Lecture 14

In last class we have seen how a deformation effects volumes, areas and curves. We then introduced material and spatial discription of fields.

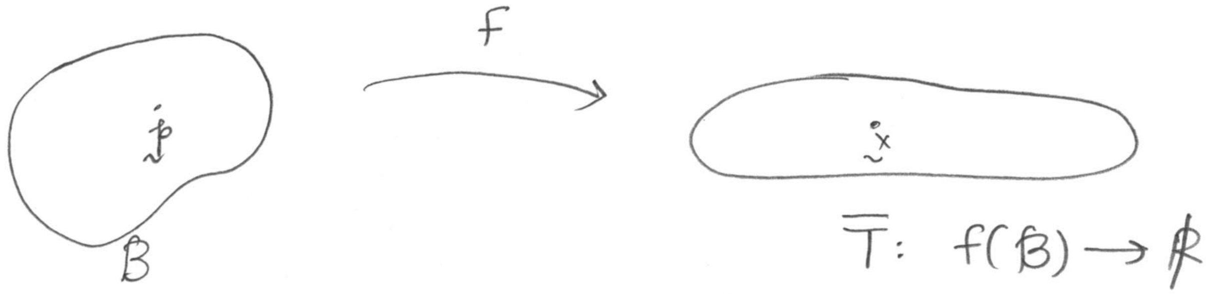
A material field $g : \mathcal{B} \to \mathbb{R}$ describes a property of material points, while a spatial field $\bar{g} : \underbrace{f(\mathcal{B})}_{\boxed{\text{deformed / current configuration}}} \longrightarrow \mathbb{R}$ describes the property at a spatial point. For example, let $\underset{\sim}{v}$ denote a vector field described materially, i.e. $\underset{\sim}{v} : \mathcal{B} \to V$.

$\underset{\sim}{v}(\underset{\sim}{p})$ is the velocity of a particle $\underset{\sim}{p}$. Its spatial description is given by a new field $\bar{\underset{\sim}{v}} : f(\mathcal{B}) \longrightarrow V$, where $\bar{v}(\underset{\sim}{x})$ is the velocity measured at a spatial point. The distinction becomes important when we discuss time-dependent deformations. It is also important when we would like to introduce differential operators with respect to spatial coordinates. So far we have only seen material fields:

Deformation f; deformation gradient F; stretch tensors

U, V ; Cauchy - Grain strain tensor $F^T F, F F^T$.

Setting :



Suppose we measure a temperature field on the deformed body, say $\overline{T} : f(B) \to \mathbb{R}$. We would like to calculate the gradient of temperature. A question you should be asking is : gradient with respect to spatial points or material points ?

$$\frac{\partial \overline{T}(\underset{\sim}{x})}{\partial x_i} \quad \text{— spatial gradient.}$$

You can construct a material temperature field as well:

$T : B \to \mathbb{R}$, where T is defined as

$T(\underset{\sim}{p}) := \overline{T}(\underset{\sim}{f}^{-1}(\underset{\sim}{x}))$ (or equivalently $\overline{T}(\underset{\sim}{x}) = T(\underset{\sim}{f}(\underset{\sim}{p})$

The quantity

$$\frac{\partial T}{\partial p_I} \quad \text{— material gradient}$$

Notation to differentiate material and spatial differential operations

	Material	Spatial
gradient	∇	grad
divergence	Div	div
curl	Curl	curl

How are ∇ and grad related ?

$$(\nabla T) = \frac{\partial T}{\partial p_I}(\underset{\sim}{p})$$

$$(\text{grad}\,\bar{T})_i = \frac{\partial \bar{T}}{\partial x_i}(\underset{\sim}{x})$$

Since $T(\underset{\sim}{x}) = \bar{T}(\underset{\sim}{f}(\underset{\sim}{p}))$, we have

$$(\text{grad}\,\bar{T})_i(\underset{\sim}{x}) = \frac{\partial \bar{T}}{\partial x_i}(\underset{\sim}{x})$$

$$= \frac{\partial \bar{T}(\underset{\sim}{f}(\underset{\sim}{p}))}{\partial p_J} \frac{\partial p_J}{\partial x_i}, \text{ where } p_J = f_J^{-1}(\underset{\sim}{x})$$

$$= \left[\nabla T(\underset{\sim}{f}(\underset{\sim}{p}))\right]_J \boxed{\frac{\partial p_J}{\partial x_i}}$$

What is this ?

