SOLUTIONS FOR HOMEWORK 5

THROUGHOUT THIS HOMEWORK, WE ASSUME THAT TRIGONOMETRIC FUNCTIONS – SUCH AS sin, cos, AND tan – ARE CONTINUOUS. OTHER COMMON CALCULUS FACTS ABOUT TRIG FUNCTIONS CAN ALSO BE USED.

14.2. (d) Let $b_n = \dfrac{n^3}{3^n}$, then $\lim_n b_n^{1/n} = \lim \dfrac{(n^{1/n})^3}{3} = \dfrac{1}{3}$, since $\lim n^{1/n} = 1$. By Root Test, $\sum b_n$ converges.

14.4. (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$ **converges**. Indeed, by Exercise 14.5(b), and the fact about p-series, $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2}$ converges. Note that $\frac{2}{n^2} \geq \frac{1}{(n+(-1)^n)^2}$ for $n \geq 2$, hence our series **converges** due to Comparison test.

(b) We are dealing with $\sum y_n$, where

$$y_n = \sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{3\sqrt{n}}.$$

$\displaystyle\sum \frac{1}{3\sqrt{n}} = +\infty$ (can use comparison with $\displaystyle\sum_n \frac{1}{n}$, plus Problem 14.6(b)), hence $\sum y_n$ **diverges** as well.

ALTERNATIVELY, one can compute the partial sums: $s_m = \displaystyle\sum_{n=1}^{m} y_n = \sqrt{m+1} - 1$, and $\lim_m s_m = +\infty$.

14.5. (a) $\displaystyle\sum_{k=j_0}^{\infty} (a_n + b_n) = \lim_m \sum_{k=j_0}^{m} (a_n + b_n) = \lim_m \sum_{k=j_0}^{m} a_n + \lim_m \sum_{k=j_0}^{m} b_n = A + B.$

(b) $\displaystyle\sum_{j=j_0}^{\infty} ka_n = \lim_m \sum_{n=j_0}^{m} ka_n = k \lim_n \sum_{j=j_0}^{m} a_n = kA.$

(c) In general, $\sum a_n b_n \neq AB$. Consider, for instance, $a_n = b_n = \frac{1}{2^n}$ (with $n \geq 0$). Then $A = B = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$, while $\displaystyle\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-1/4} = \frac{4}{3} \neq 2^2.$

14.6. (a) Find $\beta > 0$ s.t. $|b_n| \leq \beta$ for any n. By Problem 14.6(b), $\sum \beta |a_n|$ converges. For any n, $|a_n b_n| \leq \beta |a_n|$, hence, by Comparison Test, $\sum_n a_n b_n$ converges absolutely.

17.4. By Example 5 (and Exercise 8.9) from §8, $\lim \sqrt{x_n} = \sqrt{x}$ whenever $\lim x_n = x$ (for any $x \in [0, \infty)$). Now apply the sequential criterion for continuity.

17.9. (c) $\delta = \varepsilon$ with work. We need to show that, if $|x| = |x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$. For $x = 0$, the inequality is obvious. Otherwise, $|f(x) - f(0)| = |x| \left|\sin \frac{1}{x}\right| \leq |x| < \varepsilon$, which is what we want.

(d) We can take $\delta = \min\left\{\dfrac{\varepsilon}{3(|x_0|+1)^2}, 1\right\}$. If $|x - x_0| < \delta$, then $|x| < |x_0| + 1$, hence $\left|x^2 + xx_0 + x_0^2\right| < 3(|x_0|+1)^2$. Per Hint, if $|x - x_0| < \delta$, then $\left|x^3 - x_0^3\right| = |x - x_0|\left|x^2 + xx_0 + x_0^2\right| < 3(|x_0|+1)^2\delta = \varepsilon$.

17.10. (b) Use Definition 17.1 to show the lack of continuity. Let $x_n = \dfrac{2}{(2n+1)\pi}$; clearly $\lim x_n = 0$. However, $f(x_n) = \sin\left(n\pi + \dfrac{\pi}{2}\right) = (-1)^n$, so $(f(x_n))$ does not converge to $f(0) = 0$.