Problem 1

(a) Formulation :-

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial t^2} \quad , \quad x \in [0, L], \quad t \geq 0$$

$$\frac{\partial^2 T_j}{\partial x^2} = \frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2}$$

$$\therefore \frac{\partial T_j}{\partial t} = \frac{\alpha}{(\Delta x)^2}\left(T_{j+1} - 2T_j + T_{j-1}\right)$$

Using CN,

$$T_j^{n+1} - T_j^n = \underbrace{\frac{\alpha \Delta t}{2(\Delta x)^2}}_{\beta}\left(T_{j+1}^{n+1} + T_{j+1}^n - 2T_j^{n+1} - 2T_j^n + T_{j-1}^{n+1} + T_{j-1}^n\right)$$

$$-\beta T_{j+1}^{n+1} + (1+2\beta)T_j^{n+1} - \beta T_{j-1}^{n+1} = \beta T_{j+1}^n + (1-2\beta)T_j^n + \beta T_{j-1}^n$$

$$T_0^{n+1} = T_{hot} \quad \forall n$$

$\therefore$ When $j = 1$,

$$-\beta T_2^{n+1} + (1+2\beta)T_1^{n+1} - \beta T_{hot} = \beta T_2^n + (1-2\beta)T_1^n + \beta T_{hot}$$

When $j = N_x - 1$,

$$-\beta T_{N_x}^{n+1} + (1+2\beta) T_{N_x-1}^{n+1} - \beta T_{N_x-2}^{n+1} = \beta T_{N_x}^{n} + (1-2\beta) T_{N_x-1}^{n} + \beta T_{N_x-2}^{n}$$

Using the convection BC,

$$\frac{T_{N_x}^{n+1} - T_{N_x-1}^{n+1}}{\Delta x} = \gamma \left(T_\infty - T_{N_x}^{n+1} \right)$$

$$\Longrightarrow T_{N_x}^{n+1} = \frac{T_{N_x-1}^{n+1} + \gamma \Delta x T_\infty}{(1+\gamma \Delta x)} \quad \text{for all time-steps}$$

$\therefore$ The $j = N_x - 1$ equation becomes,

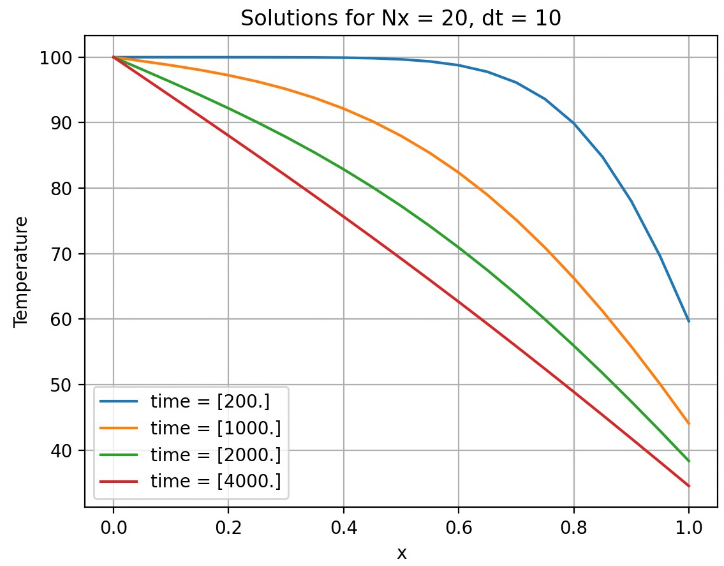
$$-\beta T_{N_x-2}^{n+1} + \left(1 + 2\beta - \frac{\beta}{1+\gamma\Delta x}\right) T_{N_x-1}^{n+1} = \beta T_{N_x}^{n} + (1-2\beta) T_{N_x-1}^{n} + \beta T_{N_x-2}^{n} + \frac{\beta \gamma \Delta x T_\infty}{(1+\gamma \Delta x)}$$

