
Problem 1.

1. (a)

$$\mathrm{tr}(\mathrm{ST}) = \mathrm{tr}(\mathrm{TS}) \tag{1}$$
$$(\mathrm{ST})_{ij} = \mathrm{S}_{ik}\mathrm{T}_{kj} \tag{2}$$
$$\mathrm{tr}(\mathrm{ST}) = (\mathrm{ST})_{ii} = \mathrm{S}_{ik}\mathrm{T}_{ki} \tag{3}$$
$$(\mathrm{TS})_{ij} = \mathrm{T}_{ik}\mathrm{S}_{kj} \tag{4}$$
$$\mathrm{tr}(\mathrm{TS}) = (\mathrm{TS})_{ii} = \mathrm{T}_{ik}S_{ki} \tag{5}$$

Summation over i and k does not depend on the order of $\mathrm{S}_{ik}, \mathrm{T}_{ki}$, so $\mathrm{S}_{ik}\mathrm{T}_{ik} = \mathrm{T}_{ik}\mathrm{S}_{ki}$

$$\boxed{\mathrm{tr}(\mathrm{ST}) = \mathrm{tr}(\mathrm{TS})} \tag{6}$$

.

(b)

$$\mathrm{curl}(\phi\mathbf{u}) = \nabla\phi \times \mathbf{u} + \phi\ \mathrm{curl}(\mathbf{u}) \tag{7}$$
$$(\mathrm{curl}(\mathbf{v}))_i = \epsilon_{ijk}\partial_j v_k \tag{8}$$
$$(\mathrm{curl}(\phi\mathbf{u}))_i = \epsilon_{ijk}\partial_j(\phi u_k) \tag{9}$$
$$\partial_j(\phi u_k) = (\partial_j\phi)u_k + \phi(\partial_j u_k) \tag{10}$$
$$(\mathrm{curl}(\phi\mathbf{u}))_i = \epsilon_{ijk}((\partial_j\phi)u_k + \phi(\partial_j u_k)) \tag{11}$$
$$= \epsilon_{ijk}(\partial_j\phi)u_k + \epsilon_{ijk}\phi(\partial_j u_k) \tag{12}$$
$$\epsilon_{ijk}(\partial_j u_k) = \phi(\mathrm{curl}\mathbf{u})_i \tag{13}$$
$$(\mathrm{curl}(\phi\mathbf{u}))_i = (\nabla\phi \times \mathbf{u})_i + \phi(\mathrm{curl}\mathbf{u})_i \tag{14}$$
$$\tag{15}$$

$$\boxed{\mathrm{curl}(\phi\mathbf{u}) = \nabla\phi \times \mathbf{u} + \phi\ \mathrm{curl}(\mathbf{u})} \tag{16}$$

(c)

$$\mathrm{div}(\phi\mathrm{T}) = \mathrm{T}\nabla\phi + \phi\mathrm{div}(\mathrm{T}) \tag{17}$$
$$(\phi\mathrm{T})_{ij} = \phi\mathrm{T}_{ij} \tag{18}$$
$$\mathrm{div}(\mathrm{T})_i = \partial_j\mathrm{T}_{ij} \tag{19}$$
$$\mathrm{div}(\phi\mathrm{T})_i = \partial_j(\phi\mathrm{T}_{ij}) \tag{20}$$
$$\partial_j(\phi\mathrm{T}_{ij}) = (\partial_j\phi)\mathrm{T}_{ij} + \phi(\partial_j\mathrm{T}_{ij}) \tag{21}$$
$$\mathrm{div}(\phi\mathrm{T})_i = (\mathrm{T}\nabla\phi)_i + \phi(\mathrm{div}(\mathrm{T})_i) \tag{22}$$
$$\tag{23}$$

$$\boxed{\mathrm{div}(\phi\mathrm{T}) = \mathrm{T}\nabla\phi + \phi\mathrm{div}(\mathrm{T})} \tag{24}$$

(d)

$$\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) = (\operatorname{div}(\mathbf{v}))\mathbf{u} + (\nabla\mathbf{u})\mathbf{v} \tag{25}$$
$$(\mathbf{u}\otimes\mathbf{v})_{ij} = u_i v_j \tag{26}$$
$$\operatorname{div}(\mathrm{T})_i = \partial_j \mathrm{T}_{ij} \tag{27}$$
$$\operatorname{div}(\mathbf{u}\otimes\mathbf{v})_i = \partial_j(u_i v_j) \tag{28}$$
$$\partial_j(u_i v_j) = u_i\partial_j v_j + v_j\partial_j u_i \tag{29}$$
$$\operatorname{div}(\mathbf{u}\otimes\mathbf{v})_i = (\operatorname{div}(\mathbf{v}))u_i + (\nabla\mathbf{u}\mathbf{v})_i \tag{30}$$
$$\tag{31}$$

$$\boxed{\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) = (\operatorname{div}(\mathbf{v}))\mathbf{u} + (\nabla\mathbf{u})\mathbf{v}} \tag{32}$$

