TAM 470
Sample Final Exam Problems

Name: ___________________________________

NetID: __________________________________

Instructions

- Do not begin the exam until instructed to do so
- This exam is closed book, closed notes; equation sheet is provided separately
- You may use a pencil/pen and calculator
- Use the back side of each page to continue your work if needed
- Show all of your work and box your final answer to receive full credit

Problem 1 (xx points):

Consider the transient heat equation with a space and time-dependent source term:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + xe^{-t} \quad \text{for } x \in [0,1], t \geq 0$$

With boundary conditions $T(0,t) = T(1,t) = 0$ and initial condition $T(x,0) = 0$.

a) Write the semi-discretized form of this PDE with constant grid spacing Δx on a grid with $N+1$ points $x_0, x_1, \dots x_N$, i.e. complete the following equation:

$$\frac{dT_j(t)}{dt} = \cdots, \qquad j = 1, 2, \dots, N-1$$

b) Write the forward Euler update scheme for interior nodes using time step Δt, i.e. complete the statement below

$$T_j^{(n+1)} = T_j^{(n)} + \cdots, \quad j = 1, 2, \dots, N-1$$

[Problem continues on next page]

c) The backward Euler update scheme for interior nodes is of the form

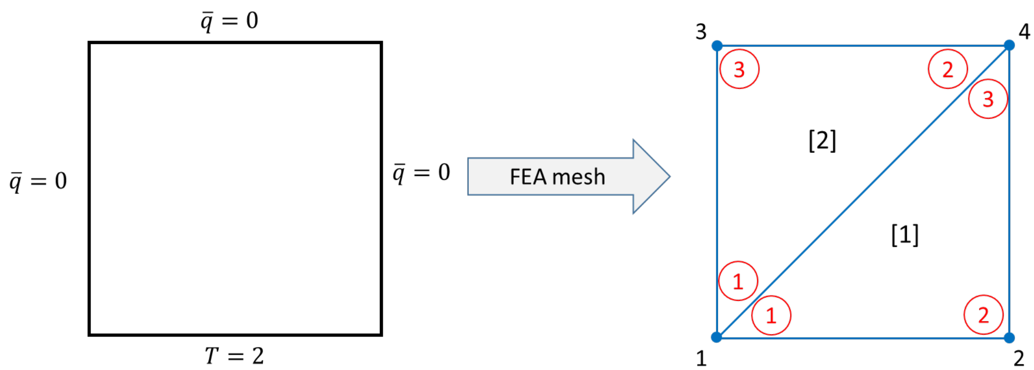
$$-\gamma T_{j+1}^{(n+1)} + (1+2\gamma)T_j^{(n+1)} - \gamma T_{j-1}^{(n+1)} = \cdots, \qquad j = 1, 2, \dots, N-1$$

Derive the scheme; write the expression for γ and the complete right-hand side of the equation above.

Problem 2 (xx points)

Consider the square domain below and a very coarse mesh using just two elements; *pay close attention to the following description of the mesh*:

1. Global node numbers are displayed **outside** the mesh and in **black**;
2. Local node numbers are displayed **inside** the mesh and are **circled in red**;
3. Element numbers are displayed **inside** each element in **brackets** [ ]



Suppose we wish to solve the 2D steady state heat equation on this mesh:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial xy}\right) + Q = 0$$

The boundary conditions are described visually on the figure: $T = 2$ on the entire bottom edge (including the bottom corners) and the remaining boundaries are insulated with $\bar{q} = 0$. The element geometry and internal heating Q are such that the element stiffness matrices k_{ij}^{e} and element force vectors F_i^e ($e = 1, 2$ to indicate the element) are:

$$k_{ij}^1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad k_{ij}^2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad F_i^1 = F_i^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Given this information, do the following:

a) Write down the LM matrix (1 column per element) to describe the element connectivity

[Problem continues on next page]

b) Compute the global stiffness K and global force vector F by assembling the element matrices and vectors; write your final answer as the system of equations $KT = F$ and include all the numerical entries in K and F, where the nodal unknowns are labeled T_1, T_2, T_3, and T_4, corresponding to global nodes 1 – 4.

c) Impose boundary conditions on the global system of equations to solve for the unknown nodal temperatures T_3 and T_4

Problem 3 (xx points)

