h}\frac{(3h)^6}{720}$

To solve for (a,b,c,d) in terms of α, we set the first 4 columns to be 0.

$\therefore\ a+b+c+d = 0 \quad - (5.1)$

$$b + 2c + 3d = (1+\alpha) \quad - (5.2)$$

$$b + 4c + 9d = 2\alpha \quad - (5.3)$$

$$b + 8c + 27d = 3\alpha \quad - (5.4)$$

Solving using sympy,

$$\boxed{a = -\frac{(11+2\alpha)}{6}, \quad b = \frac{6-\alpha}{2}, \quad c = \frac{2\alpha - 3}{2}, \quad d = \frac{2-\alpha}{6}}$$

One good choice is $\alpha = 2$ because this sets $d = 0$. Now we can get same third order accuracy using just f_0, f_1 and f_2.

(b) For the scheme to be fourth order accurate, we take the 5^{th} column as well (along with equations (5.1)-(5.4))

$$\therefore \quad b + 16c + 81d = 4\alpha \quad - (5.5)$$

Solving (5.1) – (5.5) for (α, a, b, c, d), we get.

$$\boxed{\alpha = 3 \,, \quad a = -\frac{17}{6} \,, \quad b = 3/2 \,, \quad c = 3/2 \,, \quad d = -1/6}$$

Leading term is $\frac{h^4}{120} f_0^{v} \left[\alpha - b - 32c - 243d \right]$

$$= -\frac{h^4}{20} f_0^{v}$$

$\therefore$ The scheme can be written as :-

$$\boxed{f_0' + 3f_1' = \frac{-17f_0 + 9f_1 + 9f_2 - f_3}{6h} - \frac{h^4}{20} f_0^{v} + H.O.T}$$