Problem 1.

We begin by creating the deformation mapping:

$$\mathbf{x}(\mathbf{p}) = \begin{bmatrix} (p_2 + r)\sin\left(\frac{p_1}{r}\right) \\ p_2 - (p_2 + r)\left(1 - \cos\left(\frac{p_1}{r}\right)\right) \\ p_3 \end{bmatrix} \tag{1}$$

The gradient of the deformation mapping is as follows:

$$\mathbf{F}(\mathbf{p}) = \begin{bmatrix} \frac{(p_2+r)\cos\left(\frac{p_1}{r}\right)}{r} & \sin\left(\frac{p_1}{r}\right) & 0 \\ -\frac{(p_2+r)\sin\left(\frac{p_1}{r}\right)}{r} & \cos\left(\frac{p_1}{r}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Problem 2. The determinant of the gradient is as follows:

$$\det(\mathbf{F}(\mathbf{p})) = \frac{(p_2 + r)\cos\left(\frac{p_1}{r}\right)}{r}\left(\cos\left(\frac{p_1}{r}\right)\right) - \sin\left(\frac{p_1}{r}\right)\left(-\frac{(p_2 + r)\sin\left(\frac{p_1}{r}\right)}{r}\right) \tag{3}$$

$$= \frac{p_2 + r}{r} \tag{4}$$

We evaluate the determinant of the deformation gradient at values at the mid-line, or when $p_2 = 0$:

$$\det(\mathbf{F})|_{p_2=0} = 1 \tag{5}$$

$$\tag{6}$$

A determinant of 1 results in no dilation transformation.

We evaluate the determinant of the deformation gradient at values above the mid-line, or when $p_2 > 0$:

$$\det \mathbf{F}|_{p_2=\frac{h}{2}} = \left(\frac{h}{2} + r\right)r \tag{7}$$

$$= \left(\frac{h}{2r} + 1\right) > 1 \tag{8}$$

h and r are always positive values, so the points above the midline undergo an expansion.

We evaluate the determinant of the deformation gradient at values below the mid-line, or when $p_2 < 0$:

$$\det \mathbf{F}|_{p_2=-\frac{h}{2}} = \frac{\left(r-\frac{h}{2}\right)}{r} < 1 \tag{9}$$

$$= 1 - \frac{h}{2r} < 1 \tag{10}$$

h and r are always positive values, so the points above the midline undergo a compression.

Problem 3. Evaluating the deformation gradient at point p of the surface S where p is:

$$\mathbf{p} = \left(l, \frac{h}{2}, 0\right) \tag{11}$$

$$\mathbf{F}(\mathbf{p}) = \begin{bmatrix} \frac{\left(\frac{h}{2}\right)\cos\left(\frac{l}{r}\right)}{r} & \sin\left(\frac{l}{r}\right) & 0 \\ \frac{\left(-\frac{h}{2}+r\right)\sin\left(\frac{l}{r}\right)}{r} & \cos\left(\frac{l}{r}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

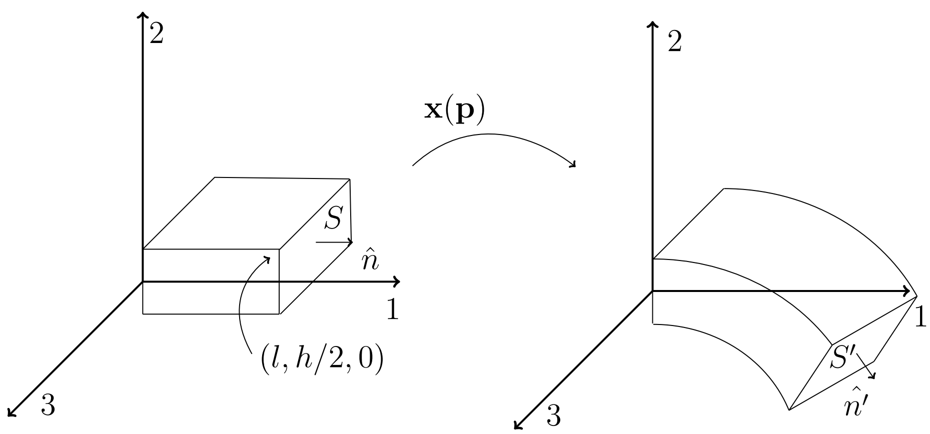

Figure 1: The surface normal vector $\hat{n}$ before and after the deformation mapping $\mathbf{x}(\mathbf{p})$

