Series (Section 14)

Goal: make sense of infinite sums $\sum_{j=k_0}^{\infty} a_j = a_{k_0} + a_{k_0+1} + \dots$.

Definition (p. 95 of textbook)

n-th partial sum: $s_n = \sum_{j=k_0}^{n} a_j$. The series $\sum_{j=k_0}^{\infty} a_j$ converges (diverges) if (s_n) does. $\sum_{j=k_0}^{\infty} a_j = \lim_n s_n$ (if $\lim s_n$ exists).

Example: $\sum_{j=0}^{\infty} r^j$. Partial sums: $s_n = \sum_{j=0}^{n} r^j$.
If $r = 1$, then $s_n = n + 1$, hence the series diverges.
If $r \neq 1$, then $s_n = \frac{1-r^{n+1}}{1-r}$. If $|r| < 1$, then $\lim s_n = \frac{1}{1-r}$. Otherwise, (s_n) diverges.

If $r \geqslant 1$, then $s_n \geqslant n$ for any n, hence $\lim s_n = +\infty$.

If $r \leqslant -1$, then $s_n \leqslant 0$ when n is odd (since $r^{n+1} \geqslant 1$). For n even, $-r^{n+1} \geqslant -r \geqslant 0$, hence $s_n \geqslant 1$. Thus, $\lim s_n$ does not exist.

Summary: $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$ for $|r| < 1$; $\sum_{j=0}^{\infty} r^j = +\infty$ for $r \geqslant 1$; $\lim s_n$ does not exist if $r \leqslant -1$.

Cauchy Criterion for convergence

Definition (14.3)

A series $\sum_j a_j$ satisfies Cauchy Criterion if $\forall \varepsilon > 0 \; \exists N$ s.t. $\left|\sum_{j=m}^{n} a_j\right| < \varepsilon$ whenever $n \geqslant m > N$.

Theorem (14.4)

A series converges iff it satisfies the Cauchy Criterion.

Proof. Let $s_n = \sum_{j=k_0}^{n} a_j$. $\sum_{j=k_0}^{\infty} a_j$ converges $\Leftrightarrow (s_n)$ converges
$\Leftrightarrow (s_n)$ is Cauchy $\Leftrightarrow \forall \varepsilon > 0 \; \exists N$ s.t. $|s_n - s_k| < \varepsilon$ when $n > k \geqslant N$.
$s_n - s_k = \sum_{j=k_0}^{n} a_j - \sum_{j=k_0}^{k} a_j = \sum_{j=m}^{n} a_j$, where $m = k + 1$.
So: $\sum_j a_j$ satisfies Cauchy Criterion $\Leftrightarrow (s_n)$ is Cauchy $\Leftrightarrow (s_n)$ converges. ■

More about convergence

Corollary (14.5)

If $\sum_j a_j$ *converges, then* $\lim_n a_n = 0$*.*

Proof. Suppose $\sum_j a_j$ converges. Fix $\varepsilon > 0$. Need to find N s.t. $|a_n| < \varepsilon$ for $n > N$. Find N s.t. $\left|\sum_{j=m}^{n} a_j\right| < \varepsilon$ when $n \geqslant m > N$ (Cauchy Criterion). Now take $m = n$. ■

If $\lim_n a_n = 0$, does $\sum_j a_j$ converge? **No!** Let $a_n = \frac{1}{n}$; $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since the sequence of partial sums (s_n) is unbounded (Lectures 6 and 7).

Comparison test for convergence (preview)

Theorem (14.6 – Comparison test)

(i) If $a_n \geqslant |b_n|$ for any n, and $\sum_n a_n$ converges, then $\sum_n b_n$ converges.
(ii) If $0 \leqslant a_n \leqslant b_n$ for any n, and $\sum_n a_n = +\infty$, then $\sum_n b_n = +\infty$.

Proof of (i). Need to check Cauchy Criterion for $\sum_n b_n$:
$\forall \varepsilon > 0\ \exists N$ s.t. $\left|\sum_{j=m}^{n} b_j\right| < \varepsilon$ if $n \geqslant m > N$.
$\sum_n a_n$ converges, hence $\exists N$ s.t. $\sum_{j=m}^{n} a_j < \varepsilon$ if $n \geqslant m > N$.
By Triangle Inequality, $\left|\sum_{j=m}^{n} b_j\right| \leqslant \sum_{j=m}^{n} |b_j| \leqslant \sum_{j=m}^{n} a_j < \varepsilon$. ∎

Absolutely convergent series

Definition (P. 96 of textbook)

A series $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges.

Corollary (14.7)

Any absolutely convergent series converges.

Example: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \ldots$ converges, but not absolutely.

$s_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$, so $s_2 \leqslant s_4 \leqslant s_6 \leqslant \ldots$.

$s_{2n-1} = \frac{1}{1} - \left(\frac{1}{2} - \frac{1}{3}\right) - \ldots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right)$, so $s_1 \geqslant s_3 \geqslant s_5 \geqslant \ldots$.

$s_{2n} = s_{2n-1} - \frac{1}{2n}$, hence $s_1 \geqslant s_3 \geqslant s_5 \geqslant \ldots \geqslant s_4 \geqslant s_2$.

$\lim_n s_{2n-1} = a$, $\lim_n s_{2n} = b$. $\lim \left(s_{2n-1} - s_{2n}\right) = 0$, hence $a = b$.

Can show: $\lim_n s_n = a$ (or b). In fact, $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2$.

