Proof of continuity of f^{-1}

Theorem 18.4. Suppose $I \subset \mathbb{R}$ is an interval, $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $J = f(I)$ is an interval; $f^{-1} : J \to I$ is strictly increasing and continuous.

Proof (from 18.5, essentially). Clearly $g = f^{-1}$ is strictly increasing. Need to show continuity.

Pick $y_0 = f(x_0) \in J$ (so $x_0 = g(y_0)$), show that g is cont. at y_0. Assume $x_0 \notin \partial I$ – that is, $\exists \varepsilon_0 > 0$ s.t. $(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset I$.
Need: if $\varepsilon \in (0, \varepsilon_0)$, then $\exists \delta > 0$ s.t. $g(y) \in (x_0 - \varepsilon, x_0 + \varepsilon)$ whenever $|y - y_0| < \delta$.

Let $y_1 = f(x_0 - \varepsilon)$, $y_2 = f(x_0 + \varepsilon)$. Let $\delta = \min\{y_2 - y_0, y_0 - y_1\}$. If $|y - y_0| < \delta$, then $y_1 < y < y_2$, hence
$x_0 - \varepsilon = g(y_1) < g(y) < g(y_2) = x_0 + \varepsilon$. ∎

Monotonicity of injective functions on intervals

Theorem (18.6)

Suppose f is a continuous $1-1$ function on an interval I. Then f is strictly monotone.

Proof. Pick $a, b \in I$, with $a < b$. Suppose $f(a) < f(b)$. Prove that f is sitrictly increasing.

(1) Suppose $c \in (a, b)$, show that $f(c) \in \big(f(a), f(b)\big)$.
If $f(c) > f(b)$, then, by IVT, $\exists x \in (a, c)$ s.t. $f(x) = f(b)$. But f is $1-1$, so no such x can exist. $f(c) < f(a)$ is ruled out similarly.
(2) Similarly, $c < a$ ($c > b$) $\Rightarrow f(c) < f(a)$ (resp. $f(c) > f(b)$).
(3) Conclusion: $f(c) < f(a)$ ($f(c) > f(a)$) if $c < a$ (resp. $c > a$).

(4) Suppose $x, y \in I$, $x < y$. Want: $f(x) < f(y)$.
- If $x < a$, then $f(x) < f(a)$, hence $f(y) > f(x)$.
- If $x > a$, then $f(x) > f(a)$, hence $f(y) > f(x)$. ∎

Section 19: Uniform continuity

Definition (21.1)

Suppose (S, d) and (S^*, d^*) are metric spaces. The function $f : S \to S^*$ is called continuous at $x \in S$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d^*(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.
f is called continuous on E ($E \subset S$) if it is continuous $\forall x \in E$ – that is, $\forall x \in E$, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d^*(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.
f is called uniformly continuous on E ($E \subset S$) if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d^*(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

Main feature of uniform continuity: δ depends only on ε, but not on the specific x.
Uniform continuity implies continuity, but not vice versa.

Examples of uniform continuity

Example. $f(x) = x^2$

(1) For $a \in (0, \infty)$, f is uniformly continuous on $[-a, a]$.

For $\varepsilon > 0$, find $\delta > 0$ s.t. $|x^2 - y^2| = |x - y| \cdot |x + y| < \varepsilon$ whenever $x, y \in [-a, a]$, $|x - y| < \delta$.

Let $\delta = \frac{\varepsilon}{2a}$. If $x, y \in [-a, a]$, $|x - y| < \delta$, then
$|x^2 - y^2| = |x - y| \cdot |x + y| < 2a\delta = \varepsilon$. ∎

(2) f is **not** uniformly continuous on $\mathbb{R}$.

Show: $\forall \delta > 0 \ \exists x, y \in \mathbb{R}$ s.t. $|x - y| < \delta$, $|x^2 - y^2| > 1$. Let $y = \frac{1}{\delta}$, $x = y + \frac{\delta}{2}$. Then $x - y = \frac{\delta}{2}$, but
$x^2 - y^2 = \left(y + \frac{\delta}{2}\right)^2 - y^2 = 2y\frac{\delta}{2} + \frac{\delta^2}{4} = 1 + \frac{\delta^2}{4} > 1$. ∎

Lipschitz functions

Definition (not in textbook)

A function $f : S \to S^*$ is Lipschitz if $\exists K > 0$ (Lipschitz constant) s.t. $\forall s, t \in S$, $d^*\big(f(s), f(t)\big) \leqslant K d(s, t)$.

Proposition (not in textbook)

Any Lipschitz function is uniformly continuous.

Proof. For $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{K}$. If $d(s, t) < \delta$, then $d^*\big(f(s), f(t)\big) < \varepsilon$. ∎

Example. $\forall a > 0$, $f(x) = \frac{1}{x}$ is Lipschitz (hence uniformly continuous) on $[a, \infty)$. If $x, y > a$, then $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \leqslant \frac{|x-y|}{a^2}$.

Uniformly continuous function which is not Lipschitz

Example. $f(x) = \sqrt{x}$ is uniformly continuous (on $[0, \infty)$), not Lipschitz.

(1) f is not Lipschitz: there is no K s.t.
$\sqrt{x} = |f(x) - f(0)| \leqslant Kx = K|x - 0|$ for any $x \geqslant 0$.

