Bounded and monotone convergence

Theorem (Bounded convergence – 33.11)

Suppose (f_n) are integrable on $[a, b]$, $|f_n| \leqslant M$ for any n, $f_n \to f$ pointwise on $[a, b]$, and f is integrable. Then $\lim_n \int_a^b f_n$ exists, and equals $\int_a^b f$.

Theorem (Monotone convergence – 33.12)

Suppose (f_n) are integrable on $[a, b]$, $f_1 \leqslant f_2 \leqslant \ldots$, $f_n \to f$ pointwise on $[a, b]$, and f is integrable. Then $\lim_n \int_a^b f_n$ exists, and equals $\int_a^b f$.

Proof of Monotone Convergence Theorem. For any n, $|f_n(x)| \leqslant M$, where $M = \max\left\{ \sup_{x\in[a,b]} |f_1(x)|, \sup_{x\in[a,b]} |f(x)| \right\}$. Apply Bounded Convergence Theorem. ∎

Example. $\lim_n \int_0^1 \frac{dx}{1+nx^3} = 0$, since the sequence of functions $f_n(x) = \frac{1}{1+nx^3}$ decreases to $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0, 1] \end{cases}$, and $\int_0^1 f = 0$.

Fundamental Theorem of Calculus I

Convention. $\int_z^z g = 0$; if $z > w$, set $\int_z^w g = -\int_w^z g$. Why?

Recall additivity of integral: $\int_z^w g + \int_w^v g = \int_z^v g$.

(1) $\int_z^z g = \int_z^z g + \int_z^z g$, hence $\int_z^z g = 0$.
Geometric interpretation: area under the graph of g.

(2) We require $\int_z^w g + \int_w^v g = \int_z^v g$. For $v = z$, obtain $\int_z^w g + \int_w^z g = 0$.

Theorem (34.1 – Fundamental Theorem of Calculus I)

Suppose $g : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and g' is integrable on $[a, b]$. Then $\int_a^b g' = g(b) - g(a)$.

Example. $\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1}$. Use $g(x) = \frac{x^{n+1}}{n+1}$.

Examples: g' need not be integrable! Let $g(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Then $g'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. g' is unbounded on $[0, 1]$, hence not integrable.

Proof of Fundamental Theorem of Calculus

Theorem (34.1 – Fundamental Theorem of Calculus I)

Suppose $g : [a, b] \to \mathbb{R}$ *is continuous, differentiable on* (a, b)*, and* g' *is integrable on* $[a, b]$*. Then* $\int_a^b g' = g(b) - g(a)$*.*

Proof. Consider a partition $P = \big(a = t_0 < t_1 < \ldots < t_n = b\big)$. For $1 \leqslant k \leqslant n$, find $x_k \in (t_{k-1}, t_k)$ s.t. $g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$ (use Mean Value Theorem). Then
$m(g', [t_{k-1}, t_k]) \leqslant g'(x_k) \leqslant M(g', [t_{k-1}, t_k])$, so
$L(g', P) \leqslant \sum_{k=1}^n g'(x_k)(t_k - t_{k-1}) = g(b) - g(a) \leqslant U(g', P)$.
$U(g') = \inf_P U(g', P) \geqslant g(b) - g(a)$,
$L(g') = \sup_P L(g', P) \leqslant g(b) - g(a)$. But $U(g') = L(g') = \int_a^b g'$, hence $\int_a^b g' = g(b) - g(a)$. ■

Integration by parts

Theorem (34.2 – Integration by parts)

If $u, v : [a, b] \to \mathbb{R}$ are continuous, differentiable on (a, b), and u', v' are integrable on $[a, b]$, then $\int_a^b uv' + \int_a^b u'v = uv\big|_a^b$.

Proof. Let $g = uv$, then $g' = u'v + uv'$. $u'v$ and uv' are integrable (as products of integrable functions). Apply FTC. ∎

Example 2, p. 293. Compute $\int_0^\pi x \cos x \, dx$.
Let $u(x) = x$, $v(x) = \sin x$. Then $u'(x) = 1$, $v'(x) = \cos x$.
$\int_0^\pi x \cos x \, dx = \int_0^\pi uv' = uv\big|_0^\pi - \int_0^\pi u'v = x \sin x\big|_0^\pi - \int_0^\pi \sin x \, dx =$
$0 + \cos x\big|_0^\pi = -2.$

Fundamental Theorem of Calculus II

Theorem (34.3 – Fundamental Theorem of Calculus II)

Suppose $f : [a, b] \to \mathbb{R}$ *is integrable. For* $x \in [a, b]$ *define* $F(x) = \int_a^x f$. *If* f *is continuous at* c, *then* F *is differentiable at* c, *with* $F'(c) = f(c)$.

