Monotone functions are integrable

Theorem (32.5)

A bounded $f : [a, b] \to \mathbb{R}$ is integrable iff $\forall \varepsilon > 0$ $\exists$ partition P s.t. $U(f, P) - L(f, P) < \varepsilon$.

Theorem (33.1)

Any monotone function on $[a, b]$ is integrable.

Proof for the case of increasing f.

Fix $\varepsilon > 0$. Need to find a partition P s.t. $U(f, P) - L(f, P) < \varepsilon$.

Find $n \in \mathbb{N}$ s.t. $\frac{(f(b)-f(a))(b-a)}{n} < \varepsilon$. Consider the "equal partition" P, with of points $t_k = a + kh$ $(0 \leqslant k \leqslant n)$, where $h = \frac{b-a}{n}$. Then $m(f, [t_{k-1}, t_k]) = f(t_{k-1})$ and $M(f, [t_{k-1}, t_k]) = f(t_k)$. $U(f, P) - L(f, P)$
$= \sum_{k=1}^{n} \big(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\big)(t_k - t_{k-1})$
$= h \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1})\big) = \frac{(f(b)-f(a))(b-a)}{n} < \varepsilon.$ ■

Continuous functions are integrable

Theorem (33.2)

Any continuous function on $[a, b]$ is integrable.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Fix $\varepsilon > 0$. Need to find a partition P s.t. $U(f, P) - L(f, P) < \varepsilon$.
f is uniformly continuous. Find $\delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ if $|x - y| < \delta$. Find $n \in \mathbb{N}$ s.t. $h = \frac{b-a}{n} < \delta$. Consider the partition P consisting of points $t_k = a + kh$ $(0 \leqslant k \leqslant n)$. We claim that, for any k, $M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b-a}$. Indeed, find $x_k, y_k \in [t_{k-1}, t_k]$ s.t. $M(f, [t_{k-1}, t_k]) = f(x_k)$, $m(f, [t_{k-1}, t_k]) = f(y_k)$. $|x_k - y_k| < \delta$, so $|f(x_k) - f(y_k)| < \frac{\varepsilon}{b-a}$.
$U(f, P) - L(f, P) =$
$\sum_{k=1}^{n} \big(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\big)(t_k - t_{k-1}) < hn\frac{\varepsilon}{b-a} = \varepsilon$, since $hn = b - a$. ■

Mesh of a partition, and its applications

Definition (32.6)

The mesh of a partition $P = \big(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\big)$ is $\mathrm{mesh}(P) = \max_{1 \leqslant k \leqslant n}(t_k - t_{k-1})$ (length of longest subinterval).

Theorem (32.7)

A bounded $f : [a, b] \to \mathbb{R}$ is integrable iff $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ whenever $\mathrm{mesh}(P) < \delta$.

Remark. If δ is as above, $\mathrm{mesh}(P) < \delta$, then $U(f, P) - \int_a^b f, \int_a^b f - L(f, P) < \varepsilon$. Roughly speaking: $\lim_{\mathrm{mesh} P \to 0} U(f, P) = \int_a^b f = \lim_{\mathrm{mesh} P \to 0} L(f, P)$.

Proof of 32.7 (optional)

Need to show: f is integrable $\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ if $\mathrm{mesh}(P) < \delta$.

Find a partition $Q = \big(a = s_0 < \ldots < s_m = b\big)$ s.t. $U(f, Q) - L(f, Q) < \frac{\varepsilon}{2}$.
Let $B = \sup_{t \in [a,b]} |f(t)|$. We claim: $\delta = \frac{\varepsilon}{8mB}$ works.

Suppose $\mathrm{mesh}(P) < \delta$, let $R = P \cup Q$. We have $U(f, R) \leqslant U(f, Q)$ and $L(f, R) \geqslant L(f, Q)$, hence $U(f, R) - L(f, R) < \frac{\varepsilon}{2}$. Need:
$U(f, P) \leqslant U(f, R) + \frac{\varepsilon}{4}$, $L(f, P) \geqslant L(f, R) + \frac{\varepsilon}{4}$.

