

# MATH 447: Real Variables - Homework #10

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**Problem 1** (1.8 a). The principle of mathematical induction can be extended as follows. A list  $P_m, P_{m+1}, \dots$  of propositions is true provided (i)  $P_m$  is true, (ii)  $P_{n+1}$  is true whenever  $P_n$  is true and  $n \geq m$ . Prove  $n^2 > n + 1$  for all integers  $n \geq 2$ .

**Solution 1.** *Proof.* We will prove the base case of  $n = 2$  first:

$$(2)^2 > (2) + 1$$

$$4 > 3$$

The above verifies the base case. The inductive hypothesis is as follows:

$$n^2 > n + 1, \quad n \geq 2$$

We want to prove the case such that  $P_{n+1}$  is true whenever  $P_n$  is true and  $n \geq m$ . The inductive step is as follows:

$$(n + 1)^2 > (n + 1) + 1 \tag{1}$$

$$n^2 + 2n + 1 > n + 2 \tag{2}$$

$$n^2 + n > 1 \tag{3}$$

$$n(n + 1) > 1 \tag{4}$$

By the inductive hypothesis,  $n^2 > n + 1, n \geq 2$ ,

$$n(n^2) > n(n + 1) > 1 \tag{5}$$

$$n^3 > 1 \tag{6}$$

The last line is true for all  $n \geq 2$ . □

**Problem 2** (2.8). Find all rational solutions of the equation  $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$ .

**Solution 2.** To find all rational solutions of the equation  $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$ , we can use the Rational Zeros Theorem.

Corollary—Rational Zeros Theorem: If a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

with integer coefficients has a rational solution  $x = \frac{p}{q}$ —where  $p$  and  $q$  are integers with no common factors and  $q \neq 0$ —, then:

- (a)  $p$  must be a factor of the constant term  $a_0$
- (b)  $q$  must be a factor of the leading coefficient  $a_n$

First, let's identify the coefficients:

- (a)  $c_8 = 1$
- (b)  $c_5 = -4$
- (c)  $c_3 = 13$
- (d)  $c_1 = -7$
- (e)  $c_0 = 1$

According to the theorem, if  $\frac{c}{d}$  is a rational solution (where  $c$  and  $d$  are integers with no common factors and  $d \neq 0$ ), then:

- (a)  $c$  must divide  $c_0 = 1$
- (b)  $d$  must divide  $c_8 = 1$

The only integers that divide 1 are 1 and -1. Therefore, the only possible rational solutions are:

- (a)  $\frac{1}{1} = 1$
- (b)  $\frac{-1}{1} = -1$

Now, we need to check if these candidates actually satisfy the equation:

For  $x = 1$ :

$$1^8 - 4(1^5) + 13(1^3) - 7(1) + 1 = 1 - 4 + 13 - 7 + 1 = 4 \neq 0$$

For  $x = -1$ :

$$(-1)^8 - 4((-1)^5) + 13((-1)^3) - 7(-1) + 1 = 1 + 4 - 13 + 7 + 1 = 0$$

Therefore, the only rational solution to the equation  $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$  is  $x = -1$ .

**Problem 3 (3.5).** (a) Show  $|b| \leq a$  if and only if  $-a \leq b \leq a$ .

(b) Prove  $||a| - |b|| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ .

**Solution 3.** *Proof.* (a) If  $|b| \leq a$ , then by definition,  $-a \leq b \leq a$ . This is because  $|b| \leq a$  implies  $b$  is within the interval  $[-a, a]$ .

(b) If  $-a \leq b \leq a$ , then  $b$  is within the interval  $[-a, a]$ . This directly implies  $|b| \leq a$  since the maximum deviation of  $b$  from zero is  $a$ .

Thus,  $|b| \leq a$  if and only if  $-a \leq b \leq a$ . □

*Proof.* We will prove this inequality using the triangle inequality and considering both possible cases. **Using the Triangle Inequality:**

(a) The triangle inequality states  $|a| = |(a - b) + b| \leq |a - b| + |b|$ .