(e)

$$\Delta(\xi\eta) = \xi\Delta\eta + \eta\Delta\xi + 2\nabla\xi\cdot\nabla\eta \tag{33}$$
$$\Delta\psi = \nabla^2\psi = \partial_i\partial_i\psi \tag{34}$$
$$\Delta(\zeta\eta) = \partial_i\partial_i(\zeta\eta) \tag{35}$$
$$\partial_i\partial_i(\zeta\eta) = \partial_i(\partial_i\zeta\eta + \zeta\partial_i\eta) \tag{36}$$
$$\partial_i(\partial_i\zeta\eta + \zeta\partial_i\eta) = \partial_i(\partial_i\zeta\eta) + \partial_i(\zeta\partial_i\eta) \tag{37}$$
$$= \partial_i\partial_i\zeta\eta + \partial_i\zeta\partial_i\eta + \partial_i\zeta\partial_i\eta + \zeta\partial_i\partial_i\eta \tag{38}$$
$$= \eta\Delta\zeta + \zeta\Delta\eta + 2\partial_i\zeta\partial_i\eta \tag{39}$$
$$\nabla\phi = \partial_i\phi\mathbf{e}_i \tag{40}$$
$$2\partial_i\zeta\partial_i\eta = 2\nabla\zeta\cdot\nabla\eta \tag{41}$$

$$\boxed{\Delta(\xi\eta) = \xi\Delta\eta + \eta\Delta\xi + 2\nabla\xi\cdot\nabla\eta} \tag{42}$$

2. (a)

$$\mathrm{I}_1(\mathrm{S}) = \operatorname{tr}(\mathrm{S}) \tag{43}$$
$$\operatorname{tr}(\mathrm{S}) = \operatorname{tr}(\mathrm{QSQ}^{\mathrm{T}}) \tag{44}$$
$$\mathrm{T} = \mathrm{Q} \tag{45}$$
$$\operatorname{tr}(\mathrm{SQ}) = \operatorname{tr}(\mathrm{QS}) \tag{46}$$
$$= \operatorname{tr}(\mathrm{QS}\underbrace{\mathrm{Q}^{\mathrm{T}}Q}_{\mathrm{I}}) = \operatorname{tr}(\mathrm{QQSQ}^{\mathrm{T}}) \tag{47}$$
$$\mathrm{T} = \mathrm{Q}^{\mathrm{T}} \tag{48}$$
$$\operatorname{tr}(\mathrm{SQ}^{\mathrm{T}}) = \operatorname{tr}(\mathrm{Q}^{\mathrm{T}}\mathrm{S}) \tag{49}$$
$$= \operatorname{tr}(\mathrm{SQ}^{\mathrm{T}}) = \operatorname{tr}(\mathrm{Q}^{T}\mathrm{QSQ}^{\mathrm{T}}) \tag{50}$$
$$\tag{51}$$

$$\boxed{\operatorname{tr}(\mathrm{SQ}^{\mathrm{T}}) = \operatorname{tr}(\mathrm{SQ}^{\mathrm{T}})} \tag{52}$$

(b)

$$\mathrm{I}_2(\mathrm{SQ}) = \frac{1}{2}[(\mathrm{tr}(\mathrm{SQ}))^2 - \mathrm{tr}((\mathrm{SQ})^2)] \tag{53}$$

$$= \frac{1}{2}[(\mathrm{tr}(\mathrm{QSQ}^{\mathrm{T}}Q))^2 - \mathrm{tr}((\mathrm{QSQ}^{\mathrm{T}}\mathrm{Q})^2)] \tag{54}$$

$$= [\mathrm{tr}(\mathrm{SQ}^2) - \mathrm{tr}((\mathrm{SQ})^2)] = [\mathrm{tr}(\mathrm{QS})^2 - \mathrm{tr}((\mathrm{QS})^2)] \tag{55}$$

$$\cancel{\mathrm{tr}(\mathrm{SQ})^2} - \mathrm{tr}(\mathrm{SQSQ}) = \cancel{\mathrm{tr}(\mathrm{QS})^2} - \mathrm{tr}(\mathrm{QSQS}) \tag{56}$$

