TAM 445 Spring 2024 Problem Set #1
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Problem 1. *Expand the following indicial expressions (all indices range from 1 to 3). Indicate the rank (number of free indices) and the number of resulting expressions.*

a) $a_i b_i$

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\begin{aligned} \text{rank} &= 0 \\ \text{num. exp.} &= 1 \end{aligned}$$

b) $a_i b_j$

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$$\begin{aligned} \text{rank} &= 2 \\ \text{num. exp.} &= 9 \end{aligned}$$

c) $\sigma_{ik} n_k$

$$\begin{aligned} \sigma_{ik} n_k &= \begin{bmatrix} \sigma_{1k} n_k \\ \sigma_{2k} n_k \\ \sigma_{3k} n_k \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11} n_k + \sigma_{12} n_2 + \sigma_{13} n_3 \\ \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{rank} &= 1 \\ \text{num. exp.} &= 3 \end{aligned}$$

d) $A_{ij} x_i x_j$ ($\boldsymbol{A}$ is symmetric, i.e. $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$)

$$A_{ij} x_i x_j = A_{11} x_1^2 + A_{22} x_2^2 + A_{33} x_3^2 + 2A_{12} x_1 x_2 + 2A_{13} x_1 x_3 + 2A_{23} x_2 x_3$$

$$\begin{aligned} \text{rank} &= 0 \\ \text{num. exp.} &= 1 \end{aligned}$$

e) $\frac{\partial u_i}{\partial z_k}\frac{\partial z_k}{\partial x_j}$

$$\frac{\partial u_i}{\partial z_k}\frac{\partial z_k}{\partial x_j} = \begin{bmatrix} \frac{\partial u_1}{\partial z_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_1}{\partial z_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial u_1}{\partial z_3}\frac{\partial z_3}{\partial x_1} & \frac{\partial u_1}{\partial z_1}\frac{\partial z_1}{\partial x_2} + \frac{\partial u_1}{\partial z_2}\frac{\partial z_2}{\partial x_2} + \frac{\partial u_1}{\partial z_3}\frac{\partial z_3}{\partial x_2} & \frac{\partial u_1}{\partial z_1}\frac{\partial z_1}{\partial x_3} + \frac{\partial u_1}{\partial z_2}\frac{\partial z_2}{\partial x_3} + \frac{\partial u_1}{\partial z_3}\frac{\partial z_3}{\partial x_3} \\ \frac{\partial u_2}{\partial z_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_2}{\partial z_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial u_2}{\partial z_3}\frac{\partial z_3}{\partial x_1} & \frac{\partial u_2}{\partial z_1}\frac{\partial z_1}{\partial x_2} + \frac{\partial u_2}{\partial z_2}\frac{\partial z_2}{\partial x_2} + \frac{\partial u_1}{\partial z_3}\frac{\partial z_3}{\partial x_2} & \frac{\partial u_2}{\partial z_1}\frac{\partial z_1}{\partial x_3} + \frac{\partial u_2}{\partial z_2}\frac{\partial z_2}{\partial x_3} + \frac{\partial u_2}{\partial z_3}\frac{\partial z_3}{\partial x_3} \\ \frac{\partial u_3}{\partial z_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_3}{\partial z_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial u_3}{\partial z_3}\frac{\partial z_3}{\partial x_1} & \frac{\partial u_3}{\partial z_1}\frac{\partial z_1}{\partial x_2} + \frac{\partial u_3}{\partial z_2}\frac{\partial z_2}{\partial x_2} + \frac{\partial u_3}{\partial z_3}\frac{\partial z_3}{\partial x_2} & \frac{\partial u_3}{\partial z_1}\frac{\partial z_1}{\partial x_3} + \frac{\partial u_3}{\partial z_2}\frac{\partial z_2}{\partial x_3} + \frac{\partial u_3}{\partial z_3}\frac{\partial z_3}{\partial x_3} \end{bmatrix}$$

