TAM 445 Continuum Mechanics - Spring 2024
Homework 5

Due: Mar 01, 2024

Notation: Uppercase bold letters denote second-order tensors, lowercase bold letters denote vectors, and Greek letters denote scalars unless stated otherwise. All vectors fields are assumed to be smooth.

1. Consider the vector field $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{x}$, i.e. $v_1 = x_1$, $v_2 = x_2$, and $v_3 = x_3$. Let $v := |\boldsymbol{v}| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$. Plot the vector field. Using the definitions of differential operators ∇, div, and curl, and indicial notation, prove the following identities.

1. $\nabla \boldsymbol{v} = \boldsymbol{I}$
2. $\nabla v = \frac{\boldsymbol{x}}{x}$
3. $\operatorname{div} \boldsymbol{v} = 3$
4. $\operatorname{div} \frac{\boldsymbol{v}}{v} = \frac{2}{x}$
5. $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = 4\boldsymbol{x}$
6. $\operatorname{curl} \boldsymbol{v} = \boldsymbol{0}$
7. $\operatorname{curl} \frac{\boldsymbol{v}}{v} = \boldsymbol{0}$

Do not use matrices.

2. Let $\phi(\boldsymbol{x})$ and $\boldsymbol{v}(\boldsymbol{x})$ denote a scalar field and a vector field, respectively.

1. Show that $\operatorname{div}(\operatorname{curl} \boldsymbol{v}) = 0$ and $\operatorname{curl}(\nabla \phi) = \boldsymbol{0}$.
2. In particular, if

$$\phi(\boldsymbol{x}) = \frac{\boldsymbol{c} \cdot \boldsymbol{x}}{x^3}, \text{ and } \boldsymbol{v}(\boldsymbol{x}) = \frac{\boldsymbol{c} \times \boldsymbol{x}}{x^3}, \quad \boldsymbol{x} \neq 0, \tag{1}$$

for a given constant vector $\boldsymbol{c}$ and $x := |\boldsymbol{x}| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$, show that

$$\begin{aligned} \nabla \phi + \operatorname{curl} \boldsymbol{v} &= \boldsymbol{0}, \\ \triangle \phi &= 0, \\ \operatorname{div} \boldsymbol{v} &= 0. \end{aligned}$$

Plot the fields ϕ and $\boldsymbol{v}$ assuming a $\boldsymbol{c}$ of your choice.