03·03·2020

Lecture 13

Some specialized deformations we discussed last class are translations & rotations which are special cases of homogeneous deformations (F is a constant field).

We know that translations and rotations preserve distances, i.e.

$$\|\underset{\sim}{p} - \underset{\sim}{q}\| = \|\underset{\sim}{f}(\underset{\sim}{p}) - \underset{\sim}{f}(\underset{\sim}{q})\| \quad \forall\ \underset{\sim}{p}, \underset{\sim}{q} \in \mathcal{B}, \qquad (1)$$

whenever f is a translation or a rotation. You may now ask if a converse statement is true:

Is every deformation that preserves distance, a rotation or a translation ?

- True.

Theorem: The following are equivalent

(a) f satisfies (1)

(b) There exists a constant $R \in \mathrm{Orth}^{+}$, and $\underset{\sim}{v} \in V$ such that

$$f(\underset{\sim}{p}) = \underset{\sim}{v} + R\underset{\sim}{p}.$$

We will now see how a deformation changes volumes of infinitesimal volume elements. An infinitesimal volume element at a point $p \in B$ is given by the three mutually orthogonal basis vectors $\varepsilon \mathbf{u}, \varepsilon \mathbf{v}, \varepsilon \mathbf{w}$:

$$\varepsilon \mathbf{u} \cdot (\varepsilon \mathbf{v} \times \varepsilon \mathbf{w}) = \varepsilon^3 \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \quad \text{— } (2)$$

(2) is transformed to

$$F(\varepsilon \mathbf{u}) \cdot \Big(F(\varepsilon \mathbf{v}) \times F(\varepsilon \mathbf{w}) \Big) = \varepsilon^3 F\mathbf{u} \cdot (F\mathbf{v} \times F\mathbf{w})$$

$$= \varepsilon^3 (\det F)\ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

Therefore, the volume is scaled by $\det F$.

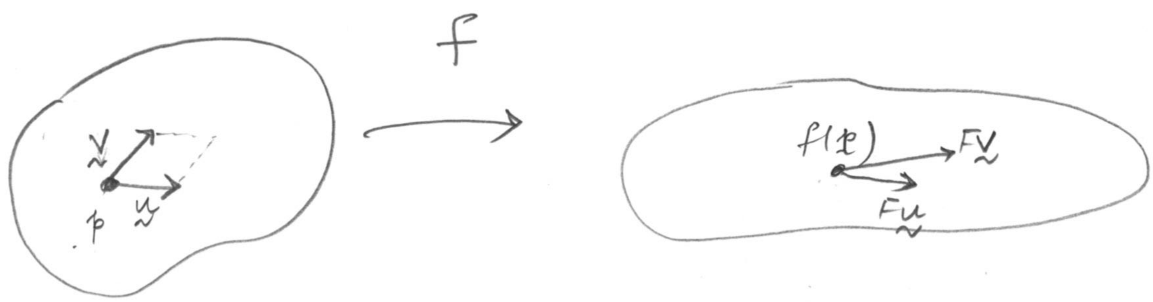
Example: Consider the shear deformation

$$f(\mathbf{p}) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$F(\mathbf{p}) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det F = 1 \implies$ Shear deformation is volume preserving.

Let us now see how areas change due to deformation.



Reference area : $\underset{\sim}{u} \times \underset{\sim}{v}$ $\longrightarrow$ A vector whose magnitude describes area and direction " the normal.

Deformed area : $F\underset{\sim}{u} \times F\underset{\sim}{v}$

$$(F\underset{\sim}{u} \times F\underset{\sim}{v})_i = \varepsilon_{ijk} (F\underset{\sim}{u})_j \, (F\underset{\sim}{v})_k$$

$$= \varepsilon_{ijk} \, F_{jL} \, u_L \, F_{kM} \, v_M \qquad — (3)$$

Multiplying (3) by F_{iN}, we have

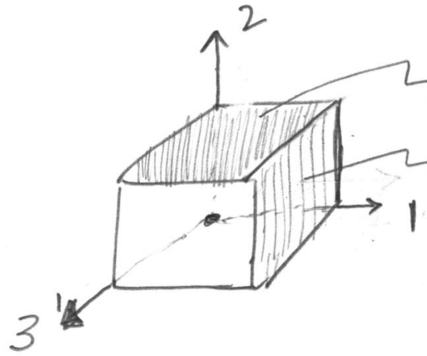
$$\left[F^T (F\underset{\sim}{u} \times F\underset{\sim}{v})\right]_N = \underbrace{\varepsilon_{ijk} \, F_{iN} F_{jL} \, F_{kM}}_{(\det F)\, \varepsilon_{NLM}} \, u_L \, v_M$$

$$= (\det F) (\underset{\sim}{u} \times \underset{\sim}{v})_N$$

$$\therefore \quad \boxed{F\underset{\sim}{u} \times F\underset{\sim}{v} = (\det F) \, F^{-T} (u \times v).} \qquad — (4)$$

