TAM 470 Equations

Taylor series

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} h^n f^n(x) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{(4)}(x) + \cdots$$

Common difference schemes

Scheme	Formula
Forward Difference	$\frac{du_j}{dx} = \frac{(u_{j+1} - u_j)}{\Delta x} + O(\Delta x)$
Backwards Difference	$\frac{du_j}{dx} = \frac{(u_j - u_{j-1})}{\Delta x} + O(\Delta x)$
Central difference for **first** derivative	$\frac{du_j}{dx} = \frac{(u_{j+1} - u_{j-1})}{2\Delta x} + O(\Delta x^2)$
Central difference for **second** derivative	$\frac{d^2u_j}{dx^2} = \frac{(u_{j+1} - 2u_j + u_{j-1})}{\Delta x^2} + O(\Delta x^2)$

Fundamental ODE schemes for initial value problems

Assuming $y' = f(y,t)$ and time step size h:

- Forward Euler: $y_{n+1} = y_n + hf(y_n, t_n)$
- Backward Euler: $y_{n+1} = y_n + hf(y_{n+1}, t_{n+1})$
- Trapezoid rule: $y_{n+1} = y_n + \frac{1}{2}h[f(y_n, t_n) + f(y_{n+1}, t_{n+1})]$

Linear stability analysis: Assuming $y' = f(y,t)$:

Single step method: Assume $f(y,t) = \lambda y$

Multi-step method: Additionally assume $y_n = \sigma^n y_0$

Require $|\sigma| \leq 1$ for stability

Nonlinear $f(y,t)$: Use $\lambda = \frac{\partial f}{\partial y}$ at time $t = 0$

For vector equation, use the most extreme eigenvalue as λ

For the forward Euler method and pure real λ, $\Delta t_{max} = \frac{2}{|\lambda|_{max}}$

Von Neumann stability analysis

Assume solution is $u(x_j, t_n) = u_j^n = \sigma^n e^{ikx_j}$

Apply spatial and time discretization of choice

Plug into scheme, find σ, require $|\sigma| \leq 1$ for stability

Modified Wave Number Analysis

Assume solution is $u(x,t) = \psi(t)e^{ikx}$

Apply spatial discretization of choice

Plug into scheme, arrange into the form $\psi'(t) = \lambda\psi$

Apply time discretization scheme of choice, find σ, require $|\sigma| \leq 1$ for stability

Eigenvalues

2D matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} = (\lambda - a)(\lambda - d) - bc = 0$$

Diagonal matrix (n x n): $\begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix} \rightarrow$ eigenvalues are the diagonal entries.

Newton's method

$$f(x) = 0 \rightarrow \; x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0} \rightarrow \; \boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^k)\right]^{-1} \boldsymbol{f}(\boldsymbol{x}^k)$$

Complex numbers

Given a complex number $z = a + bi$, its magnitude is $|z| = \sqrt{a^2 + b^2}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta, \qquad\qquad e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

<u>FEA for 1D Poisson Equation</u>

$$u''(x) + f(x) = 0 \text{ for } x \in [0, L]$$

The **general weak form** result is:

$$\int_0^L w'u'dx = \int_0^L wfdx + w(L)u'(L) - w(0)u'(0)$$

Basis functions for linear 1D element:

$$N_1^e(x) = \frac{x - x_2^e}{x_1^e - x_2^e} = -\frac{1}{\Delta x_e}(x - x_2^e), \qquad N_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e} = \frac{1}{\Delta x_e}(x - x_1^e)$$

Interpolation of the solution in the element domain:

$$u_h^e(x) = u_1^e N_1^e(x) + u_2^e N_2^e(x) = \sum_{j=1}^{2} u_j^e N_j^e(x)$$

Element stiffness:

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \frac{dN_i^e}{dx}\frac{dN_j^e}{dx}dx = \frac{1}{\Delta x_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element force vector:

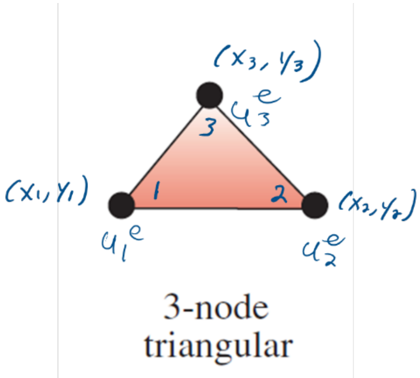
$$F_i^e = \int_e N_i^e(x)f(x)dx = \begin{bmatrix} \int_{x_1^e}^{x_2^e} N_1^e(x)f(x)dx \\ \int_{x_1^e}^{x_2^e} N_2^e(x)f(x)dx \end{bmatrix}, \qquad \boldsymbol{F}^e = \frac{1}{2}\Delta x_e f \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } f = \text{consant}$$

Additional contribution to global force vector due to boundary terms is:

$$\begin{bmatrix} -u'(0) \\ 0 \\ \vdots \\ 0 \\ u'(L) \end{bmatrix}$$

FEA for 2D Steady Heat Equation

3 Node triangle element:



$$A = \frac{1}{2}\left(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right)$$

Basis functions at the element level are

$$N_1^e = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_2^e = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$N_3^e = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

Element Stiffness matrix:

$$K_{ij}^e = \int_{\Omega_e} k\left(\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\right) d\Omega$$

Element load vector for constant Q

$$F_{Q,i}^e = \left(\frac{A}{3}\right)Q, \ i = 1,\ 2,\ 3$$

Element load vector for constant flux $\bar{q}$ on an edge:

$$F_{q,i}^e = \frac{1}{2}\bar{q}L_e, \ i = \text{nodes along edge where } \bar{q} \text{ is applied}$$