SOLUTIONS FOR HOMEWORK 6

Throughout this homework, we assume that (reverse) trigonometric and exponential functions – such as exp, sin, cos, tan, or arctan – are continuous. Other common calculus facts about these functions can also be used.

18.7 $f(x) = xe^x$ is a continuous function on $[0, 1]$. We have $f(0) = 0 < 2 < f(1) = e$, hence, by Intermediate Value Theorem, there exists $x \in (0, 1)$ with $f(x) = 2$.

21.2. Suppose first f is continuous at s_0, and U is an open set containing $f(s_0)$. Pick $\varepsilon > 0$ s.t. $\mathbf{B}^o_\varepsilon(f(s_0)) \subset U$. Find $\delta > 0$ s.t. $d^*(f(s), f(s_0)) < \varepsilon$ whenever $d(s, s_0) < \delta$. Let $V = \mathbf{B}^o_\delta(s_0)$; clearly $f(V) \subset U$.

Now suppose that for any open $U \ni f(s_0)$ there exists an open $V \ni s_0$ so that $f(V) \subset U$. Fix $\varepsilon > 0$; for $U = \mathbf{B}^o_\varepsilon(f(s_0))$ find open $V_\varepsilon \ni s_0$ s.t. $f(V_\varepsilon) \subset U$. As V_ε is open, there exists $\delta > 0$ s.t. $\mathbf{B}^o_\delta(s_0) \subset V_\varepsilon$, so $f(\mathbf{B}^o_\delta(s_0)) \subset U = \mathbf{B}^o_\varepsilon(s_0)$. As $\varepsilon > 0$ is arbitrary, the continuity of f at s_0 follows.

21.3. It suffices to show that, for any $s, t \in S$, we have $|f(s) - f(t)| \leq d(s, t)$ (it would immediately follow that we could take $\delta = \varepsilon$ in the definition of uniform continuity).

Note that, by the triangle inequality, $d(s, s_0) \leq d(t, s_0) + d(s, t)$, hence $d(s, s_0) - d(t, s_0) \leq d(s, t)$. Likewise, $d(t, s_0) - d(s, s_0) \leq d(s, t)$. Therefore,

$$\big|f(s) - f(t)\big| = \big|d(s, s_0) - d(t, s_0)\big| = \max\big\{d(s, s_0) - d(t, s_0), d(t, s_0) - d(s, s_0)\big\} \leq d(s, t).$$

21.4. <u>(i) $\Rightarrow$ (ii)</u>: follows from Exercise 21.2.

<u>(ii) $\Rightarrow$ (iii)</u>: trivial.

<u>(iii) $\Rightarrow$ (ii)</u>: We can find $a_1, b_1, a_2, b_2 \ldots \in (a, b) \cap \mathbb{Q}$ s.t. (i) $a_n < b_n$ for any n, and (ii) $\lim a_n = a$, $\lim b_n = b$. Then $(a, b) = \cup_n (a_n, b_n)$, hence also $f^{-1}\big((a, b)\big) = \cup_n f^{-1}\big((a_n, b_n)\big)$. Now recall that a union of open sets is open.

<u>(ii) $\Rightarrow$ (i)</u>: We have to show that, for any open $U \subset \mathbb{R}$, $f^{-1}(U)$ is open. In the lectures we proved that U, as an open set, is a union of open balls – which, in our case, are open intervals: $U = \cup_{(a,b)\subset U}(a, b)$. Consequently, $f^{-1}(U) = \cup_{(a,b)\subset U} f^{-1}\big((a, b)\big)$, which must be open.

21.5. As E is not compact, then either it is unbounded, or it is not closed (Heine-Borel Theorem).

(a) If E is not bounded, pick $s_0 \in \mathbb{R}^k$, and consider $f(s) = d(s_0, s)$. This function is continuous (by Exercise 21.3), and clearly unbounded .

If E is not closed, find $s_0 \in E^- \backslash E$, and let $f(s) = \frac{1}{d(s_0,s)}$. This function is continuous on E, and unbounded, since $d(s_0, s)$ can be arbitrarily small.

(b) In part (a), we constructed a function $f : E \to [0, \infty)$ which is not bounded above. Let $g(s) = \arctan(f(s))$. This function is continuous, as a composition of continuous functions; it takes values in $[0, \frac{\pi}{2})$. As $f(s)$ can be arbitrarily large, $g(s)$ can be arbitrarily close to $\frac{\pi}{2}$. However, g is never equal to $\frac{\pi}{2}$.

21.10. (d) Any continuous $f : [0,1] \to \mathbb{R}$ has to be bounded, hence it cannot be onto. Moreover, f has to attain its maximum and minimum, hence $f([0,1])$ cannot equal $(0,1)$.

Bonus problem. Is it true that any bounded continuous function on $\mathbb{R}$ is uniformly continuous?

No. Consider, for instance, $f(x) = \cos(\pi x^2)$. Clearly this function is bounded and continuous. Suppose, for the sake of contradiction, that it is uniformly continuous as well. Then there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$.
Note that $f(\sqrt{n}) = \cos(n\pi) = (-1)^n$. We know that

$$\lim_n \left(\sqrt{n+1} - \sqrt{n}\right) = \lim_n \left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_n \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0,$$

hence we can find $n \in \mathbb{N}$ s.t. $\sqrt{n+1} - \sqrt{n} < \delta$, yet

$$\left|f(\sqrt{n}) - f(\sqrt{n+1})\right| = \left|(-1)^n - (-1)^{n+1}\right| = 2 > 1.$$

This is the desired contradiction.