(2) f is uniformly continuous. It suffices to show that, $\forall x, y \geqslant 0$, we have $\left|\sqrt{x} - \sqrt{y}\right| \leqslant \sqrt{|x - y|}$. Indeed, for $\varepsilon > 0$ let $\delta = \varepsilon^2$. If $|x - y| < \delta$, then $|f(x) - f(y)| \leqslant \sqrt{|x - y|} < \sqrt{\delta} = \varepsilon$.

Without loss of generality, $x > y$, need to show: $\sqrt{x} - \sqrt{y} \leqslant \sqrt{x - y}$, or equivalently, $\sqrt{x} \leqslant \sqrt{y} + \sqrt{x - y}$.
Square both sides:
$x \leqslant \left(\sqrt{y} + \sqrt{x - y}\right)^2 = y + (x - y) + 2\sqrt{x}\sqrt{x - y} = x + 2\sqrt{x}\sqrt{x - y}$. ■

Uniformly continuous functions and Cauchy sequences

Sequential criterion of continuity: $f : S \to S^*$ is continuous iff $(f(s_n)) \subset S^*$ converges whenever $(s_n) \subset S$ converges (f maps convergent sequences to convergent sequences).

Theorem (19.4)

If $f : S \to S^*$ is uniformly continuous, then $(f(s_n))$ is Cauchy when (s_n) is Cauchy (f maps Cauchy sequences to Cauchy sequences).

Example. $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$. This is witnessed by the Cauchy sequence $x_n = \frac{1}{n}$. Then $f(x_n) = n$, so the sequence $(f(x_n))$ is not Cauchy (not even bounded).

Proof: if (s_n) is Cauchy, f is unif. cont., then $(f(s_n))$ is Cauchy.
Fix $\varepsilon > 0$, find N s.t. $d^*\big(f(s_n), f(s_m)\big) < \varepsilon$ for $n, m > N$. Find $\delta > 0$ s.t. $d^*\big(f(s), f(t)\big) < \varepsilon$ when $d(s,t) < \delta$. Find N s.t. $d(s_n, s_m) < \delta$ for $n, m > N$. This N works! ∎

Example. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, but it maps Cauchy sequences to Cauchy sequences.

A continuous function on a compact set is unif. cont.

Theorem (21.4(ii))

Suppose $(S, d), (S^*, d^*)$ are metric spaces, $f : S \to S^*$ is continuous, $E \subset S$ is compact. Then $f|_E$ is uniformly continuous.

Proof 1. Suppose, for the sake of contradiction, that $f|_E$ is not unif. cont..

Then $\exists \varepsilon > 0$ and $x_n, y_n \in E$ s.t. $d(x_n, y_n) < 1/n$, $d^*(f(x_n), f(y_n)) \geqslant \varepsilon$.

Find $n_1 < n_2 < \ldots$ s.t. $x_{n_k} \to s \in E$.

$d(s, y_{n_k}) \leqslant d(s, x_{n_k}) + d(x_{n_k}, y_{n_k})$, so $y_{n_k} \to s$.

f is cont. at s, so $f(x_{n_k}) \to f(s)$, $f(y_{n_k}) \to f(s)$, hence
$d^*(f(x_{n_k}), f(x_{n_k})) \to 0$. Yet $d^*(f(x_{n_k}), f(x_{n_k})) \geqslant \varepsilon$. ■

A continuous function on a compact set is unif. cont. II

Theorem (21.4(ii))

Suppose $(S,d),(S^*,d^*)$ are metric spaces, $f : S \to S^*$ is continuous, $E \subset S$ is compact. Then $f|_E$ is uniformly continuous.

We did not have time for Proof 2, will go over it next time.

Proof 2. For $\varepsilon > 0$, find $\delta > 0$ s.t. $d^*\big(f(s),f(t)\big) < \varepsilon$ when $d(s,t) < \delta$.
For $s \in S$ find $\delta_s > 0$ s.t. $d^*\big(f(s),f(t)\big) < \frac{\varepsilon}{2}$ when $d(s,t) < \delta_s$.
$E \subset \cup_{s\in E}\mathbf{B}^o_{\delta_s/2}(s)$, hence $\exists s_1,\ldots,s_n$ s.t. $E \subset \cup_{k=1}^n \mathbf{B}^o_{\delta_{s_k}/2}(s_k)$.
We claim that $\delta = \frac{1}{2}\min_{1\leqslant k\leqslant n}\delta_{s_k}$ works. Suppose $d(t,s) < \delta$. Find k s.t. $s \in \mathbf{B}^o_{\delta_{s_k}/2}(s_k) \Leftrightarrow d(s,s_k) < \frac{\delta_{s_k}}{2}$.

$d(t,s_k) \leqslant d(t,s) + d(s,s_k) < \delta + \frac{\delta_{s_k}}{2} \leqslant \delta_{s_k}$.
$d^*\big(f(s),f(s_k)\big), d^*\big(f(t),f(s_k)\big) < \frac{\varepsilon}{2}$.
Thus, $d^*\big(f(s),f(t)\big) \leqslant d^*\big(f(s),f(s_k)\big) + d^*\big(f(t),f(s_k)\big) < \varepsilon$ ■