Rules of differentiation: sum, product, etc.

Definition (28.1)

Suppose I is an open interval, $a \in I$. $f : I \to \mathbb{R}$ is differentiable at a if the derivative $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists, and is finite.

Theorem (part of 28.3 – read the whole theorem in textbook!)

Suppose f and g are differentiable at a. Then fg is differentiable at a, with $\left[fg\right]'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof. Let $p = fg$, then
$p(x) - p(a) = f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a) =$
$f(x)\big(g(x) - g(a)\big) + \big(f(x) - f(a)\big)g(a)$.
$\frac{p(x)-p(a)}{x-a} = f(x)\frac{g(x)-g(a)}{x-a} + g(a)\frac{f(x)-f(a)}{x-a}$.

$p'(a) = \lim_{x \to a} \frac{p(x)-p(a)}{x-a} = \lim_{x \to a} f(x) \lim_{x \to a} \frac{g(x)-g(a)}{x-a} +$
$g(a) \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f(a)g'(a) + g(a)f'(a)$. ∎

Corollary. For $m \in \mathbb{N}$, $\left[x^m\right]' = mx^{m-1}$. Proof: induction.

Chain Rule

Theorem (28.4 – Chain Rule)

Suppose f is differentiable at a, g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, with $(g \circ f)'(a) = g'(f(a))f'(a) = (g' \circ f)(a) \cdot f'(a)$.

False proof. $(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a}$
$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}$
$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$.
$f(x) \to f(a)$ (differentiability $\Rightarrow$ continuity), hence
$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a))$.

This does not work! $f(x)$ might equal $f(a)$; can't divide by 0.

On the road to Chain Rule: Caratheodory Theorem

Theorem (Caratheodory – Exercise 28.16)

Suppose I is an interval, $f : I \to \mathbb{R}$. f is differentiable at $a \in I$ iff $\exists$ function $\phi : I \to \mathbb{R}$, continuous at a, s.t. $f(x) - f(a) = \phi(x)(x - a)$ $\forall x \in I$. Then $\phi(a) = f'(a)$.

Proof: existence of $\phi \Rightarrow$ differentiability.
$\phi(a) = \lim_{x\to a} \phi(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$. ∎

Proof: differentiability $\Rightarrow$ existence of ϕ.
Define $\phi(a) = f'(a)$, $\phi(x) = \frac{f(x)-f(a)}{x-a}$ for $x \neq a$. $\lim_{x\to a} \phi(x)$ $= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) = \phi(a)$, so ϕ is continuous at a. ∎

Remark. We re-prove that, if f is differentiable at a, then it is continuous at a (as a product of two continuous functions).

Chain Rule

Theorem (28.4 – Chain Rule)

Suppose f is differentiable at a, g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, with $(g \circ f)'(a) = g'(f(a))f'(a) = (g' \circ f)(a) \cdot f'(a)$.

Proof. Let $b = f(a)$. Caratheodory: $\exists\ \phi : J \to \mathbb{R}$ and $\psi : I \to \mathbb{R}$ (I, J intervals), continuous at a and $b = f(a)$ respectively, so that
$f(x) - f(a) = \phi(x)(x-a)$, $g(y) - g(b) = \psi(y)(y-b)$.
Then $g(f(x)) - g(f(a)) = \psi\big(f(x)\big)\big(f(x) - f(a)\big) = \psi\big(f(x)\big)\phi(x)(x-a)$.
$\lim_{x\to a} \frac{g(f(x))-g(f(a))}{x-a} = \lim_{x\to a} \psi\big(f(x)\big)\phi(x) = \psi\big(f(a)\big)\phi(a) =$
$g'\big(f(a)\big)f'(a)$, since f, ϕ are continuous at a, and ψ is continuous at $b = f(a)$. ■

Examples of derivatives

(a) If f is differentiable at a, then so is f^n ($n \in \mathbb{N}$), and $\big(f^n\big)'(a) = nf^{n-1}(a)f'(a)$.
$f^n = g \circ f$, where $g(y) = y^n$. $g'(y) = ny^{n-1}$.
$\big(f^n\big)'(a) = g'\big(f(a)\big)f'(a) = nf^{n-1}(a)f'(a)$.

(b) If f is differentiable at a and $f(a) \neq 0$, then $\frac{1}{f}$ is differentiable at a, and $\big(\frac{1}{f}\big)'(a) = -\frac{f'(a)}{f^2(a)}$.
$\frac{1}{f} = g \circ f$, where $g(y) = \frac{1}{y}$. $g'(y) = -\frac{1}{y^2}$.
$\big(f^n\big)'(a) = g'\big(f(a)\big)f'(a) = -\frac{f'(a)}{f^2(a)}$.

