Graphs of functions and path connectedness

The graph of a function $f : I \to \mathbb{R}$ ($I \subset \mathbb{R}$ is an interval):
$\mathbf{G}(f) = \{(x, f(x)) : x \in I\}$.

Proposition (Example 4 from Section 22)

$\mathbf{G}(f)$ *is path connected iff* f *is continuous on interval* I.

Remark. Exercise 22.4: a discontinuous f s.t. $\mathbf{G}(f)$ is connected.

Proposition 21.2. The function $f : S \to \mathbb{R}^n : x \mapsto (f_1(x), \ldots, f_n(x))$ is continuous iff $f_i : S \to \mathbb{R}$ is continuous, for $1 \leqslant i \leqslant n$.

Proof: f is continuous on $I \Rightarrow \mathbf{G}(f)$ is path connected.
Suppose $\vec{x} = (a, f(a)), \vec{y} = (b, f(b)) \in \mathbf{G}(f)$. A path between $\vec{x}$ and $\vec{y}$:
$\gamma(t) = \Big((1-t)a + tb, f\big((1-t)a + tb\big)\Big)$ $(t \in [0, 1])$. ■

f is continuous on $I \Leftarrow \mathbf{G}(f)$ is path connected. See textbook.

A connected set which is not path connected

Consider $f : [0, \infty) \to \mathbb{R} : x \mapsto \begin{cases} \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$.

f is discontinuous at 0, hence $\mathbf{G}(f)$ is not path connected.

We shall show that that $\mathbf{G}(f)$ is connected.

Suppose, for the sake of contradiction, that $\mathbf{G}(f)$ is disconnected. Then $\exists$ open $U_1, U_2 \subset \mathbb{R}^2$ so that $\mathbf{G}(f) \subset U_1 \cup U_2$, $\mathbf{G}(f) \cap U_1 \cap U_2 = \emptyset$, while $\mathbf{G}(f) \cap U_1 \neq \emptyset$ and $\mathbf{G}(f) \cap U_2 \neq \emptyset$.

Write $\mathbf{G}(f) = E_1 \cup E_2$, where $E_1 = \{(0,0)\}$ and $E_2\{(x, \sin 1/x) : x > 0\}$. Both E_1 and E_2 are path connected, hence connected. Thus, by relabeling, we assume $E_1 \subset U_1, E_2 \subset U_2$.

Find $r > 0$ so hat $\mathbf{B}^o_r(0,0) \subset U_1$. But $\forall n \in \mathbb{N}$ $(1/(n\pi), 0) \in E_2$. Pick n so that $1/(n\pi) > r$, then $(1/(n\pi), 0)$ belongs to both $E_2 \subset U_2$ and to $\mathbf{B}^o_r(0,0) \subset U_1$. Thus, $(1/(n\pi), 0) \in E \cap U_1 \cap U_2$, a contradiction. ■

Section 23: power series

Consider series $\sum_{n=0}^{\infty} a_n x^n$, where x is a variable.
Let $\beta = \limsup |a_n|^{1/n}$, $R = \frac{1}{\beta}$. Convention: $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$.

Theorem (23.1)

The series $\sum_n a_n x^n$ converges for $|x| < R$, diverges for $|x| > R$.

R is the radius of convergence of the series.

Proof. Ratio Test (14.9): $\sum z_n$ converges absolutely when $\limsup |z_n|^{1/n} < 1$, diverges when $\limsup |z_n|^{1/n} > 1$.
For $z_n = a_n x^n$, $\limsup |z_n|^{1/n} = |x|\beta$. $\sum_n a_n x^n$ converges when $|x|\beta < 1$ ($|x| < \frac{1}{\beta} = R$), diverges when $|x|\beta > 1$ ($|x| > \frac{1}{\beta} = R$). ∎

Remark. If $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, it equals β.

The series may either converge or diverge at $\pm R$. The interval of convergence of a power series is the set of all $x \in \mathbb{R}$ for which the series converges. This is an interval: $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$.