(c)

$$\mathrm{I}_3(\mathrm{S}) = \det(\mathrm{S}) \tag{57}$$

$$\mathrm{T} = \mathrm{Q} \tag{58}$$

$$\det(\mathrm{ST}) = \det(\mathrm{S})\det(\mathrm{T}) = \det(\mathrm{T})\det(\mathrm{S}) \tag{59}$$

$$\det(\mathrm{T}) = 1,\ \mathrm{T} = \mathrm{Q},\ \det(\mathrm{Q}) = 1 \tag{60}$$

$$\det(\mathrm{S}) = \underbrace{\det(\mathrm{Q})}_{1}\det(\mathrm{S})\underbrace{\det(\mathrm{Q}^{\mathrm{T}})}_{1} \tag{61}$$

$$\tag{62}$$

$$\boxed{\det(\mathrm{S}) = \det(\mathrm{S})} \tag{63}$$

(d)

$$\mathrm{F} = \mathrm{RU},\ \mathrm{F} = \mathrm{VR} \tag{64}$$

Eigenvalues are given by the following:

$$-\lambda^3 + \mathrm{I}_1\lambda^2 - \mathrm{I}_2\lambda + \mathrm{I}_3 = 0 \tag{65}$$

$$\det(\mathbf{U} - \lambda\mathbf{I}) = \det(\mathbf{V} - \lambda\mathbf{I}) \tag{66}$$

$$\mathbf{S} = \mathbf{U} - \lambda\mathbf{I} \tag{67}$$

$$\mathbf{T} = \mathbf{R} \tag{68}$$

$$\det(\mathbf{S}) = \det(\mathbf{R}\mathbf{S}\mathbf{R}^{\mathrm{T}}) \tag{69}$$

$$\tag{70}$$

$\mathrm{RU} = \mathrm{VR},\ \mathrm{R}^{\mathrm{T}}\mathrm{RU} = \mathrm{R}^{\mathrm{T}}\mathrm{VR} \implies \mathrm{U} = \mathrm{R}^{\mathrm{T}}\mathrm{VR}$, and $\mathrm{R} \in \mathrm{Orth}$.

$$\mathbf{U} = \mathbf{R}^{-1}\mathbf{F},\ \mathbf{V} = \mathbf{F}\mathbf{R}^{-1} \tag{71}$$

$$\det(\mathbf{R}^{-1}\mathbf{F}) = \det(\mathbf{F}\mathbf{R}^{-1}) \tag{72}$$

$$\underbrace{\cancel{\det(\mathbf{R}^{-1})U}}_{1}\det(\mathbf{F}) = \det(\mathbf{F})\underbrace{\cancel{\det(\mathbf{R}^{-1})}}_{1} \tag{73}$$

$$= \det(\mathbf{F}) = \det(\mathbf{F}) \tag{74}$$

The equation above shows that the eigenvalues of **U** and **V** are equal, as the invariants of **U** and **V** are the same.

$$\det(\mathrm{U} - \lambda \mathrm{I}) = 0 \tag{75}$$
$$\mathrm{U}\mathbf{e} = \lambda_u \mathbf{e} \tag{76}$$
$$\mathrm{R}\mathrm{U}\mathbf{e} = \mathrm{R}\lambda_u \mathbf{e} \tag{77}$$
$$= \mathrm{R}\mathrm{U}\mathbf{e} = \lambda_u \mathrm{R}\mathbf{e} \tag{78}$$
$$\mathrm{V}\mathbf{e} = \lambda_u \mathrm{R}\mathbf{e} \tag{79}$$

The equation above shows that that if **e** is an eigenvector of **U**, then **Re** is an eigenvalue of **V**.

3.

$$\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{A}\forall\mathbf{Q} \in \mathrm{Orth} \iff \mathbf{A}\mathbf{Q}\mathbf{v} \tag{80}$$
$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{81}$$
$$\alpha\mathbf{I}\mathbf{v} = \lambda\mathbf{v} \tag{82}$$
$$\alpha\mathbf{v} = \lambda\mathbf{v} \tag{83}$$
$$\alpha = \lambda \tag{84}$$

