

# MATH 447: Real Variables - Homework #3

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**Problem 1** (11.5). Let  $(q_n)$  be an enumeration of all the rationals in the interval  $(0, 1]$ .

- Give the set of subsequential limits for  $(q_n)$ .
- Give the values of  $\limsup q_n$  and  $\liminf q_n$ .

**Solution 1.** 1. The set of subsequential limits for  $(q_n)$  is  $q \in \mathbb{Q} : 0 < q \leq 1$ .

2. (a)  $\limsup q_n = 1$   
(b)  $\liminf q_n = 0$

**Problem 2** (11.11). Let  $S$  be a bounded set. Prove there is an increasing sequence  $(s_n)$  of points in  $S$  such that  $\lim_{n \rightarrow \infty} s_n = \sup S$ . Compare Exercise 10.7. Note: If  $\sup S$  is in  $S$ , it's sufficient to define  $s_n = \sup S$  for all  $n$ .

**Solution 2.**  $u_n = \sup s_n$ , and  $s_n \leq \sup s_n$ . If  $S$  is bounded, then there exists a real number  $a$  such that  $s_n \leq a$ . By Theorem 11.4 in Ross, there exists a monotonic subsequence for bounded set  $S$ . We shall prove that this sequence is increasing.

*Proof.* Because  $S$  is bounded, we know that there exists  $\sup S$  by the Axiom of Completeness. We also know that monotone bounded sequences converge by Theorem 10.2 in Ross. Therefore, we can say:

$$\sup S - s_n < 1 \tag{1}$$

Where 1 is chosen arbitrarily. We can then choose a subsequence of  $S$  such that it satisfies the following conditions:

$$\sup S - s_n = \min \left( \sup S - s_{n-1}, \frac{1}{n} \right) \tag{2}$$

Choosing such  $s_n$  enforces that the sequence is increasing, and is getting closer and closer to the supremum. With the values selected, we now have a subsequence of  $S$  such that  $n_1 < n_2 < \dots < n_k$ ,  $s_{n_1} < s_{n_2} < \dots < s_{n_k}$ , and  $\lim s_{n_k} = \sup S$ .  $\square$

**Problem 3** (12.3 (d,e)). Let  $(s_n)$  and  $(t_n)$  be the following sequences that repeat in cycles of four:

- $(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$
- $(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \dots)$

Find:

- $\limsup(s_n + t_n)$
- $\limsup s_n + \limsup t_n$

**Solution 3.** 1.  $\limsup(s_n + t_n) = 3$

2.  $\limsup(s_n) + \limsup(t_n) = 4$

**Problem 4** (12.4). Show  $\lim_{N \rightarrow \infty} \sup_{n > N}(s_n + t_n) \leq \lim_{N \rightarrow \infty} \sup_{n > N} s_n + \lim_{N \rightarrow \infty} \sup_{n > N} t_n$  for bounded sequences  $(s_n)$  and  $(t_n)$ . Hint: First show:

- $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$

Then apply Exercise 9.9 (c).

Suppose there exists  $N_0$  such that  $s_n \leq t_n$  for all  $n > N_0$

Exercise 9.9 (c):

- Prove that if  $\lim s_n$  and  $\lim t_n$  exist, then  $\lim s_n \leq \lim t_n$ .

**Solution 4.** Proof. We want to find the following:

$$\limsup_{N \rightarrow \infty} s_N + t_N \leq (\limsup_{N \rightarrow \infty} s_n) + (\limsup_{N \rightarrow \infty} t_n) \quad (3)$$

We know that the following holds true, as this is the definition of  $\limsup$ .

$$\limsup_{N \rightarrow \infty} s_N = \limsup_{M \rightarrow \infty} \{s_N | N \geq M\} \quad (4)$$

We also know that any element of a sequence  $s_n$  is always less than or equal to the supremum of  $s_n$  due to the Axiom of Completeness. Then, we can state the following.

$$N > M \implies s_N \leq \sup\{s_N | N \geq M\} \quad (5)$$

Therefore, the following also holds true from above:

$$s_N + t_N \leq \sup\{s_N | N > M\} + \sup\{t_N | N > M\} \quad (6)$$

The RHS of the above is a constant, so the following is true:

$$(s_N + t_N \leq C) \quad (7)$$

If any value of  $s_n + t_n$  is always less than the RHS of above, then we can say the the supremum of  $s_n + t_n$  is also less than or equal to the RHS of above:

$$\sup\{s_N + t_N | N > M\} \leq \sup\{s_N | N > M\} + \sup\{t_N | N > M\} \quad (8)$$

