Lecture 7

Inner product of second-order tensors

Definition: The inner-product of two tensors S and T is defined as

$$S \cdot T = tr(ST^T) \quad — (1)$$

(1) is basis-independent.

How does the inner product look like in indicial notation?

$$S \cdot T = tr(ST^T) = (ST^T)_{ii}$$
$$= S_{ik} T^T_{ki}$$
$$= S_{ik} T_{ik}$$

We called $S \cdot T$ defined in (1) as the inner-product, but we haven't shown that it satisfies the properties of an inner-product:

Recall from pg 10 of lecture 2.

a) Linearity: $\langle \alpha \underset{\sim}{u} + \beta \underset{\sim}{v}, \underset{\sim}{w} \rangle = \alpha \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \beta \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

b) Commutativity: $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle$

c) $\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle > 0 \quad \forall \underset{\sim}{u} \in V$ except $\underset{\sim}{u} = \underset{\sim}{0}$.

Problem: Show that the product defined in 1 is a inner-product on the real vector space of second-order tensors.

Norm of a tensor: $|T| := \sqrt{T \cdot T}$

Inverse of a second-order tensor

A tensor S is invertible if there exists a tensor denoted by S^{-1} and referred to as the inverse of S, such that

$$SS^{-1} = S^{-1}S = I.$$

If S^{-1} exists, how does it look in terms of the components of S?

Since we showed in last class that the components of the tensor ST are obtained by matrix-matrix multiplication of $[S][T]$, it follows that

$[S^{-1}] = [S]^{-1} \Rightarrow$ From eq 5 of lecture 2 we have

$$S^{-1}_{ij} = \frac{1}{2(\det S)} \varepsilon_{pqi} \varepsilon_{mnj} S_{mp} S_{nq} \quad — 2$$

We showed in lecture 2 (pg 2) that

$$[S^{-1}] \text{ exists} \iff \det[S] \neq 0 \quad \text{— (3)}$$

From (2) and (3) it follows that

$$S^{-1} \text{ exists} \iff \det S \neq 0.$$

Properties of inverse: Let $T, S \in Lin$ such that $\det T \neq 0$, $\det S \neq 0$

(i) $T\underset{\sim}{u} = \underset{\sim}{v} \iff \underset{\sim}{u} = T^{-1}\underset{\sim}{v}$

(ii) $\det(T^{-1}) = (\det T)^{-1}$

(iii) $(T^{-1})^T = (T^T)^{-1} =: T^{-T}$

(iv) $(ST)^{-1} = T^{-1}S^{-1}$

$$Lin^+ := \{T \in Lin : \det T > 0\}$$

$$Inv := \{T \in Lin : \det T \neq 0\}.$$

Orthogonal tensors.

We have seen what orthogonal matrices are:

$$\Lambda\Lambda^T = \text{identity} \quad \text{— (4)}$$

$\hookrightarrow$ a matrix (not a tensor).

We can extend the above definition to tensors.

<u>Definition</u>: A tensor Q is orthogonal if

$$QQ^T = Q^TQ = I. \quad - (5)$$

$$\text{Orth} := \{ T \in \text{Lin} : T \text{ is orthogonal} \}$$

$$\text{Orth}^+ := \{ T \in \text{Orth} : \det T = +1 \}$$

↓

proper orthogonal tensors.

Why do we call matrices / tensors that satisfy (4)/(5) as "orthogonal"?

$$Q_{ik} Q_{jk} = \delta_{ij}$$

⇒ The rows if viewed as vectors are mutually orthogonal.

<u>Caution</u>: All orthogonal tensor have determinant ± 1. But not all tensors whose determinant is ± 1 are orthogonal!

What does (5) actually mean?

Theorem: Let $Q \in Lin$. Then the following statements are equivalent.

(i) $Q \in Orth$ i.e. $QQ^T = Q^TQ = I$

(ii) Q preserves inner product: $(Q\underset{\sim}{u})\cdot(Q\underset{\sim}{v}) = \underset{\sim}{u}\cdot\underset{\sim}{v} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in V$

(iii) Q preserves magnitudes: $\|Q\underset{\sim}{u}\| = \|\underset{\sim}{u}\| \quad \forall\, \underset{\sim}{u} \in V$

(iv) Q preserves distances: $\|Q\underset{\sim}{u} - Q\underset{\sim}{v}\| = \|\underset{\sim}{u} - \underset{\sim}{v}\| \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in V$

Proof:

$$\begin{tikzcd}
 & (i) \arrow[d, Leftrightarrow] & \\
 & (ii) \arrow[dr, Rightarrow] & \\
(iv) \arrow[ur, Rightarrow] & & (iii) \arrow[ll, Rightarrow]
\end{tikzcd}$$

