Properties of integrals

Theorem (33.3, 33.4(i))

Suppose f, g are integrable on $[a, b]$, and $c \in \mathbb{R}$. Then:

1. *cf is integrable, and $\int_a^b cf = c \int_a^b f$.*
2. *$f + g$ are integrable, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.*
3. *If $f \geqslant g$, then $\int_a^b f \geqslant \int_a^b g$.*

For $P = \big(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\big)$,
$\mathrm{mesh}(P) = \max_{1 \leqslant k \leqslant n}(t_k - t_{k-1})$ (length of longest subinterval).

Theorem 32.7. A bounded $f : [a, b] \to \mathbb{R}$ is integrable iff $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $U(f, P) - L(f, P) < \varepsilon$ whenever $\mathrm{mesh}(P) < \delta$.

For such P, $U(f, P) - \int_a^b f, \int_a^b f - L(f, P) < \varepsilon$

Proof: integrability of $f + g$

It suffices to show that $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $\mathrm{mesh}(P) < \delta$, then $L(f+g,P) > \int_a^b f + \int_a^b g - \varepsilon$, $U(f+g,P) < \int_a^b f + \int_a^b g + \varepsilon$.

Find $\delta_1 > 0$ s.t. $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$ if $\mathrm{mesh}(P) < \delta_1$.

$L(f,P) \leqslant \int_a^b f \leqslant U(f,P)$,

so $L(f,P) > \int_a^b f - \frac{\varepsilon}{2}$, $U(f,P) < \int_a^b f + \frac{\varepsilon}{2}$ if $\mathrm{mesh}(P) < \delta_1$.

Likewise, find $\delta_2 > 0$ s.t. $L(g,P) > \int_a^b g - \frac{\varepsilon}{2}$, $U(g,P) < \int_a^b g + \frac{\varepsilon}{2}$ if $\mathrm{mesh}(P) < \delta_2$.

Suppose $\mathrm{mesh}(P) < \delta := \min\{\delta_1, \delta_2\}$, then

$$L(f+g,P) \geqslant L(f,P) + L(g,P) > \left(\int_a^b f - \frac{\varepsilon}{2} \right) + \left(\int_a^b g - \frac{\varepsilon}{2} \right) \int_a^b f + \int_a^b g - \varepsilon,$$

and likewise, $U(f+g,P) < \int_a^b f + \int_a^b g + \varepsilon$.

Triangle Inequality for Integrals

Recall: $\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n} |a_i|$.

Theorem (33.5)

If f is integrable on $[a,b]$, then so is $|f|$, and $\left|\int_a^b f\right| \leqslant \int_a^b |f|$.

Lemma 1. $M(h,S) - m(h,S) = \sup_{x,y\in S} |h(x) - h(y)|$.

Proof. See below.

Lemma 2. For any function g and set S,
$M(|g|,S) - m(|g|,S) \leqslant M(g,S) - m(g,S)$.

Proof. Recall Lemma 1: for any function h,
$M(h,S) - m(h,S) = \sup_{x,y\in S} |h(x) - h(y)|$.

By the triangle inequality, $\big||g(x)| - |g(y)|\big| \leqslant \big|g(x) - g(y)\big|$, hence
$M(|g|,S) - m(|g|,S) = \sup_{x,y\in S} \big||g(x)| - |g(y)|\big| \leqslant$
$\sup_{x,y\in S} \big|g(x) - g(y)\big| = M(g,S) - m(g,S)$. ■

Proof of Lemma 1

Lemma (Lemma 1)

$M(h,S) - m(h,S) = \sup_{x,y \in S} |h(x) - h(y)|$.

Proof. $\forall x, y \in S$, $h(x) \leqslant M(h,S)$ and $h(y) \geqslant m(h,S)$, hence
$M(h,S) - m(h,S) \geqslant h(x) - h(y)$.
Likewise, $M(h,S) - m(h,S) \geqslant h(y) - h(x)$.
So, $M(h,S) - m(h,S) \geqslant \sup_{x,y \in S} |h(x) - h(y)|$.

On the other hand, fix $\varepsilon > 0$, and pick $x, y \in S$ s.t. $h(x) > M(h,S) - \varepsilon$, $h(y) < m(h,S) + \varepsilon$. Then
$|h(x) - h(y)| \geqslant h(x) - h(y) > M(h,S) - m(h,S) - 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, $M(h,S) - m(h,S) \leqslant \sup_{x,y \in S} |h(x) - h(y)|$. ∎

Triangle inequality for integrals

Theorem (33.5)

If f is integrable on $[a,b]$, then so is $|f|$, and $\left|\int_a^b f\right| \leqslant \int_a^b |f|$.

