HW-1 - MATH 447: Real Variables

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Problem 1 (1.8 a)

The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.

Prove $n^2 > n + 1$ for all integers $n \geq 2$.

We will prove the base case of $n = 2$ first:

$$(2)^2 > (2) + 1 \tag{1}$$

$$4 > 3 \tag{2}$$

The above verifies the base case. The inductive hypothesis is as follows:

$$n^2 > n + 1, \ n \geq 2 \tag{3}$$

We want to prove the case such that P_{n+1} is true whenever P_n is true and $n \geq m$. The inductive step is as follows:

$$(n+1)^2 > (n+1) + 1 \tag{4}$$

$$n^2 + 2n + 1 > n + 2 \tag{5}$$

$$n^2 + n > 1 \tag{6}$$

$$n(n+1) > 1 \tag{7}$$

By the inductive hypothesis, $n^2 > n + 1, n \geq 2$,

$$n(n^2) > n(n+1) > 1 \tag{8}$$

$$n^3 > 1 \tag{9}$$

The last line is true for all $n \geq 2$. $\square$

Problem 2 (2.8)

Find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$.

To find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$, we can use the Rational Zeros Theorem.

Corollary (Rational Zeros Theorem): If a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0 \tag{10}$$

with integer coefficients has a rational solution $x = \frac{p}{q}$ (where p and q are integers with no common factors and $q \neq 0$), then:

1. p must be a factor of the constant term a_0
2. q must be a factor of the leading coefficient a_n

First, let's identify the coefficients:

1. $c_8 = 1$
2. $c_5 = -4$
3. $c_3 = 13$
4. $c_1 = -7$
5. $c_0 = 1$

According to the theorem, if $\frac{c}{d}$ is a rational solution (where c and d are integers with no common factors and $d \neq 0$), then:

1. c must divide $c_0 = 1$
2. d must divide $c_8 = 1$

The only integers that divide 1 are 1 and -1. Therefore, the only possible rational solutions are:

1. $\frac{1}{1} = 1$
2. $\frac{-1}{1} = -1$

Now, we need to check if these candidates actually satisfy the equation:
For $x = 1$:

$$1^8 - 4(1^5) + 13(1^3) - 7(1) + 1 = 1 - 4 + 13 - 7 + 1 = 4 \neq 0 \tag{11}$$

For $x = -1$:

$$(-1)^8 - 4((-1)^5) + 13((-1)^3) - 7(-1) + 1 = 1 + 4 - 13 + 7 + 1 = 0 \tag{12}$$

Therefore, the only rational solution to the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$ is $x = -1$.

Problem 3 (3.5)

1. Show $|b| \leq a$ if and only if $-a \leq b \leq a$.
2. Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

1. Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

Proof:

Forward Direction ($|b| \leq a \Rightarrow -a \leq b \leq a$):

1. If $|b| \leq a$, then by definition, $-a \leq b \leq a$.
2. This is because $|b| \leq a$ implies b is within the interval $[-a, a]$.

Reverse Direction ($-a \leq b \leq a \Rightarrow |b| \leq a$):

1. If $-a \leq b \leq a$, then b is within the interval $[-a, a]$.
2. This directly implies $|b| \leq a$ since the maximum deviation of b from zero is a.

Thus, $|b| \leq a$ if and only if $-a \leq b \leq a$. $\square$

2. Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Proof:

We will prove this inequality using the triangle inequality and considering both possible cases.

1. **Using the Triangle Inequality:**
   (a) The triangle inequality states $|a| = |(a-b)+b| \leq |a-b| + |b|$.
   (b) Rearranging gives $|a| - |b| \leq |a-b|$.
2. **Consider the Reverse Situation:**
   (a) Similarly, $|b| = |(b-a)+a| \leq |b-a| + |a| = |a-b| + |a|$.
   (b) Rearranging gives $|b| - |a| \leq |a-b|$.
3. **Combine the Results:**
   (a) From the two inequalities, we have:
   $$|a| - |b| \leq |a-b| \quad \text{and} \quad |b| - |a| \leq |a-b| \tag{13}$$
4. **Conclusion:**
   (a) The absolute value $||a| - |b||$ is defined as:
   $$||a| - |b|| = \max(|a| - |b|, |b| - |a|) \tag{14}$$
   (b) Therefore, $||a| - |b|| \leq |a-b|$.

Thus, we have proven that $||a| - |b|| \leq |a-b|$ for all $a, b \in \mathbb{R}$. $\square$

Problem 4 (3.8)

Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Proof: We will prove this statement using a proof by contradiction.

