$\rho \ddot{u} = \operatorname{Div} T$

Setting $\mathcal{E}$ - Euclidean point space w/ a Cartesian coordinate

$$\underset{\sim}{v}(\underset{\sim}{x}): \begin{bmatrix} v_1(x_1, x_2, x_3) \\ v_2(\;''\;) \\ v_3(\;''\;) \end{bmatrix}$$

Let $\mathcal{D} \subset \mathcal{E}$ be an open subset of $\mathcal{E}$.

At every point $p \in \mathcal{D}$, there exists an arbitrary small ball that lies entirely in $\mathcal{D}$.

$\underset{\sim}{x} \in \mathcal{D}$ (x_1, x_2, x_3)

$\varphi(\underset{\sim}{x})$ - Scalar field $(\varphi: \mathcal{D} \to \mathbb{R})$

$\underset{\sim}{v}(\underset{\sim}{x})$ - vector field $(\underset{\sim}{v}: \mathcal{D} \to V)$

$T(\underset{\sim}{x})$ - tensor field $(T: \mathcal{D} \to \mathrm{Lin})$

∇(Scalar field) = vector field

∇ (vector field) = tensor field

- Gradient of φ: $\nabla\phi(\underset{\sim}{x}) = \frac{\partial\phi}{\partial x_1}\underset{\sim}{e_1} + \frac{\partial\phi}{\partial x_2}\underset{\sim}{e_2} + \frac{\partial\phi}{\partial x_3}\underset{\sim}{e_3} \to \boxed{\frac{\partial\phi}{\partial x_i}} e_i = \quad \phi_{,i} = (\nabla\phi)_i$

- " " $\underset{\sim}{v}$: $\nabla v(\underset{\sim}{x}) = \frac{\partial v_i}{\partial x_j} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$

$(\nabla \underset{\sim}{v})_{ij}$

Divergence of a vector field $= \operatorname{div}(\underset{\sim}{v}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i}{\partial x_i} = v_{i,i}$

$\operatorname{div}(\underset{\sim}{v}) = \operatorname{tr}(\nabla \underset{\sim}{v})$

div(vector field) = scalar field

div (tensor field) = vector field

Curl of a vector field

- curl (vector field) - vector field
- $(\operatorname{curl} \underset{\sim}{v})_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} \underset{\sim}{e_i}$

$$\begin{bmatrix} e_i & e_j & e_k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Laplacian does not change the order of a field $\Delta \equiv \operatorname{div}(\nabla \cdot)$

dot is scalar, vector, tensor

$\Delta\varphi = \varphi_{,ii}$

$(\Delta \underset{\sim}{v}) = \boxed{v_{i,jj}} \underset{\sim}{e_i}$, $(\Delta \underset{\sim}{v})_i = v_{i,jj}$

$\frac{\partial^2 v_i}{\partial x_j^2}$

$\operatorname{div}(T) = \frac{\partial T_{ij}}{\partial x_j} \underset{\sim}{e_i}$; $(\operatorname{div} T) = T_{ij,j}$

Divergence, Gradients & Curl

$\mathcal{E}$ - Euclidean point space

$\mathcal{D} \subset \mathcal{E}$ - open subset of $\mathcal{E}$

$\varphi : \mathcal{D} \to \mathbb{R}$

$T : \mathcal{D} \to \text{Lin}$

$\nabla S \quad \frac{dS}{dx} = ?$

$$\lim_{\varepsilon \to 0} \left[S(x_0 + \varepsilon) - S(x) - \frac{dS}{dx}\Big|_{x_0} \varepsilon \right] = 0$$