The unit normal vector of the surface S is defined as:

$$\hat{n} = (1, 0, 0) \tag{13}$$

Calculating the vector of the change in the surface area using Nanson's Formula:

$$\hat{n'} = \mathbf{F}\hat{e}_2 \times \mathbf{F}\hat{e}_3 = \det \mathbf{F} F^{-T} \hat{n} \tag{14}$$

$$\tag{15}$$

$$= \hat{n'} = \mathbf{F}\hat{e}_2 \times \mathbf{F}\hat{e}_3 = \det \mathbf{F} F^{-T} \hat{n} \tag{16}$$

$$\tag{17}$$

$$= \left(\frac{\frac{h}{2}+r}{r}\right)\left(\begin{bmatrix} \left(-\frac{h}{2}+r\right)\cos\left(\frac{l}{r}\right) & -\frac{\left(-\frac{h}{2}+r\right)\sin\left(\frac{l}{r}\right)}{r} & 0 \\ \sin\left(\frac{l}{r}\right) & \cos\left(\frac{l}{r}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} \frac{(h-2r)^2\cos\left(\frac{l}{r}\right)}{4r^2} \\ \frac{(-h+2r)\sin\left(\frac{l}{r}\right)}{2r} \\ 0 \end{bmatrix} \tag{19}$$

Calculating the magnitude of the vector of Eq (19), which is the change in the surface area of the unit vector $\hat{n}$:

$$|\mathbf{F}\hat{e}_2 \times \mathbf{F}\hat{e}_3| = \sqrt{\frac{(-h+2r)^2\sin^2\left(\frac{l}{r}\right)}{4r^2} + \frac{(h-2r)^4\cos^2\left(\frac{l}{r}\right)}{16r^4}} \tag{20}$$

Calculating the change of orientation of Eq (13) and (19):

$$\theta = \frac{\hat{n}\hat{\cdot}n'}{|\hat{n}||\hat{n'}|} \tag{21}$$

$$= \frac{(h-2r)^2\cos\left(\frac{l}{r}\right)}{r^2\sqrt{\frac{(h-2r)^2 4r^2\sin\left(\frac{l}{r}\right)+(h-2r)^2\cos^2\left(\frac{l}{r}\right)}{r^4}}} \tag{22}$$

Problem 4.

We can show that the deformation mapping Eqn (1) preserves planes by determining the change in surface area using the general point $\mathbf{p} = (a, b, c)$. Using Nanson's Formula (19) for the general point $\mathbf{p}$:

$$\hat{n'} = \mathbf{F}\hat{e}_2 \times \mathbf{F}\hat{e}_3 = \det \mathbf{F} F^{-T} \hat{n} \tag{23}$$

$$= \left(\frac{b+r}{r}\right) \left(\begin{bmatrix} (-b+r)\cos\left(\frac{a}{r}\right) & -\frac{(-b+r)\sin\left(\frac{a}{r}\right)}{r} & 0 \\ \sin\left(\frac{a}{r}\right) & \cos\left(\frac{a}{r}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} \frac{(b+r)^2 \cos\left(\frac{a}{r}\right)}{r^2} \\ \frac{(b+r)\sin\left(\frac{a}{r}\right)}{r} 0 \end{bmatrix} \tag{25}$$

Eqn (25) is only dependent upon two parameters, namely a and b. Therefore, points on a surface in the reference configuration will remain on the same plane on the deformed configuration.

Problem 5.

