SOLUTIONS FOR HOMEWORK 8

Throughout this homework, we assume that (reverse) trigonometric and exponential functions – such as exp, sin, cos, tan, or arctan – are continuous. Other common calculus facts about these functions can also be used.

20.16 (a) Find a sequence $(x_n) \subset (a, b)$, which converges to a. Then $L_i = \lim f_i(x_n)$, for $i = 1, 2$. The inequality $f_1(x_n) \leq f_2(x_n)$ holds for every n, hence $L_1 \leq L_2$.

(b) No we cannot conclude that $L_1 < L_2$. Consider, for instance, $f_1(x) = -x^2$ and $f_2(x) = x^2$, both defined on $(0, \infty)$. Then $f_1 < f_2$ everywhere, but $\lim_{x\to 0} f_1 = 0 = \lim_{x\to 0} f_2$.

20.17 We have to show that, if a sequence $(x_n) \subset (a, b)$ converges to a, then $\lim f_2(x_n) = L$. We know that $\lim f_1(x_n) = L = \lim f_3(x_n)$, and that $f_1(x_n) \leq f_2(x_n) \leq f_3(x_n)$. The equality $\lim f_2(x_n) = L$ follows by Squeeze Theorem for sequences.

23.4 (c) We have $a_n = \left[\frac{4+2(-1)^n}{5}\right]^n$. Then $a_n^{1/n} = \begin{cases} 6/5 & n \text{ even} \\ 4/5 & n \text{ odd} \end{cases}$, hence $\limsup_n |a_n|^{1/n} = \frac{6}{5}$. Therefore, the radius of convergence of the power series $\sum_n a_n x^n$ equals $\frac{5}{6}$. If $x = \pm\frac{5}{6}$, the series diverges. The interval of convergence is therefore $(-\frac{5}{6}, \frac{5}{6})$.

23.5 (b) Suppose $\limsup |a_n| = \lambda > 0$. Then $\lim u_m \geq \lambda$, where $u_m = \sup_{n\geq m} |a_n|$. Fix $c \in (0, \lambda)$. For any m we can find $n \geq m$ s.t. $|a_n| > c$. Consequently, there exist $n_1 < n_2 < \ldots$ so that $|a_{n_k}| > c$. Let $v_m = \sup_{n\geq m} |a_n|^{1/n}$, then $v_m \geq \sup_{n_k\geq m} |a_{n_k}|^{1/n_k}$. However, $|a_{n_k}|^{1/n_k} > c^{1/n_k}$, hence $v_m \geq \lim_k c^{1/n_k} = \lim_j c^{1/j} = 1$. In the notation of Theorem 23.1, $\beta = \limsup |a_n|^{1/n} = \lim_m v_m \geq 1$, hence $R = \frac{1}{\beta} \leq 1$.

Alternative proof. If $\limsup |a_n| > 0$, then the sequence (a_n) does not converge to 0, hence the series $\sum_k a_k$ diverges. Denote by I the interval of convergence of the power series $\sum_k a_k x^k$. Then $(-R, R) \subset I$, and $1 \notin I$, so $R \leq 1$.

24.10 (a) Fix $\varepsilon > 0$, and find $N \in \mathbb{N}$ so that $\big|(f_n(x)+g_n(x))-(f(x)+g(x))\big| < \varepsilon$ for $n \geq N$, and any $x \in S$. To this end, find $N_1, N_2 \in \mathbb{N}$ s.t. (i) $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for any $x \in S$, and $n \geq N_1$, and (ii) $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ for any $x \in S$, and $n \geq N_2$. If $n \geq \max\{N_1, N_2\}$, then, for $x \in S$,

$$\big|(f_n(x) + g_n(x)) - (f(x) + g(x))\big| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon.$$

24.11 (a) The uniform convergence $f_n \to f$ is evident. For $g_n \to g$, fix $\varepsilon > 0$, and find $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. Them for $n \geq N$, $\sup_{x\in\mathbb{R}} \big|g_n(x) - g(x)\big| = \dfrac{1}{n} < \varepsilon$.

(b) Let $h_n = f_n g_n$, then $h_n(x) = \frac{x}{n}$. The sequence (h_n) does not converge to $h(x) = 0$ uniformly, since, for any n, $\sup_{x\in\mathbb{R}} |h_n(x) - h(x)| = +\infty$.

24.14 (a) Clearly $f_n(0) = 0$ for any n. If $x \neq 0$, then $\lim_n f_n(x) = \lim_n \dfrac{x/n}{1/n^2 + x} = \dfrac{\lim(x/n)}{\lim(1/n^2 + x)} = \dfrac{0}{x} = 0$.

(b) The convergence is not uniform: $\sup_{x\in[0,1]} \big|f_n(x) - f(x)\big| \geq f_n\big(\frac{1}{n}\big) = \frac{1}{2}$ (in fact, equality holds), hence $\lim_n \sup_{x\in[0,1]} \big|f_n(x) - f(x)\big| \neq 0$.

(c) The convergence on $[1,\infty)$ is uniform: differentiating, one shows that f_n is decreasing on $[1,\infty)$, hence $\lim_n \sup_{x\in[1,\infty)} \big|f_n(x) - f(x)\big| = \lim_n f_n(1) = \lim_n \frac{n}{1+n^2} = 0$.

25.5 To prove that f is bounded, find N s.t. $\sup_{x\in S} |f_n(x) - f(x)| < 1$ for all $n \geq N$ (and, in particular, for $n = N$). f_N is bounded, that is, there exists $A > 0$ s.t. $|f_N(x)| < A$, for any $x \in S$. Then $|f(x)| < A + 1$ for any x, fo f is bounded.

25.9 (a) The uniform convergence follows from Weierstrass M-test, with $M_k = a^k$.

(b) [*Bonus problem*] The convergence on $(-1,1)$ is not uniform. To see this, consider the partial sums $s_k(x) = \sum_{n=0}^{k-1} x^n = \frac{1-x^k}{1-x}$, then $s(x) = \lim_k s_k(x) = \frac{1}{1-x}$ pointwise. Look at the error $e_k(x) = s(x) - s_k(x) = \frac{x^k}{1-x}$. We shall show that the sequence $\big(\sup_{x\in(-1,1)} |e_k(x)|\big)_k$ doesn't converge to 0. Note that $e_k'(x) = \frac{kx^{k-1} - (k-1)x^k}{(1-x)^2}$, which vanishes at $\frac{k-1}{k}$. Then

$$\sup_{x\in(-1,1)} |e_k(x)| \geq e_k\big(\frac{k-1}{k}\big) = k\Big(1 - \big(\frac{k-1}{k}\big)^k\Big).$$

We have $\lim_k \big(\frac{k-1}{k}\big)^k = \frac{1}{e}$, hence

$$\lim_k \sup_{x\in(-1,1)} |e_k(x)| \geq \lim_k k\Big(1-\big(\frac{k-1}{k}\big)^k\Big) = \lim_k k\cdot\lim_k \Big(1-\big(\frac{k-1}{k}\big)^k\Big) = (+\infty)\cdot\big(1-\frac{1}{e}\big) = +\infty,$$

and in particular, $\sup_{x\in(-1,1)} |e_k(x)| \neq 0$.

Alternative solution. The partial sums s_k are bounded on $(-1,1)$ (since they are polynomials), while $s(x)$ is not. Now invoke Problem 25.5.