Peano Axioms

Definition 1 (Peano Axioms)**.** The natural numbers $\mathbb{N}$ are defined by the following postulates:

(N1) $\mathbb{N}$ contains a distinguished element 1.

(N2) Every $n \in \mathbb{N}$ has its successor in $\mathbb{N}$, denoted $S(n)$.

(N3) 1 is not the successor of any element in $\mathbb{N}$.

(N4) If m and n have the same successor, then $m = n$.

(N5) If $A \subseteq \mathbb{N}$ such that $1 \in A$ and $S(n) \in A$ whenever $n \in A$, then $A = \mathbb{N}$.

Theorem 1 (Uniqueness of $\mathbb{N}$)**.** *Suppose X is a set with a distinguished element $1'$ and a successor map S', satisfying the Peano Axioms (N1-N5). Then there exists a bijection $\Phi : \mathbb{N} \to X$ such that:*

$$\Phi(1) = 1', \quad \Phi(S(n)) = S'(\Phi(n)) \, \forall n \in \mathbb{N}.$$

Mathematical Induction

Theorem 2 (Principle of Mathematical Induction)**.** *Suppose $(P_n)_{n\in\mathbb{N}}$ is a sequence of statements such that:*

1. *P_1 is true.*
2. *For any $n \in \mathbb{N}$, P_n implies P_{n+1}.*

Then P_n is true for all $n \in \mathbb{N}$.

Example 1 (Induction Proof)**.** *Prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$.*

Proof. Base case ($n = 1$):

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}. \tag{1}$$

Inductive step: Assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Then:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1). \tag{2}$$

Simplify:

$$\sum_{k=1}^{n+1} k = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}. \tag{3}$$

Thus, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ holds for all n.

Properties of Integers

Definition 2 (Addition Properties of $\mathbb{Z}$)**.** The integers $\mathbb{Z}$ satisfy:

(A1) Associativity: $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.

(A2) Commutativity: $a + b = b + a$ for all $a, b \in \mathbb{Z}$.

(A3) Neutral Element: $\exists 0 \in \mathbb{Z}$ such that $a + 0 = a$.

(A4) Existence of Opposites: For every $a \in \mathbb{Z}$, $\exists -a \in \mathbb{Z}$ such that $a + (-a) = 0$.

Theorem 3 (Uniqueness of Additive Elements)**.** 1. *The neutral element 0 is unique.*

2. *For any $a \in \mathbb{Z}$, the opposite $-a$ is unique.*

Proof. 1. Suppose 0 and $0'$ are both neutral elements. Then:

$$0 = 0 + 0' = 0'. \tag{4}$$

2. Suppose $a + x = 0$ and $a + y = 0$. Then:

$$x = x + 0 = x + (a + y) = (x + a) + y = 0 + y = y. \tag{5}$$

Thus, $-a$ is unique. □

Definition 3 (Addition Properties of $\mathbb{Z}$)**.** (A1) Associativity: $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.

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(A3) Neutral Element: $\exists 0 \in \mathbb{Z}$ such that $a + 0 = a$ for all $a \in \mathbb{Z}$.

(A4) Existence of Opposites: $\forall a \in \mathbb{Z}, \exists -a \in \mathbb{Z}$ such that $a + (-a) = 0$.

Proposition 1 (Uniqueness of 0 and $-a$). *1. The neutral element 0 is unique.*

2. For each $a \in \mathbb{Z}$, the opposite $-a$ is unique.

Definition 4 (Multiplication Properties of $\mathbb{Z}$). (M1) Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{Z}$.

(M2) Commutativity: $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Z}$.

(M3) Neutral Element: $\exists 1 \in \mathbb{Z}$ such that $1 \cdot a = a$ for all $a \in \mathbb{Z}$.

(M4) Distributive Law: $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in \mathbb{Z}$.

Proposition 2 (Multiplication by Zero). *For any $a \in \mathbb{Z}$, $0 \cdot a = 0$.*

Definition 5 (Field Properties of $\mathbb{Q}$). The rational numbers $\mathbb{Q}$ satisfy:

(M1) Inverse: $\forall a \in \mathbb{Q} \setminus \{0\}, \exists a^{-1} \in \mathbb{Q}$ such that $a \cdot a^{-1} = 1$.

If $(X, +, 0, \cdot, 1)$ satisfies (A1–A4), (M1–M4), and the distributive law, X is called a field.

Ordered Fields

Definition 6 (Ordered Fields). A field F is ordered if equipped with a linear order $\leq$ such that:

(O1) If $a \leq b$, then $a + c \leq b + c$ for all $a, b, c \in F$.

(O2) If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

Theorem 4 (Properties of Ordered Fields). *Let F be an ordered field. Then for all $a, b, c \in F$:*

(i) If $a \leq b$, then $-b \leq -a$.

(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv) $0 \leq a^2$ for all $a \in F$.

Rational Zeros Theorem

Theorem 5 (Rational Zeros Theorem). *Suppose $p(x) = c_n x^n + \ldots + c_1 x + c_0$, with $c_0, \ldots, c_n \in \mathbb{Z}$, $c_0 \neq 0$, $c_n \neq 0$. If $p(r) = 0$ for $r = \frac{c}{d}$ (where $c, d \in \mathbb{Z}$, $d \neq 0$, $\gcd(c, d) = 1$), then $c \mid c_0$ and $d \mid c_n$.*

Corollary 1 (Irrationality of $\sqrt{2}$). *No rational number r satisfies $r^2 = 2$.*

Ordered Fields and Completeness
Fields and Order

Proposition 3. *If F is a field with more than one element, then $0 \neq 1$.*

Proof. Let $x \in F$ be distinct from 0. Then $0 = x \cdot 0 \neq x \cdot 1 = x$, hence $0 \neq 1$. □

Definition 7 (Ordered Fields). A field F is called ordered if it is equipped with a linear order $\leq$ satisfying:

(O1) If $a \leq b$, then $a + c \leq b + c$ for all $a, b, c \in F$.

(O2) If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

Theorem 6 (Denseness of $\mathbb{Q}$). *The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, meaning that for any $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Sequences and Limits (Sections 7-9)
Facts about Limits

Proposition 4. *If (s_n) converges, and $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.*

Proposition 5. *If $s_n \geq 0$ for all n and $\lim s_n = s$, then $\lim \sqrt{s_n} = \sqrt{s}$.*

Definition 8 (Bounded Sequence). A sequence (s_n) is called bounded if $\exists A \in \mathbb{R}$ such that $|s_n| \leq A$ for all n.

Theorem 7. *Convergent sequences are bounded.*

Theorem 8 (Arithmetic of Limits). *Suppose $\lim s_n = s$ and $\lim t_n = t$. Then:*

1. $\lim(s_n + t_n) = s + t$,
2. $\lim(a \cdot s_n) = a \cdot s$ *for any* $a \in \mathbb{R}$,
3. $\lim(s_n \cdot t_n) = s \cdot t$,
4. *If* $t \neq 0$, *then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$.

Theorem 9 (Squeeze Theorem). *If* $a_n \leq s_n \leq b_n$ *for all* n, *and* $\lim a_n = \lim b_n = s$, *then* $\lim s_n = s$.

Theorem 10 (Basic Examples).

1. $\lim \frac{1}{n^p} = 0$ *for* $p > 0$,
2. $\lim a^n = 0$ *if* $|a| < 1$,
3. $\lim n^{1/n} = 1$,
4. $\lim a^{1/n} = 1$ *for* $a > 0$.

Definition 9 (Divergence to Infinity). We say $\lim s_n = +\infty$ if for all $A > 0$, there exists $N \in \mathbb{R}$ such that $s_n > A$ for $n > N$. Similarly, $\lim s_n = -\infty$ is defined.

Theorem 11 (Product Rule for Divergence). *If* $\lim s_n = +\infty$ *and* $\lim t_n > 0$, *then* $\lim(s_n \cdot t_n) = +\infty$.

Theorem 12. *If* $s_n > 0$ *for all* n, *then* $\lim s_n = +\infty$ *if and only if* $\lim \frac{1}{s_n} = 0$.

Definition 10 (Monotone Sequences). A sequence (s_n) is:

1. *Increasing* if $s_n \leq s_{n+1}$ for all n,
2. *Decreasing* if $s_n \geq s_{n+1}$ for all n,
3. *Monotone* if it is either increasing or decreasing.

Example 2 (Examples of Monotone Sequences).

1. $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ *is increasing because* $x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n$.
2. $y_n = \frac{(-1)^n}{n^2}$ *is not monotone because* $y_{n+1} > y_n$ *if* n *is odd, and* $y_{n+1} < y_n$ *if* n *is even.*

Definition 11 (Monotone Sequences). A sequence (s_n) is:

1. *Increasing* if $s_n \leq s_{n+1}$ for all n,
2. *Decreasing* if $s_n \geq s_{n+1}$ for all n,
3. *Monotone* if it is either increasing or decreasing.

Theorem 13 (Theorem 10.2). *Any monotone bounded sequence converges.*

Example 3 (Bounded Monotone Sequence). *Consider* $s_n = \sum_{k=0}^{n-1} \frac{1}{k!}$. *This sequence is:*

1. *Increasing, since* $s_{n+1} = s_n + \frac{1}{n!} > s_n$.
2. *Bounded, since* $s_n \leq 3$ *(using an induction-based proof that* $k! \geq 2^{k-1}$ *for* $k \geq 1$*).*

Thus, $\lim s_n = e \approx 2.71828$.

Theorem 14 (Theorem 10.4). *If* $(s_n)_{n \geq m}$ *is an unbounded increasing (decreasing) sequence, then* $\lim s_n = +\infty$ *(resp.* $\lim s_n = -\infty$*).*

Example 4 (Harmonic Sequence). *Let* $s_n = \sum_{k=1}^{n} \frac{1}{k}$. *This sequence is:*

1. *Increasing, since* $s_{n+1} = s_n + \frac{1}{n+1} > s_n$.
2. *Unbounded, as shown using a lower bound argument:*

$$s_{2^m} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^{m-1}} \geq m. \tag{6}$$

Thus, $s_n \to +\infty$.

Definition 12 (Lim Sup and Lim Inf). Let $u_N = \sup\{s_n : n > N\}$ and $v_N = \inf\{s_n : n > N\}$. Then:

$$\limsup s_n = \lim_{N \to \infty} u_N, \quad \liminf s_n = \lim_{N \to \infty} v_N. \tag{7}$$

Theorem 15 (Properties of Lim Sup and Lim Inf).

1. $\limsup s_n \geq \liminf s_n$.
2. *If* $\lim s_n$ *exists, then* $\limsup s_n = \lim s_n = \liminf s_n$.
3. *If* $\limsup s_n = \liminf s_n = s$, *then* $\lim s_n = s$.

Example 5 (Oscillating Sequence). *Let*

$$s_n = \begin{cases} \frac{1}{n}, & n \text{ even} \\ -n, & n \text{ odd} \end{cases}. \tag{8}$$

Then:

1. $\limsup s_n = 0$,
2. $\liminf s_n = -\infty$.

Theorem 16. *Any real number can be expressed as a decimal expansion $K.d_1d_2d_3\ldots$, where $K \in \{0, 1, 2, \ldots\}$ and $d_k \in \{0, \ldots, 9\}$. For instance:*

$$1 = 1.000\ldots = 0.999\ldots \tag{9}$$

Definition 13 (Lim Sup and Lim Inf). Let $u_N = \sup\{s_n : n > N\}$ and $v_N = \inf\{s_n : n > N\}$. Then:

$$\limsup s_n = \lim_{N\to\infty} u_N, \quad \liminf s_n = \lim_{N\to\infty} v_N. \tag{10}$$

Theorem 17 (Theorem 10.7).

