02·20·2020

Lecture 10

The theorem we discussed in the last class is sometimes referred to as the spectral theorem. As a result of the theorem, we can say every symmetric tensor has the following representation:

$$\boxed{S = \sum_{i=1}^{3} \lambda_i \, \hat{\underset{\sim}{u}}_i \otimes \hat{\underset{\sim}{u}}_i} \qquad \text{—— (1)}$$

<u>spectral decomposition of S.</u>

where $\hat{\underset{\sim}{u}}_i$'s denote eigenvectors with norm 1.

The set of eigenvalues is called the <u>spectrum</u> of S.

<u>Cayley-Hamilton theorem</u> Every second-order tensor satisfies its own characteristic equation:

$$T^3 - I_1(T)\,T^2 + I_2(T)\,T - I_3(T)\,I = 0 \qquad T \in \mathrm{Lin} \qquad \text{—— (2)}$$

<u>Comment</u>: If $T \in \mathrm{Sym}$, it is easy to prove the above statement. Just substitute (1) into (2)! But the C-H theorem holds for any $T \in \mathrm{Lin}$! We will not prove the general result here.

Arthur Cayley – British mathematician

William Hamilton – Irish "

Positive definite second-order tensors & Polar decomposition

Before we start...

Geometric significance of the spectral decomposition:

Recall our geometric interpretation of a tensor action as "expansion/contraction and shear". The spectral decomposition (written for only $S \in Sym$) says that there are three-mutually orthogonal directions along which vectors are either shrunk ($\lambda < 1$), expanded ($\lambda > 1$) or reflected ($\lambda < 0$). This is only for $S \in Sym$. But what about for an arbitrary $T \in Lin$?

Definition: A tensor $T \in Lin$ is positive definite if

$$T\underset{\sim}{v} \cdot \underset{\sim}{v} > 0 \quad \forall \text{ non-zero } \underset{\sim}{v} \in V$$

Q) Can a $W \in Skw$ by positive-definite?

No! - You can always find a $\underset{\sim}{v} \in V$ such that $W\underset{\sim}{v} \cdot \underset{\sim}{v} = 0$.

$\therefore$ For any $T \in Lin$, since $T = \underbrace{S}_{\frac{T+T^T}{2}} + W$ (from Lecture 6, pg 7)

$$T\underset{\sim}{v} \cdot \underset{\sim}{v} = S\underset{\sim}{v} \cdot \underset{\sim}{v}$$

<u>Notation</u>: · Psym : positive-definite symmetric tensors.

<u>Theorem</u>: Let $S \in$ Sym. Then $S \in$ Psym $\Leftrightarrow$ its eigenvalues are strictly positive.

<u>Proof</u>: ($\Rightarrow$) Assume $S \in$ Psym. This implies $S\underset{\sim}{v} \cdot \underset{\sim}{v} > 0 \;\forall\; \underset{\sim}{0} \neq \underset{\sim}{v} \in V$

From the spectral theorem, we know S has three real eigenvalues and three mutually orthogonal eigenvectors, say $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3$:

$$S\underset{\sim}{u}_i = \lambda_i \underset{\sim}{v}_i \qquad \text{(no sum)} \qquad i = 1,2,3.$$

By assumption $S\underset{\sim}{u}_i \cdot \underset{\sim}{v}_i > 0$ (no sum), which implies

$$\lambda_i u_i \cdot u_i > 0. \quad \text{(no sum)} \qquad i = 1,2,3.$$

Since $u_i \cdot v_i > 0$ (property of dot product), $\lambda_i > 0$

($\Leftarrow$) Assume all eigenvalues are positive. By the spectral decomposition theorem

$$S = \sum_{i=1}^{3} \lambda_i \hat{\underset{\sim}{v}}_i \otimes \hat{\underset{\sim}{u}}_i \quad , \text{ where } \hat{\underset{\sim}{u}}_i = \underset{\sim}{v}_i / \|\underset{\sim}{u}_i\| \quad \text{(no sum)}$$

For an arbitrary non-zero $\underset{\sim}{v} \in V$

$$S\underset{\sim}{v} = \sum_{i=1}^{3} \lambda_i (\hat{\underset{\sim}{u}}_i \cdot \underset{\sim}{v}) \hat{\underset{\sim}{u}}_i .$$

Therefore $$S\underset{\sim}{v} \cdot \underset{\sim}{v} = \sum_{i=1}^{3} \lambda_i (\hat{u}_i \cdot \underset{\sim}{v})(\hat{\underset{\sim}{v}}_i \cdot v) > 0.$$

$\square$

Psym $\xleftrightarrow{\text{Analogy}}$ Positive numbers,

in the sense that tensors in Psym have a square root.

