TAM 445 2024 HW 5
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Problem 1.

1. $\nabla \mathbf{v} = \mathbf{I}$

   Proof. We begin by using the definition of the gradient of a vector field:

$$\nabla \mathbf{v} = \frac{\partial x_i}{\partial x_j} \underline{e}_i \otimes \underline{e}_j \tag{1}$$
$$\mathbf{I} = \delta_{ij} \underline{e}_i \otimes \underline{e}_j \tag{2}$$
$$\tag{3}$$

   Applying the identity tensor to an arbitrary vector $\mathbf{u}$:

$$\mathbf{I}\underline{u} = (\delta_{ij} \underline{e}_i \otimes \underline{e}_j)\underline{u} \tag{4}$$
$$(\mathbf{I}\underline{u})_i = \mathbf{I}_{ij} u_j \tag{5}$$
$$(\delta_{ij} \underline{e}_i \otimes \underline{e}_j)\underline{u} = \delta_{ij}(\underline{e}_j \cdot \underline{u})\underline{e}_i \tag{6}$$
$$= \delta_{ij} u_j \underline{e}_i \tag{7}$$

   Using the definiton of the identity tensor and equating Eqns (5) and (7):

$$\mathbf{I}_{ij} u_j = \delta_{ij} u_j \tag{8}$$

   The identity tensor is equal to the Kronecker delta, as desired. □

2. $\nabla v = \frac{\mathbf{x}}{x}$

   Proof. Using the given definition of x and v:

$$v(\mathbf{x}) = (x_i x_i)^{1/2} \tag{9}$$

   Substituting Eqn (9) into the definition of the gradient of a scalar field:

$$\frac{\partial (x_j x_j)^{1/2}}{\partial x_i} = \frac{1}{2}(x_j x_j)^{-1/2} \left(\frac{\partial x_j}{\partial x_i} x_j + \frac{\partial x_j}{\partial x_i} x_j \right) \tag{10}$$
$$= \frac{1}{2}(x_j x_j)^{-1/2} \left(2 \frac{\partial x_j}{\partial x_i} x_j \right) \tag{11}$$
$$= \frac{\delta_{ij} x_j}{(x_j x_j)^{1/2}} \tag{12}$$
$$= \frac{x_i}{(x_j x_j)^{1/2}} \tag{13}$$
$$\tag{14}$$

   The RHS is equal to the RHS of the problem, as desired. □

3. $\text{div}(\mathbf{v}) = 3$

   Proof.

$$\text{div}(\mathbf{v}) = \text{tr}(\nabla\mathbf{v}) \tag{15}$$
$$= \text{tr}(\mathbf{I}) \tag{16}$$
$$= 3 \tag{17}$$

   Eqn (16) is implied by Eqn (8), and the trace of the identity is 3, as desired. □

4. $\text{div}\left(\frac{\mathbf{v}}{v}\right) = \frac{2}{x}$

   Proof. We begin by setting ϕ equal to the scalar field of the domain of the LHS of the given problem:

$$\phi = \frac{1}{v} = \frac{1}{(v_i v_i)^{1/2}} \tag{18}$$

   Eqn (18) is substituted into the following identity:

$$\text{div}(\phi\mathbf{v}) = \phi\ \text{div}(\mathbf{v}) + \mathbf{v} \cdot \nabla\phi \tag{19}$$
$$= \frac{1}{v}\text{tr}(\nabla\mathbf{v}) + \mathbf{v} \cdot \left(\nabla\frac{1}{v}\right) \tag{20}$$

   We observe the following statement from the given information:

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} \tag{21}$$
$$= x \tag{22}$$
$$= v \tag{23}$$

   Simplifying the first term of Eqn (20) using Eqns (21) through (23):

$$\frac{1}{v}\text{tr}(\mathbf{I}) = \frac{3}{x} \tag{24}$$

   The term inside of the parentheses in the second term of Eqn (20) is simplified as follows:

$$\nabla\frac{1}{v} \tag{25}$$
$$= \nabla(x_i x_i)^{-1/2} \tag{26}$$
$$= \frac{\partial(x_j x_j)^{-1/2}}{\partial x_i} \tag{27}$$
$$= -\frac{1}{2}(x_j x_j)^{-3/2}(\delta_{ij}x_j + \delta_{ij}x_j) \tag{28}$$
$$= -\frac{2\delta_{ij}x_j}{2(x_j x_j)^{3/2}} \tag{29}$$
$$= -\frac{x_i}{(x_j x_j)^{3/2}} \tag{30}$$
$$= -\frac{\mathbf{x}}{x^3} \tag{31}$$

