# Differentiating inverse functions (pp. 237-239)

**Set-up.** Suppose $I$ is an interval, and $f : I \to \mathbb{R}$ is a continuous strictly monotone function. Then $J = f(I)$ is an interval; $g = f^{-1} : J \to I$ is strictly monotone, and continuous (Section 18).

## Theorem (29.9)

*If $f$ is differentiable at $c \in I$, and $f'(c) \neq 0$, then $g$ is differentiable at $d = f(c)$, and $g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$.*

**A way to memorize the formula:** $\forall y \in J$, $f(g(y)) = y$. Differentiate both sides at $d$, using Chain Rule: $f'(g(d))g'(d) = 1$. This is not a proof! We need to prove $g$ is differentiable at $d$.

## Theorem (Caratheodory – Lecture 27)

*Suppose $I$ is an interval, $f : I \to \mathbb{R}$. $f$ is differentiable at $a \in I$ iff $\exists$ function $\phi : I \to \mathbb{R}$, continuous at $a$, s.t. $f(x) - f(a) = \phi(x)(x - a)$ $\forall x \in I$. Then $\phi(a) = f'(a)$.*

# Derivatives of inverse functions: proof

**Theorem 29.9.** Suppose $I$ is an interval, $f : I \to \mathbb{R}$ strictly monotone and continuous, $J = f(I)$, let $g = f^{-1} : J \to \mathbb{R}$. If $f$ is differentiable at $c \in I$, and $f'(c) \neq 0$, then $g$ is differentiable at $d = f(c)$, and $g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$.

**Proof.** By Caratheodory, we need to show: $\exists \psi : J \to \mathbb{R}$, continuous at $d$, s.t. $g(y) - g(d) = \psi(y)(y - d)\ \forall y \in J$, and $\psi(d) = \frac{1}{f'(c)}$.

$\exists\ \phi : I \to \mathbb{R}$, continuous at $c$ (with $\phi(c) = f'(c)$), s.t.
$f(x) - f(c) = \phi(x)(x - c)\ \forall x \in I$.
$y - d = f(g(y)) - f(g(d)) = \phi(g(y))\big(g(y) - g(d)\big)$.
$g$ is continuous, $c = g(d)$, hence $\phi \circ g$ is continuous at $d$.
$f'(c) = \phi(g(d)) \neq 0$, hence $\phi \circ g \neq 0$ on a neighborhood of $d$.
$g(y) - g(d) = \psi(y)(y - d)$, where $\psi(y) = \frac{1}{\phi(g(y))}$ is continuous at $d$.
$g'(d) = \frac{1}{\phi(g(d))} = \frac{1}{f'(c)}$. ∎

# Derivatives of rational powers

**Example 2, p. 238** $f(x) = x^n$ is differentiable, strictly increasing on $(0, \infty)$; $f'(x) = nx^{n-1} \neq 0$. $f\big((0, \infty)\big) = (0, \infty)$. The inverse function: $g(y) = y^{1/n}$. For $y \in (0, \infty)$,
$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n} \cdot \frac{1}{y^{1-1/n}} = \frac{1}{n} y^{1/n-1}$.

If $n$ is odd, we can view $f$ and $g$ as functions $\mathbb{R} \to \mathbb{R}$. Then $g'(y) = \frac{1}{n} y^{1/n-1}$ for $y < 0$ as well (but not for $y = 0$).

Derivative of $h(x) = x^r$, $r \in \mathbb{Q}$: $h'(x) = rx^{r-1}$. $r = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.
$h(x) = x^{m/n} = q(p(x))$, $p(x) = x^{1/n}$, $q(y) = y^m$. $h'(x) = q'(p(x))p'(x) = m\big(x^{1/n}\big)^{m-1} \cdot \frac{1}{n} x^{1/n-1} = \frac{m}{n} x^{(m-1)/n+1/n-1} = rx^{r-1}$.

# Inverse trigonometric functions

**Example 3, p. 238** $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is continuous and strictly increasing. $f\left([-\frac{\pi}{2}, \frac{\pi}{2}]\right) = [-1, 1]$.
Inverse function: $g = f^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is called arcsin.

$f'(x) = \cos x$ is positive for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
$f\left(-\frac{\pi}{2}\right) = -1$, $f\left(\frac{\pi}{2}\right) = 1$, hence $g$ is differentiable on $(-1, 1)$.
$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{\cos(g(y))}$, for $y \in (-1, 1)$.
$g(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, hence $\cos g(y) > 0$. $\sin g(y) = f(g(y)) = y$, so
$\cos g(y) = \sqrt{1 - \sin^2 g(y)} = \sqrt{1 - y^2}$.

$$\boxed{\left(\arcsin y\right)' = \frac{1}{\sqrt{1-y^2}}}$$

Similarly one can show: $\boxed{\left(\arctan y\right)' = \frac{1}{1+y^2}}$

# Integration: the goals

Goal: find the area under the graph of bounded $f : [a, b] \to \mathbb{R}$.

