Square Root Theorem: $S \in Psym$. Then there exists a unique $U \in Psym$ s.t. $U^2 = S$, where $U := \sqrt{S}$.

What does it mean for a matrix to be PDM vs NDM? What about particular matrix for square root of a NDM (negative definite matrix)?

Proof: (Existence)

$$S = \sum_{i=1}^{3} \lambda_i \, \underset{\sim}{u}_i \otimes \underset{\sim}{u}_i, \quad \lambda_i > 0$$

orthonormal

Let $$U = \sum_{i=1}^{3} \sqrt{\lambda_i} \, \underset{\sim}{u}_i \otimes \underset{\sim}{u}_i$$

Uniqueness: Assume there exists a $V \in Psym$ that satisfies $V^2 = S$

$$(S - \lambda_i I)\underset{\sim}{u}_i = \underset{\sim}{0} \quad (\text{no sum})$$

$$(V^2 - \lambda_i I)\underset{\sim}{u}_i = \underset{\sim}{0} \quad (\text{"})$$

$$(V - \sqrt{\lambda_i} I)\underbrace{\boxed{(V + \sqrt{\lambda_i} I)\underset{\sim}{u}_i}}_{v_i} = \underset{\sim}{0} \quad (\text{"})$$

$\Rightarrow \underset{\sim}{v}_i$ is the eigenvector of V with eigenvalue $\sqrt{\lambda_i}$

Polar Decomposition:

$F \in Inv$ (set of all invertible tensor $\det \neq 0$) there exist unique $U, V \in Psym$ and $R \in Orth$

s.t. $F = RU = VR$

Moreover, $\det R = 1$ or -1

According as $\det F > 0$ or < 0, and

$$U = \sqrt{F^T F}, \quad V = \sqrt{F F^T}$$

Symmetric — but why? Confirm to me research on Transposes

$$F^T F \underset{\sim}{v} \cdot \underset{\sim}{v} \overset{?}{>} 0$$

$$= F\underset{\sim}{v} \cdot F\underset{\sim}{v} > 0 \text{ provided } F\underset{\sim}{v} \neq \underset{\sim}{0}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$