TAM 445 Continuum Mechanics - Spring 2024
Final Examination

Due: May 12, 11:59 PM Central Time

1. (Tensor algebra)

a) (10 points) Derive the following differential identities.

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{S}\boldsymbol{T}) &= \operatorname{tr}(\boldsymbol{T}\boldsymbol{S}) \\
\operatorname{curl}(\phi\boldsymbol{u}) &= \nabla\phi \times \boldsymbol{u} + \phi \operatorname{curl} \boldsymbol{u} \\
\operatorname{div}(\phi\boldsymbol{T}) &= \boldsymbol{T}\nabla\phi + \phi \operatorname{div} \boldsymbol{T} \\
\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{v}) &= (\operatorname{div} \boldsymbol{v})\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{v} \\
\triangle(\xi\eta) &= \xi\triangle\eta + \eta\triangle\xi + 2\nabla\xi \cdot \nabla\eta
\end{aligned}
$$

b) (10 points) Recall that the invariants of a tensor $\boldsymbol{S}$ are

$$
I_1(\boldsymbol{S}) = \operatorname{tr} \boldsymbol{S}, \quad I_2(\boldsymbol{S}) = \frac{1}{2}[(\operatorname{tr} \boldsymbol{S})^2 - \operatorname{tr}(\boldsymbol{S}^2)], \quad I_3(\boldsymbol{S}) = \det(\boldsymbol{S}).
$$

Show that for any $\boldsymbol{Q} \in \mathrm{Orth}$, $I_k(\boldsymbol{S}) = I_k(\boldsymbol{Q}\boldsymbol{S}\boldsymbol{Q}^{\mathrm{T}})$, $k = 1, 2, 3$. Hint: Use the identity $\operatorname{tr}(\boldsymbol{S}\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{T}\boldsymbol{S})$ of Problem 1.

Let $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ and $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$ denote the right and left polar decomposition of a positive definite tensor $\boldsymbol{F}$. Using the above result, show that $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same eigenvalues. Hint: eigenvalues of a tensor are the roots of its characteristic polynomial.

In addition, show that if $\boldsymbol{e}$ is an eigenvector of $\boldsymbol{U}$, then $\boldsymbol{R}\boldsymbol{e}$ is an eigenvector of $\boldsymbol{V}$.

c) (5 points) Let $\boldsymbol{A}$ be a symmetric tensor. Show that $\boldsymbol{A}$ satisfies

$$
\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{A} \quad \forall \boldsymbol{Q} \in \mathrm{Orth}
$$

if and only if $\boldsymbol{A} = \alpha\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity tensor. Hint: For the forward implication, show that if $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{Q}\boldsymbol{v}$ is also is eigenvector for all rotations $\boldsymbol{Q}$. What does this imply?

2. (Motion) A spherical cavity of radius a_0 at time $t = 0$ in an infinite body is centered at the origin. An explosion inside the cavity at $t = 0$ produces the motion

$$
\boldsymbol{x}(\boldsymbol{X}, t) = \frac{f(R, t)}{R}\boldsymbol{X},
$$

where $R = \|\boldsymbol{X}\|$ is the magnitude of the position vector in the reference configuration. The cavity wall has a radial motion given in the above equation such that at time t the cavity is spherical with radius $a(t)$.

(a) (13 points) Show that the deformation gradient is

$$
\boldsymbol{F} = \frac{f(R, t)}{R}\boldsymbol{I} + \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\boldsymbol{X} \otimes \boldsymbol{X},
$$

and its determinant is

$$
\det \boldsymbol{F} = \left(\frac{f}{R}\right)^2 \frac{\partial f}{\partial R}.
$$

(b) (5 points) Find the velocity and acceleration fields.

(c) (7 points) Show that if the motion is restricted to be *isochoric*, i.e. volume preserving (which implies $\det \boldsymbol{F} \equiv 1$), then $f(R,t) = (R^3 + a^3 - a_0^3)^{1/3}$.

3. (Momentum balance law and stress) A rectangular body occupies the region $-a \le x_1 \le a$, $-a \le x_2 \le a$ and $-b \le x_3 \le b$ in the deformed configuration. The components of the Cauchy stress tensor in the body are given by

$$\boldsymbol{T} = \frac{c}{a^2}\begin{bmatrix} -(x_1^2 - x_2^2) & 2x_1x_2 & 0 \\ 2x_1x_2 & x_1^2 - x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where a, $b > a$ and c are positive constants.

