SOLUTIONS FOR HOMEWORK 3

11.5. (a) s is a limit of a subsequence of (q_n) iff $s \in [0,1]$. Indeed, any subsequence of (q_n) is contained in $[0,1]$, so possible limits are in this interval as well. It remains to show that, for any $s \in [0,1]$ and $\varepsilon > 0$, $|\{n : |q_n - s| < \varepsilon\}| = \infty$.

We can assume that $\varepsilon < \max\{s, 1-s\}$. Thus, one of the intervals $(s-\varepsilon, s)$, $(s, s+\varepsilon)$ lies entirely in $[0,1]$. As shown in class, this interval contains infinitely many q_n's.

(b) We claim that $\limsup q_n = 1$ and $\liminf q_n = 0$. We shall prove the equality for $\limsup$, as $\liminf$ is handled similarly.

By definition, $\limsup q_n = \lim u_N$, with $u_N = \sup_{n>N} q_n$. It suffices to show that $u_N = 1$ for any N. Indeed, 1 is clearly an upper bound for (q_n), hence $u_N \leq 1$. Suppose, for the sake of contradiction, that $u_N < 1$. Then $(u_N, 1] \cap \mathbb{Q} \subset \{q_1, \ldots, q_N\}$, so there are finitely many rationals in $(u_N, 1]$. This is impossible!

Alternatively, one can use the fact that $\limsup q_n$ ($\liminf q_n$) is the supremum (resp. infimum) of the set of subsequential limits of (q_n). As shown in (a), the latter set is $[0,1]$.

11.11. We have three cases: (1) S is not bounded above – that is, $\sup S = +\infty$; (2) $\sup S \in \mathbb{R}$, $\sup S \in S$; (3) $\sup S \in \mathbb{R}$, $\sup S \notin S$.

Case 1: We find a sequence (s_n) which is increasing, and not bounded above. Pick $s_1 \in S \cap (1, \infty)$, $s_2 \in S \cap (\max\{s_1, 2\}, \infty)$, $s_3 \in S \cap (\max\{s_2, 3\}, \infty)$. This way we obtain a strictly increasing sequence (s_n) s.t. $s_n > n$ for any n. Clearly $\lim s_n = +\infty$.

Case 2: As the book suggests, take $s_n = \sup S$ for any n. Then (s_n) is increasing, converging to s.

Case 3: We find a sequence (s_n) which strictly increases, and converges to $s = \sup S$. Note first that $(u, s) \cap S \neq \emptyset$ (otherwise u would be an upper bound for S). Pick $s_1 \in S \cap (s-1, s)$, $s_2 \in S \cap (\max\{s_1, s - \frac{1}{2}\}, \infty)$, $s_3 \in S \cap (\max\{s_2, s - \frac{1}{3}\}, \infty)$, etc.. This way we obtain a strictly increasing sequence (s_n) s.t. $s_n > s - \frac{1}{n}$ for any n. By Squeeze Theorem, $\lim s_n = s$.

12.3. (d) Let $r_n = s_n + t_n$, then $(r_n) = (2, 2, 3, 1, 2, 2, 3, 1, \ldots)$. For any k, $\sup_{n \geq k} r_n = 2$, hence $\limsup_n r_n = \lim_k (\sup_{n \geq k} r_n) = 3$.

(e) For any k, $\sup_{n \geq k} s_n = 2$, hence $\limsup_n s_n = \lim_k (\sup_{n \geq k} s_n) = 2$. Similarly, $\limsup t_n = 2$. Thus, $\limsup s_n + \limsup t_n = 4$.

12.4. For a fixed N, let $a_N = \sup_{n>N} s_n$ and $b_N = \sup_{n>N} t_n$. For $n > N$, $s_n + t_n \leq a_N + b_N$, hence, for $M \geq N$, $\sup_{n>M}(s_n + t_n) \leq \sup_{n>N}(s_n + t_n) \leq a_N + b_N$. Passing to the limit as M grows, we get:

$$\limsup(s_n + t_n) = \lim_M \sup_{n>M} (s_n + t_n) \leq a_N + b_N.$$

Now pass to the limit as N grows to obtain

$$\limsup(s_n + t_n) \leq \lim_N (a_N + b_N) = \lim_N a_N + \lim_N b_N = \limsup s_n + \limsup t_n.$$

12.12. [*Bonus problem – very little partial credit given.*] **(a)** We shall show that $\limsup \sigma_n \leq \limsup s_n$. The inequality $\liminf \sigma_n \geq \liminf s_n$ would follow by replacing s_n by $-s_n$, while $\liminf \sigma_n \leq \limsup \sigma_n$ is always true.

