Sums, products, ratios of limits (Section 9)

Theorem (Theorems 9.3, 9.4, 9.6 from textbook)

Suppose $\lim s_n = s$ *and* $\lim t_n = t$. *Then:*

- $\lim(s_n + t_n) = s + t$
- $\lim(as_n) = as$, *for any* $a \in \mathbb{R}$.
- $\lim(s_n t_n) = st$.
- *If, in addition,* $t \neq 0$, *then* $\lim \dfrac{s_n}{t_n} = \dfrac{s}{t}$.

If $t \neq 0$, then $t_n \neq 0$ for n sufficiently large.
Indeed, find $N \in \mathbb{R}$ s.t. $|t_n - t| < |t|$ for $n > N$.
By Triangle Inequality, for such n, $|t_n| \geqslant |t| - |t - t_n| > |t| - |t| = 0$.

Proof of $\lim(s_n + t_n) = s + t$, where $s = \lim s_n$, $t = \lim t_n$

We need to show that, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $|(s_n + t_n) - (s + t)| < \varepsilon$ whenever $n > N$.

Know: $|s_n - s|$, $|t_n - t|$ are small for n large. Need to find an upper estimate for $|(s_n + t_n) - (s + t)|$ in terms of $|s_n - s|$, $|t_n - t|$.

By Triangle Inequality, $|(s_n + t_n) - (s + t)| \leqslant |s_n - s| + |t_n - t|$.

Find $N_1 \in \mathbb{R}$ s.t., for any $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2}$.
Find $N_2 \in \mathbb{R}$ s.t., for any $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2}$.

For $n > N = \max\{N_1, N_2\}$,
$\big|(s_n + t_n) - (s + t)\big| \leqslant |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ■

Proof of $\lim(s_n t_n) = st$, where $s = \lim s_n$, $t = \lim t_n$

Need to show: $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{R}$ s.t. $|s_n t_n - st| < \varepsilon$ for $n > N$. Find an upper estimate for $|s_n t_n - st|$ in terms of $|s_n - s|$, $|t_n - t|$. Write $s_n t_n - st = s_n t_n - st_n + st_n - st = (s_n - s)t_n + s(t_n - t)$, hence $|s_n t_n - st| \leqslant |s_n - s|\,|t_n| + |s|\,|t_n - t|$.

(t_n) is bounded; find $A > 0$ s.t. $|t_n| \leqslant A$ for any n.

Find $N_1 \in \mathbb{R}$ s.t., for any $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2A}$.
Find $N_2 \in \mathbb{R}$ s.t., for any $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2|s|+1}$.

For $n > N = \max\{N_1, N_2\}$,
$|s_n t_n - st| \leqslant |s_n - s|\,|t_n| + |s|\,|t_n - t| < \frac{\varepsilon}{2A} \cdot A + \frac{\varepsilon}{2|s|+1} \cdot |s| < \varepsilon.$ ■

Squeeze Theorem

Theorem

If $a_n \leqslant s_n \leqslant b_n$ *for any* n*, and* $\lim a_n = s = \lim b_n$*, then* $\lim s_n = s$*.*

Corollary

If $0 \leqslant |s_n| \leqslant t_n$ *for any* n*, and* $\lim t_n = 0$*, then* $\lim s_n = 0$*.*

Proof of Theorem and Corollary: Exercise 8.5 (Homework).

Corollary

$\lim s_n = 0$ *if and only if* $\lim |s_n| = 0$*.*

Sketch of a proof: $\lim s_n = 0$ iff $\forall\, \varepsilon > 0\ \exists N \in \mathbb{R}$ s.t. $|s_n| < \varepsilon$ for $n > N$.
This is equivalent to $\lim |s_n| = 0$. ■

Basic examples (Theorem 9.7)

a. $\lim \frac{1}{n^p} = 0$ for $p > 0$.
b. $\lim a^n = 0$ if $|a| < 1$.
c. $\lim n^{1/n} = 1$.
d. $\lim a^{1/n} = 1$ if $a > 0$.

Theorem (Binomial expansion)

For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, $\left(a+b\right)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$ *is the numbers of ways to select k objects out of n (order not important). Convention:* $0! = 1$.

Examples: $\binom{n}{0} = \frac{n!}{0! \cdot n!} = 1$, $\binom{n}{1} = \frac{n!}{1! \cdot (n-1)!} = n$,
$\binom{n}{2} = \frac{n!}{2! \cdot (n-2)!} = \frac{n(n-1)}{2}, \ldots$

Basic examples: some proofs.

Proof of $\lim a^n = 0$ if $|a| < 1$.

Case of $a = 0$ is easy, so assume $0 < |a| < 1$.

