

MATH 447: Real Variables - Homework #3

Jerich Lee

December 2, 2024

Problem 1 (11.5). Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.

- Give the set of subsequential limits for (q_n) .
- Give the values of $\limsup q_n$ and $\liminf q_n$.

Solution 1. 1. The set of subsequential limits for (q_n) is $q \in \mathbb{Q} : 0 < q \leq 1$.

2. (a) $\limsup q_n = 1$
(b) $\liminf q_n = 0$

Problem 2 (11.11). Let S be a bounded set. Prove there is an increasing sequence (s_n) of points in S such that $\lim_{n \rightarrow \infty} s_n = \sup S$. Compare Exercise 10.7. Note: If $\sup S$ is in S , it's sufficient to define $s_n = \sup S$ for all n .

Solution 2. $u_n = \sup s_n$, and $s_n \leq \sup s_n$. If S is bounded, then there exists a real number a such that $s_n \leq a$. By Theorem 11.4 in Ross, there exists a monotonic subsequence for bounded set S . We shall prove that this sequence is increasing.

Proof. Because S is bounded, we know that there exists $\sup S$ by the Axiom of Completeness. We also know that monotone bounded sequences converge by Theorem 10.2 in Ross. Therefore, we can say:

$$\sup S - s_n < 1 \tag{1}$$

Where 1 is chosen arbitrarily. We can then choose a subsequence of S such that it satisfies the following conditions:

$$\sup S - s_n = \min \left(\sup S - s_{n-1}, \frac{1}{n} \right) \tag{2}$$

Choosing such s_n enforces that the sequence is increasing, and is getting closer and closer to the supremum. With the values selected, we now have a subsequence of S such that $n_1 < n_2 < \dots < n_k$, $s_{n_1} < s_{n_2} < \dots < s_{n_k}$, and $\lim s_{n_k} = \sup S$. \square

Problem 3 (12.3 (d,e)). Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:

- $(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$
- $(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \dots)$

Find:

- $\limsup(s_n + t_n)$
- $\limsup s_n + \limsup t_n$

Solution 3. 1. $\limsup(s_n + t_n) = 3$

2. $\limsup(s_n) + \limsup(t_n) = 4$

Problem 4 (12.4). Show $\lim_{N \rightarrow \infty} \sup_{n > N}(s_n + t_n) \leq \lim_{N \rightarrow \infty} \sup_{n > N} s_n + \lim_{N \rightarrow \infty} \sup_{n > N} t_n$ for bounded sequences (s_n) and (t_n) . Hint: First show:

- $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$

Then apply Exercise 9.9 (c).

Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$

Exercise 9.9 (c):

- Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Solution 4. Proof. We want to find the following:

$$\limsup_{N \rightarrow \infty} s_N + t_N \leq (\limsup_{N \rightarrow \infty} s_n) + (\limsup_{N \rightarrow \infty} t_n) \quad (3)$$

We know that the following holds true, as this is the definition of \limsup .

$$\limsup_{N \rightarrow \infty} s_N = \limsup_{M \rightarrow \infty} \{s_N | N \geq M\} \quad (4)$$

We also know that any element of a sequence s_n is always less than or equal to the supremum of s_n due to the Axiom of Completeness. Then, we can state the following.

$$N > M \implies s_N \leq \sup\{s_N | N \geq M\} \quad (5)$$

Therefore, the following also holds true from above:

$$s_N + t_N \leq \sup\{s_N | N > M\} + \sup\{t_N | N > M\} \quad (6)$$

The RHS of the above is a constant, so the following is true:

$$(s_N + t_N \leq C) \quad (7)$$

If any value of $s_n + t_n$ is always less than the RHS of above, then we can say the the supremum of $s_n + t_n$ is also less than or equal to the RHS of above:

$$\sup\{s_N + t_N | N > M\} \leq \sup\{s_N | N > M\} + \sup\{t_N | N > M\} \quad (8)$$