(b) Rearranging gives  $|a| - |b| \leq |a - b|$ .

**Consider the Reverse Situation:**

(a) Similarly,  $|b| = |(b - a) + a| \leq |b - a| + |a| = |a - b| + |a|$ .

(b) Rearranging gives  $|b| - |a| \leq |a - b|$ .

**Combine the Results:**

(a) From the two inequalities, we have:

$$|a| - |b| \leq |a - b| \quad \text{and} \quad |b| - |a| \leq |a - b|$$

**Conclusion:**

(a) The absolute value  $||a| - |b||$  is defined as:

$$||a| - |b|| = \max(|a| - |b|, |b| - |a|)$$

(b) Therefore,  $||a| - |b|| \leq |a - b|$ .

Thus, we have proven that  $||a| - |b|| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ . □

**Problem 4 (3.8).** Let  $a, b \in \mathbb{R}$ . Show if  $a \leq b_1$  for every  $b_1 > b$ , then  $a \leq b$ .

**Solution 4.** *Proof.* We will prove this statement using a proof by contradiction. Assume the hypothesis: For every  $b_1 > b$ , we have  $a \leq b_1$ . Suppose, for the sake of contradiction, that  $a > b$ . Consider  $b_1 = \frac{a+b}{2}$ . Note that:

(a)  $b_1 > b$  (because  $a > b$ )

(b)  $b_1 < a$  (because it's the midpoint between  $a$  and  $b$ )

By our initial assumption, since  $b_1 > b$ , we must have  $a \leq b_1$ . However, we also showed that  $b_1 < a$ . This is a contradiction: we can't have both  $a \leq b_1$  and  $b_1 < a$ .

Therefore, our supposition that  $a > b$  must be false. We conclude that  $a \leq b$ .

Thus, we have shown that if  $a \leq b_1$  for every  $b_1 > b$ , then  $a \leq b$ . □

**Problem 5 (4.1 r).** For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA".

$$\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$$

**Solution 5.** It is observed that the intersection of the sequences above converges to 1 as  $n$  approaches infinity. Therefore, 2, 3, and 4 are all upper bounds for the set.

**Problem 6** (4.8). Let  $S$  and  $T$  be nonempty subsets of  $\mathbb{R}$  with the following property:  $s \leq t$  for all  $s \in S$  and  $t \in T$ .

**Solution 6.** Observe  $S$  is bounded above and  $T$  is bounded below. Prove  $\sup S \leq \inf T$ . Give an example of such sets  $S$  and  $T$  where  $S \cap T$  is nonempty. Give an example of sets  $S$  and  $T$  where  $\sup S = \inf T$  and  $S \cap T$  is the empty set.

- (a) Let  $M = t$ ,  $t \in T$ . Then  $S \leq M$  for all  $s \in S$ . By Def 4.2,  $M$  is an upper bound of  $S$ , and  $S$  is bounded above.
- (b) Let  $m = s$ , such that  $s \in S$ . Then,  $m \leq t$  for all  $t \in T$ . Then,  $m$  is a lower bound of  $T$ , and  $T$  is bounded below.

To prove  $\sup S \leq \inf T$ :

- (a) By the given property, we know that  $s \leq t$  for all  $s \in S$  and all  $t \in T$ .
- (b) Let  $M = \sup S$ . By definition of supremum,  $s \leq M$  for all  $s \in S$ .
- (c) For any  $t \in T$ , we have  $s \leq t$  for all  $s \in S$ .
- (d) Therefore,  $M = \sup S \leq t$  for all  $t \in T$ .
- (e) Since  $M \leq t$  for all  $t \in T$ ,  $M$  is a lower bound for  $T$ .
- (f) By definition of infimum,  $\inf T$  is the greatest lower bound of  $T$ .
- (g) Thus,  $M \leq \inf T$ .

Therefore, we have proven that  $\sup S \leq \inf T$ .

