Bounded sets

Definition (Bounded sets)

A set E in a metric space (S, d) is called bounded if there exists $y \in S$ so that $\sup_{x \in E} d(y, x) < \infty$.

If such a y exists, then $\sup_{x \in E} d(z, x) < \infty$ for any $z \in S$. Indeed, $d(z, x) \leqslant d(y, x) + d(z, y)$ (triangle inequality), hence $\sup_{x \in E} d(z, x) \leqslant d(y, z) + \sup_{x \in E} d(y, x_k) < \infty$.

A sequence (x_k) is bounded if $\{x_1, x_2, \ldots\}$ is a bounded set – that is, for some (equivalently, any) $y \in S$ we have $\sup_k d(y, x_k) < \infty$.

Bolzano-Weierstrass for $\mathbb{R}^n$

Theorem (Bolzano-Weierstrass, 13.5)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof. Suppose $(\vec{x}^{(k)})_k$ is bounded. Write $\vec{x}^{(k)} = (x_i^{(k)})_{i=1}^n$.
$(x_1^{(k)})_k$ is bounded, hence exist $k_1 < k_2 < \dots$ s.t. $\lim_j x_1^{(k_j)} = x_1$.
$(x_2^{(k_j)})_j$ is bounded, hence exist $j_1 < j_2 < \dots$ s.t. $\lim_\ell x_2^{(k_{j_\ell})} = x_2$.
Note that $(\vec{x}^{(k_{j_\ell})})_\ell$ is a subsequence of $(\vec{x}^{(k)})_k$, and $\lim_\ell x_i^{(k_{j_\ell})} = x_i$ for $i = 1, 2$.
Repeat this procedure $n-2$ more times, obtain a subsequence $(\vec{y}^{(p)})_p$ of $(\vec{x}^{(k)})_k$, so that $\lim_p y_i^{(p)} = x_i$ for $1 \leqslant i \leqslant n$ (here $\vec{y}^{(p)} = (y_i^{(p)})_{i=1}^n$).
By Lemma (from Lecture 10), $\lim_p \vec{y}^{(p)} = \vec{x} = (x_i)_{i=1}^n$. ■

Remark: failure of Bolzano-Weierstrass for arbitrary metric spaces.
Equip $\mathbb{N}$ with discrete metric: let $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y$. The bounded sequence $x_n = n$ has no convergent subsequences (convergent sequences are eventually constant, see Lecture 10).

Interior points in metric space (S, d)

Definition (Open ball – not in textbook)

Suppose $s_0 \in S$, $r > 0$. The open ball with center s_0 and radius r is $\mathbf{B}_r^o(s_0) = \{s \in S : d(s, s_0) < r\}$.

Definition (13.6 – interior of $E \subset S$)

$s_0 \in S$ is called interior to E if $\exists\, r > 0$ s.t. $\mathbf{B}_r^o(s_0) \subset E$. The set of interior points is denoted by E^o, and called the interior of E.

If s_0 is interior to E, then it belongs to E. Thus, $E^o \subset E$.

Example. $S = \mathbb{R}$ (with usual metric), $E = [0, \infty)$. $E^o = (0, \infty)$.

$S = \mathbb{R}^2$ (with Euclidean metric), $E = \{(x, 0) : x \geqslant 0\}$. $E^o = \emptyset$.

Open sets in metric space (S, d)

Definition (13.6 – open sets in S)

$E \subset S$ is called open if $E = E^o$.

Example. $S = \mathbb{R}$. $[0, \infty)$ is not open, $(0, \infty)$ is.

Fact (13.7 – (iii) and (iv) proved in Homework 4)

1. *S is open.*
2. *$\emptyset$ is open.*
3. *A union of any collection of open sets is open.*
4. *An intersection of finitely many open sets is open.*

(iv) doesn't generalize to infinite intersections. $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subset \mathbb{R}$ is open, $\cap_n I_n = \{0\}$ is not.

More about open sets

Proposition (not in textbook)

1. *Any open ball is open.*
2. *For any E, E^o is open – that is, $(E^o)^o = E^o$.*

Proof. (1) Want: if $x \in \mathbf{B}_r^o(x_0)$, then $\exists\, \varepsilon > 0$ s.t. $\mathbf{B}_\varepsilon^o(x) \subset \mathbf{B}_r^o(x_0)$. In fact, $\varepsilon = r - d(x, x_0)$ will do: if $y \in \mathbf{B}_\varepsilon^o(x)$, then $d(x_0, y) \leqslant d(x_0, x) + d(x, y) < d(x_0, x) + \varepsilon = r$.

(2) Suppose $x_0 \in E^o$, and show that $x_0 \in (E^o)^o$. By definition of E^o, $\exists\, r > 0$ s.t. $\mathbf{B}_r^o(x_0) \subset E$. For $x \in \mathbf{B}_r^o(x_0)$ $\exists\, \varepsilon > 0$ s.t. $\mathbf{B}_\varepsilon^o(x) \subset \mathbf{B}_r^o(x_0) \subset E$. Thus, $x \in E^o$, and therefore, $\mathbf{B}_r^o(x_0) \subset E^o$. Therefore, x_0 is interior to E^o. ∎

Open sets are unions of open balls

Proposition (not in textbook)

A set is open iff it is a union of open balls.

Proof. $\Leftarrow$: a union of open sets is open.

$\Rightarrow$: if E is open, them $E = E^o$ – that is, $\forall\, x \in E\ \exists\, r = r(x) > 0$ s.t. $\mathbf{B}^o_{r(x)}(x) \subset E$. Then $E = \cup_{x \in E} \mathbf{B}^o_{r(x)}(x)$. ■

Corollary (not in textbook)

For any $x_0 \in S$, $S \backslash \{x_0\}$ is open.

