Problem 1. Determine the eigenvalues, eigenvector spaces (also known as characteristic spaces), and a spectral decomposition for each of the following tensors:

1. $$\mathbf{A} = \alpha\mathbf{I} + \beta\mathbf{m}\otimes\mathbf{m}$$

   Due to the following result from the definition of eigenvectors and eigenvalues:

   $$Ax = \lambda x$$

   A can act upon an arbitrary vector, namely m as shown below:

   $$\begin{aligned} Am &= (\alpha I + \beta m\otimes m)m \\ &= \alpha Im + \beta(m\otimes m)m \\ &= \alpha m + \beta(m\cdot m)m \\ &= \alpha m + \beta m \\ &= (\alpha+\beta)m \\ Am &= (\alpha+\beta)m \end{aligned}$$

   We know that there must be three eigenvalues because the dimension of the tensor A is 3. We can use the property than eigenvectors are mutually orthogonal to each other to determine the other eigenvectors of A. A prospective vector is n, which is orthogonal to m:

   $$\begin{aligned} An &= (\alpha I + \beta m\otimes m)n \\ &= (\alpha In + \beta(m\otimes m)n) \\ &= (\alpha n + \beta(m\cdot n)m) \\ &= \alpha n \\ An &= \alpha n \end{aligned}$$

   If m and n are both orthogonal to each other, then a vector $l = m\times n$ is also mutually orthogonal to m and n, and lives in the space of the third eigenvector of A.:

$$
\begin{aligned}
Al &= (\alpha I + \beta m \otimes m) \\
&= \alpha l + \beta (m \otimes m) l \\
&= \alpha l + \underbrace{\cancel{\beta (m \cdot l) m C}}_{0} \\
Al &= \alpha l
\end{aligned}
$$

The following lists the eigenvalues and eigenvectors of the tensor A:

$$\lambda_1 = \alpha + \beta,\ \ \lambda_2 = \alpha,\ \ \lambda_3 = \alpha$$

$$v_1 = m,\ \ v_2 = n,\ \ v_3 = m \times n$$

The spectral decomposition of a tensor $\in$ Psym is defined as the following:

$$S = \sum_{i=1}^{3} \lambda_i \underline{u}_i \otimes \underline{u}_i$$

Therefore, the tensor A can be written in terms of the eigenvalues and eigenvectors as:

$$
\begin{aligned}
A &= (\alpha + \beta) m \otimes m + \alpha n \otimes n + \alpha (m \times n) \otimes (m \times n) \\
&= \alpha m \otimes m + \beta m \otimes m + \alpha n \otimes n + \alpha (m \times n) \otimes (m \times n) \\
A &= \alpha (m \otimes m + n \otimes n + (m \times n) \otimes (m \times n)) + \beta (m \otimes m)
\end{aligned}
$$

2.

$$\mathbf{B} = \mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}$$

Applying the operation of tensor B upon the given unit vector m:

$$
\begin{aligned}
Bm &= (m \otimes n + n \otimes m) m) \\
&= (m \otimes n) m + (n \otimes m) m \\
&= \underbrace{\cancel{(n \cdot m) m C}}_{0} + \underbrace{(m \cdot m)}_{1} n \\
Bm &= n
\end{aligned}
$$

Applying the operation of tensor B upon the given unit vector n:

$$Bn = (m \otimes n + n \otimes m)n)$$

$$= (m \otimes n)n + (n \otimes m)n$$

$$= (n \cdot n)m + (m \cdot n)n$$

$$Bn = m$$

Applying the operation of tensor B upon the unit vector $m \times n$: The following lists the eigenvalues and eigenvectors of the tensor B:

$$\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 0$$

$$v_1 = n,\ v_2 = m,\ v_3 = 0$$

The spectral decomposition of a tensor $\in$ Psym is defined as the following:

$$B = \sum_{i=1}^{3} \lambda_i \underline{u}_i \otimes u_i$$

$$= n \otimes n + m \otimes m$$

Problem 2. Compute the polar decompositions $\mathbf{T} = \mathbf{RU} = \mathbf{VR}$ where $\mathbf{U}, \mathbf{V} \in$ Psym and $\mathbf{R} \in$ Orth, or a tensor $\mathbf{T}$ whose components are given by

$$\begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Write down the eigenvectors and eigenvalues of $\mathbf{U}$ and $\mathbf{V}$, and describe in words the geometric interpretation of the above decompositions.

$T^T T$ is found by multiplying the tensor T by its transpose:

$$T^T T = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To determine the eigenvalues and the eigenvalues pairs of symmetric tensor $T^T T$, the λ values of $\det(T - \lambda I) = 0$ are found.

$$\det(T - \lambda I) = 0$$

$$= \det \begin{bmatrix} 3-\lambda & \sqrt{3} & 0 \\ \sqrt{3} & 5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (3-\lambda)(5-\lambda)(1-\lambda) - \sqrt{3}(\sqrt{3}(1-\lambda))$$

$$= -\lambda^3 + 9\lambda^2 - 20\lambda + 12 = 0$$

Solving the polynomial equation, the following eigenvalues are determined:

$$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 6$$

The procedure that follows involves determining the eigenvectors for each eigenvalue. Substituting λ_1 into $T - \lambda I$:

$$\begin{bmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ v_{3,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The nullspace of $T - \lambda_1 I$ is the space of which the eigenvector that corresponds to eigenvalue λ_1 lives in:

$$2v_{1,1} + \sqrt{3}v_{1,2} = 0$$

$$\sqrt{3}v_{1,1} + 4v_{1,2} = 0$$

Setting our free variable equal to v_3, we get the following solution for eigenvalue $\lambda_1 = 1$:

$$v_1 = v_{1,3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This procedure is repeated for the remaining two eigenvalues, λ_2, and λ_3.