Square-root theorem: Let $S \in$ Psym. Then there exists a unique $U \in$ Psym such that

$$U^2 = S.$$

We write $\sqrt{S}$ for U.

Proof: (Existence) By the spectral decomposition theorem, we have

orthonormal eigenvectors.

$$S = \sum_{i=1}^{3} \lambda_i \, \hat{\underset{\sim}{u}}_i \otimes \hat{\underset{\sim}{u}}_i \quad \text{—— (3)}$$

eigenvalues

From the previous theorem, $\lambda_i > 0$ since $S \in$ Psym. Let

$$U := \sum_{i=1}^{3} \sqrt{\lambda_i} \, \hat{\underset{\sim}{u}}_i \otimes \hat{\underset{\sim}{u}}_i \quad \text{— (4)}$$

Clearly, $U^2 = S$ (check it yourself).

(Uniqueness): Assume there exists another $V \in$ Psym such that

$$V^2 = U^2 = S.$$

Since $(\underset{\sim}{S}-\lambda_i I)\hat{\underset{\sim}{u}}_i = \underset{\sim}{0}$ (no sum), we have

$$(U^2-\lambda_i I)\,\hat{u}_i = \underset{\sim}{0} \qquad \text{(no sum)} \qquad (i=1,2,3)$$

$$\Rightarrow \quad (U-\lambda_i I)\,\underbrace{(U+\lambda_i I)\,\hat{\underset{\sim}{u}}_i}_{=:\,\underset{\sim}{v}_i} = \underset{\sim}{0} \qquad (i=1,2,3)$$

$$\Rightarrow \quad (U-\lambda_i I)\,\underset{\sim}{v}_i = 0 \qquad (i=1,2,3)$$

$\Rightarrow$ $\underset{\sim}{v}_i$ is an eigenvector of U. $(i=1,2,3)$

Similarly, $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ are eigenvectors of V. Since $U \in \text{Psym}$, $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ form a basis and

$$U\underset{\sim}{v}_i = V\underset{\sim}{v}_i \qquad (i=1,2,3)$$

$$\Rightarrow \quad U = V. \qquad \square$$

<u>Polar decomposition theorem</u>: Let $F \in \underset{\frown}{\text{InV}}$ (set of all invertible tensors). Then $\exists$ unique $U, V \in \text{Psym}$, and a unique $R \in \text{Orth}$ such that

Right polar decomposition of F $\quad \boxed{F = RU = VR} \quad$ —(5) $\quad$ left polar decomposition of F

Moreover, $\det R = 1$ or -1 according as $\det F > 0$ or < 0; and

$$U = \sqrt{F^T F}, \quad V = \sqrt{F F^T}. \qquad \text{— (5)}$$

Proof: We first prove that $F^T F$ and $F F^T$ belong to Psym for (6) to be meaningful: For any non-zero $\underset{\sim}{v} \in V$

$$F^T F \underset{\sim}{v} \cdot \underset{\sim}{v} = F \underset{\sim}{v} \cdot F \underset{\sim}{v}. \qquad \text{— (7)}$$

Now, if $F\underset{\sim}{v} = \underset{\sim}{0}$ for a non-zero $\underset{\sim}{v}$, then this is equivalent to saying the following equation

$$\begin{bmatrix} & & \\ & F_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution. In the last class we have seen that this happens $\Longleftrightarrow \det F = 0$. Since $F \in \text{Inv}$, it follows that $F\underset{\sim}{v} \neq \underset{\sim}{0}$, and by the property of dot product $F\underset{\sim}{v} \cdot F\underset{\sim}{v} > 0 \implies F^T F$ is positive definite. Moreover since $(F^T F)^T = F^T F$, we have $F^T F \in$ Psym. Similarly, it can be shown that $F F^T \in$ Psym. Therefore, by the square root theorem $\sqrt{F^T F}$ and $\sqrt{F F^T}$ exist.

