Problem 1

$$u''(x) + f(x) = 0 \quad \text{on} \quad x \in [0, \pi]$$

(a) $u''(x) = -f(x) = -\sin x$

$$u'(x) = -\int \sin x \, dx + C_1$$

$$= \cos x + C_1$$

$$u(x) = \int (\cos x + C_1) \, dx + C_2$$

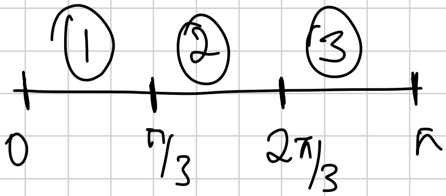
$$= \sin x + C_1 x + C_2$$

$u(0) = 1 \implies C_2 = 1$

$u'(\pi) = \frac{1}{2} \implies -1 + C_1 = \frac{1}{2} \implies C_1 = \frac{3}{2}$.

$$\therefore \; u(x) = \sin x + \frac{3x}{2} + 1$$

(b)



$\Delta x_e = \pi/3$

For 1-D linear shape functions.

$$N_1^e = \frac{-1}{\Delta x_e}(x - x_2^e) \quad , \quad N_2^e = \frac{1}{\Delta e}(x - x_1^e)$$

$$\frac{\partial N_1^e}{\partial x} = -\frac{1}{\Delta x_e} \quad , \quad \frac{\partial N_2^e}{\partial x} = \frac{1}{\Delta x_e}$$

$$\Rightarrow \quad \frac{\partial N_i^e}{\partial x} = \frac{(-1)^i}{\Delta x_e}$$

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left(\frac{\partial N_i^e}{\partial x}\right)\left(\frac{\partial N_j^e}{\partial x}\right) dx$$

$$= \int_{x_1^e}^{x_2^e} \frac{(-1)^{i+j}}{(\Delta x_e)^2}\, dx = \frac{(-1)^{i+j}}{\Delta x_e}$$

Since all elements are equal length of $\pi/3$,

$$K^{(1)} = K^{(2)} = K^{(3)} = \frac{3}{\pi}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For force vectors,

$$F^{(1)} = \int_{x_1^e}^{x_2^e} \frac{1}{\Delta x_e} \begin{bmatrix} -(x-\pi/3) \\ x \end{bmatrix} \sin x \, dx$$

$$= \begin{bmatrix} 0.173 \\ 0.327 \end{bmatrix}$$

$$F^{(2)} = \frac{1}{\Delta x_e} \int_{x_1^e}^{x_2^e} \begin{bmatrix} -(x-2\pi/3) \\ (x-\pi/3) \end{bmatrix} \sin x \, dx$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$F^{(3)} = \frac{1}{\Delta x_e} \int_{x_1^e}^{x_2^e} \begin{bmatrix} \pi - x \\ x - 2\pi/3 \end{bmatrix} \sin x \, dx = \begin{bmatrix} 0.327 \\ 0.173 \end{bmatrix}$$

Assembly :-

$$\mathbb{K} = \frac{3}{\pi} \left(\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \right)$$

$$= \frac{3}{\pi}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbb{F} = \underbrace{\begin{bmatrix} 0.173 \\ 0.327 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.327 \\ 0.173 \end{bmatrix}}_{\text{Elemental contribution}}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ u'(\pi) = 0.5 \end{bmatrix}}_{\text{Given Neuman BC}} + \underbrace{\begin{bmatrix} -u'(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{Unknown Neuman BC.}}$$

$$\Rightarrow \mathbb{F} = \begin{bmatrix} 0.173 \\ 0.827 \\ 0.827 \\ 0.673 \end{bmatrix} + \begin{bmatrix} -u'(0) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