For $\lambda_2 = 2$:

$$\begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Computing the nullspace of the system above and setting the free variable equal to $v_{2,2}$:

$$v_{2,1} + \sqrt{3}v_{2,2} = 0$$

$$\sqrt{3}v_{2,1} + 3v_{2,2} = 0$$

$$-v_{2,3} = 0$$

$$\bar{v}_2 = \begin{bmatrix} -\sqrt{3}v_{2,2} \\ v_{2,2} \\ 0 \end{bmatrix}$$

$$= v_{2,2} \begin{bmatrix} -\sqrt{3} \\ 1 \\ 0 \end{bmatrix}$$

Substituting λ_3 into $\det(T - \lambda I)$:

$$\begin{bmatrix} -3 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} v_{3,1} \\ v_{3,2} \\ v_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Computing the nullspace of the above system and setting the free variable equal to $v_{3,2}$:

$$-3v_{3,1} + \sqrt{3}v_{3,2} = 0$$

$$\sqrt{3}v_{3,1} - v_{3,2} = 0$$

$$-5v_{3,3} = 0$$

Solving for the nullspace:

$$\bar{v}_3 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ 1 \\ 0 \end{bmatrix}$$

The eigenvalue and eigenvector pairs are shown below:

$$\left([1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}], [2, \begin{bmatrix} -\sqrt{3} \\ 1 \\ 0 \end{bmatrix}], \left[6, \begin{bmatrix} \frac{\sqrt{3}}{3} \\ 0 \\ 1 \end{bmatrix}\right] \right)$$

Our next goal is to determine the polar decomposition $T = RU = VR$, where $U, V \in$ Psym and $R \in$ Orth.

U is determined by the definition of the square root theorem, where $\underline{u}_i$ are normalized eigenvectors and λ_i are the eigenvalues that were found in earlier steps:

$$U = \sum_{i=1}^{3} \sqrt{\lambda_i}\underline{u}_i \otimes u_i$$

$$= \sqrt{1}\begin{bmatrix}0\\0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\0\\1\end{bmatrix} + \sqrt{2}\begin{bmatrix}-\sqrt{3}\\1\\0\end{bmatrix} \otimes \begin{bmatrix}-\sqrt{3}\\1\\0\end{bmatrix} + \sqrt{6}\begin{bmatrix}\frac{\sqrt{3}}{3}\\1\\0\end{bmatrix} \otimes \begin{bmatrix}\frac{\sqrt{3}}{3}\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + \sqrt{2}\begin{bmatrix}3&-\sqrt{3}&0\\-\sqrt{3}&1&0\\0&0&0\end{bmatrix} + \sqrt{6}\begin{bmatrix}\frac{1}{3}&\frac{\sqrt{3}}{3}&0\\\frac{\sqrt{3}}{3}&1&0\\0&0&0\end{bmatrix}$$

$$U = \begin{bmatrix}\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & -\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & 0\\ -\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{3\sqrt{6}}{4} & 0\\ 0&0&1\end{bmatrix}$$

Finding R is straightforward using:

$$R = TU^{-1}$$

The inverse of U can be found using the following:

$$U^{-1} = \frac{1}{\det(U)}\text{adj}(U)$$

$$= \begin{bmatrix}\frac{\sqrt{6}}{24}+\frac{3\sqrt{2}}{8} & -\frac{\sqrt{6}}{8}+\frac{\sqrt{2}}{8} & 0\\ -\frac{\sqrt{6}}{8}+\frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{8}+\frac{\sqrt{6}}{8} & 0\\ 0&0&1\end{bmatrix}$$

Computing R:

$$R = TU^{-1}$$

$$= \begin{bmatrix}\sqrt{3}&1&0\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{24}+\frac{3\sqrt{2}}{8} & -\frac{\sqrt{6}}{8}+\frac{\sqrt{2}}{8} & 0\\ -\frac{\sqrt{6}}{8}+\frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{8}+\frac{\sqrt{6}}{8} & 0\\ 0&0&1\end{bmatrix}$$

$$R = \begin{bmatrix}\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0\\ -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0\\ 0&0&1\end{bmatrix}$$

Using the definition $V = RUR^T$:

$$V = RUR^T$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & -\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & 0 \\ -\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{3\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} & 0 \\ -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & -\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The tensor T can be decomposed into the the polar decomposition $T = RU$ and $T = VR$ as the following:

Right Polar Decomposition:

$$T = RU$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & -\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & 0 \\ -\frac{\sqrt{6}}{4}+\frac{3\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{3\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Left Polar Decomposition:

$$T = VR$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & -\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The geometric interpretation of the polar decomposition of tensor T consists of a rotation and a dilation, where R is a rotation matrix, taking a unit square and operating upon the square as a rigid body rotation without deformation. The tensors U and V perform stretch operations upon a unit square.