TAM 470 - HW 5 Solutions

Problem 1

$$\theta''(t) + \frac{g}{l}\theta(t) = 0 \qquad - \quad (3.1)$$

$$\theta(0) = \theta_0 \quad \text{and} \quad \theta'(0) = \omega_0 \qquad - \quad (3.2)$$

(a) Let $\theta'(t) = \beta(t) \implies \theta''(t) = \beta'(t)$

Using these in (3.1) and (3.2),

$$\beta'(t) + \frac{g}{l}\theta(t) = 0$$

$$\theta(0) = \theta_0 \quad \text{and} \quad \beta(0) = \omega_0$$

In the matrix form,

$$\boxed{\begin{bmatrix} \theta'(t) \\ \beta'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \beta(t) \end{bmatrix} \quad \text{with } \theta(0) = \theta_0 \text{ and } \beta(0) = \omega_0}$$

(b) Let λ be the eigen values.

$\det|A - I\lambda| = 0$ and $A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -g/l & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + \frac{g}{l} = 0$$

or

$$\boxed{\lambda = i\sqrt{\frac{g}{l}}, \quad -i\sqrt{\frac{g}{l}}}$$

(c) The eigen values are purely imaginary
$\Rightarrow$ The forward Euler method is
unstable. Refer to Figure 4.2 in Moin

(d) The backward Euler scheme is
unconditionally stable for all eigenvalues
with a non-positive real part.

(e) Referring to Moin, Page 59 and 60, when $\lambda_R = 0$,
$A = B \Rightarrow |\sigma| = 1$. This is called
neutral stability (i.e no error in amplitude).
Can also interpret as unconditionally stable w.r.t amplitude

(f) (i) Forward Euler formulation

$$\begin{bmatrix} \dfrac{\theta_{n+1} - \theta_n}{h} \\ \dfrac{\beta_{n+1} - \beta_n}{h} \end{bmatrix} = [A] \begin{bmatrix} \theta_n \\ \beta_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_{n+1} \\ \beta_{n+1} \end{bmatrix} = \left([I] + h[A] \right) \begin{bmatrix} \theta_n \\ \beta_n \end{bmatrix}$$

(ii) Backward Euler formulation:

$$\begin{bmatrix} \dfrac{\theta_{n+1} - \theta_n}{h} \\ \dfrac{\beta_{n+1} - \beta_n}{h} \end{bmatrix} = [A] \begin{bmatrix} \theta_{n+1} \\ \beta_{n+1} \end{bmatrix}$$

$$\begin{bmatrix} \theta_{n+1} \\ \beta_{n+1} \end{bmatrix} - h[A] \begin{bmatrix} \theta_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} \theta_n \\ \beta_n \end{bmatrix}$$

$$\Rightarrow \left[[I] - h[A] \right] \begin{bmatrix} \theta_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} \theta_n \\ \beta_n \end{bmatrix}$$

(iii) Trapezoidal rule formulation:-

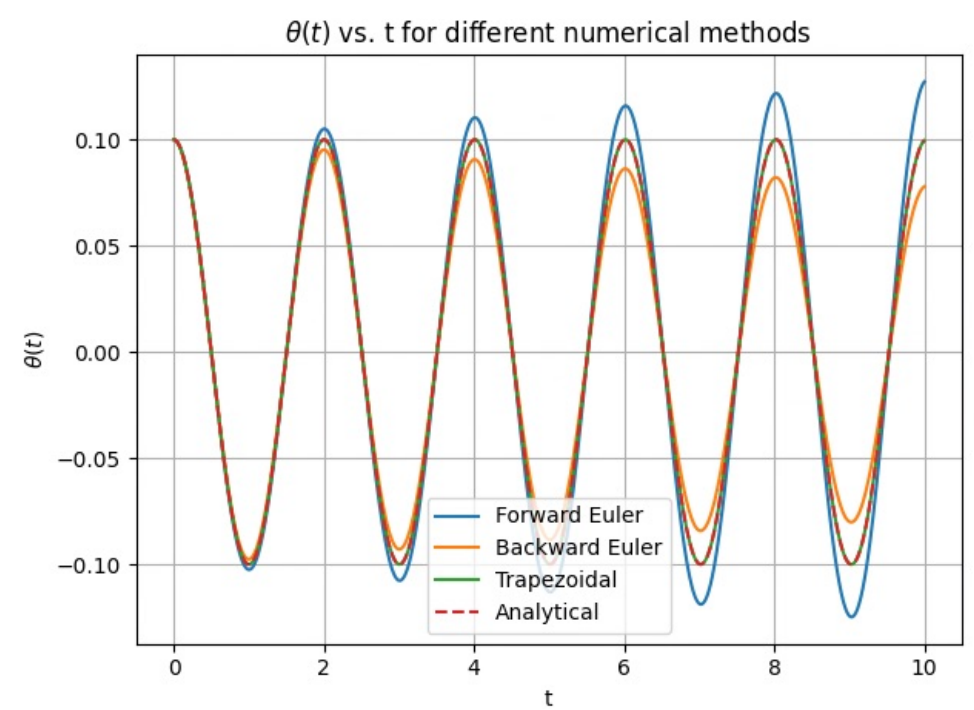
$$\begin{bmatrix} \dfrac{\theta_{n+1} - \theta_n}{h} \\ \dfrac{\beta_{n+1} - \beta_n}{h} \end{bmatrix} = [A] \begin{bmatrix} \dfrac{\theta_{n+1} + \theta_n}{2} \\ \dfrac{\beta_{n+1} + \beta_n}{2} \end{bmatrix}$$