Say $P = \big(a = t_0 < \ldots < t_N = b\big)$.
$U(f, P) = \sum_{k=1}^{N} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$.

Notation. $R = \big(a = r_0 < \ldots < r_M = b\big)$.
$R|_{[r_i, r_j]} := \big(r_i < r_{i+1} < \ldots < r_j\big)$.
$U_{[r_i, r_j]}\big(f, R|_{[r_i, r_j]}\big) = \sum_{k=i+1}^{j} M(f, [r_{k-1}, r_k])(r_k - r_{k-1})$;
$L_{[r_i, r_j]}\big(f, R|_{[r_i, r_j]}\big) = \sum_{k=i+1}^{j} m(f, [r_{k-1}, r_k])(r_k - r_{k-1})$.

$U(f, R) = \sum_{k=1}^{N} U_{[t_{k-1}, t_k]}\big(f, R|_{[t_{k-1}, t_k]}\big)$.

Proof of 32.7, part 2

If $(t_{k-1}, t_k) \cap Q = \emptyset$, then $R|_{[t_{k-1},t_k]} = \{t_{k-1}, t_k\}$, hence $U_{[t_{k-1},t_k]}(f, R|_{[t_{k-1},t_k]}) = M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$.

If $(t_{k-1}, t_k) \cap Q \neq \emptyset$, then $U_{[t_{k-1},t_k]}(f, R|_{[t_{k-1},t_k]}) \geqslant -B(t_k - t_{k-1})$, and $M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) - U_{[t_{k-1},t_k]}(f, R|_{[t_{k-1},t_k]}) \leqslant 2B(t_k - t_{k-1})$.

Let S be the set of all $k \in \{1, \dots, N\}$ for which $(t_{k-1}, t_k) \cap Q \neq \emptyset$. Then $|S| \leqslant m$.

$U(f, P) - U(f, R) = \sum_{k \in S} \Big(M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) - U_{[t_{k-1},t_k]}(f, R|_{[t_{k-1},t_k]}) \Big) \geqslant \sum_{k \in S} 2B(t_k - t_{k-1}) \leqslant 2mB\mathrm{mesh}(P) < \frac{\varepsilon}{4}$.

Similarly, $L(f, R) - L(f, P) < \frac{\varepsilon}{4}$.

Conclusion: $U(f, P) - L(f, P) = \big(U(f, P) - U(f, R)\big) + \big(U(f, R) - L(f, R)\big) + \big(L(f, R) - L(f, P)\big) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$. ∎

We can shrink the interval of integration

Proposition (Exercise 32.8)

If f is integrable on $[a,b]$, and $[c,d] \subset [a,b]$, then f is integrable on $[c,d]$.

Proof. Need to show: $\forall \varepsilon > 0 \; \exists$ partition P of $[c,d]$ s.t. $U_{[c,d]}(f,P) - L_{[c,d]}(f,P) < \varepsilon$. Find $\delta > 0$ s.t. $U(f,Q) - L(f,Q) < \varepsilon$ if Q is a partition of $[a,b]$, with $\mathrm{mesh}(Q) < \delta$.

Consider $Q = \big(a = s_0 < \ldots < s_{i-1} = c < \ldots < s_j = d < \ldots < s_m = b\big)$, with $\mathrm{mesh}(Q) < \delta$. We claim that $P = Q \cap [c,d]$ works. Indeed,
$U_{[c,d]}(f,P) - L_{[c,d]}(f,P) =$
$\sum_{\ell=i}^{j} \big(M(f,[s_{\ell-1},s_\ell]) - m(f,[s_{\ell-1},s_\ell])\big)(s_\ell - s_{\ell-1}) \leqslant$
$\sum_{\ell=1}^{m} \big(M(f,[s_{\ell-1},s_\ell]) - m(f,[s_{\ell-1},s_\ell])\big)(s_\ell - s_{\ell-1}) = U(f,Q) - L(f,Q) < \varepsilon$.
$\blacksquare$

Riemann integration

Definition (32.8)

Suppose $f : [a, b] \to \mathbb{R}$ is bounded. For a partition $P = \big(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\big)$, and $x_k \in [t_{k-1}, t_k]$, define the Riemann sum $S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$.
f is Riemann integrable if $\exists r \in \mathbb{R}$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|S - r| < \varepsilon$ when $\mathrm{mesh}(P) < \delta$. Notation: $r = \mathcal{R}\int_a^b f$ is the Riemann integral.

