Subsequential limits

Definition (Definition 11.6)

For a sequence (s_n), a subsequential limit is any limit of a subsequence (in $\mathbb{R} \cup \{\pm\infty\}$).

Theorem (Theorem 11.2)

Suppose (s_n) is a sequence.

1. *$t \in \mathbb{R}$ is a subsequential limit iff $\forall\, \varepsilon > 0$, $\{n : |s_n - t| < \varepsilon\}$ is infinite.*
2. *$t = +\infty$ ($t = -\infty$) is a subsequential limit iff (s_n) is not bounded above (resp. below).*

Theorem (Theorem 11.7)

For any sequence (s_n), $\limsup s_n$ and $\liminf s_n$ are limits of monotone subsequences.

The set of subsequential limits

Theorem (Theorem 11.8)

Suppose (s_n) is a sequence, S is its set of subsequential limits.

1. *S is non-empty.*
2. $\inf S = \liminf s_n$, $\sup S = \limsup s_n$.
3. $\lim s_n$ *exists iff* S *consists of a single point. Then* $\{\lim s_n\} = S$.

Proof. (i) S contains $\liminf s_n$ and $\limsup s_n$, hence it is non-empty.

(iii) If $\lim s_n = s$, then $\limsup s_n = s = \liminf s_n$, hence, by (ii), $\sup S = s = \inf S$. Thus, $S = \{s\}$.
If $S = \{s\}$, then $\limsup s_n = s = \liminf s_n$, hence $\lim s_n$ exists, and $= s$. ∎

Proof of Theorem 11.8(ii)

Proof: $\sup S = \limsup s_n$ **(S = set of subseq. limits of (s_n)).**

Let $s = \sup S$. Already know: $\limsup s_n \in S$, hence $s \geqslant \limsup s_n$. Need: $s \leqslant \limsup s_n$. Suppose, for contradiction, $s > \limsup s_n$ (so $s \in \mathbb{R} \cup \{+\infty\}$).

Case 1: $s \in \mathbb{R}$. Find $\varepsilon \in (0, s - \limsup s_n)$, and $t \in S$ s.t. $t > \limsup s_n + \varepsilon$. $|\{n : |s_n - t| < \varepsilon\}| = \infty$, hence, for any m, $\exists\ n > m$ s.t. $s_n > t - \varepsilon$. Thus, $u_m = \sup_{n>m} s_n > t - \varepsilon$, hence $\limsup s_n = \lim u_m \geqslant t - \varepsilon > \limsup s_n$, contradiction! $\square$

Case 2: $s = +\infty$. As $\limsup s_n < \infty$, (s_n) is bounded above: $\exists A$ s.t. $s_n \leqslant A$ for any n. Then $S \subset [-\infty, A]$, so $\sup S \leqslant A < \infty$. Again, contradiction! $\blacksquare$

Section 12: lim inf and lim sup revisited

Theorem (Theorem 12.1)

If $\lim s_n = s \in (0, \infty)$*, then, for any sequence* (t_n)*,*
$\limsup(s_n t_n) = s \limsup t_n$*.*

Convention: for $s \in (0, \infty)$, $s \cdot (+\infty) = +\infty$, $s \cdot (-\infty) = -\infty$.

The conclusion of the theorem fails if $s \notin (0, \infty)$. Some examples:

1. $s = +\infty$. Let $s_n = n$, $t_n = \frac{1}{n}$. $\limsup t_n = 0$. $\limsup(s_n t_n) = 1$.

2. $s = 0$. Let $s_n = \frac{1}{n}$, $t_n = n^2$. $\limsup t_n = +\infty = \limsup(s_n t_n)$.

3. $s = -1$. Let $s_n = -1$, $t_n = (-1)^n$. $\limsup t_n = 1 = \limsup(s_n t_n)$.

4. $s = -\infty$. Let $s_n = -n^2$, $t_n = \frac{1}{n}$. $\limsup t_n = 0$, $\limsup(s_n t_n) = -\infty$.

Proof of Theorem 12.1

Proof: if $\lim s_n = s \in (0, \infty)$**, then** $\limsup(s_n t_n) = s \limsup t_n$**.**

Let $\beta = \limsup t_n$.

Step 1: Prove that $\limsup(s_n t_n) \geqslant s\beta$.

Case 1: $\beta \in \mathbb{R}$. $\limsup t_n$ is a subsequential limit of (t_n), hence $\exists$ $n_1 < n_2 < \dots$ s.t. $t_{n_k} \to \beta$. $s_{n_k} t_{n_k} \to s\beta$, hence $s\beta$ is a subsequential limit, and $\leqslant \limsup(s_n t_n)$.

Case 2: $\beta = +\infty$. (t_n) is not bounded above, hence neither is $(s_n t_n)$ (verify this!). So, $\limsup(s_n t_n) = +\infty = s\beta$.

Case 3: $\beta = -\infty$. We always have $\limsup(s_n t_n) \geqslant -\infty = s\beta$.

Step 2: Prove that $\limsup(s_n t_n) \leqslant s\beta$. Know: $\lim \frac{1}{s_n} = \frac{1}{s}$, hence $\beta = \limsup t_n = \limsup\left(\frac{1}{s_n} \cdot (s_n t_n)\right) \geqslant \frac{1}{s} \limsup(s_n t_n)$. ∎

More about lim inf and lim sup

Theorem (Theorem 12.2)

If (s_n) is a sequence of positive numbers, then
$\liminf \frac{s_{n+1}}{s_n} \leqslant \liminf s_n^{1/n} \leqslant \limsup s_n^{1/n} \leqslant \limsup \frac{s_{n+1}}{s_n}$.

Proof of $\limsup s_n^{1/n} \leqslant \limsup \frac{s_{n+1}}{s_n}$.