We have now established $\sup(s_n + t_n) \leq \sup s_n + \sup t_n$. Now, we can use a familiar limit theorem to prove our original hypothesis.

$$\limsup_{N \rightarrow \infty} s_N + t_N \quad (9)$$

We know that the above is equal to the following from the definition of \limsup . Then, we can take the limit of both sides. From Exercise 9.9c, we know that if given N_0 s.t. $s_n \leq t_n$ for all $n > N_0$ $\lim_{n \rightarrow \infty} s_n$ and $\lim_{n \rightarrow \infty} t_n$ exist, then $\lim_{n \rightarrow \infty} s_n \leq \lim_{n \rightarrow \infty} t_n$

$$= \limsup_{M \rightarrow \infty} \{s_N + t_N | N > M\} \leq \lim_{M \rightarrow \infty} (\sup \{s_N | N > M\} + \sup \{t_N | N > M\}) \quad (10)$$

From a known limit theorem, we know that $\lim_{n \rightarrow \infty} s_n + t_n = \lim_{n \rightarrow \infty} s_n + \lim_{n \rightarrow \infty} t_n$:

$$= \limsup_{M \rightarrow \infty} \{s_N | N > M\} + \limsup_{M \rightarrow \infty} \{t_N | N > M\} \quad (11)$$

$$\limsup_{N \rightarrow \infty} (s_N + t_N) \leq \limsup_{N \rightarrow \infty} s_N + \limsup_{N \rightarrow \infty} t_N \quad (12)$$

□

Problem 5 (12.12). Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

- Show:

$$\lim_{N \rightarrow \infty} \inf_{n > N} s_n \leq \lim_{N \rightarrow \infty} \inf_{n > N} \sigma_n \leq \lim_{N \rightarrow \infty} \sup_{n > N} \sigma_n \leq \lim_{N \rightarrow \infty} \sup_{n > N} s_n \quad (13)$$

Hint: for the last inequality, show first that $M > N$ implies:

$$\sup \{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_n) + \sup \{s_n : n > N\} \quad (14)$$

- Show that if $\lim_{n \rightarrow \infty} s_n$ exists, then $\lim_{n \rightarrow \infty} \sigma_n$ exists and $\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} s_n$.
- Give an example where $\lim_{n \rightarrow \infty} \sigma_n$ exists, but $\lim_{n \rightarrow \infty} s_n$ does not exist.

Solution 5. 1. Proof.

$$\liminf_{N \rightarrow \infty} s_N \leq \liminf_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} s_N \quad (15)$$

$$\lim_{n \rightarrow \infty} s_N = s \implies s = \liminf_{N \rightarrow \infty} s_N \leq \liminf_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} s_N = s \quad (16)$$

By the Squeeze Theorem,

$$\implies \liminf_{N \rightarrow \infty} \sigma_N = \limsup_{n \rightarrow \infty} \sigma_N = s \quad (17)$$

$$\implies \lim_{N \rightarrow \infty} \sigma_N = s \quad (18)$$

We wish to find the third inequality as shown below, as the second inequality is obvious, and the first inequality can be proved similarly to the third inequality.

$$\limsup_{N \rightarrow \infty} \sigma_N \leq \limsup_{N \rightarrow \infty} s_N \quad (19)$$

We will fix L . Then, if $M > L$, we can consider $\sup \{\sigma_N | N > M\}$

$$\sigma_N = \frac{s_1 + \dots + s_N}{N} \quad (20)$$

$$= \frac{s_1 + \dots + s_N}{N} + \frac{s_{L+1} + \dots + s_N}{N} \quad (21)$$

We know that there are $N - (L + 1) + 1 = N - L$ total terms in the denominator of the second term of the RHS above. Therefore, we can state the following:

$$\sigma_N \leq \frac{s_1 + \dots + s_L}{N} + \underbrace{\frac{N - L}{N}}_{\leq 1} \sup \{s_N | N > L\} \quad (22)$$

$$\leq \frac{s_1 + \dots + s_L}{M} + \sup \{s_N | N > L\} \quad (23)$$

$$\sup \{\sigma_N | N > M\} \leq \frac{s_1 + \dots + s_L}{M} + \sup \{s_N | N > L\} \quad (24)$$