1. *If $\lim s_n$ is defined, then $\liminf s_n = \lim s_n = \limsup s_n$.*
2. *If $\liminf s_n = s = \limsup s_n$, then $\lim s_n = s$.*

Example 6. *Let*

$$s_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even,} \\ -n & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

Then:

1. $\limsup s_n = 0$,
2. $\liminf s_n = -\infty$.

Definition 14 (Cauchy Sequence). A sequence (s_n) is called Cauchy if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |s_n - s_m| < \varepsilon \text{ for } n, m > N. \tag{12}$$

Theorem 18 (Theorem 10.11). *A sequence (s_n) converges if and only if it is Cauchy.*

Lemma 1 (Lemma 10.9). *Any convergent sequence is Cauchy.*

Lemma 2 (Lemma 10.10). *Any Cauchy sequence is bounded.*

Definition 15 (Subsequence). A sequence (t_k) is a subsequence of (s_n) if there exists a strictly increasing sequence $n_1 < n_2 < \ldots$ such that $t_k = s_{n_k}$ for any k.

Example 7 (Subsequence Examples).

1. *$s_n = \frac{1}{n}$: A subsequence $t_k = \frac{1}{k^2}$, where $n_k = k^2$.*
2. *$s_n = (-1)^n + \frac{1}{n}$: The sequence diverges, but $s_{2k} = 1 + \frac{1}{2k}$ converges.*

Theorem 19 (Subsequential Limits). *Every sequence has a subsequence that converges to a limit.*

Definition 16 (Subsequence). A sequence (t_k) is a subsequence of (s_n) if there exists a strictly increasing sequence $n_1 < n_2 < \ldots$ such that $t_k = s_{n_k}$ for any k.

Lemma 3 (Subsequences of Subsequences). *Any subsequence of a subsequence of (s_n) is a subsequence of (s_n).*

Theorem 20 (Theorem 11.3). *If $\lim s_n = s$ (finite or $\pm\infty$), then any subsequence (t_k) has the same limit.*

Proof. Let $\lim s_n = s$, and let $t_k = s_{n_k}$. Then for $\varepsilon > 0$, there exists N such that $|s_n - s| < \varepsilon$ for $n > N$. Since $n_k \to \infty$, we can find K such that $n_k > N$ for $k > K$. Thus, $|t_k - s| < \varepsilon$ for $k > K$, implying $\lim t_k = s$. □

Theorem 21 (Theorem 11.4). *Every sequence has a monotone subsequence.*

Corollary 2 (Bolzano-Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

Example 8 (Divergent Sequence with Convergent Subsequence). *Let $s_n = (-1)^n\left(1 + \frac{1}{n}\right)$. This sequence is bounded but divergent. The subsequence $s_{2k} = 1 + \frac{1}{2k}$ converges to 1.*

Definition 17 (Subsequential Limit). A subsequential limit of (s_n) is any limit of a subsequence, possibly $\pm\infty$.

Theorem 22 (Theorem 11.2). *Suppose (s_n) is a sequence.*

1. $t \in \mathbb{R}$ *is a subsequential limit if and only if* $\forall \varepsilon > 0, \{n : |s_n - t| < \varepsilon\}$ *is infinite.*

2. $t = +\infty$ *(or* $t = -\infty$*) is a subsequential limit if* (s_n) *is not bounded above (or below).*

Theorem 23 (Theorem 11.7)**.** *For any sequence* (s_n)*,* $\limsup s_n$ *and* $\liminf s_n$ *are limits of monotone subsequences.*

Theorem 24 (Theorem 11.8)**.** *Let* S *be the set of subsequential limits of* (s_n)*. Then:*

1. S *is non-empty.*

2. $\inf S = \liminf s_n, \quad \sup S = \limsup s_n$.

3. $\lim s_n$ *exists if and only if* S *consists of a single point,* $S = \{\lim s_n\}$.

Definition 18 (Subsequential Limit (Definition 11.6))**.** For a sequence (s_n), a subsequential limit is any limit of a subsequence (in $\mathbb{R} \cup \{\pm\infty\}$).

Theorem 25 (Properties of Subsequential Limits (Theorem 11.2))**.** *Suppose* (s_n) *is a sequence.*

1. $t \in \mathbb{R}$ *is a subsequential limit if and only if* $\forall \varepsilon > 0, \{n : |s_n - t| < \varepsilon\}$ *is infinite.*

2. $t = +\infty$ *(*$t = -\infty$*) is a subsequential limit if* (s_n) *is not bounded above (resp. below).*

Corollary 3 (Convergent Sequences)**.** *If* $\lim s_n = s$*, then* $S = \{s\}$, $\limsup s_n = s = \liminf s_n$.

Theorem 26 (Theorem 12.1)**.** *If* $\lim s_n = s \in (0, \infty)$*, then for any sequence* (t_n)*:*

$$\limsup(s_n t_n) = s \cdot \limsup t_n. \tag{13}$$

Corollary 4 (Corollary 12.3)**.** *If* (s_n) *is a sequence of positive numbers and* $\lim \frac{s_{n+1}}{s_n}$ *exists, then:*

$$\lim s_n^{1/n} \text{ also exists, and } \lim s_n^{1/n} = \lim \frac{s_{n+1}}{s_n}. \tag{14}$$

Example 9. 1. $\lim (n!)^{1/n} = +\infty$.

2. $\lim \frac{1}{n}(n!)^{1/n} = \frac{1}{e}$.

Definition 19 (Metric (Definition 13.1))**.** A metric $d : S \times S \to [0, \infty)$ satisfies:

(D1) Non-degeneracy: $d(x, y) = 0 \iff x = y$.

(D2) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in S$.

(D3) Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in S$.

Example 10 (Metrics)**.** 1. *On* $\mathbb{R}$*:* $d(x, y) = |x - y|$.

2. *On* $\mathbb{R}^n$*:* $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

Definition 20 (Convergence (Definition 13.2))**.** A sequence $(s_n) \subset S$ converges to $s \in S$ if:

$$\lim_{n\to\infty} d(s_n, s) = 0. \tag{15}$$

Definition 21 (Cauchy Sequence (Definition 13.2))**.** A sequence $(s_n) \subset S$ is Cauchy if:

$$\forall \varepsilon > 0, \exists N \text{ such that } d(s_n, s_m) < \varepsilon \text{ for all } n, m > N. \tag{16}$$

Proposition 6 (Cauchy and Convergence)**.** *If* (s_n) *converges in* (S, d)*, then* (s_n) *is Cauchy.*

Definition 22 (Complete Metric Spaces)**.** A metric space (S, d) is complete if every Cauchy sequence in S converges to a point in S.

Example 11. *The space* $\mathbb{R}^n$ *with the Euclidean metric is complete.*

Definition 23 (Metric (Definition 13.1))**.** Suppose S is a set. A function $d : S \times S \to [0, \infty)$ is called a metric if the following hold:

(D1) Non-degeneracy: $d(x, y) = 0 \iff x = y$ (hence $d(x, y) > 0$ when $x \neq y$).

(D2) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in S$.

(D3) Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in S$.

Example 12 (Examples of Metrics)**.** 1. *On* $\mathbb{R}$*:* $d(x, y) = |x - y|$.

2. *On* $\mathbb{R}^n$*:* $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ *(Euclidean metric).*

3. Discrete Metric: For $x, y \in S$, define:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \tag{17}$$

Definition 24 (Convergence (Definition 13.2)). Suppose (S,d) is a metric space. A sequence $(s_n) \subset S$ converges to $s \in S$ if $\lim_{n\to\infty} d(s_n, s) = 0$; that is, $\forall \varepsilon > 0, \exists N$ such that $d(s_n, s) < \varepsilon$ for $n > N$.

Definition 25 (Cauchy Sequence (Definition 13.2 continued)). A sequence $(s_n) \subset S$ is called Cauchy if:

$$\forall \varepsilon > 0, \exists N \text{ such that } d(s_n, s_m) < \varepsilon \text{ for } n, m > N. \tag{18}$$

Proposition 7 (Cauchy Sequences are Convergent in Complete Spaces). *If (S,d) is complete, then every Cauchy sequence in S converges to a point in S.*

Example 13 (Completeness of $\mathbb{R}$). *$\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$ is complete. Any Cauchy sequence in $\mathbb{R}$ converges to a real number.*

Example 14 (Non-Completeness of $\mathbb{Q}$). *Consider $\mathbb{Q}$ with $d(x,y) = |x-y|$. The sequence $r_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2})$ is Cauchy in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$ because $\sqrt{2} \notin \mathbb{Q}$.*

Example 15 (Manhattan (Taxicab) Metric). *For $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the taxicab metric is defined as:*

$$d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|. \tag{19}$$

Proposition 8 (Completeness of $(\mathbb{R}^n, d_1)$). *The metric space $(\mathbb{R}^n, d_1)$ is complete.*

Definition 26 (Inner Product). For $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the inner product:

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i. \tag{20}$$

The magnitude of $\vec{x}$ is:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}. \tag{21}$$

Theorem 27 (Bunyakovsky-Cauchy-Schwarz Inequality). *For all $\vec{x}, \vec{y} \in \mathbb{R}^n$:*

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|. \tag{22}$$

Lemma 4 (Triangle Inequality Lite). *For $\vec{x}, \vec{y} \in \mathbb{R}^n$:*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|. \tag{23}$$

Definition 27 (Bounded Sets). A set E in a metric space (S,d) is bounded if there exists $y \in S$ such that:

$$\sup_{x \in E} d(y, x) < \infty. \tag{24}$$

Remark 1. *If such a y exists, then for any $z \in S$, $\sup_{x\in E} d(z,x) < \infty$. This follows from the triangle inequality:*

$$d(z,x) \leq d(y,x) + d(z,y). \tag{25}$$

Example 16. *A sequence (x_k) is bounded if the set $\{x_1, x_2, \ldots\}$ is bounded. That is, for some (or any) $y \in S$:*

$$\sup_k d(y, x_k) < \infty. \tag{26}$$

Theorem 28 (Bolzano-Weierstrass for $\mathbb{R}^n$). *Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Example 17 (Failure of Bolzano-Weierstrass in Discrete Metrics). *Consider $\mathbb{N}$ equipped with the discrete metric $d(x,y) = 1$ for $x \neq y$ and $d(x,y) = 0$ for $x = y$. The sequence $x_n = n$ is bounded but has no convergent subsequences because convergent sequences are eventually constant in discrete metrics.*

Definition 28 (Open Ball). An open ball with center s_0 and radius $r > 0$ is:

$$B_r^o(s_0) = \{s \in S : d(s, s_0) < r\}. \tag{27}$$

Definition 29 (Interior Points). A point $s_0 \in S$ is interior to $E \subset S$ if there exists $r > 0$ such that $B_r^o(s_0) \subset E$. The set of all interior points is denoted E^o, called the interior of E.

Definition 30 (Open Sets). A set $E \subset S$ is open if $E = E^o$.

Example 18. *1. In $S = \mathbb{R}$ with the usual metric, $[0, \infty)$ is not open, but $(0, \infty)$ is.*

2. In $S = \mathbb{R}^2$, the set $E = \{(x, 0) : x \geq 0\}$ has $E^o = \emptyset$.

Theorem 29 (Facts about Open Sets)**.** *1. S and $\emptyset$ are open.*

2. A union of any collection of open sets is open.

3. A finite intersection of open sets is open.

Definition 31 (Closed Sets)**.** A set $E \subset S$ is closed if $S \setminus E$ is open.

Theorem 30 (Facts about Closed Sets)**.** *1. S and $\emptyset$ are closed.*

2. An intersection of any collection of closed sets is closed.

3. A finite union of closed sets is closed.

Definition 32 (Closure)**.** The closure of $E \subset S$, denoted $\overline{E}$, is the intersection of all closed sets containing E.

