Finals Exam Prep

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So I finally finished all Final Exam practice problems that my professor gave to me. 36 of them! (Actually 32—one of them was a repeat and a couple of them were prev hw assignments) It is currently Sunday, and my final is on Tuesday. Let's go. I'm going to walkthrough all of my solutions, then analyze them with the solutions provided by my prof. Let's run it.

Problem 1 (1 (Cauchy Concentration Test))**.** Suppose $a(1) \geq a(2) \geq \cdots > 0$. Prove that

$$\sum_n a(n) \quad \text{and} \quad \sum_n 2^n a(2^n) \tag{1}$$

converge or diverge simultaneously.

Hint. Let $s(m) = \sum_{k=1}^m a(k)$. Prove first that

$$\frac{1}{2}\sum_{j=1}^n 2^j a(2^j) \leq s(2^n) \leq a(2^n) + \sum_{j=0}^n 2^j a(2^j). \tag{2}$$

This can be achieved by grouping a_k's into blocks of length $2^0, 2^1, 2^2, \ldots$.

(b) Use the result of (a) to show that $\sum_n \frac{1}{n^p}$ converges if and only if $p > 1$.

Solution 1.

Problem 2 (2)**.**

1. State the definition of a compact set
2. Give an example of an open cover of $\mathbb{R}$ which has no finite subcover
3. Consider the set $E \subset \mathbb{R}^5$, consisting of all vectors $\vec{x} = (x_1, \ldots, x_5)$ for which

$$\sum_{k=1}^5 |x_k| \leq 1 \tag{3}$$

Is E compact?

Solution 2.

Problem 3 (3)**.**

1. Use Arithmetic-Geometric means inequality to prove that, for every n,

(a) $\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}$

(b) $\left(1+\frac{1}{n}\right)^{n+1} \geq \left(1+\frac{1}{n+1}\right)^{n+2}$

2. Conclude that the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_n$ converges.

Solution 3.

Problem 4 (4)**.** Prove that the space $C(S)$ (the space of bounded continuous functions on a set S, with $d(f,g) = \sup_{x\in S} |f(x) - f(y)|$) is complete.

Solution 4.

Problem 5 (5)**.** Prove that $|\sin x - \sin y| > \frac{|x-y|}{2}$ for distinct $-\frac{\pi}{3} \leq x, y \leq \frac{\pi}{3}$.

Solution 5.

Problem 6 (6)**.** Prove that the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{x}{x^2+1}$ is Lipschitz.

Solution 6.

Problem 7 (7)**.** Suppose the function f is continuous on the interval $[1, 9]$, differentiable in its interior, and satisfies $f(1) = 3$, $f(4) = 0$, and $f(9) = 10$. Prove that there exists $c \in (1, 9)$, such that $f'(c) = 1$.

Solution 7.

Problem 8 (8)**.** Prove that the set of all real numbers of the form $a + b\sqrt{5}$ (where $a, b \in \mathbb{Q}$) is a field.

Solution 8.

Problem 9 (9)**.** Suppose (a_n) is a bounded sequence of real numbers. Denote by A the set of its subsequential limits. Prove that A is a closed subset of $\mathbb{R}$.

Solution 9.

Problem 10 (10)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and $\lim_{t\to 0} f'(t)$ exists. Prove that f' is continuous at 0.

Solution 10. So for this problem, I appealed to the Sequential Criterion for continuity (Theorem 17.1 + 17.2):

$$f : S \to S^* \text{ is continuous at } x \in S \iff f(x_n) \to f(x) \text{ whenever } x_n \to x. \tag{4}$$

The question was labeled *Difficult Problem*, which always kind of confuses bc sometimes these labeled problems can be impossible but else they are actually not bad. This problem (I think) is of the latter...

Proof.

We know that $f'(0)$ is defined on f', as $f$is diff'able everywhere.

WTF: $\lim_{n\to 0} f'(x) = f'(0)$

We know that $\lim_{x\to 0} f'(x)$ exists. So, we want to show that it equals $f'(0)$ by Equation 4.

$$\lim_{x\to 0} f'(x) = L \tag{5}$$

We know that the above implies:

$$\lim_{n\to\infty} f'(a_n) = L \; s.t. \; \lim_{n\to\infty} a_n \to 0 \tag{6}$$

$$|f'(a_n) - L| < \varepsilon \tag{7}$$

$$\frac{f'(a_n) - f'(0)}{a_n} = L \tag{8}$$

$$f'(0) = -La_n + f'(a_n) \tag{9}$$

$$f'(a_n) = La_n + f'(0) \tag{10}$$

$$|La_n + f'(0) - L| < \varepsilon \tag{11}$$

$$|L(a_n - 1) + f'(0)| < \varepsilon \tag{12}$$

$$|f'(0) - L| < \varepsilon \implies f'(0) = L \tag{13}$$

By Equation 13,

$$\lim_{x\to 0} f'(x) = f'(0) \tag{14}$$

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Problem 11 (11)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly decreasing, with $f(2) = 3$ and $f'(2) = -5$. Find $g'(3)$, where $g = f^{-1}$ (the inverse function of f).