We have now established  $\sup(s_n + t_n) \leq \sup s_n + \sup t_n$ . Now, we can use a familiar limit theorem to prove our original hypothesis.

$$\limsup_{N \rightarrow \infty} s_N + t_N \quad (9)$$

We know that the above is equal to the following from the definition of  $\limsup$ . Then, we can take the limit of both sides. From Exercise 9.9c, we know that if given  $N_0$  s.t.  $s_n \leq t_n$  for all  $n > N_0$   $\lim_{n \rightarrow \infty} s_n$  and  $\lim_{n \rightarrow \infty} t_n$  exist, then  $\lim_{n \rightarrow \infty} s_n \leq \lim_{n \rightarrow \infty} t_n$

$$= \limsup_{M \rightarrow \infty} \{s_N + t_N | N > M\} \leq \lim_{M \rightarrow \infty} (\sup \{s_N | N > M\} + \sup \{t_N | N > M\}) \quad (10)$$

From a known limit theorem, we know that  $\lim_{n \rightarrow \infty} s_n + t_n = \lim_{n \rightarrow \infty} s_n + \lim_{n \rightarrow \infty} t_n$ :

$$= \limsup_{M \rightarrow \infty} \{s_N | N > M\} + \limsup_{M \rightarrow \infty} \{t_N | N > M\} \quad (11)$$

$$\limsup_{N \rightarrow \infty} (s_N + t_N) \leq \limsup_{N \rightarrow \infty} s_N + \limsup_{N \rightarrow \infty} t_N \quad (12)$$

□

**Problem 5** (12.12). Let  $(s_n)$  be a sequence of nonnegative numbers, and for each  $n$  define  $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$ .

- Show:

$$\lim_{N \rightarrow \infty} \inf_{n > N} s_n \leq \lim_{N \rightarrow \infty} \inf_{n > N} \sigma_n \leq \lim_{N \rightarrow \infty} \sup_{n > N} \sigma_n \leq \lim_{N \rightarrow \infty} \sup_{n > N} s_n \quad (13)$$

*Hint: for the last inequality, show first that  $M > N$  implies:*

$$\sup \{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_n) + \sup \{s_n : n > N\} \quad (14)$$

- Show that if  $\lim_{n \rightarrow \infty} s_n$  exists, then  $\lim_{n \rightarrow \infty} \sigma_n$  exists and  $\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} s_n$ .
- Give an example where  $\lim_{n \rightarrow \infty} \sigma_n$  exists, but  $\lim_{n \rightarrow \infty} s_n$  does not exist.

**Solution 5.** 1. Proof.

$$\liminf_{N \rightarrow \infty} s_N \leq \liminf_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} s_N \quad (15)$$

$$\lim_{n \rightarrow \infty} s_N = s \implies s = \liminf_{N \rightarrow \infty} s_N \leq \liminf_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} s_N = s \quad (16)$$

By the Squeeze Theorem,

$$\implies \liminf_{N \rightarrow \infty} \sigma_N = \limsup_{n \rightarrow \infty} \sigma_N = s \quad (17)$$

$$\implies \lim_{N \rightarrow \infty} \sigma_N = s \quad (18)$$

We wish to find the third inequality as shown below, as the second inequality is obvious, and the first inequality can be proved similarly to the third inequality.

$$\limsup_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} s_N \quad (19)$$

We will fix  $L$ . Then, if  $M > L$ , we can consider  $\sup \{\sigma_N | N > M\}$

$$\sigma_N = \frac{s_1 + \dots + s_N}{N} \quad (20)$$

$$= \frac{s_1 + \dots + s_N}{N} + \frac{s_{L+1} + \dots + s_N}{N} \quad (21)$$

We know that there are  $N - (L + 1) + 1 = N - L$  total terms in the denominator of the second term of the RHS above. Therefore, we can state the following:

$$\sigma_N \leq \frac{s_1 + \dots + s_L}{N} + \underbrace{\frac{N - L}{N}}_{\leq 1} \sup \{s_N | N > L\} \quad (22)$$

$$\leq \frac{s_1 + \dots + s_L}{M} + \sup \{s_N | N > L\} \quad (23)$$

$$\sup \{\sigma_N | N > M\} \leq \frac{s_1 + \dots + s_L}{M} + \sup \{s_N | N > L\} \quad (24)$$

