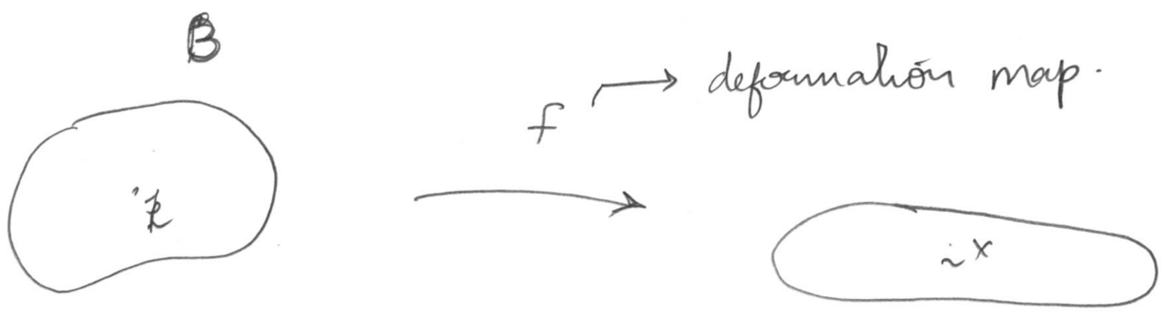
03.10.2019

Lecture 15

Infinitesimal strain

Recall the different deformation and strain tensors we discussed so far:



$$F_{iJ}(\underset{\sim}{p}) = \frac{\partial f_i}{\partial p_J} = \nabla f \quad \text{— deformation gradient}$$

$$\left.\begin{aligned} C(\underset{\sim}{p}) &:= F^T F(\underset{\sim}{p}) \\ B(\underset{\sim}{p}) &:= F F^T(\underset{\sim}{p}) \end{aligned}\right\} \text{C-G deformation tensors}$$

$$E(\underset{\sim}{p}) := \cancel{C(\underset{\sim}{p}) - I} \quad \text{— Lagrangian strain tensor.}$$

(C(p)-I)/2

Introduce a new field called the <u>displacement field</u> $u : \mathcal{B} \to \mathcal{V}$ defined as

$$u(\underset{\sim}{p}) = f(\underset{\sim}{p}) - \underset{\sim}{p},$$

which describes the displacement of particle $\underset{\sim}{p}$ from reference to current/deformed configuration.

Then

$$F_{iJ}(\mathbf{p}) := \frac{\partial f_i}{\partial p_J}(\mathbf{p})$$

$$= \frac{\partial}{\partial p_J}\left(u_i(\mathbf{p}) + p_i\right)$$

$$= \delta_{iJ} + \frac{\partial u_i}{\partial p_J}(\mathbf{p})$$

$$= I + \nabla \mathbf{u}(\mathbf{p})$$

$$\Longleftrightarrow \quad \boxed{F(\mathbf{p}) = I + \underbrace{\nabla \mathbf{u}}_{\text{Displacement gradient}}(\mathbf{p})}$$

Writing the Lagrangian strain in terms of the displacement gradient, we have

$$2E(\mathbf{p}) = F^T F - I$$

$$= (I + \nabla \mathbf{u})(I + \nabla \mathbf{u}^T) - I$$

$$= I + \nabla \mathbf{u}^T + \nabla \mathbf{u} + \nabla \mathbf{u}\nabla \mathbf{u}^T - I$$

$$= \nabla \mathbf{u}^T + \nabla \mathbf{u} + \nabla \mathbf{u}\nabla \mathbf{u}^T \quad \text{——} \ (1)$$

If the derivatives $\frac{\partial u_i}{\partial x_j}$ are small, then the Lagrangian strain can be approximated by

$$\underset{\approx}{\varepsilon} = \frac{\nabla u + \nabla u^T}{2} \quad \text{— infintesimal strain}$$

Notation: We are breaking our notational convention here as $\underset{\approx}{\varepsilon}$ is a greek letter and it is being used to describe a tensor.