Consider the one-sided difference approximation for the **second** derivative f_j'' at a grid point x_j:

$$f_j'' = \frac{2f_j - 5f_{j-1} + 4f_{j-2} - f_{j-3}}{h^2} + \tau$$

Where $f_{j+n} = f\left(x_{j+n}\right)$ and the approximation is being made on a grid of uniform spacing such that $x_{j+1} - x_j = h$ for all grid points. The variable τ denotes the truncation error.

a) Determine the leading term of the truncation error τ for this scheme

b) State the order of accuracy of the method

Problem 4 (xx points)

It is proposed to solve the convection problem

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

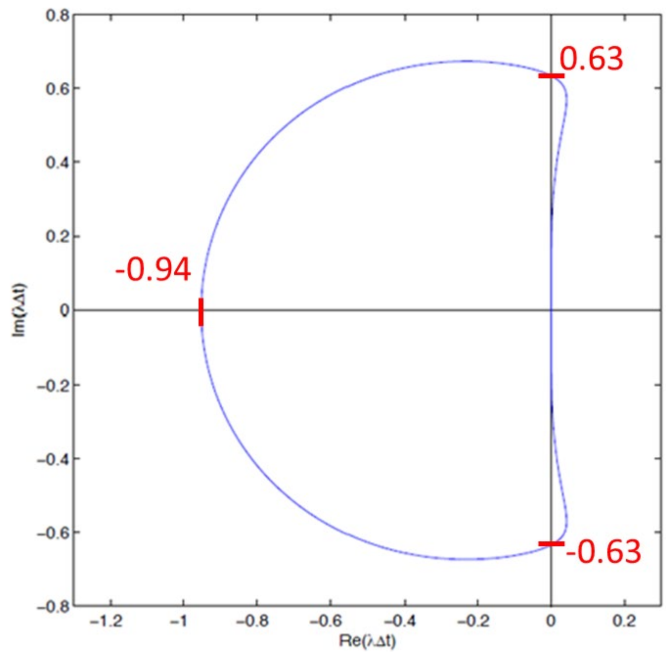
Using semi-discretization by approximating the spatial derivative term with the second order central difference scheme:

$$\frac{\partial u}{\partial x} = \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

a) Use Modified Wavenumber Analysis to cast the semi-discretized form of the equation into the form $\psi'(t) = \lambda\psi(t)$; give the expression for λ. Assume a uniform spatial grid with spacing Δx.

[Problem continues on next page]

b) It is proposed to use an explicit scheme called BDF3/EXT3 (based on backwards differentiation methods) to perform the time integration on this problem. The scheme's stability diagram is given below (the scheme is stable for $\lambda\Delta t$ **inside** the region). Find the maximum allowable CFL number (CFL = $\frac{c\Delta t}{\Delta x}$) to ensure stability, assuming a uniform spatial grid and constant c



Problem 5 (xx pts)

The following explicit scheme is proposed to solve the pure convection equation with constant convecting velocity c on a uniform space and time grid with discretizations Δx and Δt, where $u_j^{(n)} = u(x_j, t_n)$ and $\gamma = \frac{c\Delta t}{\Delta x}$:

$$u_j^{(n+1)} = u_j^{(n)} - \frac{1}{2}\gamma\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) + \frac{1}{2}\gamma^2\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}\right)$$

a) Using von Neumann stability analysis, find the expression for the amplification factor σ in terms of $\gamma, \Delta x$, k and sine and cosine functions.

b) State whether the scheme is stable or unstable for $\gamma = 1$, and justify your answer

Problem 6 (xx points)

The ODE and below is for a damped pendulum with small amplitude swings:

$$\theta''(t) + c\theta'(t) + k\theta(t) = 0$$

The variables c and k are constants that do not depend on time

a) Rewrite this 2nd order ODE as a system of two first-order ODEs such that $\mathbf{y}' = \mathbf{A}\mathbf{y}$ where $\mathbf{y} = (y_1, y_2)$ is a 2×1 vector and $\mathbf{A}$ is a 2×2 matrix of constant coefficients

b) Suppose the parameters c and k are such that the eigenvalues of the matrix $\mathbf{A}$ are

$$\lambda_1 = -2 + 0.5i \quad \text{and} \qquad \lambda_2 = -2 - 0.5i$$

Find the largest allowable time step h for stability when using forward Euler to numerically solve the problem.