MATH 447: Real Variables
Lecture Notes

December 8, 2024

Peano Axioms

Definition 1 (Peano Axioms)**.** The natural numbers $\mathbb{N}$ are defined by the following postulates:

(N1) $\mathbb{N}$ contains a distinguished element 1.

(N2) Every $n \in \mathbb{N}$ has its successor in $\mathbb{N}$, denoted $S(n)$.

(N3) 1 is not the successor of any element in $\mathbb{N}$.

(N4) If m and n have the same successor, then $m = n$.

(N5) If $A \subseteq \mathbb{N}$ such that $1 \in A$ and $S(n) \in A$ whenever $n \in A$, then $A = \mathbb{N}$.

Theorem 1 (Uniqueness of $\mathbb{N}$)**.** *Suppose X is a set with a distinguished element $1'$ and a successor map S', satisfying the Peano Axioms (N1-N5). Then there exists a bijection $\Phi : \mathbb{N} \to X$ such that:*

$$\Phi(1) = 1', \quad \Phi(S(n)) = S'(\Phi(n)) \, \forall n \in \mathbb{N}.$$

Mathematical Induction

Theorem 2 (Principle of Mathematical Induction)**.** *Suppose $(P_n)_{n \in \mathbb{N}}$ is a sequence of statements such that:*

1. *P_1 is true.*
2. *For any $n \in \mathbb{N}$, P_n implies P_{n+1}.*

Then P_n is true for all $n \in \mathbb{N}$.

Example 1 (Induction Proof)**.** *Prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$.*
Proof. *Base case ($n = 1$):*

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}.$$

Inductive step: Assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Then:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

Simplify:

$$\sum_{k=1}^{n+1} k = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

Thus, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ holds for all n.

Properties of Integers

Definition 2 (Addition Properties of $\mathbb{Z}$)**.** The integers $\mathbb{Z}$ satisfy:

(A1) **Associativity:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.

(A2) **Commutativity:** $a + b = b + a$ for all $a, b \in \mathbb{Z}$.

(A3) **Neutral Element:** $\exists 0 \in \mathbb{Z}$ such that $a + 0 = a$.

(A4) **Existence of Opposites:** For every $a \in \mathbb{Z}$, $\exists -a \in \mathbb{Z}$ such that $a + (-a) = 0$.

Theorem 3 (Uniqueness of Additive Elements)**.** *1. The neutral element* 0 *is unique.*

2. For any $a \in \mathbb{Z}$*, the opposite* $-a$ *is unique.*

Proof. 1. Suppose 0 and $0'$ are both neutral elements. Then:

$$0 = 0 + 0' = 0'.$$

2. Suppose $a + x = 0$ and $a + y = 0$. Then:

$$x = x + 0 = x + (a + y) = (x + a) + y = 0 + y = y.$$

Thus, $-a$ is unique. $\square$

Properties of $\mathbb{Z}$ and $\mathbb{Q}$

Properties of Addition on $\mathbb{Z}$

Definition 3 (Addition Properties of $\mathbb{Z}$)**.**

(A1) **Associativity:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.

(A2) **Commutativity:** $a + b = b + a$ for all $a, b \in \mathbb{Z}$.

(A3) **Neutral Element:** $\exists 0 \in \mathbb{Z}$ such that $a + 0 = a$ for all $a \in \mathbb{Z}$.

(A4) **Existence of Opposites:** $\forall a \in \mathbb{Z}, \exists -a \in \mathbb{Z}$ such that $a + (-a) = 0$.

Proposition 1 (Uniqueness of 0 and $-a$)**.** *1. The neutral element* 0 *is unique.*

2. For each $a \in \mathbb{Z}$*, the opposite* $-a$ *is unique.*

Multiplication on $\mathbb{Z}$

Definition 4 (Multiplication Properties of $\mathbb{Z}$)**.**

(M1) **Associativity:** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{Z}$.

(M2) **Commutativity:** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Z}$.

(M3) **Neutral Element:** $\exists 1 \in \mathbb{Z}$ such that $1 \cdot a = a$ for all $a \in \mathbb{Z}$.

(M4) **Distributive Law:** $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in \mathbb{Z}$.

Proposition 2 (Multiplication by Zero)**.** *For any* $a \in \mathbb{Z}$*,* $0 \cdot a = 0$.

Properties of $\mathbb{Q}$

Definition 5 (Field Properties of $\mathbb{Q}$)**.** The rational numbers $\mathbb{Q}$ satisfy:

(M1) **Inverse:** $\forall a \in \mathbb{Q} \setminus \{0\}, \exists a^{-1} \in \mathbb{Q}$ such that $a \cdot a^{-1} = 1$.

If $(X, +, 0, \cdot, 1)$ satisfies (A1–A4), (M1–M4), and the distributive law, X is called a field.

Ordered Fields

Definition 6 (Ordered Fields)**.** A field F is ordered if equipped with a linear order $\leq$ such that:

(O1) If $a \leq b$, then $a + c \leq b + c$ for all $a, b, c \in F$.

(O2) If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

Theorem 4 (Properties of Ordered Fields)**.** *Let F be an ordered field. Then for all $a, b, c \in F$:*

(i) If $a \leq b$, then $-b \leq -a$.

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv) $0 \leq a^2$ for all $a \in F$.

Rational Zeros Theorem

Theorem 5 (Rational Zeros Theorem)**.** *Suppose $p(x) = c_n x^n + \ldots + c_1 x + c_0$, with $c_0, \ldots, c_n \in \mathbb{Z}$, $c_0 \neq 0$, $c_n \neq 0$. If $p(r) = 0$ for $r = \frac{c}{d}$ (where $c, d \in \mathbb{Z}$, $d \neq 0$, $\gcd(c, d) = 1$), then $c \mid c_0$ and $d \mid c_n$.*

Corollary 1 (Irrationality of $\sqrt{2}$)**.** *No rational number r satisfies $r^2 = 2$.*

Ordered Fields and Completeness

Fields and Order

Proposition 3. *If F is a field with more than one element, then $0 \neq 1$.*

Proof. Let $x \in F$ be distinct from 0. Then $0 = x \cdot 0 \neq x \cdot 1 = x$, hence $0 \neq 1$. □

Definition 7 (Ordered Fields)**.** A field F is called ordered if it is equipped with a linear order $\leq$ satisfying:

(O1) If $a \leq b$, then $a + c \leq b + c$ for all $a, b, c \in F$.

(O2) If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

Properties of Ordered Fields

Theorem 6. *Let F be an ordered field. Then for all $a, b, c \in F$:*

(i) If $a \leq b$, then $-b \leq -a$.

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv) $0 \leq a^2$ for all $a \in F$.

Absolute Value and Distance

Definition 8 (Absolute Value)**.** For $a \in F$, the absolute value $|a|$ is defined as:

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Definition 9 (Distance)**.** The distance between $a, b \in F$ is defined as:

$$\operatorname{dist}(a, b) = |a - b|.$$

Completeness Axiom

Definition 10 (Completeness Axiom)**.** If $S \subset \mathbb{R}$ is non-empty and bounded above, then it has a unique least upper bound (supremum), denoted $\sup S$.

Archimedean Property and Denseness of $\mathbb{Q}$

Proposition 4 (Archimedean Property)**.** *If $a, b > 0$ in $\mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n \cdot a > b$.*

Theorem 7 (Denseness of $\mathbb{Q}$)**.** *The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, meaning that for any $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Sequences and Limits (Sections 7-9)

Definitions and Examples

Definition 11 (Sequence)**.** A sequence is a function $s : \{m, m+1, \dots\} \to \mathbb{R}$ (for $m \in \mathbb{Z}$). We denote the sequence as $(s_n)_{n \geq m}$, where $s_n = s(n)$.

Definition 12 (Convergence and Limit)**.** A sequence (s_n) converges to $L \in \mathbb{R}$ if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{R} \text{ such that } |s_n - L| < \varepsilon \text{ for } n > N.$$

We write $\lim_{n \to \infty} s_n = L$ or $s_n \to L$.

Proposition 5 (Uniqueness of Limits)**.** *A sequence cannot have more than one limit.*

Examples of Limits

1. $\lim_{n \to \infty} \frac{1}{n} = 0$.
2. The sequence $((-1)^n)_{n \in \mathbb{N}}$ does not converge.
3. $\lim_{n \to \infty} \frac{3n+1}{2n+1} = \frac{3}{2}$.

Facts about Limits

Proposition 6. *If (s_n) converges, and $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.*

Proposition 7. *If $s_n \geq 0$ for all n and $\lim s_n = s$, then $\lim \sqrt{s_n} = \sqrt{s}$.*

Convergent Sequences are Bounded

Definition 13 (Bounded Sequence)**.** A sequence (s_n) is called bounded if $\exists A \in \mathbb{R}$ such that $|s_n| \leq A$ for all n.

Theorem 8. *Convergent sequences are bounded.*

Arithmetic of Limits

Theorem 9 (Limits of Sums, Products, and Ratios)**.** *Suppose $\lim s_n = s$ and $\lim t_n = t$. Then:*

1. $\lim(s_n + t_n) = s + t$,
2. $\lim(a \cdot s_n) = a \cdot s$ *for any* $a \in \mathbb{R}$,
3. $\lim(s_n \cdot t_n) = s \cdot t$,
4. *If $t \neq 0$, then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$.

Sums, Products, Ratios of Limits (Section 9)

Arithmetic of Limits

Theorem 10 (Arithmetic of Limits)**.** *Suppose $\lim s_n = s$ and $\lim t_n = t$. Then:*

1. $\lim(s_n + t_n) = s + t$,
2. $\lim(a \cdot s_n) = a \cdot s$ *for any* $a \in \mathbb{R}$,
3. $\lim(s_n \cdot t_n) = s \cdot t$,
4. *If $t \neq 0$, then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$.

Theorem 11 (Squeeze Theorem)**.** *If $a_n \leq s_n \leq b_n$ for all n, and $\lim a_n = \lim b_n = s$, then $\lim s_n = s$.*

Basic Examples of Limits

Theorem 12 (Basic Examples)**.**

1. $\lim \frac{1}{n^p} = 0$ *for* $p > 0$*,*
2. $\lim a^n = 0$ *if* $|a| < 1$*,*
3. $\lim n^{1/n} = 1$*,*
4. $\lim a^{1/n} = 1$ *for* $a > 0$*.*

Diverging Sequences

Definition 14 (Divergence to Infinity)**.** We say $\lim s_n = +\infty$ if for all $A > 0$, there exists $N \in \mathbb{R}$ such that $s_n > A$ for $n > N$. Similarly, $\lim s_n = -\infty$ is defined.

Theorem 13 (Product Rule for Divergence)**.** *If* $\lim s_n = +\infty$ *and* $\lim t_n > 0$*, then* $\lim(s_n \cdot t_n) = +\infty$*.*

Theorem 14. *If* $s_n > 0$ *for all* n*, then* $\lim s_n = +\infty$ *if and only if* $\lim \frac{1}{s_n} = 0$*.*

Monotone Sequences (Section 10)

Definition 15 (Monotone Sequences)**.** A sequence (s_n) is:

- *Increasing* if $s_n \leq s_{n+1}$ for all n,
- *Decreasing* if $s_n \geq s_{n+1}$ for all n,
- *Monotone* if it is either increasing or decreasing.

Example 2 (Examples of Monotone Sequences)**.**

1. $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ *is increasing because* $x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n$*.*
2. $y_n = \frac{(-1)^n}{n^2}$ *is not monotone because* $y_{n+1} > y_n$ *if* n *is odd, and* $y_{n+1} < y_n$ *if* n *is even.*

Monotone Sequences and Convergence (Section 10)

Monotone Sequences

Definition 16 (Monotone Sequences)**.** A sequence (s_n) is:

- *Increasing* if $s_n \leq s_{n+1}$ for all n,
- *Decreasing* if $s_n \geq s_{n+1}$ for all n,
- *Monotone* if it is either increasing or decreasing.