Recall: $\lim s_n = 0$ iff $\lim |s_n| = 0$. It suffices to show that $\lim |a|^n = 0$.

Let $b = |a|^{-1} - 1$, then $b > 0$. $|a| = \frac{1}{1+b}$. $|a|^n = \frac{1}{(1+b)^n}$.

$(1+b)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} b^k > \binom{n}{1} b = nb$

$\frac{1}{(1+b)^n} \leqslant \frac{1}{n} \cdot \frac{1}{b} \xrightarrow[n]{} 0$, hence, by Squeeze Theorem, $\frac{1}{(1+b)^n} = |a|^n \xrightarrow[n]{} 0$. ■

Proof of $\lim n^{1/n} = 1$.

Let $s_n = n^{1/n} - 1$ (so $s_n > 0$); we need to prove $\lim s_n = 0$.

We have $n = \left(1 + s_n\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} s_n^k \geqslant 1 + \frac{n(n-1)}{2} s_n^2$ (here we are accounting for the terms with $k = 0$ and $k = 2$).

$n - 1 \geqslant \frac{n(n-1)}{2} s_n^2$, so $s_n \leqslant \frac{\sqrt{2}}{\sqrt{n}}$.

By (a), $\lim \frac{1}{\sqrt{n}} = 0$. By Squeeze Theorem, $\lim s_n = 0$. ■

Example

Compute $\lim \frac{n^3 - 4n}{2n^3 + 1}$**.**

$$\frac{n^3 - 4n}{2n^3 + 1} = \frac{n^3(1 - 4/n^2)}{n^3(2 + 1/n^3)} = \frac{1 - 4/n^2}{2 + 1/n^3}.$$

$$\lim\left(1 - \frac{4}{n^2}\right) = 1 - 4\lim\frac{1}{n^2} = 1,$$

$$\lim\left(2 + \frac{1}{n^3}\right) = 2 + \lim\frac{1}{n^3} = 2.$$

By the quotient rule of limits, $\frac{n^3 - 4n}{2n^3 + 1} = \frac{\lim(1 - 4/n^2)}{\lim(2 + 1/n^3)} = \mathbf{\frac{1}{2}}$.

Sequences diverging to $\pm\infty$

Definition (Definition 9.8)

We say that $\lim s_n = +\infty$ (the sequence (s_n) diverges to $+\infty$) if $\forall\, A$ $\exists N \in \mathbb{R}$ s.t. $s_n > A$ for $n > N$. $\lim s_n = -\infty$ (divergence to $-\infty$) is defined similarly.

Example: $\lim \dfrac{n^2 - n + 2}{n+1} = +\infty$.

Need to show: $\forall\, A > 0\ \exists\, N \in \mathbb{R}$ s.t. $\dfrac{n^2 - n + 2}{n+1} > A\ \forall\, n > N$.

Note that, for $n \geqslant 2$, $\dfrac{n^2 - n + 2}{n+1} > \dfrac{n^2 - n^2/2}{2n} = \dfrac{n}{4}$.

Let $N = \max\{2, 4A\}$. If $n > N$, then $\dfrac{n^2 - n + 2}{n+1} > \dfrac{n}{4} > A$. ■

When is lim equal to $+\infty$?

Theorem (Theorem 9.9 – product rule for divergence to $+\infty$)

If $\lim s_n = +\infty$, *and* $\lim t_n > 0$ *(positive real number or* $+\infty$*), then* $\lim s_n t_n = +\infty$.

Question. Suppose $\lim s_n = +\infty$, $t_n > 0\ \forall n$, and $\lim t_n = 0$. What can we say about $\lim s_n t_n$?

$\lim s_n t_n$ need not exist; if $\lim s_n t_n$ exists, it can be 0, $+\infty$, or any positive real number (try to come up with examples!).

Theorem (Theorem 9.10)

If (s_n) *are positive numbers, then* $\lim s_n = +\infty$ *iff* $\lim \dfrac{1}{s_n} = 0$.

Monotone sequences (Section 10)

This material will be repeated next time.

Definition (Increasing, decreasing, monotone sequences)

A sequence (s_n) is called increasing (decreasing) if $s_n \leqslant s_{n+1}$ (resp. $s_n \geqslant s_{n+1}$) for any n.
A sequence is monotone if it is either increasing or decreasing.

Our "increasing" sequences are sometimes called "nondecreasing." We do not require $s_n < s_{n+1}$. Similarly, our "decreasing" sequences are also referred to as "nonincreasing."

Examples. 1. $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. This sequence is increasing: $x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n$ for any n.

2. $y_n = \frac{(-1)^n}{n^2}$. This sequence is not monotone: $y_{n+1} > y_n$ if n is odd, $y_{n+1} < y_n$ if n is even.