MATH 447: Real Variables - Homework #3

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Problem 1 (11.5). *Let (q_n) be an enumeration of all the rationals in the interval $(0,1]$.*

1. Give the set of subsequential limits for (q_n).

2. Give the values of $\limsup q_n$ and $\liminf q_n$.

Solution 1.   *1. The set of subsequential limits for (q_n) is $q \in Q : 0 < q \leq 1$.*

2. (a) $\limsup q_n = 1$

(b) $\liminf q_n = 0$

Problem 2 (11.11). *Let S be a bounded set. Prove there is an increasing sequence (s_n) of points in S such that $\lim_{n \to \infty} s_n = \sup S$. Compare Exercise 10.7. Note: If $\sup S$ is in S, it's sufficient to define $s_n = \sup S$ for all n.*

Solution 2. *$u_n = \sup s_n$, and $s_n \leq \sup s_n$. If S is bounded, then there exists a real number a such that $s_n \leq a$. By Theorem 11.4 in Ross, there exists a monotonic subsequence for bounded set S. We shall prove that this sequence is increasing.*

Proof. Because S is bounded, we know that there exists $\sup S$ by the Axiom of Completeness. We also know that monotone bounded sequences converge by Theorem 10.2 in Ross. Therefore, we can say:

$$\sup S - s_n < 1 \tag{1}$$

Where 1 is chosen arbitrarily. We can then choose a subsequence of S such that it satisfies the following conditions:

$$\sup S - s_n = \min\left(\sup S - s_{n-1}, \frac{1}{n}\right) \tag{2}$$

Choosing such s_n enforces that the sequence is increasing, and is getting closer and closer to the supremum. With the the values selected, we now have a subsequence of S such that $n_1 < n_2 < \ldots n_k$, $s_{n_1} < s_{n_2} < \ldots s_{n_k}$, and $\lim S_{n_k} = \sup S$. $\qquad \square$

Problem 3 (12.3 (d,e)). *Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:*

1. $(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$

2. $(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \dots)$

Find:

1. $\limsup(s_n + t_n)$

2. $\limsup s_n + \limsup t_n$

Solution 3.   1. $\limsup(s_n + t_n) = 3$

2. $\limsup(s_n) + \limsup(t_n) = 4$

Problem 4 (12.4). *Show* $\lim_{N\to\infty} \sup_{n>N}(s_n + t_n) \leq \lim_{N\to\infty} \sup_{n>N} s_n + \lim_{N\to\infty} \sup_{n>N} t_n$ *for bounded sequences* (s_n) *and* (t_n). *Hint: First show:*

1. $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$.

Then apply Exercise 9.9 (c).
Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$
Exercise 9.9 (c):

1. *Prove that if* $\lim s_n$ *and* $\lim t_n$ *exist, then* $\lim s_n \leq \lim t_n$.

Solution 4. *Proof.* We want to find the following:

$$\limsup_{N\to\infty} s_N + t_N \leq (\limsup_{N\to\infty} s_n) + (\limsup_{N\to\infty} t_n) \tag{3}$$

We know that the following holds true, as this is the definition of $\limsup$.

$$\limsup_{N\to\infty} s_N = \limsup_{M\to\infty} \{s_N | N \geq M\} \tag{4}$$

We also know that any element of a sequence s_n is always less than or equal to the supremum of s_n due to the Axiom of Completeness. Then, we can state the following.

$$N > M \implies s_N \leq \sup\{s_N | N \geq M\} \tag{5}$$

Therefore, the following also holds true from above:

$$s_N + t_N \leq \sup\{s_N | N > M\} + \sup\{t_N | N > M\} \tag{6}$$

The RHS of the above is a constant, so the following is true:

$$(s_N + t_N \leq C) \tag{7}$$

If any value of $s_n + t_n$ is always less than the RHS of above, then we can say the the supremum of $s_n + t_n$ is also less than or equal to the RHS of above:

$$\sup\{s_N + t_N | N > M\} \leq \sup\{s_N | N > M\} + \sup\{t_N | N > M\} \tag{8}$$