Theorem 15 (Theorem 10.2)**.** *Any monotone bounded sequence converges.*

Example 3 (Bounded Monotone Sequence)**.** *Consider* $s_n = \sum_{k=0}^{n-1} \frac{1}{k!}$*. This sequence is:*

- *Increasing, since* $s_{n+1} = s_n + \frac{1}{n!} > s_n$*.*
- *Bounded, since* $s_n \leq 3$ *(using an induction-based proof that* $k! \geq 2^{k-1}$ *for* $k \geq 1$*).*

Thus, $\lim s_n = e \approx 2.71828$*.*

Unbounded Monotone Sequences

Theorem 16 (Theorem 10.4)**.** *If* $(s_n)_{n \geq m}$ *is an unbounded increasing (decreasing) sequence, then* $\lim s_n = +\infty$ *(resp.* $\lim s_n = -\infty$*).*

Example 4 (Harmonic Sequence)**.** *Let* $s_n = \sum_{k=1}^{n} \frac{1}{k}$*. This sequence is:*

- *Increasing, since* $s_{n+1} = s_n + \frac{1}{n+1} > s_n$*.*
- *Unbounded, as shown using a lower bound argument:*

$$s_{2^m} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^{m-1}} \geq m.$$

Thus, $s_n \to +\infty$*.*

Lim Sup and Lim Inf

Definition 17 (Lim Sup and Lim Inf)**.** Let $u_N = \sup\{s_n : n > N\}$ and $v_N = \inf\{s_n : n > N\}$. Then:

$$\limsup s_n = \lim_{N\to\infty} u_N, \quad \liminf s_n = \lim_{N\to\infty} v_N.$$

Theorem 17 (Properties of Lim Sup and Lim Inf)**.**

1. $\limsup s_n \geq \liminf s_n$.
2. *If* $\lim s_n$ *exists, then* $\limsup s_n = \lim s_n = \liminf s_n$.
3. *If* $\limsup s_n = \liminf s_n = s$, *then* $\lim s_n = s$.

Example 5 (Oscillating Sequence)**.** *Let* $s_n = \begin{cases} \frac{1}{n}, & n \text{ even} \\ -n, & n \text{ odd} \end{cases}$. *Then:*

- $\limsup s_n = 0$,
- $\liminf s_n = -\infty$.

Decimal Expansions

Theorem 18. *Any real number can be expressed as a decimal expansion* $K.d_1d_2d_3\ldots$, *where* $K \in \{0, 1, 2, \ldots\}$ *and* $d_k \in \{0, \ldots, 9\}$. *For instance:*

$$1 = 1.000\ldots = 0.999\ldots$$

Lim Inf, Lim Sup, and Cauchy Sequences (Sections 10-11)

Lim Inf and Lim Sup

Definition 18 (Lim Sup and Lim Inf)**.** Let $u_N = \sup\{s_n : n > N\}$ and $v_N = \inf\{s_n : n > N\}$. Then:

$$\limsup s_n = \lim_{N\to\infty} u_N, \quad \liminf s_n = \lim_{N\to\infty} v_N.$$

Theorem 19 (Theorem 10.7)**.**

1. *If* $\lim s_n$ *is defined, then* $\liminf s_n = \lim s_n = \limsup s_n$.
2. *If* $\liminf s_n = s = \limsup s_n$, *then* $\lim s_n = s$.

Example 6. *Let* $s_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even,} \\ -n & \text{if } n \text{ is odd.} \end{cases}$ *Then:*

- $\limsup s_n = 0$,
- $\liminf s_n = -\infty$.

Cauchy Sequences

Definition 19 (Cauchy Sequence)**.** A sequence (s_n) is called Cauchy if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |s_n - s_m| < \varepsilon \text{ for } n, m > N.$$

Theorem 20 (Theorem 10.11)**.** *A sequence* (s_n) *converges if and only if it is Cauchy.*

Lemma 1 (Lemma 10.9)**.** *Any convergent sequence is Cauchy.*

Lemma 2 (Lemma 10.10)**.** *Any Cauchy sequence is bounded.*

Subsequences

Definition 20 (Subsequence)**.** A sequence (t_k) is a subsequence of (s_n) if there exists a strictly increasing sequence $n_1 < n_2 < \ldots$ such that $t_k = s_{n_k}$ for any k.

Example 7 (Subsequence Examples)**.**

- $s_n = \frac{1}{n}$: *A subsequence* $t_k = \frac{1}{k^2}$, *where* $n_k = k^2$.
- $s_n = (-1)^n + \frac{1}{n}$: *The sequence diverges, but* $s_{2k} = 1 + \frac{1}{2k}$ *converges.*

Theorem 21 (Subsequential Limits)**.** *Every sequence has a subsequence that converges to a limit.*

Subsequences and Subsequential Limits (Section 11)

Subsequences

Definition 21 (Subsequence)**.** A sequence (t_k) is a subsequence of (s_n) if there exists a strictly increasing sequence $n_1 < n_2 < \dots$ such that $t_k = s_{n_k}$ for any k.

Lemma 3 (Subsequences of Subsequences)**.** *Any subsequence of a subsequence of (s_n) is a subsequence of (s_n).*

Convergence of Subsequences

Theorem 22 (Theorem 11.3)**.** *If $\lim s_n = s$ (finite or $\pm\infty$), then any subsequence (t_k) has the same limit.*

Proof. Let $\lim s_n = s$, and let $t_k = s_{n_k}$. Then for $\varepsilon > 0$, there exists N such that $|s_n - s| < \varepsilon$ for $n > N$. Since $n_k \to \infty$, we can find K such that $n_k > N$ for $k > K$. Thus, $|t_k - s| < \varepsilon$ for $k > K$, implying $\lim t_k = s$. □

Monotone Subsequences

Theorem 23 (Theorem 11.4)**.** *Every sequence has a monotone subsequence.*

Corollary 2 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Example 8 (Divergent Sequence with Convergent Subsequence)**.** *Let $s_n = (-1)^n \left(1 + \frac{1}{n}\right)$. This sequence is bounded but divergent. The subsequence $s_{2k} = 1 + \frac{1}{2k}$ converges to 1.*

Subsequential Limits

Definition 22 (Subsequential Limit)**.** A subsequential limit of (s_n) is any limit of a subsequence, possibly $\pm\infty$.

Theorem 24 (Theorem 11.2)**.** *Suppose (s_n) is a sequence.*

1. *$t \in \mathbb{R}$ is a subsequential limit if and only if $\forall \varepsilon > 0, \{n : |s_n - t| < \varepsilon\}$ is infinite.*
2. *$t = +\infty$ (or $t = -\infty$) is a subsequential limit if (s_n) is not bounded above (or below).*

Lim Inf and Lim Sup as Subsequential Limits

Theorem 25 (Theorem 11.7)**.** *For any sequence (s_n), $\limsup s_n$ and $\liminf s_n$ are limits of monotone subsequences.*

Theorem 26 (Theorem 11.8)**.** *Let S be the set of subsequential limits of (s_n). Then:*

1. *S is non-empty.*
2. $\inf S = \liminf s_n, \quad \sup S = \limsup s_n$.
3. *$\lim s_n$ exists if and only if S consists of a single point, $S = \{\lim s_n\}$.*

Lim Sup, Lim Inf, and Metric Spaces (Sections 11-13)

Subsequential Limits

Definition 23 (Subsequential Limit (Definition 11.6))**.** For a sequence (s_n), a subsequential limit is any limit of a subsequence (in $\mathbb{R} \cup \{\pm\infty\}$).

Theorem 27 (Properties of Subsequential Limits (Theorem 11.2))**.** *Suppose (s_n) is a sequence.*

1. *$t \in \mathbb{R}$ is a subsequential limit if and only if $\forall \varepsilon > 0, \{n : |s_n - t| < \varepsilon\}$ is infinite.*
2. *$t = +\infty$ ($t = -\infty$) is a subsequential limit if (s_n) is not bounded above (resp. below).*

The Set of Subsequential Limits

Theorem 28 (Theorem 11.8)**.** *Suppose (s_n) is a sequence, and S is the set of subsequential limits. Then:*

1. *S is non-empty.*
2. *$\inf S = \liminf s_n$ and $\sup S = \limsup s_n$.*
3. *$\lim s_n$ exists if and only if S consists of a single point. Then $\{\lim s_n\} = S$.*

Corollary 3 (Convergent Sequences)**.** *If $\lim s_n = s$, then $S = \{s\}$, $\limsup s_n = s = \liminf s_n$.*

Lim Sup and Lim Inf Revisited

Theorem 29 (Theorem 12.1)**.** *If* $\lim s_n = s \in (0, \infty)$*, then for any sequence* (t_n)*:*

$$\limsup(s_n t_n) = s \cdot \limsup t_n.$$

Corollary 4 (Corollary 12.3)**.** *If* (s_n) *is a sequence of positive numbers and* $\lim \frac{s_{n+1}}{s_n}$ *exists, then:*

$$\lim s_n^{1/n} \text{ also exists, and } \lim s_n^{1/n} = \lim \frac{s_{n+1}}{s_n}.$$

Example 9.

1. $\lim(n!)^{1/n} = +\infty$.
2. $\lim \frac{1}{n}(n!)^{1/n} = \frac{1}{e}$.

Metric Spaces

Definition 24 (Metric (Definition 13.1))**.** A metric $d : S \times S \to [0, \infty)$ satisfies:

(D1) **Non-degeneracy:** $d(x, y) = 0 \iff x = y$.

(D2) **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in S$.

(D3) **Triangle Inequality:** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in S$.

Example 10 (Metrics)**.**

- *On* $\mathbb{R}$*:* $d(x, y) = |x - y|$.
- *On* $\mathbb{R}^n$*:* $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Convergence in Metric Spaces

Definition 25 (Convergence (Definition 13.2))**.** A sequence $(s_n) \subset S$ converges to $s \in S$ if:

$$\lim_{n\to\infty} d(s_n, s) = 0.$$

Definition 26 (Cauchy Sequence (Definition 13.2))**.** A sequence $(s_n) \subset S$ is Cauchy if:

$$\forall \varepsilon > 0, \exists N \text{ such that } d(s_n, s_m) < \varepsilon \text{ for all } n, m > N.$$

Proposition 8 (Cauchy and Convergence)**.** *If* (s_n) *converges in* (S, d)*, then* (s_n) *is Cauchy.*

Definition 27 (Complete Metric Spaces)**.** A metric space (S, d) is complete if every Cauchy sequence in S converges to a point in S.

Example 11. *The space* $\mathbb{R}^n$ *with the Euclidean metric is complete.*

Metric Spaces (Section 13)

Definition and Examples

Definition 28 (Metric (Definition 13.1))**.** Suppose S is a set. A function $d : S \times S \to [0, \infty)$ is called a metric if the following hold:

(D1) **Non-degeneracy:** $d(x, y) = 0 \iff x = y$ (hence $d(x, y) > 0$ when $x \neq y$).

(D2) **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in S$.

(D3) **Triangle Inequality:** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in S$.

Example 12 (Examples of Metrics)**.**

- *On* $\mathbb{R}$*:* $d(x, y) = |x - y|$.
- *On* $\mathbb{R}^n$*:* $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ *(Euclidean metric).*
- *Discrete Metric: For* $x, y \in S$*, define:*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Convergence and Completeness

Definition 29 (Convergence (Definition 13.2)). Suppose (S, d) is a metric space. A sequence $(s_n) \subset S$ converges to $s \in S$ if $\lim_{n\to\infty} d(s_n, s) = 0$; that is, $\forall \varepsilon > 0, \exists N$ such that $d(s_n, s) < \varepsilon$ for $n > N$.

Definition 30 (Cauchy Sequence (Definition 13.2 continued)). A sequence $(s_n) \subset S$ is called Cauchy if:

$$\forall \varepsilon > 0, \exists N \text{ such that } d(s_n, s_m) < \varepsilon \text{ for } n, m > N.$$

Proposition 9 (Cauchy Sequences are Convergent in Complete Spaces). *If (S, d) is complete, then every Cauchy sequence in S converges to a point in S.*

Example 13 (Completeness of $\mathbb{R}$). *$\mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ is complete. Any Cauchy sequence in $\mathbb{R}$ converges to a real number.*

Example 14 (Non-Completeness of $\mathbb{Q}$). *Consider $\mathbb{Q}$ with $d(x, y) = |x - y|$. The sequence $r_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2})$ is Cauchy in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$ because $\sqrt{2} \notin \mathbb{Q}$.*

Special Metrics

Example 15 (Manhattan (Taxicab) Metric). *For $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the taxicab metric is defined as:*

$$d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|.$$

Proposition 10 (Completeness of $(\mathbb{R}^n, d_1)$). *The metric space $(\mathbb{R}^n, d_1)$ is complete.*

Inner Product and Triangle Inequality

Definition 31 (Inner Product). For $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the inner product:

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i.$$

The magnitude of $\vec{x}$ is:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}.$$

Theorem 30 (Bunyakovsky-Cauchy-Schwarz Inequality). *For all $\vec{x}, \vec{y} \in \mathbb{R}^n$:*

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|.$$

Lemma 4 (Triangle Inequality Lite). *For $\vec{x}, \vec{y} \in \mathbb{R}^n$:*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$

Metric Spaces: Bounded Sets, Open and Closed Sets, and Closure (Section 13)

Bounded Sets

Definition 32 (Bounded Sets). A set E in a metric space (S, d) is bounded if there exists $y \in S$ such that:

$$\sup_{x \in E} d(y, x) < \infty.$$

Remark 1. *If such a y exists, then for any $z \in S$, $\sup_{x \in E} d(z, x) < \infty$. This follows from the triangle inequality:*

$$d(z, x) \leq d(y, x) + d(z, y).$$

Example 16. *A sequence (x_k) is bounded if the set $\{x_1, x_2, \ldots\}$ is bounded. That is, for some (or any) $y \in S$:*

$$\sup_k d(y, x_k) < \infty.$$

Bolzano-Weierstrass Theorem

Theorem 31 (Bolzano-Weierstrass for $\mathbb{R}^n$)**.** *Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Example 17 (Failure of Bolzano-Weierstrass in Discrete Metrics)**.** *Consider $\mathbb{N}$ equipped with the discrete metric $d(x, y) = 1$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$. The sequence $x_n = n$ is bounded but has no convergent subsequences because convergent sequences are eventually constant in discrete metrics.*

Interior Points and Open Sets

Definition 33 (Open Ball)**.** An open ball with center s_0 and radius $r > 0$ is:

$$B_r^o(s_0) = \{s \in S : d(s, s_0) < r\}.$$

Definition 34 (Interior Points)**.** A point $s_0 \in S$ is interior to $E \subset S$ if there exists $r > 0$ such that $B_r^o(s_0) \subset E$. The set of all interior points is denoted E^o, called the interior of E.