An example of such sets  $S$  and  $T$  where  $S \cap T$  is nonempty:  $S = [0, 1], T = [1, 2]$

An example of sets  $S$  and  $T$  where  $\sup S = \inf T$  and  $S \cap T$  is the empty set:  $S = [0, 1), T = (1, 2]$

## Problem 7 (8.5)

**Problem 7** (8.5). (a) Consider three sequences  $(a_n)$ ,  $(b_n)$  and  $(s_n)$  such that  $a_n \leq s_n \leq b_n$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = s$ . Prove  $\lim_{n \rightarrow \infty} s_n = s$ . This is called the "squeeze lemma".

**Solution 7.** (a) Given: For all  $\varepsilon > 0$ , there exist  $N_1, N_2 \in \mathbb{N}$  such that for  $n > N_1$  and  $n > N_2$ :

- (a)  $|a_n - s| < \varepsilon$ , which implies  $s - \varepsilon < a_n < s + \varepsilon$
- (b)  $|b_n - s| < \varepsilon$ , which implies  $s - \varepsilon < b_n < s + \varepsilon$

- (b) We also know that for all  $n \in \mathbb{N}$ ,  $a_n \leq s_n \leq b_n$

(c) Combining these facts, we can conclude that for  $n > \max(N_1, N_2)$ :

$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$$

(d) This implies:

$$s - \varepsilon < s_n < s + \varepsilon$$

(e) Therefore:

$$|s_n - s| < \varepsilon$$

(f) By the definition of a limit of a sequence, this proves that  $\lim_{n \rightarrow \infty} s_n = s$

Thus, we have proven the squeeze lemma.

(g) Suppose  $(s_n)$  and  $(t_n)$  are sequences such that  $|s_n| \leq t_n$  for all  $n$  and  $\lim_{n \rightarrow \infty} t_n = 0$ .  
Prove  $\lim_{n \rightarrow \infty} s_n = 0$ .

(a) Given:  $\lim_{n \rightarrow \infty} t_n = 0$ , so for any  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n > N$ :

(b)  $|t_n - 0| < \varepsilon$

(c) This implies:  $|t_n| < \varepsilon$

(d) We know that  $|s_n| \leq t_n$  for all  $n$ , so:  $|s_n| \leq |t_n| < \varepsilon$

(e) This means:  $-\varepsilon < s_n < \varepsilon$

(f) Therefore:  $|s_n - 0| < \varepsilon$

By the definition of a limit, this proves that  $\lim_{n \rightarrow \infty} s_n = 0$ .

**Problem 8 (8.6).** Let  $(s_n)$  be a sequence in  $\mathbb{R}$ .

(a) Prove  $\lim_{n \rightarrow \infty} s_n = 0$  if and only if  $\lim_{n \rightarrow \infty} |s_n| = 0$

**Solution 8.** *Proof.* We will prove this statement in two parts:

(a) If  $\lim_{n \rightarrow \infty} s_n = 0$ , then  $\lim_{n \rightarrow \infty} |s_n| = 0$ :

(a) Given  $\lim_{n \rightarrow \infty} s_n = 0$ , for every  $\epsilon > 0$ , there exists an  $N$  such that for all  $n \geq N$ ,  
 $|s_n - 0| < \epsilon$ .

(b) This simplifies to  $|s_n| < \epsilon$ .

(c) Therefore,  $\lim_{n \rightarrow \infty} |s_n| = 0$ .

(b) If  $\lim_{n \rightarrow \infty} |s_n| = 0$ , then  $\lim_{n \rightarrow \infty} s_n = 0$ :

(a) Given  $\lim_{n \rightarrow \infty} |s_n| = 0$ , for every  $\epsilon > 0$ , there exists an  $N$  such that for all  $n \geq N$ ,  
 $|s_n| < \epsilon$ .

(b) This directly implies that  $|s_n - 0| < \epsilon$ , so  $\lim_{n \rightarrow \infty} s_n = 0$ .

Thus, we have proven that  $\lim_{n \rightarrow \infty} s_n = 0$  if and only if  $\lim_{n \rightarrow \infty} |s_n| = 0$ .