Definition 33 (Boundary)**.** The boundary of $E \subset S$ is:

$$\partial E = \overline{E} \setminus E^o. \tag{28}$$

Example 19 (Closure in $\mathbb{R}$)**.** *Let $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$. Then:*

$$\overline{E} = E \cup \{0\}. \tag{29}$$

Definition 34 (Open Ball)**.** For $s_0 \in S$ and $r > 0$, the open ball with center s_0 and radius r is:

$$B_r^o(s_0) = \{s \in S : d(s, s_0) < r\}. \tag{30}$$

Proposition 9. *A set $E \subset S$ is open if and only if it is a union of open balls.*

Definition 35 (Closed Sets)**.** A set $E \subset S$ is closed if $S \setminus E$ is open.

Proposition 10 (Properties of Closed Sets)**.** *1. S and $\emptyset$ are closed.*

2. Any intersection of closed sets is closed.

3. A finite union of closed sets is closed.

Proposition 11 (De Morgan's Laws)**.** *For any collection $\{A_i\}_{i \in I} \subset S$:*

$$S \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (S \setminus A_i), \quad S \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (S \setminus A_i). \tag{31}$$

Example 20 (Intervals in $\mathbb{R}$)**.** *1. (a, b) is open, but not closed.*

2. $[a, b]$ is closed, but not open.

3. $(a, b], [a, b)$ are neither open nor closed.

Example 21 (Discrete Metric)**.** *In a discrete metric space:*

1. Every set is both open and closed.

Definition 36 (Closure)**.** The closure of $E \subset S$, denoted $\overline{E}$, is the intersection of all closed sets containing E.

Definition 37 (Boundary)**.** The boundary of $E \subset S$ is:

$$\partial E = \overline{E} \setminus E^o. \tag{32}$$

Proposition 12. *1. $E = \overline{E}$ if and only if E is closed.*

2. $s \in \overline{E}$ if and only if s is a limit of a sequence in E.

3. $\partial E = \overline{E} \cap (S \setminus E)^-$.

Example 22 (Closure of $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$)**.** *The closure is:*

$$\overline{E} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}. \tag{33}$$

Compactness (Section 13)

Definition 38 (Compactness (Definition 13.11))**.** Suppose $E \subset S$. A family $\mathcal{U}$ of open sets is an *open cover* for E if:

$$E \subset \bigcup_{U \in \mathcal{U}} U. \tag{34}$$

A *subcover* is a subfamily of $\mathcal{U}$ which is also an open cover. E is called *compact* if every open cover has a finite subcover.

Note 1. *A cover $\mathcal{U}$ is a collection of sets, not their union. Thus, a cover is a subset of $\mathcal{P}(S)$ (the power set of S), not S.*

1. $E = [0, \infty)$ is not compact. For example:
   (a) $U_k = (-1, k)$ $(k \in \mathbb{N})$ is an open cover with no finite subcover.
2. $E = (0, 1)$ is not compact. For example:
   (a) $U_k = (1/k, 1)$ $(k \in \mathbb{N})$ is an open cover with no finite subcover.
3. $E = [a, b]$ $(a, b \in \mathbb{R})$ is compact (proof to follow).

Proposition 13 (Compactness of Finite Sets)**.** *Any finite set is compact.*

Proof. Let $E = \{e_1, \ldots, e_N\}$. For any open cover $\mathcal{U}$ of E, select $U_i \in \mathcal{U}$ containing e_i. Then $\{U_1, \ldots, U_N\}$ is a finite subcover. ☐

Proposition 14. *Any compact set is bounded.*

Proof. If E is not bounded, then for $s \in S$, the sets $B^o_k(s)$ $(k \in \mathbb{N})$ form an open cover of E with no finite subcover. ☐

Example 23 (Bounded but Not Compact)**.** *Equip $\mathbb{N}$ with the discrete metric*

$$d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases} \tag{35}$$

Then $\mathbb{N}$ is bounded, but it is not compact because the open cover $U_n = \{n\}$ $(n \in \mathbb{N})$ has no finite subcover.

Proposition 15. *1. A closed subset of a compact set is compact.*

2. A finite union of compact sets is compact.

Proposition 16. *Suppose $F_1 \supset F_2 \supset \cdots$ are closed non-empty subsets of a compact set E. Then:*

$$\bigcap_n F_n \neq \emptyset, \quad \text{and it is compact.} \tag{36}$$

Theorem 31 (Heine-Borel Theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 24 (Cantor Set)**.** *Define:*

$$F_0 = [0, 1], \quad F_1 = [0, 1/3] \cup [2/3, 1], \quad F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \ldots \tag{37}$$

The Cantor set $C = \bigcap_n F_n$ is non-empty, closed, and compact. It contains no intervals, and its interior is empty.

Definition 39 (Compactness (Definition 13.11))**.** Suppose $E \subset S$. A family $\mathcal{U}$ of open sets is an *open cover* for E if:

$$E \subset \bigcup_{U \in \mathcal{U}} U. \tag{38}$$

A *subcover* is a subfamily of $\mathcal{U}$ which is also an open cover. E is called *compact* if every open cover has a finite subcover.

Note 2. *A cover $\mathcal{U}$ is a collection of sets, not their union. Thus, a cover is a subset of $\mathcal{P}(S)$ (the power set of S), not S.*

Proposition 17. *Suppose $E \subset S$. If E is compact, then E is complete.*

Proof. Suppose E is not complete. Then there exists a Cauchy sequence $(s_n) \subset E$ which does not converge in E. For $k \in \mathbb{N}$, find n_k such that $d(s_m, s_\ell) < 2^{-k}$ for $m, \ell \geq n_k$. Construct open sets $U_k = \{s \in S : d(s, s_{n_k}) > 2^{-k}\}$, which form an open cover for E with no finite subcover. Contradiction. ☐

Corollary 5. *Any compact set is closed.*

Proposition 18 (Compactness Criterion)**.** *$E \subset S$ is compact if and only if it is complete and totally bounded.*

Definition 40 (Total Boundedness)**.** A set $E \subset S$ is called totally bounded if $\forall \varepsilon > 0$, there exist $s_1, \ldots, s_n \in S$ such that:

$$E \subset \bigcup_{i=1}^{n} B^o_\varepsilon(s_i). \tag{39}$$

Proposition 19. *A set is totally bounded if and only if any sequence in the set has a Cauchy subsequence.*

Theorem 32. *For a subset E of a metric space, the following are equivalent:*

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Example 25. *The space $\mathbb{N}$ with the discrete metric is complete and bounded but not compact.*

Theorem 33 (Heine-Borel Theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 26 (Cantor Set)**.** *The Cantor set C, constructed as:*

$$F_0 = [0,1], \quad F_1 = [0,1/3] \cup [2/3,1], \quad F_2 = \cdots, \tag{40}$$

is compact, closed, and totally bounded but has no interior.

A Note on Compactness

Definition 41 (Total Boundedness (Definition 1.1))**.** A set $S \subset E$ is called totally bounded if for every $\varepsilon > 0$, there exist $p_1, \ldots, p_n \in E$ such that:

$$S \subset \bigcup_{i=1}^{n} B^o_\varepsilon(p_i). \tag{41}$$

Proposition 20 (Intrinsic Nature of Total Boundedness (Proposition 1.2))**.** *A set $S \subset E$ is totally bounded if and only if for every $\varepsilon > 0$, there exist $q_1, \ldots, q_m \in S$ such that:*

$$S \subset \bigcup_{j=1}^{m} B^o_\varepsilon(q_j). \tag{42}$$

Proposition 21 (Total Boundedness and Cauchy Subsequences (Proposition 1.3))**.** *A set S is totally bounded if and only if any sequence in S has a Cauchy subsequence.*

Corollary 6 (Characterization of Compactness (Corollary 1.4))**.** *A set S is totally bounded and complete if and only if any sequence in S has a subsequence converging to a limit in S.*

Theorem 34 (Characterization of Compactness (Theorem 2.1))**.** *For a subset $S \subset E$, the following are equivalent:*

1. *S is compact.*
2. *Any sequence in S has a convergent subsequence.*
3. *S is complete and totally bounded.*

Theorem 35 (Heine-Borel Theorem (Theorem 3.1))**.** *A set $S \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Lemma 5 (Total Boundedness in $\mathbb{R}^n$ (Lemma 3.3))**.** *A set $S \subset \mathbb{R}^n$ is bounded if and only if it is totally bounded.*

Theorem 36 (Bolzano-Weierstrass Theorem (Theorem 3.4))**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Definition 42 (Total Boundedness)**.** A set $E \subset S$ is totally bounded if:

$$\forall \varepsilon > 0, \exists s_1, \ldots, s_n \in S \text{ such that } E \subset \bigcup_{i=1}^{n} B^o_\varepsilon(s_i). \tag{43}$$

Proposition 22. *A set E is totally bounded if and only if any sequence in E has a Cauchy subsequence.*

Proof. For $(s_i) \subset E$, construct $\{s_{i_k}\}$ with $s_{i_k} \in B^o_{2^{-k}}(x_{kj_k})$. Using the triangle inequality, show (s_{i_k}) is Cauchy. □

Theorem 37. *For a subset E of a metric space, the following are equivalent:*

1. *E is compact.*
2. *E is complete and totally bounded.*
3. *Any sequence in E has a subsequence with a limit in E.*

Proof.

1. (1) $\Longrightarrow$ (2): Shown in the last lecture.
2. (2) $\Longrightarrow$ (3): Total boundedness guarantees a Cauchy subsequence, and completeness ensures convergence.

3. (3) $\implies$ (1): Contraposition: if E is not compact, construct an open cover with no finite subcover.

$\square$

Theorem 38 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 27 (Non-Compact Set)**.** *In $\mathbb{N}$ with the discrete metric, $\mathbb{N}$ is closed and bounded but not compact.*

Definition 43 (Series and Convergence)**.** The n-th partial sum of a series $\sum_{j=k_0}^{\infty} a_j$ is:

$$s_n = \sum_{j=k_0}^{n} a_j. \tag{44}$$

The series converges if $\lim_{n\to\infty} s_n$ exists, diverges otherwise.

Example 28 (Geometric Series)**.**

$$\sum_{j=0}^{\infty} r^j = \begin{cases} \frac{1}{1-r}, & |r|< 1, \\ \infty, & r \geq 1. \end{cases} \tag{45}$$

Definition 44 (Cauchy Criterion)**.** A series $\sum_j a_j$ satisfies the Cauchy criterion if:

$$\forall \varepsilon > 0, \exists N \text{ such that } \left| \sum_{j=m}^{n} a_j \right| < \varepsilon \text{ for } n \geq m > N. \tag{46}$$

Theorem 39. *A series converges if and only if it satisfies the Cauchy criterion.*

Definition 45 (Partial Sums and Convergence of Series)**.** The n-th partial sum of a series $\sum_{j=k_0}^{\infty} a_j$ is:

$$s_n = \sum_{j=k_0}^{n} a_j. \tag{47}$$

The series $\sum_{j=k_0}^{\infty} a_j$ converges if $\lim_{n\to\infty} s_n$ exists. Otherwise, it diverges.

