MATH 447: Real Variables - Homework 10

Jerich Lee

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Problem 1 (26.6)**.** Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.

(a) Prove $s' = c$ and $c' = -s$.

(b) Prove $(s^2 + c^2)' = 0$.

(c) Prove $s^2 + c^2 = 1$.

Actually, $s(x) = \sin x$ and $c(x) = \cos x$, but you do **not** need these facts.

Solution 1. (a) (a) *Proof.* For $x \in \mathbb{R}$:

$$s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \tag{1}$$

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \tag{2}$$

$$\lim_{t\to x} \phi(t) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} \tag{3}$$

$$s(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1)}! \tag{4}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{5}$$

$$c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2x)}}{(2n)!} \tag{6}$$

$$s'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} \tag{7}$$

$$= \frac{(-1)^n x^{2n}}{(2n)!} \tag{8}$$

$$= c(x) \tag{9}$$

$\square$

(b) *Proof.*

$$c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{10}$$

$$c'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} \tag{11}$$

$$= \frac{(-1)^n x^{2n-1}}{(2n-1)!} \tag{12}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} \tag{13}$$

$$= -s(x) \tag{14}$$

$\square$

(b) *Proof.*

$$\left(s^2 + c^2\right)' = 0 \tag{15}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \tag{16}$$

$$= \sum_{k=0}^{\infty} \left(\frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^k x^{2k}}{(2k)!} \right) \tag{17}$$

$$= 0 \tag{18}$$

$\square$

(c) todo

Problem 2 (33.3)**.** A function f on $[a,b]$ is called a *step function* if there exists a partition

$$P = \{a = u_0 < u_1 < \cdots < u_m = b\} \tag{19}$$

of $[a,b]$ —not $P = \{a = u_0 < u_1 < \cdots < c_m = b\}$, as stated in the textbook— such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for x in (u_{j-1}, u_j).

(a) Show that a step function f is integrable and evaluate $\int_a^b f$.

Solution 2. *Proof.* If f is constant on every interval, then $M_i = m_i$. Then, f is monotone and bounded on (u_{j-1}, u_j), (in fact it is constant from (u_{j-1}, u_j)), so therefore it is uniformly continuous, which implies that there exists a continuous extension to the closed set $[u_{j-1}, u_j]$. Invoking Theorem (3.38) from Ross (Piecewise Monotone), we show that $f \in \mathcal{R}$. $\square$

Problem 3 (33.7)**.** Let f be a bounded function on $[a, b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

(a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$. *hint:* $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$.

(b) Show that if f is integrable on $[a, b]$, then f^2 also is integrable on $[a, b]$.

Solution 3. *Proof.* f is a bounded function on $[a, b]$, so there exists $B > 0$ *s.t.* $|f(x)| \leq B$ for all $x \in [a, b]$.

$$u(f^2, p) = \sum_{i=1}^{\infty} M_i^2 \Delta x_i \tag{20}$$

$$= M_i^2 = \sup f(x)^2 \tag{21}$$

$$= \sum_{i=1}^{n} \left(M_i^2 - m_i^2\right) \Delta x_i \tag{22}$$

$$= \sum_{i=1}^{n} \left(M_{i+m_i}\right)\left(M_i - m_i\right) \Delta x_i \tag{23}$$

$$\leq \sum_{i=1}^{n} 2B \left(M_i - m_i\right) \Delta x_i \tag{24}$$

$$= 2B \sum_{i=1}^{\infty} \left(M_i - m_i\right) \Delta x_i \tag{25}$$

$$= 2B \left(U(f, p) - L(f, p)\right) \tag{26}$$

□

Problem 4 (34.2)**.** Calculate

(a) $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt$.

Solution 4. *Proof.*

$$\int_3^{3+h} e^{t^2} \, \mathrm{d}t \tag{27}$$

$$= \lim_{h\to 0} \frac{F(3+h) - F(3)}{h} \tag{28}$$

$$\tag{29}$$

By FTC I,

$$F'(3) = e^9 \tag{30}$$

□

Problem 5 (34.5)**.** Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)\,dt \quad \text{for } x \in \mathbb{R}.$$

Show F is differentiable on $\mathbb{R}$ and compute F'.

Solution 5. *Proof.*

$$F(x) = \int_x^0 f(t)\,\mathrm{d}t + \int_0^{x+1} f(t)\,\mathrm{d}t \tag{31}$$

$$= -\int_0^{x-1} f(t)\,\mathrm{d}t + \int_0^{x+1} f(t)\,\mathrm{d}t \tag{32}$$

$$\tag{33}$$

By FTC II, we get:

$$F'(x_0) = f(x_0 + 1) - f(x_0 - 1) \tag{34}$$

□

Problem 6. A. [Bonus problem] Suppose f is a continuous non-negative function on $[a, b]$, with

$$M = \max_{x \in [a,b]} f(x). \tag{35}$$

For $n \in \mathbb{N}$, let

$$M_n = \left(\int_a^b f^n\,dt \right)^{1/n}. \tag{36}$$

Prove that $\lim M_n = M$.

Solution 6. *Proof.*

$$\left(\int_a^b f^n \right)^{\frac{1}{n}} \leq ((b-a)(M^n))^{\frac{1}{n}} \tag{37}$$

$$= \lim_{n \to \infty} \underbrace{(b-a)^{\frac{1}{n}}}_{1} M = M \tag{38}$$

□