$S : \mathbb{R}^3 \to \mathbb{R} \qquad \nabla S|_{x_0}$

$$\lim_{\varepsilon \to 0} \left[S(\underset{\sim}{x_0} + \varepsilon \underset{\sim}{u}) - S(\underset{\sim}{x_0}) - \Box(\varepsilon \underset{\sim}{u}) \right] = 0$$

unit vector

Linear transformation

input vector $\to$ output - scalar

$$\lim_{\varepsilon \to 0} \frac{S(x_0 + \varepsilon \underset{\sim}{u}) - S(x_0)}{\varepsilon} = \nabla S \cdot \underset{\sim}{u} \qquad \nabla S \hookrightarrow \frac{\partial S}{\partial x_1} \underset{\sim}{e_1} + \frac{\partial S}{\partial x_2} e_2 + \frac{\partial S}{\partial x_3} \underset{\sim}{e_3}$$

$\nabla S \cdot \underset{\sim}{u}$ - rate of change of S in the direction of u.

$$\nabla \phi = \frac{\partial \phi}{\partial x_1} \underset{\sim}{e_1}$$

$$\text{div } \underset{\sim}{V} = V_{i,i} = \text{tr}(\nabla \underset{\sim}{V})$$

$$\nabla \underset{\sim}{V} = V_{i,j}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$$

$$\Delta \Box := \text{div} \nabla \Box$$

$S : \mathbb{R} \to \mathbb{R}$

$$\frac{dS}{dx}(x_0) = \frac{S(x_0 + \varepsilon) - S(x_0)}{\varepsilon}$$

$S : \mathbb{R}^3 \to \mathbb{R}^3$

$$\lim_{\varepsilon \to 0} \left[\underset{\sim}{S}(\underset{\sim}{x_0} + \varepsilon \underset{\sim}{u}) - \underset{\sim}{S}(\underset{\sim}{x_0}) - \Box(\varepsilon \underset{\sim}{u}) \right] = \underset{\sim}{0}$$

input - vector

output - vector

$$\lim_{\varepsilon \to 0} \frac{\underset{\sim}{S}(x_0 + \varepsilon u) - \underset{\sim}{S}(x_0)}{\varepsilon} = \nabla \underset{\sim}{S}\, \underset{\sim}{u}$$