If $f(x) = x^n$ ($n \in \mathbb{N}$), then $f'(x) = nx^{n-1}$, hence $\big(x^{-n}\big)' = -\frac{nx^{n-1}}{(x^n)^2} = -nx^{-n-1}$.

Conclusion. For $k \in \mathbb{Z}$, $\boxed{\big(x^k\big)' = kx^{k-1}}$.

Example: a derivative can be discontinuous

Example (p. 168). $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Compute f'; find all points where it exists. Where is f' continuous?

f is differentiable everywhere.

For $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} + x^2\big(-\frac{1}{x^2}\big) \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

$f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} t \sin \frac{1}{t} = 0$ (use Squeeze Theorem: $-|t| \leqslant t \sin \frac{1}{t} \leqslant |t|$).

f' is discontinuous at 0, continuous everywhere else.[1]

Let $x_n = \frac{1}{n\pi}$. $x_n \to 0$; however, $f'(x_n) = \frac{2}{n\pi} \sin(n\pi) - \cos(n\pi) = (-1)^{n-1}$, so $f'(x_n) \not\to f'(0) = 0$.

[1] f differentiable $\Rightarrow$ f continuous

Criterion for min and max of a function

Theorem (29.1 – criterion for max and min)

Suppose f is defined on an open interval I, has max *or* min *at $x_0 \in I$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

We assume I is open, to avoid the possibility of x_0 being an endpoint.

Corollary

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. We know that f attains its maximum and minimum. If f attains its maximum (or minimum) at x_0, then one of the following holds:

1. $x_0 \in \{a, b\}$.
2. *f is not differentiable at x_0.*
3. $f'(x_0) = 0$.

Criterion for min and max: a proof

Theorem (29.1 – criterion for max and min)

Suppose f is defined on an open interval I, has max *or* min *at $x_0 \in I$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof of the case when f has max at x_0.

f is defined on open interval $I \ni x_0$, $f(x) \leqslant f(x_0)$ for $x \in I$. We rule out the possibility of $f'(x_0) > 0$; $f'(x_0) < 0$ is handled similarly.

Suppose, for the sake of contradiction, that $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} > 0$. Find $\delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \subset I$, and $\frac{f(x)-f(x_0)}{x-x_0} > 0$ for $0 < |x - x_0| < \delta$.

Then $f(x) > f(x_0)$ for $x \in (x_0, x_0 + \delta)$, which contradicts the assumption that we have max at x_0. ∎

Rolle's Theorem

Theorem (29.2)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b). If $f(a) = f(b)$, then $\exists x \in (a, b)$ s.t. $f'(x) = 0$.

Proof. f attains its max and min; say $f(y_0) \leqslant f(x) \leqslant f(x_0)$ for any $x \in [a, b]$. In particular, $f(y_0) \leqslant f(a) = f(b) \leqslant f(x_0)$.
If $f(y_0) = f(a) = f(b) = f(x_0)$, then f is a constant, hence $f' = 0$ on (a, b).
If $f(x_0) > f(a) = f(b)$, then $x_0 \in (a, b)$, hence $f'(x_0) = 0$.
If $f(y_0) < f(a) = f(b)$, then $y_0 \in (a, b)$, hence $f'(y_0) = 0$. ■

Mean Value Theorem

Theorem (29.2 – MVT)

If $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), then $\exists x \in (a, b)$ s.t. $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$, $g(x) = f(x) - L(x)$. g is continuous on $[a, b]$, differentiable on (a, b), $g(a) = 0 = g(b)$. By Rolle's Theorem, $\exists x \in (a, b)$ s.t. $g'(x) = 0$. Then
$f'(x) = g'(x) + L'(x) = 0 + \frac{f(b)-f(a)}{b-a}$. ■

Example. For $x, y \in \mathbb{R}$, $|\sin x - \sin y| \leqslant |x - y|$.

Apply MVT to $f(t) = \sin t$ on $[x, y]$: $\exists z \in (x, y)$ s.t.
$\frac{f(x)-f(y)}{x-y} = f'(z) = \cos z$. $|\cos z| \leqslant 1$, hence $\frac{|f(x)-f(y)|}{|x-y|} = |\cos z| \leqslant 1$. ■

Consequences of Mean Value Theorem

Corollary (29.4)

If f is differentiable on (a, b), and $f' = 0$ on (a, b), then f is a constant function.

Proof. Suppose f is not a constant. Find $x < y$ s.t. $f(x) \neq f(y)$. By MVT, $\exists z \in (x, y)$ s.t. $f'(z) = \frac{f(y)-f(x)}{y-x} \neq 0$. ∎

Corollary (29.5)

If f, g are differentiable on (a, b), and $f' = g'$ on (a, b), then $\exists c \in \mathbb{R}$ s.t. $f(x) = g(x) + c \ \forall x \in (a, b)$.

Proof. Apply Corollary 29.4 to $h = f - g$. ∎