For $j = 2$ to $N_x - 2$,

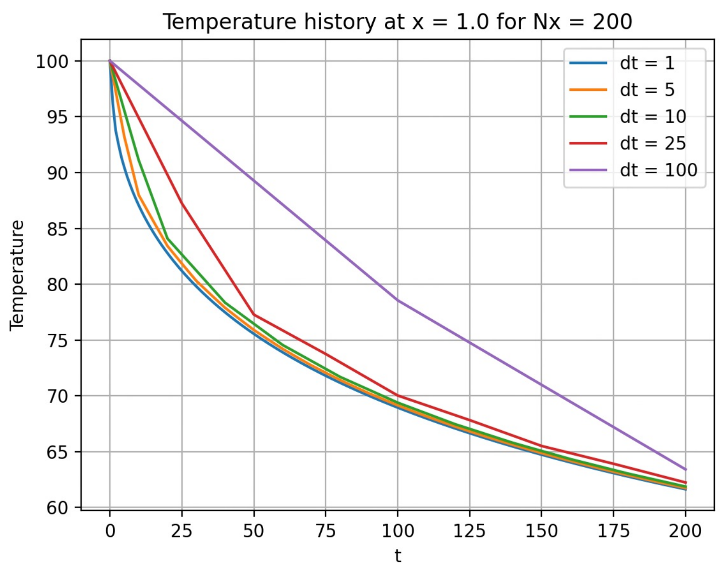
$$-\beta T_{j+1}^{n+1} + (1+2\beta) T_j^{n+1} - \beta T_{j-1}^{n+1} = \beta T_{j+1}^{n} + (1-2\beta) T_j^{n} + \beta T_{j-1}^{n}$$

T_0^{n+1} is given as T_{hot} and $T_{N_x}^{n+1} = \dfrac{T_{N_x-1}^{n+1} + \gamma \Delta x T_\infty}{(1+\gamma\Delta x)}$

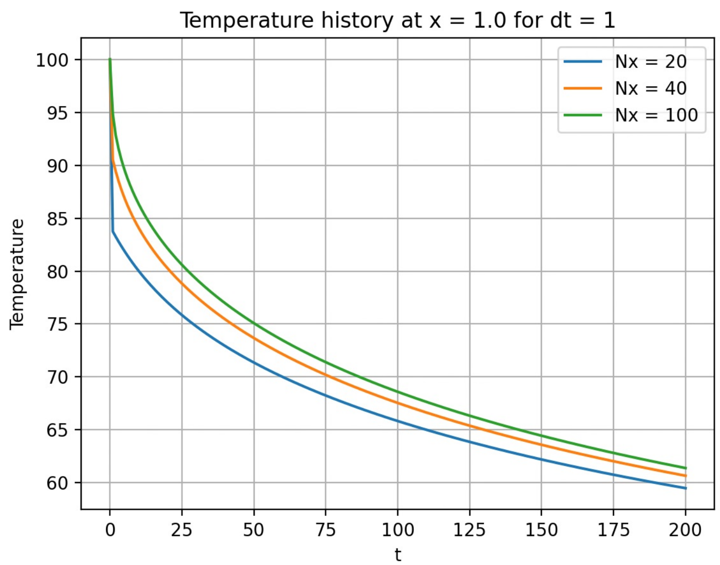
(b) (i)


Solutions for Nx = 20, dt = 10
Temperature
x
time = [200.]
time = [1000.]
time = [2000.]
time = [4000.]
100
90
80
70
60
50
40
0.0
0.2
0.4
0.6
0.8
1.0

(ii)


Temperature history at x = 1.0 for Nx = 200
Temperature
t
dt = 1
dt = 5
dt = 10
dt = 25
dt = 100
100
95
90
85
80
75
70
65
60
0
25
50
75
100
125
150
175
200

(iii)


Temperature history at x = 1.0 for dt = 1
Temperature
t
Nx = 20
Nx = 40
Nx = 100
100
95
90
85
80
75
70
65
60
0
25
50
75
100
125
150
175
200

Problem 2

Part 1 on Prairelearn.