Substituting Eqn (31) into the second term of Eqn (20):

$$\mathbf{x} \cdot \left(-\frac{\mathbf{x}}{x^3}\right) \tag{32}$$

$$= \frac{1}{x^3}(\mathbf{x} \cdot \mathbf{x}) \tag{33}$$

$$= -\frac{1}{x^3}(x^2) \tag{34}$$

$$= -\frac{1}{x} \tag{35}$$

Substituting Eqns (35) and (24) into Eqn (20):

$$\frac{3}{x} + \mathbf{x} \cdot \left(-\frac{\mathbf{x}}{x^3}\right) \tag{36}$$

$$= \frac{2}{x} \tag{37}$$

$\square$

5. $\operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = 4x$

Proof. Using the following identity:

$$\operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = \mathbf{v}(\operatorname{div}(\mathbf{v})) + (\nabla \mathbf{v})\mathbf{v} \tag{38}$$

Simplifying:

$$\tag{39}$$

$$\mathbf{v}(\operatorname{div}(\mathbf{v})) + (\nabla \mathbf{v})\mathbf{v} = 3\mathbf{v} + I\mathbf{v} \tag{40}$$

$$= 4\mathbf{v} \tag{41}$$

$$= 4\mathbf{x} \tag{42}$$

$\square$

6. $\text{curl}(\mathbf{v}) = \mathbf{0}$

Proof. We start with the definition of the curl of a vector in indicial notation:

$$\text{curl}(\mathbf{v}) = \epsilon_{kji} \frac{\partial v_i}{\partial x_j} e_k \tag{43}$$

Using Eqn (21), Eqn (43) is simplified as follows:

$$\begin{align} &= \epsilon_{kji} \frac{\partial v_i}{\partial x_j} e_k \tag{44} \\ &= \epsilon_{kji} \delta_{ij} e_k \tag{45} \\ &= \epsilon_{kii} e_k \tag{46} \\ &= \mathbf{0} \tag{47} \end{align}$$

□

7. $\text{curl}\left(\frac{\mathbf{v}}{v}\right) = \mathbf{0}$

Proof. Setting $\phi = \frac{1}{v}$, using the identity of the curl of a scalar field multiplied by a vector field, and Eqn (47):

$$\begin{align} \frac{1}{v} &= \phi \tag{48} \\ &= \underbrace{\cancel{\phi \, \text{curl}(\mathbf{v})}}_{0} + \nabla \phi \times \mathbf{v} \tag{49} \end{align}$$

Substituting Eqn (31) into Eqn (49):

$$\begin{align} &= -\left(\frac{1}{x^3}\right) \underbrace{\cancel{\mathbf{x} \times \mathbf{x}}}_{0} \tag{50} \\ &= \mathbf{0} \tag{51} \end{align}$$

□

Problem 2.

1. (a) $\mathrm{div}(\mathrm{curl}(\mathbf{v})) = 0$

   Proof.

$$\mathrm{div}(\mathrm{curl}(\mathbf{v})) = \mathrm{div}\left(\epsilon_{kji}\frac{\partial v_i}{\partial x_j}\underline{e}_k\right) \tag{52}$$

$$\tag{53}$$

   Setting $\gamma(\mathbf{v}) = \underline{e}_k$ and $\varepsilon(\mathbf{v}) = \epsilon_{kji}\frac{\partial v_i}{\partial x_j}$ and using the identity of divergence of the curl of a vector:

$$\underbrace{\cancel{\varepsilon\ \mathrm{tr}(\nabla \underline{e}_k)}}_{0} + \underbrace{\cancel{\underline{e}_k \cdot \frac{\partial \varepsilon}{\partial x_i}\underline{e}_i}}_{0} \tag{54}$$

   □

   (b) $\mathrm{curl}(\nabla\phi) = \mathbf{0}$

   Proof.

$$\mathrm{curl}(\nabla\phi) = \mathbf{0} \tag{55}$$

$$\mathrm{curl}\left(\frac{\partial \phi}{\partial x_i}\underline{e}_i\right) \tag{56}$$

$$= \nabla \times \left(\frac{\partial \phi}{\partial x_i}\underline{e}_i\right) \tag{57}$$

$$= \underbrace{\cancel{\nabla\frac{\partial \phi}{\partial x_i} \times \underline{e}_i}}_{0} + \underbrace{\cancel{\frac{\partial \phi}{\partial x_i}\mathrm{curl}(\underline{e}_i)}}_{0} \tag{58}$$

