TAM 470 / CSE 450

Homework 10

Problem 1 (20 points)

No coding required:

Consider the 1D Poisson equation on the domain $[0, \pi]$:

$$u''(x) + f(x) = 0 \text{ on } x \in [0, \pi], \tag{1}$$

Let $f(x) = \sin(x)$ and let the boundary conditions be $u(0) = 1$ and $u'(\pi) = \frac{1}{2}$.

- **(a)** (6 pts) Find the exact analytical solution for $u(x)$ by integrating the original differential equation twice and choosing constants of integration that enforce the boundary conditions.
- **(b)** (8 pts) Using a mesh of 3 elements with nodes at $(x_0, x_1, x_2, x_3) = (0, \pi/3, 2\pi/3, \pi)$, solve this problem **by hand** with the finite element method. Use the linear element interpolation functions discussed in class. Your solution should include computation of the element stiffness matrices K^e_{ij} and element force vectors F^e_i, assembly of the system of equations, and imposition of boundary conditions to solve for the unknown nodal values u_1, u_2, and u_3. You can use any results presented in class for the Galerkin method applied to the Poisson equation without re-deriving them here. You can use numerical integration to compute the element force vectors.
- **(c)** (4 pts) Plot the FEM solution $u_h(x)$ vs the exact solution $u(x)$ on the same axes (you should plot $u(x)$ on a fine grid so it is smooth, and plot $u_h(x)$ on the coarse finite element mesh).
- **(d)** (2 pts) Plot the FEM solution **derivative** $u_h'(x)$ vs the exact solution **derivative** $u'(x)$ on the same axes (you should plot $u_h'(x)$ on a fine grid so it is smooth, and plot $u'(x)$ on the coarse finite element mesh).

Problem 2 (10 points)

(8 pts, PL) Go to PrairieLearn to write an SOR solver to solve the steady-state heat equation with the domain and boundary conditions shown in Figure 1.

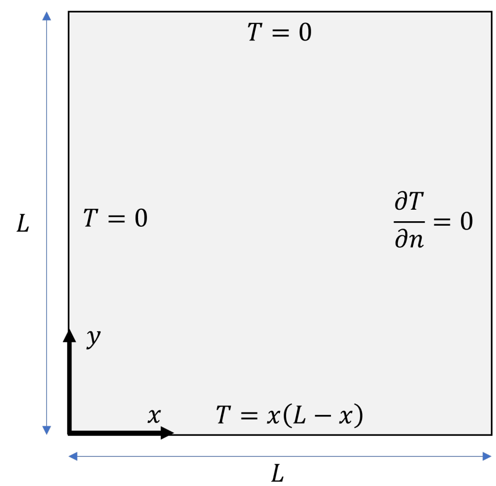

Figure 1: Domain and boundary conditions this problem.

(2 pts) On Gradescope, submit a plot of $d = |T^{k+1} - T^k|$ versus number of iterations for the Point Jacobi solver for this problem compared to the SOR solver. Plot the data on the same axes (1 total figure). Use the inputs listed below when creating the plots:

```
L = 10
N = 40
tol = 1e-6
maxiter = 10000
omega = 1.8 # Relevant for SOR solver only
```

Consult the iterative solver tutorial on PrairieLearn for syntax to create these plots.

Problem 3: 4 credit-hour students only (20 points)

No coding required

An axisymmetric problem is one where the domain, solution, and boundary conditions exhibit radial symmetry; this is appropriate for some modeling situations involving cylinders, both hollow and solid.

The axisymmetric form of the steady Poisson equation using cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + f(r) = 0, \qquad r \in (a, b) \tag{2}$$

Boundary conditions impose the solution $u(r)$ (Dirichlet) or its gradient $u_r(r)$ (Neumann) at the boundaries $r = a$ and $r = b$.

- **(a)** (6 pts) Use the weighted residual method to write down the **general** weak form of this problem. Hint: Remember the weak form should use integration by parts to reduce second derivatives to first derivatives, when possible.
- **(b)** (6 pts) Use the Galerkin finite element method to write down the global finite element formulation of this problem in the form

  $$\mathbf{K}\mathbf{u} = \mathbf{F} \tag{3}$$

  Specifically, what are the formulas for the matrices K_{ij}, and F_i in terms of the global basis functions $\phi_i(r)$, the independent variable r, fluxes (derivatives) $u_r(a)$ and $u_r(b)$, and the domain $[a, b]$?) You do not have to substitute specific forms for the basis functions $\phi_i(r)$ at this stage.
- **(c)** (8 pts) Derive the specific form of the 2×2 element matrix K^e_{ij} if linear basis functions $N^e_i(r)$ are used for each element. A single element consists of two nodes with coordinates r^e_1 and r^e_2 and the element length is $\Delta r_e = r^e_2 - r^e_1$.