MATH 447: Real Variables - Homework #9

Jerich Lee

December 22, 2024

Problem 1 (28.6b)**.** Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. See Fig. 19.3.

(b) Is f differentiable at $x = 0$? Justify your answer.

Solution 1. *Proof.* By using Definition 28.1 from Ross of the derivative, we can show that the function $f(x)$ is not differentiable at $x = 0$.

$$\frac{f(t) - f(0)}{t - 0} = \frac{t \sin(\frac{1}{t}) - 0}{t} = \sin\left(\frac{1}{t}\right) \tag{1}$$

$\sin(\frac{1}{t})$ does not tend to any limit as $t \to 0$, so the proof is done. $\square$

Problem 2 (29.3)**.** Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

(a) Show $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.

(b) Show $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.

Solution 2.

1. *Proof.* By MVT, $\exists t$ *s.t.* $\frac{f(2)-f(0)}{2-0} = f'(t)$

$$f'(t) = \frac{f(2) - f(0)}{2 - 0} = \frac{1}{2} \tag{2}$$

$\square$

2. *Proof.* From $(0, 1)$, by MVT, $\exists t_1$ *s.t.* $f'(t) = 1$. From $(1, 2)$, by MVT, $\exists t_2$ *s.t.* $f'(t) = 0$. We know that $1 > \frac{1}{7} > 0$. By IVTD (Intermediate Value property of Derivatives), $\exists t_3 \in (0, 2)$ *s.t.* $f'(t_3) = \frac{1}{7}$. $\square$

Problem 3 (29.10)**.** Let $f(x) = x^2 \sin\left(\frac{1}{x}\right) + \frac{x}{2}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show $f'(0) > 0$; see Exercise 28.4.

(b) Show f is not increasing on any open interval containing 0.

(c) Compare this example with Corollary 29.7(i).

Solution 3. We will appeal to Corollary 29.7 and determine $f'(x) < 0$ for all $x \in (a, b) \cup \{0\}$, which will prove that $f(x)$ is not increasing on any open interval containing 0. Applying Theorem 28.3 from Ross, we get the following:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + \frac{1}{2} \tag{3}$$

Discussion: The idea is to find a value $x \in (a, b)$ containing 0 so the equation above is negative. This involves two cases: $a < x < 0$ and $0 < x < b$. For the first case, we see that:

$$\frac{1}{x} = -\frac{3\pi}{2}n \tag{4}$$

$$x > a = \frac{1}{a} > \frac{1}{x} \tag{5}$$

$$\frac{1}{a} > -\frac{3\pi}{2}n \tag{6}$$

$$n > -\frac{2}{3\pi a} \tag{7}$$

Where $n \in \mathbb{N}$.

Proof. Choose $n > -\frac{2}{3\pi a}$ *s.t.* $n \in \mathbb{N}$. Let $x = -\frac{2}{3\pi n}$. Then, substituting x into Equation 3 gives us the following:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + \frac{1}{2} \tag{8}$$

$$= 2\left(-\frac{3\pi}{2}n\right)(1) - 0 + \frac{1}{2} \tag{9}$$

$$= \frac{1 - 6\pi}{2} < 0 \tag{10}$$

From the above, we were able to find $a < x < 0$ such that $f'(x) < 0$, which disproves that f is increasing from (a, b) containing 0. The second case where $0 < x < b$ is handled similarly. □

Problem 4 (29.12)**.**

(a) Show $x < \tan x$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

(b) Show $\frac{x}{\sin x}$ is a strictly increasing function on $\left(0, \frac{\pi}{2}\right)$.

(c) Show $x \leq \frac{\pi}{2} \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$.

Solution 4.

1. To prove that $x < \tan(x)$ such that $f(x) = \tan(x)$ for all $x \in (0, \frac{\pi}{2})$, it suffices to show that $f'(x) > 1$ for all $x \in (0, \frac{\pi}{2})$

   Proof.

   $$f'(x) > 1 \tag{11}$$

   $$\sec^2(x) - 1 > 0 \tag{12}$$

   $$\underbrace{\frac{1}{\cos^2(x)}}_{<1} > 1 \tag{13}$$

   □

2. To prove that $f(x) = \frac{x}{\sin(x)}$ is strictly increasing on $(0, \frac{\pi}{2})$, it suffices to show that $f'(x) > 0$ for all $x \in (0, \frac{\pi}{2})$

   Proof.

   $$f'(x) = -x(\sin(x))^{-2}\cos(x) + (\sin(x))^{-1} \tag{14}$$

   $$\frac{\sin(x) - x\cos(x)}{\sin^2(x)} > 0 \tag{15}$$

   $$\sin(x) - x\cos(x) > 0 \tag{16}$$

   Equation 16 is always true for $x \in (0, \frac{\pi}{2})$. □

3. To prove that $x \leq \frac{\pi}{2}\sin(x)$ such that $f(x) = \frac{\pi}{2}\sin(x)$ for all $x \in [0, \frac{\pi}{2}]$, it suffices to show that $g(x) = \frac{\pi}{2}\sin(x) - x \geq 0$ on the interval.

