Abel summation theorem

Theorem (26.6)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of convergence $R > 0$. If the series converges at R $(-R)$, then f is continuous at R (resp. $-R$).

Example: $1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln 2$.

Indeed: consider $g(t) = \frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$ (rad. of conv. $= 1$). Let $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$ ("formal" term by term integral; rad. of conv. $= 1$). The series diverges at -1, but converges at 1 (see Lecture 16).

For $|x| < 1$, $\int_0^x g(t)\,dt = \ln(1+x)$
$= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n\,dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = f(x)$.

f is continuous at 1, hence
$f(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \lim_{x \to 1} f(x) = \lim_{x \to 1} \ln(1+x) = \ln 2$. ∎

Abel Theorem: another example

Proposition (Alternating series)

Suppose $a_1 \geqslant a_2 \geqslant \ldots \geqslant 0$. *Then* $\sum_{k=1}^{\infty}(-1)^{k-1}a_k = a_1 - a_2 + a_3 - \ldots$ *converges iff* $\lim_k a_k = 0$.

Sketch of a proof. To show that $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ converges if $\lim_k a_k = 0$, emulate Lecture 16 (where we had $a_k = \frac{1}{k}$). ∎

Example: $1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}$.

Let $f(x) = \arctan x$. We know: $f'(x) = \frac{1}{1+x^2}$ $(x \in \mathbb{R})$. For $|x| < 1$, $f'(x) = 1 - x^2 + x^4 - \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ (radius of convergence $= 1$). For $|x| < 1$, $f(x) = \int_0^x f'(t)\,dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$.

Series converges when $x = 1$ (it's alternating). f is cont. at 1, so $\frac{\pi}{4} = \arctan 1 = \lim_{x \to 1} \arctan x = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$.

Proof of Abel's Theorem

Theorem (26.6)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of convergence $R > 0$. If the series converges at R $(-R)$, then f is continuous at R (resp. $-R$).

Lemma

If the radius of convergence of $g(x) = \sum_{n=0}^{\infty} a_n x^n$ is 1, and $\sum_{n=0}^{\infty} a_n$ converges, then g is continuous at 1.

Lemma $\Rightarrow$ Theorem. Suppose $f(R) = \sum_n a_n R^n$ converges. Consider $g(t) = f(Rt) = \sum_{n=0}^{\infty} a_n R^n t^n$ – so $f(x) = g\left(\frac{x}{R}\right)$. The series for g has rad. of conv. 1, $g(1)$ exists.

By Lemma, g is continuous at 1, hence f is continuous at R.

For $-R$, consider $g(t) = f(-Rt)$, with the same effect. ■

Proof of Lemma

Lemma. If the radius of convergence of $g(x) = \sum_{n=0}^{\infty} a_n x^n$ is 1, and $\sum_{n=0}^{\infty} a_n$ converges, then g is continuous at 1.

Replacing a_0 with $-\sum_{n=1}^{\infty} a_n$ if necessary, we can assume $g(1) = \sum_{n=0}^{\infty} a_n = 0$.

Consider the partial sums $g_i(x) = \sum_{n=0}^{i} a_n x^n$, $s_i = g_i(1) = \sum_{n=0}^{i} a_n$.
Know: $s_i \to 0$. Shall show: the sequence (g_i) is uniformly Cauchy on $[0,1]$ – that is, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\sup_{x\in[0,1]} \left|g_j(x) - g_{i-1}(x)\right| \leqslant \varepsilon$ whenever $j \geqslant i > N$. Any uniformly Cauchy sequence must converge uniformly.
$g_i \to g$ pointwise on $[0,1]$, hence uniformly.

$g_j(x) - g_{i-1}(x) = \sum_{n=i}^{j} a_n x^n = \sum_{n=i}^{j} (s_n - s_{n-1}) x^n$
$= \left(s_j x^j - s_{i-1} x^i\right) + \sum_{k=i}^{j-1} s_k \left(x^k - x^{k+1}\right)$
$= \left(s_j x^j - s_{i-1} x^i\right) + (1-x) \sum_{k=i}^{j-1} s_k x^k.$

Proof of Lemma, continued

Lemma. If the radius of convergence of $g(x) = \sum_{n=0}^{\infty} a_n x^n$ is 1, and $\sum_{n=0}^{\infty} a_n$ converges, then g is continuous at 1.