$\underset{\sim}{S} = \underset{\sim}{e} + \underset{\sim}{e_2}$

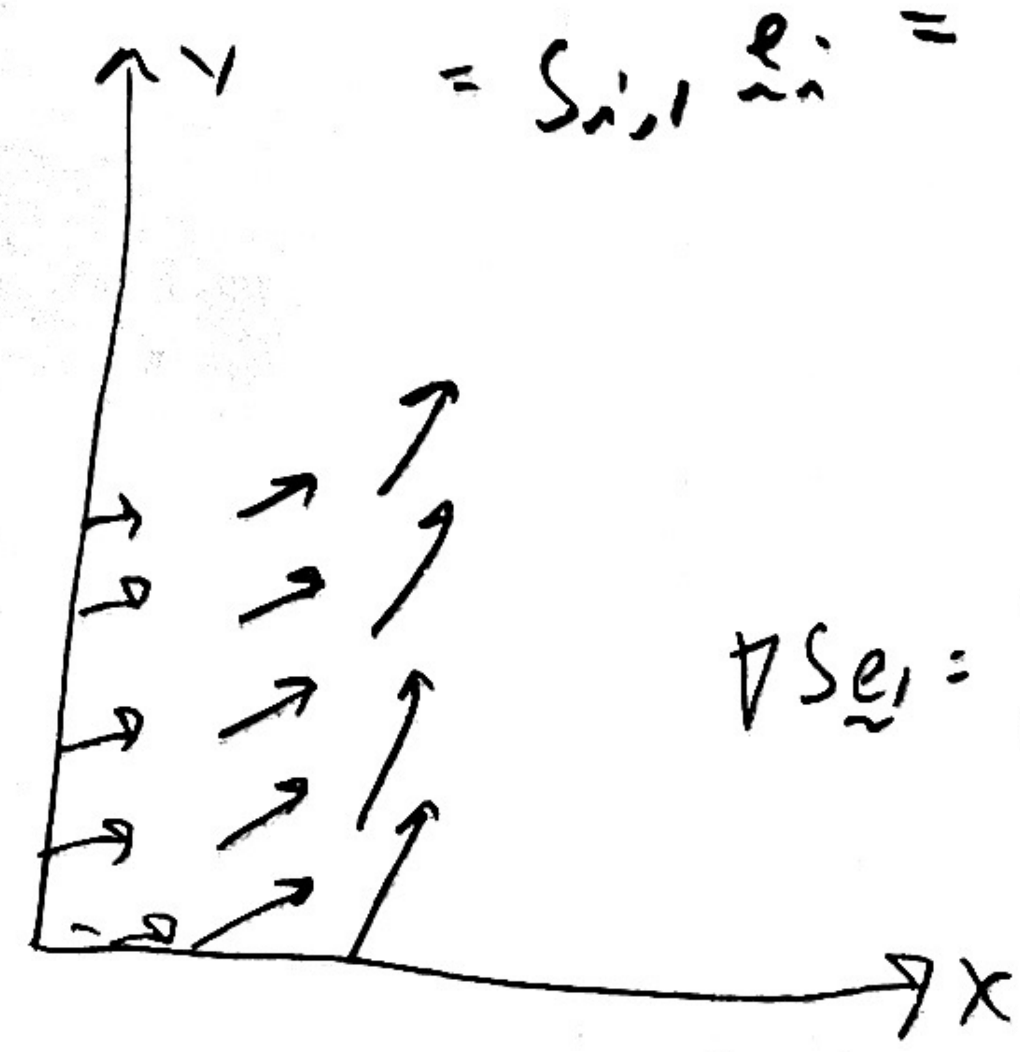
$u_0 \cdot \Box$

$\nabla \underset{\sim}{S}\, \underset{\sim}{u}$

$$(S_{i,j}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}) \underset{\sim}{e_1}$$

$$= (\underset{\sim}{e_j} \cdot \underset{\sim}{e_1}) S_{i,j}$$

$$= S_{i,1} \underset{\sim}{e_i} = \frac{\partial S_1}{\partial x_1} \underset{\sim}{e_1} + \frac{\partial S_2}{\partial x_1} \underset{\sim}{e_2} + \frac{\partial S_3}{\partial x_1} e_3$$

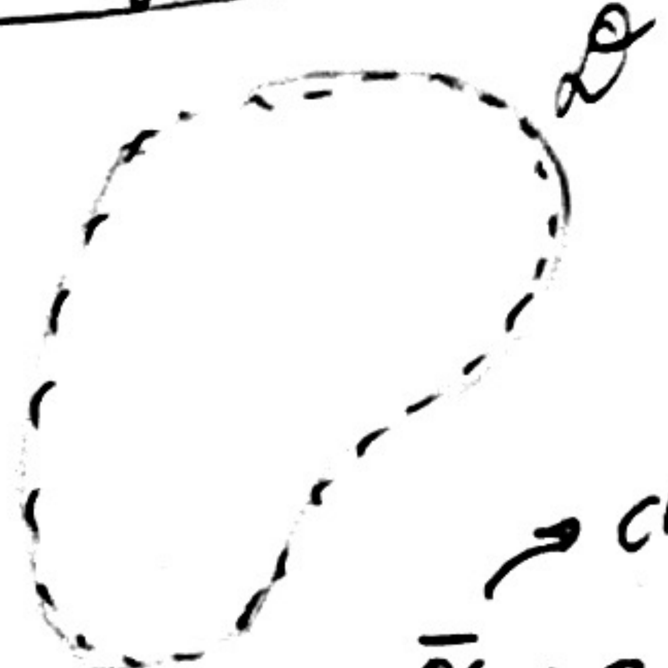


$$\nabla \underset{\sim}{S}\, \underset{\sim}{e_1} = \frac{\partial S_1}{\partial x_1} \underset{\sim}{e_1} + \frac{\partial S_2}{\partial x_1}$$