   Proof. Let $g(x) = \frac{\pi}{2}\sin(x) - x$. To analyze $g(x)$, we compute its derivative:

   $$g'(x) = \frac{\pi}{2}\cos(x) - 1 \tag{17}$$

   $$\frac{\pi}{2}\cos(x) \geq 1 \tag{18}$$

   $$\cos(x) \geq \frac{2}{\pi}. \tag{19}$$

   The inequality $\cos(x) \geq \frac{2}{\pi}$ implies that $x \leq \arccos\left(\frac{2}{\pi}\right)$, since $\cos(x)$ is decreasing on $[0, \frac{\pi}{2}]$. Thus, $g'(x) \geq 0$ for $x \in [0, \arccos\left(\frac{2}{\pi}\right)]$, and $g'(x) \leq 0$ for $x \in [\arccos\left(\frac{2}{\pi}\right), \frac{\pi}{2}]$. Therefore, $g(x)$ increases on $[0, \arccos\left(\frac{2}{\pi}\right)]$ and decreases on $[\arccos\left(\frac{2}{\pi}\right), \frac{\pi}{2}]$, reaching its maximum at $x = \arccos\left(\frac{2}{\pi}\right)$.

Now, compute $g(x)$ at the boundaries:

$$g(0) = \frac{\pi}{2}\sin(0) - 0 = 0, \tag{20}$$

$$g\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = 0. \tag{21}$$

Since $g(x) \geq 0$ on $[0, \frac{\pi}{2}]$, it follows that $x \leq \frac{\pi}{2}\sin(x)$ for all $x \in [0, \frac{\pi}{2}]$. □

Problem 5 (29.16)**.** Use Theorem **29.9** to obtain the derivative of the inverse $g = \tan^{-1} =$ arctan of f where $f(x) = \tan x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

29.9 Theorem.
Let f be a one-to-one continuous function on an open interval I, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}. \tag{22}$$

Solution 5.

$$f(x) = \tan(x) \tag{23}$$

$$f'(x) = \sec^2(x) \tag{24}$$

$$x = \arctan(y) \tag{25}$$

$$\left(f^{-1}\right)(y) = \frac{1}{\sec^2\left(\arctan(y)\right)} \tag{26}$$

Problem 6 (32.7)**.** Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show g is integrable and

$$\int_a^b f = \int_a^b g. \tag{27}$$

Hint: First reduce to the case where f is the function identically equal to 0.

Solution 6. *Proof.* Let g be a bounded function from $[a, b]$ (we can deduce the boundedness of g from the integrability of f). Let $M = \sup|g(x)|$, and let E be the finite set of points such that $g(x) \neq f(x)$. As E is finite, we are able to cover E with disjoint intervals $[u_j, v_j] \subset [a, b]$, and also make the sum of all disjoint intervals $[u_j, v_j]$ less than ε (arbitrarily small). By removing these intervals from $[a, b]$, we obtain a new set K (this set is compact, as it is bounded and closed). Using Theorem 21.4 in Ross, we can say that g is uniformly continuous on K. This implies the following:

$$s \in K, t \in K, |s - t| < \delta \implies |g(s) - g(t)| \tag{28}$$

We can then create a partition P of $[a, b]$ such that $u_j, v_j \in P$, but $(u_j, v_j) \notin P$. If x_{i-1} is not u_j, then $\Delta x_i < \delta$. We know that g is bounded for all $x \in [a, b]$, so $M_i - m_i \leq 2M$ for all i (this includes the points u_j in the finite set E), and if x_{i-1} is not one of the finite u_j, then $M_i - m_i < \varepsilon$. This implies the following:

$$U(P, f, x) - L(P, f, x) \leq [b - a]\varepsilon + 2M\varepsilon \tag{29}$$

As ε is arbitrary, this proves that g is integrable. □