Sequences and limits (Sec. 7-9)

Definition (Sequences)

A sequence is a function $s : \{m, m+1, \ldots\} \to \mathbb{R}$ ($m \in \mathbb{Z}$). Notation: $s_n = s(n)$, $\big(s_n\big)_{n \geqslant m}$.

Examples. $\big(\sin(2n)\big)_{n\geqslant 0}$, $\big(\ln(n-2)\big)_{n \geqslant 3}$.

Definition (Convergence and limit)

We say that (s_n) converges to $L \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{R}$ s.t. $|s_n - L| < \varepsilon$ for $n > N$. L is the limit of (s_n). We write $\lim_n s_n = L$, or $s_n \xrightarrow[n]{} L$.

Uniqueness of limits (pp. 37-38)

Proposition

A sequence cannot have more than one limit.

Proof. Suppose, for the sake of contradiction, that (s_n) converges to both a and b, with $a < b$. Let $\varepsilon = |b - a|/3$.
Find $M \in \mathbb{R}$ s.t. $|s_n - a| < \varepsilon$ for $n > M$.
Find $N \in \mathbb{R}$ s.t. $|s_n - b| < \varepsilon$ for $n > N$.
If $n > \max\{N, M\}$, then, by Triangle Inequality,
$|b - a| = 3\varepsilon \leqslant |b - s_n| + |a - s_n| < 2\varepsilon$, which is impossible. ∎

Examples of limits

1. $\lim \frac{1}{n} = 0$**.**

For $\varepsilon > 0$, need to find $N \in \mathbb{R}$ s.t. $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ for $n > N$.
Note that $\frac{1}{n} < \frac{1}{N}$ when $n > N$, so it suffices to select $N \in \mathbb{N}$ with $\frac{1}{N} \leqslant \varepsilon$.
This, in turn, is equivalent to $N \geqslant \varepsilon^{-1}$. We can, in fact, take $N = \varepsilon^{-1}$.
To summarize: if $N = \varepsilon^{-1}$, and $n > N$, then $\left|\frac{1}{n} - 0\right| < \varepsilon$.

2. $\left((-1)^n\right)_{n \in \mathbb{N}}$ **does not converge.**

Suppose, for contradiction, that the sequence $(-1)^n$ converges to some L.
Find $N \in \mathbb{R}$ s.t. $|(-1)^n - L| < 1$ whenever $n > N$ (use definition of convergence with $\varepsilon = 1$).
If $n > N$, then $2n > 2n - 1 \geqslant n > N$. Therefore,
$|1 - L| = |(-1)^{2n} - L| < 1$, and $|-1 - L| = |(-1)^{2n-1} - L| < 1$. Triangle Inequality:
$2 = \left|1 - (-1)\right| = \left|(1 - L) - (-1 - L)\right| \leqslant |1 - L| + |-1 - L| < 1 + 1 = 2$,
impossible!

Examples of limits

3. $\lim \frac{3n+1}{2n+1} = \frac{3}{2}$**.**

Guess. When n is large, then $\frac{3n+1}{2n+1} \approx \frac{3n}{2n} = \frac{3}{2}$, so we *conjecture* that $\lim \frac{3n+1}{2n+1} = \frac{3}{2}$.

Verification. $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| = \frac{|2(3n+1)-3(2n+1)|}{2(2n+1)} = \frac{1}{2(2n+1)}$; want to make the RHS less than ε (for $n > N$).
If $n > N$, then $\frac{1}{2(2n+1)} < \frac{1}{4n} < \frac{1}{4N}$.
Conclusion. If $N = \frac{1}{4\varepsilon}$, then $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| < \varepsilon$.

Facts about limits (Sec. 8)

Proposition

If (s_n) converges, and $s_n \geqslant a$ for all but finitely many n, then $s = \lim s_n \geqslant a$.

Proof. Suppose, for the sake of contradiction, that $s < a$.
Find $N \in \mathbb{R}$ so that $s_n \geqslant a$ for $n > N$.
Let $\varepsilon = a - s$. Find $M \in \mathbb{R}$ so that $|s_n - s| < \varepsilon$ for $n > M$.
For such n, $s - \varepsilon < s_n < s + \varepsilon = a$.
For $n > \max\{N, M\}$, $s_n < a \leqslant s_n$, contradiction! ■

Note: It may happen that $s_n > a$ for any n, but $\lim s_n = a$. Consider, for instance, $s_n = \frac{1}{n}$, $a = 0$.

If $s_n \geqslant 0\ \forall\, n$, and $\lim s_n = s$, then $\lim \sqrt{s_n} = \sqrt{s}$

Proposition (Example 5 from p. 42)

If $s_n \geqslant 0\ \forall\, n$, and $\lim s_n = s$, then $\lim \sqrt{s_n} = \sqrt{s}$.

Note that $s \geqslant 0$, so $\sqrt{s}$ makes sense.

Proof for $s > 0$. Useful trick – multiplication by the conjugate:

$$\sqrt{a} - \sqrt{b} = \left(\sqrt{a} - \sqrt{b}\right) \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}^2-\sqrt{b}^2}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}.$$

$$\left|\sqrt{s_n} - \sqrt{s}\right| = \frac{|s_n-s|}{\sqrt{s_n}+\sqrt{s}} \leqslant \frac{|s_n-s|}{\sqrt{s}}.$$

For $\varepsilon > 0$, select $N \in \mathbb{R}$ so that $|s_n - s| < \sqrt{s} \cdot \varepsilon$ for $n > N$.

For such n, $\left|\sqrt{s_n} - \sqrt{s}\right| \leqslant \frac{|s_n-s|}{\sqrt{s}} < \varepsilon$. ∎

Proof for $s = 0$.

We have to show: $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $\sqrt{s_n} < \varepsilon$ whenever $n > N$.
Note that $\sqrt{s_n} < \varepsilon$ holds iff $s_n < \varepsilon^2$. As $\lim s_n = 0$, we can find N with the property that $s_n < \varepsilon^2$ whenever $n > N$. For such n, $\sqrt{s_n} < \varepsilon$. ∎

Convergent sequences are bounded

Definition (Bounded sequences)

A sequence (s_n) is called bounded if $\exists\, A \in \mathbb{R}$ s.t. $\forall\, n$, $|s_n| \leqslant A$.

Theorem (Theorem 9.1)

Convergent sequences are bounded.

Proof. Suppose a sequence $(s_n)_{n \geqslant k}$ converges to s.
Find $N \in \mathbb{R}$ s.t. $|s_n - s| < 1$ for $n > N$.
For $n > N$, $|s_n| \leqslant |s| + |s_n - s| < |s| + 1$ (Triangle Inequality).
Let $A = \max\left\{ \max_{k \leqslant n \leqslant N} |s_n|, |s| + 1 \right\}$. For any n, $|s_n| \leqslant A$. ■

Is any bounded sequence convergent? **No!** Example: $s_n = (-1)^n$.

Sums, products, ratios of limits (Section 9)

Theorem (Theorems 9.3, 9.4, 9.6 from textbook)

Suppose $\lim s_n = s$ *and* $\lim t_n = t$. *Then:*

- $\lim(s_n + t_n) = s + t$
- $\lim(as_n) = as$, *for any* $a \in \mathbb{R}$.
- $\lim(s_n t_n) = st$.
- *If, in addition,* $t \neq 0$, *then* $\lim \dfrac{s_n}{t_n} = \dfrac{s}{t}$.

If $t \neq 0$, then $t_n \neq 0$ for n sufficiently large.
Indeed, find $N \in \mathbb{R}$ s.t. $|t_n - t| < |t|$ for $n > N$.
By Triangle Inequality, for such n, $|t_n| \geqslant |t| - |t - t_n| > |t| - |t| = 0$.

The proof will be given in the next lecture.