01·28·2020

Summary of Euclidean space: A Euclidean space is a real inner-product space obtained from the real coordinate vector space and equipping it with an inner product. Inner-product generalizes the notion of "angles". It also induces a norm which generalizes the notion of "size".

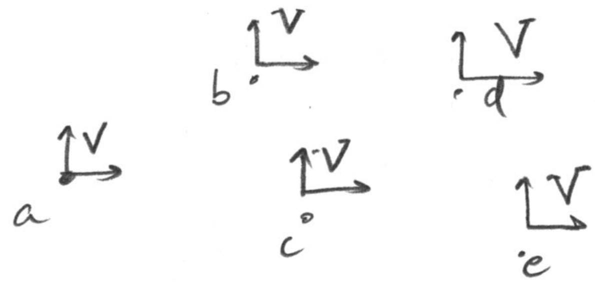
Affine point space ($\mathcal{E}$) is a collection of points and a vector space V such that for any two points $x, y \in \mathcal{E}$, there exists a unique vector $\underset{\sim}{v} \in V$ which translates x to y (This defines a translation operation: $x + \underset{\sim}{v} = y$) with the properties:

a) $x + (\underset{\sim}{v} + \underset{\sim}{w}) = (x + \underset{\sim}{v}) + \underset{\sim}{w} \quad \forall\ v, w \in V$.

(b) $x + \underset{\sim}{0} = x \quad \forall\ x \in \mathcal{E}$.

Equivalent to the translation operation, we can define a difference operation: $\underset{\sim}{v} = y - x$.

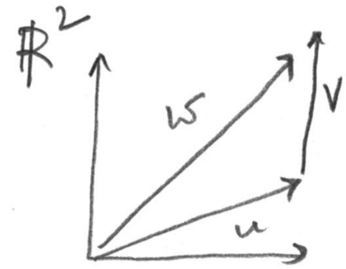
Intuitively, you can imagine an affine point space as a collection of points with a vector space V attached to each point



By definition, an affine point space has two parts: a <u>collection of points</u> + <u>a vector space</u>.

<u>CAVEAT</u>: Any vector space can also be viewed as an affine space by viewing the vectors as a collection of points, and "attaching" the same vector space to each point!

In fact, you may all have inadvertantly interpreted a vector space as an affine space while learning vector addition!



$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{w}$.

<u>Euclidean point space</u> is an affine space with its vector space chosen as the Euclidean space.

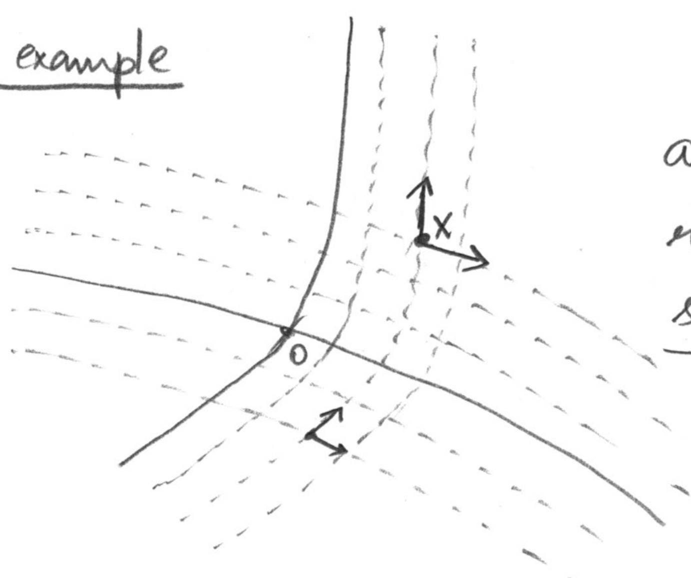
* Euclidean space is a vector space!

• " point space is an affine space!

Until this point vectors are fairly abstract entities which have certain algebraic properties. But is there a way to measure them? The moment we talk about "measurement", we should specify "relative to whom". An observer's measuring device is fixed using the notion of a <u>coordinate system</u>.

A coordinate system on the Euclidean point space is defined by first choosing a point as the "origin", and defining three coordinate curves that correspond to paths through space along which all but one of the coordinates are constant.

<u>2D example</u>



If the coordinate curves are not straight, then this results in <u>curvilinear coordinate system</u>, else it is called Cartesian coordinate system.

At each point, these coordinate curves intersect. The tangent vectors to these curves at each point defines a basis for the vector space attached to the point.

For a Cartesian coordinate system, the basis vectors do not change from point to point. In this course, we deal with only Cartesian coordinate system.

Basis coordinates in a Cartesian coordinate system are usually denoted by $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$. Any vector is represented $\underset{\sim}{u} = v_1 \underset{\sim}{e}_1 + v_2 e_2 + v_3 e_3$, or $[u_1, v_2, v_3]$, and the v_i's are called Cartesian components of $\underset{\sim}{v}$. How is the inner product represented?

Since the inner-product is bilinear, it is completely defined by the nine entities $\langle e_i, e_j \rangle$

$$\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = u_i v_j \langle e_i, e_j \rangle$$

If $\langle e_i, e_j \rangle = \delta_{ij}$, then the basis is called an orthonormal basis. In such cases,

$$\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = u_i v_i$$

<u>Note</u>: We will identify the points of the Euclidean point space with vectors of the underlying Euclidean space as

$$x - o \Longleftrightarrow \underset{\sim}{x} \in V, \text{ where } x, o \in \mathcal{E}.$$

A component of a vector u can be obtained by the operation

u_i = <u, e_i>

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<u>Tensor</u> : A Tensor is a linear transformation from one vector space to another. It is represented as

$$T : V \longrightarrow W \quad , \text{ where}$$

V, W are vector spaces.

What does "linear transformation" mean ?

$$\underset{\sim}{T} (\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha \underset{\sim}{T} \underset{\sim}{u} + \beta \underset{\sim}{T} \underset{\sim}{v} \qquad \forall \ \alpha, \beta \in \mathbb{R}; \ \underset{\sim}{u}, \underset{\sim}{v} \in V$$

Because of linear transformation property $\left(\underset{\sim}{T} \underset{\sim}{u} = \alpha_i \, T \underset{\sim}{e}_i \ , \text{where } \ \underset{\sim}{u} = \alpha_i \underset{\sim}{e}_i \right)$, $\underset{\sim}{T}$ is completely described by its action on the basis, i.e $\left\{ T\underset{\sim}{e}_1 , T\underset{\sim}{e}_2 , T\underset{\sim}{e}_3 \right\}$.

Typically, we deal with situations where $W = V$.

Therefore, T is completely described by

$$\langle T\underset{\sim}{e}_i , \underset{\sim}{e}_j \rangle .$$