(i) $\Rightarrow$ (ii) Assume $QQ^T = Q^TQ = I$. Let $\underset{\sim}{u}, \underset{\sim}{v} \in V$.

$$Q\underset{\sim}{u}\cdot Q\underset{\sim}{v} = \underset{\sim}{u}\cdot Q^T(Q\underset{\sim}{v}) \quad \text{— Definition of transpose (lecture 5)}$$

$$= \underset{\sim}{u}\cdot (Q^TQ)\underset{\sim}{v} \quad \text{— Definition of product of tensors (lecture 6, eq 3)}$$

$$= \underset{\sim}{u}\cdot I\underset{\sim}{v} \quad \text{— by assumption}$$

$$= \underset{\sim}{u}\cdot\underset{\sim}{v}$$

$\boxed{(ii) \Rightarrow (i)}$ Assume $(Q\underset{\sim}{u})\cdot(Q\underset{\sim}{v}) = \underset{\sim}{u}\cdot\underset{\sim}{v} \quad \forall\, \underset{\sim}{u},\underset{\sim}{v} \in V$.

$$\Rightarrow \underset{\sim}{u}\cdot Q^TQ\underset{\sim}{v} = \underset{\sim}{u}\cdot\underset{\sim}{v} \quad \forall\, \underset{\sim}{u},\underset{\sim}{v}\in V. \quad \text{(Defn. of transpose)}$$

$$\Rightarrow \underset{\sim}{u}\cdot(Q^TQ - I)\underset{\sim}{v} = 0 \quad \forall\, \underset{\sim}{u},\underset{\sim}{v}\in V \quad \text{(reordering)}$$

$$\Rightarrow (Q^TQ - I)\underset{\sim}{v} = \underset{\sim}{0} \quad \forall\, \underset{\sim}{v}\in V$$

$$\Rightarrow Q^TQ - I = \mathbb{O}$$

We will not prove $QQ^T = I$ as the proof is similar to what we showed for matrices - lecture 2, pg 3.

$\underline{(ii) \Rightarrow (iii)}$ Assume $Q\underset{\sim}{u}\cdot Q\underset{\sim}{v} = \underset{\sim}{u}\cdot\underset{\sim}{v} \quad \forall\, \underset{\sim}{u},\underset{\sim}{v}\in V$.

Taking $\underset{\sim}{v} = \underset{\sim}{u}$, we have

$$Q\underset{\sim}{u}\cdot Q\underset{\sim}{u} = \underset{\sim}{u}\cdot\underset{\sim}{u}$$

$$\Rightarrow \|Q\underset{\sim}{u}\|^2 = \|\underset{\sim}{u}\|^2 \quad \text{(Defn. of norm)}$$

$$\Rightarrow \|Q\underset{\sim}{u}\| = \|\underset{\sim}{u}\| \quad (\text{Since norm is } \geq 0)$$

$\underline{(iii) \Rightarrow (iv)}$ Assume $\|Q\underset{\sim}{u}\| = \|\underset{\sim}{u}\| \quad \forall\, \underset{\sim}{u}\in V$.

Consider $\|Q\underset{\sim}{u} - Q\underset{\sim}{v}\|^2$:

$$(Qu - Qv)\cdot(Qu - Qv) = Q(u-v)\cdot Q(u-v) \quad (\because Q \in \mathrm{Lin})$$

$$= \|Q(u-v)\|^2$$

$$= \|u-v\|^2$$

(iv) $\Rightarrow$ (ii) Assume $\|Qu - Qv\| = \|u - v\| \; \forall\, u, v \in V$

$$\|Qu - Qv\|^2 = (Qu - Qv)\cdot(Qu - Qv)$$

$$= Qu\cdot Qu + Qv\cdot Qv - 2\,Qu\cdot Qv$$

$$= \|Qu\|^2 + \|Qv\|^2 - 2\,Qu\cdot Qv$$

$$= \|u\|^2 + \|v\|^2 - 2\,Qu\cdot Qv \quad (\text{By assumption}) \quad — (6)$$

Similarly

$$\|u - v\|^2 = (u-v)\cdot(u-v)$$

$$= \|u\|^2 + \|v\|^2 - 2\cdot u\cdot v \quad — (7)$$

From (6), (7), and the assumption, we get

$$Qu\cdot Qv = u\cdot v.$$

<u>Theorem</u>: Orth and $Orth^+$ are closed under tensor multiplication i.e

If $T, S \in Orth$, then $TS \in Orth$

" $T, S \in Orth^+$, then $TS \in Orth^+$

Cross product