Eigenvalues & Eigenvectors

Theorem: Let $S \in Sym$. Then there exists three mutually orthogonal eigenvectors corresponding to three eigenvalues. The eigenvalues are roots of characteristic polynomial

$$\det(S-\lambda I) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3,$$

$$I_1 = \mathrm{trace}(S)$$
$$I_2 = \frac{1}{2}\left[(\mathrm{trace}(S))^2 - \mathrm{trace}(S^2)\right]$$
$$I_3 = \det(S)$$

Fundamental invariants of S. Let $\lambda_1, \lambda_2, \lambda_3$ be the three real roots

The corresponding eigenvectors are obtained by solving

$$S\underset{\sim}{u}_1 = \lambda_1 \underset{\sim}{u}_1$$

Three possibilities arise

1) λ_is are distinct: The eigenvector space of each eigenvalue is dim 1. The eigenvector space are mutually orthogonal

Moreover, the eigenvector space are — **Spectral Decomposition Theorem**

$$S = \lambda_1 \underset{\sim}{u}_1 \otimes \underset{\sim}{u}_1 + \lambda_2 \underset{\sim}{u}_2 \otimes \underset{\sim}{u}_2 + \lambda_3 \underset{\sim}{u}_3 \otimes \underset{\sim}{u}_3.$$

2) $\lambda_1 \neq \boxed{\lambda_2 = \lambda_3} = \lambda$ → dim

$\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3\}$ → normalised eigenvectors of $\lambda_1, \lambda_2, \lambda_3$

3) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

$\therefore S = \lambda I$

$I - \underset{\sim}{n}\otimes\underset{\sim}{n}$ – projection onto plane with normal $\underset{\sim}{n}$.

$\underset{\sim}{e}\otimes\underset{\sim}{e} + \underset{\sim}{f}\otimes\underset{\sim}{f} \rightarrow = I - n\otimes n$

iff: $\underset{\sim}{n} = \underset{\sim}{e}\times\underset{\sim}{f}$

$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3$

Every T in Lin satisfies its own characteristic polynomial

$$-T^3 + I_1 T - I_2 T + I_3 I = 0$$

Invariants (Scalar)

→ **Cayley-Hamilton Theorem**

$F = RU = VR$

* Analogous to polar ... numbers

$F\underset{\sim}{v} = RU\underset{\sim}{v}$

↑ Deformation Gradient (Stretching + Rotating)

Def: Positive-definite tensors PSym

$T \in Lin$ is +-definite, if

$T\underset{\sim}{v} \cdot \underset{\sim}{v} > 0 \quad \forall \underset{\sim}{v} \neq 0$

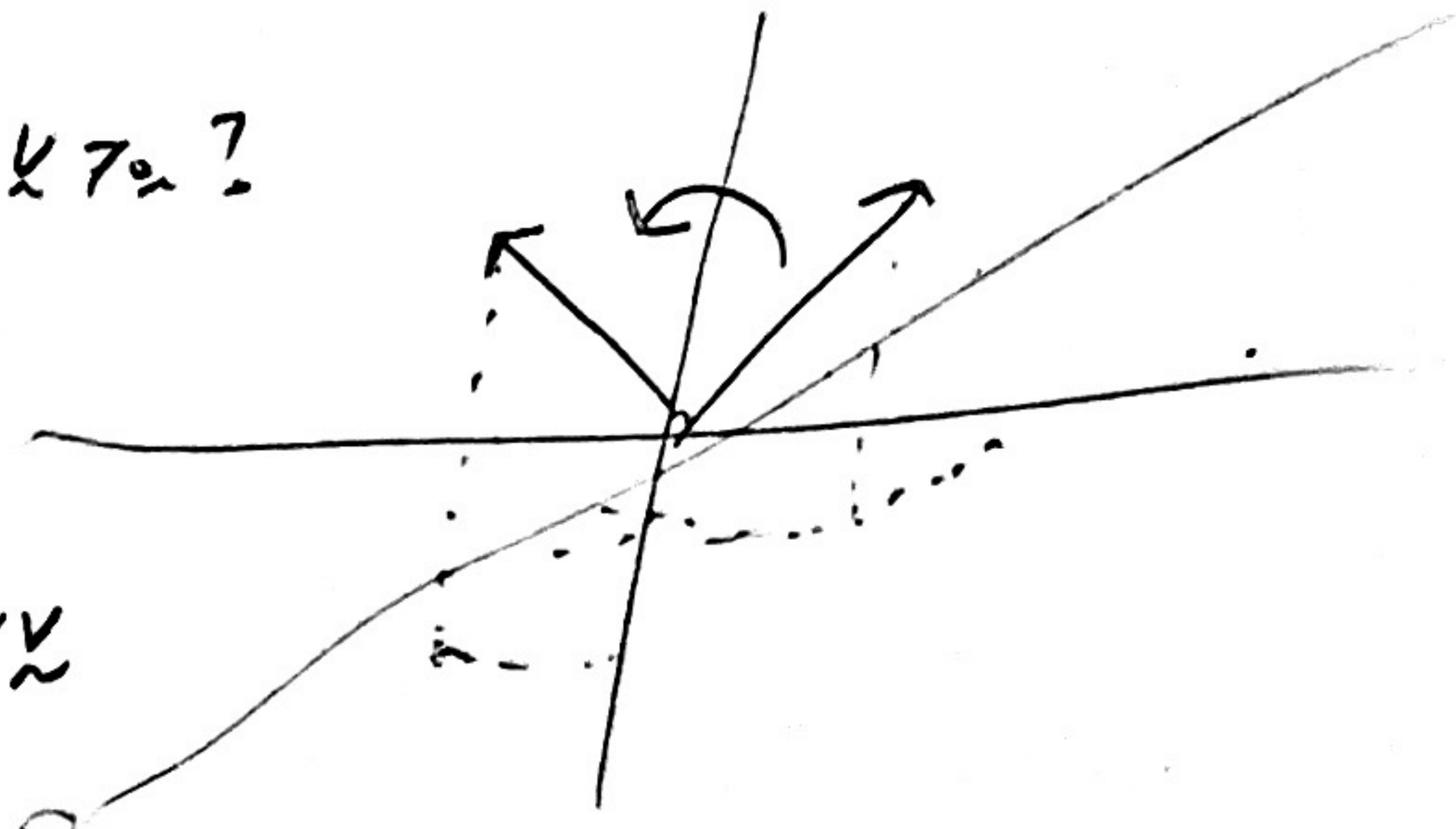
If you have a Tensor upon a vector, that resulting vector cannot have an angle > 90° with the OG tensor

