Darboux sums and integrals: reminders

If f is bounded on S, let $m(f,S) = \inf_{s\in S} f(s)$, $M(f,S) = \sup_{s\in S} f(s)$.
A partition of $[a,b]$ is $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$.
Lower Darboux sum: $L(f,P) = \sum_{k=1}^{n} m\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big)$.
Upper Darboux sum: $U(f,P) = \sum_{k=1}^{n} M\big(f,[t_{k-1},t_k]\big)\big(t_k - t_{k-1}\big)$.

Lower Darboux integral: $L(f) = \sup_P L(f,P)$.
Upper Darboux integral: $U(f) = \inf_P U(f,P)$.
The inf, sup are taken over all partitions P of $[a,b]$.

f is integrable if $L(f) = U(f)$. Denote $\int_a^b f = L(f) = U(f)$.

Theorem (Theorem 32.4)

If $f : [a,b] \to \mathbb{R}$ is bounded, then $L(f) \leqslant U(f)$.

Remark. From the definition, f is integrable when $U(f) = L(f)$. We only need to check $U(f) \leqslant L(f)$ to establish integrability.

Integrals: Example

Is $h(x) = x$ integrable on $[0, b]$? If it is, compute $\int_0^b h$.

If $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ is a partition, then, for any k,
$m\big(h, [t_{k-1}, t_k]\big) = \inf_{t \in [t_{k-1}, t_k]} h(t) = t_{k-1}$, and
$M\big(h, [t_{k-1}, t_k]\big) = \sup_{t \in [t_{k-1}, t_k]} h(t) = t_k$.
$L(h, P) = \sum_{k=1}^n t_{k-1}\big(t_k - t_{k-1}\big)$, $U(h, P) = \sum_{k=1}^n t_k\big(t_k - t_{k-1}\big)$.

To make computations easier, consider "equal" partitions
$P_n = \big(0 < \frac{b}{n} < \frac{2b}{n} < \ldots < \frac{(n-1)b}{n} < b\big)$ (that is, $t_k = \frac{kb}{n}$).
$L(h, P_n) = \sum_{k=1}^n \frac{(k-1)b}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^n (k-1) = \frac{b^2}{n^2}\frac{n(n-1)}{2} = \frac{b^2}{2}\big(1 - \frac{1}{n}\big)$.
$L(h) \geqslant \sup_n L(h, P_n) = \lim_n L(h, P_n) = \frac{b^2}{2}$. Similarly,
$U(h) \leqslant \lim_n U(h, P_n) = \frac{b^2}{2}$. However, $U(h) \geqslant L(h)$, hence
$U(h) = L(h) = \frac{b^2}{2}$. Thus, h is **integrable**, with $\int_0^b h = \frac{b^2}{2}$.
Can you interpret this geometrically?

A criterion of integrability

Theorem (32.5)

A bounded $f : [a, b] \to \mathbb{R}$ is integrable iff $\forall \varepsilon > 0$ $\exists$ partition P s.t. $U(f, P) - L(f, P) < \varepsilon$.

Recall that f is called integrable on $[a, b]$ if $L(f) = U(f)$.

Proof. **(1)** Suppose f is integrable – that is, $L(f) = U(f)$. Fix $\varepsilon > 0$. Find partitions R, Q s.t. $U(f, R) < U(f) + \frac{\varepsilon}{2}$, $L(f, Q) > L(f) - \frac{\varepsilon}{2}$. Let $P = R \cup Q$, then $L(f, P) \geqslant L(f, Q) > L(f) - \frac{\varepsilon}{2}$, and $U(f, P) < U(f) + \frac{\varepsilon}{2}$. Then $U(f, P) - L(f, P) < \varepsilon$.

(2) Suppose $\forall \varepsilon > 0$ $\exists$ partition P s.t. $U(f, P) - L(f, P) < \varepsilon$. As $U(f) \leqslant U(f, P)$ and $L(f) \geqslant L(f, P)$, we have $U(f) - L(f) < \varepsilon$. As ε is arbitrary, we conclude that $U(f) = L(f)$. ■

Monotone functions are integrable

Theorem (33.1)

Any monotone function on $[a, b]$ is integrable.

Proof for the case of increasing f.

Fix $\varepsilon > 0$. Need to find a partition P s.t. $U(f, P) - L(f, P) < \varepsilon$.

