TAM 445 Continuum Mechanics - Spring 2024
Homework 3 - Vector and tensor operations

Due: Feb 16, 2024

Notation: Uppercase bold letters denote second-order tensors, and lowercase bold letters denote vectors.

1. Prove the following identities

(a) $\boldsymbol{S}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{S}\boldsymbol{u}) \otimes \boldsymbol{v}$

(b) $(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{S} = \boldsymbol{u} \otimes (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{v})$

(c) $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}$

(d) $\boldsymbol{R} \cdot (\boldsymbol{S}\boldsymbol{T}) = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{R}) \cdot \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^{\mathrm{T}}) \cdot \boldsymbol{S}$

(e) $\boldsymbol{S} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{S}\boldsymbol{v}$

(f) $(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d})$

2. Prove the following identities related to the cross product of vectors

(a) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$

(b) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$

(c) $\det \boldsymbol{S} = \frac{\boldsymbol{S}\boldsymbol{u} \cdot (\boldsymbol{S}\boldsymbol{v} \times \boldsymbol{S}\boldsymbol{w})}{\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})}$

3. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ denote a collection of orthonormal vectors that form a basis for the three-dimensional Euclidean vector space. Let $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ represent an alternate basis of orthonormal vectors. The angles between the vectors are

	$\boldsymbol{e}'_1$	$\boldsymbol{e}'_2$	$\boldsymbol{e}'_3$
$\boldsymbol{e}_1$	$120°$	$120°$	$45°$
$\boldsymbol{e}_2$	$45°$	$135°$	$90°$
$\boldsymbol{e}_3$	$60°$	$60°$	$45°$

(a) Calculate the transformation matrix Λ whose entries are $\lambda_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$. Show that Λ is an orthogonal matrix, i.e. $\Lambda^{\mathrm{T}}\Lambda$ is the identity matrix.

(b) Let $\boldsymbol{v} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 - \boldsymbol{e}_3$. What are the components of $\boldsymbol{v}$ with respect to the two bases.

(c) Let $\boldsymbol{T} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$. Write the components of $\boldsymbol{T}$ with respect to the two bases in matrix form.

(d) Calculate the trace and determinant of the two matrices calculated in the previous step. What can you infer from your answer?

4. Calculate the orthogonal tensor that represents a rotation of $30°$ about an axis given by the unit vector $\boldsymbol{n} = \frac{1}{\sqrt{6}}\boldsymbol{e}_1 + \frac{1}{\sqrt{6}}\boldsymbol{e}_2 + \frac{2}{\sqrt{6}}\boldsymbol{e}_3$. Use the expression

$$\boldsymbol{R} = \boldsymbol{I} + (\sin\theta)\boldsymbol{W} + (1 - \cos\theta)\boldsymbol{W}^2,$$

where

$$\boldsymbol{W} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

is a skew tensor with the property $\boldsymbol{W}\boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{u}$. Make sure the matrix you report is an orthogonal matrix. Calculate $\boldsymbol{R}\boldsymbol{n}$. What can you infer from your answer?