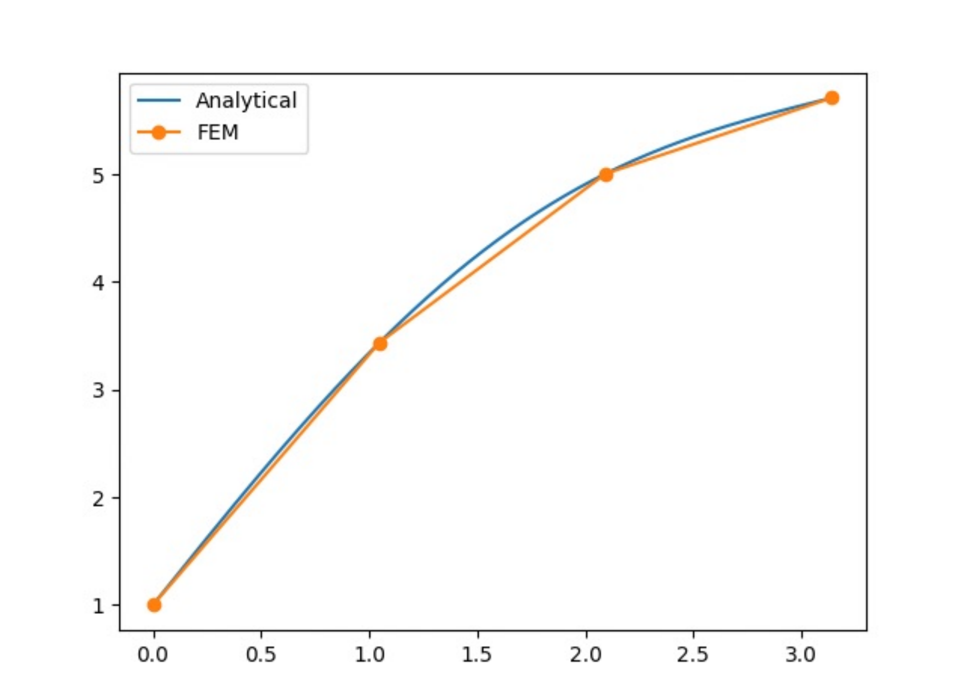
$$\mathbb{K} u = \mathbb{F}$$

$$\hat{\mathbb{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{\pi} & \frac{6}{\pi} & -\frac{3}{\pi} & 0 \\ 0 & -\frac{3}{\pi} & 6/\pi & 0 \\ 0 & 0 & 0 & 3/\pi \end{bmatrix}$$

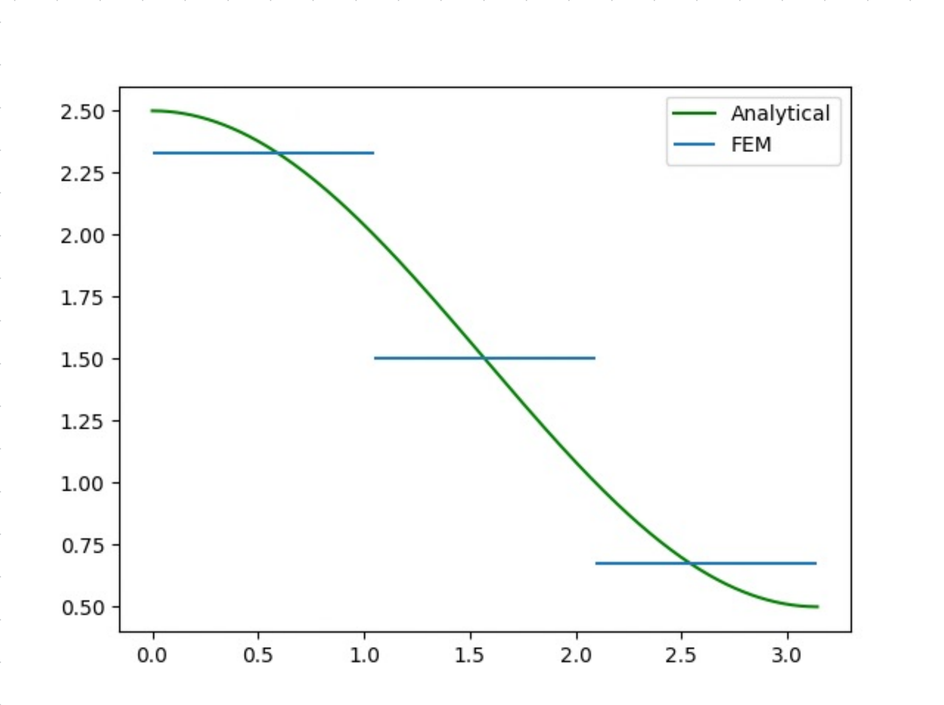
$$\hat{\mathbb{F}} = \begin{bmatrix} 1 \\ 0.827 \\ 0.827 \\ 0.673 \end{bmatrix}$$ and we solve $\hat{\mathbb{K}} u = \hat{\mathbb{F}}$

$$\rightarrow u = \begin{bmatrix} 1 \\ 3.437 \\ 5.008 \\ 5.712 \end{bmatrix}$$

(c)



