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Lecture 6

Properties of transpose of a tensor:

(i) $(T^T)^T = T$ (iv) $I^T = I$

(ii) $(S+T)^T = S^T + T^T$ (v) $0^T = 0$

(iii) $(\alpha T)^T = \alpha T^T$ (vi) $(\underset{\sim}{u} \otimes \underset{\sim}{v})^T = \underset{\sim}{v} \otimes \underset{\sim}{u}$,

Solution of (i): We want to show that the transpose of T^T is T. We can use indices to show that the components of $(T^T)^T$ are identical to those of T:

$$\left[(T^T)^T\right]_{ij} = (T^T)_{ji}$$

$$= T_{ij}.$$

Alternately, we can show using an index-free method:

For every $\underset{\sim}{u}, \underset{\sim}{v}$, taking $S = T^T$, we know

$$\underset{\sim}{u} \cdot T\underset{\sim}{v} = T\underset{\sim}{v} \cdot \underset{\sim}{u} = \underset{\sim}{v} \cdot T^T\underset{\sim}{u} \quad \text{(From the definition of the transpose)}$$

$$\overset{(S=T^T)}{=} \underset{\sim}{v} \cdot S\underset{\sim}{u} = S\underset{\sim}{u} \cdot \underset{\sim}{v}$$

$$\Longrightarrow \quad S^T = T$$

I will leave the remaining problems as exercise.

Definition: A second-order tensor T is symmetric if

$$T^T = T,$$

and it is antisymmetric/skew-symmetric if

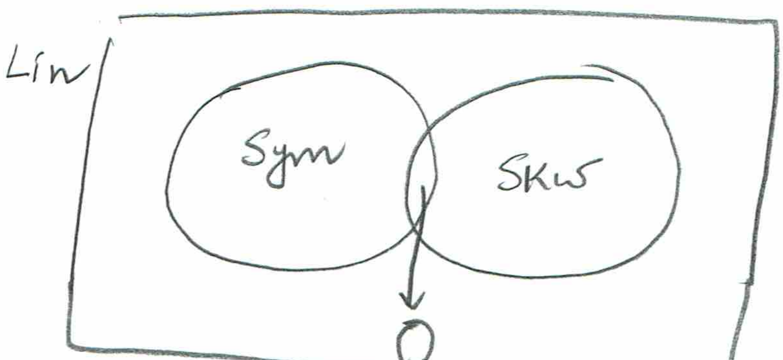
$$T^T = -T.$$

The set of all symmetric tensors is denoted by Sym

$$\mathrm{Sym} = \{ T \in \mathrm{Lin} : T = T^T \}$$

The set of all antisymmetric tensors is denoted by Skw

$$\mathrm{Skw} = \{ T \in \mathrm{Lin} : T = -T^T \}.$$



0 is the only tensor that belongs to Sym and Skw.

Theorem: Every second-order tensor T admits a unique decomposition

$$T = S + W \quad \text{---} \quad (1)$$

where $S = \frac{T+T^T}{2} \in Sym$, $W = \frac{T-T^T}{2} \in Skw$.

Proof: From the properties of the transpose, it is easy to see that $S \in Sym$, $W \in Skw$. How about uniqueness? Assume $\exists$ an alternate decomposition

$$T = \underbrace{S' + W'}_{②}, \quad S' \in Sym \text{ and } W' \in Skw.$$

Then from ① and ②

$$S + W = S' + W'$$

$$\Rightarrow \underbrace{S - S'}_{\in Sym} = \underbrace{W' - W}_{\in Skw}$$

The only element that belongs to both Sym and Skw is $0 \Rightarrow S - S' = W' - W = 0$

$$\Rightarrow S = S', \ W = W'.$$

Product of tensors

We have seen that Lin is a vector space. This implies addition of two tensors, and scalar multiplication of a tensor with a scalar result in a tensor. Here we will define an additional operation called the <u>product</u> of two tensors which results in a tensor.