Definition 35 (Open Sets)**.** A set $E \subset S$ is open if $E = E^o$.

Example 18.

- *In $S = \mathbb{R}$ with the usual metric, $[0, \infty)$ is not open, but $(0, \infty)$ is.*
- *In $S = \mathbb{R}^2$, the set $E = \{(x, 0) : x \geq 0\}$ has $E^o = \emptyset$.*

Properties of Open and Closed Sets

Theorem 32 (Facts about Open Sets)**.**

1. *S and $\emptyset$ are open.*
2. *A union of any collection of open sets is open.*
3. *A finite intersection of open sets is open.*

Definition 36 (Closed Sets)**.** A set $E \subset S$ is closed if $S \setminus E$ is open.

Theorem 33 (Facts about Closed Sets)**.**

1. *S and $\emptyset$ are closed.*
2. *An intersection of any collection of closed sets is closed.*
3. *A finite union of closed sets is closed.*

Closure and Boundary

Definition 37 (Closure)**.** The closure of $E \subset S$, denoted $\overline{E}$, is the intersection of all closed sets containing E.

Definition 38 (Boundary)**.** The boundary of $E \subset S$ is:

$$\partial E = \overline{E} \setminus E^o.$$

Example 19 (Closure in $\mathbb{R}$)**.** *Let $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$. Then:*

$$\overline{E} = E \cup \{0\}.$$

Closure, Boundary, and Open/Closed Sets (Section 13)

Open Balls and Open Sets

Definition 39 (Open Ball)**.** For $s_0 \in S$ and $r > 0$, the open ball with center s_0 and radius r is:

$$B_r^o(s_0) = \{s \in S : d(s, s_0) < r\}.$$

Proposition 11. *A set $E \subset S$ is open if and only if it is a union of open balls.*

Closed Sets and De Morgan's Laws

Definition 40 (Closed Sets)**.** A set $E \subset S$ is closed if $S \setminus E$ is open.

Proposition 12 (Properties of Closed Sets)**.**

1. *S and $\emptyset$ are closed.*

2. *Any intersection of closed sets is closed.*

3. *A finite union of closed sets is closed.*

Proposition 13 (De Morgan's Laws)**.** *For any collection $\{A_i\}_{i\in I} \subset S$:*

$$S \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I}(S \setminus A_i), \quad S \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I}(S \setminus A_i).$$

Examples of Open and Closed Sets

Example 20 (Intervals in $\mathbb{R}$)**.**

- *(a, b) is open, but not closed.*
- *$[a, b]$ is closed, but not open.*
- *$(a, b], [a, b)$ are neither open nor closed.*

Example 21 (Discrete Metric)**.** *In a discrete metric space:*

- *Every set is both open and closed.*

Closure and Boundary

Definition 41 (Closure)**.** The closure of $E \subset S$, denoted $\overline{E}$, is the intersection of all closed sets containing E.

Definition 42 (Boundary)**.** The boundary of $E \subset S$ is:

$$\partial E = \overline{E} \setminus E^o.$$

Properties of Closure and Boundary

Proposition 14.

1. *$E = \overline{E}$ if and only if E is closed.*

2. *$s \in \overline{E}$ if and only if s is a limit of a sequence in E.*

3. *$\partial E = \overline{E} \cap (S \setminus E)^-$.*

Example 22 (Closure of $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$)**.** *The closure is:*

$$\overline{E} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Compactness (Section 13)

Definition of Compactness

Definition 43 (Compactness (Definition 13.11))**.** Suppose $E \subset S$. A family $\mathcal{U}$ of open sets is an *open cover* for E if:

$$E \subset \bigcup_{U\in\mathcal{U}} U.$$

A *subcover* is a subfamily of $\mathcal{U}$ which is also an open cover. E is called *compact* if every open cover has a finite subcover.

Note 1. *A cover $\mathcal{U}$ is a collection of sets, not their union. Thus, a cover is a subset of $\mathcal{P}(S)$ (the power set of S), not S.*

Examples in $\mathbb{R}$ with Usual Metric

1. $E = [0, \infty)$ is not compact. For example:
   - $U_k = (-1, k)$ $(k \in \mathbb{N})$ is an open cover with no finite subcover.
2. $E = (0, 1)$ is not compact. For example:
   - $U_k = (1/k, 1)$ $(k \in \mathbb{N})$ is an open cover with no finite subcover.
3. $E = [a, b]$ $(a, b \in \mathbb{R})$ is compact (proof to follow).

Proposition 15 (Compactness of Finite Sets)**.** *Any finite set is compact.*

Proof. Let $E = \{e_1, \dots, e_N\}$. For any open cover $\mathcal{U}$ of E, select $U_i \in \mathcal{U}$ containing e_i. Then $\{U_1, \dots, U_N\}$ is a finite subcover. □

Compactness and Boundedness

Proposition 16. *Any compact set is bounded.*

Proof. If E is not bounded, then for $s \in S$, the sets $B_k^o(s)$ $(k \in \mathbb{N})$ form an open cover of E with no finite subcover. □

Example 23 (Bounded but Not Compact)**.** *Equip $\mathbb{N}$ with the discrete metric* $d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$ *Then $\mathbb{N}$ is bounded, but it is not compact because the open cover $U_n = \{n\}$ $(n \in \mathbb{N})$ has no finite subcover.*

Properties of Compact Sets

Proposition 17.

1. *A closed subset of a compact set is compact.*
2. *A finite union of compact sets is compact.*

Nested Sequences of Closed Sets

Proposition 18. *Suppose $F_1 \supset F_2 \supset \cdots$ are closed non-empty subsets of a compact set E. Then:*

$$\bigcap_n F_n \neq \emptyset, \quad \textit{and it is compact.}$$

Heine-Borel Theorem and Cantor Set

Theorem 34 (Heine-Borel Theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 24 (Cantor Set)**.** *Define:*

$$F_0 = [0, 1], \quad F_1 = [0, 1/3] \cup [2/3, 1], \quad F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \dots$$

The Cantor set $C = \bigcap_n F_n$ is non-empty, closed, and compact. It contains no intervals, and its interior is empty.

Compactness and Total Boundedness (Section 13)

Definition of Compactness

Definition 44 (Compactness (Definition 13.11))**.** Suppose $E \subset S$. A family $\mathcal{U}$ of open sets is an *open cover* for E if:

$$E \subset \bigcup_{U \in \mathcal{U}} U.$$

A *subcover* is a subfamily of $\mathcal{U}$ which is also an open cover. E is called *compact* if every open cover has a finite subcover.

Note 2. *A cover $\mathcal{U}$ is a collection of sets, not their union. Thus, a cover is a subset of $\mathcal{P}(S)$ (the power set of S), not S.*

Compactness and Completeness

Proposition 19. *Suppose $E \subset S$. If E is compact, then E is complete.*

Proof. Suppose E is not complete. Then there exists a Cauchy sequence $(s_n) \subset E$ which does not converge in E. For $k \in \mathbb{N}$, find n_k such that $d(s_m, s_\ell) < 2^{-k}$ for $m, \ell \geq n_k$. Construct open sets $U_k = \{s \in S : d(s, s_{n_k}) > 2^{-k}\}$, which form an open cover for E with no finite subcover. Contradiction. □

Corollary 5. *Any compact set is closed.*

Proposition 20 (Compactness Criterion)**.** *$E \subset S$ is compact if and only if it is complete and totally bounded.*

Total Boundedness

Definition 45 (Total Boundedness)**.** A set $E \subset S$ is called totally bounded if $\forall \varepsilon > 0$, there exist $s_1, \ldots, s_n \in S$ such that:

$$E \subset \bigcup_{i=1}^{n} B_\varepsilon^o(s_i).$$

Proposition 21. *A set is totally bounded if and only if any sequence in the set has a Cauchy subsequence.*

Characterization of Compactness

Theorem 35. *For a subset E of a metric space, the following are equivalent:*

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Example 25. *The space $\mathbb{N}$ with the discrete metric is complete and bounded but not compact.*

Heine-Borel Theorem

Theorem 36 (Heine-Borel Theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 26 (Cantor Set)**.** *The Cantor set C, constructed as:*

$$F_0 = [0,1], \quad F_1 = [0,1/3] \cup [2/3,1], \quad F_2 = \cdots,$$

is compact, closed, and totally bounded but has no interior.

A Note on Compactness

Total Boundedness

Definition 46 (Total Boundedness (Definition 1.1))**.** A set $S \subset E$ is called totally bounded if for every $\varepsilon > 0$, there exist $p_1, \ldots, p_n \in E$ such that:

$$S \subset \bigcup_{i=1}^{n} B_\varepsilon^o(p_i).$$

Proposition 22 (Intrinsic Nature of Total Boundedness (Proposition 1.2))**.** *A set $S \subset E$ is totally bounded if and only if for every $\varepsilon > 0$, there exist $q_1, \ldots, q_m \in S$ such that:*

$$S \subset \bigcup_{j=1}^{m} B_\varepsilon^o(q_j).$$

Proposition 23 (Total Boundedness and Cauchy Subsequences (Proposition 1.3))**.** *A set S is totally bounded if and only if any sequence in S has a Cauchy subsequence.*

Corollary 6 (Characterization of Compactness (Corollary 1.4))**.** *A set S is totally bounded and complete if and only if any sequence in S has a subsequence converging to a limit in S.*

Compactness

Theorem 37 (Characterization of Compactness (Theorem 2.1))**.** *For a subset $S \subset E$, the following are equivalent:*

1. *S is compact.*
2. *Any sequence in S has a convergent subsequence.*
3. *S is complete and totally bounded.*

Compactness in $\mathbb{R}^n$

Theorem 38 (Heine-Borel Theorem (Theorem 3.1))**.** *A set $S \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Lemma 5 (Total Boundedness in $\mathbb{R}^n$ (Lemma 3.3))**.** *A set $S \subset \mathbb{R}^n$ is bounded if and only if it is totally bounded.*

Theorem 39 (Bolzano-Weierstrass Theorem (Theorem 3.4))**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Compactness and Series (Sections 13-14)

Total Boundedness

Definition 47 (Total Boundedness)**.** A set $E \subset S$ is totally bounded if:

$$\forall \varepsilon > 0, \exists s_1, \ldots, s_n \in S \text{ such that } E \subset \bigcup_{i=1}^{n} B_\varepsilon^o(s_i).$$

Proposition 24. *A set E is totally bounded if and only if any sequence in E has a Cauchy subsequence.*

Proof. For $(s_i) \subset E$, construct $\{s_{i_k}\}$ with $s_{i_k} \in B^o_{2^{-k}}(x_{kj_k})$. Using the triangle inequality, show (s_{i_k}) is Cauchy. □

Characterization of Compactness

Theorem 40. *For a subset E of a metric space, the following are equivalent:*

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Proof.

- (1) $\implies$ (2): Shown in the last lecture.
- (2) $\implies$ (3): Total boundedness guarantees a Cauchy subsequence, and completeness ensures convergence.
- (3) $\implies$ (1): Contraposition: if E is not compact, construct an open cover with no finite subcover.

□

Compact Subsets of $\mathbb{R}^n$

Theorem 41 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 27 (Non-Compact Set)**.** *In $\mathbb{N}$ with the discrete metric, $\mathbb{N}$ is closed and bounded but not compact.*

Series

Definition 48 (Series and Convergence)**.** The n-th partial sum of a series $\sum_{j=k_0}^{\infty} a_j$ is:

$$s_n = \sum_{j=k_0}^{n} a_j.$$

The series converges if $\lim_{n\to\infty} s_n$ exists, diverges otherwise.

