Another inequality involving means

Theorem (Jensen)

If f is a convex function on an interval I, $x_1, \ldots, x_n \in I$, $t_1, \ldots, t_n \geqslant 0$, $\sum_{i=1}^n t_i = 1$, then $f\left(\sum_{i=1}^n t_i x_i\right) \leqslant \sum_{i=1}^n t_i f(x_i)$. For concave functions, the inequality is reversed.

Proposition

Suppose $r > 1$, and $x_1, \ldots, x_n \geqslant 0$. Then $\frac{x_1+\ldots+x_n}{n} \leqslant \left(\frac{x_1^r+\ldots+x_n^r}{n}\right)^{1/r}$.

If $r = 2$, we obtain the inequality between arithmetic and quadratic means: $\frac{x_1+\ldots+x_n}{n} \leqslant \left(\frac{x_1^2+\ldots+x_n^2}{n}\right)^{1/2}$.

Proof. On $[0, \infty)$, the function $f(x) = x^r$ is convex, since $f'(x) = rx^{r-1}$ is increasing. Apply Jensen's Inequality with $t_1 = \ldots = t_n = \frac{1}{n}$: $\left(\frac{x_1+\ldots+x_n}{n}\right)^r \leqslant \frac{x_1^r+\ldots+x_n^r}{n}$. Take r-th root of both sides. ■

Arithmetic versus harmonic means

Proposition (Arithmetic and harmonic means)

If $x_1, \ldots, x_n > 0$*, then* $\frac{x_1 + \ldots + x_n}{n} \geqslant \frac{n}{1/x_1 + \ldots + 1/x_n}$.

Proof. $g(x) = \frac{1}{x}$ is convex on $(0, \infty)$ $(g'' > 0)$. Let $y_i = \frac{1}{x_i}$.

Apply Jensen's Inequality with $t_1 = \ldots = t_n = \frac{1}{n}$:

$\frac{1}{n} \sum_{i=1}^{n} g(y_i) = \frac{1/y_1 + \ldots + 1/y_n}{n} \geqslant g\left(\frac{1}{n} \sum_{i=1}^{n} y_i\right) = \frac{1}{(y_1 + \ldots + y_n)/n}$.

$\frac{1/y_1 + \ldots + 1/y_n}{n} = \frac{x_1 + \ldots + x_n}{n}$, $\frac{1}{(y_1 + \ldots + y_n)/n} = \frac{n}{1/x_1 + \ldots + 1/x_n}$. ■

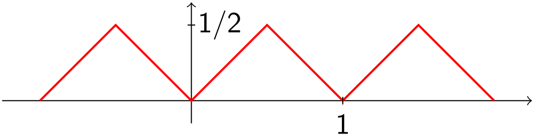
Nowhere differentiable functions

Proposition

There exists a bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$, which is differentiable nowhere.

We shall construct a 1-periodic function like this – that is, $f(x) = f(x+1)$, $\forall x \in \mathbb{R}$.

Begin by defining the sawtooth function $s : \mathbb{R} \to \mathbb{R}$: $s(x) = \phi(x - \lfloor x \rfloor)$, where, for $t \in [0,1]$, $\phi(t) = \min\{t, 1-t\}$.



$\boxed{f(x) = \sum_{k=0}^{\infty} 8^{-k} s(64^k x) \text{ has the desired properties}}$

Weierstrass Monster – the proof continues

Want to show: $f(x) = \sum_{k=0}^{\infty} 8^{-k} s(64^k x)$ is:

1. 1-periodic, bounded, uniformly continuous.
2. Differentiable nowhere.

(1) $\forall x$, $|8^{-k} s(64^k x)| \leqslant 2^{-3k-1}$, hence, by Weierstrass M-test (with $M_k = 2^{-3k-1}$, the series converges uniformly on $\mathbb{R}$.
$0 \leqslant f(x) \leqslant \sum_{k=0}^{\infty} 2^{-3k-1} = \frac{4}{7}$.
$s(x) = s(x+1)$, hence, $\forall x$,
$f(x+1) = \sum_{k=0}^{\infty} 8^{-k} s(64^k x + 64^k) = \sum_{k=0}^{\infty} 8^{-k} s(64^k x) = f(x)$.

(2) For $x \in \mathbb{R}$, $A > 0$, and $\delta > 0$, we need to find $y \in \mathbb{R} \backslash \{x\}$ with $|x - y| \leqslant \delta$, $|f(x) - f(y)| \geqslant A|x - y|$.

Lemma. For $u, v \in \mathbb{R}$, $|s(u) - s(v)| \leqslant \max\left\{|u - v|, \frac{1}{2}\right\}$. Consequently, for $k \geqslant 0$, $|s(64^k u) - s(64^k v)| \leqslant \max\left\{64^k|u - v|, \frac{1}{2}\right\}$.

