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Lecture 11

Last class we have seen that a scalar, vector or a tensor field can be represented as functions of points, vectors or ordered triple of reals.

Setting : Euclidean point space $\mathcal{E}$, with a cartesian coordinate system. Let $\mathcal{D} \subset \mathcal{E}$ be an $\underbrace{\text{open}}$ subset of $\mathcal{E}$

$\subset$ means subset

$\boxed{\text{means at every point } p \in \mathcal{D}, \ \exists \text{ an arbitrarily small ball that lies entirely in } \mathcal{D}.}$

Definitions : Given a scalar field φ, a vector field $\underset{\sim}{v}$ and a second order tensor field $\underset{\sim}{T}$ on $\mathcal{D}$ $\left(\Rightarrow \varphi : \mathcal{D} \to \mathbb{R},\ \underset{\sim}{v} : \mathcal{D} \to \mathcal{V},\ T : \mathcal{D} \to \text{Lin}\right)$, we define

- gradient of φ :
$$\nabla \varphi = \frac{\partial \varphi}{\partial x_i}(x_1, x_2, x_3)\, \underset{\sim}{e}_i$$
$$=: \varphi_{,i}\, \underset{\sim}{e}_i \qquad (\text{Notation})$$
↳ vector field

- gradient of $\underset{\sim}{v}$:
$$\nabla \underset{\sim}{v} = \frac{\partial v_i}{\partial x_j}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = v_{i,j}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$
↳ tensor field

- divergence of $\underset{\sim}{v}$:
$$\text{div}\, \underset{\sim}{v} := \text{tr}(\nabla \underset{\sim}{v}) = v_{i,i}$$
↳ scalar field

- curl of $\underset{\sim}{v}$: $\text{curl}\, \underset{\sim}{v} = \nabla \times v$

$$= \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} \underset{\sim}{e_i}$$

$$= \varepsilon_{ijk} v_{k,j} \underset{\sim}{e_i} \quad \text{— vector field}$$

- Laplacian : $\Delta(\square) = \text{div}(\nabla \square)$

$$\Delta \varphi = \text{div}(\nabla \varphi) = (\nabla\varphi)_{i,i}$$

$$\boxed{\Delta \varphi = \varphi_{,ii}} \quad \text{— scalar field}$$

Similarly $\boxed{\Delta \underset{\sim}{v} = v_{i,jj} \underset{\sim}{e_i}}$ — vector field

- divergence : $\text{div}\, T = \frac{\partial T_{ij}}{\partial x_j} \underset{\sim}{e_i} = T_{ij,j} \underset{\sim}{e_i}$.

↳ vector field

IMPORTANT:

1) Gradient increases the order of a $\overset{\text{generalized}}{\text{tensor}}$ by 1:

∇(scalar field) is a vector field
∇(vector ") is a tensor field

2) Divergence decreases the order of a $\overset{\text{generalized}}{\text{tensor}}$ by 1:

div(vector field) is a scalar field

div(tensor ") is a vector field.

3) From 1 and 2, Laplacian does not change the order.

4) curl also preserves the order. In this class we only consider curl of a vector field.

Let's now see what exactly is a gradient.

A gradient is a generalization of a derivative to higher dimension. Suppose s is $\mathbb{R}$-valued function defined on $\mathbb{R}$. i.e $s : \mathbb{R} \to \mathbb{R}$. We all know

$$\frac{ds}{dr}(r_0) = \lim_{\varepsilon \to 0} \frac{s(r_0+\varepsilon) - s(r_0)}{\varepsilon} \quad — 1$$

An alternate way of looking at 1 is

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[s(r_0+\varepsilon) - s(r_0) - \frac{ds}{dr}(r_0)\,\varepsilon \right] = 0 \quad — 2$$

$\frac{ds}{dr}(r_0)$ measures the rate at which s changes at r_0 when we take a step ε that is arbitrarily small.

Now, what if $s : \mathbb{R}^3 \to \mathbb{R}$?

In addition to taking a small step of size ε, we must also mention the direction of the step taken

now at vector!

$$\lim_{\varepsilon \to 0} \left[s(\underset{\sim}{r_0} + \varepsilon \underset{\sim}{u}) - s(\underset{\sim}{r_0}) - \square_{\underset{\sim}{r_0}}(\varepsilon \underset{\sim}{u}) \right] = 0$$

$\square_{\underset{\sim}{r_0}}$ should take as input a vector $\varepsilon \underset{\sim}{u}$ and output a scalar.