Want: $\alpha := \limsup s_n^{1/n} \leqslant L := \limsup \frac{s_{n+1}}{s_n}$.

This clearly holds for $L = +\infty$.

If $L < +\infty$, then it suffices to show that $\alpha \leqslant L_1$ whenever $L_1 > L$.

Let $u_j = \sup_{n>j} \frac{s_{n+1}}{s_n}$. As $\lim_j u_j = L < L_1$, there exists M s.t. $u_M < L_1$, hence $\frac{s_{n+1}}{s_n} < L_1$ for $n \geqslant N = M + 1$. For $m > N$,
$s_m = \frac{s_m}{s_{m-1}} \frac{s_{m-1}}{s_{m-2}} \dots \frac{s_{N+1}}{s_N} s_N < L_1^{m-N} s_N = A L_1^m$, where $A = \frac{s_N}{L_1^N} \in (0, \infty)$.
$s_m^{1/m} \leqslant L_1 A^{1/m}$, hence $\limsup s_m^{1/m} \leqslant \limsup(L_1 A^{1/m}) = L_1 \lim A^{1/m} = L_1$.
■

Even more about lim inf and lim sup

Corollary (Corollary 12.3)

If (s_n) is a sequence of positive numbers, and $\lim \frac{s_{n+1}}{s_n}$ exists, then $\lim s_n^{1/n}$ also exists, and equals $\lim \frac{s_{n+1}}{s_n}$.

Proof.

Theorem 12.2: $\liminf \frac{s_{n+1}}{s_n} \leqslant \liminf s_n^{1/n} \leqslant \limsup s_n^{1/n} \leqslant \limsup \frac{s_{n+1}}{s_n}$.

If $\lim \frac{s_{n+1}}{s_n} = s$, then $\liminf \frac{s_{n+1}}{s_n} = s = \limsup \frac{s_{n+1}}{s_n}$, hence

$\liminf s_n^{1/n} = s = \limsup s_n^{1/n}$. Then $\lim s_n^{1/n} = s$. ∎

Examples of limits

1. $\lim (n!)^{1/n} = +\infty$

Let $s_n = n!$, we are interested in $\lim s_n^{1/n}$.

$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{n!} = n+1 \xrightarrow[n]{} +\infty$, hence $\lim s_n^{1/n} = +\infty$.

2. $\lim \frac{1}{n}(n!)^{1/n} = \frac{1}{e}$

Let $s_n = \frac{n!}{n^n}$, we are interested in $\lim s_n^{1/n}$.

$\frac{s_{n+1}}{s_n} = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$.

Fact: $\lim \left(\frac{n+1}{n}\right)^n = \lim \left(1+\frac{1}{n}\right)^n = e$.

Thus, $\lim \frac{s_{n+1}}{s_n} = \frac{1}{e}$, hence $\lim s_n^{1/n} = \frac{1}{e}$.

Stirling's formula (optional) states that $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. For derivation, see `https://en.wikipedia.org/wiki/Stirling's_approximation`

So, $\frac{1}{n}\left(n!\right)^{1/n} \approx \frac{1}{e} \cdot (2\pi n)^{1/(2n)} \to \frac{1}{e}$.

Section 13: metric spaces

Definition (Definition 13.1 – metric)

Suppose S is a set. A function $d : S \times S \to [0, \infty)$ is called a metric (or distance) if the following hold:

(D1) Non-degeneracy: $d(x, y) = 0$ iff $x = y$ (hence $d(x, y) > 0$ when $x \neq y$).

(D2) Symmetry: for $x, y \in S$, $d(x, y) = d(y, x)$.

(D3) Triangle inequality: for $x, y, z \in S$, $d(x, y) + d(y, z) \geqslant d(x, z)$.

(S, d) is called a metric space.

Examples. 1. Let $S = \mathbb{R}$. Then $d(x, y) = |x - y|$ is a metric.

2. The n-dimensional Euclidean space $\mathbb{R}^n$ (p. 84): set of all n-tuples $\vec{x} = (x_1, \ldots, x_n)$ ($x_i \in \mathbb{R}$). $d(\vec{x}, \vec{y}) = \left(\sum_i (x_i - y_i)^2 \right)^{1/2}$. (D1), (D2) are easy to check; (D3) is harder.

Convergence in metric spaces

Definition (Definition 13.2)

Suppose (S, d) is a metric space.
A sequence $(s_n) \subset S$ converges to $s \in S$ if $\lim_n d(s_n, s) = 0$ – that is, $\forall \varepsilon > 0 \; \exists N$ s.t. $d(s_n, s) < \varepsilon$ for $n > N$.
A sequence $(s_n) \subset S$ is Cauchy if $\forall \varepsilon > 0 \; \exists N$ s.t. $d(s_n, s_m) < \varepsilon$ for $m, n > N$.

Note: for $(\mathbb{R}, d)$, we recover the usual definitions of "converges" and "Cauchy."

Complete metric spaces

Proposition

If (s_n) converges, then it is Cauchy.

Proof. Suppose (s_n) converges to s. Want to show (s_n) is Cauchy. Fix $\varepsilon > 0$, and find N s.t. $d(s_n, s_m) < \varepsilon$ when $n, m > N$. Find N s.t. $d(s_n, s) < \frac{\varepsilon}{2}$ if $n > N$. This N works: if $n, m > N$, then $d(s_n, s_m) \leqslant d(s_n, s) + d(s, s_m) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. ∎

Does a Cauchy sequence always converge?

Definition (Definition 13.2 continued)

A metric space (S, d) is called complete if any Cauchy sequence in S converges.

Examples. $\mathbb{R}$ is complete. Will prove: $\mathbb{R}^n$ is complete.