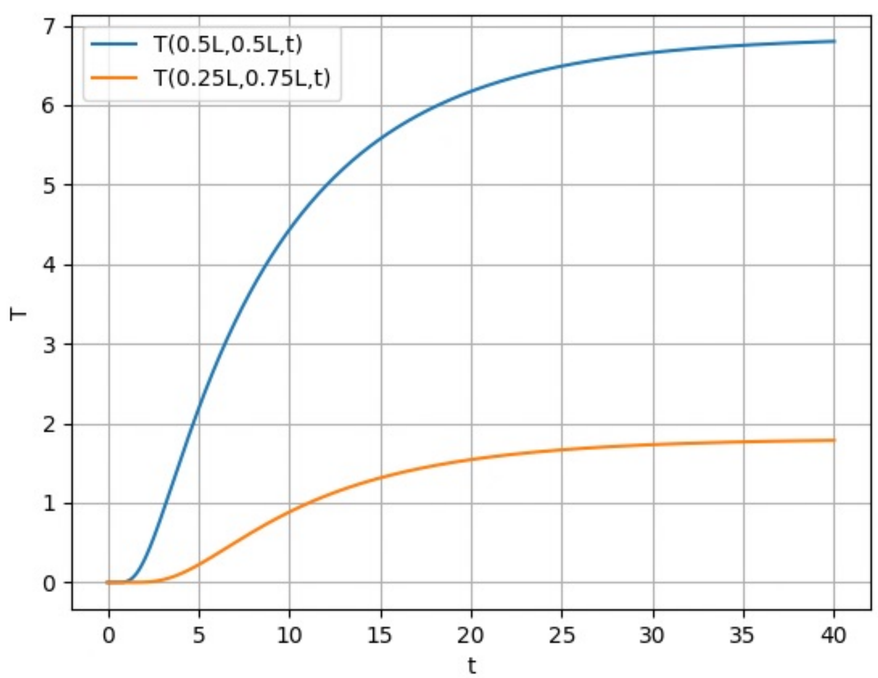
Part 2 :-

(a) $$\Delta t_{max} = \frac{1}{2\alpha\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right)}$$

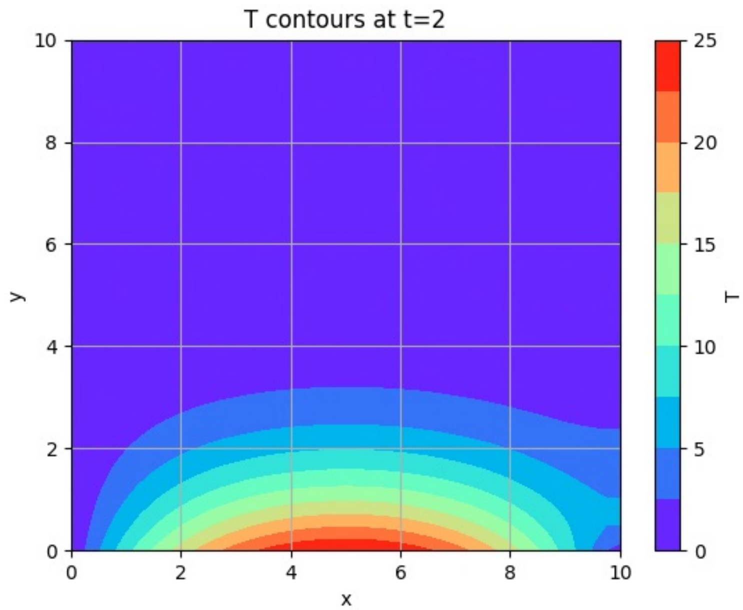
$$\alpha = 1, \quad \Delta x = \Delta y = \frac{10}{40} = 0.25$$

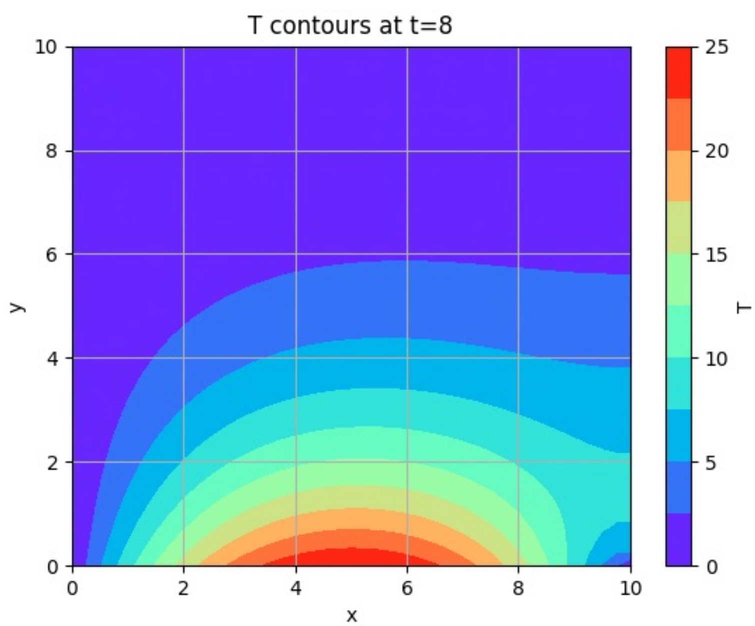
$$\therefore \Delta t_{max} = \frac{1}{2} \cdot \frac{1}{(4^2 + 4^2)} = \frac{1}{64} = 0.015625$$