(d)

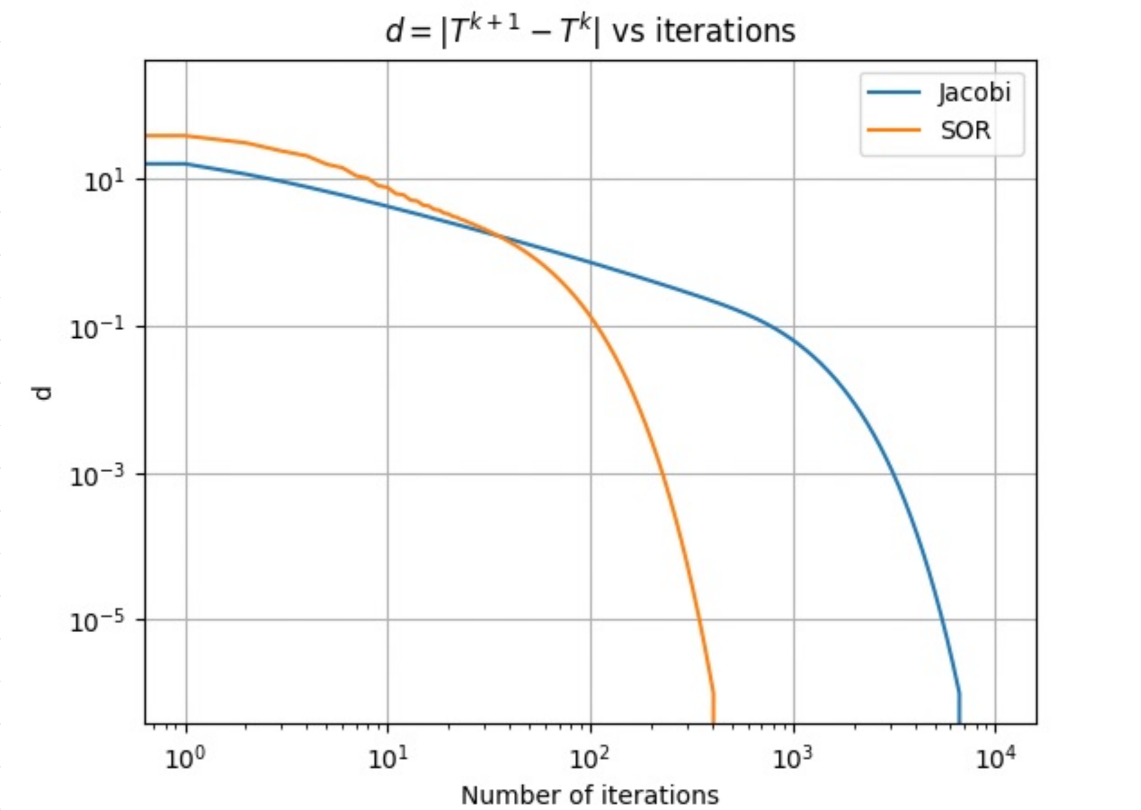


Analytical
FEM
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Problem 2

Code on PL

(b)



Problem 3

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + f(r) = 0 \quad , \quad r \in (a,b)$$

(a) For all w in the appropriate space,

$$\int_a^b \left[\frac{w}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + w f(r)\right] dr = 0$$

$$\Rightarrow \int_a^b \left[\frac{w}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right) + w f(r)\right] dr = 0$$

$$\Rightarrow \int_a^b w\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) dr + \int_a^b w f(r)\, dr = 0$$

$$\Rightarrow w\frac{\partial u}{\partial r}\Big|_a^b - \int_a^b \frac{\partial w}{\partial r}\frac{\partial u}{\partial r}\, dr + \int_a^b \frac{w}{r}\frac{\partial u}{\partial r} + \int_a^b w f(r) dr = 0$$

$$\Rightarrow \int_a^b \frac{\partial w}{\partial r}\frac{\partial u}{\partial r}\, dr - \int_a^b \frac{w}{r}\frac{\partial u}{\partial r}\, dr = w\frac{\partial u}{\partial r}\Big|_a^b + \int_a^b w f(r) dr$$