$\square_{\underset{\sim}{r_0}}$ - linear in its arguments.

$\Rightarrow$ $\square_{r_0}$ is a linear transformation $\square : V \to \mathbb{R}$.

In lecture 5, page 2 we noted that a vector can be viewed as a linear transformation from V to $\mathbb{R}$!

$$\underset{\sim}{v} : V \longrightarrow R$$

$$\underset{\sim}{v}(\underset{\sim}{w}) = \underset{\sim}{v} \cdot \underset{\sim}{w}.$$

In fact, given an inner product any linear transformation $\square_{\underset{\sim}{r_0}} : V \to \mathbb{R}$ can be "identified" with a unique vector!

The above statement is commonly referred to as the Riesz representation theorem.

Therefore, at each point $\underset{\sim}{r_0}$, we can identify $\square(r_0)$ with a unique vector $\underset{\sim}{V}_{\underset{\sim}{r_0}}$ as

$$\square_{\underset{\sim}{r_0}}(\varepsilon u) = \underset{\sim}{V}_{\underset{\sim}{r_0}} \cdot (\varepsilon \underset{\sim}{u}) \quad \text{— } (3)$$

Therefore (3) defines a vector field $v: \mathbb{R}^3 \to V$, which at any point gives the "direction and rate" of maximum change. The vector field $\underset{\sim}{V}$ is in fact the gradient of S:

$$\underset{\sim}{V} = \nabla S.$$

$$= \left(\frac{\partial S}{\partial x_1}, \frac{\partial S}{\partial x_2}, \frac{\partial S}{\partial x_3}\right)(x_1, x_2, x_3).$$

<u>Divergence theorem</u>: Let $\mathcal{D}$ be an open subset of $\mathcal{E}$ with a piecewise smooth boundary, and let $\varphi: \bar{\mathcal{D}} \to \mathbb{R}$, $\underset{\sim}{v}: \bar{\mathcal{D}} \to V$, and $T: \bar{\mathcal{D}} \to \mathrm{Lin}$ be smooth fields on $\bar{\mathcal{D}}$ $(:= \mathcal{D} \cup (\partial\mathcal{D})$, ie $\mathcal{D}$ along with its boundary$)$.

Then

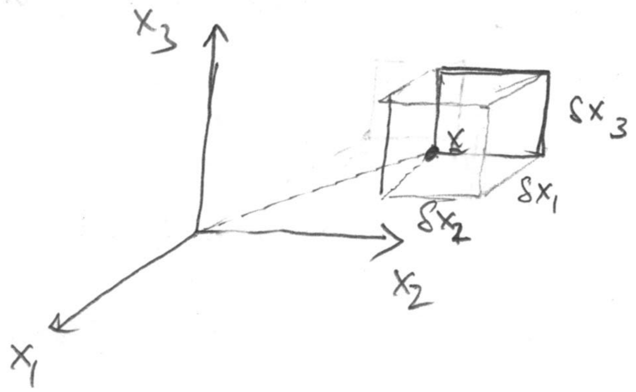
(4)
$$\begin{cases} \displaystyle\int_{\partial\mathcal{D}} \varphi\, \underset{\sim}{n}\, dA = \int_{\mathcal{D}} \nabla\varphi\, dV & \longrightarrow \text{vectors} \\ \displaystyle\int_{\partial\mathcal{D}} \underset{\sim}{v}\cdot\underset{\sim}{n}\, dA = \int_{\mathcal{D}} \operatorname{div} \underset{\sim}{v}\, dV & \text{— scalars} \\ \displaystyle\int_{\partial\mathcal{D}} T\underset{\sim}{n}\, dA = \int_{\mathcal{D}} \operatorname{div} T\, dV & \text{— vectors.} \end{cases}$$

Equality of

where $\underset{\sim}{n}$ is the outward unit normal on $\partial\mathcal{D}$.