Examples of power series (p. 189)

(1) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. $a_n = \frac{1}{n!}$, $\lim \frac{a_{n+1}}{a_n} = \lim \frac{1}{n+1} = 0$, so $\beta = 0$, $R = \infty$. Interval of convergence: $(-\infty, \infty)$. In fact, $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

(2) $\sum_{n=0}^{\infty} x^n$. $a_n = 1$, $\beta = 1$, $R = 1$. The series diverges for $x = \pm 1$, converges for $x \in (-1, 1)$. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

(3) $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$. $a_n = \frac{1}{n+1}$, $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = 1$, $R = 1$. The series diverges for $x = 1$ (harmonic series), converges for $x \in [-1, 1)$ ($x = -1$: alternating series). $\sum_{n=0}^{\infty} \frac{x^n}{n+1} = \ln(1 - x)$.

(4) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$. $a_n = \frac{1}{(n+1)^2}$, $\beta = 1$, $R = 1$. The series converges for $x \in [-1, 1]$ (comparison test at ± 1).

(5) $\sum_{n=0}^{\infty} n! x^n$. $a_n = n!$, $\beta = \lim \frac{a_{n+1}}{a_n} = \lim(n + 1) = +\infty$, $R = 0$. The series diverges $\forall x \neq 0$.

Uniform convergence

Definition (similar to 22.6)

Suppose $S \subset \mathbb{R}$. $B(S)$ is the space of bounded functions $f : S \to \mathbb{R}$, with the metric $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$.

d is indeed a metric. It is easy to see that $d(f, g) = d(g, f)$, $d(f, g) \in [0, \infty)$, with $d(f, g) = 0$ iff $f = g$.

Verify the triangle inequality. For f, g, h, show that $d(f, h) \leqslant d(f, g) + d(g, h)$.
Fix $\varepsilon > 0$. Find $x \in S$ s.t. $|f(x) - h(x)| > d(f, h) - \varepsilon$. Then $|f(x) - h(x)| \leqslant |f(x) - g(x)| + |g(x) - h(x)|$, hence
$d(f, h) - \varepsilon < |f(x) - h(x)| \leqslant |f(x) - g(x)| + |g(x) - h(x)| \leqslant d(f, g) + d(g, h)$.

Conclude: $\forall \varepsilon > 0$, $d(f, h) - \varepsilon < d(f, g) + d(g, h)$. Thus, $d(f, h) \leqslant d(f, g) + d(g, h)$. ■

Uniform convergence (Section 24)

Definition (24.1-2)

Suppose $f, f_1, f_2, \ldots$ are functions from $S \subset \mathbb{R}$ to $\mathbb{R}$.
$f_n \to f$ pointwise on S if $f_n(x) \to f(x)$, $\forall x \in S$: $\forall \varepsilon > 0, x \in S$ $\exists N = N(\varepsilon, x) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \geqslant N$.
$f_n \to f$ uniformly on S if $\forall \varepsilon > 0$ $\exists N = N(\varepsilon) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \geqslant N$, $\forall x \in S$. Equivalently, $\lim_n \sup_{x \in S} |f_n(x) - f(x)| = 0$.

Uniform convergence $\Rightarrow$ pointwise convergence.
The converse is false.

Denote by $B(S)$ the set of bounded functions on S, with the metric $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$.

If $f, f_1, f_2, \ldots$ are bounded, then $f_n \to f$ uniformly iff $\lim_n d(f_n, f) = 0$.

Examples of convergence

A sequence converging pointwise, but not uniformly.
(Example 2 from p. 194)

Let $f_n(x) = x^n$ on $[0, 1]$. Then $\lim_n f_n(x) = f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

$f_n \to f$ pointwise.

Is the convergence uniform? **No!** $\forall n$, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$.
However, for any $a \in (0, 1)$, $f_n \to f$ uniformly on $[0, a]$:
$\sup_{x\in[0,a]} |f_n(x) - f(x)| = a^n$, hence $\lim_n \sup_{x\in[0,a]} |f_n(x) - f(x)| = 0$.

