Connectedness (Section 22)

Definition (22.1)

Suppose (S, d) is a metric space. $E \subset S$ is called disconnected if $\exists$ open $U_1, U_2 \subset S$ s.t.: (i) $E \subset U_1 \cup U_2$; (ii) $(E \cap U_1) \cap (E \cap U_2) = \emptyset$; (iii) $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$.

Proposition

An open set E is disconnected if and only if $E = E_1 \cup E_2$, where E_1, E_2 are disjoint non-empty open sets.

Proof. (1) Suppose E is disconnected. Take U_1, U_2 as in the definition. Then $E_i = E \cap U_i$ have the desired properties.

(2) If E_1, E_2 exist, take $U_i = E_i$ ($i = 1, 2$). Then U_1, U_2 are as in the definition of disconnectedness. ∎

Proposition (Definition 22.1 + remark following it)

E is disconnected iff $\exists A, B \subset E$ s.t. $E = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B^- = \emptyset = A^- \cap B$.

A continuous image of a connected set is connected

Theorem (22.2)

Suppose (S, d) and (S^*, d^*) are metric spaces, $E \subset S$ is connected, and $f : S \to S^*$ is continuous. Then $f(E)$ is connected.

Proof of the contrapositive. Suppose $E \subset S$, and $f(E) \subset S^*$ is disconnected. We show that E is disconnected.

Write $f(E) = C \cup D$, where C and D are non-void, $C^- \cap D = \emptyset = C \cap D^-$.
Thus, $f^{-1}(C^-) \cap f^{-1}(D) = \emptyset = f^{-1}(C) \cap f^{-1}(D^-)$.
Let $A = f^{-1}(C) \cap E$, $B = f^{-1}(D) \cap E$. $E = A \cup B$, A and B are non-void.

$f^{-1}(C^-)$ is closed, and $A \subset f^{-1}(C) \subset f^{-1}(C^-)$. Thus, $A^- \subset f^{-1}(C^-)$.
$f^{-1}(C^-) \cap f^{-1}(D) = \emptyset$, hence $A^- \cap B = \emptyset$.
Similarly, $A \cap B^- = \emptyset$. So, E is disconnected. ∎

Path connected sets

Definition (P. 165)

A path is a continuous function $\gamma : [0, 1] \to S$, where (S, d) is a metric space.

Definition (22.4)

A set E is path connected if $\forall a, b \in E\ \exists$ path $\gamma : [0, 1] \to E$ s.t. $\gamma(0) = a$, $\gamma(1) = b$.

Theorem (22.5)

Every path connected set is connected.

Remark. A connected set need not be path connected (Exercise 22.4).

Path connected sets are connected

Theorem (22.5)

Every path connected set is connected.

Proof. We show that, if E is disconnected, then it is not path connected. Find open U_1, U_2 so that $E \subset U_1 \cup U_2$, $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$, $E \cap U_1 \cap U_2 = \emptyset$. Pick $a \in E \cap U_1$, $b \in E \cap U_2$. We show that a and b cannot be linked by a path.

Suppose, for the sake of contradiction, that there exists a continuous function $\gamma : [0,1] \to E$, with $\gamma(0) = a, \gamma(1) = b$. Let $F = \gamma([0,1])$. Then $a \in F \cap U_1, b \in F \cap U_2$, so $F \cap U_1 \neq \emptyset$ and $F \cap U_2 \neq \emptyset$. Further, $F \cap U_1 \cap U_2 \subset E \cap U_1 \cap U_2 = \emptyset$. Thus, F is disconnected. This is impossible: intervals are connected, and so are continuous images of connected sets. ∎

Path connected sets are connected

Theorem (22.5)

Every path connected set is connected.

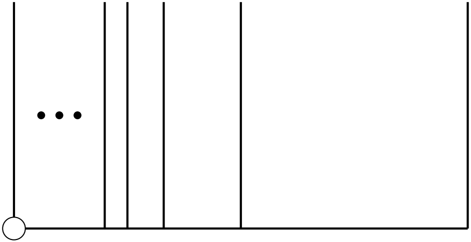
Another proof. Suppose, for contradiction, that E is path connected, but not connected. Write $E = A \cup B$, A, B non-empty, $A^- \cap B = \emptyset = A \cap B^-$. Find $a \in A, b \in B$. Let $\gamma : [0,1] \to E$ be a path s.t. $\gamma(0) = a$, $\gamma(1) = b$. Let $T = \gamma^{-1}(A)$, $S = \gamma^{-1}(B)$. Then $0 \in T, 1 \in S$, $T \cap S = \emptyset$, $T \cup S = [0,1]$. $T = \gamma^{-1}(A) \subset \gamma^{-1}(A^-)$; the latter set is closed, so $T^- \subset \gamma^{-1}(A^-)$. $\gamma^{-1}(A^-) \cap \gamma^{-1}(B) = \emptyset$, hence $T^- \cap S = \emptyset$. Similarly, $T \cap S^- = \emptyset$.