From previous slide: assume $g(1) = \sum_{n=0}^{\infty} a_n = 0$.
Let $g_i(x) = \sum_{n=0}^{i} a_n x^n$, $s_i = g_i(1) = \sum_{n=0}^{i} a_n$, then $s_i \to 0$.

Shall show: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\left|g_j(x) - g_i(x)\right| \leqslant \varepsilon$ whenever $j \geqslant i > N$, $x \in [0, 1]$.

Find N s.t. $|s_k| < \frac{\varepsilon}{3}$ for $k \geqslant N$. We have $\left|g_j(x) - g_{i-1}(x)\right| \leqslant |s_j x^j| + |s_{i-1} x^i| + (1-x) \sum_{k=i}^{j-1} |s_k| x^k$. For $j \geqslant i > N$, $|s_j x^j|, |s_{i-1} x^i| < \frac{\varepsilon}{3}$. Further, $(1-x) \sum_{k=i}^{j-1} x^{k-1} = (1-x) \cdot \frac{x^i - x^j}{1-x} \leqslant 1$, hence $(1-x) \sum_{k=i}^{j-1} |s_k| x^{k-1} < \frac{\varepsilon}{3}$. Thus, $\left|g_j(x) - g_{i-1}(x)\right| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$. ∎

Remark. We have actually shown that the power series for g converges uniformly on on $[0, 1]$. Consequently, if $\sum_{n=0}^{\infty} a_n x^n$ converges at R, then the convergence is uniform on $[0, R]$.

Convex functions (not in textbook)

Definition

A continuous function f on an interval I is called convex (concave) if $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$ $\left(f\left(\frac{x+y}{2}\right) \geqslant \frac{f(x)+f(y)}{2}\right)$ $\forall x, y \in I$.

Convex = concave up; concave = concave down.

Proposition

If f is convex (concave), then, for $x, y \in I$ and $t \in (0,1)$, we have
$f\big((1-t)x + ty\big) \leqslant (1-t)f(x) + tf(y)$
(resp. $f\big((1-t)x + ty\big) \geqslant (1-t)f(x) + tf(y)$*).*

Properties of convex functions (proof is optional)

Proposition

If f is convex, then, for $x, y \in I$ and $t \in (0,1)$, we have $f\big((1-t)x + ty\big) \leqslant (1-t)f(x) + tf(y)$.

Proof (omitted in class). By the continuity of f, enough to show that $f\big((1-t)x + ty\big) \leqslant (1-t)f(x) + tf(y)$ if $t = \frac{m}{2^n}$, $m, n \in \mathbb{N}$, $0 \leqslant m \leqslant 2^n$.

Notation: $t_{m,n} = \frac{m}{2^n}$, $x_{m,n} = (1 - t_{m,n})x + t_{m,n}y$.

Use induction on n.

Base case: $n = 1$. $m = 0$: $t_{01} = 0$, $x_{01} = x$, so
$(1-0)f(x) + 0 \cdot f(y) \leqslant f\big((1-0)x + 0 \cdot y\big)$ trivially holds.
$m = 2$: similar situation.
If $m = 1$, then $t_{11} = \frac{1}{2}$, use the definition of convexity.

Inductive step: next slide.

Convex functions: proof continues

Inductive step: suppose we have established that
$f(x_{m,n}) \leqslant (1 - t_{m,n})f(x) + t_{m,n}f(y)$ for $0 \leqslant m \leqslant 2^n$. Show that
$f(x_{m,n+1}) \leqslant (1 - t_{m,n+1})f(x) + t_{m,n+1}f(y)$ for $0 \leqslant m \leqslant 2^{n+1}$.
If m is even, then $x_{m,n+1} = x_{m/2,n}$ and $t_{m,n+1} = t_{m/2,n}$, and we are done.
If m is odd, write $m = 2k + 1$, then $t_{m,n+1} = \frac{t_{k,n}+t_{k,n+1}}{2}$. Convexity of f:
$x_{m,n+1} = \frac{x_{k,n}+x_{k+1,n}}{2}$, hence $f(x_{m,n+1}) \leqslant \frac{f(x_{k,n})+f(x_{k+1,n})}{2}$.
Induction hypothesis: $f(x_{k,n}) \leqslant (1 - t_{k,n})f(x) + t_{k,n}f(y)$,
$f(x_{k+1,n}) \leqslant (1 - t_{k+1,n})f(x) + t_{k+1,n}f(y)$, hence $f(x_{m,n+1}) \leqslant$
$\left(1 - \frac{t_{k,n}+t_{k+1,n}}{2}\right)f(x) + \frac{t_{k,n}+t_{k+1,n}}{2}f(y) = (1 - t_{m,n+1})f(x) + t_{m,n+1}f(y)$. ■

Jensen's Inequality (not in textbook)

Theorem (Jensen)

If f is a convex function on an interval I, $x_1, \ldots, x_n \in I$, $t_1, \ldots, t_n \geqslant 0$, $\sum_{i=1}^n t_i = 1$, then $f\left(\sum_{i=1}^n t_i x_i\right) \leqslant \sum_{i=1}^n t_i f(x_i)$. For concave functions, the inequality is reversed.