We have now established $\sup(s_n + t_n) \leq \sup s_n + \sup t_n$. Now, we can use a familiar limit theorem to prove our original hypothesis.

$$\limsup_{N \to \infty} s_N + t_N \tag{9}$$

We know that the above is equal to the following from the definition of $\limsup$. Then, we can take the limit of both sides. From Exercise 9.9c, we know that if if given N_0 s.t. $s_n \leq t_n$ for all $n > N_0$ $\lim_{n \to \infty} s_n$ and $\lim_{n \to \infty} t_n$ exist, then $\lim_{n \to \infty} \leq \lim_{n \to \infty} t_n$

$$= \limsup_{M \to \infty} \{s_N + t_N | N > M\} \leq \lim_{M \to \infty} (\sup \{s_N | N > M\} + \sup \{t_N | N > M\}) \tag{10}$$

From a known limit theorem, we know that $\lim_{n \to \infty} s_n + t_n = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n$:

$$= \limsup_{M \to \infty} \{s_N | N > M\} + \limsup_{M \to \infty} \{t_N | N > M\} \tag{11}$$

$$\limsup_{N \to \infty}(s_N + t_N) \leq \limsup_{N \to \infty} s_N + \limsup_{N \to \infty} t_N \tag{12}$$

$\square$

Problem 5 (12.12). *Let (s_n) be a sequence of nonnegative numbers, and for each n define* $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

1. *Show:*

$$\lim_{N \to \infty} \inf_{n > N} s_n \leq \lim_{N \to \infty} \inf_{n > N} \sigma_n \leq \lim_{N \to \infty} \sup_{n > N} \sigma_n \leq \lim_{N \to \infty} \sup_{n > N} s_n \tag{13}$$

   Hint: for the last inequality, show first that $M > N$ implies:

$$\sup \{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_n) + \sup \{s_n : n > N\} \tag{14}$$

2. *Show that if $\lim_{n \to \infty} s_n$ exists, then $\lim_{n \to \infty} \sigma_n$ exists and $\lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} s_n$.*

3. *Give an example where $\lim_{n \to \infty} \sigma_n$ exists, but $\lim_{n \to \infty} s_n$ does not exist.*

Solution 5. 1. *Proof.*

$$\liminf_{N \to \infty} s_N \leq \liminf_{N \to \infty} \sigma_N \leq \limsup_{N \to \infty} \sigma_N \leq \limsup_{N \to \infty} s_N \tag{15}$$

$$\lim_{n \to \infty} s_N = s \implies s = \liminf_{N \to \infty} s_N \leq \liminf_{N \to \infty} \sigma_N \leq \limsup_{N \to \infty} \sigma_N \leq \limsup_{N \to \infty} s_N = s \tag{16}$$

By the Squeeze Theorem,

$$\implies \liminf_{N \to \infty} \sigma_N = \limsup_{n \to \infty} \sigma_N = s \tag{17}$$

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$$\implies \lim_{N\to\infty} \sigma_N = s \tag{18}$$

We wish to find the third inequality as shown below, as the second inequality is obvious, and the first inequality can be proved similarly to the third inequality.

$$\limsup_{N\to\infty} \sigma_N \leq \limsup_{N\to\infty} s_N \tag{19}$$

We will fix L. Then, if $M > L$, we can consider $\sup\{\sigma_N | N > M\}$

$$\sigma_N = \frac{s_1 + \ldots s_N}{N} \tag{20}$$

$$= \frac{s_1 + \ldots + s_N}{N} + \frac{s_{L+1} + \ldots + s_N}{N} \tag{21}$$

We know that there are $N - (L+1) + 1 = N - L$ total terms in the denominator of the second term of the RHS above. Therefore, we can state the following:

$$\sigma_N \leq \frac{s_1 + \ldots s_L}{N} + \underbrace{\frac{N-L}{N}}_{\leq 1} \sup\{s_N | N > L\} \tag{22}$$

$$\leq \frac{s_1 + \ldots + s_L}{M} + \sup\{s_N | N > L\} \tag{23}$$

$$\sup\{\sigma_N | N > M\} \leq \frac{s_1 + \ldots + s_L}{M} + \sup\{s_N | N > L\} \tag{24}$$