Note that $0 \in T$, and $1 \in S$, so $1 \notin T^-$. Let $c = \sup T$. If $c \in T$, then $c \notin S^-$, so $\exists \delta > 0$ s.t. $(c - \delta, c + \delta) \cap S = \emptyset$, so $\sup T > c$. If $c \in S$, then $\exists \delta > 0$ s.t. $(c - \delta, c + \delta) \cap T = \emptyset$, so $\sup T < c$.

Either way we get a contradiction! ■

Example: Connected set which is not path connected

Example. Consider $E \subset \mathbb{R}^2$, with $E = E_1 \cup E_2$, where $E_1 = \{0\} \times (0, 1]$, $E_2 = (0, 1] \times \{0\} \cup \left(\cup_{n \in \mathbb{N}} \{\frac{1}{n}\} \times (0, 1]\right)$.



E is connected, but not path connected.

A connected set which is not path connected

Example. Consider $E \subset \mathbb{R}^2$, with $E = E_1 \cup E_2$, where $E_1 = \{0\} \times (0,1]$, $E_2 = (0,1] \times \{0\} \cup \left(\cup_{n \in \mathbb{N}} \{\frac{1}{n}\} \times (0,1]\right)$.
E is connected, but not path connected.

(1) E is connected.
Suppose, for the sake of contradiction, that E is not connected – that is, $\exists$ open $U_1, U_2 \subset \mathbb{R}^2$ s.t.: (i) $E \subset U_1 \cup U_2$;
(ii) $(E \cap U_1) \cap (E \cap U_2) = \emptyset$; (iii) $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$.

E_1, E_2 are path connected, hence connected. So $E_1 \subset U_1$, $E_2 \subset U_2$ (or vice versa). $\exists r > 0$ s.t. $\mathbf{B}_r^o\big((0,1)\big) \subset U_1 \subset \mathbb{R}^2 \backslash E_2$. But $\left(\frac{1}{n}, 1\right) \subset E_2 \cap \mathbf{B}_r^o\big((0,1)\big)$ for $n > \frac{1}{r}$, contradiction! ■

A connected set which is not path connected, cont'd

Example. Consider $E \subset \mathbb{R}^2$, with $E = E_1 \cup E_2$, where $E_1 = \{0\} \times (0,1]$, $E_2 = (0,1] \times \{0\} \cup \big(\cup_{n\in\mathbb{N}} \{\frac{1}{n}\} \times (0,1]\big)$.
E is connected, but not path connected.

(2) E is not path connected (optional).
Suppose, for the sake of contradiction, that $\exists$ path $\gamma : [0,1] \to E$ s.t. $\gamma(0) = (0,1)$, $\gamma(1) = (1,0)$. $\gamma([0,1])$ is compact, $(0,0) \notin \gamma([0,1])$. $\exists \varepsilon > 0$ s.t. $[0,\varepsilon] \times [0,\varepsilon] \cap \gamma([0,1]) = \emptyset$.

Write $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$, where $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}$ are continuous (see Proposition 21.2). $\gamma_1\big([0,1]\big) \in [0,1]$, $\gamma_1(0) = 0$, $\gamma_1(1) = 1$. $\Rightarrow$ $\gamma_1\big([0,1]\big) = [0,1]$. Find $x \in (0,\varepsilon)\backslash\big\{\frac{1}{n} : n \in \mathbb{N}\big\}$, and $t \in (0,1)$ s.t. $\gamma_1(t) = x$. Then $\gamma_2(t) \geqslant \varepsilon$. This is impossible: if $(x,y) \in E$, then $y = 0$.
■

Convex sets (Section 22 Example 3)

Definition

$E \subset \mathbb{R}^n$ is convex if, $\forall \vec{x}, \vec{y} \in E$, $\{(1-t)\vec{x} + t\vec{y} : 0 \leqslant t \leqslant 1\} \subset E$.