If $W \in Skw$, is it true that $W\underset{\sim}{v} \cdot \underset{\sim}{v} > 0 \quad \forall \underset{\sim}{v} \neq 0$?

$W\underset{\sim}{v} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot W^T \underset{\sim}{v} = -\underset{\sim}{v} \cdot W\underset{\sim}{v} \Rightarrow W\underset{\sim}{v} \cdot \underset{\sim}{v} = 0$

$W\underset{\sim}{v} \cdot \underset{\sim}{v} = -\underset{\sim}{v} \cdot W\underset{\sim}{v}$

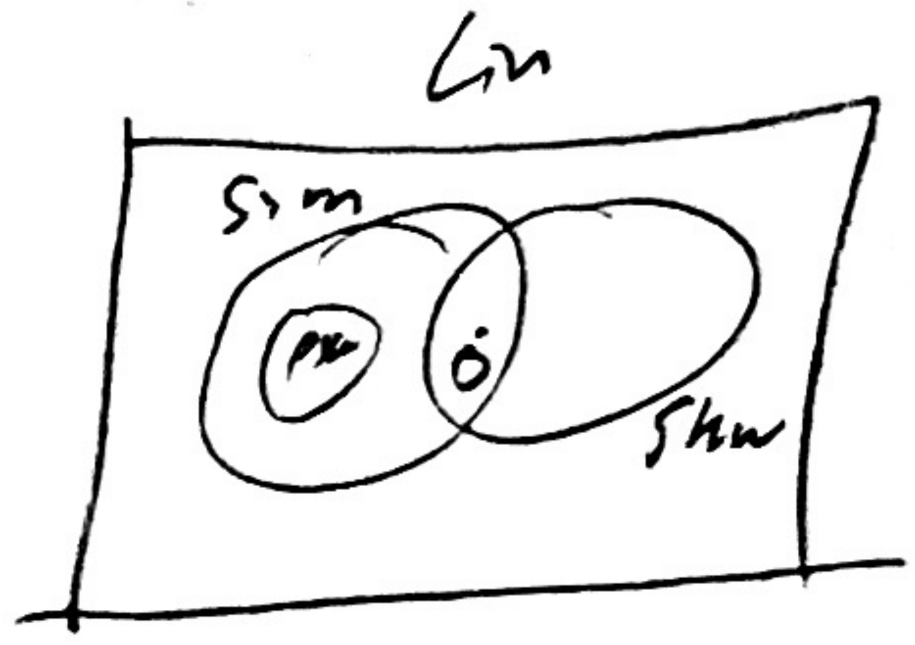
$W\underset{\sim}{v} \cdot \underset{\sim}{v} \perp W\underset{\sim}{v}\, \underset{\sim}{v} = 0$



$T = S + W$

$S = \frac{T + T^T}{2}; \; W = \frac{T - T^T}{2}$

$T\underset{\sim}{v} \cdot \underset{\sim}{v} = S\underset{\sim}{v} \cdot \underset{\sim}{v} + \cancel{W\underset{\sim}{v} \cdot \underset{\sim}{v}} \rightarrow 0$



P_{sym} - +-definite symmetric tensors

$= \{ S \in Sym : S\underset{\sim}{v} \cdot \underset{\sim}{v} > 0 \; \forall \underset{\sim}{v} \neq 0 \}$

Theorem: $S \in Psym \iff$ its eigenvalues are strictly positive

$(\Rightarrow)$ Assume $S \in PSym$.

$\lambda_i = S\underset{\sim}{u}_i \cdot \underset{\sim}{u}_i > 0$ (no sum).

$(\Leftarrow) \quad S = \sum_{i=1}^{3} \lambda_i \underset{\sim}{u}_i \otimes \underset{\sim}{u}_i \qquad \lambda_i > 0$

normalized

Show that:

$S\underset{\sim}{v} \cdot \underset{\sim}{v} > 0 \longrightarrow S\underset{\sim}{v} = \sum_{i=1}^{3} \lambda_i (\underset{\sim}{u}_i \cdot \underset{\sim}{v}) \underset{\sim}{u}_i$