Example 29 (Geometric Series)**.**

$$\sum_{j=0}^{\infty} r^j = \begin{cases} \frac{1}{1-r}, & |r|< 1, \\ \textit{diverges}, & r \geq 1 \textit{ or } r \leq -1. \end{cases} \tag{48}$$

Definition 46 (Cauchy Criterion (Definition 14.3))**.** A series $\sum_j a_j$ satisfies the Cauchy Criterion if:

$$\forall \varepsilon > 0, \exists N \text{ such that } \left| \sum_{j=m}^{n} a_j \right| < \varepsilon \text{ for } n \geq m > N. \tag{49}$$

Theorem 40 (Cauchy Criterion (Theorem 14.4))**.** *A series converges if and only if it satisfies the Cauchy Criterion.*

Corollary 7 (Necessary Condition for Convergence (Corollary 14.5))**.** *If $\sum_j a_j$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Example 30. *If $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Theorem 41 (Comparison Test (Theorem 14.6))**.** *1. If $0 \leq |b_n| \leq a_n$ and $\sum a_n$ converges, then $\sum b_n$ converges.*

2. If $0 \leq a_n \leq b_n$ and $\sum b_n = \infty$, then $\sum a_n = \infty$.

Decimal Expansions

Theorem 42 (Decimal Expansions (Theorem 16.2))**.** *Any real number $x \geq 0$ has at least one decimal expansion:*

$$x = K.d_1 d_2 d_3 \ldots = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j}, \tag{50}$$

where $K \in \mathbb{Z}$ and $d_j \in \{0, 1, \ldots, 9\}$.

Theorem 43 (Uniqueness of Decimal Expansions (Theorem 16.3))**.** *Any $x \geq 0$ has either exactly one decimal expansion or exactly two, one ending in $\ldots d000\ldots$ and the other in $\ldots [d-1]999\ldots$.*

Theorem 44 (Repeating Decimals (Theorem 16.5))**.** *A real number x is rational if and only if its decimal expansion is repeating.*

Series and Decimal Expansions (Sections 17 and 21)

Theorem 45 (Root Test)**.** *For a series $\sum_n a_n$, let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$. Then:*

1. *The series converges absolutely if $\alpha < 1$.*
2. *The series diverges if $\alpha > 1$.*
3. *If $\alpha = 1$, the test gives no information.*

Theorem 46 (Ratio Test (Theorem 14.8))**.** *For a series $\sum_n a_n$ of nonzero terms:*

1. *The series converges absolutely if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.*
2. *The series diverges if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.*
3. *If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, the test gives no information.*

Example 31. 1. *Consider $\sum_{k=1}^{\infty} \frac{k^4}{2^k}$. Using the Root Test:*

$$a_k^{1/k} = \left(k^{1/k}\right)^4 \frac{1}{2}, \quad \lim_{k\to\infty} a_k^{1/k} = \frac{1}{2} < 1, \tag{51}$$

so the series converges.

2. *The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$. The Root and Ratio Tests are inconclusive for this series.*

Theorem 47 (Repeating Decimals and Rational Numbers (Theorem 16.5))**.** *A real number x is rational if and only if its decimal expansion is repeating.*

Proof. 1. (x is rational $\implies$ repeating): Follows from performing long division.

2. (Repeating $\implies$ x is rational): Suppose $x = K.d_1 \ldots d_\ell \overline{d_{\ell+1} \ldots d_{\ell+r}}$. Then:

$$x = K + \sum_{j=1}^{\ell} \frac{d_j}{10^j} + 10^{-\ell}\left(\frac{z}{1-10^{-r}}\right), \tag{52}$$

where $z = \sum_{j=1}^{r} d_{\ell+j} 10^{-j} \in \mathbb{Q}$, so $x \in \mathbb{Q}$.

□

Definition 47 (Continuity (Definition 21.1))**.** Suppose (S, d) and (S^*, d^*) are metric spaces. The function $f : \mathrm{dom}(f) \to S^*$ (with $\mathrm{dom}(f) \subset S$) is continuous at $x \in \mathrm{dom}(f)$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d^*(f(x), f(y)) < \varepsilon \text{ whenever } d(x,y) < \delta. \tag{53}$$

f is called continuous on $E \subset S$ if it is continuous at every $x \in E$.

Theorem 48 (Sequential Criterion for Continuity (Theorem 17.1 + 17.2))**.** *$f : S \to S^*$ is continuous at $x \in S$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

Example 32 (Discontinuous Everywhere)**.** *The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$, defined as:*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}, \end{cases} \tag{54}$$

is discontinuous at every $x \in \mathbb{R}$. This is because for any $x \in \mathbb{R}$, one can find sequences $(x_n) \subset \mathbb{Q}$ and $(y_n) \not\subset \mathbb{Q}$ such that $x_n, y_n \to x$, but $f(x_n) \to 1$ and $f(y_n) \to 0$, which do not match $f(x)$.

Example 33 (Continuous at a Single Point)**.** *The modified Dirichlet function $g : \mathbb{R} \to \mathbb{R}$, defined as:*

$$g(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}, \end{cases} \tag{55}$$

is continuous only at $x = 0$. At other points, similar reasoning as the Dirichlet function applies.

Example 34 (Continuous on $\mathbb{R} \setminus \mathbb{Q}$)**.** *The Thomae function $h : \mathbb{R} \to \mathbb{R}$, defined as:*

$$h(x) = \begin{cases} \frac{1}{b} & x = \frac{a}{b},\ \gcd(a,b) = 1,\ b > 0,\ x \neq 0, \\ 0 & x \notin \mathbb{Q}, \end{cases} \tag{56}$$

is continuous at $x \notin \mathbb{Q}$ and discontinuous at $x \in \mathbb{Q}$.

Theorem 49. *Suppose f, g are continuous at x_0 in a metric space (S, d). Then the following functions are also continuous at x_0:*

1. $|f|$,
2. kf $(k \in \mathbb{R})$,
3. $f + g$,
4. $f \cdot g$,
5. f/g (if $g(x_0) \neq 0$).

Proposition 23. *If f, g are continuous at x_0, then $\max(f, g)$ and $\min(f, g)$ are continuous at x_0.*

Theorem 50. *Suppose $(S_1, d_1), (S_2, d_2), (S_3, d_3)$ are metric spaces, and $f : dom(f) \to S_2$, $g : dom(g) \to S_3$ are functions such that f is continuous at x_0, g is continuous at $f(x_0)$, and $x_0 \in dom(f)$. Then $g \circ f$ is continuous at x_0.*

Theorem 51 (Characterization of Continuity (Theorem 21.3))**.** *Suppose (S, d) and (S^*, d^*) are metric spaces. $f : S \to S^*$ is continuous if and only if $f^{-1}(U)$ is open for every open $U \subset S^*$, where:*

$$f^{-1}(U) = \{s \in S : f(s) \in U\}. \tag{57}$$

Lemma 6. *f is continuous at $s_0 \in S$ if for any open set U containing $f(s_0)$, there exists an open set V containing s_0 such that $f(V) \subset U$.*

Continuity and the Intermediate Value Theorem (Sections 18 and 21)

Theorem 52 (Theorem 21.3)**.** *Suppose (S, d) and (S^*, d^*) are metric spaces. A function $f : S \to S^*$ is continuous if and only if $f^{-1}(U)$ is open for every open $U \subset S^*$. Here:*

$$f^{-1}(U) = \{s \in S : f(s) \in U\}. \tag{58}$$

Lemma 7 (Exercise 21.2)**.** *f is continuous at $s_0 \in S$ if and only if for any open set $U \ni f(s_0)$, there exists an open set $V \ni s_0$ such that $f(V) \subset U$.*

Corollary 8 (Exercise 21.4)**.** *Suppose (S, d) is a metric space. A function $f : S \to \mathbb{R}$ is continuous if and only if $f^{-1}((a, b))$ is open whenever $a < b$.*

Theorem 53. *If $f : S \to S^*$ is continuous, and $E \subset S$ is compact, then $f(E) \subset S^*$ is compact.*

Corollary 9. *If $f : S \to \mathbb{R}$ is continuous, and $E \subset S$ is compact, then $f(E)$ is bounded. Moreover, f attains its maximum and minimum values, i.e., there exist $x, y \in E$ such that:*

$$f(x) = \sup_{e \in E} f(e), \quad f(y) = \inf_{e \in E} f(e). \tag{59}$$

Theorem 54 (Theorem 18.2)**.** *Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is continuous. Then f has the Intermediate Value Property (IVP) on I: if $a, b \in I$ with $a < b$, and y lies between $f(a)$ and $f(b)$, then there exists $x \in (a, b)$ such that $f(x) = y$.*

Corollary 10. *If I is an interval, and $f : I \to \mathbb{R}$ has the IVP, then $f(I)$ is either an interval or a single point.*

Proposition 24 (Roots of Polynomials)**.** *Any polynomial of odd degree has at least one real root.*

Proposition 25 (Existence of Fixed Points)**.** *Any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point, i.e., $x \in [0, 1]$ such that $f(x) = x$.*

Proposition 26 (Existence of m-th Roots)**.** *For any $m \in \mathbb{N}$ and $y > 0$, there exists $x > 0$ such that $x^m = y$.*

Theorem 55 (Theorem 18.4)**.** *Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $J = f(I)$ is an interval, and $f^{-1} : J \to I$ is strictly increasing and continuous.*

Corollary 11. *The function $x \mapsto x^{1/m}$, taking $[0, \infty)$ to itself, is continuous.*

Continuity and Compactness (Section 18)

Theorem 56 (Theorem 21.4(i))**.** *Suppose $f : S \to S^*$ is continuous, where (S, d) and (S^*, d^*) are metric spaces, and $E \subset S$ is compact. Then $f(E) \subset S^*$ is compact.*

Proof. Let $(U_i)_{i \in I}$ be an open cover for $f(E)$. Define $V_i = f^{-1}(U_i)$, which are open sets forming a cover for E. By compactness of E, there exist $i_1, \ldots, i_n \in I$ such that $E \subset \bigcup_{k=1}^n V_{i_k}$. It follows that $f(E) \subset \bigcup_{k=1}^n U_{i_k}$, proving compactness of $f(E)$. □

Corollary 12 (Similar to 18.1)**.** *If $f : S \to \mathbb{R}$ is continuous and $E \subset S$ is compact, then $f(E)$ is bounded. Moreover, f attains its maximum and minimum values, i.e., there exist $x, y \in E$ such that:*

$$f(x) = \sup_{e \in E} f(e), \quad f(y) = \inf_{e \in E} f(e). \tag{60}$$

Definition 48 (IVP). Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is a function. f has the Intermediate Value Property (IVP) on I if for any $a, b \in I$ with $a < b$, and any y between $f(a)$ and $f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.

Theorem 57 (Theorem 18.2). *Any continuous function has the IVP.*

Corollary 13 (18.3). *If I is an interval, and $f : I \to \mathbb{R}$ has the IVP, then $f(I)$ is either an interval or a single point.*

Proposition 27 (Roots of Polynomials). *Any polynomial of odd degree has at least one real root.*

Proposition 28 (Existence of Fixed Points). *Any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point, i.e., a point $x \in [0, 1]$ such that $f(x) = x$.*

Proposition 29 (Existence of m-th Root). *For any $m \in \mathbb{N}$ and $y > 0$, there exists $x > 0$ such that $x^m = y$.*

Theorem 58 (Theorem 18.4). *Suppose $I \subset \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is strictly increasing and continuous. Then $f(I)$ is an interval, and $f^{-1} : f(I) \to I$ is strictly increasing and continuous.*

Corollary 14. *The function $x \mapsto x^{1/m}$, taking $[0, \infty)$ to itself, is continuous.*

Theorem 59 (Theorem 18.6). *Suppose $f : I \to \mathbb{R}$ is a continuous one-to-one function on an interval I. Then f is strictly monotone.*

Sketch. For $a, b \in I$ with $a < b$, if $f(a) < f(b)$, then f is strictly increasing. Otherwise, by the IVP, there would exist $x \in (a, b)$ such that $f(x) = f(a)$, contradicting injectivity. □

Definition 49 (Uniform Continuity (Definition 21.1)). Suppose (S, d) and (S^*, d^*) are metric spaces. A function $f : S \to S^*$ is uniformly continuous on $E \subset S$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d^*(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta. \tag{61}$$

Here, δ depends only on ε and not on the specific point x.