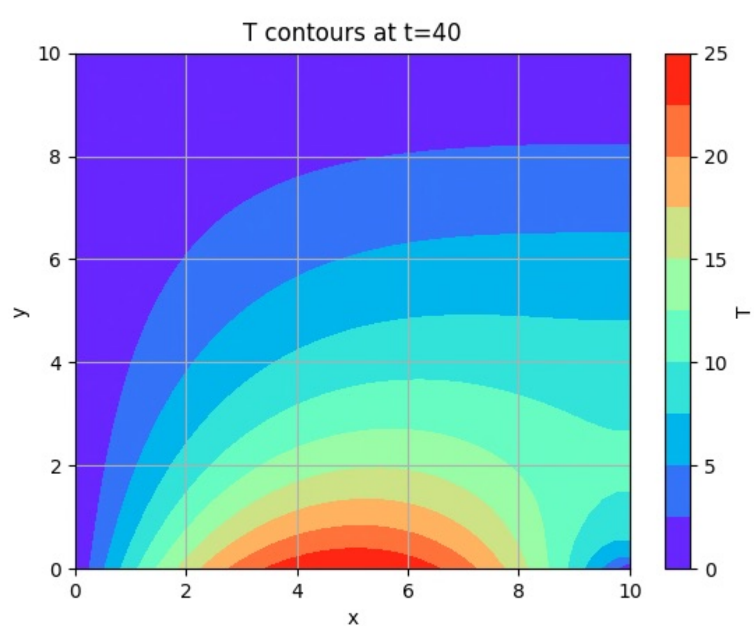
(b)



(c)







Problem 3

(a) Backward Euler (or BDF1) for x,

$$\frac{\partial T}{\partial x} = \frac{\alpha}{u(y)} \frac{\partial^2 T}{\partial y^2}$$

(We treat x to be a time-like variable)

$$\frac{\partial T_j}{\partial x} = \frac{\alpha}{(1-y_j^2)} \left(\frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta y)^2}\right)$$

$$T_j^{i+1} - T_j^i = \underbrace{\frac{\alpha \Delta x}{(1-y_j^2)(\Delta y)^2}}_{\beta_j} \left(T_{j+1}^{i+1} - 2T_j^{i+1} + T_{j-1}^{i+1} \right)$$

$$-\beta_j T_{j+1}^{i+1} + (1+2\beta_j) T_j^{i+1} - \beta_j T_{j-1}^{i+1} = T_j^i$$

$$j=1 \Rightarrow -\beta_1 T_2^{i+1} + (1+2\beta_1) T_1^{i+1} - \beta_j T_0^{i+1} = T_1^i$$

$$(1+2\beta_1) T_1^{i+1} - \beta_1 T_2^{i+1} = T_1^i + \beta_1 T_{bottom}$$

$$j = N-1 \Rightarrow -\beta_{N-1} T_{N-2}^{i+1} + (1+2\beta_{N-1}) T_{N-1}^{i+1} = T_{N-1}^i + \beta_{N-1} T_{top}$$

$$j = 2 \ldots (N-2), \Rightarrow -\beta_j T_{j-1}^{i+1} + (1+2\beta_j) T_j^{i+1} - \beta_j T_{j-1}^{i+1} = T_j$$

