MATH 447: Real Variables - Homework #2

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Problem 1 (9.12). • *Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.*

(a) Show that if $L < 1$, then $\lim s_n = 0$. Hint: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. Hint: Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

Solution 1.

Problem 2 (9.14). *Let $p > 0$. Use Exercise 9.12 to show*

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \leq 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases}$$

Hint: For the $a > 1$ case, use Exercise 9.12(b).

$\frac{1}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution 2.

Problem 3 (10.6). *(a) Let (s_n) be a sequence such that*

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Solution 3.

Problem 4 (10.8). *Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.*

Solution 4.

Problem 5 (10.10). *Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.*

(a) Find s_2, s_3 *and* s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ *for all* n.

(c) Show (s_n) *is a decreasing sequence.*

(d) Show $\lim s_n$ *exists and find* $\lim s_n$.

Solution 5.

Problem 6 (10.12)**.** *Let* $t_1 = 1$ *and* $t_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] \cdot t_n$ *for* $n \geq 1$.

(a) Show $\lim t_n$ *exists.*

(b) What do you think $\lim t_n$ *is?*

(c) Use induction to show $t_n = \frac{n+1}{2n}$.

(d) Repeat part (b).

Solution 6.