Weierstrass Monster – the proof continues

(2) For $x \in \mathbb{R}$, $A > 0$, and $\delta > 0$, we need to find $y \in \mathbb{R} \backslash \{x\}$ with $|x - y| < \delta$, $|f(x) - f(y)| \geqslant A|x - y|$.

Find $n \in \mathbb{N}$ s.t. $A < \frac{8^n}{2}$, $\delta > 64^{-n}$.
Find $m \in \mathbb{Z}$ s.t. $2 \cdot 64^n x \in [m, m+1]$.
If $2 \cdot 64^n x \in [m, m + 1/2]$, let $y = 2^{-1} 64^{-n} (m + 1)$.
If $2 \cdot 64^n x \in (m + 1/2, m + 1]$, let $y = 2^{-1} 64^{-n} m$.

$2 \cdot 64^n x, 2 \cdot 64^n y \in [m, m+1]$, hence $|x - y| \leqslant 2^{-1} 64^{-n} < \delta$. Need to show: $|f(x) - f(y)| \geqslant \frac{8^n}{2} |x - y|$.

$2 \cdot 64^n y$ is the endpoint of $[m, m+1]$ which is farthest from $2 \cdot 64^n x$.
$64^n x, 64^n y \in \left[\frac{m}{2}, \frac{m+1}{2}\right]$. Thus,
$\left| s(64^n x) - s(64^n y) \right| = \left| 64^n x) - s(64^n y) \right| = 64^n |x - y|$. Also,
$\left| 2 \cdot 64^n x - 2 \cdot 64^n y \right| \geqslant \frac{1}{2}$, hence $|x - y| \geqslant 4^{-1} 64^{-n}$.

Weierstrass Monster – the proof continues

Remains to show: $|f(y) - f(x)| > \frac{8^n}{2}|x-y|$.

$|f(y) - f(x)| \geqslant 8^{-n}\left|s(64^n x) - s(64^n y)\right| - \sum_{i=0}^{n-1} 8^{-i}\left|s(64^i x) - s(64^i y)\right| - \sum_{i=n+1}^{\infty} 8^{-i}\left|s(64^i x) - s(64^i y)\right|$.

For $i < n$, $\left|s(64^i x) - s(64^i y)\right| \leqslant 64^i|x-y|$ (Lemma), $\Rightarrow$

$\sum_{i=0}^{n-1} 8^{-i}\left|s(64^i x) - s(64^i y)\right| \leqslant \sum_{i=0}^{n-1} 8^i|x-y| < \frac{8^n}{7}|x-y|$.

$\sum_{i=n+1}^{\infty} 8^{-i}\left|s(64^i x) - s(64^i y)\right| \leqslant \frac{1}{2}\sum_{i=n+1}^{\infty} 8^{-i} = \frac{8^{-n}}{14}$.

$|x-y| \geqslant 4^{-1}64^{-n}$, hence $\frac{8^{-n}}{14} \leqslant \frac{2 \cdot 8^n}{7}|x-y|$.

$|f(x) - f(y)| > \left(1 - \frac{1}{7} - \frac{2}{7}\right)8^n|x-y| > \frac{8^n}{2}|x-y|$. ■

There are many primes

Theorem

Let $p_1 < p_2 < \dots$ be the increasing enumeration of prime numbers. Then $\sum_n \frac{1}{p_n}$ diverges.

Proof. Note that $\alpha := \sum_{n=1}^{\infty} \frac{1}{p_n^2} < \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Suppose, for the sake of contradiction, that $\sum_n \frac{1}{p_n}$ converges. Find $K \in \mathbb{N}$ s.t. $\beta := \sum_{n=K+1}^{\infty} \frac{1}{p_n} < 1 - \alpha$. Find $N \in \mathbb{N}$ s.t. $N > 2^K/(1 - \alpha - \beta)$.

Let C (B) be the set of all $x \in \{1, \dots, N\}$ which are divisible by some p_n with $n > K$ (resp. by p_n^2 for some n), and let $A = \{1, \dots, N\} \backslash (B \cup C)$.

$\{1, \dots, N\}$ contains no more than N/q numbers divisible by q, hence $|C| \leqslant \sum_{n>K} \frac{N}{p_n} = \beta N$. Likewise, $|B| \leqslant \alpha N$.

Finally, A contains only the numbers of the form $p_1^{r_1} \dots p_K^{r_K}$, with $r_i \in \{0, 1\}$. Thus, $|A| \leqslant 2^K$.

Consequently, $N = |\{1, \dots, N\}| \leqslant |A| + |B| + |C| \leqslant 2^K + (\alpha + \beta)N$, which yields the desired contradiction. ■