Theorem 60 (Sequential Criterion for Uniform Continuity (Theorem 19.4)). *If $f : S \to S^*$ is uniformly continuous, then f maps Cauchy sequences in S to Cauchy sequences in S^*.*

Example 35 (Non-Uniformly Continuous Function). *The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$, as the Cauchy sequence $x_n = \frac{1}{n}$ is mapped to $f(x_n) = n$, which is not Cauchy.*

Definition 50 (Lipschitz Continuity). A function $f : S \to S^*$ is Lipschitz if there exists $K > 0$ (called the Lipschitz constant) such that:

$$d^*(f(s), f(t)) \leq K \cdot d(s, t), \quad \forall s, t \in S. \tag{62}$$

Proposition 30. *Any Lipschitz function is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{K}$. Then if $d(s, t) < \delta$, it follows that:

$$d^*(f(s), f(t)) \leq K \cdot d(s, t) < K \cdot \frac{\varepsilon}{K} = \varepsilon. \tag{63}$$

□

Example 36. *For $a > 0$, $f(x) = \frac{1}{x}$ is Lipschitz (hence uniformly continuous) on $[a, \infty)$.*

Example 37 (Uniformly Continuous but Not Lipschitz). *The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz.*

Theorem 61 (Uniform Continuity on Compact Sets (Theorem 21.4(ii))). *If $f : S \to S^*$ is continuous, and $E \subset S$ is compact, then f is uniformly continuous on E.*

Sketch of Proof. Assume f is not uniformly continuous. Then $\exists \varepsilon > 0$ and sequences $(x_n), (y_n) \subset E$ such that $d(x_n, y_n) \to 0$ but $d^*(f(x_n), f(y_n)) \geq \varepsilon$. By compactness, (x_n) has a subsequence (x_{n_k}) converging to some $x \in E$, and (y_{n_k}) also converges to x. Continuity of f implies $d^*(f(x_{n_k}), f(y_{n_k})) \to 0$, contradicting $d^*(f(x_{n_k}), f(y_{n_k})) \geq \varepsilon$. □

Theorem 62 (Theorem 21.4(ii)). *Suppose (S, d) and (S^*, d^*) are metric spaces, and $f : S \to S^*$ is continuous. If $E \subset S$ is compact, then $f|_E$ is uniformly continuous.*

Proof. For $\varepsilon > 0$, find $\delta > 0$ such that $d^*(f(s), f(t)) < \varepsilon$ whenever $d(s, t) < \delta$. For $s \in S$, find $\delta_s > 0$ such that $d^*(f(s), f(t)) < \varepsilon/2$ whenever $d(s, t) < \delta_s$. Since E is compact:

$$E \subset \bigcup_{s \in E} B^o_{\delta_s/2}(s), \tag{64}$$

there exist $s_1, \dots, s_n$ such that $E \subset \bigcup_{k=1}^n B^o_{\delta_{s_k}/2}(s_k)$. Define $\delta = \frac{1}{2} \min_{1 \leq k \leq n} \delta_{s_k}$. For $s, t \in E$ with $d(s, t) < \delta$, choose s_k such that $s \in B^o_{\delta_{s_k}/2}(s_k)$. Then:

$$d(t, s_k) \leq d(t, s) + d(s, s_k) < \delta + \frac{\delta_{s_k}}{2} \leq \delta_{s_k}, \tag{65}$$

implying $d^*(f(s), f(t)) \leq d^*(f(s), f(s_k)) + d^*(f(t), f(s_k)) < \varepsilon$. □

Theorem 63. *Suppose $E \subset S$ is compact, $f : E \to S^*$, and S^* is complete. Then f is uniformly continuous if and only if it extends to a continuous $\tilde{f} : \overline{E} \to S^*$.*

Sketch. If f is uniformly continuous, define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$ for $x \in \overline{E}$, where $x_n \subset E$ and $x_n \to x$. The limit exists by completeness and does not depend on the sequence. Continuity of $\tilde{f}$ follows from the uniform continuity of f. □

Definition 51 (Connected and Disconnected Sets)**.** Suppose (S, d) is a metric space. A set $E \subset S$ is disconnected if there exist open sets $U_1, U_2 \subset S$ such that:

1. $E \subset U_1 \cup U_2$,
2. $(E \cap U_1) \cap (E \cap U_2) = \emptyset$,
3. $E \cap U_1 \neq \emptyset$ and $E \cap U_2 \neq \emptyset$.

A set E is connected if it is not disconnected.

Proposition 31. *E is disconnected if and only if there exist $A, B \subset E$ such that:*

$$E = A \cup B, \quad A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap \overline{B} = \emptyset, \quad \overline{A} \cap B = \emptyset. \tag{66}$$

Proposition 32. *Any interval $I \subset \mathbb{R}$ is connected.*

Proof. Suppose, for contradiction, that $I = A \cup B$, where $A, B \neq \emptyset$, $\overline{A} \cap B = \emptyset$, and $A \cap \overline{B} = \emptyset$. Choose $a \in A$, $b \in B$, with $a < b$. Define:

$$c = \sup\{x \in A : x < b\}. \tag{67}$$

Then $c \in I$ and $c < b$. If $c \in A$, there exists $\sigma > 0$ such that $(c - \sigma, c + \sigma) \subset A$, contradicting the definition of c. If $c \in B$, there exists $\sigma > 0$ such that $(c - \sigma, c + \sigma) \subset B$, contradicting $c = \sup\{x \in A : x < b\}$. □

Definition 52 (Connected Set (Definition 22.1))**.** Suppose (S, d) is a metric space. A set $E \subset S$ is called disconnected if there exist open sets $U_1, U_2 \subset S$ such that:

1. $E \subset U_1 \cup U_2$,
2. $(E \cap U_1) \cap (E \cap U_2) = \emptyset$,
3. $E \cap U_1 \neq \emptyset$ and $E \cap U_2 \neq \emptyset$.

A set E is connected if it is not disconnected.

Proposition 33. *An open set E is disconnected if and only if $E = E_1 \cup E_2$, where E_1, E_2 are disjoint, non-empty, open subsets.*

Proposition 34 (Equivalent Characterization of Connectedness)**.** *A set E is disconnected if and only if there exist $A, B \subset E$ such that:*

$$E = A \cup B, \quad A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap \overline{B} = \emptyset, \quad \overline{A} \cap B = \emptyset. \tag{68}$$

Theorem 64 (Theorem 22.2)**.** *Suppose (S, d) and (S^*, d^*) are metric spaces. If $E \subset S$ is connected and $f : S \to S^*$ is continuous, then $f(E)$ is connected.*

Sketch of Contrapositive Proof. If $f(E) \subset S^*$ is disconnected, write $f(E) = C \cup D$, where C, D are disjoint, non-empty, closed subsets. Define $A = f^{-1}(C) \cap E$, $B = f^{-1}(D) \cap E$. Then $E = A \cup B$, $A \cap \overline{B} = \emptyset$, and $\overline{A} \cap B = \emptyset$, so E is disconnected. □

Definition 53 (Path Connectedness (Definition 22.4))**.** A set $E \subset S$ is path connected if for all $a, b \in E$, there exists a continuous function $\gamma : [0, 1] \to E$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Theorem 65 (Path Connected Sets Are Connected (Theorem 22.5))**.** *Every path connected set is connected.*

Proof. If E is disconnected, then there exist open sets U_1, U_2 such that $E \subset U_1 \cup U_2$, $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$, and $(E \cap U_1) \cap (E \cap U_2) = \emptyset$. Let $a \in E \cap U_1$, $b \in E \cap U_2$. A path $\gamma : [0, 1] \to E$ with $\gamma(0) = a$, $\gamma(1) = b$ would imply $\gamma([0, 1])$ is connected, contradicting the disconnectedness of E. □

Example 38. *Consider $E \subset \mathbb{R}^2$, where:*

$$E_1 = \{(0, y) : y \in (0, 1]\}, \quad E_2 = \{(x, 0) : x \in (0, 1]\} \cup \bigcup_{n\in\mathbb{N}} \{(1/n, y) : y \in (0, 1]\}. \tag{69}$$

Then $E = E_1 \cup E_2$ is connected but not path connected.

Definition 54 (Convex Sets)**.** A set $E \subset \mathbb{R}^n$ is convex if for all $\vec{x}, \vec{y} \in E$ and $t \in [0, 1]$, the point:

$$\vec{z} = (1 - t)\vec{x} + t\vec{y} \in E. \tag{70}$$

Proposition 35. *Any convex set is path connected.*

Proposition 36. *The graph of a function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is path connected if and only if f is continuous.*

Definition 55 (Graph of a Function)**.** The graph of a function $f : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is:

$$G(f) = \{(x, f(x)) : x \in I\}. \tag{71}$$

Proposition 37 (Example 4 from Section 22)**.** *$G(f)$ is path connected if and only if f is continuous on I.*

Example 39 (Discontinuous f with Connected $G(f)$)**.** *Exercise 22.4 describes a function f such that $G(f)$ is connected but f is discontinuous.*

Proposition 38 (Multivariate Continuity)**.** *The function $f : S \to \mathbb{R}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$, is continuous if and only if each $f_i : S \to \mathbb{R}$ is continuous for $1 \leq i \leq n$.*

Sketch. If f is continuous, then $G(f)$ is path connected. For $\vec{x} = (a, f(a))$, $\vec{y} = (b, f(b)) \in G(f)$, define a path:

$$\gamma(t) = ((1-t)a + tb, f((1-t)a + tb)), \quad t \in [0, 1]. \tag{72}$$

If $G(f)$ is path connected, continuity of f follows from the textbook proof. □

Definition 56 (Power Series)**.** A power series is a series of the form:

$$\sum_{n=0}^{\infty} a_n x^n, \tag{73}$$

where x is a variable.

Theorem 66 (Radius of Convergence)**.** *Let $\beta = \limsup |a_n|^{1/n}$ and $R = 1/\beta$. The series:*

$$\sum_{n=0}^{\infty} a_n x^n \tag{74}$$

converges for $|x| < R$, diverges for $|x| > R$. R is called the radius of convergence.

Remark 2. *If $\lim |a_{n+1}/a_n|$ exists, it equals β. The series may converge or diverge at $\pm R$. The interval of convergence is one of:*

$$(-R, R),\ [-R, R),\ (-R, R],\ or [-R, R]. \tag{75}$$

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$: $a_n = \frac{1}{n!}$, $\beta = 0$, $R = \infty$. Interval: $(-\infty, \infty)$.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \tag{76}$$

2. $\sum_{n=0}^{\infty} x^n$: $a_n = 1$, $\beta = 1$, $R = 1$. Diverges for $x = \pm 1$. Interval: $(-1, 1)$.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{77}$$

3. $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$: $a_n = \frac{1}{n+1}$, $\beta = 1$, $R = 1$. Diverges at $x = 1$, converges for $x \in [-1, 1)$.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = \ln(1-x). \tag{78}$$

4. $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$: $a_n = \frac{1}{(n+1)^2}$, $\beta = 1$, $R = 1$. Converges for $x \in [-1, 1]$.
5. $\sum_{n=0}^{\infty} n!\, x^n$: $a_n = n!$, $\beta = \infty$, $R = 0$. Diverges for all $x \neq 0$.