Using the result from above:

$$\mathbf{Q}\mathbf{Q}\mathbf{v} = \lambda\mathbf{Q}\mathbf{v} \tag{85}$$
$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{Q}\mathbf{v} = \mathbf{Q}^{\mathrm{T}}\alpha\mathbf{Q}\mathbf{v} \tag{86}$$
$$= \mathbf{Q}\mathbf{v} = \mathbf{Q}^{\mathrm{T}}\alpha\mathbf{Q}\mathbf{v} \tag{87}$$
$$\mathbf{Q}\mathbf{v} = \alpha\mathbf{I}\mathbf{v} \tag{88}$$
$$\mathbf{Q}\mathbf{v} = \mathbf{A}\mathbf{v} \tag{89}$$
$$\mathbf{Q} = \mathbf{A} \tag{90}$$
$$\underbrace{\cancel{\mathbf{Q}^{\mathrm{T}}\mathbf{Q}}}_{\mathbf{I}}\mathbf{Q} = \mathbf{Q} \tag{91}$$
$$\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{A} \tag{92}$$
$$= \mathbf{Q}^{\mathrm{T}}\alpha\mathbf{I}\mathbf{Q} = \alpha\mathbf{I} \tag{93}$$
$$\tag{94}$$

$$\boxed{\alpha\mathbf{I} = \alpha\mathbf{I}} \tag{95}$$

Problem 2.

1. (a) First, we will compute the derivative of R with respect to $\mathbf{X}_i$, as we will need these values for the main derivation of the deformation gradient:

$$\mathbf{x}(\mathbf{X},t) = \frac{f(R,t)}{R}\mathbf{X},\ \ R = |\mathbf{X}| \tag{96}$$

$$R = \sqrt{\mathbf{X}_1^2 + \mathbf{X}_2^2 + \mathbf{X}_3^2} \tag{97}$$

$$= \sqrt{\mathbf{X}_i\mathbf{X}_i} \tag{98}$$

$$\frac{\partial R}{\partial \mathbf{X}_i} = \frac{\partial(\mathbf{X}_i\mathbf{X}_i)^{1/2}}{\partial \mathbf{X}_k} \tag{99}$$

$$= \frac{1}{2}(\mathbf{X}_i\mathbf{X}_i)^{-\frac{1}{2}}\left(\frac{\partial \mathbf{X}_i}{\partial \mathbf{X}_k}\mathbf{X}_i + \frac{\partial \mathbf{X}_i}{\partial \mathbf{X}_K}\mathbf{X}_i\right) \tag{100}$$

$$= \frac{2\delta_{ik}\mathbf{X}_i}{2\sqrt{\mathbf{X}_i\mathbf{X}_i}} \tag{101}$$

$$= \frac{\mathbf{X}_i\mathbf{I}}{R} \tag{102}$$

We will compute the deformation gradient using the product rule:

$$\mathbf{F} = \left(\frac{f(R,t)}{R}\mathbf{X}_j\right)_i = \frac{f(R,t)}{R}\underbrace{\cancel{\frac{\partial \mathbf{X}_j}{\partial \mathbf{X}_i}}}_{\delta_{ij}} + \frac{\partial}{\partial \mathbf{X}_i}\left(\frac{f(R,t)}{R}\right)\mathbf{X}_j \tag{103}$$

$$= \frac{f(R,t)}{R}\mathbf{I} + \frac{\partial}{\partial R}(f(R,t)R^{-1})\mathbf{X}_j \tag{104}$$

$$= \frac{f(R,t)}{R}\mathbf{I} + \mathbf{X}_j\left(\frac{\partial f(R,t)}{\partial R}\right)\frac{\partial R}{\partial \mathbf{X}_j}\frac{1}{R} + \left(\frac{\partial R^{-1}}{\partial R}\right)\frac{\partial R}{\partial \mathbf{X}_j}f(R,t)\mathbf{X}_j \tag{105}$$

$$= \frac{f(R,t)}{R}\mathbf{I} + \frac{\partial f(R,t)}{\partial R}\frac{(\mathbf{X}_i)\mathbf{I}}{R}\frac{1}{R}\mathbf{X}_j + \left(-\frac{1}{R^2}\left(\frac{\mathbf{X}_i\mathbf{I}}{R}\right)\right)f(R,t)\mathbf{X}_j \tag{106}$$

$$\tag{107}$$

$$\boxed{\mathbf{F} = \frac{f(R,t)}{R}\mathbf{I} + \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X}\otimes\mathbf{X}} \tag{108}$$

(b) The complete deformation gradient is given as follows:

$$\mathbf{F} = \begin{bmatrix} \frac{f}{R} + \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X}_1^2 & \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X}_1\mathbf{X}_2 & \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X_1X_3} \\ \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X_1X_3} & \frac{f}{R} + \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X}_2^2 & \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X_2X_3} \\ \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X_1X_3} & \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X_2X_3} & \frac{f}{R} + \frac{1}{R^2}\left(\frac{\partial f}{\partial R} - \frac{f}{R}\right)\mathbf{X}_3^2 \end{bmatrix} \tag{109}$$