Proof. $S \backslash \{x_0\} = \cup_{x \neq x_0} \mathbf{B}^o_{d(x,x_0)}(x)$. ■

Closed sets in metric space (S, d)

Definition (13.6 – closed sets in S)

$E \subset S$ is called closed if $S \setminus E$ is open.

Fact (obtained from 13.7 using de Morgan's laws)

1. *S is closed.*
2. *$\emptyset$ is closed.*
3. *An intersection of any collection of closed sets is closed.*
4. *A union of finitely many closed sets is closed (does not generalize to infinite unions).*

Fact (some de Morgan's laws, proved in Homework 4)

Suppose $(A_i)_{i \in I}$ are subsets of S. Then
$S \setminus (\cup_i A_i) = \cap_i (S \setminus A_i)$, $S \setminus (\cap_i A_i) = \cup_i (S \setminus A_i)$.

Examples of open and closed sets

Example: finite sets are closed.
Conisder $E = \{x_1, \dots, x_n\} \subset S$.
For $1 \leqslant i \leqslant n$, $S \backslash \{x_i\}$ is open, hence $\{x_i\}$ is closed.
$E = \{x_1\} \cup \dots \cup \{x_n\}$ is closed, as a finite union of closed sets.

Example: intervals in $\mathbb{R}$. Suppose $a < b$.
(a, b) is open, not closed. $[a, b]$ is closed, not open.
$(a, b]$, $[a, b)$ are neither.

Example: discrete metric. For $x, y \in S$, $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$.

Describe open and closed sets.
Hint. Suppose $s \in S$. Is $\{s\}$ closed? open?
Every set is both open and closed!
Moral of the story: sometimes a set can be both open and closed.

Being open or closed depends on the ambient space

Suppose (S, d) is a metric space, and $E \subset S$.
Being open or closed is a property of the position of E inside of S, and not of E itself.
Example: Suppose d is the usual metric on $\mathbb{R}$, which is inherited by $E = \mathbb{Q}$.
If $S = \mathbb{Q}$ itself: E is both open and closed in S.
If $S = \mathbb{R}$: E is neither open nor closed in S.
Indeed, if E were closed, then $\mathbb{R} \backslash \mathbb{Q}$ would have to be open.
That is, for any $x_0 \in \mathbb{R} \backslash \mathbb{Q}$, there would exist $r > 0$ s.t. $(x_0 - r, x_0 + r) = \mathbf{B}_r^o(x_0) \subset \mathbb{R} \backslash \mathbb{Q}$. This is impossible, due to the denseness of rationals.
The possibility of E being open is ruled out similarly.

Closure and boundary of a set

Definition (13.6 – closure)

The closure of $E \subset S$ (denote by by E^-) is the intersection of all closed sets containing E.

Observations. (1) $E \subset E^-$.
(2) E^- is closed, as an intersection of closed sets; this is the smallest closed set containing E.

Definition (13.6 – boundary)

The boundary of $E \subset S$ is $\partial E = E^- \backslash E^o$.

Descriptions of closure and boundary

Proposition (13.9)

a. *$E = E^-$ iff E is closed.*
b. *E is closed iff the limit of any sequence of points in E is in E.*
c. *$s \in E^-$ iff it is a limit of a sequence of points in E.*
d. *$\partial E = E^- \cap (S \setminus E)^-$.*

Proof. (a) is clear.
(c) $\Rightarrow$ (b): E^- is the set of limits of sequences from E. (b) implies that $E^- = E$, hence by (a), E is closed. ■

More about E^-

Lemma (not in textbook)

$x_0 \notin E^-$ *iff* $\exists\, r > 0$ *s.t.* $\mathbf{B}_r^o(x_0) \cap E = \emptyset$.

Proof. $\Leftarrow$: $S \backslash \mathbf{B}_r^o(x_0)$ is closed, contains E. E^- is the smallest closed set containing E, hence $E^- \subset S \backslash \mathbf{B}_r^o(x_0)$.

$\Rightarrow$: If $x_0 \notin E^-$, then x_0 belongs to open set $F = S \backslash E^-$. $F = F^o$, hence $\mathbf{B}_r^o(x_0) \subset F$ for some F. ∎

Proof: $s \in E^-$ iff it is a limit of a sequence of points in E.
If $s \notin E^-$, then $\exists r$ s.t. $\mathbf{B}_r^o(s) \cap E = \emptyset$. No sequence $(s_n) \subset E$ can converge to s, since $d(s_n, s) \geqslant r$.
If $s \in E^-$, then for any $n \in \mathbb{N}$ we can find $s_n \in \mathbf{B}_{1/n}^o(s) \cap E$; then $\lim s_n = s$. ∎

Further examples

Example. Find the closure of $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$.
$E^- = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

Recall: $s \in E^-$ iff $\mathbf{B}_r^o(s) \cap E \neq \emptyset$, $\forall r > 0$. Clearly all points of E has this property, as does 0 (due to the Archimedean Property of reals).
If $s < 0$, then $\mathbf{B}_{|s|}^o(s) \cap E = \emptyset$, so $s \notin E^-$.
If $s > 1$, then $\mathbf{B}_{s-1}^o(s) \cap E = \emptyset$, so $s \notin E^-$.
If $s \in (0,1)\backslash E$, let $n = \lfloor \frac{1}{s} \rfloor$, then $\frac{1}{n+1} < s < \frac{1}{n}$. Thus, $\mathbf{B}_r^o(s) \cap E = \emptyset$, for $r = \min\left\{\frac{1}{n} - s, s - \frac{1}{n+1}\right\}$.
Conclude: $E^- = E \cup \{0\}$.