Find $n \in \mathbb{N}$ s.t. $\frac{(f(b)-f(a))(b-a)}{n} < \varepsilon$. Consider the "equal partition" P, with of points $t_k = a + kh$ $(0 \leqslant k \leqslant n)$, where $h = \frac{b-a}{n}$. Then $m(f, [t_{k-1}, t_k]) = f(t_{k-1})$ and $M(f, [t_{k-1}, t_k]) = f(t_k)$. $U(f, P) - L(f, P)$
$= \sum_{k=1}^{n} \big(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\big)(t_k - t_{k-1})$
$= h \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1})\big) = \frac{(f(b)-f(a))(b-a)}{n} < \varepsilon$. ■

Continuous functions are integrable

Theorem (33.2)

Any continuous function on $[a, b]$ is integrable.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Fix $\varepsilon > 0$. Need to find a partition P s.t. $U(f, P) - L(f, P) < \varepsilon$.
f is uniformly continuous. Find $\delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ if $|x - y| < \delta$. Find $n \in \mathbb{N}$ s.t. $h = \frac{b-a}{n} < \delta$. Consider the partition P consisting of points $t_k = a + kh$ $(0 \leqslant k \leqslant n)$. We claim that, for any k, $M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b-a}$. Indeed, find $x_k, y_k \in [t_{k-1}, t_k]$ s.t. $M(f, [t_{k-1}, t_k]) = f(x_k)$, $m(f, [t_{k-1}, t_k]) = f(y_k)$. $|x_k - y_k| < \delta$, so $|f(x_k) - f(y_k)| < \frac{\varepsilon}{b-a}$.
$U(f, P) - L(f, P) =$
$\sum_{k=1}^{n} \big(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\big)(t_k - t_{k-1}) < hn\frac{\varepsilon}{b-a} = \varepsilon$, since $hn = b - a$. $\blacksquare$

Mesh of a partition, and its applications

Definition (32.6)

The mesh of a partition $P = \big(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\big)$ is $\mathrm{mesh}(P) = \max_{1 \leqslant k \leqslant n}(t_k - t_{k-1})$ (length of longest subinterval).

Theorem (32.7)

A bounded $f : [a, b] \to \mathbb{R}$ is integrable iff $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ whenever $\mathrm{mesh}(P) < \delta$.

Remark. If δ is as above, $\mathrm{mesh}(P) < \delta$, then $U(f, P) - \int_a^b f, \int_a^b f - L(f, P) < \varepsilon$. Roughly speaking: $\lim_{\mathrm{mesh} P \to 0} U(f, P) = \int_a^b f = \lim_{\mathrm{mesh} P \to 0} L(f, P)$.

Proof of 32.7 (optional)

Need to show: f is integrable $\Rightarrow \forall\varepsilon > 0\ \exists\delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ if $\mathrm{mesh}(P) < \delta$.

Find a partition $Q = \big(a = s_0 < \ldots < s_m = b\big)$ s.t. $U(f, Q) - L(f, Q) < \frac{\varepsilon}{2}$. Let $B = \sup_{t\in[a,b]} |f(t)|$. We claim: $\delta = \frac{\varepsilon}{8mB}$ works.

Suppose $\mathrm{mesh}(P) < \delta$, let $R = P \cup Q$. We have $U(f, R) \leqslant U(f, Q)$ and $L(f, R) \geqslant L(f, Q)$, hence $U(f, R) - L(f, R) < \frac{\varepsilon}{2}$. Need: $U(f, P) \leqslant U(f, R) + \frac{\varepsilon}{4}$, $L(f, P) \geqslant L(f, R) + \frac{\varepsilon}{4}$.

Say $P = \big(a = t_0 < \ldots < t_N = b\big)$.
$U(f, P) = \sum_{k=1}^{N} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$.

Notation. $R = \big(a = r_0 < \ldots < r_M = b\big)$.
$R|_{[r_i,r_j]} := \big(r_i < r_{i+1} < \ldots < r_j\big)$.
$U_{[r_i,r_j]}\big(f, R|_{[r_i,r_j]}\big) = \sum_{k=i+1}^{j} M(f, [r_{k-1}, r_k])(r_k - r_{k-1})$;
$L_{[r_i,r_j]}\big(f, R|_{[r_i,r_j]}\big) = \sum_{k=i+1}^{j} m(f, [r_{k-1}, r_k])(r_k - r_{k-1})$.

$U(f, R) = \sum_{k=1}^{N} U_{[t_{k-1},t_k]}\big(f, R|_{[t_{k-1},t_k]}\big)$.

Proof of 32.7, part 2

If $(t_{k-1}, t_k) \cap Q = \emptyset$, then $R|_{[t_{k-1},t_k]} = \{t_{k-1}, t_k\}$, hence $U_{[t_{k-1},t_k]}\big(f, R|_{[t_{k-1},t_k]}\big) = M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$.

If $(t_{k-1}, t_k) \cap Q \neq \emptyset$, then $U_{[t_{k-1},t_k]}\big(f, R|_{[t_{k-1},t_k]}\big) \geqslant -B(t_k - t_{k-1})$, and $M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) - U_{[t_{k-1},t_k]}\big(f, R|_{[t_{k-1},t_k]}\big) \leqslant 2B(t_k - t_{k-1})$.

Let S be the set of all $k \in \{1, \ldots, N\}$ for which $(t_{k-1}, t_k) \cap Q \neq \emptyset$. Then $|S| \leqslant m$.

$U(f, P) - U(f, R) = \sum_{k\in S} \Big(M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) - U_{[t_{k-1},t_k]}\big(f, R|_{[t_{k-1},t_k]}\big)\Big) \geqslant \sum_{k\in S} 2B(t_k - t_{k-1}) \leqslant 2mB\mathrm{mesh}(P) < \frac{\varepsilon}{4}$.

