TAM 470 / CSE 450

Homework 9

Problem 1 (20 points)

Consider the 1D transient heat equation for a domain of length L:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{for } x \in [0, L], \quad t \geq 0 \tag{1}$$

Suppose we wish to use this equation to model heat transfer in a metallic bar that is insulated along its sides. The bar begins at a uniform temperature T_{hot}. At time $t = 0$, the right end is exposed to air cooling via convection while the left end is maintained at a fixed temperature of T_{hot}. This situation corresponds to the following initial and boundary conditions:

$$T(x, 0) = T_{hot} \quad \text{for } x \in (0, L] \tag{2}$$

$$T(0, t) = T_{hot}, \quad t \geq 0 \tag{3}$$

$$\frac{\partial T}{\partial x}(L, t) = \gamma(T_\infty - T(L, t)), \quad t \geq 0 \tag{4}$$

For the convection condition, T_∞ represents the ambient air temperature and γ represents the ratio of convection heat transfer coefficient to thermal conductivity, $\frac{h}{k}$.

(a) (14 pts, PL) Go to PrairieLearn to implement the Crank-Nicolson scheme (trapezoid rule for time integration) for this problem. This scheme was derived in class and you do not need to repeat the derivation.

Your code should assume general parameters T_{hot}, T_∞, α, γ, L, and t_{end}, and define the space and time discretization using inputs `Nx` and `dt` to define space and time discretization, respectively. **Your code should use the first order difference scheme for the spatial derivative in the convection boundary condition**.

(b) Assume the following parameters: $L = 1.0$ m, $\alpha = 1.0 \times 10^{-4}$ m^2/s, $\gamma = 5$ m^{-1}, $T_{hot} = 100^o$ C, $T_\infty = 20^o$ C. Make the following plots (total of 3 plots)

- (2 pts) A plot of temperature T vs position x at times $t = 200, 1000, 2000,$ and 4000 using $N_x = 20$ and $dt = 10$ (4 total curves on the same axes).
- (2 pts) A plot of temperature T vs time t for $t \in [0, 200]$ at $x = L$ using $N_x = 100$ and time step values $dt = 100, 25, 10, 5, 1$ (5 total curves on the same axes).
- (2 pts) A plot of temperature T vs time t $t \in [0, 200]$ at $x = L$ using $dt = 1$ and $N_x = 20, 40, 100$ (3 total curves on the same axes).

Problem 2 (20 points)

Part 1 (12 pts, PL):

Consider the transient heat equation on a square domain with $\alpha = 1$:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{5}$$

The boundary conditions are shown in Figure 1. Note that:

- The Dirichlet conditions on each edge **include** the corner points.
- The Neumann condition on the right edge **excludes** the corner points.

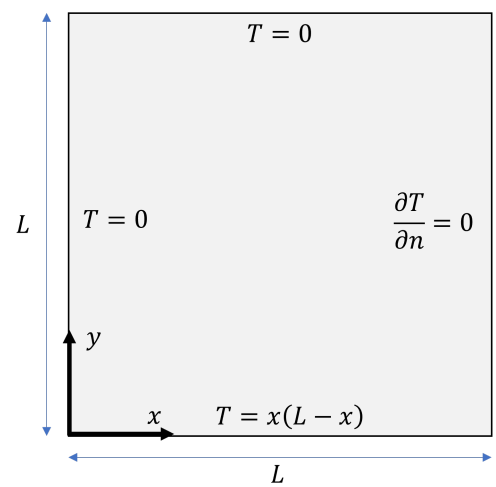

Figure 1: Domain and boundary conditions this problem.

The initial conditions for $T(x, y, t)$ are:

$$T(x, y, 0) = \begin{cases} x(L - x) & \text{if } y = 0, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Go to PrairieLearn to write a code that solves this problem numerically using:

- Semi-discretization with the second order central difference scheme for the spatial derivatives.
- A first-order one-sided difference approximation for the insulated boundary condition on the right edge.
- Forward Euler for the time integration scheme.