Rearranging,

$$\left[[I] - \frac{h}{2}[A] \right] \begin{bmatrix} \theta_{n+1} \\ \beta_{n+1} \end{bmatrix} = \left[[I] + \frac{h}{2}[A] \right] \begin{bmatrix} \theta_n \\ \beta_n \end{bmatrix}$$

where $[I]$ is the (2×2) identity matrix.

(g) The $\theta(t)$ vs t plot is shown below.



→ The amplitude of forward euler is increasing which demonstrates unstable behavior.

→ We see the dampening behavior of Backward Euler. (Stable but not accurate).

→ Trapezoid rule captures the oscillatory behavior and is stable. The plot overlaps with analytical solution the most.

Problem 2

$$y' = e^{\sin y} - ty, \quad y(0) = 1$$

$$f(y,t) = e^{\sin y} - ty$$

$$\frac{\partial f}{\partial y} = f_y = e^{\sin y} \cdot \cos y - t$$

(a) Trapezoidal rule,

$$\frac{y_{n+1} - y_n}{h} = \frac{f(y_{n+1}, t_{n+1}) + f(y_n, t_n)}{2}$$

$\Rightarrow$ Bring all terms to LHS,

$$F(y_{n+1}) = y_{n+1} - y_n - \frac{h}{2} f(y_{n+1}, t_{n+1}) - \frac{h}{2} f(y_n, t_n) = 0$$

$$F(y_{n+1}) = y_{n+1} - y_n - \frac{h}{2}\left(e^{\sin y_{n+1}} - t_{n+1} y_{n+1} + e^{\sin y_n} - t_n y_n\right) = 0$$

(b) $$DF(y_{n+1}) = \frac{\partial F}{\partial y_{n+1}} = 1 - \frac{h}{2} f_y(y_{n+1}, t_{n+1})$$

$$\therefore DF(y_{n+1}) = 1 - \frac{h}{2}\left(e^{\sin y_{n+1}} \cdot \cos y_{n+1} - t_{n+1}\right)$$

(c) Newton-Raphson:

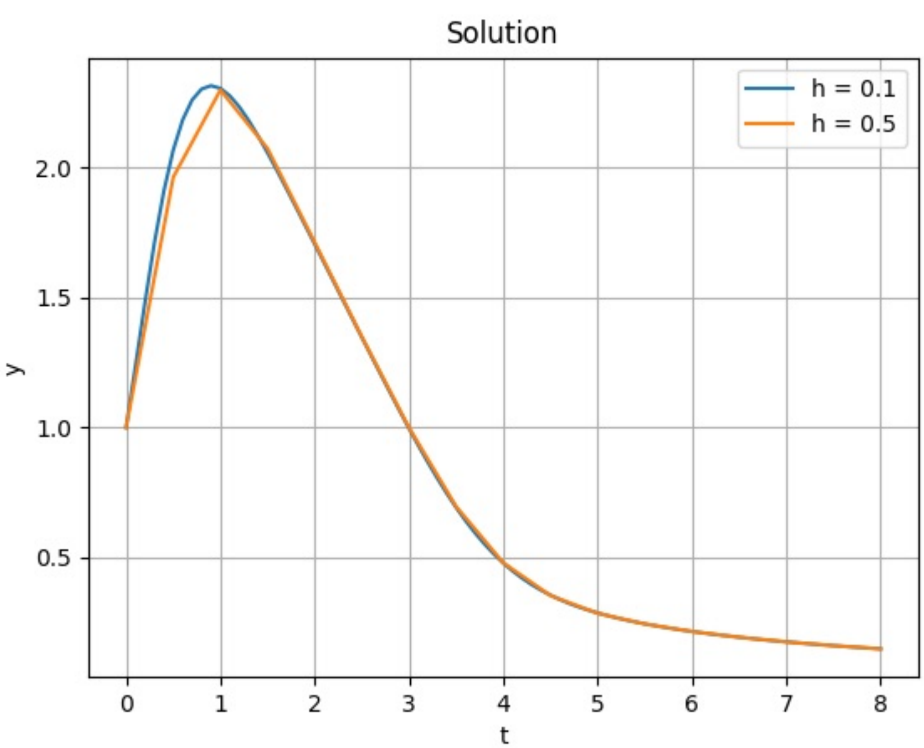
$$y_{n+1}^{k+1} = y_{n+1}^{k} - \frac{F(y_{n+1}^k)}{DF(y_{n+1}^k)}$$

$$\therefore y_{n+1}^{k+1} = y_{n+1}^{k} - \frac{\left(y_{n+1}^k - y_n - \frac{h}{2}\left(e^{\sin y_{n+1}^k} - t_{n+1} y_{n+1}^k + e^{\sin y_n} - t_n y_n\right)\right)}{1 - \frac{h}{2}\left(e^{\sin y_{n+1}^k} \cos y_{n+1}^k - t_{n+1}\right)}$$

Note that y_n is not updated and we can just write y_n instead of y_n^k as it does not change with 'k' iteration counter.