Definition 57 (Uniform Convergence (Definition 24.1-2))**.** A sequence $f_n \to f$ pointwise on S if:

$$\forall x \in S, \forall \varepsilon > 0, \exists N \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N. \tag{79}$$

It converges uniformly if:

$$\forall \varepsilon > 0, \exists N \text{ such that } \sup_{x \in S} |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N. \tag{80}$$

Theorem 67 (Preservation of Continuity (Theorem 24.3))**.** *If $f_n \to f$ uniformly on S and each f_n is continuous, then f is continuous.*

Example 40. *Consider $f_n(x) = n^2 x^n (1-x)$ on $[0, 1]$. It converges pointwise to $f(x) = 0$, but not uniformly.*

Definition 58 (Pointwise and Uniform Convergence (24.1-2))**.** Suppose $f, f_1, f_2, \dots$ are functions $S \to \mathbb{R}$.

1. $f_n \to f$ pointwise on S if:
$$\forall x \in S, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N. \tag{81}$$

2. $f_n \to f$ uniformly on S if:
$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N, \forall x \in S. \tag{82}$$
Equivalently, $\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$.

Theorem 68 (Preservation of Continuity (24.3))**.** *If $f_n \to f$ uniformly on S, and each f_n is continuous at $x_0 \in S$, then f is continuous at x_0.*

Sketch of Proof. Using an $\varepsilon/3$ argument, fix $\varepsilon > 0$. For $f_n \to f$ uniformly, find n such that $|f_n(x) - f(x)| < \varepsilon/3$. By continuity of f_n, there exists $\delta > 0$ such that $|f_n(x_0) - f_n(x)| < \varepsilon/3$ for $|x - x_0| < \delta$. Combine inequalities to conclude $|f(x_0) - f(x)| < \varepsilon$. □

Definition 59 (Uniformly Cauchy (25.3))**.** A sequence (f_n) of functions $S \to \mathbb{R}$ is uniformly Cauchy if:
$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |f_i(x) - f_j(x)| < \varepsilon \, \forall x \in S \text{ for } i, j \geq N. \tag{83}$$
Equivalently, $\sup_{x\in S} |f_i(x) - f_j(x)| < \varepsilon$ for $i, j \geq N$.

Theorem 69 (Uniformly Cauchy $\iff$ Uniform Convergence (25.4))**.** *A sequence (f_n) is uniformly Cauchy if and only if it converges uniformly to some f.*

Definition 60 (Convergence of Series)**.** A series $\sum_{n=1}^{\infty} g_n(x)$ converges (uniformly) if the sequence of partial sums $s_k(x) = \sum_{n=1}^{k} g_n(x)$ converges (uniformly).

Theorem 70 (Uniform Convergence Preserves Continuity (25.5))**.** *If $g_n : S \to \mathbb{R}$ are continuous and $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on S, then $\sum_{n=1}^{\infty} g_n(x)$ is continuous.*

Theorem 71 (Weierstrass M-Test (25.7))**.** *Suppose $M_1, M_2, \dots \geq 0$ and $\sum_{k=1}^{\infty} M_k < \infty$. If $|g_k(x)| \leq M_k$ for all $x \in S$ and k, then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on S.*

Corollary 15. *A power series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly (to a continuous function) on $[-b, b]$ if $b < R$, where $R = (\limsup |a_k|^{1/k})^{-1}$.*

Remark 3. *Convergence need not be uniform on $(-R, R)$. For example, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ converges on $(-1, 1)$, but not uniformly because the partial sums are bounded while $\frac{1}{1-x}$ is not.*

Definition 61 (Limit (20.1, slightly modified))**.** Suppose $S \subset \mathbb{R}$, $a \in S^-$, $f : S \to \mathbb{R}$, and $L \in \mathbb{R} \cup \{\pm\infty\}$. Then:
$$\lim_{x\to a, S} f = L \tag{84}$$
if $\lim f(x_n) = L$ for any sequence $(x_n) \subset S$ with $\lim x_n = a$. Such sequences (x_n) exist because $a \in S^-$.

Proposition 39 (Connection Between Limits and Continuity)**.** *If $a \in S$, then $f : S \to \mathbb{R}$ is continuous at a if and only if $\lim_{x\to a,S} f = f(a)$.*

1. **Usual Limit**: Let I be an interval, a be interior to I, and $S = I \setminus \{a\}$. Write $\lim_{x\to a} f$ instead of $\lim_{x\to a,S} f$.
2. **One-Sided Limit**: For $S = (a, b)$, write $\lim_{x\to a^+} f$ (right-hand limit). Define $\lim_{x\to a^-} f$ similarly.

Theorem 72 (Equivalent Definition of Limits (20.6, Simplified))**.** *Suppose $a \in S^-$. For $f : S \to \mathbb{R}$ and $L \in \mathbb{R}$, the following are equivalent:*

1. *$\lim_{x\to a,S} f = L$.*
2. *For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in (a - \delta, a + \delta) \cap S \setminus \{a\}$.*

Theorem 73 (Limit Operations (20.4))**.** *Suppose $\lim_{x\to a,S} f_1 = L_1$ and $\lim_{x\to a,S} f_2 = L_2$. Then:*

1. *$\lim_{x\to a,S}(f_1 + f_2) = L_1 + L_2$,*
2. *$\lim_{x\to a,S}(f_1 \cdot f_2) = L_1 \cdot L_2$,*
3. *If $L_2 \neq 0$, $\lim_{x\to a,S} \frac{f_1}{f_2} = \frac{L_1}{L_2}$.*

Theorem 74 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ for all $x \in S$, and $\lim_{x\to a,S} f = \lim_{x\to a,S} h = L$, then $\lim_{x\to a,S} g = L$.*

Definition 62 (Derivative (28.1))**.** Suppose I is an open interval, $a \in I$, and $f : I \to \mathbb{R}$. The derivative of f at a is:
$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}, \tag{85}$$
if the limit exists and is finite.

Theorem 75 (Product Rule). *If f and g are differentiable at a, then $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*

Theorem 76 (Chain Rule (28.4)). *If f is differentiable at a, and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a, with:*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a). \tag{86}$$

Theorem 77 (Carathéodory (Exercise 28.16)). *Suppose I is an interval, $f : I \to \mathbb{R}$. f is differentiable at $a \in I$ if and only if there exists a function $\phi : I \to \mathbb{R}$, continuous at a, such that:*

$$f(x) - f(a) = \phi(x) \cdot (x - a), \quad \forall x \in I, \tag{87}$$

and $\phi(a) = f'(a)$.

Definition 63 (Derivative (28.1)). Suppose I is an open interval and $a \in I$. A function $f : I \to \mathbb{R}$ is differentiable at a if the derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{88}$$

exists and is finite.

Theorem 78 (Product Rule (28.3)). *Suppose f and g are differentiable at a. Then fg is differentiable at a, with:*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a). \tag{89}$$

Corollary 16. *If $f(x) = x^m$ for $m \in \mathbb{N}$, then:*

$$(x^m)' = mx^{m-1}. \tag{90}$$

Theorem 79 (Chain Rule (28.4)). *Suppose f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, with:*

$$(g \circ f)'(a) = g'(f(a))f'(a). \tag{91}$$

1. If $f(x) = x^n$ for $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.
2. If $f(x) = x^{-n}$ for $n \in \mathbb{N}$, then $f'(x) = -nx^{-n-1}$.
3. For $f(x) = 1/g(x)$, if $g(a) \neq 0$, then:

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}. \tag{92}$$

Theorem 80 (Extrema Criterion (29.1)). *Suppose f is defined on an open interval I and has a maximum or minimum at $x_0 \in I$. If f is differentiable at x_0, then:*

$$f'(x_0) = 0. \tag{93}$$

Corollary 17. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and attains its maximum or minimum at x_0. Then one of the following holds:*

1. *$x_0 \in \{a, b\}$,*
2. *f is not differentiable at x_0,*
3. *$f'(x_0) = 0$.*

Theorem 81 (Rolle's Theorem (29.2)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that:*

$$f'(c) = 0. \tag{94}$$

Theorem 82 (Mean Value Theorem (29.3)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b). Then there exists $c \in (a, b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{95}$$

Corollary 18 (Constant Function (29.4)). *If f is differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function.*

Corollary 19 (Equality of Derivatives (29.5)). *If f and g are differentiable on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) - g(x) = c$ for some constant c.*

Theorem 83 (Rolle's Theorem (29.2)). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that:*

$$f'(x) = 0. \tag{96}$$

Proof. The function f attains its maximum and minimum on $[a,b]$. Let $x_0, y_0 \in [a,b]$ such that $f(y_0) \leq f(x) \leq f(x_0)$ for all $x \in [a,b]$. If $f(y_0) = f(a) = f(b) = f(x_0)$, then f is constant, so $f' = 0$ on (a,b). Otherwise:

1. If $f(x_0) > f(a) = f(b)$, then $x_0 \in (a,b)$, and $f'(x_0) = 0$.
2. If $f(y_0) < f(a) = f(b)$, then $y_0 \in (a,b)$, and $f'(y_0) = 0$.

□

Theorem 84 (Mean Value Theorem (29.3)). *Suppose $f : [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b). Then there exists $x \in (a,b)$ such that:*

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \tag{97}$$

Proof. Define $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ and $g(x) = f(x) - L(x)$. The function g is continuous on $[a,b]$, differentiable on (a,b), and $g(a) = g(b) = 0$. By Rolle's Theorem, there exists $x \in (a,b)$ such that $g'(x) = 0$. Thus:

$$f'(x) = g'(x) + L'(x) = 0 + \frac{f(b) - f(a)}{b - a}. \tag{98}$$

□

Example 41 (MVT Application). *For $x, y \in \mathbb{R}$, $|\sin x - \sin y| \leq |x - y|$. Apply MVT to $f(t) = \sin t$ on $[x,y]$: $\exists z \in (x,y)$ such that:*

$$\frac{f(x) - f(y)}{x - y} = f'(z) = \cos z. \tag{99}$$

Since $|\cos z| \leq 1$, $\left|\frac{f(x)-f(y)}{x-y}\right| = |\cos z| \leq 1$, hence $|\sin x - \sin y| \leq |x - y|$.

Corollary 20 (Constant Functions (29.4)). *If f is differentiable on (a,b) and $f' = 0$ on (a,b), then f is constant.*

Proof. If f is not constant, then there exist $x < y$ such that $f(x) \neq f(y)$. By MVT, $\exists z \in (x,y)$ such that:

$$f'(z) = \frac{f(y) - f(x)}{y - x} \neq 0, \tag{100}$$

contradicting $f'(z) = 0$. □

Corollary 21 (Equality of Derivatives (29.5)). *If f, g are differentiable on (a,b) and $f' = g'$ on (a,b), then $\exists c \in \mathbb{R}$ such that $f(x) - g(x) = c$ for all $x \in (a,b)$.*

Proof. Define $h(x) = f(x) - g(x)$. Then $h' = f' - g' = 0$. By Corollary 29.4, h is constant. □

Definition 64 (Monotonicity (29.6)). A function f on an interval I is:

1. Increasing if $f(x_1) \leq f(x_2)$ for $x_1 < x_2$,
2. Strictly increasing if $f(x_1) < f(x_2)$ for $x_1 < x_2$.

Corollary 22 (Monotonicity and Derivatives (29.7)). *Suppose f is differentiable on (a,b):*

1. *f is increasing if and only if $f' \geq 0$ on (a,b),*
2. *If $f' > 0$ on (a,b), then f is strictly increasing.*

Example 42 (Bernoulli's Inequality). *If $n \in \mathbb{N}$ and $x > -1$, then:*

$$(1 + x)^n \geq 1 + nx. \tag{101}$$

Let $f(x) = (1+x)^n - (1+nx)$ and show $f(x) \geq 0$ for $x > -1$. By differentiating $f(x)$, we conclude $f(x)$ is increasing and achieves its minimum at $x = 0$, where $f(0) = 0$.