The mid-plane is described as the point $\mathbf{p} = (l, 0, l)$. Computing the Lagrangian strain tensor with $\mathbf{p}$ as the input:

$$\mathbf{E} = \frac{\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}}{2} \tag{26}$$

$$= \frac{\left(\begin{bmatrix} \cos\left(\frac{l}{r}\right) & \sin\left(\frac{l}{r}\right) & 0 \\ -\sin\left(\frac{l}{r}\right) & \cos\left(\frac{l}{r}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{l}{r}\right) & -\sin\left(\frac{l}{r}\right) & 0 \\ \sin\left(\frac{l}{r}\right) & \cos\left(\frac{l}{r}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)}{2} \tag{27}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Eqn (28) tells us that there is no strain on the mid-plane after the deformation of the reference configuration. This makes sense because the mid-plane can be treated as the neutral axis passing through the centroid of the reference configuration.

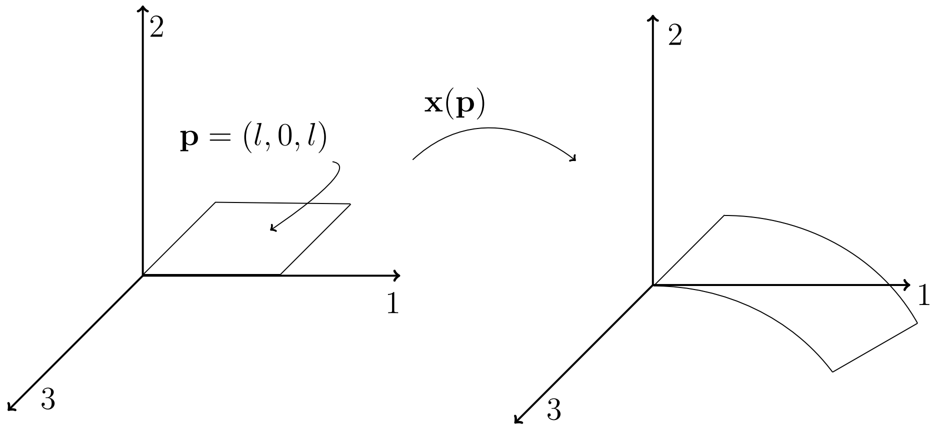

Figure 2: The mid-plane represented by point $\mathbf{p}$

Problem 6.

We will compute the infinitesimal strain tensor on the mid-plane $\mathbf{p} = (l, 0, l)$. First, we will compute the displacement field $\mathbf{u}(\mathbf{p}) = \mathbf{x}(\mathbf{p}) - \mathbf{p}$:

$$\mathbf{u}(\mathbf{p}) = \mathbf{x}(\mathbf{p}) - \mathbf{p} \tag{29}$$

$$= \begin{bmatrix} r\sin\left(\frac{l}{r}\right) \\ r\left(\cos\left(\frac{l}{r}\right) - 1\right) \\ l \end{bmatrix} - \begin{bmatrix} l \\ 0 \\ l \end{bmatrix} \tag{30}$$

$$= \begin{bmatrix} -l + r\sin\left(\frac{l}{r}\right) \\ -r\left(1 - \cos\left(\frac{l}{r}\right)\right) \\ 0 \end{bmatrix} \tag{31}$$

Computing the infinitesimal strain tensor:

$$\epsilon = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}}{2} \tag{32}$$

$$= \frac{\left(\begin{bmatrix} \cos\left(\frac{l}{r}\right) - 1 & \sin\left(\frac{l}{r}\right) & 0 \\ -\sin\left(\frac{l}{r}\right) & \cos\left(\frac{l}{r}\right) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \cos\left(\frac{l}{r}\right) - 1 & -\sin\left(\frac{l}{r}\right) & 0 \\ \sin\left(\frac{l}{r}\right) & \cos\left(\frac{l}{r}\right) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)}{2} \tag{33}$$

$$= \begin{bmatrix} \cos\left(\frac{l}{r}\right) - 1 & 0 & 0 \\ 0 & \cos\left(\frac{l}{r}\right) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

The infinitesimal strain tensor (34) is non-zero on the mid-plane, while the Lagrangian strain tensor (28) is zero on the mid-plane. The infinitesimal strain tensor shows that when $r \gg l$, then $\epsilon \to 0$, meaning that when the radius of curvature is much larger than the length of the surface along axis 1, the infinitesimal strain approaches zero. Therefore, the infinitesimal strain tensor is a good approximation of the Lagrangian strain tensor when $r \gg l$.