Matrix form,

$$\begin{bmatrix} (1+2\beta_1) & -\beta_1 & 0 & \cdots & & \\ -\beta_2 & (1+2\beta_2) & -\beta_2 & 0 & \cdots & \\ \vdots & \vdots & & \ddots & & \\ \vdots & \vdots & & -\beta_{N-2} & (1+2\beta_{N-2}) & -\beta_{N-2} \\ \vdots & \vdots & & 0 & -\beta_{N-1} & (1+2\beta_{N-1}) \end{bmatrix} \begin{bmatrix} T_1^{i+1} \\ T_2^{i+1} \\ \vdots \\ T_{N-1}^{i+1} \end{bmatrix}$$

$$= \begin{bmatrix} T_1^i + \beta_1 T_{bottom} \\ T_2^i \\ \vdots \\ T_{N-2}^i \\ T_{N-1}^i + \beta_{N-1} T_{top} \end{bmatrix}$$

When $T_{top} = T_{bottom} = 0$, then,

$$[D]\,\bar{T}^{i+1} = \bar{T}^i \Rightarrow \bar{T}^{i+1} = [D^{-1}]\,\bar{T}^i,$$

where $[D] = B\left[-\beta_j, (1+2\beta_j), -\beta_j\right]$

(b) BDF 2

$$\frac{\partial T_j}{\partial x} = \frac{\alpha}{(1-y_j^2)} \frac{\left(T_{j+1} - 2T_j + T_{j-1}\right)}{(\Delta y)^2}$$

$$T_j^{i+1} - \frac{4}{3} T_j^i + \frac{T_j^{i-1}}{3} = \frac{2\alpha \Delta x}{3(1-y_j^2)(\Delta y)^2} \left(T_{j+1}^{i+1} - 2T_j^{i+1} + T_{j-1}^{i+1}\right)$$

Let $\gamma_j = \dfrac{2\alpha\Delta x}{3(1-y_j^2)(\Delta y)^2}$

$$\therefore\ -\gamma_j T_{j+1}^{i+1} + (1+2\gamma_j) T_j^{i+1} - \gamma_j T_{j-1}^{i+1} = \frac{4}{3} T_j^i - \frac{T_j^{i-1}}{3}$$

Case $j=1$,

$$-\gamma_1 T_2^{i+1} + (1+2\gamma_1) T_1^{i+1} = \frac{4}{3} T_1^i - \frac{T_1^{i-1}}{3} + \gamma_1 T_{botto}$$

Case $j = N-1$

$$\Rightarrow\ -\gamma_{N-1} T_{N-2}^{i+1} + (1+2\gamma_{N-1}) T_{N-1}^{i+1} = \frac{4}{3} T_{N-1}^i - \frac{T_{N-1}^{i-1}}{3} + \gamma_{N-1} T_{top}$$

For $j = 2, 3 \ldots (N-2)$

$$-\gamma_j T_{j+1}^{i+1} + (1+2\gamma_j) T_j^{i+1} - \gamma_j T_{j-1}^{i+1} = \frac{4}{3} T_j^i - \frac{T_j^{i-1}}{3}$$

Matrix form,

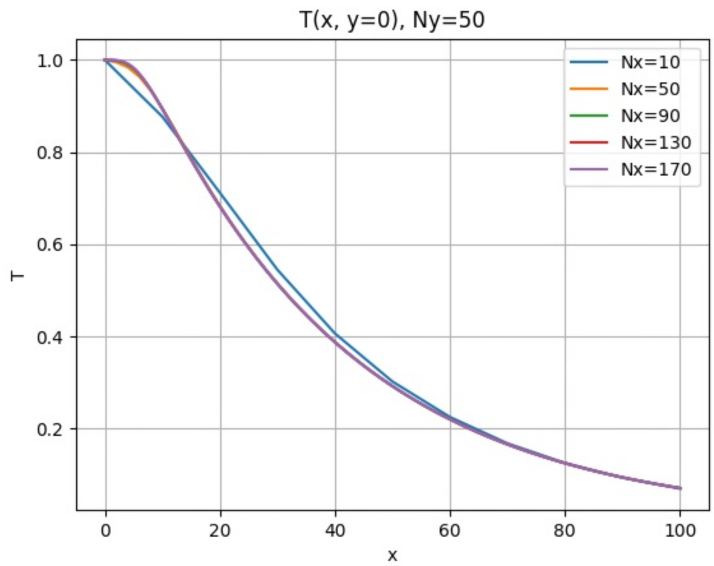
$$\begin{bmatrix} (1+2r_1), & -r_1, & 0 & \cdots & \cdots & 0 \\ -r_2, & (1+2r_2), & -r_2, & 0 & \cdots & \\ \vdots & & \ddots & & & \\ \vdots & & & -r_{N-2} & (1+2r_{N-2}) & -r_{N-2} \\ \vdots & & & & -r_{N-1} & (1+2r_{N-1}) \end{bmatrix} \begin{bmatrix} T_1^{i+1} \\ T_2^{i+1} \\ \vdots \\ T_{N-1}^{i+1} \end{bmatrix}$$

