TAM 470 - HW 1 Solutions

Problem 1

Quadratic polynomial $\Rightarrow$ need 3 points

$$\{x_0, x_1, x_2\} = \{0, 1/2, 1\}$$

$$y = f(x) = \sin(\pi x)$$

$$\Rightarrow \{y_0, y_1, y_2\} = \{0, 1, 0\}$$

$$p(x) = \cancelto{0}{y_0} L_0(x) + y_1 L_1(x) + \cancelto{0}{y_2} L_2(x)$$

$$= L_1(x)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-1)}{-1/4} = -4x(x-1)$$

$$\therefore p(x) = L_1(x) = -4x(x-1) \Rightarrow p'(x) = -8x+4$$

(a) $p(1/4) = -4(1/4)(1/4-1) = 3/4$

(b) $p'(1/4) = -8(1/4)+4 = 2$

Problem 2

Given :- $n+1 = 5 \implies n = 4$ cubic functions.

$$(x_0, y_0) = (0,0)$$
$$(x_1, y_1) = (1,3)$$
$$(x_2, y_2) = (2,7)$$
$$(x_3, y_3) = (3,-1)$$
$$(x_4, y_4) = (4,0)$$

The spacing b/w x_i is constant for $i = 0$ to 4.

$$\therefore \Delta_i = x_{i+1} - x_i = h = 1$$

(a) Using equation (1.7) from Moin, and using $\Delta_i = \Delta_{i-1} = \Delta_{i+1} = h$ (constant spacing),

$$\left\{ \frac{h}{6} g''(x_{i-1}) + \frac{2h}{3} g''(x_i) + \frac{h}{6} g''(x_{i+1}) = \frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h} \right\} \quad -(3.1)$$

For free run-out spline we have $g''(x_0) = g''(x_4) = 0$ and using $h = 1$, Setting up the 5x5 matrix for $g''(x_i)$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g''(x_0) \\ g''(x_1) \\ g''(x_2) \\ g''(x_3) \\ g''(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \\ 9 \\ 0 \end{bmatrix}$$

End conditions

Solving,

$$\boxed{\begin{bmatrix} g''(x_0) \\ g''(x_1) \\ g''(x_2) \\ g''(x_3) \\ g''(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 7.71428 \\ -24.8571 \\ 19.7143 \\ 0 \end{bmatrix}}$$

(b) The point $x = 2.4$ lies in interval $[2,3]$ $\therefore$ $x_i = x_2 = 2$, $x_{i+1} = x_3 = 3$

$y_i = y_2 = 7$, $y_{i+1} = y_3 = -1$

We have $i = 2$ and $\Delta_i = h = 1$.

$$\therefore g_2(x) = \frac{g''(x_2)}{6}\left[(x_3 - x)^3 - (x_3 - x)\right]$$

$$+$$

$$\frac{g''(x_3)}{6}\left[(x - x_2)^3 - (x - x_2)\right]$$

$$+$$

$$y_2(x_3 - x) + y_3(x - x_2)$$

$$\therefore g_2(x) = -4.4125\left[(3-x)^3 - (3-x)\right]$$

$$+$$

$$3.2857\left[(x-2)^3 - (x-2)\right]$$

$$+$$

$$7(3-x) - 1(x-2)$$

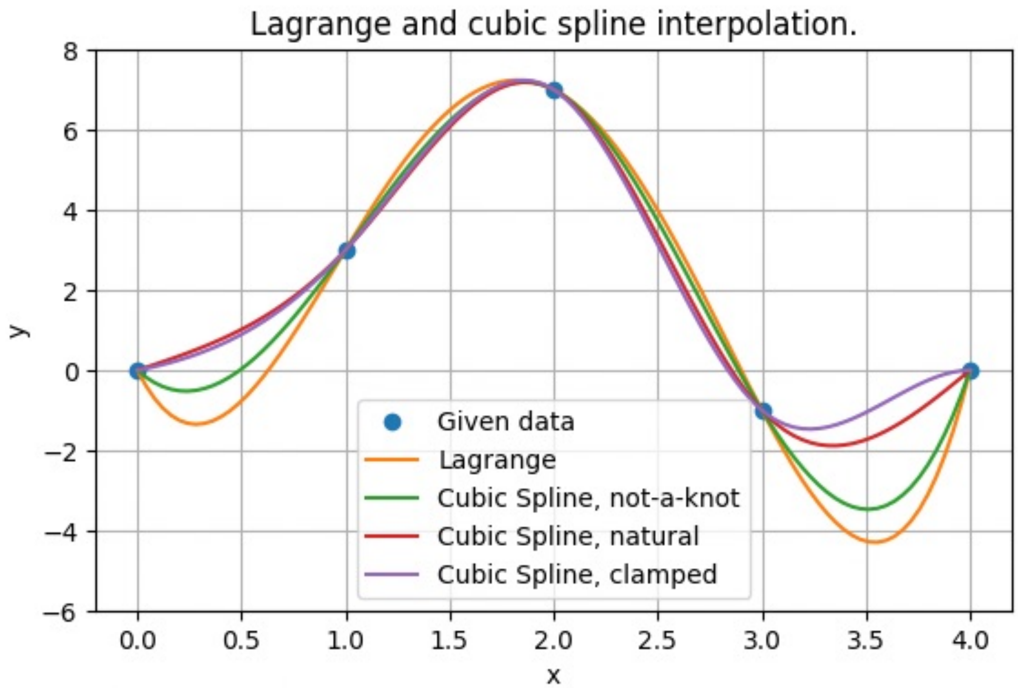
Using calculator,

$$\boxed{g_2(x = 2.4) = 4.28685}$$

This value should match with part (c) as well.

