MATH 447: Real Variables - Homework #10

Jerich Lee

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Problem 1 (26.6)**.** Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.

(a) Prove $s' = c$ and $c' = -s$.

(b) Prove $(s^2 + c^2)' = 0$.

(c) Prove $s^2 + c^2 = 1$.

Actually, $s(x) = \sin x$ and $c(x) = \cos x$, but you do **not** need these facts.

Solution 1.

Problem 2 (33.3)**.** A function f on $[a, b]$ is called a *step function* if there exists a partition

$$P = \{a = u_0 < u_1 < \cdots < u_m = b\}$$

of $[a, b]$ —not $P = \{a = u_0 < u_1 < \cdots < c_m = b\}$, as stated in the textbook— such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for x in (u_{j-1}, u_j).

(a) Show that a step function f is integrable and evaluate $\int_a^b f$.

Solution 2.

Problem 3 (33.7)**.** Let f be a bounded function on $[a, b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

(a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$. *Hint:* $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$.

(b) Show that if f is integrable on $[a, b]$, then f^2 also is integrable on $[a, b]$.

Solution 3.

Problem 4 (34.2)**.** Calculate

(a) $\lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt$.

Solution 4.

Problem 5 (34.5)**.** Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)\,dt \quad \text{for } x \in \mathbb{R}.$$

Show F is differentiable on $\mathbb{R}$ and compute F'.

Solution 5.

Problem 6. A. [Bonus problem] Suppose f is a continuous non-negative function on $[a, b]$, with

$$M = \max_{x \in [a,b]} f(x).$$

For $n \in \mathbb{N}$, let

$$M_n = \left(\int_a^b f^n\,dt \right)^{1/n}.$$

Prove that $\lim M_n = M$.

Solution 6.