Computing the determinant:

$$\det(\mathbf{F}) = \frac{1}{R^5}\left[R^2\mathbf{F}^3 + Rf^2\frac{\partial \mathbf{F}}{\partial R}(\mathbf{X}_1^2 + \mathbf{X}_2^2 + \mathbf{X}_3^2) - \mathbf{F}(\mathbf{X}_1^2 + \mathbf{X}_2^2 + \mathbf{X}_3^2)\right] \tag{110}$$

$$\tag{111}$$

Substituting $R^2 = \mathbf{X}_1^2 + \mathbf{X}_2^2 + \mathbf{X}_3^2$:

$$\frac{1}{R^5}\left[R^2\mathbf{F}^3 + R^3 f^2\frac{\partial \mathbf{F}}{\partial R}R^2\mathbf{F}^3\right] \tag{112}$$

$$= \frac{1}{R^5}R^3\mathbf{F}^2\frac{\partial F}{\partial R} = \frac{\mathbf{F}^2}{R^2}\frac{\partial \mathbf{F}}{\partial R} \tag{113}$$

$$\tag{114}$$

$$\boxed{\det(\mathbf{F}) = \left(\frac{f}{R}\right)^2\frac{\partial f}{\partial R}} \tag{115}$$

2. To compute the material velocity and acceleration fields, we simply take the time derivatives:

$$\mathbf{x}(\mathbf{X}, t) = \frac{f(R,t)}{R}\mathbf{X} \tag{116}$$

$$\tag{117}$$

$$\boxed{\begin{aligned}\mathbf{v} &= \frac{\mathbf{X}}{R}\dot{f} \quad (118)\\ \mathbf{a} &= \frac{\mathbf{X}}{R}\ddot{f} \quad (119)\end{aligned}}$$

To compute the spatial velocity and acceleration fields, we find and substitute the inverse map $\mathbf{X} = \frac{\mathbf{x}R}{f}$ into the material velocity field, and the spatial acceleration field is found by applying the formula below:

$$\mathbf{a}_s = \mathbf{v}_s + \text{grad}(\mathbf{v}_s)\mathbf{v_s} \tag{120}$$

Computing the spatial fields:

$$\mathbf{v}_s = \left(\frac{\mathbf{x}R}{fR}\right)\dot{f} = \left(\frac{\mathbf{x}}{f}\right)\dot{f} \tag{121}$$

$$\mathbf{a}_s = \frac{\mathbf{x}\ddot{f}}{f} + \mathbf{x}\dot{f}(-f^{-2}) + \text{grad}(\mathbf{v}_s)\mathbf{v}_s \tag{122}$$

$$\tag{123}$$

$$\boxed{\begin{aligned}\mathbf{v}_s &= \left(\frac{\mathbf{x}}{f}\right)\dot{f} \qquad (124)\\ \mathbf{a}_s = \frac{\mathbf{x}\ddot{f}}{f} &- \frac{\mathbf{x}\dot{f}}{f^2} + \frac{\mathbf{x}\dot{f}^2}{f^2} \qquad (125)\end{aligned}}$$

3. Performing separation of variables:

$$\det(\mathbf{F}) = \frac{f^2}{R^2}\frac{\partial \mathbf{F}}{\partial R} = 1 \implies \mathbf{F}^2 \partial \mathbf{F} = R^2 \partial \mathbf{F} \tag{126}$$

$$\frac{1}{3}\mathbf{F}^3 = \frac{1}{3}R^3 + p(t) + C \tag{127}$$

$$\mathbf{x}(\mathbf{X}, t) = \frac{f(R,t)}{R}\mathbf{X}, \mathbf{X} = a_0 \tag{128}$$

Using the initial conditions such that $\mathbf{X} = a_0, R = a_0$, this implies that $x(a_0, t) = f(a_0, t)$ and $\mathbf{x}(a_0, t) = \mathbf{F}(a_0, t) = a(t)$:

$$\mathbf{F}^3 = R^3 + p(t) + C \tag{129}$$

$$= F(a_0, t)^3 = a(t)^3 = a_0^3 + p(t) + C, p(t) = a(t)^3, C = -a_0^3 \tag{130}$$

$$f(R,t)^3 = R^3 + a(t)^3 - a_0^3 \tag{131}$$

$$\tag{132}$$

$$\boxed{f(R,t) = (R^3 + a(t)^3 - a_0^3)^{\frac{1}{3}}} \tag{133}$$

Problem 3.

