Rolle's Theorem

Theorem (29.1 – criterion for max and min)

Suppose f is defined on an open interval I, has max *or* min *at $x_0 \in I$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Theorem (29.2)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b). If $f(a) = f(b)$, then $\exists x \in (a, b)$ s.t. $f'(x) = 0$.

Proof. f attains its max and min; say $f(y_0) \leqslant f(x) \leqslant f(x_0)$ for any $x \in [a, b]$. In particular, $f(y_0) \leqslant f(a) = f(b) \leqslant f(x_0)$.
If $f(y_0) = f(a) = f(b) = f(x_0)$, then f is a constant, hence $f' = 0$ on (a, b).
If $f(x_0) > f(a) = f(b)$, then $x_0 \in (a, b)$, hence $f'(x_0) = 0$.
If $f(y_0) < f(a) = f(b)$, then $y_0 \in (a, b)$, hence $f'(y_0) = 0$. ∎

Mean Value Theorem

Theorem (29.2 – MVT)

If $f : [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b), then $\exists x \in (a,b)$ s.t. $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$, $g(x) = f(x) - L(x)$. g is continuous on $[a,b]$, differentiable on (a,b), $g(a) = 0 = g(b)$. By Rolle's Theorem, $\exists x \in (a,b)$ s.t. $g'(x) = 0$. Then
$f'(x) = g'(x) + L'(x) = 0 + \frac{f(b)-f(a)}{b-a}$. ■

Example. For $x, y \in \mathbb{R}$, $|\sin x - \sin y| \leqslant |x-y|$.

Apply MVT to $f(t) = \sin t$ on $[x,y]$: $\exists z \in (x,y)$ s.t.
$\frac{f(x)-f(y)}{x-y} = f'(z) = \cos z$. $|\cos z| \leqslant 1$, hence $\frac{|f(x)-f(y)|}{|x-y|} = |\cos z| \leqslant 1$. ■

Consequences of Mean Value Theorem

Corollary (29.4)

If f is differentiable on (a, b), and $f' = 0$ on (a, b), then f is a constant function.

Proof. Suppose f is not a constant. Find $x < y$ s.t. $f(x) \neq f(y)$. By MVT, $\exists z \in (x, y)$ s.t. $f'(z) = \frac{f(y)-f(x)}{y-x} \neq 0$. ■

Corollary (29.5)

If f, g are differentiable on (a, b), and $f' = g'$ on (a, b), then $\exists c \in \mathbb{R}$ s.t. $f(x) = g(x) + c \ \forall x \in (a, b)$.

Proof. Apply Corollary 29.4 to $h = f - g$. ■

Increasing and decreasing functions

Definition (29.6)

A function f on an interval I is called increasing (strictly increasing) if $f(x_1) \leqslant f(x_2)$ $(f(x_1) < f(x_2))$ when $x_1 < x_2$.

Corollary (29.7)

Suppose f is a differentiable function on (a, b).

1. *f is increasing iff $f' \geqslant 0$ on (a, b).*
2. *If $f' > 0$ on (a, b), then f is strictly increasing.*

Remark. f strictly increasing $\not\Rightarrow$ $f' > 0$ on (a, b). For instance, $f(x) = x^3$ is strictly increasing on $(-1, 1)$; however, $f'(x) = 3x^2$, so $f'(0) = 0$.

Increasing and decreasing functions, continued

Corollary 29.7(i) Suppose f is differentiable on (a, b). Then f is increasing iff $f' \geqslant 0$ on (a, b).

Proof. **(1)** Suppose f is increasing on (a, b). For $c \in (a, b)$, $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. Right hand side is $\geqslant 0$.

(2) Suppose $f' \geqslant 0$ on (a, b). If $x_1 < x_2$, use MVT to find $x \in (x_1, x_2)$ s.t. $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \geqslant 0$, hence $f(x_2) \geqslant f(x_1)$. ∎

Corollary (not in the textbook)

Suppose f is continuous on $[a, b]$, and differentiable on (a, b).
(i) f is increasing on $[a, b]$ iff $f' \geqslant 0$ on (a, b).
(ii) If $f' > 0$ on (a, b), then f is strictly increasing on $[a, b]$.

Remark. Suppose $f'(c) > 0$. Is f "locally increasing" at c – that is, does there exist $\delta > 0$ s.t. f increases on $(c - \delta, c + \delta)$? No! See Exercise 29.10.

