TAM 445 Spring 2024 Problem Set #2
Jerich Lee. (02/8)

In the following problems, let $e_i \quad (i = 1, 2, 3)$ denote three orthonormal basis vectors for a Euclidean vector space V equipped with the standard inner product $u \cdot v = u_i v_i$.

Problem 1. *Show that the set of nine tensors $\{e_i \otimes e_j : i, j = 1, 2, 3\}$ forms a basis for the real vector space of second-order tensors.*

If e_i and e_j represents three orthonormal basis vectors each $\in V$, then this implies that there exists f_i and $g_i \in V'$ such that

$$f_i(e_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j; \end{cases} = \delta_{ij} \quad \text{and} \quad g_j(e_i) = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k; \end{cases} = \delta_{ji}$$

Suppose that there exists a list of scalars $\{a_{j,k}\}$ such that

$$a_{i,j}(e_i \otimes e_j) = 0 \tag{1}$$

We can also state the following:

$$(e_i \otimes e_j)(f_M \otimes g_N) = \begin{cases} 1, & \text{if } i = M; \\ 0, & \text{if } j = N; \end{cases} \tag{2}$$

Multiplying both sides of Equation 2 by 1, we get the following:

$$a_{i,j}(e_i \otimes e_j)(f_M \otimes g_N) = 0(f_M \otimes g_N)$$
$$a_{M,N} = 0$$

This shows that the scalar coefficient $a_{M,N}$ is equal to 0, which implies that the original list of tensors $e_i \otimes e_j$ is linearly independent, which implies that this set is also a basis.

Problem 2. *a) Show that an example that the dyadic product is not commutative. In other words,*

$$\boldsymbol{u} \otimes \boldsymbol{v} = \boldsymbol{v} \otimes \boldsymbol{u} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{1}$$

is not true.

We can provide a counterexample to prove this statement. Suppose that the dyadic product is commutative, and we define vectors

$$\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \underline{\boldsymbol{u}} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \underline{\boldsymbol{w}} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

The following is the definition of a tensor product:

$$(\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{v}})\underline{\boldsymbol{w}} = (\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}$$

$$(\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{v}})\underline{\boldsymbol{w}} = (\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{w}})\underline{\boldsymbol{u}} = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}7\\8\\9\end{bmatrix}$$

$$(\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{v}})\underline{\boldsymbol{w}} = (\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{w}})\underline{\boldsymbol{u}} = \left(\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}7\\8\\9\end{bmatrix}\right) \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}200\\250\\300\end{bmatrix}$$

$$(\underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{u}})\underline{\boldsymbol{w}} = (\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{w}})\underline{\boldsymbol{v}} = \left(\begin{bmatrix}4\\5\\6\end{bmatrix} \cdot \begin{bmatrix}7\\8\\9\end{bmatrix}\right) \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}122\\244\\366\end{bmatrix}$$

$$\begin{bmatrix}200\\250\\300\end{bmatrix} \neq \begin{bmatrix}122\\244\\366\end{bmatrix}$$

Therefore, the tensor product is not commutative.

Problem 3. *b) Consider a vector $\boldsymbol{n} \in V$ with $\|n\| = 1$. Such vectors are referred to as unit vectors. Examine how the tensor*

$$\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} \tag{2}$$

operates on vectors. Describe in words, the geometric significance of the above tensor.

The tensor $\boldsymbol{n} \otimes \boldsymbol{n}$ operating upon some vector $\boldsymbol{v}$ can be visualized in the framework of the definition of a tensor product:

$$(\boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{v} = (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}$$

The expression $(\boldsymbol{n} \cdot \boldsymbol{v})$ can be thought of as the magnitude of the projection of the vector $\boldsymbol{v}$ in the direction of the normal vector $\boldsymbol{n}$, in the direction of the normal vector $\boldsymbol{n}$. Therefore, when this value is subtracted from the identity matrix, the result is the projection of the vector v that is perpendicular to the normal vector $\boldsymbol{n}$.

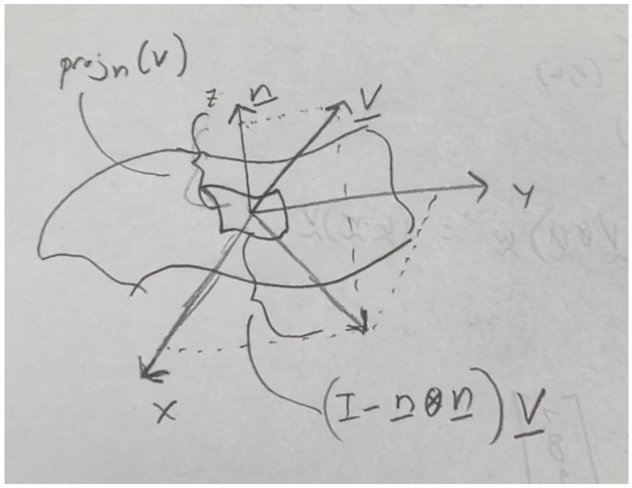

Figure 1:

Problem 4. *c) Let* e *and* $\boldsymbol{f}$ *be orthogonal unit vectors. Describe the geometric nature of the tensor* $e \otimes e + \boldsymbol{f} \otimes \boldsymbol{f}$

The geometric nature of $e \otimes e + f \otimes f$ has the following effect upon vector $\boldsymbol{v}$:

$$
\begin{aligned}
&(e \otimes e + \boldsymbol{f} \otimes \boldsymbol{f})\boldsymbol{v} \\
&\qquad = (e \otimes e)\boldsymbol{v} + (\boldsymbol{f} \otimes \boldsymbol{f})\boldsymbol{v} \\
&\qquad = (e \cdot \boldsymbol{v})e + (\boldsymbol{f} \cdot \boldsymbol{v})\boldsymbol{f}
\end{aligned}
$$

This has the geometric interpretation of capturing the projections of a vector $\boldsymbol{v}$ in the direction of e and $\boldsymbol{f}$, and removing the other components that the complete vector $\boldsymbol{v}$ may also contain.

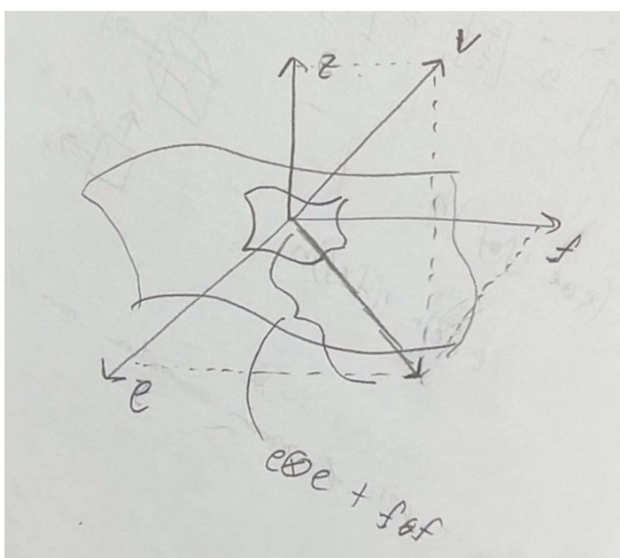

Figure 2: