Limits (Section 20)

Definition (20.1, slightly modified)

Suppose $S \subset \mathbb{R}$, $a \in S^-$, $f : S \to \mathbb{R}$, $L \in \mathbb{R} \cup \{\pm\infty\}$. Then $\lim_{x \to a^S} f = L$ (limit of f at a along S) if $\lim f(x_n) = L$ **for any** sequence $(x_n) \subset S$, with $\lim x_n = a$.

Such sequences (x_n) exist, due to $a \in S^-$.

Proposition (Connection between limits and continuity)

If $a \in S$, then $f : S \to \mathbb{R}$ is continuous at a iff $\lim_{x \to a^S} f = f(a)$.

Proof. f is continuous at a iff $f(x_n) \to f(a)$ for any sequences $(x_n) \subset S$ which converges to a. ■

Common set-ups for limits

"Usual" limit. I is an interval, a is interior to I, $S = I \backslash \{a\}$, $f : I \backslash \{a\} \to \mathbb{R}$. Instead of $\lim_{x \to a^S} f$, simply write $\lim_{x \to a} f$. $\lim_{x \to a} f = L$ if $\lim f(x_n) = L$ for any sequence $(x_n) \subset I \backslash \{a\}$, with $\lim x_n = a$.

Example. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x + 1) = 2$.

One-sided limit. $S = (a, b)$ $(b > a)$. Instead of $\lim_{x \to a^S} f$, write $\lim_{x \to a^+} f$ (right-hand limit). The left-hand limit $\lim_{x \to a^-} f$ is defined similarly.

Example. $\lim_{x \to 0^+} \frac{1}{x} = +\infty$. Indeed, if $x_n > 0$ $\forall n$, and $x_n \to 0$, then $\frac{1}{x_n} \to +\infty$.

Equivalent definition

Theorem (20.6, simplified)

Suppose $a \subset S^-$. *For* $f : S \to \mathbb{R}$ *and* $L \in \mathbb{R}$, *TFAE:*

1. $\lim_{x \to a^S} f = L$.
2. $\forall \varepsilon > 0\ \exists \delta > 0$ *s.t.* $|f(x) - L| < \varepsilon$ *whenever* $x \in (a - \delta, a + \delta) \cap S$.

Corollary

Suppose a *is interior to the interval* I. *For* $f : I \backslash \{a\} \to \mathbb{R}$ *and* $L \in \mathbb{R}$, *TFAE:*

1. $\lim_{x \to a} f = L$.
2. $\forall \varepsilon > 0\ \exists \delta > 0$ *s.t.* $|f(x) - L| < \varepsilon$ *whenever* $x \in \big((a - \delta, a + \delta) \cap I\big) \backslash \{a\}$.

If f *is defined at* a, *then it is continuous at* a *iff* $\lim_{x \to a} f = f(a)$.

Useful theorems about limits

Theorem (20.4 – sums, products, ratios)

Suppose $\lim_{x \to a^S} f_1 = L_1$, $\lim_{x \to a^S} f_2 = L_2$, $L_1, L_2 \in \mathbb{R}$. *Then*

- $\lim_{x \to a^S} \left(f_1 + f_2\right) = L_1 + L_2$.
- $\lim_{x \to a^S} f_1 f_2 = L_1 L_2$.
- *If* $L_2 \neq 0$, *then* $\lim_{x \to a^S} \frac{f_1}{f_2} = \frac{L_1}{L_2}$.

Theorem (Squeeze Theorem for functions - Homework)

Consider $f, g, h : S \to \mathbb{R}$, *so that* $f \leqslant g \leqslant h$. *If* $\lim_{x \to a^S} f = L = \lim_{x \to a^S} h$, *then* $\lim_{x \to a^S} g = L$.

More theorems about limits

Theorem (20.5 – compositions of functions)

Suppose $\lim_{x \to a^S} f = L \in \mathbb{R}$, *and* g *is defined on* $f(S) \cup L$, *continuous at* L. *Then* $\lim_{x \to a^S} g \circ f = g(L)$.

Proof. Suppose $(x_n) \subset S$, and $x_n \to a$. Prove that $g(f(x_n)) \to g(L)$. Fix $\varepsilon > 0$. Need to find $N \in \mathbb{N}$ s.t. $\big|g(f(x_n)) - g(L)\big| < \varepsilon \ \forall n \geqslant N$. Find $\delta > 0$ s.t. $|g(y) - g(L)| < \varepsilon$ if $|y - L| < \delta$. Find N s.t. $|f(x_n) - L| < \delta$ if $n \geqslant N$. This N works for us!

Theorem (20.10 – when are one sided limits the same)

$\lim_{x \to a} f = L$ *iff* $\lim_{x \to a^+} f = L = \lim_{x \to a^-} f$.

Differentiation (Section 28)

Definition (28.1)

Suppose I is an open interval, $a \in I$. $f : I \to \mathbb{R}$ is differentiable at a if the derivative $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, and is finite.

