Review: interchanging integration with limit/sum

Proposition (Exercise 33.9 – Lecture 32)

Suppose (f_n) is a sequence of integrable functions on $[a, b]$, converging uniformly to f. Then $\lim_n \int_a^b f_n$ exists, f is integrable, with $\int_a^b f = \lim_n \int_a^b f_n$.

Uniform convergence cannot, in general, be replaced by pointwise convergence (see examples in Lecture 32).

Corollary

If $g_0, g_1, \ldots$ are integrable on $[a, b]$, and $f = \sum_{n=0}^{\infty} g_n$ converges uniformly, then f is integrable, with $\int_a^b f = \sum_{n=0}^{\infty} \int_a^b g_n$.

Proof. Let $f_k = \sum_{n=0}^{k} g_n$, then $f_k \to f$ uniformly, hence $\int_a^b f_k \to \int_a^b f$. However, $\int_a^b f_k = \sum_{n=0}^{k} \int_a^b g_n$. ■

Interchanging differentiation with limit/sum

Example. It may happen that $f_n \to f$ uniformly, but (f_n') does not converge at all (not even pointwise).

Take $f_n(x) = \frac{1}{n}\sin(n^2 x)$. Then $f_n \to 0$ uniformly on $\mathbb{R}$.
However, $f_n'(x) = n\cos(n^2 x)$. If $\frac{x}{\pi}$ is rational, then the sequence $\left(f_n'(x)\right)$ has no real limit. Indeed, write $x = \frac{p}{q}\pi$. If $n = mq$, then
$f_n'(x) = mq\cos(m^2 qp\pi) = mq(-1)^{m^2 qp}$. $\lim_{m\to\infty} \left|f_{mq}'\left(\frac{p}{q}\pi\right)\right| = +\infty$.

Thus, the sequence (f_n') does not converge pointwise to any function on any interval. Indeed, for any interval $[c, d]$, $\left[\frac{c}{\pi}, \frac{d}{\pi}\right]$ contains a rational $\frac{p}{q}$, hence $x = \frac{p}{q}\pi \in [c, d]$.

Example. $\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$ converges uniformly on $\mathbb{R}$ (apply Weierstrass M-test with $M_n = \frac{1}{n^2}$). However, $\sum_{n=1}^{\infty} \left(\frac{\sin(n^2 x)}{n^2}\right)'$ does not converge pointwise, on any interval.

Review of power series (§23, Lecture 25)

For a power series $\sum_n a_n x^n$, let $\beta = \limsup |a_n|^{1/n}$.
Radius of convergence $R = \frac{1}{\beta}$.

Theorem (26.1)

The power series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R_1, R_1]$ for $R_1 < R$.

Proof. Weierstrass M-test: $\sum_n g_n$ converges uniformly if $|g_n| \leqslant M_n$, $\sum_n M_n < \infty$. Use $M_n = |a_n| R_1^n$. ■

Corollary (26.2)

The power series $\sum_{n=0}^{\infty} a_n x^n$ converges to a continuous function on $(-R, R)$.

Proof. The uniform limit of continuous function is continuous, hence $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $[-R_1, R_1]$ for any $R_1 \in (0, R)$. ■

Differentiation and integration of power series (§26)

Lemma (26.3)

If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the same is true for $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

Proof for $\sum_{n=1}^{\infty} n a_n x^{n-1}$. The series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$ have the same interval of convergence. For the latter series, $\limsup \left| n a_n \right|^{1/n} = \limsup n^{1/n} |a_n|^{1/n} = \beta$, since $\lim n^{1/n} = 1$. ∎

Remark. Intervals of convergence may be different!
For instance, take $a_n = \frac{1}{n+1}$.
$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ has interval of convergence $[-1, 1)$.
$\sum_{n=1}^{\infty} \frac{n x^{n-1}}{n+1}$ has interval of convergence $(-1, 1)$.
$\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2} = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ has interval of convergence $[-1, 1]$.

Integration of power series

Theorem (26.4 – Interchange of summation and integration)

Suppose a power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has radius of convergence R. Then for $|x| < R$, $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

Proof for $0 < x < R$. Let $f_k(t) = \sum_{n=0}^{k} a_n t^n$. $f_k \to f$ uniformly on $[0, x]$. Further, $\int_0^x f_k = \sum_{n=0}^{k} \frac{a_n}{n+1} x^{n+1}$, hence $\lim_k \int_0^x f_k = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. We can interchange integration and uniform limit, hence
$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \lim_k \int_0^x f_k = \int_0^x \lim_k f_k = \int_0^x f$. ∎

Example (p. 211): $f(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $R = 1$.
$-\ln(1-x) = \int_0^x f = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for $|x| < 1$.
Let $u = -x$. $\ln(1+u) = \frac{u}{1} - \frac{u^2}{2} + \frac{u^3}{3} - \dots$ for $|u| < 1$.

Differentiation of power series

Theorem (26.5 – Interchange of summation and differentiation)

Suppose a power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has radius of convergence R. Then for $|t| < R$, $f'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$.

Proof. The series $g(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ has radius of convergence R.
Theorem 26.4: for $|x| < R$, $\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$.
Differentiate both sides, applying Fundamental Theorem of Calculus:
$f'(x) = g(x)$. ■

Example (p. 211). $f(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $R = 1$.
$f'(t) = \frac{1}{(1-t)^2} = \sum_{n=1}^{\infty} n t^{n-1}$.

Exercise. Compute $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots$.

Use the preceding example, with $t = \frac{1}{2}$: $4 = \frac{1}{(1-1/2)^2} = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$, hence $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{4}{2} = \mathbf{2}$.

Power series: another example

Exercise. Consider the power series $g(t) = \sum_{n=1}^{\infty} n^2 t^n$.
(a) Find the interval of convergence of this series.
(b) Find a closed formula for g.

The radius of convergence is the same as for $\sum_n n^2 t^{n-1}$, which, in turn, is the same as for $\sum_n nt^n$ – that is, 1. For $t = \pm 1$, the series diverges, hence the interval of convergence is $(-1, 1)$.

We have $\frac{1}{(1-t)^2} = \sum_{n=1}^{\infty} nt^{n-1}$. Differentiate:
$\frac{2}{(1-t)^3} = \sum_{n=2}^{\infty} n(n-1)t^{n-2}$, hence $\frac{2t^2}{(1-t)^3} = \sum_{n=1}^{\infty} n(n-1)t^n$. We also know that $\frac{t}{(1-t)^2} = \sum_{n=1}^{\infty} nt^n$.
As $n^2 = n(n-1) + n$,
$g(t) = \sum_{n=1}^{\infty} \big(n(n-1) + n\big)t^n = \sum_{n=1}^{\infty} n(n-1)t^n + \sum_{n=1}^{\infty} nt^n = \frac{2t^2}{(1-t)^3} + \frac{t}{(1-t)^2} = \boldsymbol{\frac{t+t^2}{(1-t)^3}}$.

Theorem (Abel's Theorem, 26.6)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of convergence $R > 0$. If the series converges at R $(-R)$, then f is continuous at R (resp. $-R$).