02.12.2020

Lecture 8

Definition: The cross product of two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is a vector, denoted as $\underset{\sim}{u} \times \underset{\sim}{v}$, whose components are given by

$$(\underset{\sim}{u} \times \underset{\sim}{v})_i := \varepsilon_{ijk} u_j v_k . \quad \text{——} \quad (1)$$

Properties of cross product:

(i) $\underset{\sim}{v} \times \underset{\sim}{u} = - \underset{\sim}{u} \times \underset{\sim}{v}$ (Anticommutativity)

(ii) $\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \times \underset{\sim}{v} + \underset{\sim}{u} \times \underset{\sim}{w}$ (distributivity with respect to addition)

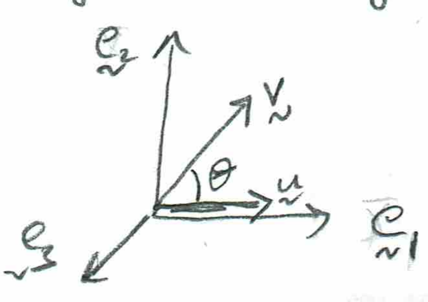
(iii) $(\lambda \underset{\sim}{u}) \times \underset{\sim}{v} = \lambda (\underset{\sim}{u} \times \underset{\sim}{v})$

$\boxed{\text{Cross product is not associative}}$!

(iv) $(\underset{\sim}{u} \times \underset{\sim}{v}) \times \underset{\sim}{w} = (\underset{\sim}{u} \cdot \underset{\sim}{w}) \underset{\sim}{v} - (\underset{\sim}{v} \cdot \underset{\sim}{w}) \underset{\sim}{u}$

$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = (\underset{\sim}{u} \cdot \underset{\sim}{w}) \underset{\sim}{v} - (\underset{\sim}{u} \cdot \underset{\sim}{v}) \underset{\sim}{w}$

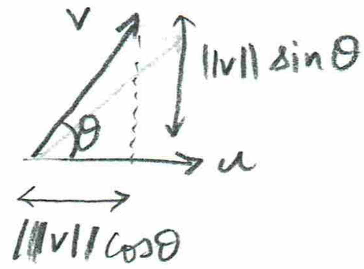
What is the geometric significance of cross product?



Assume $u = u_1 e_1$, $v = v_1 e_1 + v_2 e_2$. Then

$$u \times v = \varepsilon_{ijk} u_j v_k e_i$$

$$= u_1 v_2 \, e_3$$



$$= \|u\| \, \|v\| \sin\theta \; e_3 \quad \to (2)$$

Notice if we had chosen our basis such that e_3 was pointing into the plane of the paper, then (2) would have resulted in a vector that is antiparallel to the vector we currently obtained. This suggests that the definition (1) assumes a "handedness" of the basis. In particular, definition in (1) assumes the basis is <u>right-handed</u>, which will always be the case for us from now onwards.

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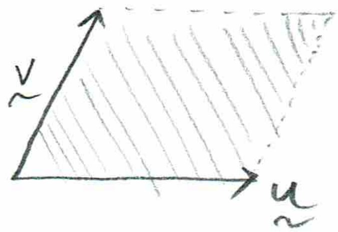
definition (1) is sometimes also stated as

$$u \times v = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Continuing with the geometrical significance we note that the norm of $\underset{\sim}{u} \times \underset{\sim}{v}$ is given by

$$\|\underset{\sim}{u} \times \underset{\sim}{v}\| = \|\underset{\sim}{u}\| \, \|\underset{\sim}{v}\| \sin\theta$$

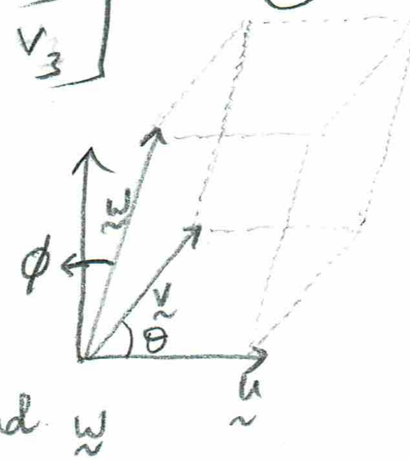
which is equal to the area of the parallelogram:



Now, consider the dot product

$$(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} = \varepsilon_{ijk} \, u_j v_k w_i$$

$$= \det \begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \text{—— (3)}$$



= Volume of the parallelopiped spanned by $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$

$$= \underbrace{\left(\|\underset{\sim}{u}\| \, \|\underset{\sim}{v}\| \sin\theta\right)}_{\|\underset{\sim}{u} \times \underset{\sim}{v}\|} \|w\| \cos\phi .$$

For a right-handed basis $(\underset{\sim}{e}_1 \times \underset{\sim}{e}_2) \cdot \underset{\sim}{e}_3 = 1.$

A comment on change of basis:

Recall that a new basis $\{\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3'\}$ is obtained from a basis $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ using constants λ_{ij}'s that form an orthogonal matrix. Assume $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ is right-handed.

