SOLUTIONS FOR HOMEWORK 7

Throughout this homework, we assume that (reverse) trigonometric and exponential functions – such as exp, sin, cos, tan, or arctan – are continuous. Other common calculus facts about these functions can also be used.

22.3 Suppose, for the sake of contradiction, that E^- is not connected. Then we can write E^- as a disjoint union of nonempty sets A and B, so that $A^- \cap B = \emptyset = A \cap B^-$. Let $A_0 = E \cap A$ and $B_0 = E \cap B$, then $E = A_0 \cup B_0$. Further, $A_0^- \cap B_0 \subset A^- \cap B = \emptyset$. Similarly, $A_0 \cap B_0^- = \emptyset$. As E is connected, either A_0 or B_0 must be empty. After relabeling, assume $A_0 = A \cap E = \emptyset$ – that is, $A \subset E^- \backslash E$. Note that then, $E \subset B$. Pick $a \in A$; as it belongs to E^-, it is a limit of a sequence $(x_n) \subset E$. However, $(x_n) \subset B$, hence $a = \lim x_n \in B^-$, which contradicts our assumption that $A \cap B^- = \emptyset$.

Alternative solution. Suppose, for the sake of contradiction, that E^- is not connected. Then there exist open sets U_1, U_2 so that $E^- \subset U_1 \cup U_2$, $E^- \cap U_1 \neq \emptyset$, $E^- \cap U_2 \neq \emptyset$, and $E^- \cap U_1 \cap U_2 = \emptyset$. As E is connected, it belongs to either U_1 or U_2; by re-labeling, assume $E \subset U_1$. Pick $x \in E^- \cap U_2$. Then there exists $r > 0$ so that $\mathbf{B}_r^o(x) \subset U_2$. However, as x belongs to the closure of E, we have $\mathbf{B}_r^o(x) \cap U_1 \neq \emptyset$. This contradicts $E^- \cap U_1 \cap U_2 = \emptyset$.

A. Prove that an intersection of convex sets in $\mathbb{R}^n$ is convex.

Suppose $\mathcal{E}$ is a family of convex sets. Show that $F = \cap_{E \in \mathcal{E}} E$ is convex – that is, if $a, b \in F$ and $t \in (0,1)$, $(1-t)a + tb \in F$. a and b belong to each of the sets E, hence, by convexity, the same is true for $(1-t)a + tb$. Therefore, $(1-t)a + tb \in \cap_{E \in \mathcal{E}} E = F$.

B. On the metric space $\mathbb{R}^n$ (with the Euclidean metric d), denote by P_i $(1 \leq i \leq n)$ the projection onto the i-th coordinate. Specifically, $P_i : \mathbb{R}^n \to \mathbb{R}$ takes $\vec{x} = (x_1, \ldots, x_n)$ to x_i. Prove that P_i is Lipschitz.

For $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots y_n) \in \mathbb{R}^n$, $|x_i - y_i| = \big|P_i(\vec{x}) - P_i(\vec{y})\big| \leq \big(\sum_j |x_j - y_j|^2\big)^{1/2} = d\big(\vec{x}, \vec{y}\big)$. This, P_i is Lipschitz with constant 1.

C. Denote by ℓ_1 the set of all absolutely convergent series: the elements of ℓ_1 are sequences $a = (a_i)_{i=1}^\infty$ with $\sum_{i=1}^\infty |a_i| < \infty$. For $a = (a_i)_{i=1}^\infty$ and $b = (b_i)_{i=1}^\infty$, define $d(a,b) = \displaystyle\sum_{i=1}^\infty |a_i - b_i|$.

(a) Prove that d is a metric.

(b) Prove that the function $f : \ell_1 \to \mathbb{R} : (a_i) \mapsto \sum_{i=1}^{\infty} a_i$ is Lipschitz.

(c) Determine whether the function $g : \mathbb{R} \to \ell_1$, taking $t \in \mathbb{R}$ to the sequence $(t^2/2^i)_{i=1}^{\infty}$, is uniformly continuous.

(a) Clearly $d(a, b) \geq 0$, with $d(a, b) = 0$ iff $a_i = b_i$ for any i, in other words, iff $a = b$. Also, $d(a, b) = d(b, a)$. To establish the triangle inequality, consider $a = (aI =_i)$, $b = (b_i)$, and $c = (c_i)$. Then

$$\begin{aligned} d(a, c) &= \sum_{i=1}^{\infty} |a_i - c_i| \leq \sum_{i=1}^{\infty} \big(|a_i - b_i| + |b_i - c_i|\big) \\ &= \sum_{i=1}^{\infty} |a_i - c_i| + \sum_{i=1}^{\infty} |a_i - c_i| = d(a, b) + d(b, c). \end{aligned}$$

(b) For $a = (a_i)$ and $b = (b_i)$,

$$|f(a) - f(b)| = \sum_{i=1}^{\infty} (a_i - b_i) \leq \sum_{i=1}^{\infty} |a_i - b_i| = d(a, b),$$

hence f is Lipschitz with constant 1.

(c) g is not uniformly continuous. We shall show that, for any $\delta > 0$ there exist $t, s \in \mathbb{R}$ so that $|t - s| < \delta$, while $d(g(t), g(s)) \geq 1$. Note first that, for $t, s \in \mathbb{R}$,

$$d(g(t), g(s)) = \sum_{i=1}^{\infty} \left| \frac{t^2}{2^i} - \frac{s^2}{2^i} \right| = |t^2 - s^2| = |t - s| \cdot |t + s|.$$

Take $s = \dfrac{1}{\delta}, t = s + \dfrac{\delta}{2}$ to produce a desired example.