From (1), we have

$$E = \underset{\approx}{\varepsilon} + \frac{\nabla u \nabla u^T}{2} \quad \text{— (2)}$$

Note: $\underset{\approx}{\varepsilon}$ in linear in the gradients of $\underset{\sim}{u}$, i.e if we have two displacement maps $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ then for $\underset{\sim}{u} = \underset{\sim}{u}_1 + \underset{\sim}{u}_2$, the corresponding $\underset{\approx}{\varepsilon}$ is given by

$$\underset{\approx}{\varepsilon} = \frac{\nabla \underset{\sim}{u} + \nabla u^T}{2}$$

$$= \underset{\approx}{\varepsilon}_1 + \underset{\approx}{\varepsilon}_2 \quad \text{where} \quad \underset{\approx}{\varepsilon}_i = \frac{\nabla \underset{\sim}{u}_i + \nabla \underset{\sim}{u}_i^T}{2}$$

Caution: Approximating E by $\underset{\approx}{\varepsilon}$ is reasonable only under "small" displacement gradients. In future lectures, we will discuss the uses of infinitesimal strain. For now, we note that if $\frac{\partial u_i}{\partial x_j}$ are not small, then $\underset{\approx}{\varepsilon}$ can sometimes be unphysical.

Example 1: Consider a rigid deformation map

$$f(\underset{\sim}{p}) = Q\underset{\sim}{p}, \quad Q \in \mathrm{Orth}^+$$

Calculate the corresponding Lagrangian and infitesimal strains.

Since $F(p) \equiv Q$, it follows that

$E(p) \equiv \frac{F^T F - I}{2} = 0$. On the other hand,

$$\underset{\approx}{\varepsilon}(p) = \frac{\nabla u + \nabla u^T}{2}$$

$$= \frac{Q + Q^T}{2} - I \qquad \left(\because \begin{array}{l} u = Q\underset{\sim}{p} - \underset{\sim}{p} \\ \nabla u = Q - I \end{array} \right)$$

For the purpose of discussion let's assume be basis to be oriented such that Q is a clockwise rotation of angle θ about the z-axis $\Longrightarrow$

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$\underset{\approx}{\varepsilon} = \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\boxed{\text{small } \theta}$, $\cos\theta \simeq 1$ and $\underset{\approx}{\varepsilon} \approx 0$. For large θ, ε_{11} and ε_{22} are not small!.

$\boxed{\text{For large rotations, infinitesimal strain can be quite large!}}$

<u>Example</u> 2 : In this example, we ask ourselves are there any displacement/deformation maps for which $\underset{\approx}{\varepsilon} = 0$ while $E \neq 0$.

For a given $W \in \mathrm{Skw}$, consider a displacement map

$$\underset{\sim}{u}(\underset{\sim}{p}) = W\underset{\sim}{p} \quad \left(\Rightarrow f(\underset{\sim}{p}) = W\underset{\sim}{p} + \underset{\sim}{p} \right)$$

The deformation gradient $F(p) = W + I$, and $\nabla u = W$.

$$\Rightarrow \quad \underset{\approx}{\varepsilon} = \frac{W + W^T}{2} = 0.$$

On the other hand

$$E(p) = \frac{(W+I)^T(W+I) - I}{2}$$

$$= \frac{W^TW + W^T + W}{2}$$

$$= \frac{W^TW}{2}.$$

$$= \frac{1}{2}\begin{bmatrix} \beta^2+\gamma^2 & -\alpha\beta & -\alpha\gamma \\ -\alpha\beta & \gamma^2+\alpha^2 & -\beta\gamma \\ -\alpha\gamma & -\beta\gamma & \alpha^2+\beta^2 \end{bmatrix}, \text{ where } W = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}$$

If $\alpha = \beta = 0$, then

$$E(p) = \frac{1}{2}\begin{bmatrix} \gamma^2 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The map $u(p) = Wp$ is called infinitesimal rigid displacement.