MATH 447: Real Variables - Homework #10

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Problem 1 (1.8 a)**.** The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$. Prove $n^2 > n + 1$ for all integers $n \geq 2$.

Solution 1. *Proof.* We will prove the base case of $n = 2$ first:

$$(2)^2 > (2) + 1 \tag{1}$$

$$4 > 3 \tag{2}$$

The above verifies the base case. The inductive hypothesis is as follows:

$$n^2 > n + 1, \ n \geq 2 \tag{3}$$

We want to prove the case such that P_{n+1} is true whenever P_n is true and $n \geq m$. The inductive step is as follows:

$$(n + 1)^2 > (n + 1) + 1 \tag{4}$$
$$n^2 + 2n + 1 > n + 2 \tag{5}$$
$$n^2 + n > 1 \tag{6}$$
$$n(n + 1) > 1 \tag{7}$$

By the inductive hypothesis, $n^2 > n + 1, n \geq 2$,

$$n(n^2) > n(n + 1) > 1 \tag{8}$$
$$n^3 > 1 \tag{9}$$

The last line is true for all $n \geq 2$. □

Problem 2 (2.8)**.** Find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$.

Solution 2. To find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$, we can use the Rational Zeros Theorem.

Corollary—Rational Zeros Theorem: If a polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0 \tag{10}$$

with integer coefficients has a rational solution $x = \frac{p}{q}$ —where p and q are integers with no common factors and $q \neq 0$—, then:

(a) p must be a factor of the constant term a_0

(b) q must be a factor of the leading coefficient a_n

First, let's identify the coefficients:

(a) $c_8 = 1$

(b) $c_5 = -4$

(c) $c_3 = 13$

(d) $c_1 = -7$

(e) $c_0 = 1$

According to the theorem, if $\frac{c}{d}$ is a rational solution (where c and d are integers with no common factors and $d \neq 0$), then:

(a) c must divide $c_0 = 1$

(b) d must divide $c_8 = 1$

The only integers that divide 1 are 1 and -1. Therefore, the only possible rational solutions are:

(a) $\frac{1}{1} = 1$

(b) $\frac{-1}{1} = -1$

Now, we need to check if these candidates actually satisfy the equation:
For $x = 1$:

$$1^8 - 4(1^5) + 13(1^3) - 7(1) + 1 = 1 - 4 + 13 - 7 + 1 = 4 \neq 0 \tag{11}$$

For $x = -1$:

$$(-1)^8 - 4((-1)^5) + 13((-1)^3) - 7(-1) + 1 = 1 + 4 - 13 + 7 + 1 = 0 \tag{12}$$

Therefore, the only rational solution to the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$ is $x = -1$.

Problem 3 (3.5). (a) Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

(b) Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Solution 3. *Proof.* (a) If $|b| \leq a$, then by definition, $-a \leq b \leq a$. This is because $|b| \leq a$ implies b is within the interval $[-a, a]$.

(b) If $-a \leq b \leq a$, then b is within the interval $[-a, a]$. This directly implies $|b| \leq a$ since the maximum deviation of b from zero is a.

Thus, $|b| \leq a$ if and only if $-a \leq b \leq a$. $\square$

Proof. We will prove this inequality using the triangle inequality and considering both possible cases. **Using the Triangle Inequality:**

(a) The triangle inequality states $|a| = |(a - b) + b| \leq |a - b| + |b|$.

(b) Rearranging gives $|a| - |b| \leq |a - b|$.

Consider the Reverse Situation:

(a) Similarly, $|b| = |(b - a) + a| \leq |b - a| + |a| = |a - b| + |a|$.

(b) Rearranging gives $|b| - |a| \leq |a - b|$.

Combine the Results:

(a) From the two inequalities, we have:

$$|a| - |b| \leq |a - b| \quad \text{and} \quad |b| - |a| \leq |a - b| \tag{13}$$

Conclusion:

(a) The absolute value $||a| - |b||$ is defined as:

$$||a| - |b|| = \max(|a| - |b|, |b| - |a|) \tag{14}$$

(b) Therefore, $||a| - |b|| \leq |a - b|$.

