TAM 470 / CSE 450
Homework 8

Problem 1 (10 points)

Consider the 1D transient heat equation for $T(x,t)$:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

Suppose we wish to approximate the spatial derivative with the fourth-order central difference scheme (assume uniform grid spacing and α is constant throughout the domain):

$$\frac{\partial^2 T}{\partial x^2} = \frac{-T_{j-2} + 16T_{j-1} - 30T_j + 16T_{j+1} - T_{j+2}}{12\Delta x^2} + O(\Delta x^4) \tag{2}$$

Do the following:

(a) (5 pts) Use von Neumann stability analysis to find the maximum allowable time step for stability when using forward Euler for time integration.

(b) (5 pts) Repeat part (a), but instead using modified wavenumber analysis.

Problem 2 (10 points)

The Du Fort-Frenkel scheme for the heat equation is:

$$(1 + 2\gamma)\phi_j^{(n+1)} = (1 - 2\gamma)\phi_j^{(n-1)} + 2\gamma\phi_{j+1}^{(n)} + 2\gamma\phi_{j-1}^{(n)} \tag{3}$$

where $\gamma = \frac{\alpha \Delta t}{\Delta x^2}$. The scheme is unique in that it is explicit, but also unconditionally stable. The catch is that it is also inconsistent: for a given time step, the error actually **increases** when Δx is decreased (you can read more about this scheme in the Moin textbook, Section 5.6, if desired).

(a) (4 pts) Use von Neumann stability analysis to find expressions for the amplification factors σ_1 and σ_2 (there are two for this multistep method) as a function of γ and wavenumber $k\Delta x$ for this scheme.

(b) (6 pts) Show that the scheme is unconditionally stable by plotting the magnitudes of each von Neumann amplification factor vs wavenumber $k\Delta x$ for different values of γ. Make sure you explicitly state/argue why these plots imply unconditional stability of the method.

Problem 3 (10 points)

The 1D convection-diffusion equation (with convection in the $+x$ direction) is

$$\frac{\partial T}{\partial t} + c\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{for } x \in [0, L], \quad t \geq 0 \tag{4}$$

(a) (4 pts) Show that for homogeneous boundary conditions $T(0,t) = T(L,t) = 0$, semi-discretization using second order central difference schemes for the spatial derivatives leads to a system of ODEs of the form $\frac{d\mathbf{T}}{dt} = \mathbf{AT}$, where $\mathbf{T}$ represents internal (non-boundary) nodal temperatures $[T_1, T_2, ..., T_{N-1}]$ and $\mathbf{A}$ is a banded tridiagonal matrix:

$$\mathbf{A} = B\left[\frac{c}{2\Delta x} + \frac{\alpha}{\Delta x^2}, -2\frac{\alpha}{\Delta x^2}, -\frac{c}{2\Delta x} + \frac{\alpha}{\Delta x^2}\right] \tag{5}$$

(b) (3 pts) The eigenvalues λ_j of $\mathbf{A}$ can be shown to be:

$$\lambda_j = -2\frac{\alpha}{\Delta x^2} + 2\sqrt{\left(\frac{\alpha}{\Delta x^2}\right)^2 - \left(\frac{c}{2\Delta x}\right)^2} \cos\frac{\pi j}{N}, \quad j = 1, 2, ..., N-1 \tag{6}$$

For $L = 1$, $N = 50$ ($\Delta x = \frac{L}{N}$), $\alpha = 0.001$, and $c = 0.08$, what is the maximum allowable value of Δt for stability when using the **forward Euler** scheme for time integration?

(c) (3 pts) Based on the eigenvalues λ_j, and assuming that the convection and diffusion constant ranges are $\alpha \geq 0$ and $c \geq 0$, derive a condition that would allow the **leapfrog** time integration scheme to be conditionally stable for this choice of spatial discretization.

Problem 4 (15 points)

(a) (10 pts, PL) Go to PrairieLearn to write a function that solves the convection-diffusion PDE (Equation 4) using forward Euler for time integration for the following initial profile:

$$T(x,0) = \begin{cases} 1 - (10x-1)^2 & \text{for } 0 \leq x \leq 0.2, \\ 0 & \text{for } 0.2 < x \leq 1 \end{cases} \tag{7}$$

The boundary conditions are homogeneous: $T(0,t) = T(1,t) = 0$.

The exact solution to Equation 4 **for pure convection** (i.e. $\alpha = 0$) given the initial profile defined in Equation 7 is

$$T_{exact}(x,t) = \begin{cases} 1 - [10(x-ct)-1]^2 & \text{for } 0 \leq x - ct \leq 0.2, \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(b) (5 pts) Use your code from Problem 4 to produce a plots of the the exact solution with no diffusion (Equation 8) and the numerical solution **with** diffusion (use $\alpha = 0.001$, $N = 50$, $c = 0.08$, and time step defined by $\alpha\frac{\Delta t}{\Delta x^2} = 0.4$). Create the plots for $t = 4$ and $t = 8$, and include both the exact solution (no diffusion) and numerical solution (with diffusion) on the same axes in each figure (**two total plots**).