Strategy:

• Split the subgraph of $f$ into thin vertical strips.

• Estimate the area of each strip from above, and from below (by using a rectangle).

• Add areas of these rectangles, find some kind of a limit.

# Intergration: definitons (Section 32)

If $f$ is bounded on $S$, let $m(f,S) = \inf_{s\in S} f(s)$, $M(f,S) = \sup_{s\in S} f(s)$.
A partition of $[a,b]$ is $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$.
Lower Darboux sum: $L(f,P) = \sum_{k=1}^{n} m\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big)$.
Upper Darboux sum: $U(f,P) = \sum_{k=1}^{n} M\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big)$.

Lower Darboux integral: $L(f) = \sup_P L(f,P)$.
Upper Darboux integral: $U(f) = \inf_P U(f,P)$.
The inf, sup are taken over all partitions $P$ of $[a,b]$.

For any $P$, $(b-a)m\big(f,[a,b]\big) \leqslant L(f,P) \leqslant U(f,P) \leqslant (b-a)M\big(f,[a,b]\big)$, hence $L(f)$, $U(f)$ are finite.

**Definition (p. 270)**

$f$ is integrable if $L(f) = U(f)$. Denote $\int_a^b f = L(f) = U(f)$.

# Intergrals: Example 1

Is $f(x) = c$ integrable on $[a, b]$? If it is, compute the integral.

If $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ is a partition, then, for any $k$,
$m\big(f, [t_{k-1}, t_k]\big) = \inf_{t \in [t_{k-1}, t_k]} f(t) = c$, and
$M\big(f, [t_{k-1}, t_k]\big) = \sup_{t \in [t_{k-1}, t_k]} f(t) = c$.

$L(f, P) = \sum_{k=1}^{n} m\big(f, [t_{k-1}, t_k]\big)\big(t_k - t_{k-1}\big)$
$= c \sum_{k=1}^{n} \big(t_k - t_{k-1}\big) = c(b - a) = U(f, P)$.

$L(f) = \sup_P L(f, P) = c(b - a)$. $U(f) = \inf_P U(f, P) = c(b - a)$.
$U(f) = L(f)$, hence $f$ is **integrable** on $[a, b]$, and $\int_a^b f = c(b - a)$. Can you find a geometric interpretation?

# Intergrals: Example 2

Let $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Is $g$ integrable on $[0, 1]$? If it is, compute the integral.

If $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ is a partition, then, for any $k$,
$m\big(g, [t_{k-1}, t_k]\big) = \inf_{t \in [t_{k-1}, t_k]} g(t) = 0$, and
$M\big(g, [t_{k-1}, t_k]\big) = \sup_{t \in [t_{k-1}, t_k]} g(t) = 1$.
$L(g, P) = \sum_{k=1}^{n} m\big(g, [t_{k-1}, t_k]\big)\big(t_k - t_{k-1}\big) = 0$.
$U(g, P) = \sum_{k=1}^{n} M\big(g, [t_{k-1}, t_k]\big)\big(t_k - t_{k-1}\big) = \sum_{k=1}^{n} \big(t_k - t_{k-1}\big) = 1$.

$L(g) = \sup_P L(g, P) = 0$. $U(g) = \inf_P U(g, P) = 1$.
$U(g) \neq L(g)$, hence $g$ is **not integrable** on $[0, 1]$.

---

We shall show that $U(f) \geqslant L(f)$, for any bounded $f : [a, b] \to \mathbb{R}$.

# More about Darboux sums

**Aim.** We want to show that $U(f) \geqslant L(f)$, for any bounded $f : [a,b] \to \mathbb{R}$.

**Notation.** For partitions $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ and $Q = \{a = s_0 < s_1 < \ldots < s_m = b\}$, we say that $P \subset Q$ if $\{t_1, \ldots, t_n\} \subset \{s_1, \ldots, s_m\}$. We say $Q$ is a refinement of $P$.