Proof in the convex case. Use induction on n. Base case $(n = 2)$ has been established.

Induction step: suppose $f\left(\sum_{i=1}^n t_i x_i\right) \leqslant \sum_{i=1}^n t_i f(x_i)$ (for all appropriate (t_i) and (x_i)), and show that $f\left(\sum_{i=1}^{n+1} t_i x_i\right) \leqslant \sum_{i=1}^{n+1} t_i f(x_i)$.
For $1 \leqslant i \leqslant n$ let $s_i = \frac{t_i}{1-t_{n+1}}$, then $\sum_{i=1}^n s_i = 1$. Also let $x = \sum_{i=1}^n s_i x_i$.
$f(x) \leqslant \sum_{i=1}^n s_i f(x_i)$. $\sum_{i=1}^{n+1} t_i x_i = (1 - t_{n+1})x + t_{n+1}x_{n+1}$, so
$f\left(\sum_{i=1}^{n+1} t_i x_i\right) \leqslant (1 - t_{n+1})f(x) + t_{n+1}f(x_{n+1}) \leqslant$
$(1 - t_{n+1})\sum_{i=1}^n s_i f(x_i) + t_{n+1}f(x_{n+1}) = \sum_{i=1}^{n+1} t_i f(x_i)$. ■

Which functions are convex?

Proposition

If f is differentiable on an interval I, and f' is increasing (decreasing), then f is convex (resp. concave).

Proof in the convex case. Suppose $x_1, x_2 \in I$, $x_1 < x_2$. Let $x = \frac{x_1+x_2}{2}$, and $y = \frac{x_2-x_1}{2}$. By MVT, $\exists$ $z_1 \in (x_1, x)$ and $z_2 \in (x, x_2)$ s.t.
$f'(z_1) = \frac{f(x)-f(x_1)}{y}$, $f'(z_2) = \frac{f(x_2)-f(x)}{y}$. $z_1 < z_2$, hence
$f'(z_1) = \frac{f(x)-f(x_1)}{y} \leqslant f'(z_2) = \frac{f(x_2)-f(x)}{y}$.
$f(x) - f(x_1) \leqslant f(x_2) - f(x)$, hence $f(x) \leqslant \frac{f(x_1)+f(x_2)}{2}$. ■

Criteria for convexity and concavity; examples

Proposition

If f is differentiable on an interval I, and f' is increasing (decreasing), then f is convex (resp. concave).

Corollary

If f is twice differentiable on an interval I, and $f'' \geqslant 0$ ($f'' \leqslant 0$), then f is convex (resp. concave).

Proof in the convex case. If $f'' \geqslant 0$, then f' is increasing, hence f is convex. ∎

Examples. **(1)** $f(x) = e^x$ is convex on $\mathbb{R}$. $f'(x) = e^x$ is increasing.

(2) $g(x) = \ln x$ is concave on $(0, \infty)$. $g'(x) = \frac{1}{x}$ is decreasing.

Arithmetic-Geometric Means Inequality

Proposition

If $x_1, \ldots, x_n > 0$, $t_1, \ldots, t_n > 0$, *and* $\sum_{i=1}^n t_i = 1$, *then* $\sum_{i=1}^n t_i x_i \geqslant \prod_{i=1}^n x_i^{t_i}$.

Proof. $g(x) = \ln(x)$ is concave on $(0, \infty)$, hence $\ln\left(\sum_{i=1}^n t_i x_i\right) \geqslant \sum_{i=1}^n t_i \ln x_i$. Exponentiate both sides. ■

Corollary (Arithmetic-Geometric Means Inequality)

If $x_1, \ldots, x_n > 0$, *then* $\frac{x_1 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 \ldots x_n}$.

Proof. Consider $t_1 = \ldots = t_n = \frac{1}{n}$. ■