V - vector space

$S, T \in Lin$

$ST \in Lin$, and it is defined as

$$(ST)\underset{\sim}{u} = S(T\underset{\sim}{u}). \quad — ③$$

Convince yourself that ST is in fact a linear transformation.

How do the components of ST look like in terms of the components of S and T?

Recall $(ST)_{ij} = (ST)\underset{\sim}{e}_j \cdot e_i$

$$= S(T\underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i \quad (\text{From } ③) \quad — ④$$

Since $T\underset{\sim}{e}_j = (T\underset{\sim}{e}_j \cdot \underset{\sim}{e}_k)\underset{\sim}{e}_k$ (Recall $\underset{\sim}{v} = (\underset{\sim}{v} \cdot \underset{\sim}{e}_i)\underset{\sim}{e}_i$)

$$\Rightarrow \quad T\underset{\sim}{e}_j = T_{kj}\underset{\sim}{e}_k \quad — (5)$$

Substituting (5) in (4), we have

$$(ST)_{ij} = S(T_{kj}\underset{\sim}{e}_k) \cdot \underset{\sim}{e}_i$$

$$= T_{kj} \, S\underset{\sim}{e}_k \cdot \underset{\sim}{e}_i$$

$$= \underbrace{T_{kj} \, S_{ik}}$$

This is nothing but matrix vector multiplication.

The product of two tensors is not commutative i.e it is <u>not true</u> that $ST = TS \;\; \forall S, T \in Lin$

Construct a counter example!

<u>Some properties of product of tensors</u>

(i) $(RS)T = R(ST)$ — product is associative

(ii) $T(S+R) = TS + TR$ — distributive w.r.t addition

(iii) $(\alpha S)(\beta T) = (\alpha\beta)(ST)$

(iv) I is the <u>unit</u> element under multiplication i.e. $$IS = SI = S$$

(v) $(ST)^T = T^T S^T$

(v) $S(\underset{\sim}{u} \otimes \underset{\sim}{v}) = S\underset{\sim}{u} \otimes \underset{\sim}{v}$; $(u \otimes v)S = \underset{\sim}{u} \otimes S^T \underset{\sim}{v}$.

(vi) $(a \otimes b)(c \otimes d) = (b \cdot c)\, a \otimes d$.

Determinant and trace of a tensor

The determinant of a tensor is defined as the determinant of its matrix with respect to an orthonormal basis. But this definition depends on the choice of the basis.

It turns out that the determinant (just like the inner product) is independent of the choice of the basis!

From previous class, recall the transformation of the components of a tensors under a change of basis :

$$[T'_{ij}] = \Lambda [T_{ij}] \Lambda^T$$

$$\Rightarrow \det [T'_{ij}] = (\det \Lambda) (\det [T_{ij}]) (\det \Lambda^T)$$

$$= \det [T_{ij}]$$ (Since Λ is an orthogonal matrix, its determinant is 1)

→ Show that $\det(u \otimes v) = 0$

Another important property of a tensor that is independent of the choice of basis is the <u>trace</u>.
The trace of T is denoted by $tr(T)$, and defined as

$$tr(T) := T_{ii} \quad (6)$$ (i.e. the sum of the diagonal elements of the matrix of T)

(6) is a basis-dependent definition. For an alternate basis

$$tr(T) = T'_{ii}$$

$$= \Lambda_{ik} T_{kl} \Lambda_{il} \qquad \left([T'] = \Lambda [T] \Lambda^T \right)$$

$$= \big(\underbrace{\Lambda_{ik} \Lambda_{il}}_{\delta_{kl}} \big) T_{kl}$$

$$= T_{ll} \, .$$

Properties of trace :

(i) Linearity $\quad tr(\alpha S + \beta T) = \alpha \, tr(S) + \beta \, tr(T)$

(ii) $tr(\underset{\sim}{u} \otimes \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{v}$

(iii) $tr(T^T) = tr(T)$

(iv) $tr(ST) = tr(TS)$

(v) $tr(I) = 3$

(vi) $tr(0) = 0$.