Since $\underset{\sim}{e}_i' = \lambda_{ij}\underset{\sim}{e}_j$,

$$(\underset{\sim}{e}_1' \times \underset{\sim}{e}_2')\cdot \underset{\sim}{e}_3' = \varepsilon_{ijk}\Big(\overbrace{\lambda_{1j}}^{(\underset{\sim}{e}_1')_j}\ \overbrace{\lambda_{2k}}^{(\underset{\sim}{e}_2')_k}\Big)\lambda_{3i}$$

$$= \det\begin{bmatrix} \lambda_{31} & \lambda_{32} & \lambda_{33} \\ \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{bmatrix}$$

$$= \det \Lambda .$$

This implies the new basis is also right-handed $\Leftrightarrow \det \Lambda = 1$.

<u>Problem</u>: Show that $\underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{0} \Leftrightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ are either parallel or antiparallel.

<u>Problem</u>: Show that $\underset{\sim}{u} \times \underset{\sim}{v}$ is perpendicular to $\underset{\sim}{u}$ and v.

<u>Problem</u>: $(\underset{\sim}{u} \times \underset{\sim}{v})\cdot \underset{\sim}{w} = (\underset{\sim}{w} \times \underset{\sim}{u})\cdot \underset{\sim}{v} = (\underset{\sim}{v} \times \underset{\sim}{w})\cdot \underset{\sim}{u}$ —— (4)

Relationship between skew-symmetric tensors and the cross product:

Recall that a $W \in \mathrm{Skw}$ has only three independent components. We will now show that W can be "represented" as a vector.

<u>Theorem</u>: Let $W \in \text{Skw}$. There exists a unique $\underset{\sim}{w} \in V$ such that

$$W\underset{\sim}{u} = \underset{\sim}{w} \times \underset{\sim}{u} \qquad \forall\, \underset{\sim}{u} \in V \quad \text{— } (5)$$

In fact, $\underset{\sim}{w}$ is given by

$$w_i = -\frac{1}{2}\,\varepsilon_{ijk}\,W_{jk}. \quad \text{— } (6)$$

Conversely, for a given $\underset{\sim}{w} \in V$, $\exists!\ W \in \text{Skw}$ such that (5) holds, and W is given by

$$W_{ij} = -\varepsilon_{ijk}\,w_k$$

Before we prove the above theorem we note the following

- $\underset{\sim}{w}$ is called the axial vector of W
- If $W = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$, then $w = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
- $W\underset{\sim}{w} = \underset{\sim}{0}$

Proof: We will first check that the candidate for $\underset{\sim}{\omega}$ given in (6) satisfies (5). This will prove existence of $\underset{\sim}{\omega}$. First, note that

$$(W\underset{\sim}{u})_i = W_{ij}\, u_j \quad \text{——} \quad (7)$$

On the other hand

$$(\underset{\sim}{\omega} \times \underset{\sim}{u})_i = \varepsilon_{ijk}\, \omega_j\, u_k \quad (\text{definition of cross product})$$

$$= \varepsilon_{ijk}\left(-\frac{1}{2}\varepsilon_{jlm} W_{lm}\right) u_k$$

$$= \frac{1}{2}\varepsilon_{jik}\,\varepsilon_{jlm}\, W_{lm}\, u_k$$

$$= \frac{1}{2}\left(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}\right) W_{lm}\, u_k$$

$$= \frac{1}{2}\left(W_{ik}\, u_k - W_{ki}\, u_k\right)$$

$$= W_{ik}\, u_k \quad \text{——} \quad (8)$$

From (7) and (8), we see that $\underset{\sim}{\omega}$ satisfies (5).

Uniqueness: Assume $\exists$ another $\underset{\sim}{\omega}' \in V$ such that $W\underset{\sim}{u} = \underset{\sim}{\omega}' \times \underset{\sim}{u} \;\; \forall\, \underset{\sim}{u} \in V$. Since $W\underset{\sim}{u} = \underset{\sim}{\omega} \times \underset{\sim}{u}$, we have

$$\underset{\sim}{\omega}' \times \underset{\sim}{u} = \underset{\sim}{\omega} \times \underset{\sim}{u} \quad \forall\, \underset{\sim}{u} \in V$$

$$\Rightarrow (\tilde{w}' - \tilde{w}) \times \tilde{u} = 0 \quad \forall \, \tilde{u} \in V$$

~~Taking $\tilde{u} = \tilde{w}' - \tilde{w}$, we conclude that $\tilde{w} = \tilde{w}'$.~~

Converse is left as an exercise.

Eigenvalues and eigenvectors of second-order tensors

A scalar ω is an eigenvalue of a tensor S if there exists a non-zero vector $\tilde{u}$ such that

$$S\tilde{u} = \omega \tilde{u},$$

in which case $\tilde{u}$ is an eigenvector.

- If $\tilde{u}$ is an eigenvector with eigenvalue λ then $\alpha \tilde{u}$ is also an eigenvector with the same eigenvalue.

- If two eigenvectors $\tilde{u}$ and $\tilde{v}$ have a common eigenvalue, then all linear combinations of $\tilde{u}$ and $\tilde{v}$ have the same eigenvalue:

  If $S\tilde{u} = \omega \tilde{u}$ and $S\tilde{v} = \omega \tilde{v}$, then $S(\lambda \tilde{u} + \beta \tilde{v}) = \omega(\lambda \tilde{u} + \beta \tilde{v})$.

$\boxed{\therefore \text{All eigenvectors of an eigenvalue form a vector space (after including } \tilde{0})}$