Thus, we have proven that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$. $\square$

Problem 4 (3.8)**.** Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Solution 4. *Proof.* We will prove this statement using a proof by contradiction. Assume the hypothesis: For every $b_1 > b$, we have $a \leq b_1$. Suppose, for the sake of contradiction, that $a > b$. Consider $b_1 = \frac{a+b}{2}$. Note that:

(a) $b_1 > b$ (because $a > b$)

(b) $b_1 < a$ (because it's the midpoint between a and b)

By our initial assumption, since $b_1 > b$, we must have $a \leq b_1$. However, we also showed that $b_1 < a$. This is a contradiction: we can't have both $a \leq b_1$ and $b_1 < a$.

Therefore, our supposition that $a > b$ must be false. We conclude that $a \leq b$.

Thus, we have shown that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$. $\square$

Problem 5 (4.1 r)**.** For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA".

$\cap_{n=1}^{\infty}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$

Solution 5. It is observed that the intersection of the sequences above converges to 1 as n approaches infinity. Therefore, 2, 3, and 4 are all upper bounds for the set.

Problem 6 (4.8)**.** Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.

Solution 6. Observe S is bounded above and T is bounded below. Prove sup $S \leq$ inf T Give an example of such sets S and T where $S \cap T$ is nonempty. Give an example of sets S and T where sup $S =$ inf T and $S \cap T$ is the empty set.

(a) Let $M = t,\ t \in T$. Then $S \leq M$ for all $s \in S$. By Def 4.2, M is an upper bound of S, and S is bounded above.

(b) Let $m = s$, such that $s \in S$. Then, $m \leq t$ for all $t \in T$. Then, m is a lower bound of T, and T is bounded below.

To prove $\sup S \leq \inf T$:

(a) By the given property, we know that $s \leq t$ for all $s \in S$ and all $t \in T$.

(b) Let $M = \sup S$. By definition of supremum, $s \leq M$ for all $s \in S$.

(c) For any $t \in T$, we have $s \leq t$ for all $s \in S$.

(d) Therefore, $M = \sup S \leq t$ for all $t \in T$.

(e) Since $M \leq t$ for all $t \in T$, M is a lower bound for T.

(f) By definition of infimum, $\inf T$ is the greatest lower bound of T.

(g) Thus, $M \leq \inf T$.

Therefore, we have proven that $\sup S \leq \inf T$.

An example of such sets S and T where $S \cap T$ is nonempty: $S = [0, 1], T = [1, 2]$

An example of sets S and T where sup $S =$ inf T and $S \cap T$ is the empty set: $S = [0, 1), T = (1, 2]$

Problem 7 (8.5)

Problem 7 (8.5)**.** (a) Consider three sequences $(a_n), (b_n)$ and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. Prove $\lim_{n\to\infty} s_n = s$. This is called the "squeeze lemma".

Solution 7. (a) Given: For all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that for $n > N_1$ and $n > N_2$:

   (a) $|a_n - s| < \varepsilon$, which implies $s - \varepsilon < a_n < s + \varepsilon$
   (b) $|b_n - s| < \varepsilon$, which implies $s - \varepsilon < b_n < s + \varepsilon$

(b) We also know that for all $n \in \mathbb{N}$, $a_n \leq s_n \leq b_n$

(c) Combining these facts, we can conclude that for $n > \max(N_1, N_2)$:

$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon \tag{15}$$

(d) This implies:

$$s - \varepsilon < s_n < s + \varepsilon \tag{16}$$

(e) Therefore:

$$|s_n - s| < \varepsilon \tag{17}$$

(f) By the definition of a limit of a sequence, this proves that $\lim_{n\to\infty} s_n = s$
Thus, we have proven the squeeze lemma.

(g) Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim_{n\to\infty} t_n = 0$. Prove $\lim_{n\to\infty} s_n = 0$.

(a) Given: $\lim_{n\to\infty} t_n = 0$, so for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$:

(b) $|t_n - 0| < \varepsilon$

(c) This implies: $|t_n| < \varepsilon$

(d) We know that $|s_n| \leq t_n$ for all n, so: $|s_n| \leq |t_n| < \varepsilon$

(e) This means: $-\varepsilon < s_n < \varepsilon$

(f) Therefore: $|s_n - 0| < \varepsilon$

By the definition of a limit, this proves that $\lim_{n\to\infty} s_n = 0$.