Remark. F is called the indefinite integral of f (with basepoint a).
F is Lipschitz. Indeed, $\exists B \geqslant 0$ s.t. $|f| \leqslant B$. We claim that
$|F(x) - F(y)| \leqslant B|x - y|$, for $x, y \in [a, b]$.
By relabeling, can assume $y < x$. Then $F(x) - F(y) = \int_y^x f$. Write
$-B \leqslant f \leqslant B$, and integrate.

Example. $G(x) = \int_2^{x^2} \sin(t^2)\,dt$. Find $G'(x)$.

Write $G(x) = F(x^2)$, where $F(y) = \int_2^y \sin(t^2)\,dt$. The function $\sin(t^2)$ is continuous for any t, hence $F'(y) = \sin(y^2)$. By Chain Rule,
$G'(x) = 2xF'(x^2) = 2x\sin(x^4)$.

Fundamental Theorem of Calculus II is sharp

Theorem (34.3 – Fundamental Theorem of Calculus II)

Suppose $f : [a, b] \to \mathbb{R}$ is integrable. For $x \in [a, b]$ define $F(x) = \int_a^x f$. If f is continuous at c, then F is differentiable at c, with $F'(c) = f(c)$.

(1) If f is not continuous at c, then $F'(c)$ need not exist.
Let $f(x) = \mathrm{sgn}(x)$[1]. Then $F(x) = \int_{-1}^x f = |x| - 1$ (verify!). f is continuous everywhere except at 0. F is not differentiable at 0.

(2) If f is not continuous at c, and $F'(c)$ exists, then it may happen that $f(c) \neq F'(c)$.
Define $f(x) = 2$ if $x = 1$, $f(x) = 0$ otherwise. Then $F(z) = \int_0^z f = 0$, for any z. $F'(z) = 0 \Rightarrow F'(1) = 0 \neq f(1)$.

[1] $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$, $f(x) = 0$ if $x = 0$

Proof of Fundamental Theorem of Calculus II

FTC II. If f is continuous at c, then F is differentiable at c, with $F'(c) = f(c)$.

Proof. Need to show: $f(c) = \lim_{x\to c} \frac{F(x)-F(c)}{x-c}$. Fix $\varepsilon > 0$. Need to find $\delta > 0$ s.t. $\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| \leqslant \varepsilon$ if $0 < |x-c| < \delta$.

$F(x) - F(c) = \int_c^x f(t)\,dt$. Write $(x-c)f(c) = \int_c^x f(c)\,dt$, hence
$\frac{F(x)-F(c)}{x-c} - f(c) = \frac{1}{x-c}\left(\int_c^x f(t)\,dt - \int_c^x f(c)\,dt\right) = \frac{1}{x-c}\int_c^x \big(f(t)-f(c)\big)\,dt.$
Pick $\delta > 0$ s.t. $|f(t)-f(c)| \leqslant \varepsilon$ whenever $|t-c| < \delta$. This δ works.
Indeed, if $x \in (c, c+\delta)$, then
$-\varepsilon(x-c) = \int_c^x (-\varepsilon)\,dt \leqslant \int_c^x \big(f(t)-f(c)\big)\,dt \leqslant \int_c^x \varepsilon\,dt = \varepsilon(x-c)$, so
$\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| \leqslant \varepsilon.$
The case of $x \in (c-\delta, c)$ is handled similarly. ∎

Change of variable for integrals

Theorem (34.4 – Change of variable)

Suppose J, I are open intervals, $u : J \to I$, u' is continuous, $f : I \to \mathbb{R}$ is continuous. Then, for $a, b \in J$, $\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(t)\,dt$.

Mnemonic: $t = u(x)$, $dt = u'(x)\,dx$.

Proof. Fix $c \in I$, let $F(v) = \int_c^v f$. Then $F'(v) = f(v)$. Let $g = F \circ u$ – that is, $g(x) = \int_c^{u(x)} f$. By Chain Rule,
$g'(x) = F'(u(x))u'(x) = f(u(x))u'(x)$.
$\int_a^b f(u(x))u'(x)\,dx = \int_a^b g'(x)\,dx = g(b) - g(a) = F(u(b)) - F(u(a)) =$
$\int_c^{u(b)} f - \int_c^{u(a)} f = \int_{u(a)}^{u(b)} f$. ■

Example. Find $A = \int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. Let $u(x) = \sqrt{x}$, then $u'(x) = \frac{1}{2\sqrt{x}}$, so $f(t) = 2\sin t$. $A = \int_1^2 2\sin t\,dt = 2(\cos 1 - \cos 2)$.