TAM 470 - HW 8 Solutions

Problem 1

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{-T_{j-2} + 16T_{j-1} - 30T_j + 16T_{j+1} - T_{j+2}}{12(\Delta x)^2}$$

(a) von-Neumann,

$$T_j^n = \sigma^n e^{ikx_j}$$

Substituting above,

$$\left(\frac{\sigma^{n+1} - \sigma^n}{\Delta t}\right) e^{ikx_j} = \frac{\alpha \sigma^n e^{ikx_j}}{12(\Delta x)^2}\left[-\left(e^{ik2\Delta x} + e^{-2ki\Delta x}\right) + 16\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - 30\right]$$

$$\Rightarrow \frac{\sigma - 1}{\Delta t} = \frac{\alpha}{12(\Delta x)^2}\left[32\cos(k\Delta x) - 2\cos(2k\Delta x) - 30\right]$$

$$\rightarrow \sigma = 1 + \alpha \Delta t\left[32\cos(k\Delta x) - 2\cos(2k\Delta x) - 30\right] / 12(\Delta x)^2$$

For stability, $|\sigma| \leq 1$

$$\Rightarrow \Delta t \leq \frac{2 \times 12(\Delta x)^2}{\left|\alpha\left[32\cos(k\Delta x) - 2\cos(2k\Delta x) - 30\right]\right|}$$

The denominator attains maximum absolute value when $k\Delta x = (2m-1)\pi$, $m = 1, 2 \ldots$

$$\Rightarrow \quad \Delta t \leq \frac{2 \times 12 (\Delta x)^2}{\alpha (32 + 2 + 30)}$$

$$\Rightarrow \quad \boxed{\Delta t \leq \frac{3 (\Delta x)^2}{8 \alpha}}$$

(b) Modified wave-number analysis.

Let $T = \psi(t)\, e^{ikx}$

$$\therefore \quad \frac{d\psi}{dt} = -\alpha k^2 \psi$$

Discrete form, $T_j = \psi(t)\, e^{ikx_j}$

$$\therefore \quad \frac{\partial T_j}{\partial t} = \frac{\alpha}{12} \left[\frac{-T_{j-2} + 16 T_{j-1} - 30 T_j + 16 T_{j+1} - T_{j+2}}{(\Delta x)^2} \right]$$

$$\rightarrow \cancel{e^{ikx_j}} \frac{d\psi}{dt} = \frac{\alpha \psi}{12 (\Delta x)^2} \left[-\left(e^{-ik2\Delta x} + e^{ik2\Delta x}\right) + 16 \left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - 30 \right] \cancel{e^{ikx_j}}$$

$$= \frac{\alpha}{12 (\Delta x)^2} \left[32 \cos(k\Delta x) - 2 \cos(2k\Delta x) - 30 \right] \psi$$

$$= -\alpha (k')^2 \psi$$

$$(k')^2 = \frac{1}{12(\Delta x)^2}\left[-32\cos(k\Delta x) + 2\cos(2k\Delta x) + 30\right]$$

Recalling the ODE stability theory, we have $\lambda = -(k')^2\alpha$. Since $(k')^2$ and α are real,

$$\Delta t \leq \frac{2}{|\alpha (k')^2|}$$

$$(k')^2\Big|_{max} = \frac{64}{12(\Delta x)^2} \quad \left(\text{attained when } k\Delta x = (2m-1)\pi\right)$$

$$\therefore \boxed{\Delta t \leq \frac{3(\Delta x)^2}{8\alpha}}$$

Problem 2

$$(1+2r)\,\phi_j^{(n+1)} = (1-2r)\,\phi_j^{n-1} + 2r\,\phi_{j+1}^{(n)} + 2r\,\phi_{j-1}^{(n)}$$

Let $\phi_j^{(n)} = \sigma^n e^{ikx_j}$

$$\Rightarrow (1+2r)\,\sigma^{n+1} e^{ikx_j} = (1-2r)\,\sigma^{n-1} e^{ikx_j} + 2r\,\sigma^n e^{ik(x_j+\Delta x)} + 2r\,e^{ik(x_j-\Delta x)}\sigma^n$$

$$\rightarrow (1+2r)\,\sigma = \frac{(1-2r)}{\sigma} + 2r\left(e^{ik\Delta x} + e^{-ik\Delta x}\right)$$

