- Product of Tensors
- Trace & determinant of a tensor
- inner-product of two tensors
- inverse of a tensor
- Orthogonal tensors

$S, T \in Lin(V)$ $(ST)\underset{\sim}{u} = S(T\underset{\sim}{u})$

$[ST] = [S][T]$

$(ST)_{ij} = S_{ik} T_{kj}$

$$\det([T_{ij}]) = \det([T_{ij}'])$$

$$tr(T) := T_{ii} = T_{11} + T_{22} + T_{33} \; ; \; T_{ii} = T_{ii}'$$

$$[T_{ij}'] = \Lambda [T_{ij}] \Lambda^T$$

$$\det([T_{ij}']) = \det(\Lambda) \det([T_{ij}]) (\det \Lambda^T)$$

$$\det(\underline{u} \otimes \underline{v}) = 0$$

$$T_{ii} = T_{ii}'$$

$$T_{ii}' = \Lambda_{ik} T_{kj} (\Lambda^T)_{ji}$$

$$= \Lambda_{ik} T_{kj} \Lambda_{ij}$$

$$= \Lambda^T_{ji} \Lambda_{ik} T_{kj}$$

$$= \delta_{jk} T_{kj}$$

$$= T_{jj}$$

1) $tr(\alpha S + \beta T) = \alpha\, tr(S) + \beta\, tr(T)$

linear function

$\det(A+B) \neq \det(A) + \det(B)$

2) $tr(\underline{u} \otimes \underline{v}) = \underline{u} \cdot \underline{v}$

3) $tr(I) = 3$

4) $tr(T^T) = tr(T)$

5) $tr(0) = 0$

6) $tr(ST) = tr(TS)$

$$(\underline{u} \otimes \underline{v}) w = (\underline{v} \cdot \underline{w}) \underline{u}$$

Inner Product of two tensors

Bilinear in both mappings

$\langle \underline{u}, \underline{w} \rangle$

Satisfies all a, b, c, it indeed is an inner product.

a) $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle \longleftrightarrow \alpha \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle$

$: \langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

b) commutativity

c) $\langle \underline{u}, \underline{u} \rangle > 0$

$S \cdot T = tr(ST^T)$ $\quad S \cdot T = tr(ST^T)$

$= (ST^T)_{ii}$ $\quad = S_{ik} T_{ik}$

$= S_{ik} T^T_{ki}$

Norm of a tensor: $|T| := \sqrt{T \cdot T}$

Inverse of a tensor: $S \in Lin$ is invertible if there exists another tensor T s.t. $ST = TS = I$