We know that the RHS above is independent of  $N$ , so we take the limit of both sides above. Doing so results in the following:

$$\limsup_{N \rightarrow \infty} \sup \sigma_N = \limsup_{M \rightarrow \infty} \sigma_N \{N | N > M\} \quad (25)$$

$$\leq \lim_{M \rightarrow \infty} \left( \underbrace{\frac{s_1 + \dots + s_L}{M}}_{\text{goes to 0 as } M \rightarrow \text{infinity}} + \sup \{s_N | N > L\} \right) \quad (26)$$

$$\limsup_{N \rightarrow \infty} \sigma_N \leq \sup \{s_N | N > L\} \quad (27)$$

□

2. Proof.

$$|s_n - L| < \varepsilon \quad (28)$$

$$\sigma_n = \frac{1}{n} \sum_{i=1}^n s_n \quad (29)$$

$$\lim \frac{1}{n} \sum_{i=1}^n s_n \quad (30)$$

$$n > N \implies \sum_{i=1}^n s_n \quad (31)$$

$$\implies |ns_n - nL| \quad (32)$$

$$\frac{1}{n} |n(s_n - L)| < \varepsilon \quad (33)$$

$$= |(s_n - L)| < \varepsilon \quad (34)$$

$$= |s_n - L| < \varepsilon \quad (35)$$

$$= \lim \sigma_n = L = \lim s_n \quad (36)$$

□

Proof.

Suppose  $s_n = (-1)^n$ . Then  $\lim s_n = \text{DNE}$ . But we know:

$$\frac{-1 \text{ or } 0}{N} \xrightarrow[N \rightarrow \infty]{} 0 \quad (37)$$

Therefore, the  $\lim_{n \rightarrow \infty} \sigma_n$  exists if  $\lim_{n \rightarrow \infty} s_n = \text{DNE}$ . □

**Problem 6** (12.13). Let  $(s_n)$  be a bounded sequence in  $\mathbb{R}$ . Let  $A$  be the set of  $a \in \mathbb{R}$  such that  $\{n \in \mathbb{N} : s_n < a\}$  is finite, i.e., all but finitely many  $s_n$  are  $\geq a$ . Let  $B$  be the set of  $b \in \mathbb{R}$  such that  $\{n \in \mathbb{N} : s_n > b\}$  is finite. Prove  $\sup A = \lim_{N \rightarrow \infty} \inf_{n > N} s_n$  and  $\inf B = \lim_{N \rightarrow \infty} \sup_{n > N} s_n$ .

**Solution 6.** From Theorem 11.7, there exists a monotonic subsequence whose limit is  $\limsup s_n$  and  $\liminf s_n$ . So  $\limsup s_n$  and  $\liminf s_n$  exist. From Theorem 10.1, we know that these limits converge because  $s_n$  is bounded. Our goal is to show that  $\limsup s_n = \inf B$ , and  $\liminf s_n = \sup A$ , for sets  $A = \{s_n : s_n < a\}$ , and  $B = \{s_n : s_n > b\}$ , where  $A, B$  both have finite cardinality.

*Proof.* Choose  $N_1$  s.t.  $n > N_1 \implies \{n : |s_n - a| < \varepsilon\}$  is infinite. By Theorem 11.2,  $a$  is a subsequential limit. Choose  $N_2$  s.t.  $n > N_2 \implies \{n : |s_n - b| < \varepsilon\}$  is infinite. By 11.2,  $b$  is also a subsequential limit. To prove that  $a = \sup A$ , choose  $N = \min \{d(s_N, a)\}$ . Then

$N = n + 1 \implies s_{n+1} > a$ , so  $s_n$  is not the least upper bound of  $A$ . So  $a = \sup A$ . A similar argument follows for  $\inf B = b$ . We know that  $a < b$ , or else  $B$  would be infinite, contradictory to  $B$  being finite in the given. Let  $S = \{a, b\}$ . Then, by Theorem 11.8,  $\inf S = \liminf s_n$ ,  $\sup S = \limsup s_n$ . This implies  $\sup A = \liminf s_n$ ,  $\inf B = \limsup s_n$ .  $\square$