Example 28 (Geometric Series)**.**

$$\sum_{j=0}^{\infty} r^j = \begin{cases} \frac{1}{1-r}, & |r| < 1, \\ \infty, & r \geq 1. \end{cases}$$

Cauchy Criterion for Convergence

Definition 49 (Cauchy Criterion)**.** A series $\sum_j a_j$ satisfies the Cauchy criterion if:

$$\forall \varepsilon > 0, \exists N \text{ such that } \left| \sum_{j=m}^{n} a_j \right| < \varepsilon \text{ for } n \geq m > N.$$

Theorem 42. *A series converges if and only if it satisfies the Cauchy criterion.*

Comparison Test for Convergence

Theorem 43 (Comparison Test)**.**

1. *If* $|b_n| \leq a_n$ *and* $\sum a_n$ *converges, then* $\sum b_n$ *converges.*
2. *If* $0 \leq a_n \leq b_n$ *and* $\sum b_n = \infty$, *then* $\sum a_n = \infty$.

Series and Decimal Expansions (Sections 14 and 16)

Series

Definition 50 (Partial Sums and Convergence of Series)**.** The n-th partial sum of a series $\sum_{j=k_0}^{\infty} a_j$ is:

$$s_n = \sum_{j=k_0}^{n} a_j.$$

The series $\sum_{j=k_0}^{\infty} a_j$ converges if $\lim_{n\to\infty} s_n$ exists. Otherwise, it diverges.

Example 29 (Geometric Series)**.**

$$\sum_{j=0}^{\infty} r^j = \begin{cases} \frac{1}{1-r}, & |r| < 1, \\ \textit{diverges}, & r \geq 1 \textit{ or } r \leq -1. \end{cases}$$

Cauchy Criterion for Convergence

Definition 51 (Cauchy Criterion (Definition 14.3))**.** A series $\sum_j a_j$ satisfies the Cauchy Criterion if:

$$\forall \varepsilon > 0, \exists N \text{ such that } \left| \sum_{j=m}^{n} a_j \right| < \varepsilon \text{ for } n \geq m > N.$$

Theorem 44 (Cauchy Criterion (Theorem 14.4))**.** *A series converges if and only if it satisfies the Cauchy Criterion.*

Properties of Convergence

Corollary 7 (Necessary Condition for Convergence (Corollary 14.5))**.** *If* $\sum_j a_j$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Example 30. *If* $a_n = \frac{1}{n}$, *then* $\lim_{n\to\infty} a_n = 0$, *but* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Tests for Convergence

Theorem 45 (Comparison Test (Theorem 14.6))**.**

1. *If* $0 \leq |b_n| \leq a_n$ *and* $\sum a_n$ *converges, then* $\sum b_n$ *converges.*
2. *If* $0 \leq a_n \leq b_n$ *and* $\sum b_n = \infty$, *then* $\sum a_n = \infty$.

Theorem 46 (Root Test (Theorem 14.9))**.** *For a series* $\sum_n a_n$, *let* $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$. *Then:*

1. *The series converges absolutely if* $\alpha < 1$.
2. *The series diverges if* $\alpha > 1$.
3. *If* $\alpha = 1$, *the test gives no information.*

Theorem 47 (Ratio Test (Theorem 14.8)). *For a series $\sum_n a_n$ of nonzero terms:*

1. *The series converges absolutely if* $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. *The series diverges if* $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$.
3. *If* $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, *the test gives no information.*

Decimal Expansions

Theorem 48 (Decimal Expansions (Theorem 16.2)). *Any real number $x \geq 0$ has at least one decimal expansion:*

$$x = K.d_1d_2d_3\ldots = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j},$$

where $K \in \mathbb{Z}$ and $d_j \in \{0, 1, \ldots, 9\}$.

Theorem 49 (Uniqueness of Decimal Expansions (Theorem 16.3)). *Any $x \geq 0$ has either exactly one decimal expansion or exactly two, one ending in $\ldots d000\ldots$ and the other in $\ldots[d-1]999\ldots$.*

Theorem 50 (Repeating Decimals (Theorem 16.5)). *A real number x is rational if and only if its decimal expansion is repeating.*

Series and Decimal Expansions (Sections 17 and 21)

Root and Ratio Tests for Series Convergence

Theorem 51 (Root Test). *For a series $\sum_n a_n$, let $\alpha = \limsup_{n\to\infty}|a_n|^{1/n}$. Then:*

1. *The series converges absolutely if $\alpha < 1$.*
2. *The series diverges if $\alpha > 1$.*
3. *If $\alpha = 1$, the test gives no information.*

Theorem 52 (Ratio Test (Theorem 14.8)). *For a series $\sum_n a_n$ of nonzero terms:*

1. *The series converges absolutely if* $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. *The series diverges if* $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$.
3. *If* $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, *the test gives no information.*

Example 31.

- *Consider $\sum_{k=1}^{\infty} \frac{k^4}{2^k}$. Using the Root Test:*

  $$a_k^{1/k} = \left(k^{1/k}\right)^4 \frac{1}{2}, \quad \lim_{k\to\infty} a_k^{1/k} = \frac{1}{2} < 1,$$

  so the series converges.
- *The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$. The Root and Ratio Tests are inconclusive for this series.*

Decimal Expansions

Definition 52 (Decimal Expansion). For $x \in [0, \infty)$, the decimal expansion of x is:

$$x = K.d_1d_2d_3\ldots = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j},$$

where $K \in \{0, 1, 2, \ldots\}$ and $d_1, d_2, \ldots \in \{0, 1, \ldots, 9\}$.

Theorem 53 (Existence of Decimal Expansions (Theorem 16.2)). *Any real number $x \geq 0$ has at least one decimal expansion.*

Theorem 54 (Uniqueness of Decimal Expansions (Theorem 16.3)). *Any $x \geq 0$ has either exactly one decimal expansion or exactly two:*

- *One ending in $\ldots d000\ldots$, where $d \in \{1, \ldots, 9\}$,*
- *Another ending in $\ldots(d-1)999\ldots$.*

For example, $\frac{1}{2} = 0.5000\ldots = 0.4999\ldots$.

Repeating Decimal Expansions

Definition 53 (Repeating Decimals (Definition 16.4)). A repeating decimal expansion is one of the form:

$$K.d_1 \ldots d_\ell d_{\ell+1} \ldots d_{\ell+r} = K.d_1 \ldots d_\ell \overline{d_{\ell+1} \ldots d_{\ell+r}},$$

where the sequence $d_{\ell+1} \ldots d_{\ell+r}$ repeats.

Theorem 55 (Repeating Decimals and Rational Numbers (Theorem 16.5)). *A real number x is rational if and only if its decimal expansion is repeating.*

Proof.

- (x is rational $\implies$ repeating): Follows from performing long division.
- (Repeating $\implies$ x is rational): Suppose $x = K.d_1 \ldots d_\ell \overline{d_{\ell+1} \ldots d_{\ell+r}}$. Then:

$$x = K + \sum_{j=1}^{\ell} \frac{d_j}{10^j} + 10^{-\ell} \left(\frac{z}{1 - 10^{-r}} \right),$$

where $z = \sum_{j=1}^{r} d_{\ell+j} 10^{-j} \in \mathbb{Q}$, so $x \in \mathbb{Q}$.

□

Continuity in Metric Spaces (Sections 17 and 21)

Definition of Continuity

Definition 54 (Continuity (Definition 21.1)). Suppose (S, d) and (S^*, d^*) are metric spaces. The function $f : \mathrm{dom}(f) \to S^*$ (with $\mathrm{dom}(f) \subset S$) is continuous at $x \in \mathrm{dom}(f)$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d^*(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta.$$

f is called continuous on $E \subset S$ if it is continuous at every $x \in E$.

Theorem 56 (Sequential Criterion for Continuity (Theorem 17.1 + 17.2)). *$f : S \to S^*$ is continuous at $x \in S$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

Examples of Continuity and Discontinuity

Example 32 (Discontinuous Everywhere). *The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$, defined as:*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

is discontinuous at every $x \in \mathbb{R}$. This is because for any $x \in \mathbb{R}$, one can find sequences $(x_n) \subset \mathbb{Q}$ and $(y_n) \not\subset \mathbb{Q}$ such that $x_n, y_n \to x$, but $f(x_n) \to 1$ and $f(y_n) \to 0$, which do not match $f(x)$.

Example 33 (Continuous at a Single Point). *The modified Dirichlet function $g : \mathbb{R} \to \mathbb{R}$, defined as:*

$$g(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

is continuous only at $x = 0$. At other points, similar reasoning as the Dirichlet function applies.

Example 34 (Continuous on $\mathbb{R} \setminus \mathbb{Q}$). *The Thomae function $h : \mathbb{R} \to \mathbb{R}$, defined as:*

$$h(x) = \begin{cases} \frac{1}{b} & x = \frac{a}{b},\ \gcd(a, b) = 1,\ b > 0,\ x \neq 0, \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

is continuous at $x \notin \mathbb{Q}$ and discontinuous at $x \in \mathbb{Q}$.

Operations on Continuous Functions

Theorem 57. *Suppose f, g are continuous at x_0 in a metric space (S, d). Then the following functions are also continuous at x_0:*

- $|f|$,
- kf *($k \in \mathbb{R}$),*
- $f + g$,
- $f \cdot g$,
- f/g *(if $g(x_0) \neq 0$).*

Proposition 25. *If f, g are continuous at x_0, then $\max(f, g)$ and $\min(f, g)$ are continuous at x_0.*

Composition of Continuous Functions

Theorem 58. *Suppose $(S_1, d_1), (S_2, d_2), (S_3, d_3)$ are metric spaces, and $f : dom(f) \to S_2$, $g : dom(g) \to S_3$ are functions such that f is continuous at x_0, g is continuous at $f(x_0)$, and $x_0 \in dom(f)$. Then $g \circ f$ is continuous at x_0.*

Characterization of Continuity

Theorem 59 (Characterization of Continuity (Theorem 21.3))**.** *Suppose (S, d) and (S^*, d^*) are metric spaces. $f : S \to S^*$ is continuous if and only if $f^{-1}(U)$ is open for every open $U \subset S^*$, where:*

$$f^{-1}(U) = \{s \in S : f(s) \in U\}.$$

Lemma 6. *f is continuous at $s_0 \in S$ if for any open set U containing $f(s_0)$, there exists an open set V containing s_0 such that $f(V) \subset U$.*

Continuity and the Intermediate Value Theorem (Sections 18 and 21)

Another Characterization of Continuity

Theorem 60 (Theorem 21.3)**.** *Suppose (S, d) and (S^*, d^*) are metric spaces. A function $f : S \to S^*$ is continuous if and only if $f^{-1}(U)$ is open for every open $U \subset S^*$. Here:*

$$f^{-1}(U) = \{s \in S : f(s) \in U\}.$$

Lemma 7 (Exercise 21.2)**.** *f is continuous at $s_0 \in S$ if and only if for any open set $U \ni f(s_0)$, there exists an open set $V \ni s_0$ such that $f(V) \subset U$.*

Corollary 8 (Exercise 21.4)**.** *Suppose (S, d) is a metric space. A function $f : S \to \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open whenever $a < b$.*

Continuous Image of a Compact Set

Theorem 61. *If $f : S \to S^*$ is continuous, and $E \subset S$ is compact, then $f(E) \subset S^*$ is compact.*

Corollary 9. *If $f : S \to \mathbb{R}$ is continuous, and $E \subset S$ is compact, then $f(E)$ is bounded. Moreover, f attains its maximum and minimum values, i.e., there exist $x, y \in E$ such that:*

$$f(x) = \sup_{e \in E} f(e), \quad f(y) = \inf_{e \in E} f(e).$$

Intermediate Value Theorem (IVT)

Theorem 62 (Theorem 18.2)**.** *Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is continuous. Then f has the Intermediate Value Property (IVP) on I: if $a, b \in I$ with $a < b$, and y lies between $f(a)$ and $f(b)$, then there exists $x \in (a, b)$ such that $f(x) = y$.*

Corollary 10. *If I is an interval, and $f : I \to \mathbb{R}$ has the IVP, then $f(I)$ is either an interval or a single point.*

Applications of IVT

Proposition 26 (Roots of Polynomials)**.** *Any polynomial of odd degree has at least one real root.*

Proposition 27 (Existence of Fixed Points)**.** *Any continuous function $f : [0,1] \to [0,1]$ has a fixed point, i.e., $x \in [0,1]$ such that $f(x) = x$.*

Proposition 28 (Existence of m-th Roots)**.** *For any $m \in \mathbb{N}$ and $y > 0$, there exists $x > 0$ such that $x^m = y$.*

Continuity of Inverse Functions

Theorem 63 (Theorem 18.4)**.** *Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $J = f(I)$ is an interval, and $f^{-1} : J \to I$ is strictly increasing and continuous.*

Corollary 11. *The function $x \mapsto x^{1/m}$, taking $[0, \infty)$ to itself, is continuous.*

Continuity and Compactness (Section 18)

Continuous Image of a Compact Set

Theorem 64 (Theorem 21.4(i))**.** *Suppose $f : S \to S^*$ is continuous, where (S, d) and (S^*, d^*) are metric spaces, and $E \subset S$ is compact. Then $f(E) \subset S^*$ is compact.*

Proof. Let $(U_i)_{i \in I}$ be an open cover for $f(E)$. Define $V_i = f^{-1}(U_i)$, which are open sets forming a cover for E. By compactness of E, there exist $i_1, \ldots, i_n \in I$ such that $E \subset \bigcup_{k=1}^n V_{i_k}$. It follows that $f(E) \subset \bigcup_{k=1}^n U_{i_k}$, proving compactness of $f(E)$. □

Maximum and Minimum of Continuous Functions

Corollary 12 (Similar to 18.1)**.** *If $f : S \to \mathbb{R}$ is continuous and $E \subset S$ is compact, then $f(E)$ is bounded. Moreover, f attains its maximum and minimum values, i.e., there exist $x, y \in E$ such that:*

$$f(x) = \sup_{e \in E} f(e), \quad f(y) = \inf_{e \in E} f(e).$$

Intermediate Value Property

Definition 55 (IVP)**.** Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is a function. f has the Intermediate Value Property (IVP) on I if for any $a, b \in I$ with $a < b$, and any y between $f(a)$ and $f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.