Solution 11. $f : \mathbb{R} \to \mathbb{R}$ is cont., strictly decreasing, $f(2) = 3, f'(2) = -5$. By the Derivative of an Inverse Function,

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))} \tag{15}$$

$$g'(3) = \boxed{\frac{1}{-5}} \tag{16}$$

Problem 12 (12)**.** Compute $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n+5j}$.

Solution 12.

Problem 13 (13)**.** Suppose the functions f and g are integrable on $[a, b]$.

1. Prove that the inequalities $\int_a^b (tf+g)^2 \geq 0$ and $\int_a^b (tf-g)^2 \geq 0$ holds for any t.
2. Prove that, for any $t > 0$, we have $2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$.
3. Show that, if $\int_a^b f^2 = 0$, then $\int_a^b fg = 0$.
4. Prove Bunyakovsky-Cauchy-Schwarz Inequality for Integrals:

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right)\cdot\left(\int_a^b g^2\right) \tag{17}$$

Solution 13. This one was a monster. Prob the hardest problem in the entire pset. Lots of duh moments.

1. By the Linearity and Comparison of Integrals, we know that

$$g \leq f \implies \int_a^b g \leq \int_a^b g \tag{18}$$

Let $g = 0$ and $f = (tf+g)^2$. We know that $f \geq g$, as the square of a integrand is always positive. By Equation 18, $\int_a^b (tf+g)^2 \geq 0$ and $\int_a^b (tf-g)^2 \geq 0$.

2. Using the result of part 1:

$$\int_a^b (tf+g)^2 \geq 0 \tag{19}$$

$$= \int_a^b t^2f^2 + 2tfg + g^2 \tag{20}$$

$$= \int_a^b t^2f^2 + 2\int_a^b tfg + \int_a^b g \geq 0 \tag{21}$$

$$= t^2\int_a^b f^2 + 2t\int_a^b fg + \int_a^b g^2 \tag{22}$$

$$= -2\int_a^b fg \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2 \tag{23}$$

The other case, $\int_a^b (tf-g)^2 \geq$ is handled similarly to obtain the absolute value of $2\left|\int_a^b fg\right|$.

3. *Proof.* Suppose by contradiction, $fg = y > 0$ for some $x \in [a, b]$. By continuity of the interval of integrable functions, $\exists$ interval $[c, d] \subset [a, b]$ *s.t.* $f \geq \frac{y}{2}$ on $[c, d]$. Then,

$$\int_a^b f^2 = \int_a^b f^2 + \int_c^d f^2 + \int_d^b f^2 \geq \frac{y}{2}(d-c) > 0 \tag{24}$$

Which is a contradiction. $\square$

Problem 14 (14)**.** Recall that the metric d on $\mathbb{R}^n$ is defined as follows: for $\vec{a}, \vec{b} \in \mathbb{R}^n$, $d(\vec{a}, \vec{b}) = \left\|\vec{a} - \vec{b}\right\|$, where, for $\vec{c} = (c_1, \ldots, c_n)$, $\|\vec{c}\| = (\sum_i |c_i|)^{\frac{1}{2}}$. Suppose $\vec{x}, \vec{y} \in \mathbb{R}^n$. Prove that the function $\phi : \mathbb{R} \to \mathbb{R} : t \mapsto \left\|\vec{x} + t\vec{y}\right\|$ is convex.

Solution 14.

Problem 15 (15)**.** Suppose E is a compact subset of a metric space (S, d).

1. Prove that any $x \in S$ has a *nearest point* in E—i.e.,the point y such that $d(x, y) = \inf \{d(x, s) : s \in E\}$.
2. Give an example of x and E where the nearest point is not unique—i.e.,there exists distinct $y, z \in E$ such that $d(x, y) = d(x, z) = \inf \{d(x, s) : s \in E\}$.
3. Prove that, if $S \in \mathbb{R}^n$ with its usual Euclidean metric, and $E \subset S$ is compact and convex, then, for any $x \in S$, the point nearest to it in E is unique.

Solution 15.

Problem 16 (16)**.** The function $g : \mathbb{R} \to \mathbb{R}$ is defined by setting

$$g(x) = \begin{cases} x^3, & \text{if } x \in \mathbb{Q}; \\ -x^3, & \text{if } x \neq \mathbb{Q}. \end{cases} \tag{25}$$

Compute the derivative g' at all points where it exists.

Solution 16.