$$= 0 \underset{\sim}{e_1} + \underset{\sim}{e_2} = e_2$$

Divergence Theorem

$\bar{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$



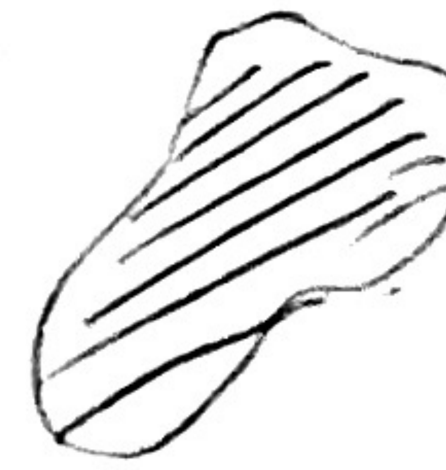
$(0,1)$

→ closure of $\mathcal{D}$

$\phi : \bar{\mathcal{D}} \to \mathbb{R}$
$\underset{\sim}{v} : \bar{\mathcal{D}} \to V$
$T : \mathcal{D} \to Lin$

$\int :$ $\quad \int_{\mathcal{D}} \nabla\phi \, dV \hat{=} \int_{\mathcal{D}} \sum \nabla\phi \, V$

$\int_S (\text{curl}\, \underset{\sim}{v}) \cdot \underset{\sim}{n} \, dA$



$= \int_{\partial S} \underset{\sim}{v} \cdot \underset{\sim}{t} \, dt$ — tangent

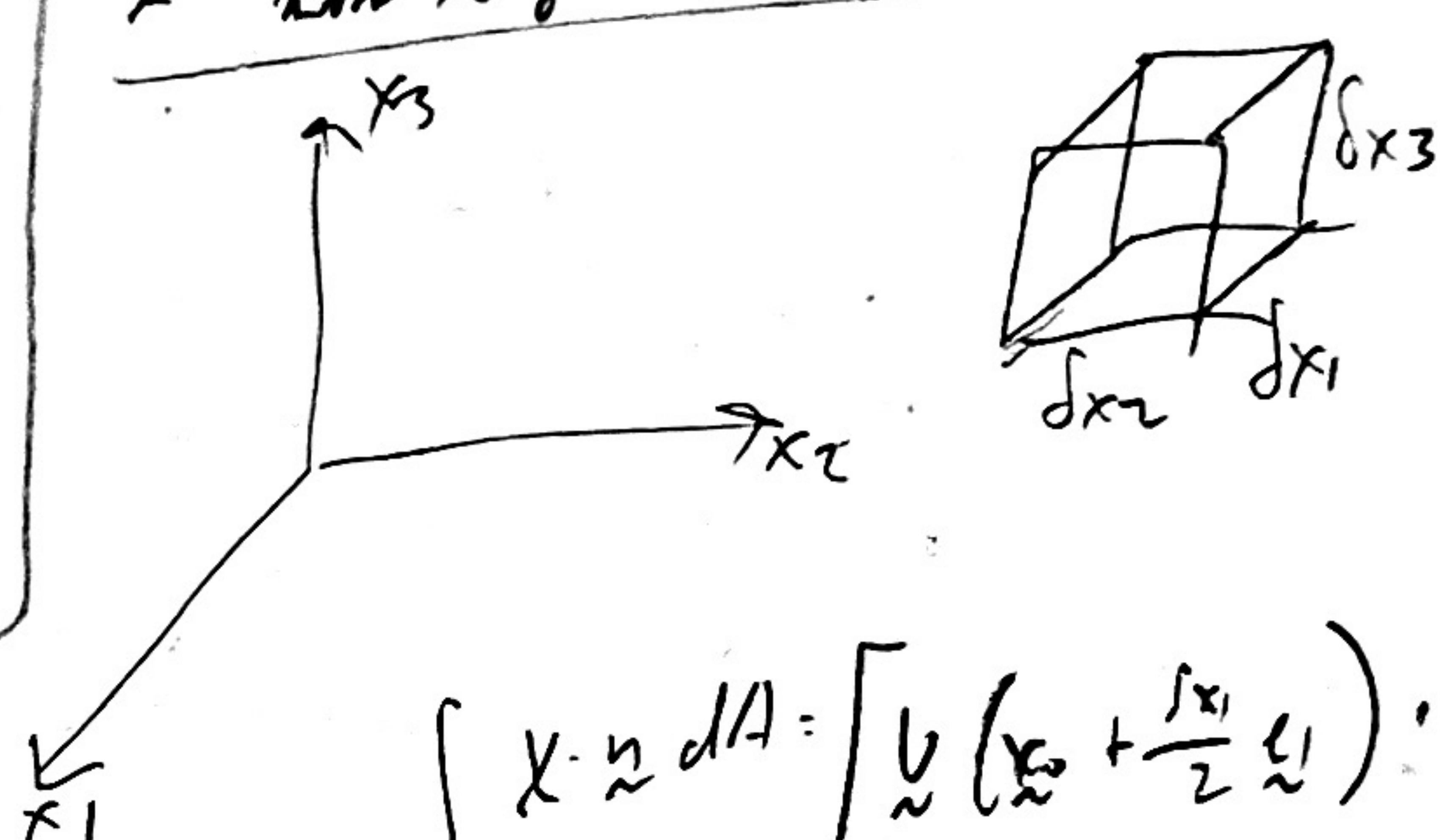
$\text{div}(\text{curl}\, \underset{\sim}{v}) = 0$
$\text{curl}(\nabla\phi) = 0$

