SOLUTIONS FOR HOMEWORK 1

Bonus problem: existence of $\sqrt{2}$. [*little partial credit*] Use the completeness of $\mathbb{R}$ to show the existence of $x > 0$ with $x^2 = 2$ (this is x is denoted by $\sqrt{2}$). Specifically, consider $S = \{t \in \mathbb{R} : t > 0, t^2 < 2\}$. Clearly, S is non-empty ($1 \in S$). Further, 2 is an upper bound for S. Indeed, suppose $t \in S$, then $(2-t)(2+t) = 4-x^2 > 4-2 > 0$. Clearly $2 + t > 0$, hence $2 - t > 0$.

Let $x = \sup S$. Prove that $x^2 = 2$, by establishing that (i) $x^2 \leq 2$, and (ii) $x^2 \geq 2$. Once these inequalities are established, we can conclude that $x^2 = 2$.

Hint. For (i), suppose, for the sake of contradiction, that $x^2 > 2$. Use the Archimedean property to find $n \in \mathbb{N}$ so that $\frac{x^2-2}{2x} > \frac{1}{n}$. What can you say about $y = x - \frac{1}{n}$?
For (ii), begin by noting that $x \geq 1$. Supposing, for the sake of contradiction, that $x^2 < 2$, consider $n \in \mathbb{N}$ so that $\frac{2-x^2}{4x} > \frac{1}{n}$; look at $z = x + \frac{1}{n}$.

(i) $y^2 = \left(x - \frac{1}{n}\right)^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2} > x^2 - 2x \cdot \frac{x^2-2}{2x} + \frac{1}{n^2} > x^2 - (x^2-2) + \frac{1}{n^2} > 2$, hence y is an upper bound for S This contradicts x being the least upper bound.

(ii) $z^2 = \left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} < x^2 + \frac{4x}{n} < 2$, hence x is not an upper bound for S. This again gives a contradiction.

1.8(a). We use induction to show that, for any $n \in \mathbb{N}$, $n \geq 2$, the statement $P_n : n^2 > n + 1$ holds.

Basis for induction. P_2 holds: $2^2 > 2 + 1$.

Inductive step. For $n \geq 2$, show that, if P_n holds, then P_{n+1} holds as well.

We have $(n+1)^2 = n^2 + 2n + 1$. If P_n holds – that is, if $n^2 > n + 1$, then
$(n+1)^2 = n^2 + 2n + 1 > (n+1) + (2n+1) = 3n + 2 > n + 2$, hence P_{n+1} holds.

2.8. The rational roots of the polynomial $p(x) = x^8 - 4x^5 + 13x^3 - 7x + 1$ are intergers, which divide 1. Thus, the only possible roots are ± 1. We check whether they are indeed roots using brute force: plug ± 1 into p. We have $p(1) = 1 - 4 + 13 - 7 + 1 = 4 \neq 0$, and $p(-1) = 1 + 4 - 13 + 7 + 1 = 0$. So, -1 is the only rational zero of p.

3.5. **(a)** If either $|b| \leq a$ or $-a \leq b \leq a$ holds, then $a \geq 0$. Now consider two cases – (i) $b \geq 0$, and (ii) $b < 0$.

Case (i): $b \geq 0$. Then
$|b| \leq a \Leftrightarrow b \leq a$ [Definition of $|\cdot|$]
$\Leftrightarrow -a \leq b \leq a$ [since $b \geq 0 \geq -a$] .

Case (ii): $b < 0$. Then
$|b| \leq a \Leftrightarrow -b \leq a$ [Definition of $|\cdot|$]

$\Leftrightarrow -a \le b$ [Theorem 3.2(i)]
$\Leftrightarrow -a \le b \le a$ [since $b \le 0 \le a$] .

(b) By (a), it suffices to show that $-|a-b| \le |a| - |b| \le |a-b|$.
The right hand side follows from the triangle inequality: $|a| = |(a-b)+b| \le |a-b|+|b|$.
The left hand side is derived similarly: $|b| = |a+(b-a)| \le |a|+|b-a| = |a|+|a-b|$, so $-|a-b| \le |a|-|b|$.

3.8. Suppose, for the sake of contradiction, that $a \le b_1$ whenever $b_1 > b$, yet $a > b$. Let $b_1 = (a+b)/2$. Then

$$b_1 = \frac{a}{2} + \frac{b}{2} > \frac{b}{2} + \frac{b}{2} = b, \text{ and } b_1 = \frac{a}{2} + \frac{b}{2} < \frac{a}{2} + \frac{a}{2} = a,$$

which contradicts our assumption that $b_1 > b$ implies $b_1 < a$.

4.1. (r) We have $S = \cap_n \Big(1 - \dfrac{1}{n}, 1 + \dfrac{1}{n}\Big)$. Then $\boldsymbol{\sup S = 1}$. In fact, we shall show that $S = \{1\}$. Indeed, if $u > 1$, then there exists $n \in \mathbb{N}$ so that $\dfrac{1}{n} < u-1$. Therefore, $u \notin \Big(1-\dfrac{1}{n}, 1+\dfrac{1}{n}\Big)$, hence also $u \notin S$. Similarly, if $u < 1$, then $u \notin S$. It is clear that $1 \in \Big(1-\dfrac{1}{n}, 1+\dfrac{1}{n}\Big)$ for any n, which implies $1 \in S$. Thus, $S = \{1\}$. Consequently, $x \in \mathbb{R}$ works as an upper bound iff $x \ge 1$.

Remark. We are only asked to give three upper bounds for S. As $S \subset (0,2)$, any three numbers from $[2, \infty)$ will suffice.

4.8. (a) Every $t \in T$ is an upper bound for S; similarly, any $s \in S$ is a lower bound for T.

(b) Suppose, for the sake of contradiction,that $s_0 := \sup S > \inf T =: t_0$. Let $a = (s_0 + t_0)/2$. As $s_0 > a$, a is not an upper bound for S. Therefore, there exists $s \in S$ with $s > a$. Similarly, there exists $t \in T$ with $t < a$. By the associativity of order, $s > t$, which contradicts our assumption.

(c) Let $S = (-\infty, 0]$ and $T = [0, \infty)$. Many other examples are possible.

(d) Let $S = (-\infty, 0)$ and $T = (0, \infty)$. Once again, many other examples are possible.

8.5. (a) Fix $\varepsilon > 0$. Find $M, N \in \mathbb{R}$ so that $|a_n - s| < \varepsilon$ for $n > M$, and $|b_n - s| < \varepsilon$ for $n > N$. Let $K = \max\{M, N\}$. For $n > K$, we have $s - \varepsilon < a_n, b_n < s + \varepsilon$, and therefore,

$$s - \varepsilon < a_n \le s \le b_n < s + \varepsilon.$$

In particular, $|s_n - s| < \varepsilon$ for $n > K$.

(b) Apply Part (a), with $a_n = -t_n$ and $b_n = t_n$.

8.6. (a) From the definition of convergence, $\lim_n s_n = 0$ if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{R}$ so that $|s_n| < \varepsilon$ for any $n > N$. The latter condition is equivalent to $\lim |s_n| = 0$.

(b) $|s_n| = 1$ for any n, hence $\lim |s_n| = 1$. In class, we proved that the sequences (s_n) diverges.