SOLUTIONS FOR HOMEWORK 2

9.12. **(a)** Pick $a \in (L, 1)$. By the definition of the limit, there exists $N \in \mathbb{N}$ s.t. $\left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < a - L$. In particular, for such n, $\left|\frac{s_{n+1}}{s_n}\right| < L + (a - L) = a$.

Use induction to show that, for $n \geq N$, $|s_n| \leq a^{n-N}|s_N|$. Indeed, the basis (the case of $n = N$) is clear. For the step, suppose $n \geq N$, and $|s_n| \leq a^{n-N}|s_N|$. Then $|s_{n+1}| = \frac{|s_{n+1}|}{|s_n|} \cdot |s_n| < a^{n-N}|s_N| \cdot a \leq a^{(n+1)-N}|s_N|$.

To conclude the proof, recall that $\lim a^n = 0$ (Theorem 9.7(b)), and invoke Squeeze Theorem.

(b) Per Hint, let $t_n = \frac{1}{s_n}$. Then $\lim\left|\frac{t_{n+1}}{t_n}\right| = \lim\left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{L} < 1$, hence by Part (a), $\lim|t_n| = 0$. By Theorem 9.10, $\lim|s_n| = \lim\frac{1}{|t_n|} = +\infty$.

9.14. Let $s_n = a^n/n^p$. By Theorem 9.7(a), $\lim\frac{1}{n^p} = 0$. If $|a| \leq 1$, then $|s_n| \leq \frac{1}{n^p}$, hence $\lim s_n = 0$ (Exercise 9.5(b)).

If $|a| > 1$, then

$$\lim\left|\frac{s_{n+1}}{s_n}\right| = \lim\left|\frac{a^{n+1}/a^n}{(n+1)^p/n^p}\right| = |a|\Big(\lim\frac{n+1}{n}\Big)^{-p} = |a|\Big(1 + \lim\frac{1}{n}\Big)^{-p} = |a|,$$

hence, by Exercise 9.12(b), $\lim|s_n| = +\infty$.

If $a > 1$, then $s_n > 0$ for any n, hence $\lim s_n = +\infty$.

If $a < -1$, then $s_n > 0$ if n is even, and $s_n < 0$ if n is odd. Suppose, for the sake of contradiction, that $s = \lim s_n$ exists. Then $s = \lim_k s_{2k} \geq 0$, and also, $s = \lim_k s_{2k-1} \leq 0$.

Thus, if $\lim s_n$ exists, it has to equal 0. This, in turn, would imply $\lim|s_n| = 0$, which is not true: $|a| > 1$, hence $\lim|a_n| = \lim\frac{|a|^n}{n^p} = +\infty$. Thus, $\lim s_n$ does not exist.

10.6 **(a)** Note that, if $m > n$, then $\big|s_m - s_n\big| = \big|\sum_{k=n}^{m-1}(s_{k+1} - s_k)\big| \leq \sum_{k=n}^{m-1}|s_{k+1} - s_k| < \sum_{k=n}^{m-1}\frac{1}{2^k} = \frac{1}{2^n}\Big(1 + \ldots\frac{1}{2^{m+n-1}}\Big) = \frac{1}{2^n} \cdot \frac{1 - 1/2^{m-n}}{1 - 1/2} < \frac{1}{2^{n-1}}$.

For $\varepsilon > 0$, find $N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$ (this is possible, since $\lim_N \frac{1}{2^N} = 0$). Then, if $m, n > N$, we have $|s_n - s_m| < \varepsilon$ for $n, m > N$. Indeed, without loss of generality we can assume that $m > n$. By the reasoning above, $\big|s_m - s_n\big| < \frac{1}{2^N} < \varepsilon$.

(b) No, the conclusion of (a) need not hold. Indeed, consider $s_n = \sum_{k=1}^{n} \frac{1}{k}$. We have $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$. In Lecture 6 we showed that (s_n) is unbounded, hence divergent.

10.8 We need to show that $\sigma_{n+1} - \sigma_n \geq 0$ for any n. We have $\sigma_{n+1} - \sigma_n = \frac{1}{n+1}\big(s_1 + \ldots + s_n + s_{n+1}\big) - \frac{1}{n}\big(s_1 + \ldots + s_n\big) = \frac{1}{n+1} s_{n+1} - \Big(\frac{1}{n} - \frac{1}{n+1}\Big)\big(s_1 + \ldots + s_n\big)$.

Writing $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, we obtain $\sigma_{n+1} - \sigma_n = \frac{1}{n+1}\Big(s_{n+1} - \frac{1}{n}\big(s_1 + \ldots + s_n\big)\Big) = \frac{1}{n(n+1)}\Big(n s_{n+1} - \sum_{k=1}^{n} s_k\Big) = \frac{1}{n(n+1)} \sum_{k=1}^{n} (s_{n+1} - s_k)$.

As (s_n) is increasing, $s_{n+1} - s_k \geq 0$ for any k, hence the summands on the right are non-negative. Thus, $\sigma_{n+1} - \sigma_n \geq 0$.

10.10 (a) $s_1 = 1$, $s_2 = \frac{2}{3}$, $s_3 = \frac{5}{9}$, $s_4 = \frac{14}{27}$.

(b) Basis for induction. $s_1 = 1 > \frac{1}{2}$.

Inductive step. We need to show that, if $s_n > \frac{1}{2}$, then $s_{n+1} > \frac{1}{2}$.

$s_{n+1} = \frac{1}{3}\big(1 + s_n\big) > \frac{1}{3}\big(1 + \frac{1}{2}\big) = \frac{1}{2}$.

(c) For $n \geq 1$, $s_n - s_{n+1} = s_n - \frac{1}{3}\big(1 + s_n\big) = \frac{2}{3}s_n - \frac{1}{3} = \frac{2}{3}\big(s_n - \frac{1}{2}\big)$; the right hand side is positive, by (b).

(d) (s_n) is decreasing and bounded below, hence convergent. Let $s = \lim s_n$. Passing to the limit in the equation $s_{n+1} = \frac{1}{3}\big(1+s_n\big)$, obtain $s = \frac{1}{3}\big(1+s\big)$. Solve this equation to obtain $\boldsymbol{s = \frac{1}{2}}$.

10.12. (a) (t_n) is a decreasing sequence of positive numbers. As (t_n) is bounded below (by 0), it must be bounded, hence convergent.

(b) We shall see that $\lim t_n = \frac{1}{2}$

(c) The basis for induction is straightforward. For the inductive step, we shall show that, if $t_n = \frac{n+1}{2n}$, then $t_{n+1} = \frac{n+2}{2(n+1)}$.

Note that $1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - 1}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}$, hence, in the light of the induction hypothesis, $t_{n+1} = \frac{n(n+2)}{(n+1)^2} t_n = \frac{n(n+2)}{(n+1)^2} \cdot \frac{n+1}{2n} = \frac{n+2}{2(n+1)}$, as we claimed.

(d) $\lim \frac{n+1}{2n} = \lim \frac{1 + 1/n}{2} = \frac{\lim(1 + 1/n)}{2} = \frac{(1 + \lim 1/n)}{2} = \frac{1}{2}$, hence $\boldsymbol{\lim t_n = \frac{1}{2}}$.