Examples

Examples. (1) $\sin x \leqslant x$ for $x \geqslant 0$. Consider $f(x) = x - \sin x$. Want to show: $f \geqslant 0$ on $[0, \infty)$.
f is differentiable (hence continuous) on $\mathbb{R}$. $f'(x) = 1 - \cos x \geqslant 0$, hence f is increasing on $\mathbb{R}$. If $x \geqslant 0$, then $f(x) \geqslant f(0) = 0$. ■

(2) Bernoulli Inequality. If $n \in \mathbb{N}$, and $x > -1$, then $(1 + x)^n \geqslant 1 + nx$. Let $f(x) = (1 + x)^n - (1 + nx)$, and show that $f \geqslant 0$ on $(-1, \infty)$. The function f is differentiable (hence continuous) on $(-1, \infty)$.
$f'(x) = n\big((1 + x)^{n-1} - 1\big)$; $f' < 0$ on $(-1, 0)$, and $f' > 0$ on $(0, \infty)$.
Thus, f decreases on $(-1, 0]$, increases on $[0, \infty)$.
If $x \in (-1, 0]$, then $f(x) \geqslant f(0) = 0$. If $x \in (0, \infty)$, then $f(x) \geqslant f(0) = 0$.
Thus, for any $x \in (-1, \infty)$, $f(x) \geqslant f(0) = 0$. ■

Inermediate Value Theorem for Derivatives

Theorem (29.8 – IVT for Derivatives, due to Darboux)

Suppose f is differentiable on (a, b), $a < x_1 < x_2 < b$, and c lies between $f'(x_1)$ and $f'(x_2)$. Then $\exists\ y \in (x_1, x_2)$ s.t. $f'(y) = c$.

Theorem (Intermediate Value Theorem, Lecture 19)

Suppose g is continuous on $[x_1, x_2]$, and c lies between $g(x_1)$ and $g(x_2)$. Then $\exists\ y \in (x_1, x_2)$ s.t. $g(y) = c$.

Remark. f' need not be continuous! Consider

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

So, we cannot apply IVT to prove IVTD.

Intermediate Value Theorem for Derivatives

Theorem (29.8 – IVT for Derivatives, due to Darboux)

Suppose f is differentiable on (a, b), $a < x_1 < x_2 < b$, and c lies between $f'(x_1)$ and $f'(x_2)$. Then $\exists\ y \in (x_1, x_2)$ s.t. $f'(y) = c$.

Proof. Say $f'(x_1) < c < f'(x_2)$. $g(x) = f(x) - cx$ attains its min on $[x_1, x_2]$ at some y. $\lim_{x\to x_1} \frac{g(x)-g(x_1)}{x-x_1} = g'(x_1) = f'(x_1) - c < 0$, hence $\exists\delta > 0$ s.t. $g(x) < g(x_1)$ for $x \in (x_1, x_1 + \delta)$. Thus, $y \neq x_1$. Similarly, $y \neq x_2$. $y \in (x_1, x_2)$, so $g'(y) = 0$, and $f'(y) = c$. ∎

IVT for Derivatives: applications

Examples. (1) Suppose f is differentiable on $\mathbb{R}$, $f(0) = 7$, $f(2) = 1$, $f(5) = 4$. Prove that there exists $x \in \mathbb{R}$ s.t. $f'(x) = -1$.

(i) Apply MVT to $[0, 2]$: $\exists x_1 \in (0, 2)$ s.t. $f'(x_1) = \frac{f(2)-f(0)}{2-0} = -3$.

(ii) Apply MVT to $[2, 5]$: $\exists x_2 \in (2, 5)$ s.t. $f'(x_2) = \frac{f(5)-f(2)}{5-2} = 1$.

(iii) Apply IVTD to (x_1, x_2), with $-1 \in \big(f'(x_1), f'(x_2)\big)$.

(2) There is **no** function f, differentiable on $\mathbb{R}$, so that $f'(x) = \mathrm{sign}(x)$ (recall that $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x < 0$, and $\mathrm{sign}(0) = 0$). Apply IVTD with $x_1 = 0$, $x_2 = 1$, $c = \frac{1}{2}$: there is no $x \in (x_1, x_2)$ s.t. $\mathrm{sign}(x) = c$.

Differentiating inverse functions (pp. 237-239)

Set-up. Suppose I is an interval, and $f : I \to \mathbb{R}$ is a continuous strictly monotone function. Then $J = f(I)$ is an interval; $g = f^{-1} : J \to I$ is strictly monotone, and continuous (Section 18).

Theorem (29.9)

If f is differentiable at $c \in I$, and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$, and $g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$.

A way to memorize the formula: $\forall y \in J$, $f(g(y)) = y$. Differentiate both sides at d, using Chain Rule: $f'(g(d))g'(d) = 1$. This is not a proof! We need to prove g is differentiable at d.