TAM 470 / CSE 450
Homework 4

Problem 1 (10 points)

This problem is to be completed on PrairieLearn.

Consider the ODE:

$$\frac{dy}{dt} = -0.2y - 2\cos(2t)y^2 \tag{1}$$

with initial condition $y(0) = 1$.

Write a function that solves this problem for a specified time step value h and number of steps N using the forward Euler method.

Problem 2 (10 points)

Referencing the ODE and initial conditions from Problem 1, answer the following:

(a) (4 pts) Use linear stability analysis to estimate the maximum allowable initial time step to compute a stable solution using the forward Euler method.

(b) (4 pts) Use your code solution from Problem 1 to compute the solution for $t \in [0, 7]$ using time steps of $h = 0.2\left(\frac{1}{2}\right)^m$ for $m = 0, 1, 2, 3, 4, 5$ (i.e. 6 distinct numerical solutions based on different time step values). Plot all solutions on the same axes, with the horizontal axis ranging from 0 to 7 and the vertical axis ranging from 0 to 1.4. Be sure to include a legend to clearly indicate the solution that each curve on the plot represents.

(c) (2 pts) Using the time step definition $h = 0.2\left(\frac{1}{2}\right)^m$, what is the minimum value of m that you believe achieves convergence of the solution? Justify your answer by suggesting a convergence criteria and showing that your proposed value of m achieves it. Note there is not a single "right" answer to this question; your answer will be graded on how appropriate your convergence criteria is, and how you demonstrate it has been achieved for your proposed value of m.

Problem 3 (10 points)

The ODE below is for a damped pendulum with small amplitude swings:

$$\theta''(t) + c\theta'(t) + \frac{g}{l}\theta(t) = 0 \tag{2}$$

The variable c is a constant that does not depend on time and g and l retain their definitions from Problem 3.

(a) (3 pts) Rewrite (2) as a system of two first-order ODEs such that $\mathbf{y}' = \mathbf{A}\mathbf{y}$ where $\mathbf{y} = (\theta(t), \theta'(t))$ and $\mathbf{A}$ is a matrix of constant coefficients.

(b) (3 pts) Find an expression for the eigenvalues of $\mathbf{A}$ in terms of the constants c, g, and l.

(c) (2 pts) for $c = 4$ and $\frac{g}{l} = 2$, find the largest allowable time step required for stability of the forward Euler scheme for this equation.

(d) (2 pts) for $c = 4$ and $\frac{g}{l} = 5$, find the largest allowable time step required for stability of the forward Euler scheme for this equation.

Problem 4 (10 points)

Consider the ODE below:

$$u''(t) + [u(t)]^2 u'(t) + t^2 u(t) = 0 \tag{3}$$

(a) (6 pts) Rewrite (3) as a system of two first order ODEs in the form $\mathbf{y}' = \mathbf{f}(\mathbf{y}, t)$ with initial condition $\mathbf{y}(0) = (u(0), u'(0))$.

(b) (4 pts) Estimate the maximum allowable time step for stability at time $t = 0$ for the forward Euler method if the initial condition is $\mathbf{y}(0) = (1, 0)$.