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Lecture 9

Eigenvalues are sometimes also referred to as principal values, and eigenvectors as principal directions. The problem of solving eigenvectors and eigenvalues of a second order tensor is much easier when the tensor is symmetric, and this is the usual situation in mechanics contexts. Therefore, we will only study eigenvalues and eigenvectors of tensors in Sym

we always assume orthonormal

Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ denote the $\boxed{\text{basis}}$ of the three-dimensional Euclidean vector space. Let $\underset{\sim}{e}_1$ be an eigenvector of a tensor $S \in$ Sym. How does S look like with respect to the above basis?

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{bmatrix},$$

where λ is the eigenvalue corresponding to $\underset{\sim}{e}_1$.

Why? Lets prove the following statement

. $\underset{\sim}{e}_1$ is an eigenvector of S —— ①

$$\Longleftrightarrow \quad [S_{ij}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{bmatrix},$$ where λ is the eigenvalue of $\underset{\sim}{e}_1$.

($\Longleftarrow$) $S\underset{\sim}{e}_1 = \lambda \underset{\sim}{e}_1$ (obvious!)

($\Longrightarrow$) Assume $\underset{\sim}{e}_1$ is the eigenvector of S with eigenvalue λ. This implies

$$S\underset{\sim}{e}_1 = \lambda \underset{\sim}{e}_1$$

We know $S_{ij} = S\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i$. For $i = 2,3$ and $j=1$, clearly

$$\begin{aligned} S_{i1} &= S\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_i \\ &= \lambda \underset{\sim}{e}_1 \cdot e_i \\ &= 0 \quad (\because i \neq 1), \end{aligned}$$

while $S_{11} = S\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1 = \lambda \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1 = \lambda$. Since S is symmetric, ① follows.

<u>Theorem</u>: Eigenvectors associated with <u>distinct</u> eigenvalues of $S \in \text{Sym}$ are orthogonal.

<u>Proof</u>: Let $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ denote two eigenvectors of S with eigenvalues λ_1 and λ_2 respectively, with $\lambda_1 \neq \lambda_2$. We will show that $\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 = 0$. By assumption

$$S\underset{\sim}{u}_1 = \lambda_1 \underset{\sim}{u}_1, \qquad \text{— (2a)}$$

$$S\underset{\sim}{u}_2 = \lambda_2 \underset{\sim}{u}_2 \qquad \text{— (2b)}$$

Dot product of (2a) with $\underset{\sim}{u}_2$ results in

$$S\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 = \lambda_1 \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2$$

$$\Rightarrow \underset{\sim}{u}_1 \cdot S^T \underset{\sim}{u}_2 = \lambda_1 \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 \qquad \left(\begin{array}{c}\text{Definition of}\\ \text{transpose}\end{array}\right)$$

$$\Rightarrow \underset{\sim}{u}_1 \cdot S\underset{\sim}{u}_2 = \lambda_1 \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 \qquad (\because S \in \text{Sym})$$

$$\Rightarrow \underset{\sim}{u}_1 \cdot \lambda_2 \underset{\sim}{u}_2 = \lambda_1 \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2$$

$$\Rightarrow (\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2)(\lambda_1 - \lambda_2) = 0$$

Since $\lambda_1 \neq \lambda_2$, we have $\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 = 0$. $\square$

How to compute eigenvalues and eigenvectors, i.e find the pairs $(\underset{\sim}{u} \neq 0, \lambda)$ such that

$$S\underset{\sim}{u} = \lambda \underset{\sim}{u},$$

or in other words $(S - \lambda I)\underset{\sim}{u} = 0$. In indicial notation, we have three equations:

$$\left(S_{ij} - \lambda \delta_{ij} \right) u_j = 0 \quad \text{———} (3)$$

A trivial solution of (3) is $u_1 = u_2 = u_3 = 0$. Recall that eigenvector, by definition is a nonzero vector. Therefore, we look for non-trivial solutions of (3). It is a well-known fact that (3) has non-trivial solutions

$$\Longleftrightarrow \quad \boxed{\det (S - \lambda I) = 0} \quad — (4)$$

(4) results in a cubic polynomial for λ. By the fundamental theorem of algebra, a cubic polynomial has three roots, not necessarily in $\mathbb{R}$, but surely in $\mathbb{C}$ (complex numbers). But whenever $S \in Sym$, we will show that the three roots are real.

complex non-real roots occur as conjugates

CAUTION : By 'three roots' we don't necessarily mean three distinct roots. For example, the two roots of the $x^2-4x+4=0$ have 2, 2.

Some examples :

a) $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 4 & -1 \end{bmatrix} \notin \text{Sym}$

$$\det(T-\lambda I) = \det \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 3-\lambda & -2 \\ 0 & 4 & -1-\lambda \end{bmatrix}$$

$$= -\lambda\left[(\lambda+1)(\lambda-3)+8\right]$$

$$= -\lambda\left[\lambda^2-2\lambda+5\right]$$

Roots : $0,\ 1 \pm 2i \Rightarrow$ All eigenvalues are not real.

b) $T = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Eigenvalues : $8, -1, -1$.

(c) $T = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Eigenvalues : $2, 3, 6$

How about obtaining eigenvector corresponding to an eigenvalue?