Note that the Riemann integral is unique.

Theorem (32.9)

A bounded $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff it is (Darboux) integrable. In this case, $\mathcal{R}\int_a^b f = \int_a^b f$.

Corollary 32.8. If the Riemann sums S_n correspond to partitions P_n, and $\lim_n \mathrm{mesh}(P_n) = 0$, then $\lim_n S_n = \int_a^b f$.

Proof: equivalence of Riemann and Darboux integrability

Remark. Any Riemann integrable function is bounded.

Proof: Darboux $\Rightarrow$ Riemann. Suppose $f : [a, b] \to \mathbb{R}$ is integrable. For $\varepsilon > 0$ find $\delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ if $\mathrm{mesh}(P) < \delta$. Note that $U(f, P) \geqslant \int_a^b f \geqslant L(f, P)$, hence $U(f, P) - \int_a^b f, \int_a^b f - L(f, P) < \varepsilon$.

For any Riemann sum S, $U(f, P) \geqslant S \geqslant L(f, P)$. If $\mathrm{mesh}(P) < \delta$, then $\int_a^b f + \varepsilon > U(f, P) \geqslant S \geqslant L(f, P) > \int_a^b f - \varepsilon$, hence $\left|S - \int_a^b f\right| < \varepsilon$. Thus, f is Riemann integrable, with $\mathcal{R} \int_a^b f = \int_a^b f$. ∎

Proof: equivalence of Riemann and Darboux integrability

Proof: Riemann $\Rightarrow$ Darboux (optional).

Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Fix a partition P; note that $U(f, P) = \sup S$, and $L(f, P) = \inf S$, where the sup and inf run over all Riemann sums corresponding to the partition P.

Fix $\varepsilon > 0$. Find $\delta > 0$ s.t. $\left|\mathcal{R}\int_a^b f - S\right| < \varepsilon$ whenever $\mathrm{mesh}(P) < \delta$. For such P, $S \in \left(\mathcal{R}\int_a^b f - \varepsilon, \mathcal{R}\int_a^b f + \varepsilon\right)$, hence $\mathcal{R}\int_a^b f - \varepsilon \leqslant L(f, P) \leqslant U(f, P) \leqslant \mathcal{R}\int_a^b f + \varepsilon$. In particular, $U(f) - L(f) \leqslant U(f, P) - L(f, P) \leqslant 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, we obtain $U(f) = L(f)$, hence f is integrable. Further, $\mathcal{R}\int_a^b f - \varepsilon \leqslant L(f) \leqslant U(f) \leqslant \mathcal{R}\int_a^b f + \varepsilon$ for any $\varepsilon > 0$, hence $L(f) = U(f) = \mathcal{R}\int_a^b f$. ∎

Some properties of integrable functions

Proposition (Exercise 32.7 – Homework 9)

If f is integrable on $[a, b]$, and $f = g$ except for finitely many points on $[a, b]$, then g is integrable on $[a, b]$, and $\int_a^b f = \int_a^b g$.

Remark. Can we replace "finitely many" by "countably many?" No! On $[0, 1]$, consider $f(x) = 0$, and $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

Exercise. Suppose $\phi(x) = \begin{cases} x & x \in \{\sqrt{2}, \sqrt{3}, \sqrt{5}\} \\ 1 & x \notin \{\sqrt{2}, \sqrt{3}, \sqrt{5}\} \end{cases}$. Is ϕ integrable on $[0, 3]$? If it is, find $\int_0^3 \phi$.

Let $h(x) = 1$. $\phi = h$ except at three points. h is integrable, hence so is ϕ.
$\int_0^3 \phi = \int_0^3 h = 3$.