Ratio Test for convergence

Root Test: For a series $\sum_n a_n$ let $\alpha = \limsup_n |a_n|^{1/n}$. The series:

1. Converges absolutely if $\alpha < 1$;
2. Diverges if $\alpha > 1$.
3. If $\alpha = 1$, the test gives no information.

Theorem (14.8 – Ratio Test)

A series $\sum_n a_n$ of non-zero terms:

1. *Converges absolutely if* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$;
2. *Diverges if* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.
3. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leqslant 1 \leqslant \limsup \left|\frac{a_{n+1}}{a_n}\right|$, *the test gives no information.*

Proof; examples

Partial proof, using Root Test. Let $\alpha = \limsup_n |a_n|^{1/n}$. We know:
$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leqslant \alpha \leqslant \limsup \left|\frac{a_{n+1}}{a_n}\right|$.
Case (i): $\alpha < 1 \Rightarrow$ series converges.
Case (ii): $\alpha > 1 \Rightarrow$ series diverges. ■

Examples: **(1)** $\sum_{k=1}^{\infty} \frac{k^4}{2^k}$ converges (absolutely). $a_k = \frac{k^4}{2^k}$.
Can use Root Test: $a_k^{1/k} = \frac{(k^{1/k})^4}{2}$, hence $\lim a_k^{1/k} = \frac{1}{2} < 1$.
Can also use Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{1}{2}\left(1 + \frac{1}{k}\right)^2$, hence $\lim \frac{a_{k+1}}{a_k} = \frac{1}{2} < 1$.

(2) $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges iff $p > 1$ (p-series). Ratio and Root tests are inconclusive.

Decimal expansions (Section 16) – almost no proofs!

For $x \in [0, \infty)$, we consider a decimal expansion $x = K.d_1 d_2 d_3 \ldots = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j}$, with $K = \{0, 1, 2, \ldots\}$ and $d_1, d_2, \ldots \in \{0, 1, \ldots, 9\}$. $\sum_j \frac{d_j}{10^j}$ converges (compare to $\sum_j \frac{1}{10^{j-1}}$).

Theorem (16.2)

Any real number has at least one decimal expansion.

The integer part of z $\lfloor z \rfloor$ = greatest integer $\leqslant z$. Let $S = \{n \in \mathbb{Z} : n \leqslant z\}$, $\lfloor z \rfloor = \sup S$. Clearly $\lfloor z \rfloor \leqslant z$. Can show: $\lfloor z \rfloor \in \mathbb{Z}$.

Idea of the proof: $K := \lfloor x \rfloor$, then $0 \leqslant x - K < 1$.

$d_1 := \lfloor 10(x - K) \rfloor$, so $K + \frac{d_1}{10} \leqslant x < K + \frac{d_1+1}{10} \Rightarrow \left| x - \left(K + \frac{d_1}{10} \right) \right| < \frac{1}{10}$.

$d_2 := \lfloor 10^2 \left(x - K - \frac{d_1}{10} \right) \rfloor$, so $K + \frac{d_1}{10} + \frac{d_2}{10^2} \leqslant x < K + \frac{d_1}{10} + \frac{d_2+1}{10^2} \Rightarrow \left| x - \left(K + \frac{d_1}{10} + \frac{d_2}{10^2} \right) \right| < \frac{1}{10^2}$.

Continue in this manner.

More about decimal expansions

Theorem (16.3)

Any $x \geqslant 0$ has either exactly one decimal expansion, or exactly two – one ending in $**d000\dots$ ($d \in \{1, \dots, 9\}$, another ending in $**[d-1]999\dots$.

For instance, $\frac{1}{2} = 0.5000\dots = 0.4999\dots$.

Definition (16.4)

A repeating decimal expansion is one of the form
$K.d_1 \dots d_\ell \overline{d_{\ell+1} \dots d_{\ell+r}} = K.d_1 \dots d_\ell d_{\ell+1} \dots d_{\ell+r} d_{\ell+1} \dots d_{\ell+r} \dots$.

Theorem (16.5)

$x \in \mathbb{Q}$ iff the decimal explansion of x is repeating.

Repeating decimal expansions

Theorem (16.5)

$x \in \mathbb{Q}$ iff the decimal explansion of x is repeating.

x has repeating expansion $\Rightarrow x \in \mathbb{Q}$.

Suppose $x = K.d_1 \ldots d_\ell \overline{d_{\ell+1} \ldots d_{\ell+r}}$. Let $y = 0.\overline{d_{\ell+1} \ldots d_{\ell+r}} = \sum_{j=1}^{r} d_{\ell+j}10^{-j} + 10^{-r} \cdot \sum_{j=1}^{r} d_{\ell+j}10^{-j} + 10^{-2r} \sum_{j=1}^{r} d_{\ell+j}10^{-j} + \ldots = \sum_{i=0}^{\infty} 10^{-ir} z$, where $z = \sum_{j=1}^{r} d_{\ell+j}10^{-j} \in \mathbb{Q}$. So, $y = \frac{z}{1-10^{-r}} \in \mathbb{Q}$.

$x = K + \sum_{j=1}^{\infty} \frac{d_j}{10^j} = K + \sum_{j=1}^{\ell} \frac{d_j}{10^j} + 10^{-\ell} y \in \mathbb{Q}$. ■

$x \in \mathbb{Q} \Rightarrow x$ has repeating expansion: uses long division (see textbook).