Jan 21, 2020

# Lecture 1

## TAM 445 : Continuum Mechanics

### What is mechanics ?

Study of motion of "bodies" when subjected to external loads.

1) Continuum mechanics is a branch of mechanics where bodies are modeled as continuous masses as opposed to discrete particles.

Mechanics

| Classical | Quantum |
|---|---|
| – Newtonian mechanics | – Describes nature at the smallest scale. |
| – Lagrangian " | |
| – Hamiltonian " | |

2) Continuum mechanics involves the application of the principles of classical mechanics to material bodies approximated as continuous media.

But why do we need to model bodies as continua ?

- tracking particles is computationally very expensive!

## 3) Language of continuum mechanics

While tracking a large ($\sim 10^{23}$) number of particles is compitionally very expensive, the language to describe them is quite simple — one needs to specify the positions, velocities and external forces on the atoms / particles.

But how do we describe the evolution of a continuous blob of matter, and the forces on this blob?

- space and time
- vectors and tensors

We need to have a consistent description of the above quantities irrespective of who is describing them.

- A continuum body
- Deformation, a.k.a kinematics
- Forces, stress tensor
- Balance laws: Conservation of mass,
  Conservation of momentum.
- Laws of thermodynamics

- Zeroth law (thermal equilibrium and concept of temperature)
- First law (conservation of energy)
- second law (Entropy).

- Constitutive laws.

- Partial differential equations describing fluids and solids. Examples from solid mechanics, fluid mechanics and heat transfer

## 1. Space, time and frames of reference

A unified theory of mechanics was presented for the first time in 1687 by Isaac Newton in his publication Philosophiae Naturalis Principia Mathematica.

Newton's laws directly translated from Latin:

I) Every body remains in a state, resting or moving uniformly, except insofar as forces on it compel it to change its state

II) The rate of change of momentum $\propto$ to the motive force impressed, and is made in the direction of the straight line in which the force is impressed.

III) To every action there is always opposed an equal reaction.

$$F^{ext} = \frac{d}{dt}(m v) \quad \text{— Newton's second law.}$$

The above laws are meaningless if you don't really know what space and time are.

" .... resting or moving ...." relative to what?

" .... rate of change ----" relative to whom?

Newton introduced absolute space and time in the Scholium (a chapter with explanatory comments) to the Principia. But then Newton recognized that it is not possible to work directly with absolute space and time as they cannot be detected by mere mortals like us. So he introduced relative space and relative time.

A frame of reference is a rigid physical object (such as earth, laboratory or fixed stars) relative to which space is described, and a clock to measure time.

Recall that the three laws are described with respect to the absolute space and time. But then Newton realized that the laws don't change in all frames which are moving uniformly relative to the absolute frame. Such frames are called <u>inertial frames of reference</u>.

Some commentary on inertial frames and relativistic space time ......

## 2. Scalars Vectors and Tensors

Before we start, let us first learn indicial notation which will be your bread and butter for the rest of the semester.

2.1 <u>Indicial notation</u> is a scheme whereby a list of real numbers is represented by a <u>single symbol</u> with <u>indices</u>.

<u>Eg</u>: $a_i$ denotes $a_1, a_2, a_3$.

Typically the range of the index is determined by the dimensionality of space. In this course, we always deal with three-dimensional spaces, so indices range from one to three.

Similarly, $a_{ij}$ denotes $3^2 = 9$ real numbers

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ , represented as a matrix.

## Summation and dummy indices:

Consider the following sum

$$S = a_1x_1 + a_2x_2 + a_3x_3 \quad \text{—} \quad (1)$$

$$= \sum_{i=1}^{3} a_i x_i \quad \text{—} \quad (2)$$

Clearly (2) is a shorter version of (1). Indicial notation allows us to arrive at an even more compact version. Notice that the index $i$ is dummy as any letter could have been chosen, for example $\sum_{m=1}^{3} a_m x_m$.

$$S = a_i x_i \underbrace{:=}_{\text{definition}} \sum_{i=1}^{3} a_i x_i$$

Whenever an index appears $\boxed{\text{twice}}$ in a $\boxed{\text{product}}$, it is a dummy index over which a sum is applied.

This is called Einstein's summation convention.

Examples:

$a_i x_i$ means $a_1 x_1 + a_2 x_2 + a_3 x_3$

$\sigma_{ii}$ means $\sigma_{11} + \sigma_{22} + \sigma_{33}$

CAUTION: The dummy index (on which a summation is enforced) appears only twice in a product. So

$a_i b_i x_i \longrightarrow$ ~~DOES~~ DOES NOT MAKE SENSE!

If the intent was to write $\sum_{i=1}^{3} a_i b_i x_i$, then indicial notation cannot help here.