Theorem 85 (Derivative of an Inverse Function (29.9)). *Suppose I is an interval, $f : I \to \mathbb{R}$ is a continuous, strictly monotone function. Let $J = f(I)$, and $g = f^{-1} : J \to I$. If f is differentiable at $c \in I$, and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$, and:*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}. \tag{102}$$

Proof Sketch. Using Carathéodory's Theorem:

$$f(x) - f(c) = \phi(x)(x - c), \quad \phi(c) = f'(c), \tag{103}$$

where ϕ is continuous at c. For $y = f(g(y))$, differentiate both sides to find $g'(d) = 1/\phi(g(d))$. □

Example 43 (Derivative of Rational Powers). *Let $f(x) = x^n$ (strictly increasing on $(0, \infty)$) with $f'(x) = nx^{n-1}$. The inverse function is $g(y) = y^{1/n}$. For $y > 0$:*

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n} y^{1/n-1}. \tag{104}$$

If $n \in \mathbb{Z}$ is odd, extend f, g to $\mathbb{R}$. Then $g'(y) = \frac{1}{n} y^{1/n-1}$ for $y < 0$.

Example 44 (Derivative of $h(x) = x^r$, $r \in \mathbb{Q}$). *Write $r = m/n$, $h(x) = x^{m/n}$. Use the chain rule:*

$$h'(x) = \frac{m}{n} x^{r-1}. \tag{105}$$

Example 45 (Arcsine). *For $f(x) = \sin x$ on $[-\pi/2, \pi/2]$, $g = \arcsin : [-1, 1] \to [-\pi/2, \pi/2]$. Since $f'(x) = \cos x$, for $y \in (-1, 1)$:*

$$(\arcsin y)' = \frac{1}{\sqrt{1 - y^2}}. \tag{106}$$

Example 46 (Arctangent). *For $f(x) = \tan x$ on $(-\pi/2, \pi/2)$, $g = \arctan : \mathbb{R} \to (-\pi/2, \pi/2)$. Since $f'(x) = 1 + x^2$, for $y \in \mathbb{R}$:*

$$(\arctan y)' = \frac{1}{1 + y^2}. \tag{107}$$

Definition 65 (Darboux Sums). Let $f : [a, b] \to \mathbb{R}$ be bounded.

1. Partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ gives subintervals $[t_{k-1}, t_k]$.
2. Lower Darboux sum:

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}), \tag{108}$$

   where $m(f, [t_{k-1}, t_k]) = \inf_{x \in [t_{k-1}, t_k]} f(x)$.

3. Upper Darboux sum:

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}), \tag{109}$$

   where $M(f, [t_{k-1}, t_k]) = \sup_{x \in [t_{k-1}, t_k]} f(x)$.

Definition 66 (Integrability). f is integrable on $[a, b]$ if:

$$\sup_P L(f, P) = \inf_P U(f, P), \tag{110}$$

denoted $\int_a^b f(x)dx$.

Example 47 (Constant Function). *If $f(x) = c$, then:*

$$\int_a^b f(x)dx = c(b - a). \tag{111}$$

Example 48 (Discontinuous Function). *Let $g(x) = 1$ if $x \in \mathbb{Q}$, $g(x) = 0$ otherwise. Then:*

$$\sup_P L(g, P) = 0, \quad \inf_P U(g, P) = 1. \tag{112}$$

Since $\sup_P L \neq \inf_P U$, g is not integrable.

Example 49 (Linear Function). *If $h(x) = x$, then:*

$$\int_0^b h(x)dx = \frac{b^2}{2}. \tag{113}$$

Definition 67 (Darboux Sums). Let $f : [a, b] \to \mathbb{R}$ be bounded. For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$:

1. Lower Darboux sum:

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}), \tag{114}$$

   where $m(f, [t_{k-1}, t_k]) = \inf_{x \in [t_{k-1}, t_k]} f(x)$.

2. Upper Darboux sum:

$$U(f,P)=\sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k-t_{k-1}), \tag{115}$$

where $M(f,[t_{k-1},t_k])=\sup_{x\in[t_{k-1},t_k]} f(x)$.

Definition 68 (Integrability)**.** The lower Darboux integral is $L(f)=\sup_P L(f,P)$, and the upper Darboux integral is $U(f)=\inf_P U(f,P)$. f is integrable if $L(f)=U(f)$, denoted:

$$\int_a^b f = L(f) = U(f). \tag{116}$$

Theorem 86 (32.4)**.** *If $f:[a,b]\to\mathbb{R}$ is bounded, then $L(f)\le U(f)$.*

Example 50 (Linear Function)**.** *Is $h(x)=x$ integrable on $[0,b]$? Compute $\int_0^b h(x)dx$.*
For $P=\{0,\frac{b}{n},\frac{2b}{n},\dots,b\}$:

$$L(h,P_n)=\frac{b^2}{2}\left(1-\frac{1}{n}\right),\quad U(h,P_n)=\frac{b^2}{2}. \tag{117}$$

Thus:

$$L(h)\ge\sup_n L(h,P_n)=\frac{b^2}{2},\quad U(h)\le\lim_n U(h,P_n)=\frac{b^2}{2}. \tag{118}$$

Since $L(h)=U(h)$, $h(x)$ is integrable with:

$$\int_0^b h(x)dx=\frac{b^2}{2}. \tag{119}$$

Theorem 87 (32.5)**.** *A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if for all $\varepsilon>0$, there exists a partition P such that:*

$$U(f,P)-L(f,P)<\varepsilon. \tag{120}$$

Theorem 88 (33.1)**.** *Any monotone function on $[a,b]$ is integrable.*

Theorem 89 (33.2)**.** *Any continuous function on $[a,b]$ is integrable.*

Definition 69 (Mesh (32.6))**.** The mesh of a partition $P=\{t_0,t_1,\dots,t_n\}$ is:

$$\text{mesh}(P)=\max_{1\le k\le n}(t_k-t_{k-1}). \tag{121}$$

Theorem 90 (32.7)**.** *A bounded $f:[a,b]\to\mathbb{R}$ is integrable if and only if for all $\varepsilon>0$, there exists $\delta>0$ such that:*

$$U(f,P)-L(f,P)<\varepsilon \quad \textit{whenever mesh}(P)<\delta. \tag{122}$$

Definition 70 (Riemann Integral (32.8))**.** Let $f:[a,b]\to\mathbb{R}$ be bounded. For a partition $P=\{t_0,t_1,\dots,t_n\}$ and $x_k\in[t_{k-1},t_k]$, define the Riemann sum:

$$S=\sum_{k=1}^{n} f(x_k)(t_k-t_{k-1}). \tag{123}$$

f is Riemann integrable if there exists $r\in\mathbb{R}$ such that for all $\varepsilon>0$, there exists $\delta>0$ such that:

$$|S-r|<\varepsilon \quad \text{whenever mesh}(P)<\delta. \tag{124}$$

Theorem 91 (32.9)**.** *A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. In this case:*

$$\int_a^b f = R\int_a^b f. \tag{125}$$

Theorem 92 (Integrability Criterion (32.5))**.** *A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if:*

$$\forall\varepsilon>0,\exists \textit{ a partition } P \textit{ such that } U(f,P)-L(f,P)<\varepsilon. \tag{126}$$

Theorem 93 (Monotone Functions Are Integrable (33.1))**.** *Any monotone function on $[a,b]$ is integrable.*

Proof. Assume f is increasing. Fix $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that:

$$\frac{(f(b) - f(a))(b - a)}{n} < \varepsilon. \tag{127}$$

Consider the partition P with $t_k = a + kh$ for $0 \leq k \leq n$, where $h = \frac{b-a}{n}$. Then:

$$U(f, P) - L(f, P) = h \sum_{k=1}^{n} (f(t_k) - f(t_{k-1})) = \frac{(f(b) - f(a))(b - a)}{n} < \varepsilon. \tag{128}$$

□

Theorem 94 (Continuous Functions Are Integrable (33.2))**.** *Any continuous function on $[a, b]$ is integrable.*

Proof. Fix $\varepsilon > 0$. By uniform continuity, $\exists \delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta$. Choose $n \in \mathbb{N}$ such that $h = \frac{b-a}{n} < \delta$, and partition P with $t_k = a + kh$. Then:

$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b - a}. \tag{129}$$

Thus:

$$U(f, P) - L(f, P) < hn \cdot \frac{\varepsilon}{b - a} = \varepsilon. \tag{130}$$

□

Definition 71 (Mesh (32.6))**.** The mesh of a partition $P = \{t_0, t_1, \ldots, t_n\}$ is:

$$\text{mesh}(P) = \max_{1 \leq k \leq n} (t_k - t_{k-1}). \tag{131}$$

Theorem 95 (Integrability and Mesh (32.7))**.** *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if:*

$$\forall \varepsilon > 0, \exists \delta > 0 \textit{ such that } U(f, P) - L(f, P) < \varepsilon \textit{ whenever mesh}(P) < \delta. \tag{132}$$

Definition 72 (Riemann Integral (32.8))**.** Let $f : [a, b] \to \mathbb{R}$ be bounded. For a partition $P = \{t_0, t_1, \ldots, t_n\}$ and $x_k \in [t_{k-1}, t_k]$, define the Riemann sum:

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}). \tag{133}$$

f is Riemann integrable if:

$$\exists r \in \mathbb{R} \text{ such that } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |S - r| < \varepsilon \text{ whenever mesh}(P) < \delta. \tag{134}$$

Theorem 96 (Equivalence of Riemann and Darboux Integrability (32.9))**.** *A bounded $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. In this case:*

$$R \int_a^b f = \int_a^b f. \tag{135}$$

Proposition 40 (Exercise 32.7)**.** *If f is integrable on $[a, b]$, and $f = g$ except at finitely many points, then g is integrable on $[a, b]$ and:*

$$\int_a^b f = \int_a^b g. \tag{136}$$

Remark 4. *The statement fails if the set of exceptions is countably infinite. For example:*

$$f(x) = 0, \quad g(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases} \tag{137}$$

f is integrable, but g is not.

Theorem 97 (Linearity and Comparison of Integrals (33.3, 33.4(i)))**.** *Suppose f, g are integrable on $[a, b]$, and $c \in \mathbb{R}$. Then:*

1. *cf is integrable, and:*

$$\int_a^b cf = c \int_a^b f. \tag{138}$$

2. *$f+g$ is integrable, and:*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g. \tag{139}$$

3. *If $f \geq g$, then:*

$$\int_a^b f \geq \int_a^b g. \tag{140}$$

Theorem 98 (Triangle Inequality for Integrals (33.5))**.** *If f is integrable on $[a,b]$, then $|f|$ is integrable, and:*

$$\left|\int_a^b f\right| \leq \int_a^b |f|. \tag{141}$$

Proposition 41. *If f is integrable on $[a,b]$, then f^2 is integrable.*

Corollary 23. *If f, g are integrable on $[a,b]$, then fg is integrable.*

Proof. Express fg as:

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right). \tag{142}$$

Since $f+g$ and $f-g$ are integrable, their squares are integrable, and hence fg is integrable. □

Theorem 99 (Piecewise Monotone and Continuous Functions (33.8))**.** *Suppose $f : [a,b] \to \mathbb{R}$ is either:*

1. *Piecewise monotone and bounded, or*

2. *Piecewise continuous.*

Then f is integrable.

