# TAM 445 Continuum Mechanics - Spring 2024
# Homework 6 - Deformation and the strain tensor

Due: Mar 8, 2024

Notation: Uppercase bold letters denote second-order tensors, lowercase bold letters denote vectors, and Greek letters denote scalars unless stated otherwise. All vectors fields are assumed to be smooth.

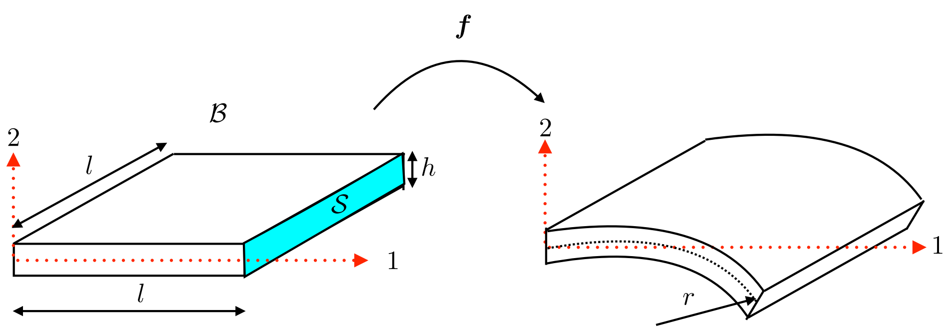
**1.** This problem relates to your experience of bending papers of various thicknesses. You may recall that as the thickness reduces, it gets progressively easier to bend a paper. In this problem, we will study the deformation of a paper of a given thickness $h$, and in-plane dimension $l$, i.e.

$$\mathcal{B} = [0, l] \times [-h/2, h/2] \times [0, l].$$

Consider a deformation of the form

$$\begin{aligned} x_1(\boldsymbol{p}) &= (p_2 + r) \sin \frac{p_1}{r}, \\ x_2(\boldsymbol{p}) &= p_2 - (p_2 + r) \left(1 - \cos \frac{p_1}{r}\right), \\ x_3(\boldsymbol{p}) &= p_3, \end{aligned}$$

which results in a curved surface with constant (independent of $\boldsymbol{p}$) radius of curvature $r$ as shown in the following figure




1. Calculate the deformation gradient $\boldsymbol{F}(\boldsymbol{p}) := \nabla \boldsymbol{x}$.

2. Calculate the determinant of $\boldsymbol{F}$, and show that the paper experiences expansion above the mid-plane and contraction below it, where the mid-plane is given by $\{\boldsymbol{p} \in \mathcal{B} : p_2 = 0\}$.

3. Determine the orientation of the outward unit normal to the surface $\mathcal{S} = \{(l, p_2, p_3) : p_3 \in (0, l), p_2 \in (-h/2, h/2)\}$ of the paper in the deformed configuration. What is the change in the cross-sectional area of $\mathcal{S}$?

4. Use the above result to show that cross-sectional planes in the reference configuration remain plane in the deformed configuration.

5. In the limit $h \to 0$, the body $\mathcal{B}$ may be modeled as a surface described by the mid-plane. Compute the Lagrangian strain tensor $\boldsymbol{E} = (\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I})/2$ on the mid-plane. Assuming stress (which we haven't covered yet) is proportional to strain, comment on the physical significance of your result.

6. Compute the infinitesimal strain tensor $\boldsymbol{\epsilon} = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})/2$, where $\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{x}(\boldsymbol{p}) - \boldsymbol{p}$ is the displacement field. How does $\boldsymbol{\epsilon}$ on the mid-plane compare to the Lagrangian strain on the mid-plane, and under what conditions is it a good approximation of the Lagrangian strain?