Eq (4) is referred to as Nanson's formula, named after British mathematician Edward J. Nanson (1850-1936).

Example: Shear deformation $f(\underset{\sim}{p}) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$.



$\hat{\underset{\sim}{n}} = \underset{\sim}{e}_3 \times \underset{\sim}{e}_1$

$\hat{\underset{\sim}{m}} = \underset{\sim}{e}_2 \times \underset{\sim}{e}_3$ (Area in the reference configuration)

$$F\underset{\sim}{e}_3 \times F\underset{\sim}{e}_1 = (\det F)\, F^{-T} \hat{\underset{\sim}{n}}$$

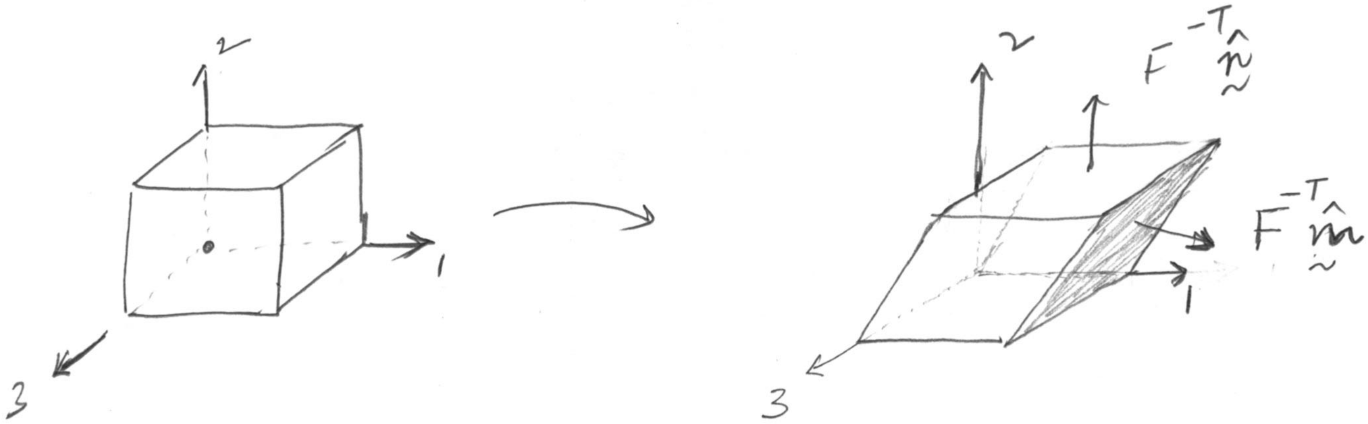
$$= F^{-T} \hat{n} \qquad (\because \text{shear is volume preserving})$$

$$= \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-T} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$F\underset{\sim}{e}_2 \times F\underset{\sim}{e}_3 = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\gamma \\ 0 \end{bmatrix}$$



The unit area described by $\hat{\underset{\sim}{m}}$ transform to an area of magnitude $\sqrt{1+\gamma^2}$, and direction along $\begin{bmatrix} 1 \\ -\gamma \\ 0 \end{bmatrix}$.

<u>Material and spatial tensor fields</u>

A deformation maps the reference configuration (also referred to as a body) to a deformed configuration. Any property of the body may now be described using either the reference configuration or the deformed configuration. Consider a scalar field g that describes temperature.

$$g : f(\mathcal{B}) \to \mathbb{R} \quad \text{vs} \quad \tilde{g} : \mathcal{B} \to \mathbb{R}$$

$$g(\underset{\sim}{x}) \quad \text{vs} \quad \tilde{g}(\underset{\sim}{p})$$

<u>Relation</u>: $g(f(\underset{\sim}{p})) = \tilde{g}(\underset{\sim}{p})$

Spatial / Eulerian description. — Material / reference / Lagrangian description