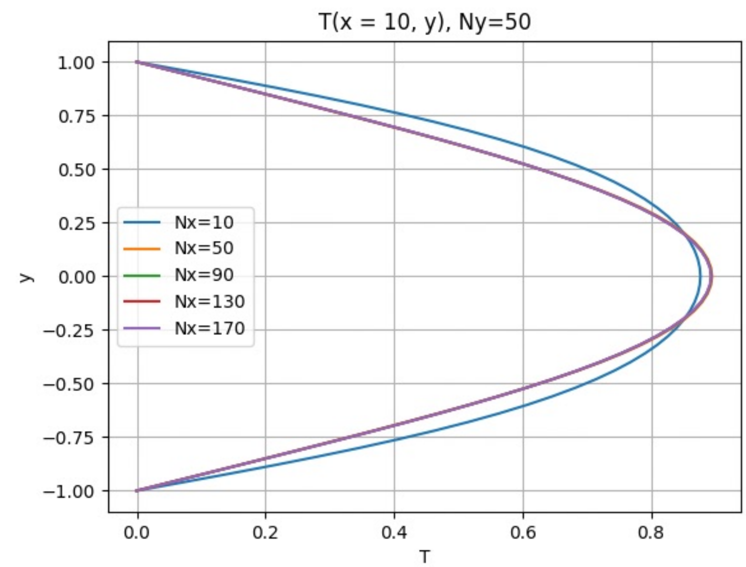
$$= \underbrace{\frac{4}{3}\begin{bmatrix} T_1^{i} \\ T_2^{i} \\ \vdots \\ T_{N-1}^{i} \end{bmatrix}}_{i} - \underbrace{\frac{1}{3}\begin{bmatrix} T_1^{i-1} \\ T_2^{i-1} \\ \vdots \\ T_{N-1}^{i-1} \end{bmatrix}}_{i-1} + \underbrace{\begin{bmatrix} r_1 T_{bottom} \\ 0 \\ 0 \\ \vdots \\ 0 \\ r_{N-1} T_{top} \end{bmatrix}}_{BC}$$

(c) Complete on PrairieLearn.

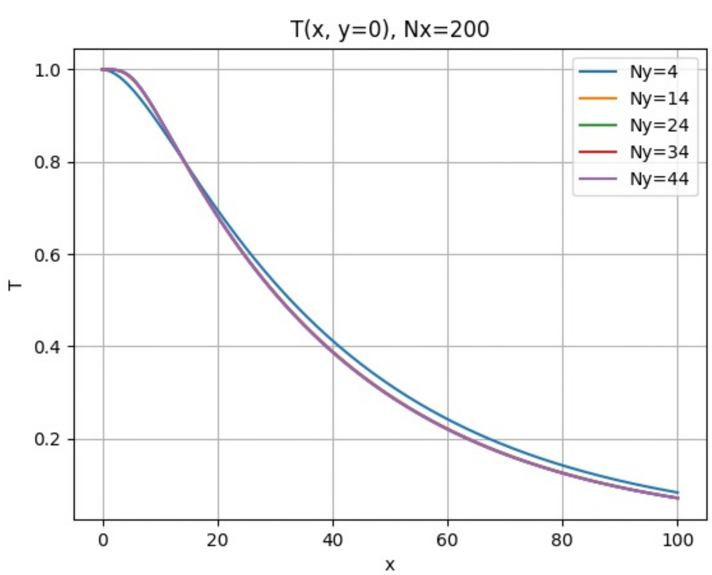
(d) 4 plots are shown below, 2 of them have 2 have fixed N_x and fixed N_y.

