TAM 445 Continuum Mechanics - Spring 2024
Homework 10 - Constitutive relations

Due: May 01, 2024

1. (20 points) A material undergoes a homogeneous, time-dependent, simple shear motion with deformation gradient:

$$\boldsymbol{F} = \begin{bmatrix} 1 & \gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\gamma(t) = \dot{\gamma} t$ is the shear parameter and the shear rate $\dot{\gamma}$ is constant. Consider the following two cases:

a) (12 points) The material is elastic, incompressible and rubber-like with a Helmholtz free energy density given by $\psi = c_1(\operatorname{tr}\boldsymbol{B} - 3)$, where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the left Cauchy–Green deformation tensor, and c_1 is a material constant. A material of this type is called neo-Hookean.

   i) (6 points) For constant temperature conditions, show that the Cauchy stress for a neo-Hookean material is given by $\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{B}$, where p is the pressure, $\boldsymbol{I}$ is the identity tensor, $\mu = 2\rho_0 c_1$ is the shear modulus and ρ_0 is the reference mass density. The term $-p\boldsymbol{I}$ may seem strange to you. Since the material is incompressible it is possible to apply an arbitrary hydrostatic pressure without deforming the material. This means that the material's constitutive relation does not uniquely determine the hydrostatic part of the stress. The pressure is usually obtained as part of the solution to a particular boundary-value problem. You may assume that the Cauchy stress is of the form $\boldsymbol{T} = -p\boldsymbol{I} + \square$, and show that $\square = \mu\boldsymbol{B}$.

   ii) (6 points) Compute the Cauchy stress due to the imposed simple shear. Present your results as a 3×3 matrix of the components of $\boldsymbol{T}$. Explicitly show the time dependence.

b) (8 points) The material is a Newtonian fluid for which the Cauchy stress is given by $\boldsymbol{T}(\boldsymbol{x}, t) = -p\boldsymbol{I} + 2\mu\boldsymbol{D}_{\mathrm{s}}$, where μ is the shear viscosity and $\boldsymbol{D}_{\mathrm{s}} = \frac{1}{2}(\operatorname{grad}\boldsymbol{v}_{\mathrm{s}} + \operatorname{grad}\boldsymbol{v}_{\mathrm{s}}^{\mathrm{T}})$ is the stretch rate tensor. Compute the Cauchy stress due to the imposed simple shear motion.