Theorem 65 (Theorem 18.2)**.** *Any continuous function has the IVP.*

Applications of IVP

Corollary 13 (18.3)**.** *If I is an interval, and $f : I \to \mathbb{R}$ has the IVP, then $f(I)$ is either an interval or a single point.*

Proposition 29 (Roots of Polynomials)**.** *Any polynomial of odd degree has at least one real root.*

Proposition 30 (Existence of Fixed Points)**.** *Any continuous function $f : [0,1] \to [0,1]$ has a fixed point, i.e., a point $x \in [0,1]$ such that $f(x) = x$.*

Proposition 31 (Existence of m-th Root)**.** *For any $m \in \mathbb{N}$ and $y > 0$, there exists $x > 0$ such that $x^m = y$.*

Continuity of Inverse Functions

Theorem 66 (Theorem 18.4)**.** *Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $f(I)$ is an interval, and $f^{-1} : f(I) \to I$ is strictly increasing and continuous.*

Corollary 14. *The function $x \mapsto x^{1/m}$, taking $[0, \infty)$ to itself, is continuous.*

Monotonicity of Injective Functions

Theorem 67 (Theorem 18.6)**.** *Suppose $f : I \to \mathbb{R}$ is a continuous one-to-one function on an interval I. Then f is strictly monotone.*

Sketch. For $a, b \in I$ with $a < b$, if $f(a) < f(b)$, then f is strictly increasing. Otherwise, by the IVP, there would exist $x \in (a, b)$ such that $f(x) = f(a)$, contradicting injectivity. □

Uniform Continuity and Lipschitz Functions (Sections 18-19)

Uniform Continuity

Definition 56 (Uniform Continuity (Definition 21.1))**.** Suppose (S, d) and (S^*, d^*) are metric spaces. A function $f : S \to S^*$ is uniformly continuous on $E \subset S$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d^*(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta.$$

Here, δ depends only on ε and not on the specific point x.

Theorem 68 (Sequential Criterion for Uniform Continuity (Theorem 19.4))**.** *If $f : S \to S^*$ is uniformly continuous, then f maps Cauchy sequences in S to Cauchy sequences in S^*.*

Example 35 (Non-Uniformly Continuous Function)**.** *The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$, as the Cauchy sequence $x_n = \frac{1}{n}$ is mapped to $f(x_n) = n$, which is not Cauchy.*

Lipschitz Functions

Definition 57 (Lipschitz Continuity)**.** A function $f : S \to S^*$ is Lipschitz if there exists $K > 0$ (called the Lipschitz constant) such that:

$$d^*(f(s), f(t)) \leq K \cdot d(s, t), \quad \forall s, t \in S.$$

Proposition 32. *Any Lipschitz function is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{K}$. Then if $d(s, t) < \delta$, it follows that:

$$d^*(f(s), f(t)) \leq K \cdot d(s, t) < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

$\square$

Example 36. *For $a > 0$, $f(x) = \frac{1}{x}$ is Lipschitz (hence uniformly continuous) on $[a, \infty)$.*

Example 37 (Uniformly Continuous but Not Lipschitz)**.** *The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz.*

Uniform Continuity on Compact Sets

Theorem 69 (Uniform Continuity on Compact Sets (Theorem 21.4(ii)))**.** *If $f : S \to S^*$ is continuous, and $E \subset S$ is compact, then f is uniformly continuous on E.*

Sketch of Proof. Assume f is not uniformly continuous. Then $\exists \varepsilon > 0$ and sequences $(x_n), (y_n) \subset E$ such that $d(x_n, y_n) \to 0$ but $d^*(f(x_n), f(y_n)) \geq \varepsilon$. By compactness, (x_n) has a subsequence (x_{n_k}) converging to some $x \in E$, and (y_{n_k}) also converges to x. Continuity of f implies $d^*(f(x_{n_k}), f(y_{n_k})) \to 0$, contradicting $d^*(f(x_{n_k}), f(y_{n_k})) \geq \varepsilon$. $\square$

Uniform Continuity and Connectedness (Section 22)

Uniform Continuity on Compact Sets

Theorem 70 (Theorem 21.4(ii))**.** *Suppose (S, d) and (S^*, d^*) are metric spaces, and $f : S \to S^*$ is continuous. If $E \subset S$ is compact, then $f|_E$ is uniformly continuous.*

Proof. For $\varepsilon > 0$, find $\delta > 0$ such that $d^*(f(s), f(t)) < \varepsilon$ whenever $d(s, t) < \delta$. For $s \in S$, find $\delta_s > 0$ such that $d^*(f(s), f(t)) < \varepsilon/2$ whenever $d(s, t) < \delta_s$. Since E is compact:

$$E \subset \bigcup_{s \in E} B^o_{\delta_s/2}(s),$$

there exist $s_1, \ldots, s_n$ such that $E \subset \bigcup_{k=1}^n B^o_{\delta_{s_k}/2}(s_k)$. Define $\delta = \frac{1}{2} \min_{1 \leq k \leq n} \delta_{s_k}$. For $s, t \in E$ with $d(s, t) < \delta$, choose s_k such that $s \in B^o_{\delta_{s_k}/2}(s_k)$. Then:

$$d(t, s_k) \leq d(t, s) + d(s, s_k) < \delta + \frac{\delta_{s_k}}{2} \leq \delta_{s_k},$$

implying $d^*(f(s), f(t)) \leq d^*(f(s), f(s_k)) + d^*(f(t), f(s_k)) < \varepsilon$. $\square$

Extension of Uniformly Continuous Functions

Theorem 71. *Suppose $E \subset S$ is compact, $f : E \to S^*$, and S^* is complete. Then f is uniformly continuous if and only if it extends to a continuous $\tilde{f} : \overline{E} \to S^*$.*

Sketch. If f is uniformly continuous, define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$ for $x \in \overline{E}$, where $x_n \subset E$ and $x_n \to x$. The limit exists by completeness and does not depend on the sequence. Continuity of $\tilde{f}$ follows from the uniform continuity of f. □

Connectedness

Definition 58 (Connected and Disconnected Sets)**.** Suppose (S, d) is a metric space. A set $E \subset S$ is disconnected if there exist open sets $U_1, U_2 \subset S$ such that:

1. $E \subset U_1 \cup U_2$,
2. $(E \cap U_1) \cap (E \cap U_2) = \emptyset$,
3. $E \cap U_1 \neq \emptyset$ and $E \cap U_2 \neq \emptyset$.

A set E is connected if it is not disconnected.

Proposition 33. *E is disconnected if and only if there exist $A, B \subset E$ such that:*

$$E = A \cup B, \quad A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap \overline{B} = \emptyset, \quad \overline{A} \cap B = \emptyset.$$

Connectedness of Intervals

Proposition 34. *Any interval $I \subset \mathbb{R}$ is connected.*

Proof. Suppose, for contradiction, that $I = A \cup B$, where $A, B \neq \emptyset$, $\overline{A} \cap B = \emptyset$, and $A \cap \overline{B} = \emptyset$. Choose $a \in A$, $b \in B$, with $a < b$. Define:

$$c = \sup\{x \in A : x < b\}.$$

Then $c \in I$ and $c < b$. If $c \in A$, there exists $\sigma > 0$ such that $(c - \sigma, c + \sigma) \subset A$, contradicting the definition of c. If $c \in B$, there exists $\sigma > 0$ such that $(c - \sigma, c + \sigma) \subset B$, contradicting $c = \sup\{x \in A : x < b\}$. □

Connectedness and Path Connectedness (Section 22)

Connectedness

Definition 59 (Connected Set (Definition 22.1))**.** Suppose (S, d) is a metric space. A set $E \subset S$ is called disconnected if there exist open sets $U_1, U_2 \subset S$ such that:

1. $E \subset U_1 \cup U_2$,
2. $(E \cap U_1) \cap (E \cap U_2) = \emptyset$,
3. $E \cap U_1 \neq \emptyset$ and $E \cap U_2 \neq \emptyset$.

A set E is connected if it is not disconnected.

Proposition 35. *An open set E is disconnected if and only if $E = E_1 \cup E_2$, where E_1, E_2 are disjoint, non-empty, open subsets.*

Proposition 36 (Equivalent Characterization of Connectedness)**.** *A set E is disconnected if and only if there exist $A, B \subset E$ such that:*

$$E = A \cup B, \quad A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap \overline{B} = \emptyset, \quad \overline{A} \cap B = \emptyset.$$

Continuous Images of Connected Sets

Theorem 72 (Theorem 22.2)**.** *Suppose (S, d) and (S^*, d^*) are metric spaces. If $E \subset S$ is connected and $f : S \to S^*$ is continuous, then $f(E)$ is connected.*

Sketch of Contrapositive Proof. If $f(E) \subset S^*$ is disconnected, write $f(E) = C \cup D$, where C, D are disjoint, non-empty, closed subsets. Define $A = f^{-1}(C) \cap E$, $B = f^{-1}(D) \cap E$. Then $E = A \cup B$, $A \cap \overline{B} = \emptyset$, and $\overline{A} \cap B = \emptyset$, so E is disconnected. □

Path Connectedness

Definition 60 (Path Connectedness (Definition 22.4))**.** A set $E \subset S$ is path connected if for all $a, b \in E$, there exists a continuous function $\gamma : [0,1] \to E$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Theorem 73 (Path Connected Sets Are Connected (Theorem 22.5))**.** *Every path connected set is connected.*

Proof. If E is disconnected, then there exist open sets U_1, U_2 such that $E \subset U_1 \cup U_2$, $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$, and $(E \cap U_1) \cap (E \cap U_2) = \emptyset$. Let $a \in E \cap U_1$, $b \in E \cap U_2$. A path $\gamma : [0,1] \to E$ with $\gamma(0) = a$, $\gamma(1) = b$ would imply $\gamma([0,1])$ is connected, contradicting the disconnectedness of E. □

Connected but Not Path Connected Sets

Example 38. *Consider $E \subset \mathbb{R}^2$, where:*

$$E_1 = \{(0,y) : y \in (0,1]\}, \quad E_2 = \{(x,0) : x \in (0,1]\} \cup \bigcup_{n \in \mathbb{N}} \{(1/n, y) : y \in (0,1]\}.$$

Then $E = E_1 \cup E_2$ is connected but not path connected.

Convex Sets

Definition 61 (Convex Sets)**.** A set $E \subset \mathbb{R}^n$ is convex if for all $\vec{x}, \vec{y} \in E$ and $t \in [0,1]$, the point:

$$\vec{z} = (1-t)\vec{x} + t\vec{y} \in E.$$

Proposition 37. *Any convex set is path connected.*

Path Connectedness of Graphs

Proposition 38. *The graph of a function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is path connected if and only if f is continuous.*

Graphs of Functions and Path Connectedness (Sections 22-23-24)

Graphs and Path Connectedness

Definition 62 (Graph of a Function)**.** The graph of a function $f : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is:

$$G(f) = \{(x, f(x)) : x \in I\}.$$

Proposition 39 (Example 4 from Section 22)**.** *$G(f)$ is path connected if and only if f is continuous on I.*

Example 39 (Discontinuous f with Connected $G(f)$)**.** *Exercise 22.4 describes a function f such that $G(f)$ is connected but f is discontinuous.*

Proposition 40 (Multivariate Continuity)**.** *The function $f : S \to \mathbb{R}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$, is continuous if and only if each $f_i : S \to \mathbb{R}$ is continuous for $1 \leq i \leq n$.*

Sketch. If f is continuous, then $G(f)$ is path connected. For $\vec{x} = (a, f(a))$, $\vec{y} = (b, f(b)) \in G(f)$, define a path:

$$\gamma(t) = ((1-t)a + tb, f((1-t)a + tb)), \quad t \in [0,1].$$

If $G(f)$ is path connected, continuity of f follows from the textbook proof. □

Power Series

Definition 63 (Power Series)**.** A power series is a series of the form:

$$\sum_{n=0}^{\infty} a_n x^n,$$

where x is a variable.