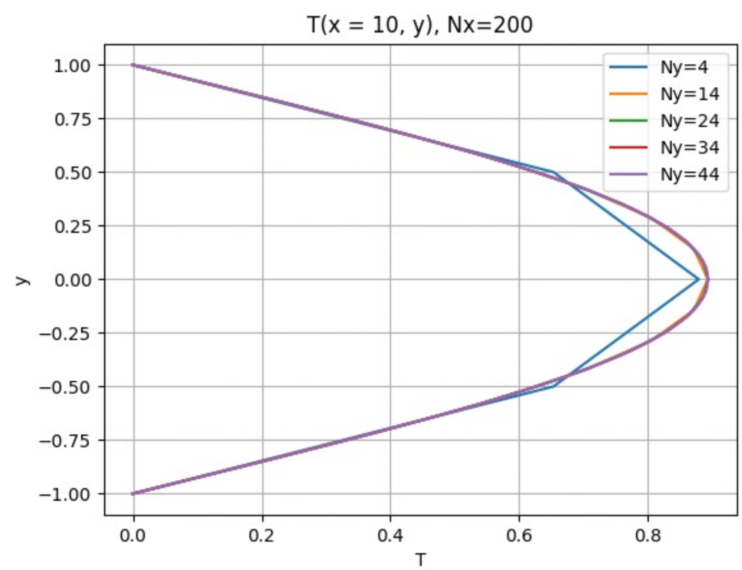
Fixed N_y,





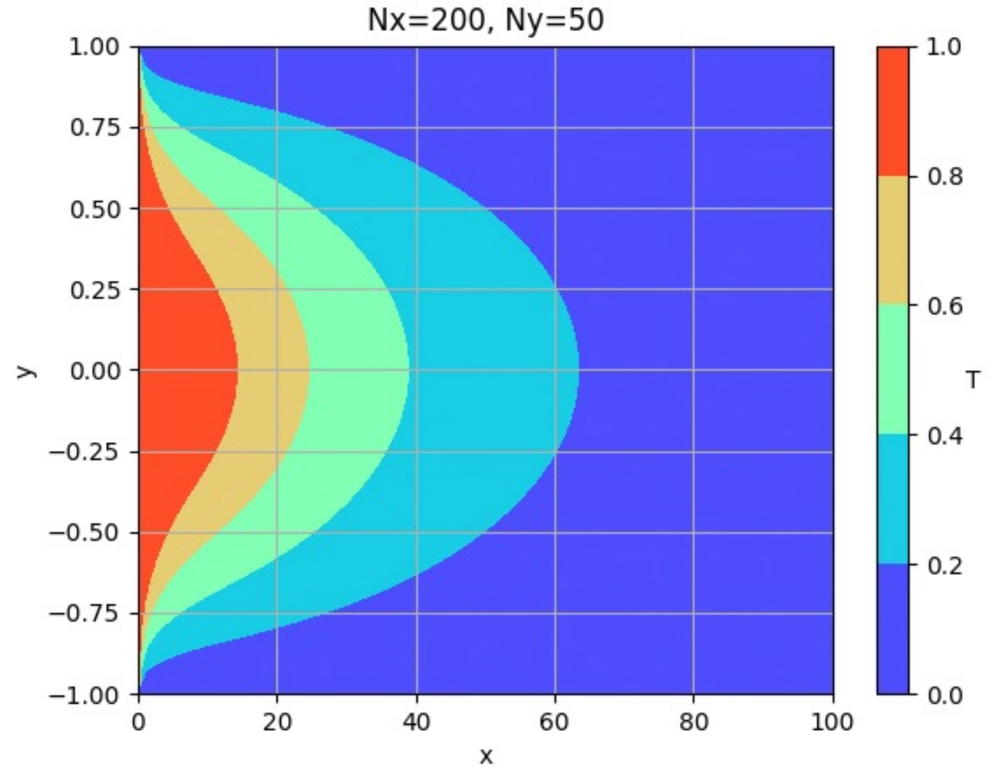
Fixed N_x,





From the plots, the solutions look converged for $N_x = 50$ and $N_y = 24$.

(e)



Nx=200, Ny=50
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
0
20
40
60
80
100
x
1.0
0.8
0.6
0.4
0.2
0.0
T