Let λ be an eigenvalue, i.e it satisfies (4). Then we know $\exists$ a non-trivial $\underset{\sim}{u}$ that satisfies $S\underset{\sim}{u} = \lambda \underset{\sim}{u}$. Of course, if $\underset{\sim}{u}$ satisfies $S\underset{\sim}{u} = \lambda \underset{\sim}{u}$, then any scaled version of $\underset{\sim}{u}$, say $\beta \underset{\sim}{u}$ also satisfies: $S(\beta\underset{\sim}{u}) = \lambda(\beta\underset{\sim}{u})$. Therefore, we look for a unit vector that satisfies $S\underset{\sim}{u} = \lambda \underset{\sim}{u}$. Let's revisit example (b):

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 8 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad -(5)$$

$$\Rightarrow \left.\begin{aligned} -5u_1 + 2u_2 + 4u_3 &= 0 \\ 2u_1 - 8u_2 + 2u_3 &= 0 \\ 4u_1 + 2u_2 - 5u_3 &= 0 \end{aligned}\right\} (6)$$

$\underline{2(6b) - (6c)}$: $\quad -18u_2 + 9u_3 = 0 \Rightarrow u_3 = 2u_2$.

Substituting in (6a): $-5u_1 + 10u_2 = 0 \Rightarrow u_1 = 2u_2$

Therefore, $u = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\lambda = 8$ is a eigenvector-value pair.

How about for the repeated eigenvalue -1:

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad — (7)$$

$$\Rightarrow \quad 4v_1 + 2v_2 + 4v_3 = 0$$

$$2v_1 + v_2 + 2v_3 = 0$$

$$4v_1 + 2v_2 + 4v_3 = 0$$

$$\Rightarrow \quad 2v_1 + v_2 + 2v_3 = 0 \quad — \quad (8) \rightarrow$$ Defines a hyper plane

The eigenvectors of eigenvalue -1 form a vector subspace of $\boxed{\text{dimension } 2}$! On the other hand eigenvectors of eigenvalue 8 form a vector subspace of dimension 1 (after including $\underset{\sim}{0}$): $\left\{ \underset{\sim}{v} \in V : v = \lambda \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$.

In fact, the plane (defined by (8)), which describes all the eigenvectors with eigenvalue -1 is perpendicular to $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, the eigenvector of eigenvalue 8 !

<u>Example C</u> : eigenvalue - vector pairs

$$\left.\begin{array}{llll} 2 - & u_1 = -1, & v_2 = 1, & v_3 = 0 \\ 3 - & v_1 = 1, & v_2 = 1, & v_3 = 1 \\ 6 - & v_1 = -1, & v_2 = -1, & v_3 = 2 \end{array}\right\}$$ Mutually orthogonal!

<u>Summary</u>:

Examples (b) and (c) clearly demonstrate that the eigen vector spaces of distinct eigenvalues are mutually orthogonal.

<u>Theorem</u>: Let $S \in Sym$. There exist three mutually orthogonal eigenvectors corresponding to at most three distinct eigenvalues. The eigenvalues are the roots of the <u>characteristic polynomial</u>

$$\det(S - \lambda I) = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 ,$$

whose coefficients

$$I_1 = \operatorname{tr}(S) = S_{ii}$$

$$I_2 = \frac{1}{2}\left[(\operatorname{tr} S)^2 - \operatorname{tr}(S^2) \right] = \frac{1}{2}\left[S_{ii} S_{jj} - S_{ij} S_{ji} \right]$$

$$I_3 = \det(S)$$

are the <u>fundamental invariants</u> of S. The characteristic polynomial has three <u>real</u> roots (not necessarily distinct), which are denoted by λ_i.

The corresponding eigenvectors are sol

The components of an eigenvector corresponding to λ_i are obtained by solving

$$(S - \lambda_i I)\, \underset{\sim}{u_i} = \underset{\sim}{0}$$

Three possibilities arise:

<u>Case 1</u>: (All eigenvalues are distinct). The eigenvector space of each eigenvalue is of dimension 1. Moreover the three vector spaces are mutually orthogonal.

<u>Case 2</u>: (Two eigenvalues are distinct, i.e. $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda$) The eigenvector space of λ is of dimension 2, and it is orthogonal to the eigenvector space of λ_1 which is of dimension 1.

<u>Case 3</u>: (All eigenvalues are equal): Every non-zero vector in V is an eigenvector. In fact

$$S = \lambda \underline{I}.$$

We will not prove the above theorem. But we can prove why the characteristic polynomial of a symmetric tensor always has three real roots:

Since the roots of a polynomials occur as complex conjugate pairs, it follows that the characteristic polynomial should have atleast one real root, i.e $\exists$ at least one real eigenvalue, say λ_0. Let $\underset{\sim}{u}$ be the corresponding eigenvector. Reorient the basis such that e_1 is parallel to $\underset{\sim}{u}$. The S has the representation

$$S = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{bmatrix}.$$

<u>Note</u>: The characteristic polynomial still looks the same, no matter which basis we choose! This is because its coefficients depend on trace and determinant of S which are independent of the choice of the basis.

$$\det(S - \lambda \underline{I}) = \det \begin{bmatrix} \lambda_0 - \lambda & 0 & 0 \\ 0 & S_{22} - \lambda & S_{23} \\ 0 & S_{32} & S_{33} - \lambda \end{bmatrix}$$

$$= (\lambda_0 - \lambda)\left[\lambda^2 - (S_{22}+S_{33})\lambda + S_{22}S_{33} - S_{23}^{\;2}\right]$$

Roots: $\lambda = \lambda_0$

$$\lambda = \frac{1}{2}(S_{22}+S_{33}) \pm \sqrt{\underbrace{(S_{22}-S_{33})^2 + 4S_{23}^{\;2}}_{\geqslant 0}}$$

$\Rightarrow$ All roots are real.