T is called inverse of S denoted by S^{-1}

$$[S^{-1}] = [S]^{-1}$$

Recall that the inverse of a matrix iff

$\det[S] \neq 0$

$\therefore S^{-1}$ exists $\iff \det S \neq 0$

Properties: $T, S \in Lin$, $\det(T)$

1) $T\underline{u} = \underline{v} \iff$

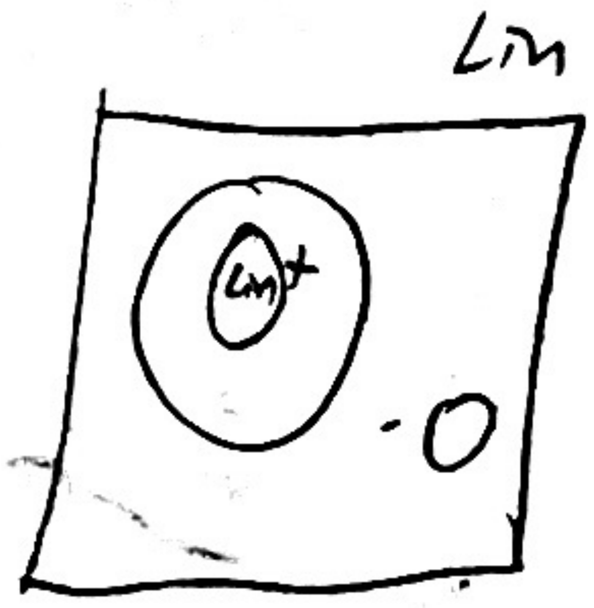
2) $\det(T^{-1}) = \dfrac{1}{\det T}$

3) $(ST)^{-1} = T^{-1} S^{-1}$

$Lin^+ = \{T \in Lin : \det T > 0\}$

$Inv = \{T \in Lin : \det T \neq 0\}$

not subspace
Closed under
Tensor Multiplication



Orthogonal Tensor

$QQ^T = Q^T Q = I$

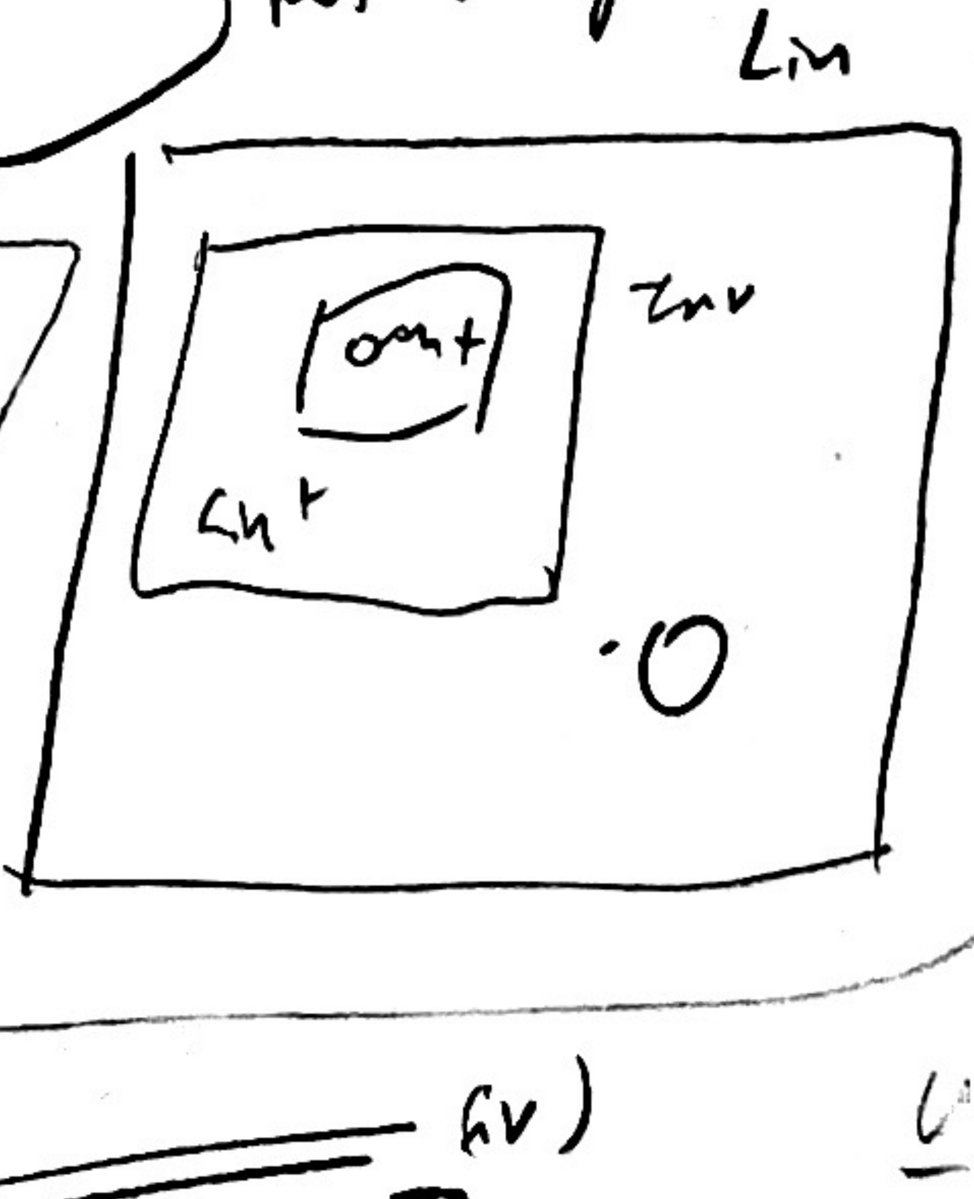
A tensor is orthogonal if

Rot + Inv

$Orth = \{T \in Lin : TT^T = T^T T = I\}$

Rot only

$Orth^+ := \{T \in Orth : \det T = 1\}$



Theorem: $Q \in Lin$. The following statements are equivalent

i) $Q \in Orth$, i.e. $QQ^T = Q^T Q = I$

ii) Q " preserves

iii) Q preserves magnitude

iv) Q " dilatives

(i) $\Longleftarrow$ (iv)

(i) $\Updownarrow$ (ii) $\Longrightarrow$ (iii) $\nearrow$ (iv)

$Orth \subset Inv$

(i) $\Rightarrow$ (ii): $\underline{u}, \underline{v} \in V$

$Q\underline{u} \cdot Q\underline{v}$

assume $QQ^T = Q^T Q = I$

ST $Q\underline{u} \cdot Q\underline{v} = \underline{u} \cdot \underline{v}$

$Q\underline{u} \cdot Q\underline{v} = \underline{v}\,(Q^T Q)\,\underline{v} = \underline{u} \cdot \underline{v}$

ii) $\Rightarrow$ i) $Q\underline{u} \cdot Q\underline{v} = \underline{u} \cdot \underline{v} \quad \forall \underline{u}, \underline{v} \in V$

$Q^T Q = I$

$(Q^T Q - I)\underline{u} \cdot \underline{v} = 0$

$(Q^T Q - I)\underline{u} \cdot \underline{v} = 0$

$\forall \underline{u}, \underline{v} \Rightarrow Q^T Q - I = 0$