$\int_{\partial\mathcal{D}} (\underset{\sim}{v} \cdot \underset{\sim}{n}) dA = \int_{\mathcal{D}} (\text{div}\, \underset{\sim}{v}) dV$

$= \int_{\mathcal{D}} \text{div}\, T \, dV$

$\int_{\partial\mathcal{D}} T n \, dA =$

A non-rigorous proof of the div Theorem



plus, but is negative

$\int_{\partial\mathcal{D}} \underset{\sim}{v} \cdot \underset{\sim}{n} \, dA = \left[\underset{\sim}{v}\left(\underset{\sim}{x}_0 + \frac{\delta x_1}{2} \underset{\sim}{e}_1\right) \cdot \underset{\sim}{e}_1 - \underset{\sim}{v}\left(\underset{\sim}{x}_0 - \frac{\delta x_1}{2} \underset{\sim}{e}_1\right) \cdot \underset{\sim}{e}_1 \right] (\delta x_2)(\ldots$

function

$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$

Identities: ϕ, ψ – scalar fields
$\underset{\sim}{u}, \underset{\sim}{v}$ – vector "
T – tensor "

c) $\nabla($

a) $\nabla(\phi\psi) = \nabla\phi\,\psi + \nabla\psi\,\phi$

$(\phi\psi)_{,i} = \phi_{,i}\psi + \phi\psi_{,i}$
$= (\nabla\phi\,\psi + \phi\,\nabla\psi)_i$

b) $\text{div}(\phi \underset{\sim}{u}) =$

b) $\underbrace{\text{div}(\phi \underset{\sim}{u})}_{\text{Scalar}} = \phi \, \text{div}\, \underset{\sim}{u} + \nabla\phi \cdot \underset{\sim}{u}$

c) $\text{curl}(\phi \underset{\sim}{u}) = \phi \, \text{curl}\, \underset{\sim}{u} + \nabla\phi \times \underset{\sim}{u}$

d) $\text{div}(\underset{\sim}{u} \times \underset{\sim}{v}) = \underset{\sim}{v}\,(\text{curl}\, \underset{\sim}{u})$

f) $\text{div}(T\underset{\sim}{v}) = \text{div}\, T^T \cdot \underset{\sim}{v} + T \cdot (\nabla \underset{\sim}{v})^T$

g) $\text{div}(\underset{\sim}{u} \otimes \underset{\sim}{v})$
$= \underset{\sim}{u} \, \text{div}\, \underset{\sim}{v} + (\nabla \underset{\sim}{u}) \underset{\sim}{v}$

h) $\nabla(u \cdot v) = \nabla u^T \underset{\sim}{v} + \nabla v^T \underset{\sim}{u}$

What is the nature of the operation of a vector onto a scalar field?