$f_1, f_2, \ldots$ are continuous on $[0, 1]$, but f is not. We can lose continuity when convergence is pointwise, but not when it is uniform!

Uniform convergence preserves continuity

Theorem (24.3)

Suppose $f_n \to f$ uniformly on S, and $\forall n$, f_n is continuous at $x_0 \in S$. Then f is continuous at x_0.

Proof by "$\frac{\varepsilon}{3}$ argument". Fix $\varepsilon > 0$. Need to find $\delta > 0$ s.t. $|f(x_0) - f(x)| < \varepsilon$ whenever $x \in S$, $|x - x_0| < \delta$.
Find n s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$, $\forall x \in S$. f_n is continuous at x_0, hence $\exists \delta > 0$ s.t. $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$ whenever $x \in S$, $|x - x_0| < \delta$.
This δ works for us: if $|x - x_0| < \delta$, then
$|f(x_0) - f(x)| \leqslant |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)| < 3\frac{\varepsilon}{3} = \varepsilon$.
∎

More examples of convergence

$f_n(x) = \frac{x}{1+nx^2}$. Does the sequence (f_n) converge pointwise on $\mathbb{R}$? If yes, find the limit, and determine whether the convergence is uniform.
(Example 7 from p. 198)

$f_n(0) = 0$ for any n. If $x \neq 0$, then $f_n(x) = \frac{x/n}{1/n+x^2}$, so $\lim_n f_n(x) = 0$.
$f_n \to f$ pointwise, where $f(x) = 0$.

$f_n \to f$ uniformly iff $\lim_n \sup_{x\in\mathbb{R}} \left|f_n(x) - f(x)\right| = 0$.

$\sup_{x\in\mathbb{R}} \left|f_n(x) - f(x)\right| = \sup_{x\in\mathbb{R}} \frac{|x|}{1+nx^2}$.
AGM Inequality: for $a, b \geqslant 0, \sqrt{ab} \leqslant \frac{a+b}{2}$.
Take $a = 1$, $b = nx^2$: $\sqrt{n}|x| \leqslant \frac{1+nx^2}{2}$, hence $\frac{|x|}{1+nx^2} \leqslant \frac{1}{2\sqrt{n}}$.
$\lim_n \frac{1}{2\sqrt{n}} = 0$, hence $\lim_n \sup_{x\in\mathbb{R}} \left|f_n(x) - f(x)\right| = 0$.

Conclusion: $f_n \to f$ uniformly ($f(x) = 0$).

More examples of convergence

$f_n(x) = n^2 x^n (1-x)$. Does the sequence (f_n) converge pointwise on $[0,1]$? If yes, find the limit, and determine whether the convergence is uniform. (Example 8 from p. 198)

$f_n(0) = f_n(1) = 0$. Recall Exercise 9.12: if $\lim_n \left|\frac{s_{n+1}}{s_n}\right| < 1$, then $\lim s_n = 0$. Take $s_n = f_n(x) = n^2 x^n (1-x)$ $(0 < x < 1)$, then $\lim_n \left|\frac{s_{n+1}}{s_n}\right| = x$, hence $\lim f_n(x) = 0$.

$f_n \to f$ pointwise, where $f(x) = 0$.
$f_n \to f$ uniformly iff $\lim_n \sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = 0$.

To find $\sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = \max_{0\leqslant x\leqslant 1} f_n(x)$, differentiate:
$f_n'(x) = n^2\big(nx^{n-1} - (n+1)x^n\big) = n^2(n+1)x^{n-1}\big(x - \frac{n}{n+1}\big)$.
$f_n\big(\frac{n}{n+1}\big) = n^2\big(\frac{n}{n+1}\big)\frac{1}{n+1} = \frac{n^{n+1}}{(n+1)^n} = \frac{n^2}{n+1}\cdot\big(\frac{n}{n+1}\big)^n$.
$\lim_n \big(\frac{n}{n+1}\big)^n = \frac{1}{e}$, hence $\lim_n \sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = +\infty$.

Conclusion: $f_n \to f$ pointwise, but not uniformly $(f(x) = 0)$.