Root Test for convergence

Theorem (14.9 – Root Test)

For a series $\sum_n a_n$ let $\alpha = \limsup_n |a_n|^{1/n}$. The series:

1. *Converges absolutely if $\alpha < 1$;*
2. *Diverges if $\alpha > 1$.*
3. *If $\alpha = 1$, the test gives no information.*

Proof of (i). $\lim_N \sup_{n>N} |a_n|^{1/n} = \alpha < 1$. Pick $c \in (\alpha, 1)$. Pick N s.t. $|a_n|^{1/n} < c$ for $n > N$. For $n > N$, $|a_n| < c^n$. $\sum_n c^n$ converges, hence so does $\sum_n |a_n|$ (Comparison Theorem). ■

Proof of (iii). $\sum_n \frac{1}{n}$ diverges (to $+\infty$).

On the other hand, let $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Then $\lim_n a_n^{1/n} = 1$, and $s_k = \sum_{n=1}^{k} a_n = 1 - \frac{1}{k+1} \to 1$. ■

Ratio Test for convergence

Theorem (14.8 – Ratio Test)

A series $\sum_n a_n$ of non-zero terms:

1. *Converges absolutely if* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$*;*
2. *Diverges if* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$*.*
3. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leqslant 1 \leqslant \limsup \left|\frac{a_{n+1}}{a_n}\right|$*, the test gives no information.*

Partial proof, using Root Test. Let $\alpha = \limsup_n |a_n|^{1/n}$. We know:
$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leqslant \alpha \leqslant \limsup \left|\frac{a_{n+1}}{a_n}\right|$.
Case (i): $\alpha < 1 \Rightarrow$ series converges.
Case (ii): $\alpha > 1 \Rightarrow$ series diverges. ∎

Examples of series

Examples: **(1)** $\sum_{k=1}^{\infty} \frac{k^4}{2^k}$ converges (absolutely). $a_k = \frac{k^4}{2^k}$.

Can use Root Test: $a_k^{1/k} = \frac{(k^{1/k})^4}{2}$, hence $\lim a_k^{1/k} = \frac{1}{2} < 1$.

Can also use Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{1}{2}\left(1 + \frac{1}{k}\right)^2$, hence $\lim \frac{a_{k+1}}{a_k} = \frac{1}{2} < 1$.

(2) $\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \frac{2}{1\cdot 3} + \frac{2}{2\cdot 4} + \ldots$ converges. $\frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2}$.

$s_n = \sum_{k=1}^{n} \frac{2}{k(k+2)} = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$, hence $\lim_n s_n = \frac{3}{2}$.

(3) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (comparison with (2)). In fact, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Fact (p-series – p. 97)

$\sum_k \frac{1}{k^p}$ *converges iff* $p > 1$.

Decimal expansions (Section 16) – almost no proofs!

For $x \in [0, \infty)$, we consider a decimal expansion $x = K.d_1 d_2 d_3 \ldots = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j}$, with $K = \{0, 1, 2, \ldots\}$ and $d_1, d_2, \ldots \in \{0, 1, \ldots, 9\}$. $\sum_j \frac{d_j}{10^j}$ converges (compare to $\sum_j \frac{1}{10^{j-1}}$).

Theorem (16.2)

Any real number has at least one decimal expansion.

The integer part of z $\lfloor z \rfloor$ = greatest integer $\leqslant z$. Let $S = \{n \in \mathbb{Z} : n \leqslant z\}$. Let $\lfloor z \rfloor = \sup S$. Clearly $\lfloor z \rfloor \leqslant z$. Can show: $\lfloor z \rfloor \in \mathbb{Z}$.

Idea of the proof: $K = \lfloor x \rfloor$. $d_1 = \lfloor 10(x - K) \rfloor$, $d_2 = \lfloor 10^2 \big(x - K - \frac{d_1}{10}\big) \rfloor$, etc..

More about decimal expansions

Theorem (16.3)

Any $x \geqslant 0$ has either exactly one decimal expansion, or exactly two – one ending in $**d000\ldots$ ($d \in \{1, \ldots, 9\}$, another ending in $**[d-1]999\ldots$.

For instance, $\frac{1}{2} = 0.5000\ldots = 0.4999\ldots$.

Definition (16.4)

A repeating decimal expansion is one of the form
$K.d_1 \ldots d_\ell \overline{d_{\ell+1} \ldots d_{\ell+r}} = K.d_1 \ldots d_\ell d_{\ell+1} \ldots d_{\ell+r} d_{\ell+1} \ldots d_{\ell+r} \ldots$.

Theorem (16.5)

$x \in \mathbb{Q}$ iff the decimal explansion of x is repeating.

Repeating decimal expansions

Theorem (16.5)

$x \in \mathbb{Q}$ iff the decimal explansion of x is repeating.

x has repeating expansion $\Rightarrow x \in \mathbb{Q}$.

Suppose $x = K.d_1 \dots d_\ell \overline{d_{\ell+1} \dots d_{\ell+r}}$. Let
$y = 0.\overline{d_{\ell+1} \dots d_{\ell+r}} = \sum_{j=1}^{r} d_{\ell+j} 10^{-j} + 10^{-r} \cdot \sum_{j=1}^{r} d_{\ell+j} 10^{-j} +$
$10^{-2r} \sum_{j=1}^{r} d_{\ell+j} 10^{-j} + \dots = \sum_{i=0}^{\infty} 10^{-ir} z$, where
$z = \sum_{j=1}^{r} d_{\ell+j} 10^{-j} \in \mathbb{Q}$. So, $y = \frac{z}{1-10^{-r}} \in \mathbb{Q}$.
$x = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j} = K + \sum_{j=1}^{\ell} \frac{d_j}{10^j} + 10^{-\ell} y \in \mathbb{Q}$. ■

$x \in \mathbb{Q} \Rightarrow x$ has repeating expansion: uses long division (see textbook).