Compactness

Definition (13.11)

Suppose $E \subset S$. A family $\mathcal{U}$ of open sets is an open cover for E is $E \subset \cup_{U \in \mathcal{U}} U$. A subcover is a subfamily of $\mathcal{U}$ which is also an open cover. E is called compact if any open cover has finite subcover.

Note. A cover $\mathcal{U}$ is a collection of sets, not their union. In other words, a cover is a subset not of S, but of $\mathcal{P}(S)$ (the power set of S).

Examples involving subsets of $\mathbb{R}$, with usual metric.

(1) $E = [0, \infty)$. The sets $(-2, 2)$ and $(1, \infty)$ form an open cover of E: they are open, and $E \subset (-2, 2) \cup (1, \infty)$.

The sets $U_k = (-1, k)$ ($k \in \mathbb{N}$) form another open cover of E, as $E \subset \cup_{k=1}^{n} (-1, k)$. Note that this open cover has no finite subcover. Indeed, suppose, for the sake of contradiction, that $U_{k_1}, \ldots, U_{k_m}$ form a finite subcover – that is, $[0, \infty) \subset \cup_{j=1}^{m} U_{k_j}$. Let $N = \max_{1 \leqslant j \leqslant m} k_j$, then $[0, \infty) \subset \cup_{j=1}^{m} U_{k_j} = (-1, N)$, contradiction!

Conclusion: $E = [0, \infty)$ **is not compact.** In fact: compact $\Rightarrow$ bounded.

Compactness – more examples

Examples involving subsets of $\mathbb{R}$, with usual metric.

(2) $(0,1)$ **is not compact.** Indeed, for $k \in \mathbb{N}$, $U_k = (1/k, 1)$ is open, and $E \subset \cup_k U_k$. We claim that this open cover has no finite subcover. Suppose, for the sake of contradiction, that $E \subset \cup_{j=1}^m U_{k_j}$. Let $N = \max_{1 \leqslant j \leqslant m} k_j$, then $(0,1) \subset \cup_{j=1}^m U_{k_j} = (1 - 1/N, 1)$, contradiction!

(3) $[a, b]$ is compact (to be proved).

Proposition (not in textbook)

Any finite set is compact.

Proof. Suppose $\mathcal{U}$ is an open cover of $E = \{e_1, \ldots, e_N\}$. For each i find $U_i \in \mathcal{U}$ so that $e_i \in U_i$. Then $U_1, \ldots, U_N$ is the desired finite subcover. ■

Compactness: bounded sets

Proposition (not in textbook)

Any compact set is bounded.

Recall: $E \subset S$ is bounded if $\exists s_0 \in S$ s.t. $\sup_{e \in E} d(s_0, e) < \infty$ (equivalently, $\forall s \in S$, $\sup_{e \in E} d(s, e) < \infty$).

Proof. If E is not bounded, then, for $s \in S$, the sets $\mathbf{B}^o_k(s)$ form an open cover of E, with no finite subcover. ■

Example: bounded set which is not compact.

Equip $\mathbb{N}$ is discrete metric: for $x, y \in \mathbb{N}$, $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$.

Then $\mathbb{N}$ itself is bounded. To show lack of compactness, we exhibit an open cover with no finite subcover. For $n \in \mathbb{N}$ let $U_n = \mathbf{B}^o_{1/2}(n) = \{n\}$. The sets $(u_n)_{n \in \mathbb{N}}$ form an open cover without finite subcover.

Properties of compact sets

Proposition

1. *A closed subset of a compact set is compact.*
2. *A finite union of compact sets is compact.*

Proof of (i). Suppose E is a compact subset of a metric space S, and $F \subset E$ is closed. Need to show: if $\{U : U \in \mathcal{U}\}$ is an open cover of F, then it has a finite subcover. The set $U_0 = S \backslash F$ is open.
$\{U_0\} \cup \{U : U \in \mathcal{U}\}$ is an open cover of E. Indeed, if $e \in E \backslash F$, then $e \in U_0$, while if $e \in F$, then $e \in \cup(U : U \in \mathcal{U})$.
E is compact, hence this open cover has finite subcover, consisting of $U_0, U_1, \ldots U_n$ (with $U_1, \ldots U_n \in \mathcal{U}$). $F \cap U_0 = \emptyset$, hence $U_1, \ldots U_n$ cover F. ■

Proof of (ii): Exercise 13.12 (Homework 4).

Nested sequences of closed sets

Proposition (not in textbook)

Suppose $F_1 \supset F_2 \supset \dots$ are closed non-void subsets of a compact set E. Then $\cap_n F_n$ is non-empty, compact.

Does E have to be compact? **Yes!** Let $E = \mathbb{R}$, $F_n = [n, \infty)$.

Proof. The sets $U_n = S \backslash F_n$ ($n \in \mathbb{N}$) are open. If $\cap_n F_n = \emptyset$, then (U_n) is an open cover of E. However, there is no finite subcover, since $F_m \cap (\cup_{j<m} U_j) = \emptyset$.

$\cap_n F_n$ is compact, since it is a closed subset of a compact set. ■

Example: the Cantor "middle third" set

Heine-Borel Theorem (to be proved): a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded. $\Rightarrow [0,1]$ is compact.

Let $F_0 = [0,1]$, $F_1 = F_0 \backslash (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$,

$F_2 = \left([0, \frac{1}{3}] \backslash (\frac{1}{9}, \frac{2}{9})\right) \cup \left([\frac{2}{3}, 1] \backslash (\frac{7}{9}, \frac{8}{9})\right) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$,

etc.. $F_0 \supset F_1 \supset F_2 \supset \ldots$. The Cantor set: $\mathcal{C} = \cap_n F_n$ is non-empty and closed, hence compact.

F_n is the union of of 2^n disjoint closed intervals, each of length 3^{-n}. Thus $\mathcal{C}$ contains no intervals, hence its interior is empty. However, there is a bijection between $\mathcal{C}$ and $\mathbb{R}$ (one has "as many points as the other"). We'll return to $\mathcal{C}$ in Section 14 (series).