Definition of subsequences

Definition (Definition 11.1)

A sequence (t_k) is a subsequence of (s_n) if there exists a strictly increasing sequence $n_1 < n_2 < \dots$ so that $t_k = s_{n_k}$ for any k.

Example. Suppose $s_n = \frac{1}{n}$. Subsequences: $t_k = \frac{1}{k^2}$ $(n_k = k^2)$
$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \dots$ (reciprocals of prime numbers). $n_k = k$-th prime.
Not subsequences: $\frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{1}{8}, \dots$ (one term of (s_n) is repeated),
$\frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \dots$ (order is changed).

Example. $s_n = (-1)^n + \frac{1}{n}$. Then (s_n) diverges. $s_{2k} = 1 + \frac{1}{2k}$, so (s_{2k}) converges. To be proved later: every sequence has a subsequence with limit.

Subsequences: alternative definition

Recall: a sequence $(s_n)_{n\geqslant m}$ is a function $s:\{m,m+1,\dots\}\to\mathbb{R}$;
$s_n=s(n)$.
A subsequence is a composition $s\circ\sigma$, where
$\sigma:\{\ell,\ell+1,\dots\}\to\{m,m+1,\dots\}$ is strictly increasing:
$\sigma(k)=n_k$, so $s\circ\sigma(k)=s(\sigma(k))=s_{n_k}$.

Lemma (Subsequences of subsequences – Exercise 11.6)

Any subsequence of a subsequence of (s_n) is a subsequence of (s_n).

Proof. A subsequence of s is a function $s\circ\sigma$. A further subsequence of that is $(s\circ\sigma)\circ\tau=s\circ(\sigma\circ\tau)$, where $\tau:\{j,j+1,\dots\}\to\{\ell,\ell+1,\dots\}$ is strictly increasing.
But $\sigma\circ\tau$ is strictly increasing as well. ∎

Convergence of subsequences

Theorem (Theorem 11.3, essentially)

If a sequence (s_n) has a limit (real or $\pm\infty$), then any subsequence has the same limit.

Proof for the case of $\lim s_n = s \in \mathbb{R}$. Let $t_k = s_{n_k}$, with $n_1 < n_2 < \dots$.
Want: $\forall \varepsilon > 0\ \exists K$ s.t. $|t_k - s| < \varepsilon$ whenever $k > K$.
Find N s.t. $|s_n - s| < \varepsilon$ for $n > N$.
$\lim_k n_k = +\infty$, hence $\exists\, K$ s.t. $n_K > N$.
If $k > K$, then $n_k > N$, hence $|s_{n_k} - s| < \varepsilon$. But $s_{n_k} = t_k$. ■

Existence of subsequences with limit

Theorem (Theorem 11.4)

Every sequence has a monotone subsequence.

Recall: any monotone sequence has limit (in $\mathbb{R} \cup \{\pm\infty\}$); any bounded monotone sequence converges.

Corollary (Theorem 11.5, Bolzano-Weierstrass)

Any bounded sequence has a convergent subsequence.

Example. $s_n = (-1)^n\left(1 + \frac{1}{n}\right)$. This sequence is bounded, divergent. Indeed, $-2 \leqslant s_n \leqslant 2$. Suppose for contradiction $s_n \to s$. Find N s.t. $|s_n - s| < 1$ for $n > N$. If $n > N$ is even, then $1 < s_n < s + 1$, and $-1 > s_{n+1} > s - 1$, hence $2 = (1 - (-1)) < (s + 1) - (s - 1) = 2$, impossible!
Pass to subsequence with $n_k = 2k$: $s_{2k} = 1 + \frac{1}{2k} \longrightarrow 1$.

Proof: every sequence has monotone subsequence

Consider sequence $(s_n)_{n \geqslant 1}$. We say that n is dominant if $s_m < s_n$ whenever $m > n$.
Example: $s_n = (-1)^n + \frac{1}{n}$. n is dominant iff it is even.
Either there are infinitely many dominant terms, or only finitely many.