$A_{ij} y_j$ means
$$A_{11} y_1 + A_{12} y_2 + A_{13} y_3,$$
$$A_{21} y_1 + A_{22} y_2 + A_{23} y_3,$$
$$A_{31} y_1 + A_{32} y_2 + A_{33} y_3.$$

i.e $A_{ij} y_j$ represents three numbers. In order to see how many numbers an expression represents, count the number of indices that occur only once, and raise it to the power of 3.

$$A_{ij} x_i y_j = A_{11} x_1 y_1 + A_{12} x_1 y_2 + A_{13} x_1 y_3 + A_{21} x_2 y_1 + A_{22} x_2 y_2 + A_{23} x_2 y_3 + A_{31} x_3 y_1 + A_{32} x_3 y_2 + A_{33} x_3 y_3 .$$

Free indices : An index that appears only once in a product term of an expression/equation is referred to as a free index.

In the expression $A_{ij} y_j$ or in the equation $A_{ij} y_j = x_i$, $i$ is a free index. The former represents three numbers while the latter represents three equations.

CAUTION : In an equation, free indices on the l.h.s should be the same as the free indices on the r.h.s. Similarly, free indices of each expression in a sum should be the same. For example,

$A_{ij} x_j = b_k$ or $a_i + b_j$ make no sense.

$A_{ij} x_j = c$ is fine as $c$ is just a constant.

$D_{ijk} x_k = A_{ij}$ represents $3^2 = 9$ equations

Expressions with two free indices can be represented as a square matrix, while those with one free index can be represented as a column or a row matrix.

## Kronecker delta

( Leopold Kronecker (1823-1891))

$$\delta_{ij} := \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

$\delta_{ij}$ in matrix form is just the identity

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\delta_{ij}$ usually appears in the context of differentiating an indexed symbol with respect of its "components".

$\frac{\partial x_i}{\partial x_j}$ has two free indices $\Rightarrow$ it represents $3^2$ numbers

$$\begin{bmatrix} \partial x_1/\partial x_1 & \partial x_1/\partial x_2 & \partial x_1/\partial x_3 \\ \partial x_2/\partial x_1 & \partial x_2/\partial x_2 & \partial x_2/\partial x_3 \\ \partial x_3/\partial x_1 & \partial x_3/\partial x_2 & \partial x_3/\partial x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

**Exercise**: $a_i \delta_{ij} = a_j$

**Solution**: $a_i \delta_{ij} = a_1 \delta_{1j} + a_2 \delta_{2j} + a_3 \delta_{3j}$

(sum over the dummy index $i$)

$$= \begin{cases} a_1 & \text{if } j=1 \\ a_2 & \text{if } j=2 \\ a_3 & \text{if } j=3 \end{cases}$$

$$= a_j$$

**Exercise**: $\delta_{ii} = 3$, $\delta_{ij} A_{ij} = A_{11} + A_{22} + A_{33}$

## Permutation on the Levi-Civita symbol

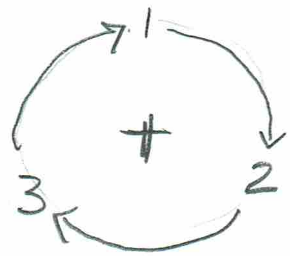
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ form an even permutation of } 1,2,3 \\ -1 & \text{" } i,j,k \text{ " " odd " " "} \\ 0 & \text{" " do not form a " " "} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$$

$$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{113} = \cdots = \varepsilon_{333} = 0$$

A convenient way to remember the definition of $\varepsilon$ is




<u>Useful identities</u>:

1. $$\varepsilon_{ijk}\,\delta_{ij} = \varepsilon_{iik} = 0 \qquad \text{— (3)}$$

$$\varepsilon_{ijk}\,\varepsilon_{mjk} = 2\,\delta_{im} \qquad \text{— (4)}$$

$$\varepsilon_{ijk}\,\varepsilon_{ijk} = 6 \qquad \text{— (5)}$$

2. $$\varepsilon_{mnp}\,(\det A) = \varepsilon_{ijk}\,A_{im}\,A_{jn}\,A_{kp} \qquad \text{— (6)}$$

$$= \varepsilon_{ijk}\,A_{mi}\,A_{nj}\,A_{pk} \qquad \text{— (7)}$$

(6) and (7) $\Rightarrow$ $\det A = \det A^T$.

Multiplying (6) with $\frac{1}{6}\varepsilon_{mnp}$, and using (5)

$$\det A = \frac{1}{6}\varepsilon_{mnp}\,\varepsilon_{ijk}\,A_{im}\,A_{jn}\,A_{kp}$$

$$= \varepsilon_{ijk}\,A_{i1}\,A_{j2}\,A_{k3} \qquad \text{— (8)}$$