Theorem 74 (Radius of Convergence)**.** *Let $\beta = \limsup|a_n|^{1/n}$ and $R = 1/\beta$. The series:*

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $|x| < R$, diverges for $|x| > R$. R is called the radius of convergence.

Remark 2. *If $\lim|a_{n+1}/a_n|$ exists, it equals β. The series may converge or diverge at $\pm R$. The interval of convergence is one of:*

$$(-R, R),\ [-R, R),\ (-R, R],\ or\,[-R, R].$$

Examples of Power Series

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$: $a_n = \frac{1}{n!}$, $\beta = 0$, $R = \infty$. Interval: $(-\infty, \infty)$.
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$
2. $\sum_{n=0}^{\infty} x^n$: $a_n = 1$, $\beta = 1$, $R = 1$. Diverges for $x = \pm 1$. Interval: $(-1, 1)$.
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$
3. $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$: $a_n = \frac{1}{n+1}$, $\beta = 1$, $R = 1$. Diverges at $x = 1$, converges for $x \in [-1, 1)$.
$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = \ln(1-x).$$
4. $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$: $a_n = \frac{1}{(n+1)^2}$, $\beta = 1$, $R = 1$. Converges for $x \in [-1, 1]$.
5. $\sum_{n=0}^{\infty} n!\,x^n$: $a_n = n!$, $\beta = \infty$, $R = 0$. Diverges for all $x \neq 0$.

Uniform Convergence

Definition 64 (Uniform Convergence (Definition 24.1-2))**.** A sequence $f_n \to f$ pointwise on S if:

$$\forall x \in S,\ \forall \varepsilon > 0, \exists N \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N.$$

It converges uniformly if:

$$\forall \varepsilon > 0, \exists N \text{ such that } \sup_{x \in S}|f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N.$$

Theorem 75 (Preservation of Continuity (Theorem 24.3))**.** *If $f_n \to f$ uniformly on S and each f_n is continuous, then f is continuous.*

Example 40. *Consider $f_n(x) = n^2 x^n (1-x)$ on $[0, 1]$. It converges pointwise to $f(x) = 0$, but not uniformly.*

Uniform Convergence and Series of Functions (Sections 24-25)

Uniform Convergence

Definition 65 (Pointwise and Uniform Convergence (24.1-2))**.** Suppose $f, f_1, f_2, \ldots$ are functions $S \to \mathbb{R}$.

- $f_n \to f$ pointwise on S if:
$$\forall x \in S,\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N.$$
- $f_n \to f$ uniformly on S if:
$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N,\ \forall x \in S.$$
Equivalently, $\lim_{n\to\infty} \sup_{x \in S}|f_n(x) - f(x)| = 0$.

Theorem 76 (Preservation of Continuity (24.3))**.** *If $f_n \to f$ uniformly on S, and each f_n is continuous at $x_0 \in S$, then f is continuous at x_0.*

Sketch of Proof. Using an $\varepsilon/3$ argument, fix $\varepsilon > 0$. For $f_n \to f$ uniformly, find n such that $|f_n(x) - f(x)| < \varepsilon/3$. By continuity of f_n, there exists $\delta > 0$ such that $|f_n(x_0) - f_n(x)| < \varepsilon/3$ for $|x - x_0| < \delta$. Combine inequalities to conclude $|f(x_0) - f(x)| < \varepsilon$. □

Uniformly Cauchy Sequences

Definition 66 (Uniformly Cauchy (25.3))**.** A sequence (f_n) of functions $S \to \mathbb{R}$ is uniformly Cauchy if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |f_i(x) - f_j(x)| < \varepsilon \, \forall x \in S \text{ for } i, j \geq N.$$

Equivalently, $\sup_{x \in S} |f_i(x) - f_j(x)| < \varepsilon$ for $i, j \geq N$.

Theorem 77 (Uniformly Cauchy $\iff$ Uniform Convergence (25.4))**.** *A sequence (f_n) is uniformly Cauchy if and only if it converges uniformly to some f.*

Series of Functions

Definition 67 (Convergence of Series)**.** A series $\sum_{n=1}^{\infty} g_n(x)$ converges (uniformly) if the sequence of partial sums $s_k(x) = \sum_{n=1}^{k} g_n(x)$ converges (uniformly).

Theorem 78 (Uniform Convergence Preserves Continuity (25.5))**.** *If $g_n : S \to \mathbb{R}$ are continuous and $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on S, then $\sum_{n=1}^{\infty} g_n(x)$ is continuous.*

Weierstrass M-Test

Theorem 79 (Weierstrass M-Test (25.7))**.** *Suppose $M_1, M_2, \ldots \geq 0$ and $\sum_{k=1}^{\infty} M_k < \infty$. If $|g_k(x)| \leq M_k$ for all $x \in S$ and k, then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on S.*

Corollary 15. *A power series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly (to a continuous function) on $[-b, b]$ if $b < R$, where $R = (\limsup |a_k|^{1/k})^{-1}$.*

Remark 3. *Convergence need not be uniform on $(-R, R)$. For example, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ converges on $(-1, 1)$, but not uniformly because the partial sums are bounded while $\frac{1}{1-x}$ is not.*

Limits and Differentiation (Sections 20, 28-29)

Limits

Definition 68 (Limit (20.1, slightly modified))**.** Suppose $S \subset \mathbb{R}$, $a \in S^-$, $f : S \to \mathbb{R}$, and $L \in \mathbb{R} \cup \{\pm\infty\}$. Then:

$$\lim_{x \to a, S} f = L$$

if $\lim f(x_n) = L$ for any sequence $(x_n) \subset S$ with $\lim x_n = a$. Such sequences (x_n) exist because $a \in S^-$.

Proposition 41 (Connection Between Limits and Continuity)**.** *If $a \in S$, then $f : S \to \mathbb{R}$ is continuous at a if and only if $\lim_{x \to a, S} f = f(a)$.*

Common Set-Ups for Limits

- **Usual Limit**: Let I be an interval, a be interior to I, and $S = I \setminus \{a\}$. Write $\lim_{x \to a} f$ instead of $\lim_{x \to a, S} f$.
- **One-Sided Limit**: For $S = (a, b)$, write $\lim_{x \to a^+} f$ (right-hand limit). Define $\lim_{x \to a^-} f$ similarly.

Useful Theorems About Limits

Theorem 80 (Equivalent Definition of Limits (20.6, Simplified))**.** *Suppose $a \in S^-$. For $f : S \to \mathbb{R}$ and $L \in \mathbb{R}$, the following are equivalent:*

1. $\lim_{x \to a, S} f = L$.
2. *For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in (a - \delta, a + \delta) \cap S \setminus \{a\}$.*

Theorem 81 (Limit Operations (20.4))**.** *Suppose $\lim_{x \to a, S} f_1 = L_1$ and $\lim_{x \to a, S} f_2 = L_2$. Then:*

1. $\lim_{x \to a, S} (f_1 + f_2) = L_1 + L_2$,
2. $\lim_{x \to a, S} (f_1 \cdot f_2) = L_1 \cdot L_2$,
3. *If $L_2 \neq 0$, $\lim_{x \to a, S} \frac{f_1}{f_2} = \frac{L_1}{L_2}$.*

Theorem 82 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ for all $x \in S$, and $\lim_{x \to a, S} f = \lim_{x \to a, S} h = L$, then $\lim_{x \to a, S} g = L$.*

Differentiation

Definition 69 (Derivative (28.1))**.** Suppose I is an open interval, $a \in I$, and $f : I \to \mathbb{R}$. The derivative of f at a is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if the limit exists and is finite.

Rules of Differentiation

Theorem 83 (Product Rule)**.** *If f and g are differentiable at a, then $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*

Theorem 84 (Chain Rule (28.4))**.** *If f is differentiable at a, and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a, with:*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Carathéodory's Theorem

Theorem 85 (Carathéodory (Exercise 28.16))**.** *Suppose I is an interval, $f : I \to \mathbb{R}$. f is differentiable at $a \in I$ if and only if there exists a function $\phi : I \to \mathbb{R}$, continuous at a, such that:*

$$f(x) - f(a) = \phi(x) \cdot (x - a), \quad \forall x \in I,$$

and $\phi(a) = f'(a)$.

Rules of Differentiation and Mean Value Theorem (Sections 28-29)

Definition of Derivative

Definition 70 (Derivative (28.1))**.** Suppose I is an open interval and $a \in I$. A function $f : I \to \mathbb{R}$ is differentiable at a if the derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite.

Rules of Differentiation

Theorem 86 (Product Rule (28.3))**.** *Suppose f and g are differentiable at a. Then fg is differentiable at a, with:*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Corollary 16. *If $f(x) = x^m$ for $m \in \mathbb{N}$, then:*

$$(x^m)' = mx^{m-1}.$$

Theorem 87 (Chain Rule (28.4))**.** *Suppose f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, with:*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Examples of Differentiation

1. If $f(x) = x^n$ for $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.
2. If $f(x) = x^{-n}$ for $n \in \mathbb{N}$, then $f'(x) = -nx^{-n-1}$.
3. For $f(x) = 1/g(x)$, if $g(a) \neq 0$, then:

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}.$$

Criterion for Extrema

Theorem 88 (Extrema Criterion (29.1)). *Suppose f is defined on an open interval I and has a maximum or minimum at $x_0 \in I$. If f is differentiable at x_0, then:*

$$f'(x_0) = 0.$$

Corollary 17. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and attains its maximum or minimum at x_0. Then one of the following holds:*

1. *$x_0 \in \{a, b\}$,*
2. *f is not differentiable at x_0,*
3. *$f'(x_0) = 0$.*

Rolle's Theorem

Theorem 89 (Rolle's Theorem (29.2)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that:*

$$f'(c) = 0.$$

Mean Value Theorem

Theorem 90 (Mean Value Theorem (29.3)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b). Then there exists $c \in (a, b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Examples and Consequences of MVT

Corollary 18 (Constant Function (29.4)). *If f is differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function.*

Corollary 19 (Equality of Derivatives (29.5)). *If f and g are differentiable on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) - g(x) = c$ for some constant c.*

Rolle's Theorem, Mean Value Theorem, and Applications (Section 29)

Rolle's Theorem

Theorem 91 (Rolle's Theorem (29.2)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that:*

$$f'(x) = 0.$$

Proof. The function f attains its maximum and minimum on $[a, b]$. Let $x_0, y_0 \in [a, b]$ such that $f(y_0) \le f(x) \le f(x_0)$ for all $x \in [a, b]$. If $f(y_0) = f(a) = f(b) = f(x_0)$, then f is constant, so $f' = 0$ on (a, b). Otherwise:

- If $f(x_0) > f(a) = f(b)$, then $x_0 \in (a, b)$, and $f'(x_0) = 0$.
- If $f(y_0) < f(a) = f(b)$, then $y_0 \in (a, b)$, and $f'(y_0) = 0$.

□

Mean Value Theorem

Theorem 92 (Mean Value Theorem (29.3)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b). Then there exists $x \in (a, b)$ such that:*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$ and $g(x) = f(x) - L(x)$. The function g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b) = 0$. By Rolle's Theorem, there exists $x \in (a, b)$ such that $g'(x) = 0$. Thus:

$$f'(x) = g'(x) + L'(x) = 0 + \frac{f(b) - f(a)}{b - a}.$$

□

Example 41 (MVT Application). *For $x, y \in \mathbb{R}$, $|\sin x - \sin y| \leq |x - y|$. Apply MVT to $f(t) = \sin t$ on $[x, y]$: $\exists z \in (x, y)$ such that:*

$$\frac{f(x) - f(y)}{x - y} = f'(z) = \cos z.$$

Since $|\cos z| \leq 1$, $\left|\frac{f(x)-f(y)}{x-y}\right| = |\cos z| \leq 1$, hence $|\sin x - \sin y| \leq |x - y|$.