We know that the RHS above is independent of N, so we take the limit of both sides above. Doing so results in the following:

$$\limsup_{N\to\infty} \sup \sigma_N = \limsup_{M\to\infty} \sigma_N \{N | N > M\} \tag{25}$$

$$\leq \lim_{M\to\infty} (\underbrace{\frac{s_1 + \ldots + s_L}{M}}_{\text{goes to 0 as } M \to infinity} + \sup\{s_N | N > L\}) \tag{26}$$

$$\limsup_{N\to\infty} \sigma_N \leq \sup\{s_N | N > L\} \tag{27}$$

$\square$

2. Proof.

$$|s_n - L| < \varepsilon \tag{28}$$

$$\sigma_n = \frac{1}{n}\Sigma_{i=1}^n s_n \tag{29}$$

$$\lim \frac{1}{n}\Sigma_{i=1}^n s_n \tag{30}$$

$$n > N \implies \Sigma_{i=1}^n s_n \tag{31}$$

$$\implies |ns_n - nL| \tag{32}$$

$$\frac{1}{n}|n(s_n - L)| < \varepsilon \tag{33}$$

$$= |(s_n - L)| < \varepsilon \tag{34}$$

$$= |s_n - L| < \varepsilon \tag{35}$$

$$= \lim \sigma_n = L = \lim s_n \tag{36}$$

$\square$

Proof.

Suppose $s_n = (-1)^n$. Then $\lim s_n =$ DNE. But we know:

$$\frac{-1 \text{ or } 0}{N} \underset{N\to\infty}{\rightarrow} 0 \tag{37}$$

Therefore, the $\lim_{n\to\infty} \sigma_n$ exists if $\lim_{n\to\infty} s_n =$ DNE. $\square$

Problem 6 (12.13)**.** *Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many s_n are $\geq a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \lim_{N\to\infty} \inf_{n>N} s_n$ and $\inf B = \lim_{N\to\infty} \sup_{n>N} s_n$.*

Solution 6. *From Theorem 11.7, there exists a monotonic subsequence whose limit is $\limsup s_n$ and $\liminf s_n$. So $\limsup s_n$ and $\liminf s_n$ exist. From Theorem 10.1, we know that these limits converge because s_n is bounded. Our goal is to show that $\limsup s_n = \inf B$, and $\liminf s_n = \sup A$, for sets $A = \{s_n : s_n < a\}$, and $B = \{s_n : s_n > b\}$, where A, B both have finite cardinality.*

Proof. Choose N_1 s.t. $n > N_1 \implies \{n : |s_n - a| < \varepsilon\}$ is infinite. By Theorem 11.2, a is a subsequential limit. Choose N_2 s.t. $n > N_2 \implies \{n : |s_n - b| < \varepsilon\}$ is infinite. By 11.2, b is also a subsequential limit. To prove that $a = \sup A$, choose $N = \min\{d(s_N, a)\}$. Then

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$N = n + 1 \implies s_{n+1} > a$, so s_n is not the least upper bound of A. So $a = \sup A$. A similar argument follows for $\inf B = b$. We know that $a < b$, or else B would be infinite, contradictory to B being finite in the given. Let $S = \{a, b\}$. Then, by Theorem 11.8, $\inf S = \liminf s_n, \sup S = \limsup s_n$. This implies $\sup A = \liminf s_n, \inf B = \limsup s_n$. $\square$