Problem 3

Plot :-



Problem 4

(a) Using the data points from problem 3 but with clamped end conditions,

$$g'(0) = 1 \quad \text{and} \quad g'(4) = 0$$

Note that the end conditions are with respect to first derivatives. To solve the matrix for g'' values, first we need to express above g' conditions interme of g''.

Start with (1.6) in Moin,

$$g_i(x) = \frac{g''(x_i)}{6}\left[\frac{(x_{i+1}-x)^3}{\Delta_i} - \Delta_i (x_{i+1}-x)\right]$$

$$+$$

$$\frac{g''(x_{i+1})}{6}\left[\frac{(x-x_i)^3}{\Delta_i} - \Delta_i (x-x_i)\right]$$

$$+$$

$$f(x_i)\left[\frac{x_{i+1}-x}{\Delta_i}\right] + f(x_{i+1})\left[\frac{x-x_i}{\Delta_i}\right]$$

Differentiate w.r.t x,

$$\left.\begin{aligned} g_i'(x) = {} & \frac{g''(x_i)}{6}\left[\frac{-3(x_{i+1}-x)^2}{\Delta_i} + \Delta_i\right] \\ & + \\ & \frac{g''(x_{i+1})}{6}\left[\frac{3(x-x_i)^2}{\Delta_i} - \Delta_i\right] \\ & + \\ & -\frac{f(x_i)}{\Delta_i} + \frac{f(x_{i+1})}{\Delta_i} \end{aligned}\right\} \rightarrow \underline{(5.1)}$$

When $x=0$, we are looking at the first interval i.e $i=0$

$$\therefore \; x_i = x_0 = 0, \quad x_{i+1} = x_1 = 1$$

$$f(x_i) = y_i = y_0 = 0, \quad f(x_{i+1}) = y_{i+1} = y_1 = 3$$

$$\Delta_i = \Delta_0 = 1$$

Enforcing $g'(0) = 1$ using (5.1),

$$\Rightarrow \frac{g''(0)}{6}\left[-3+1\right] + \frac{g''(1)}{6}\left[-1\right] + 3 = 1$$

$$\Rightarrow \left\{ \frac{g''(0)}{3} + \frac{g''(1)}{6} = 2 \right\} \longrightarrow (5.2)$$

Now similarly with the last interval,

i.e $\quad i = 3,$

$$x_i = x_3 = 3, \quad x_{i+1} = x_4 = 4$$

$$f(x_i) = y_i = y_3 = -1, \quad f(x_{i+1}) = y_{i+1} = y_4 = 0$$

$$\Delta_3 = 1$$

Enforcing $g'(4) = 0$ using (5.1),

$$\frac{g''(3)}{6}\left[0 + 1\right] + \frac{g''(4)}{6}\left[3 - 1\right] + 1 = 0$$

$$\Rightarrow \left\{ \frac{g''(3)}{6} + \frac{g''(4)}{3} = -1 \right\} \longrightarrow (5.3)$$

Recall that we have 5 unknowns,

i.e $\{g''(0), g''(1), g''(2), g''(3), g''(4)\}$.

(5.2) and (5.3) are 2 equations.

We can get 3 more using (1.7) Main.

Since we have constant Δ_i, we use

$$\Delta_i = \Delta_{i-1} = \Delta_{i+1} = h = 1$$

$\therefore$ (1.7) Main reduces to,

$$\frac{g''(x_{i-1})}{6} + \frac{2}{3} g''(x_i) + \frac{g''(x_{i+1})}{6} = y_{i+1} - 2y_i + y_{i-1}$$

Using $i = 1, 2, 3$, we obtain 3 equations. Combining this with (5.2) and (5.3) we get the system as,

$$\begin{bmatrix} 1/3 & 1/6 & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 0 & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 0 & 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} g''(0) \\ g''(1) \\ g''(2) \\ g''(3) \\ g''(4) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -12 \\ 9 \\ -1 \end{bmatrix}$$

Solving yields,

$$\boxed{\begin{bmatrix} g''(0) \\ g''(1) \\ g''(2) \\ g''(3) \\ g''(4) \end{bmatrix} = \begin{bmatrix} 2.3214 \\ 7.3571 \\ -25.750 \\ 23.643 \\ -14.821 \end{bmatrix}}$$

(b) To get $g(x=2.4)$, we use $i=2$

$$x_i = x_2 = 2, \quad x_{i+1} = x_3 = 3$$

$$y_i = y_2 = 7, \quad y_{i+1} = y_3 = -1 \quad \text{and } \Delta_i = 1$$

Using Moin (1.6),

$$g_2(x) = \frac{g''(2)}{6}\left[(3-x)^3 - (3-x)\right]$$

$$+$$

$$\frac{g''(3)}{6}\left[(x-2)^3 - (x-2)\right]$$

$$+$$

$$7(3-x) - 1(x-2)$$

Using solution from part (a), and using

$x = 2.4$,

$$\boxed{g_2(x=2.4) = 4.1240}$$

(c) The values from scipy and part (b) will match.