Corollaries of MVT

Corollary 20 (Constant Functions (29.4)). *If f is differentiable on (a, b) and $f' = 0$ on (a, b), then f is constant.*

Proof. If f is not constant, then there exist $x < y$ such that $f(x) \neq f(y)$. By MVT, $\exists z \in (x, y)$ such that:

$$f'(z) = \frac{f(y) - f(x)}{y - x} \neq 0,$$

contradicting $f'(z) = 0$. □

Corollary 21 (Equality of Derivatives (29.5)). *If f, g are differentiable on (a, b) and $f' = g'$ on (a, b), then $\exists c \in \mathbb{R}$ such that $f(x) - g(x) = c$ for all $x \in (a, b)$.*

Proof. Define $h(x) = f(x) - g(x)$. Then $h' = f' - g' = 0$. By Corollary 29.4, h is constant. □

Increasing and Decreasing Functions

Definition 71 (Monotonicity (29.6)). A function f on an interval I is:

- **Increasing** if $f(x_1) \leq f(x_2)$ for $x_1 < x_2$,
- **Strictly increasing** if $f(x_1) < f(x_2)$ for $x_1 < x_2$.

Corollary 22 (Monotonicity and Derivatives (29.7)). *Suppose f is differentiable on (a, b):*

1. *f is increasing if and only if $f' \geq 0$ on (a, b),*
2. *If $f' > 0$ on (a, b), then f is strictly increasing.*

Example 42 (Bernoulli's Inequality). *If $n \in \mathbb{N}$ and $x > -1$, then:*

$$(1 + x)^n \geq 1 + nx.$$

Let $f(x) = (1 + x)^n - (1 + nx)$ and show $f(x) \geq 0$ for $x > -1$. By differentiating $f(x)$, we conclude $f(x)$ is increasing and achieves its minimum at $x = 0$, where $f(0) = 0$.

Differentiating Inverse Functions and Integration (Sections 29, 32)

Differentiating Inverse Functions

Theorem 93 (Derivative of an Inverse Function (29.9)). *Suppose I is an interval, $f : I \to \mathbb{R}$ is a continuous, strictly monotone function. Let $J = f(I)$, and $g = f^{-1} : J \to I$. If f is differentiable at $c \in I$, and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$, and:*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$

Proof Sketch. Using Carathéodory's Theorem:

$$f(x) - f(c) = \phi(x)(x - c), \quad \phi(c) = f'(c),$$

where ϕ is continuous at c. For $y = f(g(y))$, differentiate both sides to find $g'(d) = 1/\phi(g(d))$. □

Derivatives of Rational Powers

Example 43 (Derivative of Rational Powers). *Let $f(x) = x^n$ (strictly increasing on $(0, \infty)$) with $f'(x) = nx^{n-1}$. The inverse function is $g(y) = y^{1/n}$. For $y > 0$:*

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n} y^{1/n-1}.$$

If $n \in \mathbb{Z}$ is odd, extend f, g to $\mathbb{R}$. Then $g'(y) = \frac{1}{n} y^{1/n-1}$ for $y < 0$.

Example 44 (Derivative of $h(x) = x^r$, $r \in \mathbb{Q}$). *Write $r = m/n$, $h(x) = x^{m/n}$. Use the chain rule:*

$$h'(x) = \frac{m}{n} x^{r-1}.$$

Inverse Trigonometric Functions

Example 45 (Arcsine)**.** *For* $f(x) = \sin x$ *on* $[-\pi/2, \pi/2]$*,* $g = \arcsin : [-1,1] \to [-\pi/2, \pi/2]$*. Since* $f'(x) = \cos x$*, for* $y \in (-1,1)$*:*

$$(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}.$$

Example 46 (Arctangent)**.** *For* $f(x) = \tan x$ *on* $(-\pi/2, \pi/2)$*,* $g = \arctan : \mathbb{R} \to (-\pi/2, \pi/2)$*. Since* $f'(x) = 1 + x^2$*, for* $y \in \mathbb{R}$*:*

$$(\arctan y)' = \frac{1}{1+y^2}.$$

Integration: Concepts and Definitions

Definition 72 (Darboux Sums)**.** Let $f : [a,b] \to \mathbb{R}$ be bounded.

- Partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ gives subintervals $[t_{k-1}, t_k]$.
- Lower Darboux sum:
$$L(f,P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}),$$
where $m(f, [t_{k-1}, t_k]) = \inf_{x \in [t_{k-1}, t_k]} f(x)$.
- Upper Darboux sum:
$$U(f,P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}),$$
where $M(f, [t_{k-1}, t_k]) = \sup_{x \in [t_{k-1}, t_k]} f(x)$.

Definition 73 (Integrability)**.** f is integrable on $[a,b]$ if:

$$\sup_P L(f,P) = \inf_P U(f,P),$$

denoted $\int_a^b f(x)dx$.

Examples of Integrability

Example 47 (Constant Function)**.** *If* $f(x) = c$*, then:*

$$\int_a^b f(x)dx = c(b-a).$$

Example 48 (Discontinuous Function)**.** *Let* $g(x) = 1$ *if* $x \in \mathbb{Q}$*,* $g(x) = 0$ *otherwise. Then:*

$$\sup_P L(g,P) = 0, \quad \inf_P U(g,P) = 1.$$

Since $\sup_P L \neq \inf_P U$*,* g *is not integrable.*

Example 49 (Linear Function)**.** *If* $h(x) = x$*, then:*

$$\int_0^b h(x)dx = \frac{b^2}{2}.$$

Darboux Sums, Integrability, and Riemann Integration (Sections 32-33)

Darboux Sums and Integrals

Definition 74 (Darboux Sums)**.** Let $f : [a,b] \to \mathbb{R}$ be bounded. For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$:

- Lower Darboux sum:
$$L(f,P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}),$$
where $m(f, [t_{k-1}, t_k]) = \inf_{x \in [t_{k-1}, t_k]} f(x)$.

- Upper Darboux sum:

$$U(f,P)=\sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k-t_{k-1}),$$

where $M(f,[t_{k-1},t_k])=\sup_{x\in[t_{k-1},t_k]} f(x)$.

Definition 75 (Integrability)**.** The lower Darboux integral is $L(f)=\sup_P L(f,P)$, and the upper Darboux integral is $U(f)=\inf_P U(f,P)$. f is integrable if $L(f)=U(f)$, denoted:

$$\int_a^b f = L(f) = U(f).$$

Theorem 94 (32.4)**.** *If $f:[a,b]\to\mathbb{R}$ is bounded, then $L(f)\le U(f)$.*

Integrals: Example

Example 50 (Linear Function)**.** *Is $h(x)=x$ integrable on $[0,b]$? Compute $\int_0^b h(x)dx$.*

For $P=\{0,\frac{b}{n},\frac{2b}{n},\dots,b\}$:

$$L(h,P_n)=\frac{b^2}{2}\left(1-\frac{1}{n}\right),\quad U(h,P_n)=\frac{b^2}{2}.$$

Thus:

$$L(h)\ge \sup_n L(h,P_n)=\frac{b^2}{2},\quad U(h)\le \lim_n U(h,P_n)=\frac{b^2}{2}.$$

Since $L(h)=U(h)$, $h(x)$ is integrable with:

$$\int_0^b h(x)dx=\frac{b^2}{2}.$$

Criterion for Integrability

Theorem 95 (32.5)**.** *A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if for all $\varepsilon>0$, there exists a partition P such that:*

$$U(f,P)-L(f,P)<\varepsilon.$$

Monotone and Continuous Functions

Theorem 96 (33.1)**.** *Any monotone function on $[a,b]$ is integrable.*

Theorem 97 (33.2)**.** *Any continuous function on $[a,b]$ is integrable.*

Mesh of a Partition

Definition 76 (Mesh (32.6))**.** The mesh of a partition $P=\{t_0,t_1,\dots,t_n\}$ is:

$$\text{mesh}(P)=\max_{1\le k\le n}(t_k-t_{k-1}).$$

Theorem 98 (32.7)**.** *A bounded $f:[a,b]\to\mathbb{R}$ is integrable if and only if for all $\varepsilon>0$, there exists $\delta>0$ such that:*

$$U(f,P)-L(f,P)<\varepsilon \quad \textit{whenever mesh}(P)<\delta.$$

Riemann Integration

Definition 77 (Riemann Integral (32.8))**.** Let $f:[a,b]\to\mathbb{R}$ be bounded. For a partition $P=\{t_0,t_1,\dots,t_n\}$ and $x_k\in[t_{k-1},t_k]$, define the Riemann sum:

$$S=\sum_{k=1}^{n} f(x_k)(t_k-t_{k-1}).$$

f is Riemann integrable if there exists $r\in\mathbb{R}$ such that for all $\varepsilon>0$, there exists $\delta>0$ such that:

$$|S-r|<\varepsilon \quad \text{whenever mesh}(P)<\delta.$$

Theorem 99 (32.9)**.** *A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. In this case:*

$$\int_a^b f = R\int_a^b f.$$

Integrability and Riemann Integration (Sections 32-33)

Monotone and Continuous Functions

Theorem 100 (Integrability Criterion (32.5))**.** *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if:*

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ such that } U(f, P) - L(f, P) < \varepsilon.$$

Theorem 101 (Monotone Functions Are Integrable (33.1))**.** *Any monotone function on $[a, b]$ is integrable.*

Proof. Assume f is increasing. Fix $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that:

$$\frac{(f(b) - f(a))(b - a)}{n} < \varepsilon.$$

Consider the partition P with $t_k = a + kh$ for $0 \leq k \leq n$, where $h = \frac{b-a}{n}$. Then:

$$U(f, P) - L(f, P) = h \sum_{k=1}^{n} (f(t_k) - f(t_{k-1})) = \frac{(f(b) - f(a))(b - a)}{n} < \varepsilon.$$

□

Theorem 102 (Continuous Functions Are Integrable (33.2))**.** *Any continuous function on $[a, b]$ is integrable.*

Proof. Fix $\varepsilon > 0$. By uniform continuity, $\exists \delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta$. Choose $n \in \mathbb{N}$ such that $h = \frac{b-a}{n} < \delta$, and partition P with $t_k = a + kh$. Then:

$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b - a}.$$

Thus:

$$U(f, P) - L(f, P) < hn \cdot \frac{\varepsilon}{b - a} = \varepsilon.$$

□

Mesh of a Partition

Definition 78 (Mesh (32.6))**.** The mesh of a partition $P = \{t_0, t_1, \ldots, t_n\}$ is:

$$\text{mesh}(P) = \max_{1 \leq k \leq n} (t_k - t_{k-1}).$$

Theorem 103 (Integrability and Mesh (32.7))**.** *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if:*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } U(f, P) - L(f, P) < \varepsilon \text{ whenever } \text{mesh}(P) < \delta.$$

Riemann Integration

Definition 79 (Riemann Integral (32.8))**.** Let $f : [a, b] \to \mathbb{R}$ be bounded. For a partition $P = \{t_0, t_1, \ldots, t_n\}$ and $x_k \in [t_{k-1}, t_k]$, define the Riemann sum:

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}).$$

f is Riemann integrable if:

$$\exists r \in \mathbb{R} \text{ such that } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |S - r| < \varepsilon \text{ whenever } \text{mesh}(P) < \delta.$$

Theorem 104 (Equivalence of Riemann and Darboux Integrability (32.9))**.** *A bounded $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. In this case:*

$$R \int_a^b f = \int_a^b f.$$

Properties of Integrable Functions

Proposition 42 (Exercise 32.7)**.** *If f is integrable on $[a, b]$, and $f = g$ except at finitely many points, then g is integrable on $[a, b]$ and:*

$$\int_a^b f = \int_a^b g.$$

Remark 4. *The statement fails if the set of exceptions is countably infinite. For example:*

$$f(x) = 0, \quad g(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

f is integrable, but g is not.

Properties of Integrals and Convergence Theorems (Section 33)

Properties of Integrals

Theorem 105 (Linearity and Comparison of Integrals (33.3, 33.4(i)))**.** *Suppose f, g are integrable on $[a, b]$, and $c \in \mathbb{R}$. Then:*

1. *cf is integrable, and:*
$$\int_a^b cf = c \int_a^b f.$$
2. *$f + g$ is integrable, and:*
$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$
3. *If $f \geq g$, then:*
$$\int_a^b f \geq \int_a^b g.$$

Theorem 106 (Triangle Inequality for Integrals (33.5))**.** *If f is integrable on $[a, b]$, then $|f|$ is integrable, and:*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Integrability of Products and Piecewise Functions

Proposition 43. *If f is integrable on $[a, b]$, then f^2 is integrable.*

Corollary 23. *If f, g are integrable on $[a, b]$, then fg is integrable.*

Proof. Express fg as:

$$fg = \frac{1}{4} \left((f + g)^2 - (f - g)^2 \right).$$

Since $f + g$ and $f - g$ are integrable, their squares are integrable, and hence fg is integrable. □

Theorem 107 (Piecewise Monotone and Continuous Functions (33.8))**.** *Suppose $f : [a, b] \to \mathbb{R}$ is either:*

1. *Piecewise monotone and bounded, or*
2. *Piecewise continuous.*

Then f is integrable.