Part 2 (8 pts):

For $L = 10$, $N_x = N_y = 40$, and an ending simulation time of $t_{end} = 40$:

1. (2 pts) Compute the maximum allowable time step for stability when using the forward Euler method for this specific discretization.
2. (3 pts) On the same axes, plot $T(0.5L, 0.5L, t)$ and $T(0.25L, 0.75L, t)$ for $t \in [0, 40]$ using a time step equal to the maximum allowable for the forward Euler method for this problem.
3. (3 pts) Using the maximum allowable time step for the forward Euler method, make contour plots of the solution at $t = 2$, $t = 8$, and $t = 40$ (3 separate plots). Suggested plotting syntax is shown on PrairieLearn.

Problem 3: 4 credit-hour students only (20 points)

Consider the problem of hot fluid flowing in a channel with walls at $y = \pm 1$. The flow is in the x-direction. The incoming temperature is $T^0(y) = 1$. For $x > 0$, the wall temperature is 0. A schematic for the problem is below:

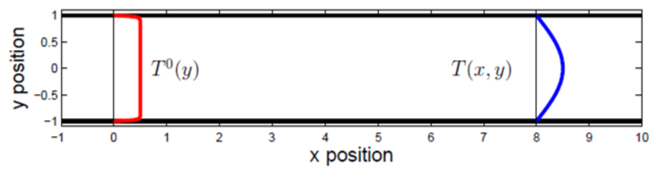

Figure 2: Schematic for the problem

The temperature for this problem is governed by the PDE

$$u(y)\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}, \quad T(x = 0, y) = T^0(y), \quad T(x, y \pm 1) = 0. \tag{7}$$

Here, the convecting velocity in the x-direction is $u(y) = 1 - y^2$, which is the velocity profile for laminar viscous flow.

In this problem, x is the time-like variable and y is the space variable. The problem can be solved numerically with semi-discretization in y and by applying a time integration scheme to x (you simply 'time-march' the x variable from 0 to L, where L represents a desired domain length). The notation for the numerical solution is $T_j^i = T(x_i, y_j)$, where $i = 0, 1, 2, ..., N_x$ and $j = 0, 1, 2, ..., N_y$. Grid spacing is uniform in each coordinate direction, but spacings Δx and Δy need not be equal.

In all schemes below, use the second order central difference scheme for the y discretization:

(a) (2 pts) Using the **backward Euler scheme** for the x discretization, write down the solution update scheme at each "time step" in the form of a system of equations, $\mathbf{A}\mathbf{T}^{i+1} = \mathbf{T}^i$, where $\mathbf{T}^i = [T_1^i, T_2^i, ..., T_{N_y-1}^i]$ is the solution for the non-boundary nodes at position x_i. What is the form of the matrix $\mathbf{A}$?

(b) (2 pts) Using the **BDF2 scheme** for the x discretization, write down the solution update scheme at each "time step" in the form of a system of equations, $\mathbf{B}\mathbf{T}^{i+1} = c_1\mathbf{T}^i + c_2\mathbf{T}^{i-1}$, where $\mathbf{T}^i = [T_1^i, T_2^i, ..., T_{N_y-1}^i]$ is the solution for the non-boundary nodes at position x_i. What is the form of the matrix $\mathbf{B}$, and what are the coefficients c_1 and c_2?

(c) (10 pts, PL) Follow instructions on PrairieLearn to implement a code to numerically solve Equation 7 using backward Euler for the first step and BDF2 for subsequent steps.

(d) (4 pts) Use your solution code to solve for $T(x, y)$ on the domain $[0, 100] \times [-1, 1]$ with $\alpha = 0.01$. Demonstrate convergence of the solution by plotting the numerical solutions for $T(10, y)$ and $T(x, 0)$ for several values of Δx and Δy (Fix Δx while varying Δy, and vice versa, to isolate the error influence of each parameter). Based on these plots, state the grid spacings Δx and Δy that you feel achieve convergence.

(e) (2 pts) Produce a color contour plot of the converged solution on $[0, 100] \times [-1, 1]$. Suggested plotting syntax is given on PrairieLearn.