We know $\quad x_i = f_i(p_1, p_2, p_3) \quad \text{—— (1)}$

and $\quad F_{iJ}(\underset{\sim}{p}) = \dfrac{\partial x_i}{\partial p_J}(\underset{\sim}{p}) = \dfrac{\partial f_i}{\partial p_J}(\underset{\sim}{p}) \quad \text{—— (2)}$

Inverting (1), we have

$$\underset{\sim}{p} = \underset{\sim}{f}^{-1}(\underset{\sim}{x})$$

(Since a deformation, by definition, is invertible)

$$\Longleftrightarrow \quad p_I = f_I^{-1}(x_1, x_2, x_3) \quad \text{—— (3)}$$

Take the derivative of (3) with respect to x_j

$$G_{Ij}(\underset{\sim}{x}) := \frac{\partial p_I}{\partial x_j}(\underset{\sim}{x}) = \frac{\partial f_I^{-1}}{\partial x_j}(\underset{\sim}{x}) \quad \text{—— (4)}$$

How are the tensors in (2) and (4) related?

$$\underset{\sim}{x} = \underset{\sim}{f}(\underset{\sim}{f}^{-1}(\underset{\sim}{x}))$$

$$\text{grad}\, \underset{\sim}{x} = \text{grad}\, f(f^{-1}(\underset{\sim}{x}))$$

$$\frac{\partial x_i}{\partial x_j}(\underset{\sim}{x}) = \frac{\partial}{\partial x_j} f_i(\underset{\sim}{p})\Big|_{\underset{\sim}{p} = \underset{\sim}{f}^{-1}(\underset{\sim}{x})}$$

$$= \frac{\partial f_i}{\partial p_L}\Big|_{\underset{\sim}{p} = \underset{\sim}{f}^{-1}(\underset{\sim}{x})} \frac{\partial}{\partial x_j} f_L^{-1}(\underset{\sim}{x})$$

$$\cancel{\delta_{i1} = (F_{iL}\, G_{Lj})(\underset{\sim}{f}^{-1}(\underset{\sim}{x}))}$$

$$\delta_{ij}(\underset{\sim}{x}) = F_{iL}(\underset{\sim}{p})\Big|_{\underset{\sim}{p}=\underset{\sim}{f}^{-1}(\underset{\sim}{x})} G_{Lj}(\underset{\sim}{x})$$

$$I \equiv F(\underset{\sim}{p})\Big|_{\underset{\sim}{p}=\underset{\sim}{f}^{-1}(\underset{\sim}{x})} G(\underset{\sim}{x})$$

$$\therefore \boxed{G(\underset{\sim}{x}) = F^{-1}(\underset{\sim}{p})\Big|_{\underset{\sim}{p}=\underset{\sim}{f}^{-1}(\underset{\sim}{x})}}$$

Let us now go back to our calculation of grad T :

$$(\text{grad}\,\bar{T})_i(\underset{\sim}{x}) = (\nabla T)_j\Big|_{\underset{\sim}{p}=\underset{\sim}{f}^{-1}(\underset{\sim}{x})} G_{Ji}$$

$$\boxed{\text{grad}\,\bar{T} = \cancel{(\nabla T\, F^{-1})}\Big|_{\underset{\sim}{p}=\underset{\sim}{f}^{-1}(\underset{\sim}{x})}} .$$

F^{-T} \nabla T

whenever we write grad$(\square)$, it means $\square$ is a spatial field. Similarly $\nabla(\square)$ implicitly assumes $\square$ is a material field. In other words, we will use the notation grad T to denote grad $\bar{T}$, i.e drop the bar.

<u>Theorem</u> Let f be a deformation, and let φ be a smooth scalar field on $f(\mathcal{B})$. Then given any part $\mathcal{P} \subset \mathcal{B}$,

$$\int_{f(\mathcal{P})} \varphi(\underset{\sim}{x})\, dV_x = \int_{\mathcal{P}} \varphi(f(\underset{\sim}{p}))\, \det F(\underset{\sim}{p})\, dV_p \,,$$

$$\int_{\partial f(\mathcal{P})} \varphi(\underset{\sim}{x})\, m(\underset{\sim}{x})\, dA_x = \int_{\partial \mathcal{P}} \varphi(f(\underset{\sim}{p}))\, H(\underset{\sim}{p})\, \underset{\sim}{n}(\underset{\sim}{p})\, dA_p$$

where $H(\underset{\sim}{p}) = (\det F)\, F^{-T}(p)$.