Problem 17 (17)**.** Compute $\lim_n \int_0^1 e^{-nt^2}\,\mathrm{d}t$

Solution 17.

Problem 18 (18)**.** Suppose f and g are continuous functions on $[0, 1]$, which coincide at all rational points. Prove that $f = g$ everywhere.

Solution 18.

Problem 19 (19)**.** Compute the following integrals:

1. $\int_0^1 t\sqrt{1 + t^2}\,\mathrm{d}t$
2. $\int_0^1 t \sin t\,\mathrm{d}t$

Solution 19.

Problem 20 (20)**.**

1. Give an example fo a continuous map $f : S \to S^*$ such that there exists an open set $U \subset S$ with the property that $f(U)$ is not open in S^*.
2. Suppose (S, d) is a compact metric space, and $f : S \to S^*$ is continuous and bijective. Prove that, for every open $U \subset S$, $f(U)$ is open.

Solution 20.

Problem 21 (21)**.** Determine whether the following series converge:

Solution 21. 1. $\sum_{k=1}^{\infty} \frac{k+\sqrt{k}\sin k}{k^3+1}$

2. $\sum_{k=1}^{\infty} \frac{(-4)^k}{k^4}$

3. $\sum_{k=1}^{\infty} \frac{2k^2-1}{k^3+1}$

Problem 22 (22)**.** The sequence (a_n) is defined by $a_1 = 5, a_{n+1} = \sqrt{2a_n + 3}$ for $n \geq 1$. Determine whether this sequence converges. If it does, find its limit.

Solution 22.

Problem 23 (23)**.** Define the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \tag{26}$$

1. Prove that f is not differentiable at 0

2. Prove that there is not continuous function $g : [-1,1] \to \mathbb{R}$ satisfying $\int_0^x g(t)\,\mathrm{d}t = f(x)$ for any $x \in [-1,1]$.

Solution 23.

Problem 24 (24 (Alternating Series Theorem))**.** Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Prove that $\sum_{k=1}^{\infty}(-1)^{k-1}a_k = a_1 - a_2 + a_3 - \ldots$ converges iff $\lim_k a_k = 0$.

Solution 24.

Problem 25 (25)**.** Denote by $\mathbb{T}$ the unit circle in $\mathbb{R}^2$—i.e.,$\mathbb{T} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Consider $f : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, \sin t)$. Prove that

1. f is continuous

2. $f([0, 2\pi)] = \mathbb{T}$

3. $f^{-1} : \mathbb{T} \to [0, 2\pi)$ is not continuous

Solution 25.

Problem 26 (26)**.** Suppose S is a non-empty metric space. Prove that S is connected iff it has exactly two subsets which are both open and closed—$\varnothing$ and itself.

Solution 26. $\sum_{k=1}^{n}(k-1)$

Problem 27 (22.4)**.** Consider the following subset of $\mathbb{R}^2$:

$$E = \left\{ \left(x, \sin\frac{1}{x} \right) : x \in (0,1] \right\}; \tag{27}$$

(a) Determine its closure E^-. See Fig. 19.4.

(b) Show E^- is connected.

(c) Show E^- is not path-connected.

E is simply the graph of $g(x) = \sin \frac{1}{x}$ along the interval $(0, 1]$.

Solution 27.

Problem 28 (26.5)**.** Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$.

1. Show $f'(x) = f(x)$.
2. Do **not** use the fact that $f(x) = e^x$; this is true but has not been established at this point in the text.

Solution 28.

Problem 29 (26.6)**.** Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.

(a) Prove $s'(x) = c(x)$ and $c'(x) = -s(x)$.

(b) Prove $(s^2(x) + c^2(x))' = 0$.

(c) Prove $s^2(x) + c^2(x) = 1$.

Solution 29.

Problem 30 (29.5)**.** Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$.

$$ \tag{28} $$

1. Prove that f is a constant function.

Solution 30.

Problem 31 (32.2)**.** Let

$$f(x) \begin{cases} x & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases} \tag{29}$$

(a) Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.

(b) Is f integrable on $[0, b]$?

Solution 31.

Problem 32 (33.5)**.** Show

$$\left| \int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\, dx \right| \leq \frac{16\pi^3}{3}. \tag{30}$$

Solution 32.

Problem 33 (34.2)**.** (a) Calculate

$$\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt. \tag{31}$$

(b) Calculate

$$\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt. \tag{32}$$

Solution 33.

Problem 34 (34.5)**.** Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)\, dt \quad \text{for } x \in \mathbb{R}. \tag{33}$$

1. Show that F is differentiable on $\mathbb{R}$.
2. Compute F'.

Solution 34.

Problem 35 (38.3)**.** Show that there is a differentiable function on $\mathbb{R}$ whose derivative is nowhere differentiable.

Solution 35.