(b) Using above weak form, use Galerkin approximations:-

$$w = \sum_{i=1}^{N} w_i \phi_i \quad \text{and} \quad u = \sum_{j=1}^{N} u_j \phi_j$$

Taking term by term from weak form,

$$\int_a^b \frac{\partial w}{\partial r} \frac{\partial u}{\partial r} dr = \int_a^b \sum_i \sum_j w_i u_j \phi_i' \phi_j' \, dr = \sum_i w_i \int_a^b \sum_j u_j \phi_i' \phi_j' \, dr$$

$$\int_a^b \frac{w}{r} \frac{\partial u}{\partial r} = \int_a^b \frac{1}{r} \sum_i \sum_j w_i u_j \phi_i \phi_j' \, dr = \sum_i w_i \int_a^b \frac{1}{r} \sum_j u_j \phi_i \phi_j' \, dr$$

$$\int_a^b w f(r) dr = \sum_i w_i \int_a^b \phi_i f(r) dr$$

$$w \frac{\partial u}{\partial r} \Big|_a^b = \sum_i w_i \left[\delta_{r=b} u'(b) - \delta_{r=a} u'(a) \right]$$

Combining,

$$\sum_i w_i \left[\sum_j u_j \int_a^b \left(\phi_i' \phi_j' - \frac{\phi_i \phi_j'}{r} \right) dr \right] = \sum_i w_i \left[\delta_{r=b} u'(b) - \delta_{r=a} u'(a) + \int_a^b \phi_i f(r) dr \right]$$

$$\Rightarrow \sum_i \sum_j K_{ij} u_j = \delta_{r=b} u'(b) - \delta_{r=a} u'(a) + \int_a^b \sum_i \phi_i f \, dr$$

$$\text{where} \left\{ \begin{aligned} K_{ij} &= \int_a^b \left(\phi_i' \phi_j' - \frac{\phi_i \phi_j'}{r} \right) dr \\ F_i &= \delta_{r=b} u'(b) - \delta_{r=a} u'(a) + \int_a^b \phi_i f \, dr \end{aligned} \right\}$$

(c) For $\mathbb{K}$, $N_1^e = \frac{-1}{\Delta r^e}(r - r_2^e)$, $N_2^e = \frac{1}{\Delta r^e}(r - r_1^e)$.

$$\mathbb{K}_{ij} = \int_{r_1^e}^{r_2^e} \left(N_i^{e\prime} N_j^{e\prime} - \frac{1}{r} N_i^e N_j^e \right) dr$$

In Matrix form,

$$[\mathbb{K}] = \frac{1}{\Delta r_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \underbrace{\int_{r_1^e}^{r_2^e} \frac{1}{r} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \begin{bmatrix} N_1^{e\prime} & N_2^{e\prime} \end{bmatrix} dr}$$

Consider 2nd term.

$$= \frac{1}{\Delta r_e^2} \int_{r_1^e}^{r_2^e} \begin{bmatrix} \left(1 - \frac{r_2^e}{r}\right) & -\left(1 - \frac{r_2^e}{r}\right) \\ -\left(1 - \frac{r_1^e}{r}\right) & \left(1 - \frac{r_1^e}{r}\right) \end{bmatrix} dr$$

$$= \frac{1}{\Delta_e^2} \begin{bmatrix} \Delta r_e & -\Delta r_e \\ -\Delta r_e & \Delta r_e \end{bmatrix} + \frac{1}{\Delta r_e^2} \cdot \ln\left(\frac{r_2^e}{r_1^e}\right) \begin{bmatrix} -r_2^e & r_2^e \\ r_1^e & -r_1^e \end{bmatrix}$$

$$\therefore [\mathbb{K}] = \frac{1}{\Delta r_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \left(\frac{1}{\Delta r_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\ln\left(\frac{r_2^e}{r_1^e}\right)}{\Delta r_e^2}\begin{bmatrix} -r_2^e & r_2^e \\ r_1^e & -r_1^e \end{bmatrix} \right)$$

$$= \frac{\ln\left(r_2^e / r_1^e\right)}{\Delta r_e^2}\begin{bmatrix} r_2^e & -r_2^e \\ r_1^e & -r_1^e \end{bmatrix}$$