Proposition

Any convex set is path connected.

Example. $\mathbf{B}_r(\vec{x}_0) \subset \mathbb{R}^n$ is convex.

We handle the case of $\vec{x}_0 = \vec{0}$, $r = 1$. Need to show: if $\|\vec{x}\|, \|\vec{y}\| \leqslant 1$ and $0 < t < 1$, then $\|(1-t)\vec{x} + t\vec{y}\| \leqslant 1$.

$\|(1-t)\vec{x} + t\vec{y}\|^2 = \langle (1-t)\vec{x} + t\vec{y}, (1-t)\vec{x} + t\vec{y} \rangle =$
$(1-t)^2\|\vec{x}\|^2 + t^2\|\vec{y}\|^2 + 2t(1-t)\langle \vec{x}, \vec{y} \rangle$.

Bunyakovsky-Cauchy-Schwarz: $\left|\langle \vec{x}, \vec{y} \rangle\right| \leqslant \|\vec{x}\|\|\vec{y}\|$.

$\|(1-t)\vec{x} + t\vec{y}\|^2 \leqslant (1-t)^2\|\vec{x}\|^2 + t^2\|\vec{y}\|^2 + 2t(1-t)\|\vec{x}\|\|\vec{y}\| =$
$\big((1-t)\|\vec{x}\| + t\|\vec{y}\|\big)^2 \leqslant 1.$ ∎

Graphs of functions and path connectedness

The graph of a function $f : I \to \mathbb{R}$ ($I \subset \mathbb{R}$ is an interval):
$\mathbf{G}(f) = \big\{\big(x, f(x)\big) : x \in I\big\}$.

Proposition (Example 4 from Section 22)

$\mathbf{G}(f)$ *is path connected iff* f *is continuous on interval* I.

Remark. Exercise 22.4: a discontinuous f s.t. $\mathbf{G}(f)$ is connected.

Proposition 21.2. The function $f : S \to \mathbb{R}^n : x \mapsto (f_1(x), \ldots, f_n(x))$ is continuous iff $f_i : S \to \mathbb{R}$ is continuous, for $1 \leqslant i \leqslant n$.

Proof: f is continuous on $I \Rightarrow \mathbf{G}(f)$ is path connected.
Suppose $\vec{x} = \big(a, f(a)\big), \vec{y} = \big(b, f(b)\big) \in \mathbf{G}(f)$. A path between $\vec{x}$ and $\vec{y}$:
$\gamma(t) = \Big((1-t)a + tb, f\big((1-t)a + tb\big)\Big)$ $(t \in [0,1])$. ■

f is continuous on $I \Leftarrow \mathbf{G}(f)$ is path connected. See textbook.

A connected set which is not path connected

Consider $f : [0, \infty) \to \mathbb{R} : x \mapsto \begin{cases} \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$.

f is discontinuous at 0, hence $\mathbf{G}(f)$ is not path connected.

We shall show that that $\mathbf{G}(f)$ is connected.

Suppose, for the sake of contradiction, that $\mathbf{G}(f)$ is disconnected. Then $\exists$ open $U_1, U_2 \subset \mathbb{R}^2$ so that $\mathbf{G}(f) \subset U_1 \cup U_2$, $\mathbf{G}(f) \cap U_1 \cap U_2 = \emptyset$, while $\mathbf{G}(f) \cap U_1 \neq \emptyset$ and $\mathbf{G}(f) \cap U_2 \neq \emptyset$.

Write $\mathbf{G}(f) = E_1 \cup E_2$, where $E_1 = \{(0,0)\}$ and $E_2 \{(x, \sin 1/x) : x > 0\}$. Both E_1 and E_2 are path connected, hence connected. Thus, by relabeling, we assume $E_1 \subset U_1, E_2 \subset U_2$.

Find $r > 0$ so hat $\mathbf{B}^o_r(0,0) \subset U_1$. But $\forall n \in \mathbb{N}$ $(1/(n\pi), 0) \in E_2$. Pick n so that $1/(n\pi) > r$, then $(1/(n\pi), 0)$ belongs to both $E_2 \subset U_2$ and to $\mathbf{B}^o_r(0,0) \subset U_1$. Thus, $(1/(n\pi), 0) \in E \cap U_1 \cap U_2$, a contradiction. ■