Similarly, $L(f, R) - L(f, P) < \frac{\varepsilon}{4}$.

Conclusion: $U(f, P) - L(f, P) = \big(U(f, P) - U(f, R)\big) + \big(U(f, R) - L(f, R)\big) + \big(L(f, R) - L(f, P)\big) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$. ∎

We can shrink the interval of integration

Proposition (Exercise 32.8)

If f is integrable on $[a,b]$, and $[c,d] \subset [a,b]$, then f is integrable on $[c,d]$.

Proof. Need to show: $\forall \varepsilon > 0\ \exists$ partition P of $[c,d]$ s.t. $U_{[c,d]}(f,P) - L_{[c,d]}(f,P) < \varepsilon$. Find $\delta > 0$ s.t. $U(f,Q) - L(f,Q) < \varepsilon$ if Q is a partition of $[a,b]$, with $\mathrm{mesh}(Q) < \delta$.

Consider $Q = \big(a = s_0 < \ldots < s_{i-1} = c < \ldots < s_j = d < \ldots < s_m = b\big)$, with $\mathrm{mesh}(Q) < \delta$. We claim that $P = Q \cap [c,d]$ works. Indeed,
$U_{[c,d]}(f,P) - L_{[c,d]}(f,P) =$
$\sum_{\ell=i}^{j} \big(M(f,[s_{\ell-1},s_\ell]) - m(f,[s_{\ell-1},s_\ell])\big)(s_\ell - s_{\ell-1}) \leqslant$
$\sum_{\ell=1}^{m} \big(M(f,[s_{\ell-1},s_\ell]) - m(f,[s_{\ell-1},s_\ell])\big)(s_\ell - s_{\ell-1}) = U(f,Q) - L(f,Q) < \varepsilon.$
■

Riemann integration

Definition (32.8)

Suppose $f : [a, b] \to \mathbb{R}$ is bounded. For a partition $P = \big(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\big)$, and $x_k \in [t_{k-1}, t_k]$, define the Riemann sum $S = \sum_{k=1}^n f(x_k)(t_k - t_{k-1})$.
f is Riemann integrable if $\exists r \in \mathbb{R}$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|S - r| < \varepsilon$ when $\mathrm{mesh}(P) < \delta$. Notation: $r = \mathcal{R} \int_a^b f$ is the Riemann integral.

Note that the Riemann integral is unique.

Theorem (32.9)

A bounded $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff it is (Darboux) integrable. In this case, $\mathcal{R} \int_a^b f = \int_a^b f$.

Corollary 32.8. If the Riemann sums S_n correspond to partitions P_n, and $\lim_n \mathrm{mesh}(P_n) = 0$, then $\lim_n S_n = \int_a^b f$.

Proof: equivalence of Riemann and Darboux integrability

Remark. Any Riemann integrable function is bounded.

Proof: Darboux $\Rightarrow$ Riemann. Suppose $f : [a, b] \to \mathbb{R}$ is integrable. For $\varepsilon > 0$ find $\delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ if $\mathrm{mesh}(P) < \delta$. Note that $U(f, P) \geqslant \int_a^b f \geqslant L(f, P)$, hence $U(f, P) - \int_a^b f, \int_a^b f - L(f, P) < \varepsilon$.

For any Riemann sum S, $U(f, P) \geqslant S \geqslant L(f, P)$. If $\mathrm{mesh}(P) < \delta$, then $\int_a^b f + \varepsilon > U(f, P) \geqslant S \geqslant L(f, P) > \int_a^b f - \varepsilon$, hence $\left|S - \int_a^b f\right| < \varepsilon$. Thus, f is Riemann integrable, with $\mathcal{R}\int_a^b f = \int_a^b f$. ∎

Proof: equivalence of Riemann and Darboux integrability

Proof: Riemann $\Rightarrow$ Darboux (optional).
Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Fix a partition P; note that $U(f, P) = \sup S$, and $L(f, P) = \inf S$, where the sup and inf run over all Riemann sums corresponding to the partition P.

Fix $\varepsilon > 0$. Find $\delta > 0$ s.t. $\left|\mathcal{R}\int_a^b f - S\right| < \varepsilon$ whenever $\mathrm{mesh}(P) < \delta$. For such P, $S \in \left(\mathcal{R}\int_a^b f - \varepsilon, \mathcal{R}\int_a^b f + \varepsilon\right)$, hence
$\mathcal{R}\int_a^b f - \varepsilon \leqslant L(f, P) \leqslant U(f, P) \leqslant \mathcal{R}\int_a^b f + \varepsilon$. In particular,
$U(f) - L(f) \leqslant U(f, P) - L(f, P) \leqslant 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, we obtain
$U(f) = L(f)$, hence f is integrable. Further,
$\mathcal{R}\int_a^b f - \varepsilon \leqslant L(f) \leqslant U(f) \leqslant \mathcal{R}\int_a^b f + \varepsilon$ for any $\varepsilon > 0$, hence
$L(f) = U(f) = \mathcal{R}\int_a^b f$. ■