Case 1. Inf. many dominant terms, $n_1 < n_2 < \ldots$.
$s_{n_k} > s_m$ if $m > n_k$, hence $s_{n_1} > s_{n_2} > \ldots$.
Found a (strictly) decreasing subsequence!

Case 2: Fin. many dominant terms. Find $N \in \mathbb{N}$ s.t. no $n \geqslant N$ is dominant. Let $n_1 = N$; it is not dominant, so find $n_2 > n_1$ s.t. $s_{n_2} \geqslant s_{n_1}$. n_2 is not dominant, so find $n_3 > n_2$ s.t. $s_{n_3} \geqslant s_{n_2}$. $s_{n_1} \leqslant s_{n_2} \leqslant \ldots$, so we have found an increasing sequence! ∎

Subsequential limits

Definition (Definition 11.6)

For a sequence (s_n), a subsequential limit is any limit of a subsequence (in $\mathbb{R} \cup \{\pm\infty\}$).

Theorem (Theorem 11.2)

Suppose (s_n) is a sequence.

1. *$t \in \mathbb{R}$ is a subsequential limit iff $\forall \varepsilon > 0$, $\{n : |s_n - t| < \varepsilon\}$ is infinite.*
2. *$t = +\infty$ $(t = -\infty)$ is a subsequential limit iff (s_n) is not bounded above (resp. below).*

Moreover, in both (1) and (2), $\exists\ n_1 < n_2 < \ldots$ s.t. the sequence $(s_{n_k})_k$ is monotone, and $s_{n_k} \xrightarrow[k]{} t$.

Subsequential limits: a partial proof of Theorem 11.2

$\Rightarrow$. Prove: if $t \in \mathbb{R}$ is a subseq. limit, then $\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = \infty$.
Suppose $n_1 < n_2 < \ldots$, $s_{n_k} \to t$. Find $K \in \mathbb{N}$ s.t. $|s_{n_k} - t| < \varepsilon$ whenever $k > K$. Then $|s_n - t| < \varepsilon$ if $n \in \{n_k : k > K\}$ (inf. set).

$\Leftarrow$. Prove: If $t \in \mathbb{R}$, $\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = \infty$, then $\exists$ monot. subseqeunce converging to t.
First find $n_1 < n_2 < \ldots$ s.t. $s_{n_k} \to t$. Pick n_1 s.t. $|s_{n_1} - t| < 2^{-1}$.
Pick $n_2 > n_1$ s.t. $|s_{n_2} - t| < 2^{-2}$. This is possible: $\{n : |s_n - t| < 2^{-2}\}$ is infinite, hence meets $\{n_1 + 1, n_1 + 2, \ldots\}$.
This way obtain $n_1 < n_2 < \ldots$, $|s_{n_k} - t| < 2^{-k}$. Then $s_{n_k} \to t$.
(s_{n_k}) has a monotone subsequence. This is still a subsequence of (s_n), and it converges to t. $\blacksquare$

Example: an "exotic" sequence

Observation. If $a < b$, then $|(a,b) \cap \mathbb{Q}| = \infty$.
Proof. Suppose, for the sake of contradiction, that $|(a,b) \cap \mathbb{Q}| < \infty$ – that is, $(a,b) \cap \mathbb{Q} = \{c_1, \ldots, c_n\}$, with $c_1 < \ldots < c_n$. $(c_1, c_2) \cap \mathbb{Q} = \emptyset$, contradicting denseness of rationals (Section 4, p. 25). ∎

Example 3, p. 70. $\exists$ sequence (r_n) so that any $s \in \mathbb{R} \cup \{\pm\infty\}$ is a limit of a subsequence of (r_n).

$\mathbb{Q}$ can be enumerated (Picture 11.1): $\mathbb{Q} = \{r_1, r_2, \ldots\}$ (one can arrange for each rational number to be listed exactly once).