1.

$$\mathrm{div}(\mathbf{T}) = \frac{\partial \mathbf{T}_{11}}{\partial x_1} + \frac{\partial \mathbf{T}_{12}}{\partial x_2} + \frac{\partial \mathbf{T}_{13}}{\partial x_3} \tag{134}$$

$$\mathbf{T}_{11} = -\frac{c}{a^2}(x_1^2 - x_2^2),\ \mathbf{T}_{12} = \frac{c}{a^2}2x_1x_2,\ \mathbf{T}_{13} = 0 \tag{135}$$

$$\frac{\partial \mathbf{T}_{11}}{\partial x_1} = -\frac{c}{a^2}2x,\ \frac{\partial \mathbf{T}_{12}}{\partial x} = \frac{c}{a^2}(2x_1),\ \frac{\partial \mathbf{T}_{13}}{\partial x_3} = 0 \tag{136}$$

$$\mathrm{div}(\mathbf{T})_1 = -\frac{2cx_1}{a^2} + \frac{2cx_1}{a^2} = 0 \tag{137}$$

$$\mathrm{div}(\mathbf{T})_2 = \frac{\partial \mathbf{T}_{21}}{\partial x_1} + \frac{\partial \mathbf{T}_{22}}{\partial x_2} + \frac{\partial \mathbf{T}_{23}}{\partial x_3} \tag{138}$$

$$\mathbf{T}_{21} = \frac{c}{a^2}2x_1x_2,\ \mathbf{T}_{22} = \frac{c}{a^2}(x_1^2 - x_2^2),\ \mathbf{T}_{23} = 0 \tag{139}$$

$$\mathrm{div}(\mathbf{T})_2 = -\frac{2cx_2}{a^2} + \frac{2cx_2}{a^2} = 0 \tag{140}$$

$$\mathrm{div}(\mathbf{T})_3 = \frac{\partial \mathbf{T}_{31}}{\partial x_1} + \frac{\partial \mathbf{T}_{32}}{\partial x_2} + \frac{\partial \mathbf{T}_{33}}{\partial x_3} \tag{141}$$

$$\mathbf{T}_{31} = 0,\ \mathbf{T}_{32} = 0,\ \mathbf{T}_{33} = 0 \tag{142}$$

$$\frac{\partial \mathbf{T}_{31}}{\partial x_1} = 0,\ \frac{\partial \mathbf{T}_{32}}{\partial x_2} = 0,\ \frac{\partial \mathbf{T}_{33}}{\partial x_3} = 0 \tag{143}$$

$$\mathrm{div}(\mathbf{T}) = 0 \tag{144}$$

$$\tag{145}$$

$$\boxed{\mathrm{div}(\mathbf{T}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \tag{146}$$

2. Computing the traction for each face on the rectangular body:
   For the face that corresponds to $x_1 = a,\ x_1 = -a$:

$$\hat{\mathbf{n}} = \begin{bmatrix} 1, 0, 0 \end{bmatrix} \text{ for } x_1 = a \tag{147}$$

$$\hat{\mathbf{n}} = \begin{bmatrix} -1, 0, 0 \end{bmatrix} \text{ for } x_1 = -a \tag{148}$$

$$\boxed{\begin{aligned} t(1,0,0) &= \frac{c}{a^2}\begin{bmatrix} -(a^2 - x_2^2) \\ 2ax_2 \\ 0 \end{bmatrix} \qquad (149) \\ t(-1,0,0) &= \frac{c}{a^2}\begin{bmatrix} a^2 - x_2^2 \\ -2ax_2 \\ 0 \end{bmatrix} \qquad (150) \end{aligned}}$$

For the face that corresponds to $x_2 = a,\ x_2 = -a$:

$$\hat{\mathbf{n}} = \begin{bmatrix}0, 1, 0\end{bmatrix} \text{ for } x_2 = a \tag{151}$$

$$\hat{\mathbf{n}} = \begin{bmatrix}0, -1, 0\end{bmatrix} \text{ for } x_2 = -a \tag{152}$$

$$\boxed{\begin{aligned} t(0,1,0) &= \frac{c}{a^2}\begin{bmatrix} 2ax_1 \\ x_1^2 - a^2 \\ 0 \end{bmatrix} && (153) \\ t(0,-1,0) &= \frac{c}{a^2}\begin{bmatrix} -2ax_1 \\ -x_1^2 + a^2 \\ 0 \end{bmatrix} && (154) \end{aligned}}$$