Lemma 2. For any function g and set S,
$M(|g|,S) - m(|g|,S) \leqslant M(g,S) - m(g,S)$.

Proof of Theorem: $|f|$ is integrable. Fix $\varepsilon > 0$. Need a partition P s.t. $U(|f|,P) - L(|f|,P) < \varepsilon$.
Any $P = (t_k)_{k=0}^n$ with $U(f,P) - L(f,P) < \varepsilon$ will do. Indeed,
$U(|f|,P) - L(|f|,P) = \sum_{k=1}^n \big(M(|f|,[t_{k-1},t_k]) - m(|f|,[t_{k-1},t_k])\big)(t_k - t_{k-1}) \leqslant \sum_{k=1}^n \big(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\big)(t_k - t_{k-1}) = U(f,P) - L(f,P) < \varepsilon$. ■

Proof of Theorem: $\left|\int_a^b f\right| \leqslant \int_a^b |f|$. Applying Theorem 33.4(i) to $-|f| \leqslant f \leqslant |f|$, obtain $-\int_a^b |f| \leqslant \int_a^b f \leqslant \int_a^b |f|$. ■

Products of integrable functions; etc.

Proposition (Exercise 33.7 – Homework 9)

If f is integrable on $[a, b]$, then so is f^2.

Corollary (Exercise 33.8)

If f, g are integrable on $[a, b]$, then so is fg.

Proof. $fg = \frac{1}{4}\big((f+g)^2 - (f-g)^2\big)$. $f+g, f-g$ are integrable, then so are $(f+g)^2, (f-g)^2$. Thus, fg is integrable as well. ■

Theorem (33.4(ii))

If f is continuous on $[a, b]$, $f \geqslant 0$, and $\int_a^b f = 0$, then $f = 0$ on $[a, b]$.

Proof. Suppose, for the sake of contradiction, that $f(x) = y > 0$, for some $x \in [a, b]$. By continuity, $\exists$ interval $[c, d] \subset [a, b]$ s.t. $f \geqslant y/2$ on $[c, d]$. Then $\int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f \geqslant \frac{y}{2}(d-c) > 0$, a contradiction! ■

Piecewise monotone, piecewise continuous functions

Definition (33.7)

A function $f : [a, b] \to \mathbb{R}$ is called piecewise monotone (piecewise continuous) if $\exists$ partition $\big(a = t_0 < t_1 < \ldots < t_n = b\big)$ s.t. f is monotone and bounded (resp. uniformly continuous) on (t_{i-1}, t_i) for each i.

Theorem (33.8)

If $f : [a, b] \to \mathbb{R}$ is either (i) piecewise monotone and bounded, or (ii) piecewise continuous, then it is integrable.

Proof for piecewise continuity. By the additivity of the integral (Theorem 33.6), it suffices to show that f is integrable on $[t_{i-1}, t_i]$, for $1 \leqslant i \leqslant n$. f is uniformly continuous on (t_{i-1}, t_i), hence it extends to a continuous function g_i on $[t_{i-1}, t_i]$; this g_i is integrable. But $f\big|_{[t_{i-1}, t_i]}$ differs from g_i at no more than two points (the endpoints), hence f is inegrable on $[t_{i-1}, t_i]$ as well. ∎

Interchanging $\int$ and lim

Proposition (Exercise 33.9)

Suppose (f_n) is a sequence of integrable functions on $[a, b]$, converging uniformly to f. Then f is integrable, $\lim_n \int_a^b f_n$ exists, and $\int_a^b f = \lim_n \int_a^b f_n$.

Observation. If $\sup_{x\in[a,b]} \left|g(x) - h(x)\right| \leqslant \sigma$, then, $\forall$ partition P, $\left|U(g, P) - U(h, P)\right|, \left|L(g, P) - L(h, P)\right| \leqslant \sigma(b - a)$.