We know that the RHS above is independent of N , so we take the limit of both sides above. Doing so results in the following:

$$\limsup_{N \rightarrow \infty} \sup \sigma_N = \limsup_{M \rightarrow \infty} \sigma_N \{N | N > M\} \quad (25)$$

$$\leq \lim_{M \rightarrow \infty} \left(\underbrace{\frac{s_1 + \dots + s_L}{M}}_{\text{goes to 0 as } M \rightarrow \text{infinity}} + \sup \{s_N | N > L\} \right) \quad (26)$$

$$\limsup_{N \rightarrow \infty} \sigma_N \leq \sup \{s_N | N > L\} \quad (27)$$

□

2. Proof.

$$|s_n - L| < \varepsilon \quad (28)$$

$$\sigma_n = \frac{1}{n} \sum_{i=1}^n s_n \quad (29)$$

$$\lim \frac{1}{n} \sum_{i=1}^n s_n \quad (30)$$

$$n > N \implies \sum_{i=1}^n s_n \quad (31)$$

$$\implies |ns_n - nL| \quad (32)$$

$$\frac{1}{n} |n(s_n - L)| < \varepsilon \quad (33)$$

$$= |(s_n - L)| < \varepsilon \quad (34)$$

$$= |s_n - L| < \varepsilon \quad (35)$$

$$= \lim \sigma_n = L = \lim s_n \quad (36)$$

□

Proof.

Suppose $s_n = (-1)^n$. Then $\lim s_n = \text{DNE}$. But we know:

$$\frac{-1 \text{ or } 0}{N} \xrightarrow[N \rightarrow \infty]{} 0 \quad (37)$$

Therefore, the $\lim_{n \rightarrow \infty} \sigma_n$ exists if $\lim_{n \rightarrow \infty} s_n = \text{DNE}$. □

Problem 6 (12.13). Let (s_n) be a bounded sequence in \mathbb{R} . Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many s_n are $\geq a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \lim_{N \rightarrow \infty} \inf_{n > N} s_n$ and $\inf B = \lim_{N \rightarrow \infty} \sup_{n > N} s_n$.

Solution 6. From Theorem 11.7, there exists a monotonic subsequence whose limit is $\limsup s_n$ and $\liminf s_n$. So $\limsup s_n$ and $\liminf s_n$ exist. From Theorem 10.1, we know that these limits converge because s_n is bounded. Our goal is to show that $\limsup s_n = \inf B$, and $\liminf s_n = \sup A$, for sets $A = \{s_n : s_n < a\}$, and $B = \{s_n : s_n > b\}$, where A, B both have finite cardinality.

Proof. Choose N_1 s.t. $n > N_1 \implies \{n : |s_n - a| < \varepsilon\}$ is infinite. By Theorem 11.2, a is a subsequential limit. Choose N_2 s.t. $n > N_2 \implies \{n : |s_n - b| < \varepsilon\}$ is infinite. By 11.2, b is also a subsequential limit. To prove that $a = \sup A$, choose $N = \min \{d(s_N, a)\}$. Then

$N = n + 1 \implies s_{n+1} > a$, so s_n is not the least upper bound of A . So $a = \sup A$. A similar argument follows for $\inf B = b$. We know that $a < b$, or else B would be infinite, contradictory to B being finite in the given. Let $S = \{a, b\}$. Then, by Theorem 11.8, $\inf S = \liminf s_n$, $\sup S = \limsup s_n$. This implies $\sup A = \liminf s_n$, $\inf B = \limsup s_n$. \square