(c) Observe that if  $s_n = (-1)^n$ , then  $\lim_{n \rightarrow \infty} |s_n|$  exists, but  $\lim_{n \rightarrow \infty} s_n$  does not exist.

**Observation:** For the sequence  $s_n = (-1)^n$ , we can see that  $s_n$  alternates between -1 and 1.

To prove that  $\lim_{n \rightarrow \infty} s_n$  does not exist, we can establish two subsequences:

(a)  $s_{n_1}$ : the subsequence of even terms, where  $s_{n_1} = 1$  for all  $n$

(b)  $s_{n_2}$ : the subsequence of odd terms, where  $s_{n_2} = -1$  for all  $n$

Clearly,  $\lim_{n \rightarrow \infty} s_{n_1} = 1$  and  $\lim_{n \rightarrow \infty} s_{n_2} = -1$

Since these two subsequences converge to different values, we can conclude that  $\lim_{n \rightarrow \infty} s_n$  does not exist, demonstrating that the sequence is divergent.

However,  $\lim_{n \rightarrow \infty} |s_n| = 1$  does exist, as  $|s_n| = 1$  for all  $n$ .

□

**Problem 9 (Bonus).** Use the completeness of  $\mathbb{R}$  to show the existence of  $x > 0$  with  $x^2 = 2$ . Specifically, consider  $S = \{t \in \mathbb{R} : t > 0, t^2 < 2\}$ . Clearly,  $S$  is nonempty ( $1 \in S$ ). Further, 2 is an upper bound for  $S$ . Indeed, suppose  $t \in S$ , then  $(2 - t)(2 + t) = 4 - t^2 > 4 - 2 > 0$ . Clearly,  $2 + t > 0$ , hence  $2 - t > 0$ . Let  $x = \sup S$ . Prove that  $x^2 = 2$ , by establishing that (i)  $x^2 \leq 2$ , and (ii)  $x^2 \geq 2$ . Once these inequalities are established, we can conclude that  $x^2 = 2$ .

**Solution 9. Proof.** (a) Let  $\alpha = \sup S$ . We will prove that  $\alpha^2 = 2$  by showing that  $\alpha^2 \leq 2$  and  $\alpha^2 \geq 2$ .

(b) First, let's prove  $\alpha^2 \leq 2$ :

(a) Assume  $\alpha^2 > 2$ .

(b) For  $n \in \mathbb{N}$ , consider  $(\alpha - \frac{1}{n})^2$ :

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$$

(c) Since  $\alpha^2 > 2$ , we have:

$$2 < \alpha^2 - \frac{2\alpha}{n} \tag{7}$$

$$2 - \alpha^2 < -\frac{2\alpha}{n} \tag{8}$$

$$\alpha^2 - 2 > \frac{2\alpha}{n} \tag{9}$$

$$\frac{\alpha^2 - 2}{2\alpha} > \frac{1}{n} \tag{10}$$

(d) Choose  $n_0 \in \mathbb{N}$  such that  $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$ .

(e) Then:

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - (\alpha^2 - 2) = 2$$

(f) This contradicts the fact that  $\alpha$  is an upper bound for  $S$ .

(g) Therefore, our assumption must be false, and  $\alpha^2 \leq 2$ .

(c) Now, let's prove  $\alpha^2 \geq 2$ :

(a) Assume  $\alpha^2 < 2$ .

(b) For  $n \in \mathbb{N}$ , consider  $(\alpha + \frac{1}{n})^2$ :

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha + 1}{n}.$$

(c) Choose  $n_0 \in \mathbb{N}$  such that  $\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$ .

(d) Then:

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2.$$

(e) This means  $\alpha + \frac{1}{n_0} \in S$ , contradicting  $\alpha$  as an upper bound for  $S$ .

(f) Therefore, our assumption must be false, and  $\alpha^2 \geq 2$ .

(d) Since we have shown  $\alpha^2 \leq 2$  and  $\alpha^2 \geq 2$ , we can conclude that  $\alpha^2 = 2$ .

Thus, we have proven the existence of a real number  $\alpha > 0$  such that  $\alpha^2 = 2$ .

□