Example. $f(x) = \frac{1}{x}$. For $a \in \mathbb{R}\backslash\{0\}$, $f'(a) = \lim_{x\to a} \frac{1/x-1/a}{x-a}$
$= \lim_{x\to a} \frac{(a-x)/xa}{x-a} = -\lim_{x\to a} \frac{1}{xa} = -\frac{1}{a^2}$.

Example 2, p. 224. $g(x) = \sqrt{x}$. For $a > 0$,
$g'(a) = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$.

Example 3, p. 224. $h(x) = x^n$ ($n \in \mathbb{N}$). Recall that
$x^n - a^n = (x-a)\big(x^{n-1} + xa^{n-2} + \ldots + x^{n-2}a + a^{n-2}\big)$.
For $a \in \mathbb{R}$, $h'(a) = \lim_{x\to a} \frac{x^n-a^n}{x-a} =$
$\lim_{x\to a} \left(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1}\right) = na^{n-1}$.

Differentiability implies continuity

Definition (28.1)

Suppose I is an open interval, $a \in I$. $f : I \to \mathbb{R}$ is differentiable at a if the derivative $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists, and is finite.

Theorem (28.2)

If f is differentiable at a, then it is continuous at a.

Proof. For $x \neq a$, $f(x) - f(a) = (x-a)\frac{f(x)-f(a)}{x-a}$, hence
$\lim_{x \to a} \big(f(x) - f(a)\big) = \lim_{x \to a}(x-a) \cdot \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = 0 \cdot f'(a) = 0$.
Thus, $\lim_{x \to a} f(x) = f(a)$, so f is continuous at a. ∎

Continuity doesn't imply differentiability.

$f(x) = |x|$ is continuous at $a = 0$, but not differentiable.

$\frac{f(x)-f(0)}{x-0} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$, so $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ does not exist.

Another example of differentiation

Let $f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geqslant 0 \end{cases}$. Where is f differentiable? Find the derivative.

$a < 0$: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{0}{x-a} = 0$.

$a > 0$: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^2-a^2}{x-a} = 2a$.

What happens when $a = 0$? If $x \neq 0$, then $\frac{f(x)-f(0)}{x-0} = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$. So,

$\left|\frac{f(x)-f(0)}{x-0}\right| \leqslant |x|$.

By Squeeze Theorem, $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0$.

Conclusion: $f(x) = \begin{cases} 0 & x \leqslant 0 \\ 2x & x > 0 \end{cases}$.

Rules of differentiation: sum, product, etc.

Theorem (part of 28.3 – read the whole theorem in textbook!)

Suppose f and g are differentiable at a. Then fg is differentiable at a, with $\big[fg\big]'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof. Let $p = fg$, then

$p(x) - p(a) = f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a) =$
$f(x)\big(g(x) - g(a)\big) + \big(f(x) - f(a)\big)g(a).$
$\frac{p(x)-p(a)}{x-a} = f(x)\frac{g(x)-g(a)}{x-a} + g(a)\frac{f(x)-f(a)}{x-a}.$

$p'(a) = \lim_{x\to a} \frac{p(x)-p(a)}{x-a} = \lim_{x\to a} f(x) \lim_{x\to a} \frac{g(x)-g(a)}{x-a} +$
$g(a) \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f(a)g'(a) + g(a)f'(a).$ ■

Corollary. For $m \in \mathbb{N}$, $\big[x^m\big]' = mx^{m-1}$. Proof: induction.

Chain Rule

Theorem (28.4 – Chain Rule)

Suppose f is differentiable at a, g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, with $(g \circ f)'(a) = g'(f(a))f'(a) = (g' \circ f)(a) \cdot f'(a)$.

False proof. $(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a}$
$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}$
$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$.
$f(x) \to f(a)$ (differentiability $\Rightarrow$ continuity), hence
$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a))$.

This does not work! $f(x)$ might equal $f(a)$; can't divide by 0.

On the road to Chain Rule: Caratheodory Theorem

Theorem (Caratheodory – Exercise 28.16)

Suppose I is an interval, $f : I \to \mathbb{R}$. f is differentiable at $a \in I$ iff $\exists$ function $\phi : I \to \mathbb{R}$, continuous at a, s.t. $f(x) - f(a) = \phi(x)(x-a)$ $\forall x \in I$. Then $\phi(a) = f'(a)$.

Proof: existence of $\phi \Rightarrow$ differentiability.
$\phi(a) = \lim_{x\to a} \phi(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$. ■

Proof: differentiability $\Rightarrow$ existence of ϕ.
Define $\phi(a) = f'(a)$, $\phi(x) = \frac{f(x)-f(a)}{x-a}$ for $x \neq a$. $\lim_{x\to a} \phi(x)$
$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) = \phi(a)$, so ϕ is continuous at a. ■

Remark. We re-prove that, if f is differentiable at a, then it is continuous at a (as a product of two continuous functions).