A non-rigorous proof of (4b) :

First assume $\mathcal{D}$ is a cube of size $\delta x_1, \delta x_2, \delta x_3$ with a corner located at $\underset{\sim}{x}$:



$$\int_{\partial\mathcal{D}} \underset{\sim}{v}\cdot\underset{\sim}{n}\, dA = (\delta x_2)(\delta x_3)\Big[\underset{\sim}{v}(x_1,x_2,x_3)\cdot(-\underset{\sim}{e}_1) + \underset{\sim}{v}(x_1+\delta x_1, x_2, x_3)\cdot \underset{\sim}{e}_1\Big]$$
$$+ (\delta x_1)(\delta x_2)\Big[\underset{\sim}{v}(x_1,x_2,x_3)\cdot(-e_3) + \underset{\sim}{v}(x_1,x_2,x_3+\delta x_3)\cdot \underset{\sim}{e}_3\Big]$$

$$+ \quad (\delta x_1)(\delta x_3)\left[\underset{\sim}{V}(x_1,x_2,x_3)\cdot(-\underset{\sim}{e}_2) + \underset{\sim}{V}(x_1,x_2+\delta x_2,x_3)\cdot \underset{\sim}{e}_2 \right] \quad \to (4)$$

Dividing (4) by $\delta V = (\delta x_1)(\delta x_2)(\delta x_3)$:

$$\frac{1}{\delta V}\int_{\partial \mathcal{D}} \underset{\sim}{V}\cdot \underset{\sim}{n}\, dA = \frac{\underset{\sim}{V}(x_1+\delta x_1,x_2,x_3) - \underset{\sim}{V}(x_1,x_2,x_3)}{\delta x_1}\cdot \underset{\sim}{e}_1$$

$$+ \frac{\underset{\sim}{V}(x_1,x_2,x_3+\delta x_3) - \underset{\sim}{V}(x_1,x_2,x_3)}{\delta x_3}\cdot \underset{\sim}{e}_3$$

$$+ \frac{\underset{\sim}{V}(x_1,x_2+\delta x_2,x_3) - \underset{\sim}{V}(x_1,x_2,x_3)}{\delta x_2}\cdot \underset{\sim}{e}_2$$

Taking the limit $\delta x_i \to 0$, we have

$$\lim_{\delta x_i \to 0} \frac{1}{\delta V}\int_{\partial \mathcal{D}} \underset{\sim}{V}\cdot \underset{\sim}{n}\, dA = \frac{\partial V_i}{\partial x_i}$$

$$= \operatorname{div} \underset{\sim}{V} \quad — (5)$$

For an arbitrary shaped $\mathcal{D}$, we may approximate $\mathcal{D}$ using a stack of cubes $\mathcal{D}_i$, and write

$$\int_{\partial \mathcal{D}} \underset{\sim}{V}\cdot \underset{\sim}{n}\, dA = \sum_i \int_{\partial \mathcal{D}_i} \underset{\sim}{V}\cdot \underset{\sim}{n}\, dA \underset{\text{From}(5)}{=} \sum_i (\operatorname{div} \underset{\sim}{V})\, \Delta V$$

$$= \int_{\mathcal{D}} (\operatorname{div} \underset{\sim}{v})\, dV$$

Identities: φ, ψ - scalar fields
$\underset{\sim}{u}, \underset{\sim}{v}$ - vector fields, T- tensor field

(a) $\nabla(\phi\psi) = \varphi \nabla\psi + \psi \nabla\varphi$

(b) $\operatorname{div}(\varphi \underset{\sim}{u}) = \varphi \operatorname{div} \underset{\sim}{u} + \underset{\sim}{u} \cdot \nabla\phi$

(c) $\nabla\times (\varphi \underset{\sim}{u}) = \nabla\varphi \times \underset{\sim}{u} + \varphi \nabla\times \underset{\sim}{u}$

(d) $\operatorname{div}(\underset{\sim}{u} \times \underset{\sim}{v}) = \underset{\sim}{v} \cdot \nabla\times \underset{\sim}{u} - \underset{\sim}{u} \cdot \nabla\times \underset{\sim}{v}$

(e) $\operatorname{div}(\varphi T) = T \nabla\varphi + \varphi \operatorname{div} T$

(f) $\operatorname{div}(T \underset{\sim}{v}) = (\operatorname{div} T^T) \cdot \underset{\sim}{v} + T \cdot (\nabla \underset{\sim}{v})^T$

(g) $\operatorname{div}(\underset{\sim}{u} \otimes \underset{\sim}{v}) = \underset{\sim}{u} (\operatorname{div} \underset{\sim}{v}) + (\nabla \underset{\sim}{u}) \underset{\sim}{v}$

(h) $\nabla(\underset{\sim}{u} \cdot \underset{\sim}{v}) = (\nabla \underset{\sim}{u})^T \underset{\sim}{v} + (\nabla \underset{\sim}{v})^T \underset{\sim}{w}$