   □

2. (a) $\nabla\phi + \mathrm{curl}(\mathbf{v}) = \mathbf{0}$

   Proof. We begin by calculating $\nabla\phi$:

$$\phi = \frac{c_j x_j}{(x_k x_k)^{3/2}} \tag{59}$$

$$\nabla\phi = \frac{\partial(c_j x_j (x_k x_k)^{-3/2})}{\partial x_i}\underline{e}_k \tag{60}$$

$$= c_j\frac{\partial x_j}{\partial x_i}(x_k x_k)^{-3/2}\underline{e}_k + -\frac{3}{2}c_j x_j (x_k x_k)^{-5/2}\left(2\frac{\partial x_k}{\partial x_i}x_k\right)\underline{e}_k \tag{61}$$

$$= \frac{c_i}{(x_k x_k)^{3/2}}\underline{e}_i - \frac{3 c_j x_j x_i}{(x_k x_k)^{5/2}}\underline{e}_i \tag{62}$$

Calculating $\mathbf{v}$ in indicial notation:

$$\mathbf{v}(\mathbf{x}) = \frac{\mathbf{c} \times \mathbf{v}}{x^3} \tag{63}$$
$$= \frac{\epsilon_{ijk} c_j x_k}{(x_s x_s)^{3/2}} \underline{e}_i \tag{64}$$

Calculating curl($\mathbf{v}$):

$$\nabla \times \mathbf{v} = \epsilon_{lnm} \frac{\partial(\epsilon_{ijk} c_j x_k (x_s x_s)^{-3/2})}{\partial x_n} \delta_{\mathrm{im}} \underline{e}_l \tag{65}$$
$$\tag{66}$$

Taking the partial derivative of Eqn (65) using the product rule for the first term:

$$= \epsilon_{lnm} \epsilon_{ijk} c_j \delta_{kn} (x_s x_s)^{-3/2} \delta_{im} \underline{e}_l \tag{67}$$
$$\epsilon_{lni} \epsilon_{jni} c_j (x_s x_s)^{-3/2} \underline{e}_l \tag{68}$$
$$= 2\delta_{lj} c_j (x_s x_s)^{-3/2} \underline{e}_l \tag{69}$$
$$= \frac{2c_l}{(x_s x_s)^{3/2}} \tag{70}$$

Taking the partial derivative of Eqn (65) using the product rule for the second term:

$$-\frac{3}{2} \epsilon_{lnm} \epsilon_{ijk} c_j x_k (x_s x_s)^{-5/2} (2\delta_{sn} x_s) \delta_{im} e_l \tag{71}$$
$$= -\frac{3}{2} \epsilon_{lni} \epsilon_{ijk} c_j x_k (x_s x_s)^{-5/2} 2 x_n e_l \tag{72}$$
$$= -3 \epsilon_{lni} \epsilon_{jki} c_j x_k x_n (x_s x_s)^{-5/2} \underline{e}_l \tag{73}$$
$$(\delta_{lj} \delta_{nk} - \delta_{lk} \delta_{nj})(-3 c_j x_k x_n (x_s x_s)^{-5/2}) \underline{e}_l \tag{74}$$
$$= \delta_{lj} \delta_{nk} (-3 c_j x_k x_n (x_s x_s)^{-5/2}) \underline{e}_l - \delta_{lk} \delta_{nj} (-3 c_j x_k x_n (x_s x_s)^{-5/2}) \underline{e}_l \tag{75}$$
$$= -3 c_l x_k x_k (x_s x_s)^{-5/2} e_l + 3 c_n x_l x_n (x_s x_s)^{-5/2} \underline{e}_l \tag{76}$$
$$= -\frac{3c_l}{(x_s x_s)^{3/2}} + \frac{3 c_n x_n x_l}{(x_s x_s)^{5/2}} + \frac{2c_l}{(x_s x_s)^{3/2}} \tag{77}$$
$$= -\frac{c_l}{(x_s x_s)^{3/2}} + \frac{3 c_n x_n x_l}{(x_s x_s)^{5/2}} \tag{78}$$

Adding Eqns (62) and (78) together:

$$\nabla \phi + \mathrm{curl}(\mathbf{v}) = \mathbf{0} \tag{79}$$
$$= \frac{c_i}{(x_k x_k)^{3/2}} \underline{e}_i - \frac{3 c_j x_j x_i}{(x_k x_k)^{5/2}} \underline{e}_i - \frac{c_l}{(x_s x_s)^{3/2}} \underline{e}_l + \frac{3 c_n x_n x_l}{(x_s x_s)^{5/2}} \underline{e}_l \tag{80}$$
$$= 0 \tag{81}$$