$$\begin{aligned} \text{rank} &= 2 \\ \text{num. exp.} &= 9 \end{aligned}$$

f) $\sigma_{ij,j} + \rho b_i = \rho a_i$, where $\sigma_{ij,j} := \frac{\partial \sigma_{ij}}{\partial x_j}$

$$\begin{aligned} \partial_j \sigma_{ij} + \rho b_i \rho a_i &= \rho a_i \\ &= \begin{bmatrix} (\partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13}) + \rho b_1 \\ (\partial_1\sigma_{21} + \partial_2\sigma_{22} + \partial_3\sigma_{23}) + \rho b_2 \\ (\partial_1\sigma_{31} + \sigma_2\sigma_{32} + \partial_3\sigma_{33}) + \rho b_3 \end{bmatrix} = \begin{bmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{rank} &= 1 \\ \text{num. exp.} &= 3 \end{aligned}$$

Problem 2. *Simplify the following indicial expressions as much as possible (all indices range from 1 to 3).*

a) $\delta_{mm}\delta_{nn}$

$$\begin{aligned} \delta_{mm}\delta_{nn} &= (3)(3) \\ &= 9 \end{aligned}$$

b) $X_I \delta_{IK} \delta_{JK}$

$$\begin{aligned} X_I \delta_{IK} \delta_{JK} &= X_K \delta_{JK} \\ &= X_J \end{aligned}$$

c) $B_{ij}\delta_{ij}$ ($\boldsymbol{B}$ is anti-symmetric, i.e. $\boldsymbol{B} = -\boldsymbol{B}^{\mathrm{T}}$)

$$\begin{aligned} B_{ij}\delta_{ij} &= -B_{ji}\delta_{ij} \\ &= B_{kk} \end{aligned}$$

d) $[A_{ij}B_{jk} - 2A_{im}B_{mk}]\delta_{ik}$

$$\begin{aligned} [A_{ij}B_{jk} - 2A_{im}B_{mk}]\delta_{ik} &= A_{ij}B_{jk}\delta_{ik} - 2A_{im}B_{mk}\delta_{ik} \\ &= A_{ij}B_{ji} - 2A_{im}B_{mi} \\ &= A_{ij}B_{jk} - 2A_{kj}B_{ji} \\ &= -A_{ij}B_{ji} \end{aligned}$$

e) Substitute $A_{ij} = B_{ik}C_{kj}$ into $\phi = A_{mk}C_{mk}$

$$\begin{aligned}\phi &= A_{ij}C_{ij}\\ &= (B_{ik}C_{kj})C_{ij}\end{aligned}$$

f) $\epsilon_{ijk}a_ia_ja_k$

$$\begin{aligned}
\epsilon_{ijk}a_ia_ja_k =\\
\epsilon_{111}a_1a_1a_1 + \epsilon_{112}a_1a_1a_2 + \epsilon_{113}a_1a_1a_3\\
+\epsilon_{121}a_1a_2a_1 + \epsilon_{122}a_1a_2a_2 + \epsilon_{123}a_1a_2a_3\\
+\epsilon_{131}a_1a_3a_1 + \epsilon_{132}a_1a_3a_2 + \epsilon_{133}a_1a_3a_3\\
+\epsilon_{211}a_2a_1a_1 + \epsilon_{212}a_2a_1a_2 + \epsilon_{213}a_2a_1a_3\\
+\epsilon_{221}a_2a_2a_1 + \epsilon_{222}a_2a_2a_2 + \epsilon_{223}a_2a_2a_3\\
+\epsilon_{231}a_2a_3a_1 + \epsilon_{232}a_2a_3a_2 + \epsilon_{233}a_2a_3a_3\\
+\epsilon_{311}a_3a_1a_1 + \epsilon_{312}a_3a_1a_2 + \epsilon_{313}a_3a_1a_3\\
+\epsilon_{321}a_3a_2a_1 + \epsilon_{322}a_3a_2a_2 + \epsilon_{323}a_3a_2a_3\\
+\epsilon_{331}a_3a_3a_1 + \epsilon_{332}a_3a_3a_2 + \epsilon_{333}a_3a_3a_3\\
= 0
\end{aligned}$$

g) $(x_mx_mx_iA_{ij})_{,k}$, where $\square, k$ denotes derivative with respect to x_k.