Proof. Partition $[a,b]$ such that f is monotone or uniformly continuous on each subinterval. On each subinterval, f is integrable. By additivity of the integral, f is integrable on $[a,b]$. □

Proposition 42. *Suppose (f_n) is a sequence of integrable functions on $[a,b]$ that converges uniformly to f. Then f is integrable, and:*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. \tag{143}$$

Theorem 100 (Bounded Convergence (33.11))**.** *Suppose (f_n) are integrable on $[a,b]$, $|f_n| \leq M$ for all n, $f_n \to f$ pointwise on $[a,b]$, and f is integrable. Then:*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f. \tag{144}$$

Theorem 101 (Monotone Convergence (33.12))**.** *Suppose (f_n) are integrable on $[a,b]$, $f_1 \leq f_2 \leq \cdots$, $f_n \to f$ pointwise on $[a,b]$, and f is integrable. Then:*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f. \tag{145}$$

Example 51 (Application of Monotone Convergence)**.** *Let $f_n(x) = \frac{1}{1+nx^3}$ on $[0,1]$. Then:*

$$\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx = 0, \tag{146}$$

where

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \in (0,1]. \end{cases} \tag{147}$$

Theorem 102 (Fundamental Theorem of Calculus I (34.1))**.** *Suppose $g : [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b), and g' is integrable on $[a,b]$. Then:*

$$\int_a^b g'(x)\,dx = g(b) - g(a). \tag{148}$$

Example 52. *Compute $\int_a^b x^n\,dx$. Use $g(x) = \frac{x^{n+1}}{n+1}$, so:*

$$\int_a^b x^n\,dx = \frac{b^{n+1} - a^{n+1}}{n+1}. \tag{149}$$

Proof. Partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. By the Mean Value Theorem:

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}, \tag{150}$$

for some $x_k \in (t_{k-1}, t_k)$. Then:

$$L(g', P) \leq \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) = g(b) - g(a) \leq U(g', P). \tag{151}$$

Thus $\int_a^b g'(x)\,dx = g(b) - g(a)$. □

Theorem 103 (Integration by Parts (34.2))**.** *Suppose $u, v : [a,b] \to \mathbb{R}$ are continuous, differentiable on (a,b), and u', v' are integrable on $[a,b]$. Then:*

$$\int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx = u(b)v(b) - u(a)v(a). \tag{152}$$

Example 53. *Compute $\int_0^\pi x\cos x\,dx$. Let $u(x) = x$, $v'(x) = \cos x$:*

$$\int_0^\pi x\cos x\,dx = x\sin x\big|_0^\pi - \int_0^\pi \sin x\,dx = 0 - (-2) = -2. \tag{153}$$

Theorem 104 (Fundamental Theorem of Calculus II (34.3))**.** *Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Define:*

$$F(x) = \int_a^x f(t)\,dt. \tag{154}$$

If f is continuous at c, then F is differentiable at c, with $F'(c) = f(c)$.

Example 54. *Let $G(x) = \int_{x^2}^2 \sin(t^2)\,dt$. Then $G'(x) = -2x\sin(x^4)$ by the Chain Rule.*

Proof. Let $F(x) = \int_a^x f(t)\,dt$. Then:

$$F'(c) = \lim_{x\to c} \frac{F(x) - F(c)}{x - c} = \lim_{x\to c} \frac{\int_c^x f(t)\,dt}{x - c}. \tag{155}$$

Since f is continuous at c, $|f(t) - f(c)| \leq \varepsilon$ for $|t - c| < \delta$, and thus:

$$\lim_{x\to c} \frac{\int_c^x f(t)\,dt}{x - c} = f(c). \tag{156}$$

□

Theorem 105 (Change of Variable (34.4))**.** *Suppose $u : J \to I$, u' is continuous, and $f : I \to \mathbb{R}$ is continuous. Then for $a, b \in J$:*

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(t)\,dt. \tag{157}$$

Example 55. *Compute $\int_1^4 \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$. Let $u(x) = \sqrt{x}$, then $u'(x) = \frac{1}{2\sqrt{x}}$:*

$$\int_1^4 \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = \int_1^2 2\sin t\,dt = 2(-\cos t)\big|_1^2 = 2(\cos 1 - \cos 2). \tag{158}$$

Proposition 43 (Exercise 33.9, Lecture 32)**.** *Suppose (f_n) is a sequence of integrable functions on $[a,b]$ converging uniformly to f. Then:*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f. \tag{159}$$

Corollary 24. *If g_n are integrable on $[a,b]$, and $f = \sum_{n=0}^\infty g_n$ converges uniformly, then f is integrable, and:*

$$\int_a^b f = \sum_{n=0}^\infty \int_a^b g_n. \tag{160}$$

Example 56. *Let $f_n(x) = \frac{1}{n}\sin(n^2 x)$. Then $f_n \to 0$ uniformly on $\mathbb{R}$. However:*

$$f_n'(x) = n\cos(n^2 x). \tag{161}$$

If $x = \frac{p}{q}\pi$, then $f_n'(x)$ does not converge, even pointwise.

Definition 73 (Radius of Convergence)**.** For a power series $\sum_{n=0}^{\infty} a_n x^n$, let:

$$\beta = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{162}$$

The radius of convergence is $R = \frac{1}{\beta}$.

Theorem 106 (Uniform Convergence (26.1))**.** *The series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R_1, R_1]$ for $R_1 < R$.*

Corollary 25. *The series $\sum_{n=0}^{\infty} a_n x^n$ converges to a continuous function on $(-R, R)$.*

Lemma 8 (Differentiation and Integration (26.3))**.** *If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then:*

1. *$\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R,*
2. *$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ has radius of convergence R.*

Theorem 107 (Integration of Power Series (26.4))**.** *Suppose $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has radius of convergence R. Then for $|x| < R$:*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}. \tag{163}$$

Example 57. *For $f(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ $(R = 1)$:*

$$-\ln(1-x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x| < 1. \tag{164}$$

Theorem 108 (Differentiation of Power Series (26.5))**.** *Suppose $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has radius of convergence R. Then for $|t| < R$:*

$$f'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}. \tag{165}$$

Theorem 109 (Abel's Theorem (26.6))**.** *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. If the series converges at R (or $-R$), then f is continuous at R (or $-R$).*

Theorem 110 (Abel's Theorem (26.6))**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. If the series converges at R (or $-R$), then f is continuous at R (or $-R$.*

Example 58.

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2. \tag{166}$$

Let $g(t) = \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$ $(R = 1)$, and $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$. The series diverges at -1 but converges at 1:

$$\ln(1+x) = \int_0^x g(t)\,dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k. \tag{167}$$

Thus:

$$f(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2. \tag{168}$$

Alternating Series

Proposition 44 (Alternating Series Test)**.** *Suppose $a_1 \geq a_2 \geq \cdots \geq 0$. Then:*

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k = a_1 - a_2 + a_3 - \cdots \tag{169}$$

converges if and only if $\lim_{k\to\infty} a_k = 0$.

Example 59.

$$1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4}. \tag{170}$$

Let $f(x)=\arctan x$, *so* $f'(x)=\frac{1}{1+x^2}$. *For* $|x|<1$:

$$f'(x)=\sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad f(x)=\int_0^x f'(t)\,dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}. \tag{171}$$

At $x=1$:

$$\frac{\pi}{4}=\arctan 1=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}. \tag{172}$$

Definition 74 (Convexity)**.** A function f on I is convex if:

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}, \quad \forall x,y\in I. \tag{173}$$

Proposition 45. *If f is convex, then:*

$$f((1-t)x+ty)\leq(1-t)f(x)+tf(y), \quad \forall t\in(0,1). \tag{174}$$

Proposition 46. *If f is differentiable on I, and f' is increasing, then f is convex.*

Corollary 26. *If f is twice differentiable on I, and $f''\geq 0$, then f is convex.*

Example 60.

1. *$f(x)=e^x$ is convex on $\mathbb{R}$ because $f''(x)=e^x>0$.*
2. *$g(x)=\ln x$ is concave on $(0,\infty)$ because $g''(x)=-\frac{1}{x^2}<0$.*

Theorem 111 (Jensen's Inequality)**.** *If f is convex on I, $x_1,\dots,x_n\in I$, $t_1,\dots,t_n\geq 0$, and $\sum_{i=1}^n t_i=1$, then:*

$$f\left(\sum_{i=1}^n t_i x_i\right)\leq\sum_{i=1}^n t_i f(x_i). \tag{175}$$

Proposition 47 (Arithmetic-Geometric Means Inequality)**.** *If $x_1,\dots,x_n>0$, $t_1,\dots,t_n>0$, and $\sum_{i=1}^n t_i=1$, then:*

$$\sum_{i=1}^n t_i x_i\geq\prod_{i=1}^n x_i^{t_i}. \tag{176}$$

Corollary 27 (Special Case)**.** *If $x_1,\dots,x_n>0$, then:*

$$\frac{x_1+\cdots+x_n}{n}\geq\sqrt[n]{x_1\cdots x_n}. \tag{177}$$

Theorem 112 (Jensen's Inequality)**.** *Let f be a convex function on an interval I, and let $x_1,\dots,x_n\in I$ with $t_1,\dots,t_n\geq 0$ and $\sum_{i=1}^n t_i=1$. Then:*

$$f\left(\sum_{i=1}^n t_i x_i\right)\leq\sum_{i=1}^n t_i f(x_i). \tag{178}$$

If f is concave, the inequality is reversed.

Proposition 48 (Power Mean Inequality)**.** *Suppose $r>1$ and $x_1,\dots,x_n\geq 0$. Then:*

$$\frac{x_1+\cdots+x_n}{n}\leq\left(\frac{x_1^r+\cdots+x_n^r}{n}\right)^{1/r}. \tag{179}$$

If $r=2$, this gives the inequality between arithmetic and quadratic means:

$$\frac{x_1+\cdots+x_n}{n}\leq\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}. \tag{180}$$

Proof. On $[0,\infty)$, $f(x)=x^r$ is convex because $f'(x)=rx^{r-1}$ is increasing. Apply Jensen's Inequality with $t_i=\frac{1}{n}$:

$$\left(\frac{x_1+\cdots+x_n}{n}\right)^r\leq\frac{x_1^r+\cdots+x_n^r}{n}. \tag{181}$$

Taking the r-th root gives the result. □

Proposition 49 (Arithmetic-Harmonic Mean Inequality)**.** *If* $x_1, \dots, x_n > 0$*, then:*

$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}. \tag{182}$$

Proof. Let $g(x) = \frac{1}{x}$, which is convex on $(0, \infty)$. Let $y_i = \frac{1}{x_i}$. By Jensen's Inequality with $t_i = \frac{1}{n}$:

$$\frac{1}{n} \sum_{i=1}^{n} g(y_i) = \frac{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}{n} \geq g\left(\frac{1}{n} \sum_{i=1}^{n} y_i\right) = \frac{n}{x_1 + \cdots + x_n}. \tag{183}$$

□

Proposition 50. *There exists a bounded, uniformly continuous function* $f : \mathbb{R} \to \mathbb{R}$ *that is differentiable nowhere.*

Sketch. Construct a 1-periodic function $f(x) = \sum_{k=0}^{\infty} 8^{-k} s(64^k x)$, where $s(x)$ is the sawtooth function:

$$s(x) = \phi(x - \lfloor x \rfloor), \quad \phi(t) = \min\{t, 1 - t\}. \tag{184}$$

1. f is bounded and uniformly continuous by the Weierstrass M-test.
2. For any $x, \delta > 0, A > 0$, there exists y with $|x - y| \leq \delta$ and $|f(x) - f(y)| \geq A|x - y|$, showing f is nowhere differentiable.

□

Theorem 113. *Let* $p_1 < p_2 < \cdots$ *be the increasing sequence of prime numbers. Then:*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \quad \textit{diverges.} \tag{185}$$

Proof. Assume $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges. Let $\alpha = \sum_{n=1}^{\infty} \frac{1}{p_{2n}^2} < 1$, and $\beta = \sum_{n=K+1}^{\infty} \frac{1}{p_n} < 1 - \alpha$ for some K. Consider N such that $N > 2K/(1 - \alpha - \beta)$. Counting arguments on $\{1, 2, \dots, N\}$ lead to a contradiction. □

0.0.1 parallelogram inequality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$