Jerich Lee. (2024-04-26)

Problem 1.

1. (a)

$$\psi = c_1(\mathrm{tr}(\mathrm{B}) - 3), \mathrm{B} = \mathrm{F}\mathrm{F}^{\mathrm{T}} \tag{1}$$
$$\tag{2}$$

Suppose T is of the form:

$$\mathrm{T} = -p\mathrm{I} + \square \tag{3}$$

The First Piola-Kirchhoff stress tensor, where $\mathrm{J} = \det \mathrm{F}$ is:

$$\mathrm{P} = \mathrm{J}\mathrm{T}\mathrm{F}^{-\mathrm{T}} \tag{4}$$

Another form of the Piola-Kirchhoff stress tensor, where $\hat{\mathrm{W}} = \rho_0 \psi$ is computed by the following:

$$\mathrm{P}^e = \frac{\partial \hat{\mathrm{W}}(\mathrm{T}, \mathrm{F})}{\partial \mathrm{F}} \tag{5}$$

Converting Eq 1 to indicial notation:

$$\begin{aligned} \psi &= c_1(\mathrm{tr}(\mathrm{F}_{iJ} F_{Ji}) - 3) && (6)\\ &= c_1(\mathrm{F} \cdot \mathrm{F} - 3) && (7)\\ & c_1((\mathrm{F}\mathrm{F}^{\mathrm{T}})_{ii} - 3) && (8)\\ &= c_1(\mathrm{F}_{iJ}\mathrm{F}_{iJ} - 3) && (9)\\ & && (10) \end{aligned}$$

Applying Eq 4 to the above:

$$\begin{aligned} \mathrm{P}^e &= \frac{\partial \hat{\mathrm{W}}(\mathrm{T}, \mathrm{F})}{\partial \mathrm{F}} && (11)\\ &= c_1 \left(\mathrm{F}_{iJ} \frac{\partial \mathrm{F}_{iJ}}{\partial \mathrm{F}_{kL}} + \mathrm{F}_{iJ} \frac{\partial \mathrm{F}_{iJ}}{\partial \mathrm{F}_{kL}} \right) && (12)\\ &= c_1 \left(\mathrm{F}_{iJ}\delta_{ik}\delta_{JL} + \mathrm{F}_{iJ}\delta_{ik}\delta_{JL} \right) && (13)\\ & c_1 \rho_0 2\mathrm{F}_{kL} && (14) \end{aligned}$$

Substituting the above into the Cauchy Stress tensor:

$$
\begin{aligned}
\mathrm{T} &= \frac{1}{\mathrm{J}}\mathrm{P}\mathrm{F}^{\mathrm{T}} && (15)\\
&= c_1\rho_0 2\mathrm{F}\mathrm{F}^{\mathrm{T}} && (16)\\
&= \underbrace{c_1\rho_0 2}_{\mu}\,\mathrm{B} && (17)\\
&= \mu\mathrm{B} && (18)
\end{aligned}
$$

Therefore, we have shown that T is of the form:

$$
\mathrm{T} = -p\mathrm{I} + \mu\mathrm{B} \tag{19}
$$

(b) The Cauchy stress due to the imposed simple shear is the following:

$$
\begin{aligned}
\mathrm{T} &= -p\mathrm{I} + \mu\mathrm{B} && (20)\\
&= -p\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathrm{I}} + \mu\underbrace{\begin{bmatrix}1 & \gamma(t) & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ \gamma(t) & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathrm{B}} && (21)\\
&= \begin{bmatrix}\mu^2(\gamma^2(t)+1)-p & \mu^2\gamma(t) & 0\\ \mu^2\gamma(t) & \mu^2-p & 0\\ 0 & 0 & \mu^2-p\end{bmatrix} && (22)
\end{aligned}
$$

2. The Cauchy stress tensor due to the imposed simple shear motion is given as the following:

Computing the deformation map:

$$
\mathbf{x} = \begin{bmatrix}\mathbf{X}_1 + \mathbf{X}_2\gamma(t)\\ \mathbf{X}_2\\ \mathbf{X}_3\end{bmatrix} \tag{23}
$$

Computing $\mathbf{v}$:

$$
\mathbf{v} = \begin{bmatrix}\mathbf{X}_2\dot{\gamma}\\ 0\\ 0\end{bmatrix} \tag{24}
$$

Converting the above into the spatial description:

$$\mathbf{v}_s = \begin{bmatrix} \mathbf{x}_2\dot{\gamma} \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

Computing the gradient of $\mathbf{v}_s$:

$$\text{grad}\mathbf{v}_s = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

The stretch rate tensor is given by:

$$\mathrm{D}_s = \frac{1}{2}\text{grad}\mathbf{v}_s + \text{grad}\mathbf{v}_s^{\mathrm{T}} \tag{27}$$

$$= \frac{1}{2}\left(\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \tag{28}$$

Substituting the above into the definition of the Cauchy stress for a Newtonian fluid:

$$\mathrm{T} = -p\mathrm{I} + 2\mu\mathrm{D}_s \tag{29}$$

$$= -p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \left(\frac{1}{2}\text{grad}\mathbf{v}_s + \text{grad}\mathbf{v}_s^{\mathrm{T}} \right) \tag{30}$$

$$= -p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu\frac{1}{2} \left(\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \tag{31}$$

$$= \begin{bmatrix} -p & \mu\dot{\gamma} & 0 \\ \mu\dot{\gamma} & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \tag{32}$$