**Lemma (32.2; proof is optional)**

*Suppose $f : [a,b] \to \mathbb{R}$ is bounded. If $P$ and $Q$ are partitions of $[a,b]$, and $P \subset Q$, then $L(f,P) \leqslant L(f,Q)$, and $U(f,P) \geqslant U(f,Q)$.*

**Lemma (32.3)**

*If $P$ and $Q$ are partitions of $[a,b]$, and $f : [a,b] \to \mathbb{R}$ is bounded, then $L(f,P) \leqslant U(f,Q)$.*

**Proof.** $L(f,P) \leqslant L(f,P \cup Q) \leqslant U(f,P \cup Q) \leqslant U(f,Q)$. ∎

# Proof of $L(f,P) \leqslant L(f,Q)$, Lemma 32.2 (optional)

Induction on $|Q|$: suffices to prove $L(f,P) \leqslant L(f,Q)$ for $|Q| = |P| + 1$:
$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$,
$Q = \{a = t_0 < \ldots < t_{j-1} < u < t_j < \ldots < t_n = b\}$.

$$\begin{aligned} L(f,P) = &\sum_{k=1}^{j-1} m\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big) \\ &+ m\big(f,[t_{j-1},t_j]\big)\big(t_j - t_{j-1}\big) \\ &+ \sum_{k=j}^{n} m\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big). \end{aligned}$$

$$\begin{aligned} L(f,Q) = &\sum_{k=1}^{j-1} m\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big) \\ &+ m\big(f,[t_{j-1},u]\big)\big(u - t_{j-1}\big) + m\big(f,[u,t_j]\big)\big(t_j - u\big) \\ &+ \sum_{k=j}^{n} m\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big). \end{aligned}$$

# Properties of $U$ and $L$

If $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$,
$Q = \{a = t_0 < \ldots < t_{j-1} < u < t_j < \ldots < t_n = b\}$, then

$$L(f,Q) - L(f,P) = m\big(f,[t_{j-1},u]\big)\big(u - t_{j-1}\big) + m\big(f,[u,t_j]\big)\big(t_j - u\big)$$
$$- m\big(f,[t_{j-1},t_j]\big)\big(t_j - t_{j-1}\big) \geqslant$$
$$m\big(f,[t_{j-1},t_j]\big)\big(u - t_{j-1}\big) + m\big(f,[t_{j-1},t_j]\big)\big(t_j - u\big)$$
$$- m\big(f,[t_{j-1},t_j]\big)\big(t_j - t_{j-1}\big) = 0.$$

Thus, $L(f,P) \leqslant L(f,Q)$. ■

**Theorem (Theorem 32.4)**

*If $f : [a,b] \to \mathbb{R}$ is bounded, then $L(f) \leqslant U(f)$.*

**Proof.** $L(f) = \sup_Q L(f,Q)$. $\forall P, Q$, $L(f,Q) \leqslant U(f,P)$. Thus, $L(f) = \sup_Q L(f,Q) \leqslant U(f,P)$, for any $P$. $L(f) \leqslant \inf_P U(f,P) = U(f)$. ■

**Remark.** From the definition, $f$ is integrable when $U(f) = L(f)$. We only need to check $U(f) \leqslant L(f)$ to establish integrability.

# Integrals: Example

Is $h(x) = x$ integrable on $[0, b]$? If it is, compute $\int_0^b h$.

If $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ is a partition, then, for any $k$,
$m\big(h, [t_{k-1}, t_k]\big) = \inf_{t \in [t_{k-1}, t_k]} h(t) = t_{k-1}$, and
$M\big(h, [t_{k-1}, t_k]\big) = \sup_{t \in [t_{k-1}, t_k]} h(t) = t_k$.
$L(h, P) = \sum_{k=1}^n t_{k-1}\big(t_k - t_{k-1}\big)$, $U(h, P) = \sum_{k=1}^n t_k\big(t_k - t_{k-1}\big)$.

To make computations easier, consider "equal" partitions
$P_n = \big(0 < \frac{b}{n} < \frac{2b}{n} < \ldots < \frac{(n-1)b}{n} < b\big)$ (that is, $t_k = \frac{kb}{n}$).
$L(h, P_n) = \sum_{k=1}^n \frac{(k-1)b}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^n (k-1) = \frac{b^2}{n^2}\frac{n(n-1)}{2} = \frac{b^2}{2}\big(1 - \frac{1}{n}\big)$.
$L(h) \geqslant \sup_n L(h, P_n) = \lim_n L(h, P_n) = \frac{b^2}{2}$. Similarly,
$U(h) \leqslant \lim_n U(h, P_n) = \frac{b^2}{2}$. However, $U(h) \geqslant L(h)$, hence
$U(h) = L(h) = \frac{b^2}{2}$. Thus, $h$ is **integrable**, with $\int_0^b h = \frac{b^2}{2}$.
Can you interpret this geometrically?