$$= \frac{(1-2r)}{\sigma} + 2r\cdot 2\cos(k\Delta x)$$

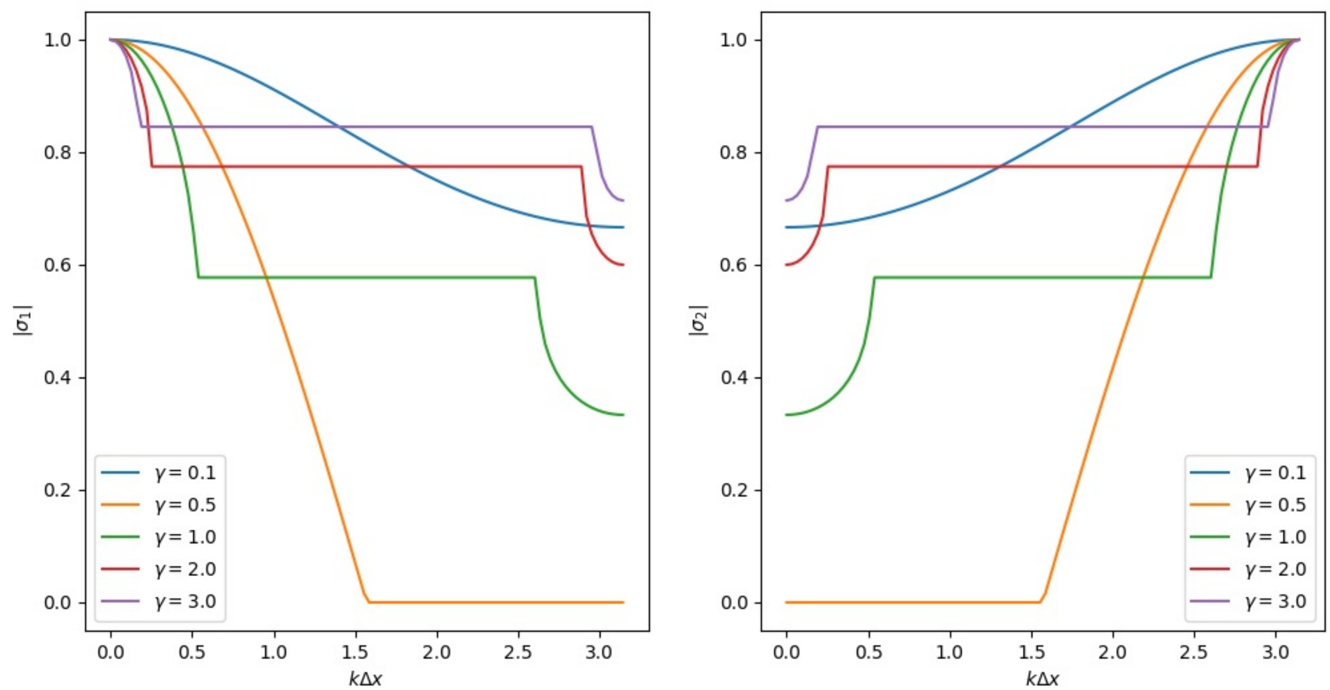
$$(1+2r)\sigma^2 - 4r\sigma\cos(k\Delta x) - (1-2r) = 0$$

$$\sigma = \frac{4r\cos(k\Delta x) \pm \sqrt{16r^2\cos^2(k\Delta x) + 4(1-4r^2)}}{2(1+2r)}$$

$$= \frac{4r\cos k\Delta x \pm 2\sqrt{4r^2\cos^2(k\Delta x) + 1 - 4r^2}}{2(1+2r)}$$

$$= \frac{2r\cos(k\Delta x) \pm \sqrt{1-4r^2\sin^2(k\Delta x)}}{(1+2r)}$$

(b)



Problem 3

$$\frac{\partial T}{\partial t} = -c\frac{\partial T}{\partial x} + \alpha\frac{\partial^2 T}{\partial x^2}, \quad x \in [0, L], \; t \geq 0$$

(a) Semi-discretization yields,

$$\frac{\partial T_j}{\partial t} = -c\left(\frac{T_{j+1} - T_{j-1}}{2\Delta x}\right) + \alpha\left(\frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2}\right)$$

$$= \left(\frac{-c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)T_{j+1} - \frac{2\alpha T_j}{(\Delta x)^2} + \left(\frac{+c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)T_{j-1}$$

Assuming x has been discretized to $N_x + 1$ grid points, we have $j = 1, 2 \ldots (N_x - 1)$

Consider $j = 1$,

$$\frac{\partial T_1}{\partial t} = \left(\frac{-c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)T_2 - \frac{2\alpha}{(\Delta x)^2}T_1 + \left(\frac{+c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)T_0$$

Consider $j = N_x - 1$

$$\frac{\partial T_{N_x-1}}{\partial t} = \left(\frac{-c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)T_{N_x} - \frac{2\alpha}{(\Delta x)^2}T_{N_x-1} + \left(\frac{+c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)T_{N_x-2}$$