1. Assume the hypothesis: For every $b_1 > b$, we have $a \leq b_1$.
2. Suppose, for the sake of contradiction, that $a > b$.
3. Consider $b_1 = \frac{a+b}{2}$. Note that:
   (a) $b_1 > b$ (because $a > b$)
   (b) $b_1 < a$ (because it's the midpoint between a and b)
4. By our initial assumption, since $b_1 > b$, we must have $a \leq b_1$.
5. However, we also showed that $b_1 < a$.
6. This is a contradiction: we can't have both $a \leq b_1$ and $b_1 < a$.

Therefore, our supposition that $a > b$ must be false. We conclude that $a \leq b$.
Thus, we have shown that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$. $\square$

Problem 5 (4.1 r)

For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA".

$\cap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

It is observed that the intersection of the sequences above converges to 1 as n approaches infinity. Therefore, 2, 3, and 4 are all upper bounds for the set.

Problem 6 (4.8)

Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.

1. Observe S is bounded above and T is bounded below.
2. Prove $\sup S \leq \inf T$
3. Give an example of such sets S and T where $S \cap T$ is nonempty.
4. Give an example of sets S and T where $\sup S = \inf T$ and $S \cap T$ is the empty set.

1. (a) Let $M = t,\ t \in T$. Then $S \leq M$ for all $s \in S$. By Def 4.2, M is an upper bound of S, and S is bounded above.
   (b) Let $m = s$, such that $s \in S$. Then, $m \leq t$ for all $t \in T$. Then, m is a lower bound of T, and T is bounded below.
2. To prove $\sup S \leq \inf T$:
   (a) By the given property, we know that $s \leq t$ for all $s \in S$ and all $t \in T$.
   (b) Let $M = \sup S$. By definition of supremum, $s \leq M$ for all $s \in S$.
   (c) For any $t \in T$, we have $s \leq t$ for all $s \in S$.
   (d) Therefore, $M = \sup S \leq t$ for all $t \in T$.
   (e) Since $M \leq t$ for all $t \in T$, M is a lower bound for T.
   (f) By definition of infimum, $\inf T$ is the greatest lower bound of T.
   (g) Thus, $M \leq \inf T$.

   Therefore, we have proven that $\sup S \leq \inf T$. $\square$
3. An example of such sets S and T where $S \cap T$ is nonempty: $S = [0, 1], T = [1, 2]$
4. An example of sets S and T where $\sup S = \inf T$ and $S \cap T$ is the empty set: $S = [0, 1), T = (1, 2]$

Problem 7 (8.5)

1. Consider three sequences $(a_n), (b_n)$ and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. Prove $\lim_{n\to\infty} s_n = s$. This is called the "squeeze lemma".

Proof:

1. Given: For all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that for $n > N_1$ and $n > N_2$:
   (a) $|a_n - s| < \varepsilon$, which implies $s - \varepsilon < a_n < s + \varepsilon$
   (b) $|b_n - s| < \varepsilon$, which implies $s - \varepsilon < b_n < s + \varepsilon$
2. We also know that for all $n \in \mathbb{N}$, $a_n \leq s_n \leq b_n$
3. Combining these facts, we can conclude that for $n > \max(N_1, N_2)$:
$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon \tag{15}$$
4. This implies:
$$s - \varepsilon < s_n < s + \varepsilon \tag{16}$$
5. Therefore:
$$|s_n - s| < \varepsilon \tag{17}$$

6. By the definition of a limit of a sequence, this proves that $\lim_{n\to\infty} s_n = s$

Thus, we have proven the squeeze lemma. $\square$

7. Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim_{n\to\infty} t_n = 0$. Prove $\lim_{n\to\infty} s_n = 0$.

1. Given: $\lim_{n\to\infty} t_n = 0$, so for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$:
2. $|t_n - 0| < \varepsilon$
3. This implies: $|t_n| < \varepsilon$
4. We know that $|s_n| \leq t_n$ for all n, so: $|s_n| \leq |t_n| < \varepsilon$
5. This means: $-\varepsilon < s_n < \varepsilon$
6. Therefore: $|s_n - 0| < \varepsilon$

By the definition of a limit, this proves that $\lim_{n\to\infty} s_n = 0$. $\square$

Problem 8 (8.6)

Let (s_n) be a sequence in $\mathbb{R}$.

