Total boundedness

$E \subset S$ is totally bounded if $\forall \varepsilon > 0\ \exists s_1, \ldots, s_n \in S$ s.t. $E \subset \cup_{i=1}^{n} \mathbf{B}_{\varepsilon}^{o}(s_i)$.

Proposition (not in textbook)

E is totally bounded iff any sequence in E has a Cauchy subsequence.

Proof: E t.b. $\Rightarrow$ any sequence in E has a Cauchy subsequence.
Suppose $(s_i) \subset EE$; find Cauchy subsequence.

Find $x_{11}, \ldots x_{1N_1} \in S$ so that $E \subset \cup_{j=1}^{N_1} \mathbf{B}_{2^{-2}}^{o}(x_{1j})$. Find $j_1 \in \{1, \ldots, N_1\}$ so that $I_1 = \{i : s_i \in \mathbf{B}_{2^{-2}}^{o}(x_{1j_1})\}$ is infinite. Pick $i_1 := \min I_1$.

Find $x_{21}, \ldots x_{2N_2} \in S$ so that $E \cap \mathbf{B}_{2^{-2}}^{o}(x_{1j_1}) \subset \cup_{j=1}^{N_2} \mathbf{B}_{2^{-3}}^{o}(x_{2j})$. Find $j_2 \in \{1, \ldots, N_2\}$ so that $I_2 = \{i \in I_1 : s_i \in \mathbf{B}_{2^{-3}}^{o}(x_{2j_2})\}$ is infinite. Pick $i_2 := \min(I_2 \backslash \{i_1\})$.

Note: $s_{i_1} \in \mathbf{B}_{2^{-2}}^{o}(x_{1j_1})$, $s_{i_2} \in \mathbf{B}_{2^{-2}}^{o}(x_{1j_1}) \cap \mathbf{B}_{2^{-3}}^{0}(x_{2j_2})$.

Proceed to obtain $i_1 < i_2 < \ldots$ s.t. $s_{i_k} \in \mathbf{B}_{2^{-1-\ell}}^{o}(x_{\ell j_\ell})$ for $k \geqslant \ell$. For $k \geqslant \ell$, $d(s_{i_k}, s_{i_\ell}) \leqslant d(s_{i_k}, x_{\ell j_\ell}) + d(x_{\ell j_\ell}, s_{i_\ell}) < 2 \cdot 2^{-1-\ell} = 2^{-\ell}$. Thus, (s_{i_k}) is Cauchy. ∎

Compactness versus convergence of sequences

Theorem (Characterization of compactness; not in textbook)

For a subset E of a metric space, the following are equivalent:

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Proof of (1) $\Rightarrow$ (2). Last lecture. ∎

Proof of (2) $\Rightarrow$ (3). Suppose E is totally bounded and complete. Any sequence has a Cauchy subsequence (total boundedness), which has limit in E (completeness). ∎

Proof of (3) $\Rightarrow$ (2). If E is not totally bounded, then it contains a sequence with no Cauchy subsequences, hence no convergent subsequences. If E is not complete, then it has a Cauchy sequence with no limit in E, hence with no convergent subsequences (Homework 4). ∎

Proof of (3) $\Rightarrow$ (1) (or (2) $\Rightarrow$ (1)): omitted for lack of time.

Compactness and convergence of sequences (redundant)

Lemma (Established for $\mathbb{R}$, true for any metric space)

A sequence (s_n) in a metric space (S, d) has a subsequence converging to s iff, for any $\varepsilon > 0$, $|\{n \in \mathbb{N} : d(s_n, s) < \varepsilon\}| = \infty$.

Proof of Characterization of Compactness, $(1) \Rightarrow (3)$. Shall show the contrapositive: $\neg(3) \Rightarrow \neg(1)$.
Suppose a sequence $(s_n) \subset E$ has no subsequence converging to a limit in E. Construct an open cover for E without a finite subcover.

For each $x \in E$, find $r(x) > 0$ s.t. $|\mathbf{B}^o_{r(x)}(x) \cap \{s_n : n \in \mathbb{N}\}| < \infty$. Then $\{\mathbf{B}^o_{r(x)}(x) : x \in E\}$ is an open cover for E. There is no finite subcover: for any finite collection $(x_i)_{i=1}^m$, $\cup_{i=1}^m \mathbf{B}^o_{r(x_i)}(x_i)$ contains only finitely many s_n's.
■

Compact subsets of $\mathbb{R}^n$

Theorem (13.13 – Heine-Borel)