$\square$

(b) $\Delta\phi = 0$

Proof.

$$\Delta\phi = 0 \tag{82}$$

$$\phi(\mathbf{x}) = \frac{\mathbf{c}\cdot\mathbf{x}}{x^3} \tag{83}$$

$$\nabla\phi = \left(\frac{c_i}{(x_k x_k)^{3/2}} - \frac{3c_j x_j x_i}{(x_k x_k)^{5/2}}\right)\underline{e}_i \tag{84}$$

$$\nabla(\nabla\phi) = \nabla\left(\frac{c_i}{(x_k x_k)^{3/2}}\right) - \nabla\left(\frac{3c_j x_j x_i}{(x_k x_k)^{5/2}}\right) \tag{85}$$

$$= \nabla(c_i(x_k x_k)^{-3/2}) - \nabla(3c_j x_j x_i (x_k x_k)^{-5/2}) \tag{86}$$

$$\frac{\partial c_i(x_k x_k)^{-3/2}}{\partial x_j} - \frac{\partial(3c_j x_j x_i(x_k x_k)^{-5/2})}{\partial x_j} \tag{87}$$

Simplifying further:

$$= -\frac{3}{2}c_i(x_k x_k)^{-5/2}(2\delta_{kj}x_k) - 3c_j(3)x_i(x_k x_k)^{-5/2} \tag{88}$$

$$= -3c_i(x_k x_k)^{-5/2}x_j = \frac{9c_j x_i}{(x_k x_k)^{5/2}} + \frac{3c_j x_i}{(x_k x_k)^{5/2}} \tag{89}$$

$$-\frac{3c_i x_j}{(x_k x_k)^{5/2}} = 3c_j x_j x_i\left(-\frac{5}{2}\right)(x_k x_k)^{-7/2}(2\delta_{kj}x_k) \tag{90}$$

$$= -3c_i x_j \tag{91}$$

$$(x_k x_k)^{5/2} = -\frac{15c_j(x_j x_j)x_i}{(x_k x_k)^{7/2}} \tag{92}$$

$$= \frac{-3c_i x_j}{(x_k x_k)^{5/2}} + \frac{3c_j x_i}{(x_k x_k)^{5/2}} \tag{93}$$

The index $i = j$ when applying the trace function upon a tensor, so the following holds true:

$$\mathrm{tr}\left(\frac{-3c_i x_j}{(x_k x_k)^{5/2}} + \frac{3c_j x_i}{(x_k x_k)^{5/2}}\right) \tag{94}$$

$$= 0 \tag{95}$$

□

(c) div($\mathbf{v}$)

$$\operatorname{div}(\mathbf{v}) = 0 \tag{96}$$

$$\operatorname{div}(\epsilon_{ijk} c_j x_k (x_s x_s)^{-3/2}) \underline{e}_i \tag{97}$$

$$= tr(\nabla \mathbf{v}) \tag{98}$$

$$= \frac{\partial \epsilon_{ijk} c_j x_k (x_s x_s^{-3/2})}{\partial x_j} \underline{e}_i \otimes \underline{e}_j \tag{99}$$

$$= \epsilon_{ijk} c_j \delta_{kj} (x_s x_s)^{-3/2} + \epsilon_{ijk} c_j x_k \left(-\frac{3}{2}\right) (x_s x_s)^{5/2} (2\delta_{sj} x_s) \tag{100}$$

$$= \underbrace{\cancel{\epsilon_{jkk} c_j (x_s x_s)^{-3/2}}}_{0} - \frac{3\epsilon_{ijk} c_j x_k x_j}{(x_s x_s)^{5/2}} \underline{e}_i \otimes \underline{e}_j \tag{101}$$

Taking the trace of Eqn (101):

$$\operatorname{tr}(\nabla \mathbf{v}) = -\frac{3\epsilon_{ijk} c_j x_k x_j}{(x_s x_s)^{5/2}} \delta_{ij} \tag{102}$$