Uniqueness of U, V and R: Assume $U, V \in$ Psym are given by (6) and $R \in$ Orth exists such that (5) holds.

Assume $\exists$ $\tilde{U}, \tilde{V} \in Psym$ and $\tilde{R} \in Orth$ such that

$$F = \tilde{R}\tilde{U} = \tilde{V}\tilde{R}.$$

This implies

$$
\begin{aligned}
F^T F &= (\tilde{R}\tilde{U})^T (\tilde{R}\tilde{U}) \\
&= \tilde{U}^T \underbrace{\tilde{R}^T \tilde{R}}_{=I} \tilde{U} \\
&= \tilde{U}^2 \quad \Rightarrow \quad \tilde{U} = \sqrt{F^T F}
\end{aligned}
$$

By the uniqueness of the square root theorem $\tilde{U} = U$. Similarly, we can show that (fill up the gap here!) $\tilde{V} = V$. Since

$$R = FU^{-1} \qquad \left(U^{-1} \text{ exists since } \det U > 0 \right.$$
$$\left. U \text{ being in } Psym \right)$$

$$\tilde{R} = F\tilde{U}^{-1},$$

and $U = \tilde{U}$, it follows that $R = \tilde{R}$.

<u>Existence</u>: Assume U, is given by (6a). Then $U, V \in Psym$ by construction. Define

$$R := FU^{-1}.$$

Lets now check if $R \in Orth$:

$$R^T R = (FU^{-1})^T (FU^{-1})$$
$$= U^{-T} F^T F U^{-1} \qquad (\because U = U^T)$$
$$= U^{-T} U U U^{-1}$$
$$= I$$

$\Rightarrow R \in \text{Orth}$.

We will now construct V such that $F = VR$.

Let $V = RUR^T \in \text{Psym}$ (check! - exercise)

$$VR = (RUR^T)\, R$$
$$= RU\underbrace{(R^T R)}_{I}$$
$$= RU$$
$$= F.$$

$\square$

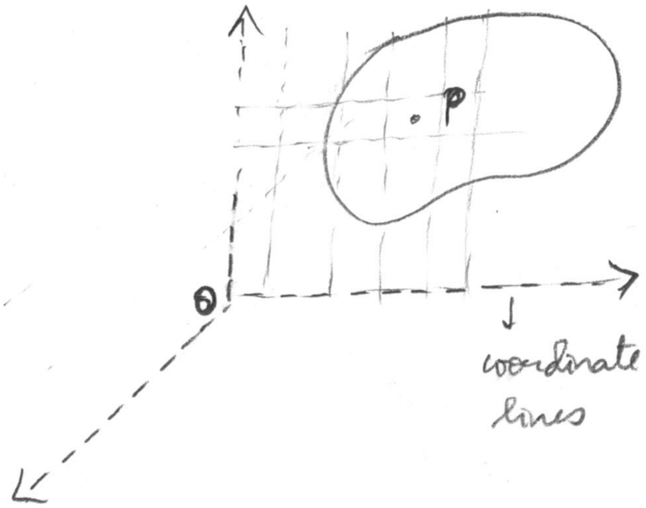
Tensor fields

From lecture 4, we have been living in Euclidean vector spaces. Let us now revisit Euclidean point space ($\mathcal{E}$):

→ coordinate system
→ cartesian coordinate system
→ "translation operator" etc.

} Recall

Typically, a continuum body is modeled as an open subset of the Euclidean point space. ($\mathcal{B} \subset \mathcal{E}$).



The point p in $\mathcal{E}$ has coordinates (x_1, x_2, x_3). Moreover, the point p may be identified with a vector $\underset{\sim}{x} := p - O$.

A scalar-valued function on the body can be represented as $f(p)$ or $\hat{f}(\underset{\sim}{x})$ or $\tilde{f}(x_1, x_2, x_3)$. These functions are representing the same property, but are expressed as functions of <u>points</u>, <u>vectors</u> or <u>ordered triple of reals</u>.

We will not make a fuss about using different notation such as f, $\hat{f}$ or $\tilde{f}$. Instead, we will use the same symbol $f(p)$, $f(\underset{\sim}{x})$ or $f(x_1, x_2, x_3)$ and the domain of definition is to be understood from the context.