Proof. Partition $[a, b]$ such that f is monotone or uniformly continuous on each subinterval. On each subinterval, f is integrable. By additivity of the integral, f is integrable on $[a, b]$. □

Convergence and Interchange of Limits and Integrals

Proposition 44. *Suppose (f_n) is a sequence of integrable functions on $[a, b]$ that converges uniformly to f. Then f is integrable, and:*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

Convergence Theorems

Theorem 108 (Bounded Convergence (33.11))**.** *Suppose* (f_n) *are integrable on* $[a, b]$, $|f_n| \leq M$ *for all* n, $f_n \to f$ *pointwise on* $[a, b]$, *and* f *is integrable. Then:*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Theorem 109 (Monotone Convergence (33.12))**.** *Suppose* (f_n) *are integrable on* $[a, b]$, $f_1 \leq f_2 \leq \cdots$, $f_n \to f$ *pointwise on* $[a, b]$, *and* f *is integrable. Then:*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Example 51 (Application of Monotone Convergence)**.** *Let* $f_n(x) = \frac{1}{1+nx^3}$ *on* $[0, 1]$. *Then:*

$$\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx = 0,$$

where $f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \in (0, 1]. \end{cases}$

Fundamental Theorems of Calculus and Change of Variable (Section 34)

Fundamental Theorem of Calculus I

Theorem 110 (Fundamental Theorem of Calculus I (34.1))**.** *Suppose* $g : [a, b] \to \mathbb{R}$ *is continuous, differentiable on* (a, b), *and* g' *is integrable on* $[a, b]$. *Then:*

$$\int_a^b g'(x)\, dx = g(b) - g(a).$$

Example 52. *Compute* $\int_a^b x^n\, dx$. *Use* $g(x) = \frac{x^{n+1}}{n+1}$, *so:*

$$\int_a^b x^n\, dx = \frac{b^{n+1} - a^{n+1}}{n + 1}.$$

Proof. Partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. By the Mean Value Theorem:

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}},$$

for some $x_k \in (t_{k-1}, t_k)$. Then:

$$L(g', P) \leq \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) = g(b) - g(a) \leq U(g', P).$$

Thus $\int_a^b g'(x)\, dx = g(b) - g(a)$. $\square$

Integration by Parts

Theorem 111 (Integration by Parts (34.2))**.** *Suppose* $u, v : [a, b] \to \mathbb{R}$ *are continuous, differentiable on* (a, b), *and* u', v' *are integrable on* $[a, b]$. *Then:*

$$\int_a^b u(x)v'(x)\, dx + \int_a^b u'(x)v(x)\, dx = u(b)v(b) - u(a)v(a).$$

Example 53. *Compute* $\int_0^\pi x \cos x\, dx$. *Let* $u(x) = x$, $v'(x) = \cos x$*:*

$$\int_0^\pi x \cos x\, dx = x \sin x|_0^\pi - \int_0^\pi \sin x\, dx = 0 - (-2) = -2.$$

Fundamental Theorem of Calculus II

Theorem 112 (Fundamental Theorem of Calculus II (34.3))**.** *Suppose $f : [a, b] \to \mathbb{R}$ is integrable. Define:*

$$F(x) = \int_a^x f(t)\, dt.$$

If f is continuous at c, then F is differentiable at c, with $F'(c) = f(c)$.

Example 54. *Let $G(x) = \int_{x^2}^2 \sin(t^2)\, dt$. Then $G'(x) = -2x \sin(x^4)$ by the Chain Rule.*

Proof. Let $F(x) = \int_a^x f(t)\, dt$. Then:

$$F'(c) = \lim_{x \to c} \frac{F(x) - F(c)}{x - c} = \lim_{x \to c} \frac{\int_c^x f(t)\, dt}{x - c}.$$

Since f is continuous at c, $|f(t) - f(c)| \leq \varepsilon$ for $|t - c| < \delta$, and thus:

$$\lim_{x \to c} \frac{\int_c^x f(t)\, dt}{x - c} = f(c).$$

□

Change of Variable in Integrals

Theorem 113 (Change of Variable (34.4))**.** *Suppose $u : J \to I$, u' is continuous, and $f : I \to \mathbb{R}$ is continuous. Then for $a, b \in J$:*

$$\int_a^b f(u(x))u'(x)\, dx = \int_{u(a)}^{u(b)} f(t)\, dt.$$

Example 55. *Compute $\int_1^4 \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$. Let $u(x) = \sqrt{x}$, then $u'(x) = \frac{1}{2\sqrt{x}}$:*

$$\int_1^4 \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx = \int_1^2 2 \sin t\, dt = 2(-\cos t)|_1^2 = 2(\cos 1 - \cos 2).$$

Interchanging Integration, Differentiation, and Power Series (Section 26)

Interchanging Integration with Limits and Sums

Proposition 45 (Exercise 33.9, Lecture 32)**.** *Suppose (f_n) is a sequence of integrable functions on $[a, b]$ converging uniformly to f. Then:*

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f.$$

Corollary 24. *If g_n are integrable on $[a, b]$, and $f = \sum_{n=0}^\infty g_n$ converges uniformly, then f is integrable, and:*

$$\int_a^b f = \sum_{n=0}^\infty \int_a^b g_n.$$

Interchanging Differentiation with Limits and Sums

Example 56. *Let $f_n(x) = \frac{1}{n} \sin(n^2 x)$. Then $f_n \to 0$ uniformly on $\mathbb{R}$. However:*

$$f_n'(x) = n \cos(n^2 x).$$

If $x = \frac{p}{q}\pi$, then $f_n'(x)$ does not converge, even pointwise.

Power Series and Radius of Convergence

Definition 80 (Radius of Convergence)**.** For a power series $\sum_{n=0}^\infty a_n x^n$, let:

$$\beta = \limsup_{n \to \infty} |a_n|^{1/n}.$$

The radius of convergence is $R = \frac{1}{\beta}$.

Theorem 114 (Uniform Convergence (26.1))**.** *The series $\sum_{n=0}^\infty a_n x^n$ converges uniformly on $[-R_1, R_1]$ for $R_1 < R$.*

Corollary 25. *The series $\sum_{n=0}^\infty a_n x^n$ converges to a continuous function on $(-R, R)$.*

Differentiation and Integration of Power Series

Lemma 8 (Differentiation and Integration (26.3)). *If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then:*

1. $\sum_{n=1}^{\infty} n a_n x^{n-1}$ *has radius of convergence R,*
2. $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ *has radius of convergence R.*

Theorem 115 (Integration of Power Series (26.4)). *Suppose $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has radius of convergence R. Then for $|x| < R$:*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Example 57. *For $f(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ ($R = 1$):*

$$-\ln(1-x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x| < 1.$$

Theorem 116 (Differentiation of Power Series (26.5)). *Suppose $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has radius of convergence R. Then for $|t| < R$:*

$$f'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}.$$

Abel's Theorem

Theorem 117 (Abel's Theorem (26.6)). *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. If the series converges at R (or $-R$), then f is continuous at R (or $-R$).*

Abel's Theorem, Convexity, and Inequalities (Section 26)

Abel Summation Theorem

Theorem 118 (Abel's Theorem (26.6)). *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. If the series converges at R (or $-R$), then f is continuous at R (or $-R$.*

Example 58.

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2.$$

Let $g(t) = \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$ ($R = 1$), and $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$. The series diverges at -1 but converges at 1:

$$\ln(1+x) = \int_0^x g(t)\,dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k.$$

Thus:

$$f(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2.$$

Alternating Series

Proposition 46 (Alternating Series Test). *Suppose $a_1 \geq a_2 \geq \cdots \geq 0$. Then:*

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k = a_1 - a_2 + a_3 - \cdots$$

converges if and only if $\lim_{k \to \infty} a_k = 0$.

Example 59.

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

Let $f(x) = \arctan x$, so $f'(x) = \frac{1}{1+x^2}$. For $|x| < 1$:

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad f(x) = \int_0^x f'(t)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

At $x = 1$:

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Convex Functions

Definition 81 (Convexity)**.** A function f on I is convex if:

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}, \quad \forall x, y \in I.$$

Proposition 47. *If f is convex, then:*

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y), \quad \forall t \in (0,1).$$

Criteria for Convexity

Proposition 48. *If f is differentiable on I, and f' is increasing, then f is convex.*

Corollary 26. *If f is twice differentiable on I, and $f'' \ge 0$, then f is convex.*

Example 60.

- *$f(x) = e^x$ is convex on $\mathbb{R}$ because $f''(x) = e^x > 0$.*
- *$g(x) = \ln x$ is concave on $(0, \infty)$ because $g''(x) = -\frac{1}{x^2} < 0$.*

Inequalities

Theorem 119 (Jensen's Inequality)**.** *If f is convex on I, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \ge 0$, and $\sum_{i=1}^n t_i = 1$, then:*

$$f\left(\sum_{i=1}^n t_i x_i\right) \le \sum_{i=1}^n t_i f(x_i).$$

Proposition 49 (Arithmetic-Geometric Means Inequality)**.** *If $x_1, \dots, x_n > 0$, $t_1, \dots, t_n > 0$, and $\sum_{i=1}^n t_i = 1$, then:*

$$\sum_{i=1}^n t_i x_i \ge \prod_{i=1}^n x_i^{t_i}.$$

Corollary 27 (Special Case)**.** *If $x_1, \dots, x_n > 0$, then:*

$$\frac{x_1 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 \cdots x_n}.$$

Convexity, Inequalities, and Nowhere Differentiable Functions (Section 36)

Jensen's Inequality for Convex Functions

Theorem 120 (Jensen's Inequality)**.** *Let f be a convex function on an interval I, and let $x_1, \dots, x_n \in I$ with $t_1, \dots, t_n \ge 0$ and $\sum_{i=1}^n t_i = 1$. Then:*

$$f\left(\sum_{i=1}^n t_i x_i\right) \le \sum_{i=1}^n t_i f(x_i).$$

If f is concave, the inequality is reversed.

Inequalities Between Means

Proposition 50 (Power Mean Inequality)**.** *Suppose $r > 1$ and $x_1, \ldots, x_n \geq 0$. Then:*

$$\frac{x_1 + \cdots + x_n}{n} \leq \left(\frac{x_1^r + \cdots + x_n^r}{n} \right)^{1/r}.$$

If $r = 2$, this gives the inequality between arithmetic and quadratic means:

$$\frac{x_1 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

Proof. On $[0, \infty)$, $f(x) = x^r$ is convex because $f'(x) = rx^{r-1}$ is increasing. Apply Jensen's Inequality with $t_i = \frac{1}{n}$:

$$\left(\frac{x_1 + \cdots + x_n}{n} \right)^r \leq \frac{x_1^r + \cdots + x_n^r}{n}.$$

Taking the r-th root gives the result. □

Arithmetic and Harmonic Means

Proposition 51 (Arithmetic-Harmonic Mean Inequality)**.** *If $x_1, \ldots, x_n > 0$, then:*

$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

Proof. Let $g(x) = \frac{1}{x}$, which is convex on $(0, \infty)$. Let $y_i = \frac{1}{x_i}$. By Jensen's Inequality with $t_i = \frac{1}{n}$:

$$\frac{1}{n} \sum_{i=1}^n g(y_i) = \frac{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}{n} \geq g\left(\frac{1}{n} \sum_{i=1}^n y_i \right) = \frac{n}{x_1 + \cdots + x_n}.$$

□

Nowhere Differentiable Functions

Proposition 52. *There exists a bounded, uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable nowhere.*

Sketch. Construct a 1-periodic function $f(x) = \sum_{k=0}^{\infty} 8^{-k} s(64^k x)$, where $s(x)$ is the sawtooth function:

$$s(x) = \phi(x - \lfloor x \rfloor), \quad \phi(t) = \min\{t, 1 - t\}.$$

1. f is bounded and uniformly continuous by the Weierstrass M-test.
2. For any $x, \delta > 0, A > 0$, there exists y with $|x - y| \leq \delta$ and $|f(x) - f(y)| \geq A|x - y|$, showing f is nowhere differentiable.

□

Infinite Primes and Divergence of Series

Theorem 121. *Let $p_1 < p_2 < \cdots$ be the increasing sequence of prime numbers. Then:*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \quad \textit{diverges.}$$

Proof. Assume $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges. Let $\alpha = \sum_{n=1}^{\infty} \frac{1}{p_{2n}^2} < 1$, and $\beta = \sum_{n=K+1}^{\infty} \frac{1}{p_n} < 1 - \alpha$ for some K. Consider N such that $N > 2K/(1 - \alpha - \beta)$. Counting arguments on $\{1, 2, \ldots, N\}$ lead to a contradiction. □