From the definition of $\limsup \sigma_n$, it suffices to show: if $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ so that $\sigma_n \leq \varepsilon + \limsup s_n$ for any $n \geq N$.

Let $u_k = \sup_{n>k} s_n$, then the sequence (u_k) is decreasing, with $\lim_k u_k = \limsup s_n$. Thus, there exists K so that $u_k < \limsup s_n + \varepsilon/2$ for $k \geq K$. Thus, $s_n < \limsup s_n + \varepsilon/2$ for $n > K$.

Now let $A = \sum_{k=1}^{K} s_k - K\Big(\limsup s_n + \frac{\varepsilon}{2}\Big)$. Find $M > K$ so that $A/M < \varepsilon/2$. Them for $n > M$, we can write

$$\sigma_n = \frac{1}{n}\Big(\sum_{k=1}^{K} s_k - K\Big(\limsup s_n + \frac{\varepsilon}{2}\Big) + \frac{1}{n}\Big(K\Big(\limsup s_n + \frac{\varepsilon}{2}\Big)\Big) + \sum_{k=K+1}^{n} s_k\Big)$$

As $n > M$, we have $\frac{1}{n}\Big(\sum_{k=1}^{K} s_k - K\Big(\limsup s_n + \frac{\varepsilon}{2}\Big)\Big) \leq \frac{A}{M} < \frac{\varepsilon}{2}$. On the other hand, we can view $K\Big(\limsup s_n + \frac{\varepsilon}{2}\Big) + \sum_{k=K+1}^{n} s_k$ as a sum of n terms, each of them not exceeding $\limsup s_n + \frac{\varepsilon}{2}$; the same must also be true for their average. Thus, $\sigma_n < \frac{\varepsilon}{2} + \Big(\limsup s_n + \frac{\varepsilon}{2}\Big) = \limsup s_n + \varepsilon$, as desired.

(b) If $\lim s_n$ exists, then $\liminf s_n = \limsup s_n$, hence, by (a), $\liminf \sigma_n = \limsup \sigma_n$, which implies the existence of $\lim \sigma_n$.

(c) Take $s_n = (-1)^{n-1}$ for $n \in \mathbb{N}$. Then clearly (s_n) diverges. However, $\sigma_n = 0$ if n is even, and $\sigma_n = 1/n$ if n is odd. Either way, $0 \leq \sigma_n \leq 1/n$, hence $\lim_n \sigma_n = 0$, by Squeeze Theorem.

12.13. We shall show that $\inf B = s$, where $s = \limsup s_n$; the other statement is established similarly.

(1) Prove first that any $b > s$ actually belongs to B; this would imply $\inf B \leq s$.

Consider $b > s$. Recall that $s = \lim u_j$, where $u_j = \sup_{n>j} s_n$ form a decreasing sequence. There exists N s.t. $|u_j - s| < b - s$ for $j \geq N$. By the triangle inequality, $u_N \leq s + |u_N - s| < s + (b - s) = b$, hence $\sup_{n>N} s_n < b$. Therefore, if $s_n > b$, then $n \leq N$; there are only finitely many such n's.

(2) In the other direction, show that no number $t < s$ can belong to B; this would imply that s is a lower bound for B, hence, from part (1), s must be the greatest lower bound.

Suppose, for the sake of contradiction, that $\{n : s_n > t\}$ is finite. This set can be listed as $\{n_1, \ldots, n_M\}$. Let $N = \max\{n_1, \ldots, n_M\}$; for $n > N$, $s_n \leq t$, and consequently, for $j \geq N$, $u_j = \sup_{n>j} s_n \leq t$. Then $\limsup s_n = \lim u_j \leq t < s$, a contradiction!

Alternatively, one can also use the fact that $s = \limsup s_n$ is a subsequential limit of (s_n), hence, for every $\varepsilon > 0$, the set $\{n : |s_n - s| < \varepsilon\}$ is infinite. Taking $\varepsilon = s - t$, we conclude that $s_n > s - \varepsilon = t$ for infinitely many values of n, hence $t \notin B$.