For the face that corresponds to $x_3 = b,\ x_3 = -b$:

$$\hat{\mathbf{n}} = \begin{bmatrix}0, 0, 1\end{bmatrix} \text{ for } x_3 = b \tag{155}$$

$$\hat{\mathbf{n}} = \begin{bmatrix}0, 0, -1\end{bmatrix} \text{ for } x_3 = -b \tag{156}$$

$$\boxed{\begin{aligned} t(0,0,1) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} && (157) \\ t(0,0,-1) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} && (158) \end{aligned}}$$

3.

$$x_1^2 + x_2^2 + x_3^2 = a^2,\ \hat{\mathbf{n}} = \left(\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}\right) \tag{159}$$

$$\mathbf{T} = \frac{c}{a^2}\begin{bmatrix} -(x_1^2 - x_2^2) & 2x_1x_2 & 0 \\ 2x_1x_2 & x_1^2 - x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{160}$$

$$t(\hat{\mathbf{n}}) = \mathbf{T} \cdot \begin{bmatrix} \frac{x_1}{a} \\ \frac{x_2}{a} \\ \frac{x_3}{a} \end{bmatrix} \tag{161}$$

$$t(\hat{\mathbf{n}}) = \frac{c}{a^3}\begin{bmatrix} -(x_1^2 - x_2^2) & 2x_1x_2 & 0 \\ 2x_1x_2 & x_1^2 - x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{162}$$

$$= \frac{c}{a^3}\begin{bmatrix} -(x_1^2 - x_2^2)x_1 + 2x_1x_2x_2 \\ 2x_1x_2x_1 + (x_1^2 + x_2^2)x_2 \\ 0 \end{bmatrix} \tag{163}$$

$$\tag{164}$$

$$\boxed{t(\hat{\mathbf{n}}) = \frac{c}{a^3}\begin{bmatrix} -x_1^3 + 2x_1 + x_2^2 \\ 3x_1^2x_2 - x_2^3 \\ 0 \end{bmatrix}} \tag{165}$$

4.

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0 \tag{166}$$

$$\lambda_1 = \frac{c(x_1^2 + x_2^2)}{a^2} \tag{167}$$

$$\lambda_2 = 0 \tag{168}$$

$$\lambda_3 = -\frac{c(x_1^2 + x_2^2)}{a^2} \tag{169}$$

$$\tau_{\max} = \frac{\lambda_1 - \lambda_3}{2} \tag{170}$$

$$= \frac{\left(\frac{c(x_1^2+x_2^2)}{a^2} - \frac{-c(x_1^2+x_2^2)}{a^2}\right)}{2} \tag{171}$$

$$\tag{172}$$

$$\boxed{\tau_{\max} = \frac{c(x_1^2 + x_2^2)}{a^2}} \tag{173}$$

Problem 4.

1.

$$\mathbf{T}_s(\mathbf{x}, t) = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^{\mathrm{T}}|_{\mathbf{X}=f^{-1}(\mathbf{x},t)} \tag{174}$$
$$\psi(\mathbf{F}) = \psi(\mathbf{QF}), \;\; \mathbf{Q} \in \mathrm{Orth} \tag{175}$$
$$\mathbf{F} = \mathbf{RU} \tag{176}$$
$$\psi(\mathbf{QRU}) = \psi(\mathbf{F}) \tag{177}$$
$$\tag{178}$$

Choosing $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$:

$$\psi(\mathbf{U}) = \psi(\mathbf{F}) \tag{179}$$
$$\mathbf{U} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{C} \tag{180}$$
$$\therefore \psi(\mathbf{F}) = \psi(\mathbf{C}) \tag{181}$$

2.

$$\mathbf{T} = \rho_s \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^{\mathrm{T}} \tag{182}$$
$$\mathbf{T}_{ij} = \rho_s \frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{C}_{LM}} (\mathbf{F}^{\mathrm{T}})_{Kj} \tag{183}$$
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \iff \mathbf{C}_{LM}\mathbf{F}_{PL}\mathbf{F}_{PM} \tag{184}$$
$$\frac{\partial \mathbf{C}_{LM}}{\partial \mathbf{F}_{iK}} = \frac{\partial \mathbf{F_{pL}}}{\partial \mathbf{F}_{iK}} \mathbf{F}_{PM} + \mathbf{F}_{pL} \frac{\partial \mathbf{F}_{PM}}{\partial \mathbf{F}_{iK}} \tag{185}$$
$$= \delta_{iP}\delta_{KL}\mathbf{F}_{PM} + \mathbf{F}_{PL}\delta_{iP}\delta_{KM} \tag{186}$$
$$= \mathbf{F}_{iM}\delta_{KL} + \mathbf{F}_{iL}\delta_{KM} \tag{187}$$
$$\therefore \mathbf{T}_{ij} = \rho_s \left[\frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{C}_{KM}} \mathbf{F}_{iM}\mathbf{F}_{jk} + \frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{C}_{LK}} \mathbf{F}_{iL}\mathbf{F}_{jk} \right] \tag{188}$$
$$= 2\rho_s \mathbf{F}_{iM} \left(\frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{C}} \right)_{KM} \mathbf{F}^{\mathrm{T}}_{Kj} \tag{189}$$
$$\tag{190}$$