(d) Implement on PL

(e)



Problem 3

$$y' = \lambda(y - \cos t) - \sin t \quad , \quad y(0) = y_0$$

$$y_{exact} = e^{\lambda t}(y_0 - 1) + \cos t$$

(a) Let the RHS be $f = \lambda(y - \cos t) - \sin t$

Trapezoid rule $\rightarrow$ $\dfrac{y_{n+1} - y_n}{h} = \dfrac{1}{2}(f_n + f_{n+1})$

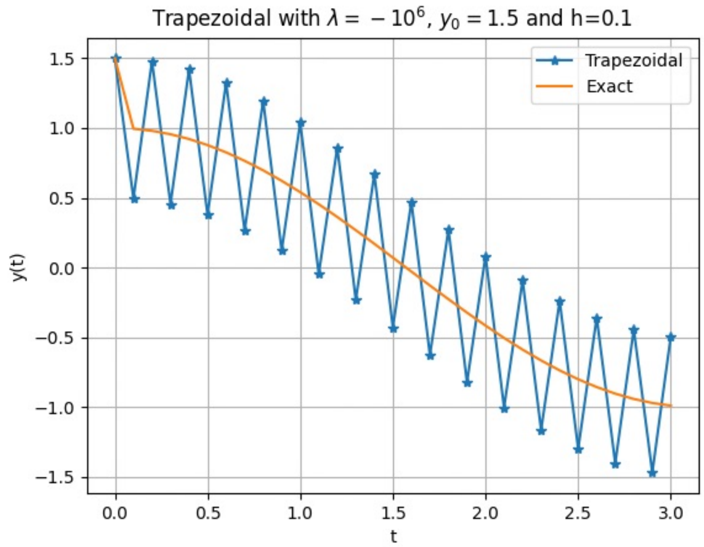
$$\therefore \quad \frac{y_{n+1} - y_n}{h} = \frac{1}{2}\Big[\lambda(y_{n+1} + y_n - \cos t_n - \cos t_{n+1}) - \sin t_n - \sin t_{n+1}\Big]$$

Rearranging.

$$\boxed{y_{n+1}\left[1 - \frac{\lambda h}{2}\right] = \left[1 + \frac{\lambda h}{2}\right] y_n - \frac{\lambda h}{2}(\cos t_n + \cos t_{n+1}) - \frac{h}{2}(\sin t_n + \sin t_{n+1})}$$

(b) We use the expression from (a) with $\lambda = -10^6$, $h = 0.1$, $t \in [0, 3]$ and $y_0 = 1.5$

Plot shown below.



(c) $|h\lambda| >> 1$ and $|\lambda| >> 1$

Recall trapezoidal formula from (a),

$$\cancel{\lambda h}\; y_{n+1}\left[\frac{1}{\lambda h}-\frac{1}{2}\right] = \cancel{\lambda h}\left[\left(\frac{1}{\lambda h}+\frac{1}{2}\right)y_n - \frac{1}{2}\left(\cos t_n+\cos t_{n+1}\right) - \frac{1}{2\lambda}\left(\sin t_n + \sin t_{n+1}\right)\right]$$

$$|h\lambda| >> 1 \implies \frac{1}{|h\lambda|} << 1 \to 0$$

$$|\lambda| >> 1 \implies \frac{1}{|\lambda|} << 1 \to 0$$

Using above approximations,

$$-\frac{y_{n+1}}{2} = \frac{y_n}{2} - \frac{1}{2}\left(\cos t_n + \cos t_{n+1}\right)$$

$$\longrightarrow \quad \underline{\underline{y_{n+1} \simeq -y_n + \cos t_n + \cos t_{n+1}}}$$

(d) From (c),

$$y_{n+1} - \cos t_{n+1} \simeq -y_n + \cos t_n$$

$$n=0,\; y_1 - \cos t_1 \simeq -y_0 + 1 = (-1)\left(y_0 - 1\right)$$

$$n=1,\; y_2 - \cos t_2 \simeq -y_1 + \cos t_1 = (-1)\left(y_1 - \cos t_1\right)$$
$$(-1)^2\left(y_0 - 1\right)$$

$$n=2,\; y_3 - \cos t_3 \simeq -y_2 + \cos t_2 = (-1)\left[y_2 - \cos t_2\right]$$
$$(-1)^3\left(y_0 - 1\right)$$

When $n = (k-1)$, where k is some integer,

$$y_k - \cos t_k \approx (-1)^k (y_0 - 1)$$

or

$$y_n \approx (-1)^n (y_0 - 1) + \cos t_n$$

(e) The oscillating behavior is due to the term $(-1)^n (y_0 - 1)$ which changes sign for each increment of n.

The oscillation will not occur with $y_0 = 1$ as this makes that oscillatory term 0 and $y_n \approx \cos t_n$.