$$\begin{aligned}
(x_mx_mx_iA_{ij})_{,k} &= \partial_k x_mx_mx_iA_{ij}\\
&= \frac{\partial(x_mx_mx_iA_{ij})}{\partial x_k}\\
&= \delta_{mk}x_mx_iA_{ij} + \delta_{mk}x_mx_iA_{ij} + \delta_{ik}x_mx_mA_{ij} + \frac{\partial A_{ij}}{x_k}x_mx_mx_i\\
&= 2x_kx_iA_{ij} + x_mx_mA_{kj} + \frac{\partial A_{ij}}{x_k}x_mx_mx_i
\end{aligned}$$

Problem 3. *Write out the following expressions in indicial notation whenever possible.*

a) $A_{11} + A_{22} + A_{33}$

$$A_{11} + A_{22} + A_{33} = A_{ii}$$

b) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ where $\boldsymbol{A}$ is a 3×3 matrix

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = A_{pi}A_{pj}$$

c) $A_{11}^2 + A_{22}^2 + A_{33}^2$

Not possible.

d) $B_{i1}\frac{\partial c_1}{\partial x_j} + B_{i2}\frac{\partial c_2}{\partial x_j} + B_{i3}\frac{\partial c_3}{\partial x_j}$

$$B_{i1}\frac{\partial c_1}{\partial x_j} + B_{i2}\frac{\partial c_2}{\partial x_j} + B_{i3}\frac{\partial c_3}{\partial x_j} = B_{ik}\frac{\partial c_k}{\partial x_j}$$

e) $(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$

$$(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) = u_i u_i v_k v_k$$

f) $A_{11} = B_{11}C_{11} + B_{12}C_{21}$

$$A_{ij} = B_{ik}C_{kj}$$

Problem 4. *Show that* $\frac{\partial A_{ip}^{-1}}{\partial A_{mn}} = -A_{im}^{-1}A_{np}^{-1}$, *where* $\boldsymbol{A}$ *is a square matrix.*

$$A_{kj}A_{ik}^{-1} = \delta_{ij}$$

$$\frac{\partial A_{kj}A_{ik}^{-1}}{\partial A_{mn}} = \frac{\partial \delta_{ij}}{\partial A_{mn}} = 0$$

$$\frac{\partial A_{kj}}{\partial A_{mn}}A_{ik}^{-1} + A_{kj}\frac{\partial A_{ik}^{-1}}{\partial A_{mn}} = 0$$

$$A_{kj}\frac{\partial A_{ik}^{-1}}{\partial A_{mn}} = -\frac{\partial A_{kj}}{\partial A_{mn}}A_{ik}^{-1}$$

$$A_{kj}\frac{\partial A_{ik}^{-1}}{\partial A_{mn}} = -\delta_{km}\delta_{jn}A_{ik}^{-1}$$

$$(A_{jp}^{-1})A_{kj}\frac{\partial A_{ik}^{-1}}{\partial A_{mn}} = -\delta_{km}\delta_{jn}A_{ik}^{-1}(A_{jp}^{-1})$$

$$\delta_{pk}\frac{\partial A_{ik}^{-1}}{\partial A_{mn}} = -\delta_{km}\delta_{jn}A_{ik}^{-1}(A_{jp}^{-1})$$

$$\frac{\partial A_{ip}^{-1}}{\partial A_{mn}} = -\delta_{km}\delta_{jn}A_{ik}^{-1}(A_{jp}^{-1})$$

$$\frac{\partial A_{ip}^{-1}}{\partial A_{mn}} = -\delta_{jn}A_{im}^{-1}(A_{jp}^{-1})$$

$$\frac{\partial A_{ip}^{-1}}{\partial A_{mn}} = -A_{im}^{-1}(A_{ip}^{-1})$$