$$= -\frac{3\epsilon_{iik} c_j x_j x_k}{(x_s x_s)^{5/2}} \tag{103}$$

$$= 0 \tag{104}$$

3. Plots, where $\mathbf{c} = (1, 1, 1)$:

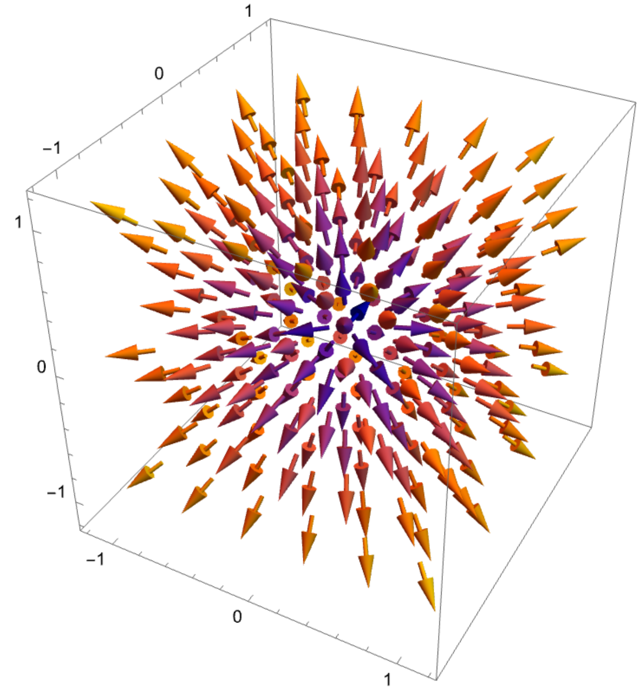

Figure 1: Plot of $\mathbf{v}(\mathbf{x})$ in Problem 1

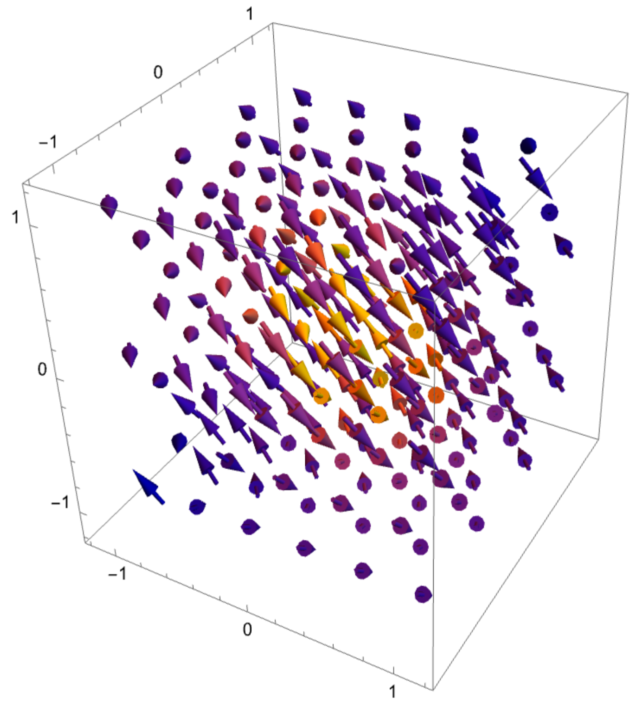

Figure 2: Plot of $\mathbf{v}(\mathbf{x})$ in Problem 2

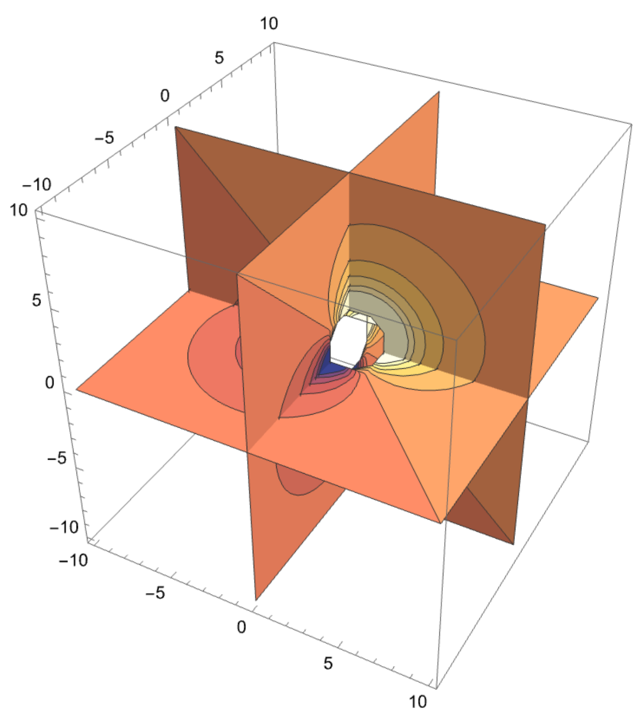

Figure 3: Plot of $\phi(\mathbf{x})$ in Problem 2