$\{r_1, r_2, \ldots\}$ is not bounded above (below), hence has a monotone sequence diverging to $+\infty$ (resp. $-\infty$).

$\forall\, t \in \mathbb{R}$ and $\varepsilon > 0$, $|\{n : |r_n - t| < \varepsilon\}| = \infty$, hence $\exists$ monotone subsequence converging to t. ∎

$\liminf$ and $\limsup$ as subsequential limits

Theorem (Theorem 11.7)

For any sequence (s_n), $\limsup s_n$ and $\liminf s_n$ are limits of monotone subsequences.

Proof: $\limsup s_n = t \in \mathbb{R}$ is a subsequential limit.
Need: $\forall\, \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = \infty$. For contradiction, suppose otherwise: $\exists\, \varepsilon > 0$ s.t. $|\{n : |s_n - t| < \varepsilon\}| < \infty$.
Let $N = \max\{n : |s_n - t| < \varepsilon\}$, then $|s_n - t| \geqslant \varepsilon \;\forall\, n > N$. For such n, either $s_n \geqslant t + \varepsilon$, or $s_n \leqslant t - \varepsilon$.

Case 1. $|\{n : s_n \geqslant t + \varepsilon\}| = \infty$. Then $\forall\, m\; \exists\, n > m$ s.t. $s_n \geqslant t + \varepsilon$.
$u_m = \sup_{n>m} s_n \geqslant t + \varepsilon$, hence $\limsup s_n = \lim u_m \geqslant t + \varepsilon$.

Case 2. $|\{n : s_n \geqslant t + \varepsilon\}| < \infty$. Then $\exists\, m$ s.t. $s_n \leqslant t - \varepsilon$ for $n > m$.
Then $u_m \leqslant t - \varepsilon$, so $\limsup s_n = \lim u_m \leqslant t - \varepsilon$.

Either way, we contradict $\limsup s_n = t$. ∎

The set of subsequential limits

Theorem (Theorem 11.8)

Suppose (s_n) is a sequence, S is its set of subsequential limits.

1. *S is non-empty.*
2. $\inf S = \liminf s_n$, $\sup S = \limsup s_n$.
3. $\lim s_n$ *exists iff* S *consists of a single point. Then* $\{\lim s_n\} = S$.

The proof will be given in the next lecture

Proof. (i) S contains $\liminf s_n$ and $\limsup s_n$, hence it is non-empty.

(iii) If $\lim s_n = s$, then $\limsup s_n = s = \liminf s_n$, hence, by (ii), $\sup S = s = \inf S$. Thus, $S = \{s\}$.
If $S = \{s\}$, then $\limsup s_n = s = \liminf s_n$, hence $\lim s_n$ exists, and $= s$. ■

Proof of Theorem 11.8(ii)

Proof: $\sup S = \limsup s_n$ **($S =$ set of subseq. limits of (s_n)).**

Let $s = \sup S$. Already know: $\limsup s_n \in S$, hence $s \geqslant \limsup s_n$. Need: $s \leqslant \limsup s_n$. Suppose, for contradiction, $s > \limsup s_n$ (so $s \in \mathbb{R} \cup \{+\infty\}$).

Case 1: $s \in \mathbb{R}$. Find $\varepsilon \in (0, s - \limsup s_n)$, and $t \in S$ s.t. $t > \limsup s_n + \varepsilon$. $|\{n : |s_n - t| < \varepsilon\}| = \infty$, hence, for any m, $\exists\ n > m$ s.t. $s_n > t - \varepsilon$. Thus, $u_m = \sup_{n>m} s_n > t - \varepsilon$, hence $\limsup s_n = \lim u_m \geqslant t - \varepsilon > \limsup s_n$, contradiction! $\square$

Case 2: $s = +\infty$. As $\limsup s_n < \infty$, (s_n) is bounded above: $\exists A$ s.t. $s_n \leqslant A$ for any n. Then $S \subset [-\infty, A]$, so $\sup S \leqslant A < \infty$. Again, contradiction! $\blacksquare$