1. (6 points) Show that $\boldsymbol{T}$ satisfies the balance of linear momentum in the static case with no body force, i.e. $\operatorname{div} \boldsymbol{T} = \boldsymbol{0}$.

2. (6 points) Determine the tractions that must be applied to the six faces of the body in order for the body to be in equilibrium.

3. (6 points) Calculate the traction distribution on the sphere $x_1^2 + x_2^2 + x_3^2 = a^2$.

4. (7 points) The principal values (eigenvalues) of the stress tensor (principal stresses) are denoted λ_i $(i = 1, 2, 3)$, such that $\lambda_1 \ge \lambda_2 \ge \lambda_3$. These give the (algebraically) maximum and minimum normal stresses at a point. It can be shown that the maximum shear stress is given by $\tau_{\max} = (\lambda_1 - \lambda_3)/2$. Calculate the principal stresses of $\boldsymbol{T}$ as a function of position. Then find the maximum value of $\tau_{\max}$ in the domain of the body.

4. (Constitutive relations) The Cauchy stress in a hyperelastic material undergoing a motion $f(\boldsymbol{X}, t)$ is given by

$$\boldsymbol{T}_{\mathrm{s}}(\boldsymbol{x}, t) = \rho \frac{\partial \psi}{\partial \boldsymbol{F}} \boldsymbol{F}^{\mathrm{T}} \bigg|_{\boldsymbol{X}=f^{-1}(\boldsymbol{x},t)}, \tag{1}$$

where $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t)$, $\boldsymbol{F}(\boldsymbol{X})$ is the deformation gradient, $\psi(\boldsymbol{F})$ is the Helmholtz free energy density, and $\rho(\boldsymbol{X}, t)$ is the material density field. The objective of this problem is to obtain constraints on the free energy density and the Cauchy stress from the *invariance under superposed rigid body motion* (ISRBM) and material symmetry.

(a) (5 points) Recall that from ISRBM, ψ has to satisfy the condition

$$\psi(\boldsymbol{F}) = \psi(\boldsymbol{QF}) \text{ for all rotations } \boldsymbol{Q}. \tag{2}$$

Show that a consequence of ISRBM is that the free energy density can always be expressed as a function $\tilde{\psi}$ of the right Cauchy–Green stretch tensor $\boldsymbol{C}$. Hint: Choose an appropriate $\boldsymbol{Q}$ while examining Equation (2).

(b) (8 points) From (a), it follows that $\psi(\boldsymbol{F}) = \tilde{\psi}(\boldsymbol{C})$. Using this relation and Equation (1), show that

$$\boldsymbol{T}_{\mathrm{s}}(\boldsymbol{x}, t) = 2\rho \boldsymbol{F} \frac{\partial \tilde{\psi}}{\partial \boldsymbol{C}} \boldsymbol{F}^{\mathrm{T}} \bigg|_{\boldsymbol{X}=f^{-1}(\boldsymbol{x},t)}.$$

(c) (4 points) If the material is isotropic, additional restrictions follow. In particular, the free energy density can always be expressed as a function $\overline{\psi}$ of the three invariants of $\boldsymbol{C}$. Show that such a free energy is indeed invariant under symmetry transformations of isotropic materials. Hint: The material symmetry group of isotropic materials is the set of all rotations. See how $\boldsymbol{C}$ transforms under symmetry transformations.

The tensor $\boldsymbol{C}$ in this part of the problem can be replaced by the left Cauchy–Green stretch tensor $\boldsymbol{B}$. Why?

(d) (8 points) From (c), we know that $\psi(\boldsymbol{F}) = \overline{\psi}(I_1(\boldsymbol{B}), I_2(\boldsymbol{B}), I_3(\boldsymbol{B}))$. Show that the Cauchy stress in an isotropic hyperelastic material is of the form

$$\boldsymbol{T} = \eta_0 \boldsymbol{I} + \eta_1 \boldsymbol{B} + \eta_2 \boldsymbol{B}^2,$$

where η_is are scalar-valued functions of the three invariants of $\boldsymbol{B}$. Hint: Start with Equation (1) with ψ replaced by $\overline{\psi}$ and use chain rule. Obtain the η_is as derivatives of $\overline{\psi}$.