1. Prove $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} |s_n| = 0$

Proof: We will prove this statement in two parts:

1. If $\lim_{n\to\infty} s_n = 0$, then $\lim_{n\to\infty} |s_n| = 0$:
   (a) Given $\lim_{n\to\infty} s_n = 0$, for every $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|s_n - 0| < \epsilon$.
   (b) This simplifies to $|s_n| < \epsilon$.
   (c) Therefore, $\lim_{n\to\infty} |s_n| = 0$.
2. If $\lim_{n\to\infty} |s_n| = 0$, then $\lim_{n\to\infty} s_n = 0$:
   (a) Given $\lim_{n\to\infty} |s_n| = 0$, for every $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|s_n| < \epsilon$.
   (b) This directly implies that $|s_n - 0| < \epsilon$, so $\lim_{n\to\infty} s_n = 0$.

Thus, we have proven that $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} |s_n| = 0$. $\square$

3. Observe that if $s_n = (-1)^n$, then $\lim_{n\to\infty} |s_n|$ exists, but $\lim_{n\to\infty} s_n$ does not exist.

Observation: For the sequence $s_n = (-1)^n$, we can see that s_n alternates between -1 and 1.
To prove that $\lim_{n\to\infty} s_n$ does not exist, we can establish two subsequences:

1. s_{n_1}: the subsequence of even terms, where $s_{n_1} = 1$ for all n
2. s_{n_2}: the subsequence of odd terms, where $s_{n_2} = -1$ for all n

Clearly, $\lim_{n\to\infty} s_{n_1} = 1$ and $\lim_{n\to\infty} s_{n_2} = -1$
Since these two subsequences converge to different values, we can conclude that $\lim_{n\to\infty} s_n$ does not exist, demonstrating that the sequence is divergent.
However, $\lim_{n\to\infty} |s_n| = 1$ does exist, as $|s_n| = 1$ for all n.

Problem 9 (Bonus)

Use the completeness of $\mathbb{R}$ to show the existence of $x > 0$ with $x^2 = 2$. Specifically, consider $S = \{t \in \mathbb{R} : t > 0, t^2 < 2\}$. Clearly, S is nonempty ($1 \in S$). Further, 2 is an upper bound for S. Indeed, suppose $t \in S$, then $(2-t)(2+t) = 4 - x^2 > 4 - 2 > 0$. Clearly, $2 + t > 0$, hence $2 - t > 0$. Let $x = \sup S$. Prove that $x^2 = 2$, by establishing that (i) $x^2 \leq 2$, and (ii) $x^2 \geq 2$. Once these inequalities are established, we can conclude that $x^2 = 2$.

Proof:

1. Let $\alpha = \sup S$. We will prove that $\alpha^2 = 2$ by showing that $\alpha^2 \leq 2$ and $\alpha^2 \geq 2$.

2. First, let's prove $\alpha^2 \leq 2$:

   (a) Assume $\alpha^2 > 2$.

   (b) For $n \in \mathbb{N}$, consider $(\alpha - \frac{1}{n})^2$:

   $$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} \tag{18}$$

   (c) Since $\alpha^2 > 2$, we have:

   $$2 < \alpha^2 - \frac{2\alpha}{n} \tag{19}$$

   $$2 - \alpha^2 < -\frac{2\alpha}{n} \tag{20}$$

   $$\alpha^2 - 2 > \frac{2\alpha}{n} \tag{21}$$

   $$\frac{\alpha^2 - 2}{2\alpha} > \frac{1}{n} \tag{22}$$

   (d) Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$.

   (e) Then:

   $$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - (\alpha^2 - 2) = 2 \tag{23}$$

   (f) This contradicts the fact that α is an upper bound for S.

   (g) Therefore, our assumption must be false, and $\alpha^2 \leq 2$.

3. Now, let's prove $\alpha^2 \geq 2$:

   (a) Assume $\alpha^2 < 2$.

   (b) For $n \in \mathbb{N}$, consider $(\alpha + \frac{1}{n})^2$:

   $$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha + 1}{n}. \tag{24}$$

   (c) Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$.

   (d) Then:

   $$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2. \tag{25}$$

   (e) This means $\alpha + \frac{1}{n_0} \in S$, contradicting α as an upper bound for S.

   (f) Therefore, our assumption must be false, and $\alpha^2 \geq 2$.

4. Since we have shown $\alpha^2 \leq 2$ and $\alpha^2 \geq 2$, we can conclude that $\alpha^2 = 2$.

Thus, we have proven the existence of a real number $\alpha > 0$ such that $\alpha^2 = 2$. □