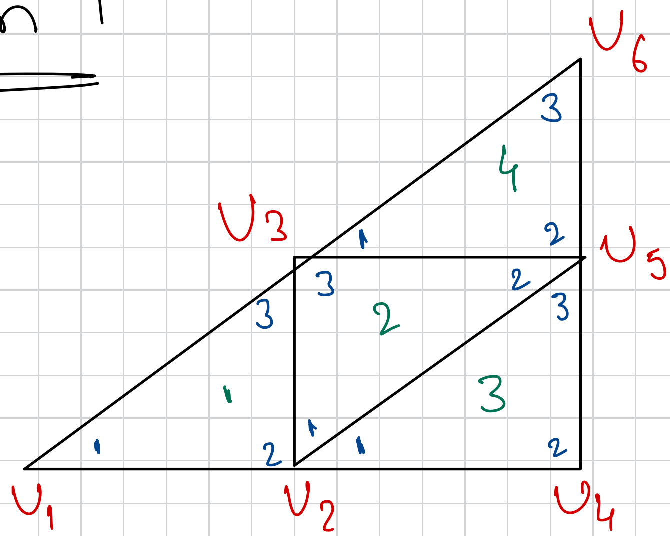
HW 11 Solutions

Problem 1



(a) From the local numbering as above,

$$LM = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^T$$

(b) Elements (1), (3) and (4) are congruent (same shape and area) $\Rightarrow$ They will have same stiffness w.r.t local node numbers $\Rightarrow k^{(1)} = k^{(3)} = k^{(4)}$

$$\frac{\partial N_1^e}{\partial x} = \frac{1}{2A}(y_2 - y_3), \quad \frac{\partial N_1^e}{\partial y} = \frac{-1}{2A}(x_2 - x_3)$$

$$\frac{\partial N_2^e}{\partial x} = \frac{1}{2A}(y_3 - y_1), \quad \frac{\partial N_2^e}{\partial y} = \frac{-1}{2A}(x_3 - x_1)$$

$$\frac{\partial N_3^e}{\partial x} = \frac{1}{2A}(y_1 - y_2), \quad \frac{\partial N_3^e}{\partial y} = \frac{-1}{2A}(x_1 - x_2)$$

$$A_1 = A_2 = A_3 = A_4 = A = \frac{1}{8} \quad \left(\text{All } \Delta^{\text{les}} \text{ have equal area}\right)$$

For any Δ^{e},

$$K^{(e)} = \frac{1}{4A^e} \begin{bmatrix} (y_2-y_3)^2 + (x_2-x_3)^2 & \text{symmetric} & \text{symmetric} \\ (y_3-y_1)(y_2-y_3) + (x_3-x_1)(x_2-x_3) & (y_3-y_1)^2 + (x_3-x_1)^2 & \text{symmetric} \\ (y_2-y_3)(y_1-y_2) + (x_2-x_3)(x_1-x_2) & (y_3-y_1)(y_1-y_2) + (x_3-x_1)(x_1-x_2) & (y_1-y_2)^2 + (x_1-x_2)^2 \end{bmatrix}$$

For element (1).

$x_1 = 0 \qquad y_1 = 0$

$x_2 = 1/2 \qquad y_2 = 0$

$x_3 = 1/2 \qquad y_3 = 1/2$

$$K^{(1)} = \frac{1}{4\left(\frac{1}{8}\right)} \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \left(\frac{1}{4}+\frac{1}{4}\right) & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{array}{c} \\ (1,2,3) \\ (2,4,5) \\ (3,5,6) \end{array} \begin{array}{c} \begin{array}{ccc} (1,2,3) & (2,4,5) & (3,5,6) \end{array} \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & +\frac{1}{2} \end{bmatrix} \end{array} = K^{(3)} = K^{(4)}$$

Global numbers for elements $\{1,3,4\}$.

For element (2),

$x_1 = 1/2 \qquad y_1 = 0$

$x_2 = 1 \qquad y_2 = 1/2$

$x_3 = 1/2 \qquad y_3 = 1/2$

$$K^{(2)} = \frac{1}{4(1/8)} \begin{bmatrix} 1/4 & 0 & -1/4 \\ 0 & 1/4 & -1/4 \\ -1/4 & -1/4 & 1/2 \end{bmatrix} = \begin{array}{c} \\ 2 \\ 5 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 5 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix} \end{array}$$

For the force vector,

$$F^{(1)} = F^{(2)} = F^{(3)} = F^{(4)} = \frac{Af}{3} = \frac{f}{24} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) Assembley,

$$
\mathbb{K} = \begin{array}{c|c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{bmatrix}
1/2 & -1/2 & 0 & 0 & 0 & 0 \\
-1/2 & 1+1/2+1/2 & -1/2-1/2 & -1/2 & 0 & 0 \\
0 & -1/2-1/2 & 1/2+1+1/2 & 0 & -1/2-1/2 & 0 \\
0 & -1/2 & 0 & 1 & -1/2 & 0 \\
0 & 0 & -1/2-1/2 & -1/2 & (1+1/2+1/2) & -1/2 \\
0 & 0 & 0 & 0 & -1/2 & +1/2
\end{bmatrix}
\end{array}
$$

$$
= \frac{1}{2} \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 4 & -2 & -1 & 0 & 0 \\
0 & -2 & 4 & 0 & -2 & 0 \\
0 & -1 & 0 & 2 & -1 & 0 \\
0 & 0 & -2 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & -1 & +1
\end{bmatrix}
$$

Similarly,

$$
F = \frac{f}{24} \left[1, \ 3, \ 3, \ 1, \ 3, \ 1 \right]^T
$$

(d) $\mathbb{K}\underline{u} = f$, with $U_4 = U_5 = U_6 = 0$.

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = f\begin{bmatrix} 1/24 \\ 1/8 \\ 1/8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Solving,

$$U = \begin{bmatrix} 0.3125f \\ 0.2292f \\ 0.1771f \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 2

Part (1) $\rightarrow$ PL

Part (2) :