Proof (for U; L is handled similarly).
For any set S, $M(g, S) - M(h, S) \leqslant \sigma$.
For a partition $P = (t_k)_{k=0}^n$, $U(g, P) - U(h, P) =$
$\sum_{k=1}^n \Big(M(g, [t_{k-1}, t_k]) - M(h, [t_{k-1}, t_k])\Big)(t_k - t_{k-1}) \leqslant$
$\sigma \sum_{k=1}^n (t_k - t_{k-1}) = \sigma(b - a)$,
and similarly, $U(h, P) - U(g, P) \leqslant \sigma(b - a)$.
Thus, $\left|U(g, P) - U(h, P)\right| \leqslant \sigma(b - a)$. ∎

Interchanging $\int$ and lim

Proposition (Exercise 33.9)

Suppose (f_n) *is a sequence of integrable functions on* $[a,b]$*, converging uniformly to* f*. Then* f *is integrable,* $\lim_n \int_a^b f_n$ *exists, and* $\int_a^b f = \lim_n \int_a^b f_n$.

Proof of Proposition. For $\varepsilon > 0$ find N s.t.
$\sup_{x\in[a,b]} \left|f_n(x) - f(x)\right| \leqslant \frac{\varepsilon}{2(b-a)}$ for $n \geqslant N$.
By Observation, for $n \geqslant N$, $U(f,P) \leqslant U(f_n,P) + \frac{\varepsilon}{2}$, for any P.
$U(f) = \inf_P U(f,P) \leqslant \inf_P U(f_n,P) + \frac{\varepsilon}{2} = \int_a^b f_n + \frac{\varepsilon}{2}$.
$U(f) \leqslant \liminf_n \int_a^b f_n + \frac{\varepsilon}{2}$. ε is arbitrary, hence $U(f) \leqslant \liminf_n \int_a^b f_n$.
Similarly, $L(f) \geqslant \limsup_n \int_a^b f_n$, so
$\liminf_n \int_a^b f_n \geqslant U(f) \geqslant L(f) \geqslant \limsup_n \int_a^b f_n$. $\liminf_n \leqslant \limsup_n$, hence
$U(f) = L(f) = \liminf_n \int_a^b f_n = \limsup_n \int_a^b f_n$.
So, $\int_a^b f$ exists, and equals $\lim_n \int_a^b f_n$ (which also exists). ∎

Can't replace uniform convergence with pointwise

In examples below, limits of functions are pointwise.

Example (1) $\lim \int f_n$ **exists,** $\lim f_n$ **is not integrable.** Let $\{r_1, r_2, \dots\} = \mathbb{Q} \cap [0,1]$. Let $f_n(x) = 1$ if $x \in \{r_1, \dots, r_n\}$, $f_n(x) = 0$ otherwise. f_n is integrable, with $\int_0^1 f_n = 0$. Let $f(x) = \lim f_n(x)$: $f(x) = 1$ if $x \in \mathbb{Q}$, $f(x) = 0$ if $x \notin \mathbb{Q}$. Not integrable!

Example (2) $\lim \int f_n \neq \int \lim f_n$**.** Let $f_n(x) = n$ if $0 < x < \frac{1}{n}$, $f_n(x) = 0$ for $x \notin \left(0, \frac{1}{n}\right)$. $\forall n$, $\int_0^1 f_n = 1$. However, $f(x) = \lim_n f_n(x) = 0$ for any x, and $\int_0^1 f = 0$.

Bounded and monotone convergence

Theorem (Bounded convergence – 33.11)

Suppose (f_n) are integrable on $[a,b]$, $|f_n| \leqslant M$ for any n, $f_n \to f$ pointwise on $[a,b]$, and f is integrable. Then $\lim_n \int_a^b f_n$ exists, and equals $\int_a^b f$.

Theorem (Monotone convergence – 33.12)

Suppose (f_n) are integrable on $[a,b]$, $f_1 \leqslant f_2 \leqslant \ldots$, $f_n \to f$ pointwise on $[a,b]$, and f is integrable. Then $\lim_n \int_a^b f_n$ exists, and equals $\int_a^b f$.

Proof of Monotone Convergence Theorem. For any n, $|f_n(x)| \leqslant M$, where $M = \max\left\{ \sup_{x\in[a,b]} |f_1(x)|, \sup_{x\in[a,b]} |f(x)| \right\}$. Apply Bounded Convergence Theorem. ■

Example. $\lim_n \int_0^1 \frac{dx}{1+nx^3} = 0$, since the sequence of functions

$f_n(x) = \frac{1}{1+nx^3}$ decreases to $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$, and $\int_0^1 f = 0$.