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof of HB: closed and bounded $\Rightarrow$ compact. Recall: compact $\Leftrightarrow$ complete and totally bounded.
Suppose $E \subset \mathbb{R}^n$ is closed and bounded.
$\mathbb{R}^n$ is complete, E is closed $\Rightarrow$ E is complete (Homework 4).
Remains to show: E is totally bounded – that is,
$\forall\ \varepsilon > 0\ \exists\ \vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ s.t. $E \subset \cup_{j=1}^{k} \mathbf{B}_{\varepsilon}^{o}(\vec{x}^{(j)})$.
Pick $A > 0$ s.t. $\forall \vec{x} = (x_i)_{i=1}^{n} \in E$ we have $\|\vec{x}\| < A$. Write $\vec{x} = (x_i)_{i=1}^{n} \in E$, then $|x_i| < A$. Find $N \in \mathbb{N}$ s.t. $\frac{A}{N} < \frac{\varepsilon}{\sqrt{n}}$. Let $\vec{x}^{(j)}$ be vectors with coordinates $p\frac{A}{N}$, with $p \in \{-N, \ldots, N\}$.
For $\vec{x} \in E$, find j s.t. $|x_i^{(j)} - x_i| < \frac{\varepsilon}{\sqrt{n}}$, for $1 \leqslant i \leqslant n$. Then
$d(\vec{x}, \vec{x}^{(j)}) = \left(\sum_{i=1}^{n} (x_i^{(j)} - x_i)^2 \right)^{1/2} < \varepsilon$. ∎

Remark. $\mathbb{N}$, with $0-1$ metric, is closed and bounded, but not compact.

Series (Section 14)

Goal: make sense of infinite sums $\sum_{j=k_0}^{\infty} a_j = a_{k_0} + a_{k_0+1} + \dots$.

Definition (p. 95 of textbook)

n-th partial sum: $s_n = \sum_{j=k_0}^{n} a_j$. The series $\sum_{j=k_0}^{\infty} a_j$ converges (diverges) if (s_n) does. $\sum_{j=k_0}^{\infty} a_j = \lim_n s_n$ (if $\lim s_n$ exists).

Example: $\sum_{j=0}^{\infty} r^j$. Partial sums: $s_n = \sum_{j=0}^{n} r^j$.
If $r = 1$, then $s_n = n + 1$, hence the series diverges.
If $r \neq 1$, then $s_n = \frac{1-r^{n+1}}{1-r}$. If $|r| < 1$, then $\lim s_n = \frac{1}{1-r}$. Otherwise, (s_n) diverges.

If $r \geqslant 1$, then $s_n \geqslant n$ for any n, hence $\lim s_n = +\infty$.

If $r \leqslant -1$, then $s_n \leqslant 0$ when n is odd (since $r^{n+1} \geqslant 1$). For n even, $-r^{n+1} \geqslant -r \geqslant 0$, hence $s_n \geqslant 1$. Thus, $\lim s_n$ does not exist.

Summary: $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$ for $|r| < 1$; $\sum_{j=0}^{\infty} r^j = +\infty$ for $r \geqslant 1$; $\lim s_n$ does not exist if $r \leqslant -1$.

Cauchy Criterion for convergence

Definition (14.3)

A series $\sum_j a_j$ satisfies Cauchy Criterion if $\forall \varepsilon > 0$ $\exists N$ s.t. $\left|\sum_{j=m}^{n} a_j\right| < \varepsilon$ whenever $n \geqslant m > N$.

Theorem (14.4)

A series converges iff it satisfies the Cauchy Criterion.

Proof. Let $s_n = \sum_{j=k_0}^{n} a_j$. $\sum_{j=k_0}^{\infty} a_j$ converges $\Leftrightarrow$ (s_n) converges
$\Leftrightarrow$ (s_n) is Cauchy $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists N$ s.t. $|s_n - s_k| < \varepsilon$ when $n > k \geqslant N$.
$s_n - s_k = \sum_{j=k_0}^{n} a_j - \sum_{j=k_0}^{k} a_j = \sum_{j=m}^{n} a_j$, where $m = k + 1$.
So: $\sum_j a_j$ satisfies Cauchy Criterion $\Leftrightarrow$ (s_n) is Cauchy $\Leftrightarrow$ (s_n) converges. ■

More about convergence

Corollary (14.5)

If $\sum_j a_j$ converges, then $\lim_n a_n = 0$.

Proof. Suppose $\sum_j a_j$ converges. Fix $\varepsilon > 0$. Need to find N s.t. $|a_n| < \varepsilon$ for $n > N$. Find N s.t. $\left|\sum_{j=m}^{n} a_j\right| < \varepsilon$ when $n \geqslant m > N$ (Cauchy Criterion). Now take $m = n$. ∎

If $\lim_n a_n = 0$, does $\sum_j a_j$ converge? **No!** Let $a_n = \frac{1}{n}$; $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since the sequence of partial sums (s_n) is unbounded (Lecture 6).

Comparison test for convergence

Theorem (14.6 – Comparison test)

(i) If $a_n \geqslant |b_n|$ for any n, and $\sum_n a_n$ converges, then $\sum_n b_n$ converges.
(ii) If $0 \leqslant a_n \leqslant b_n$ for any n, and $\sum_n a_n = +\infty$, then $\sum_n b_n = +\infty$.

Proof of (i). Need to check Cauchy Criterion for $\sum_n b_n$:
$\forall \varepsilon > 0 \; \exists N$ s.t. $\left| \sum_{j=m}^{n} b_j \right| < \varepsilon$ if $n \geqslant m > N$.
$\sum_n a_n$ converges, hence $\exists N$ s.t. $\sum_{j=m}^{n} a_j < \varepsilon$ if $n \geqslant m > N$.
By Triangle Inequality, $\left| \sum_{j=m}^{n} b_j \right| \leqslant \sum_{j=m}^{n} |b_j| \leqslant \sum_{j=m}^{n} a_j < \varepsilon$. ∎