$$\boxed{\mathbf{T} = 2\rho_s \mathbf{F} \frac{\partial \bar{\bar{\psi}}}{\partial C} \mathbf{F}^{\mathrm{T}}} \tag{191}$$

3.

$$\bar{\psi}(\mathbf{C}) = \bar{\bar{\psi}}(I_1(\mathbf{C}), \;\; I_2(\mathbf{C}), \;\; I_3(\mathbf{C})) \tag{192}$$
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{193}$$
$$= (\mathbf{VR})^{\mathrm{T}}(\mathbf{VR}) \tag{194}$$
$$= \mathbf{R}^{\mathrm{T}}\mathbf{BR} \tag{195}$$

We can use $\mathbf{B}$ instead of $\mathbf{C}$ because the invariants of $\mathbf{C}$ are the same as the invariants of $\mathbf{B}$. Therefore, we can use $\bar{\bar{\psi}}(I_1(\mathbf{B}),\ I_2(\mathbf{B}),\ I_3(\mathbf{B}))$

$$\mathbf{T} = \rho_s \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^{\mathrm{T}} \tag{196}$$

$$\mathbf{T}_{ij} = \rho_s \frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{B}_{pq}} \frac{\partial \mathbf{B}_{pq}}{\partial \mathbf{F}_{iL}} (\mathbf{F}^{\mathrm{T}})_{bj} \tag{197}$$

$$\tag{198}$$

Simplifying each term in the equation above:

$$\frac{\partial \mathbf{B}_{pq}}{\partial \mathbf{F}_{iL}} = \frac{\partial \bar{\bar{\psi}}}{\partial I_l(\mathbf{B})} \frac{\partial I_l(\mathbf{B})}{\partial \mathbf{B}_{pq}} \tag{199}$$

$$= \delta_{ip} \mathbf{F}_{qL} + \delta_{iq} \mathbf{F}_{PL} \tag{200}$$

$$\rho_s \frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{B}_{pq}} = \begin{cases} \frac{\partial I_1}{\partial \mathbf{B}_{pq}} = \delta_{pq} \\ \frac{\partial I_2}{\partial \mathbf{B}_{pq}} = I_1 \delta_{pq} - \mathbf{B}_{pq} \\ \frac{\partial I_3}{\partial \mathbf{B}_{pq}} = I_3 \mathbf{B}^{-1}_{qp} \end{cases} \tag{201}$$

$$\therefore \mathbf{T}_{ij} = \rho_s \left[2 \frac{\partial \bar{\bar{\psi}}}{\partial I_1} \mathbf{F}_{iL} \mathbf{F}^{\mathrm{T}}_{Lj} + 2 \frac{\partial \bar{\bar{\psi}}}{\partial I_2} (I_1(\mathbf{B}) \mathbf{F}_{iL} \mathbf{F}^{\mathrm{T}}_{Lj} - \mathbf{B}_{iq} \mathbf{F}_{qL} \mathbf{F}^{\mathrm{T}}_{Lj}) + \frac{\partial \bar{\bar{\psi}}}{\partial I_3} I_3(\mathbf{B}) \underbrace{(\mathbf{B}^{-1}_{qi} \mathbf{F}_{qL} + \mathbf{B}^{-1}_{pi} \mathbf{F}_{pL}) \mathbf{F}^{\mathrm{T}}_{Lj}}_{2(\mathbf{B}^{\mathrm{T}}\mathbf{B})_{ij} = 2\mathbf{I}(\mathbf{B} \in \mathrm{Psym})} \right] \tag{202}$$

$$= \eta_0 \mathbf{I} + \eta_2 \mathbf{B} + \eta_2 \mathbf{B}^2 \tag{203}$$

Where $\eta_i(I_1(\mathbf{B}),\ I_2(\mathbf{B}),\ I_3(\mathbf{B})$.