For homogeneous Dirichlet conditions,

$T_0 = T_{N_x} = 0$

$\therefore$ Above equations become,

$$\frac{\partial T_1}{\partial t} = -\frac{2\alpha}{(\Delta x)^2} T_1 + \left(-\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) T_2$$

$$\frac{\partial T_{N_x-1}}{\partial t} = \left(+\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) T_{N_x-2} - \frac{2\alpha}{(\Delta x)^2} T_{N_x-1}$$

and,

$$\frac{\partial T_k}{\partial t} = \left(+\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) T_{k-1} - \frac{2\alpha}{(\Delta x)^2} T_k + \left(-\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) T_{k+1},$$

$k = 2, 3 \ldots (N_x - 2)$

Let $\bar{T} = [T_1, T_2, T_3 \ldots T_{N_x-1}]^T$, then

$$\frac{\partial \bar{T}}{\partial t} = \begin{bmatrix} -\frac{2\alpha}{(\Delta x)^2} & \left(-\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) & 0 & 0 & \cdots & & \\ \left(+\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) & -\frac{2\alpha}{(\Delta x)^2} & \left(-\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right) & 0 & 0 & 0 & \\ & & \ddots & & & & \\ & & & & & +\frac{C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2} & -\frac{2\alpha}{(\Delta x)^2} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N_x-1} \end{bmatrix}$$

$\therefore \dfrac{\partial \bar{T}}{\partial t} = [A]\bar{T}$, where $[A]$ is the tridiagonal matrix shown above,

$$[A] = B\left[\frac{+c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}, \quad \frac{2\alpha}{(\Delta x)^2}, \quad \frac{-C}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right]$$

(b) $$\lambda_j = -\frac{2\alpha}{(\Delta x)^2} + 2\sqrt{\left(\frac{\alpha}{(\Delta x)^2}\right)^2 - \left(\frac{C}{2\Delta x}\right)^2}\cos\left(\frac{\pi j}{N}\right),$$

$j = 1, 2 \ldots N-1$

$N = 50$, $\quad N = \dfrac{L}{\Delta x} \Rightarrow \Delta x = \dfrac{L}{N}$

$$\lambda_j = -\frac{2\alpha}{L^2}N^2 + \frac{2N}{L}\sqrt{\frac{\alpha^2 N^2}{L^2} - \frac{C^2}{4}}\cos\frac{\pi j}{N}$$

Using the given values,

$\lambda_{49} = -7.994$ (i.e a real negative number)

Forward Euler is conditionally stable and

$$h \leq \frac{2}{|\lambda|_{max}} \Rightarrow \underline{h \leq 0.2502}$$

(c) Recall the above Eigen-values.

$$\lambda_j = -\frac{2\alpha}{(\Delta x)^2} + 2\sqrt{\left(\frac{\alpha}{(\Delta x)^2}\right)^2 - \left(\frac{c}{2\Delta x}\right)^2 \cos\left(\frac{\pi j}{N}\right)}$$

For leapfrog to be stable

$Re(\lambda_j) = 0$

case (i) :- $\left[\left(\frac{\alpha}{\Delta x^2}\right)^2 - \left(\frac{c}{2\Delta x}\right)^2\right] > 0$

$\Rightarrow \lambda_j \in \mathbb{R} \Rightarrow$ Unstable

case (ii) $\left[\left(\frac{\alpha}{\Delta x^2}\right)^2 - \left(\frac{c}{2\Delta x}\right)^2\right] < 0$

$$\Rightarrow \lambda_j = -\frac{2\alpha}{(\Delta x)^2} + 2i\sqrt{\left(\frac{c}{2\Delta x}\right)^2 - \left(\frac{\alpha}{\Delta x^2}\right)^2 \cos\left(\frac{\pi j}{N}\right)}$$

$Re(\lambda_j) = 0 \Rightarrow \alpha = 0 \Rightarrow$ No diffusion.

$\therefore \lambda_j = \cancel{2} i \frac{c}{\cancel{2}\Delta x} \cos\left(\frac{\pi j}{N}\right) = i\omega \Rightarrow \omega_j = \frac{c}{\Delta x}\cos\left(\frac{\pi j}{N}\right)$

For stability, $|\omega_{max}\Delta t| \leq 1$

$$\Rightarrow \boxed{\Delta t \leq \frac{\Delta x}{c}\sec\left(\frac{\pi}{N}\right)}$$

Problem 4

(a) The formulation is in last problem,

$$\frac{dT_j}{dt} = B\left[\left(+\frac{c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right), \frac{-2\alpha}{(\Delta x)^2}, \left(-\frac{c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)\right]\begin{bmatrix} T_{j-1} \\ T_j \\ T_{j+1} \end{bmatrix}$$

$$T_j^{n+1} = T_j^n + \Delta t \cdot B\left[\left(+\frac{c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right), \frac{-2\alpha}{(\Delta x)^2}, \left(-\frac{c}{2\Delta x} + \frac{\alpha}{(\Delta x)^2}\right)\right]\begin{bmatrix} T_{j-1}^n \\ T_j^n \\ T_{j+1}^n \end{bmatrix}$$

(b)