Problem 8 (8.6)**.** Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} |s_n| = 0$

Solution 8. *Proof.* We will prove this statement in two parts:

(a) If $\lim_{n\to\infty} s_n = 0$, then $\lim_{n\to\infty} |s_n| = 0$:

   (a) Given $\lim_{n\to\infty} s_n = 0$, for every $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|s_n - 0| < \epsilon$.
   (b) This simplifies to $|s_n| < \epsilon$.
   (c) Therefore, $\lim_{n\to\infty} |s_n| = 0$.

(b) If $\lim_{n\to\infty} |s_n| = 0$, then $\lim_{n\to\infty} s_n = 0$:

   (a) Given $\lim_{n\to\infty} |s_n| = 0$, for every $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|s_n| < \epsilon$.

   (b) This directly implies that $|s_n - 0| < \epsilon$, so $\lim_{n\to\infty} s_n = 0$.

Thus, we have proven that $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} |s_n| = 0$.

(c) Observe that if $s_n = (-1)^n$, then $\lim_{n\to\infty} |s_n|$ exists, but $\lim_{n\to\infty} s_n$ does not exist.

Observation: For the sequence $s_n = (-1)^n$, we can see that s_n alternates between -1 and 1.

To prove that $\lim_{n\to\infty} s_n$ does not exist, we can establish two subsequences:

(a) s_{n_1}: the subsequence of even terms, where $s_{n_1} = 1$ for all n

(b) s_{n_2}: the subsequence of odd terms, where $s_{n_2} = -1$ for all n

Clearly, $\lim_{n\to\infty} s_{n_1} = 1$ and $\lim_{n\to\infty} s_{n_2} = -1$

Since these two subsequences converge to different values, we can conclude that $\lim_{n\to\infty} s_n$ does not exist, demonstrating that the sequence is divergent.

However, $\lim_{n\to\infty} |s_n| = 1$ does exist, as $|s_n| = 1$ for all n.

□

Problem 9 (Bonus)**.** Use the completeness of $\mathbb{R}$ to show the existence of $x > 0$ with $x^2 = 2$. Specifically, consider $S = \{t \in \mathbb{R} : t > 0, t^2 < 2\}$. Clearly, S is nonempty ($1 \in S$). Further, 2 is an upper bound for S. Indeed, suppose $t \in S$, then $(2-t)(2+t) = 4 - x^2 > 4 - 2 > 0$. Clearly, $2 + t > 0$, hence $2 - t > 0$. Let $x = \sup S$. Prove that $x^2 = 2$, by establishing that (i) $x^2 \leq 2$, and (ii) $x^2 \geq 2$. Once these inequalities are established, we can conclude that $x^2 = 2$.

Solution 9. *Proof.* (a) Let $\alpha = \sup S$. We will prove that $\alpha^2 = 2$ by showing that $\alpha^2 \leq 2$ and $\alpha^2 \geq 2$.

(b) First, let's prove $\alpha^2 \leq 2$:

   (a) Assume $\alpha^2 > 2$.

   (b) For $n \in \mathbb{N}$, consider $(\alpha - \frac{1}{n})^2$:

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} \tag{18}$$

   (c) Since $\alpha^2 > 2$, we have:

$$2 < \alpha^2 - \frac{2\alpha}{n} \tag{19}$$

$$2 - \alpha^2 < -\frac{2\alpha}{n} \tag{20}$$

$$\alpha^2 - 2 > \frac{2\alpha}{n} \tag{21}$$

$$\frac{\alpha^2 - 2}{2\alpha} > \frac{1}{n} \tag{22}$$

(d) Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$.

(e) Then:

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - (\alpha^2 - 2) = 2 \tag{23}$$

(f) This contradicts the fact that α is an upper bound for S.

(g) Therefore, our assumption must be false, and $\alpha^2 \leq 2$.

(c) Now, let's prove $\alpha^2 \geq 2$:

(a) Assume $\alpha^2 < 2$.

(b) For $n \in \mathbb{N}$, consider $(\alpha + \frac{1}{n})^2$:

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha + 1}{n}. \tag{24}$$

(c) Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$.

(d) Then:

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2. \tag{25}$$

(e) This means $\alpha + \frac{1}{n_0} \in S$, contradicting α as an upper bound for S.

(f) Therefore, our assumption must be false, and $\alpha^2 \geq 2$.

(d) Since we have shown $\alpha^2 \leq 2$ and $\alpha^2 \geq 2$, we can conclude that $\alpha^2 = 2$.

Thus, we have proven the existence of a real number $\alpha > 0$ such that $\alpha^2 = 2$.

□