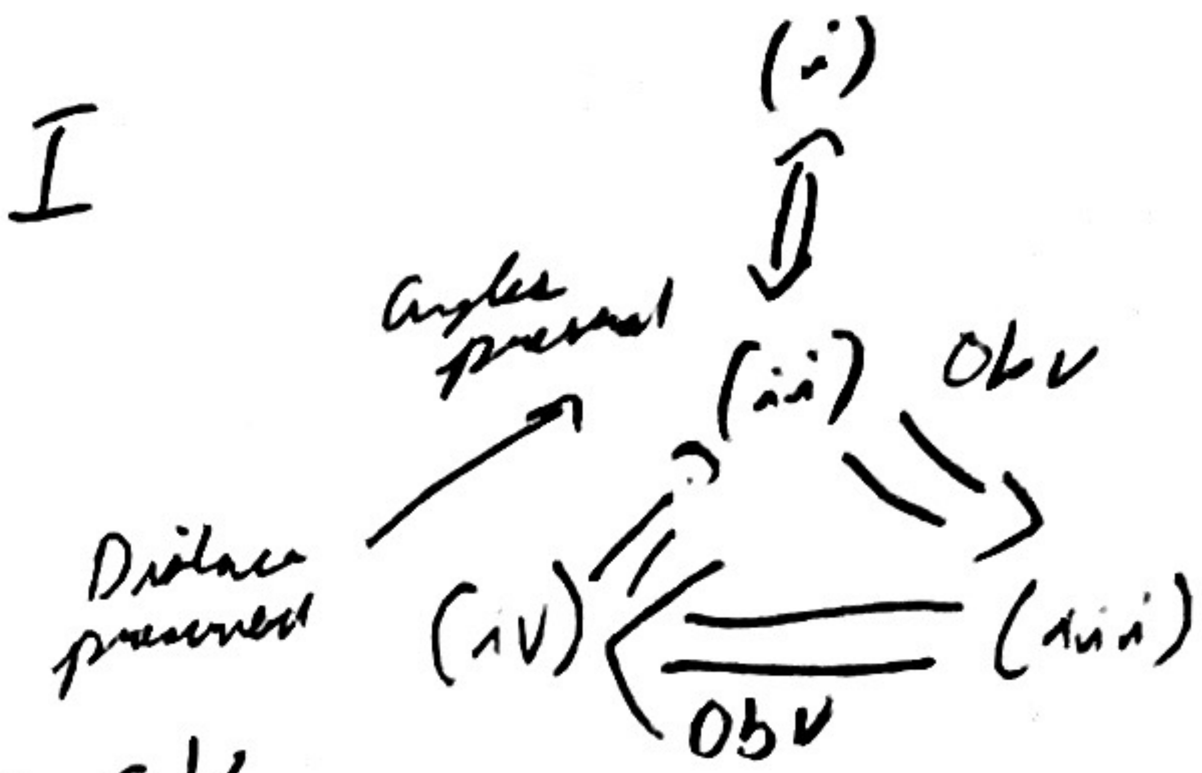
Orthogonal Tensors

i) $Q \in \mathrm{Orth}$, i.e. $QQ^T = I$

ii) $Q\underset{\sim}{u} \cdot Q\underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{v} \; \forall \underset{\sim}{u}, \underset{\sim}{v} \in V$

iii) $\| Q\underset{\sim}{x} \| = \| \underset{\sim}{x} \| \quad \forall \underset{\sim}{v} \in V$

iv) $\| Q\underset{\sim}{u} - Q\underset{\sim}{v} \| = \| \underset{\sim}{u} - \underset{\sim}{v} \|, \; \forall \underset{\sim}{u}, \underset{\sim}{v} \in V$



$T^T\underset{\sim}{u} \cdot x = \underset{\sim}{u} \cdot Tx$

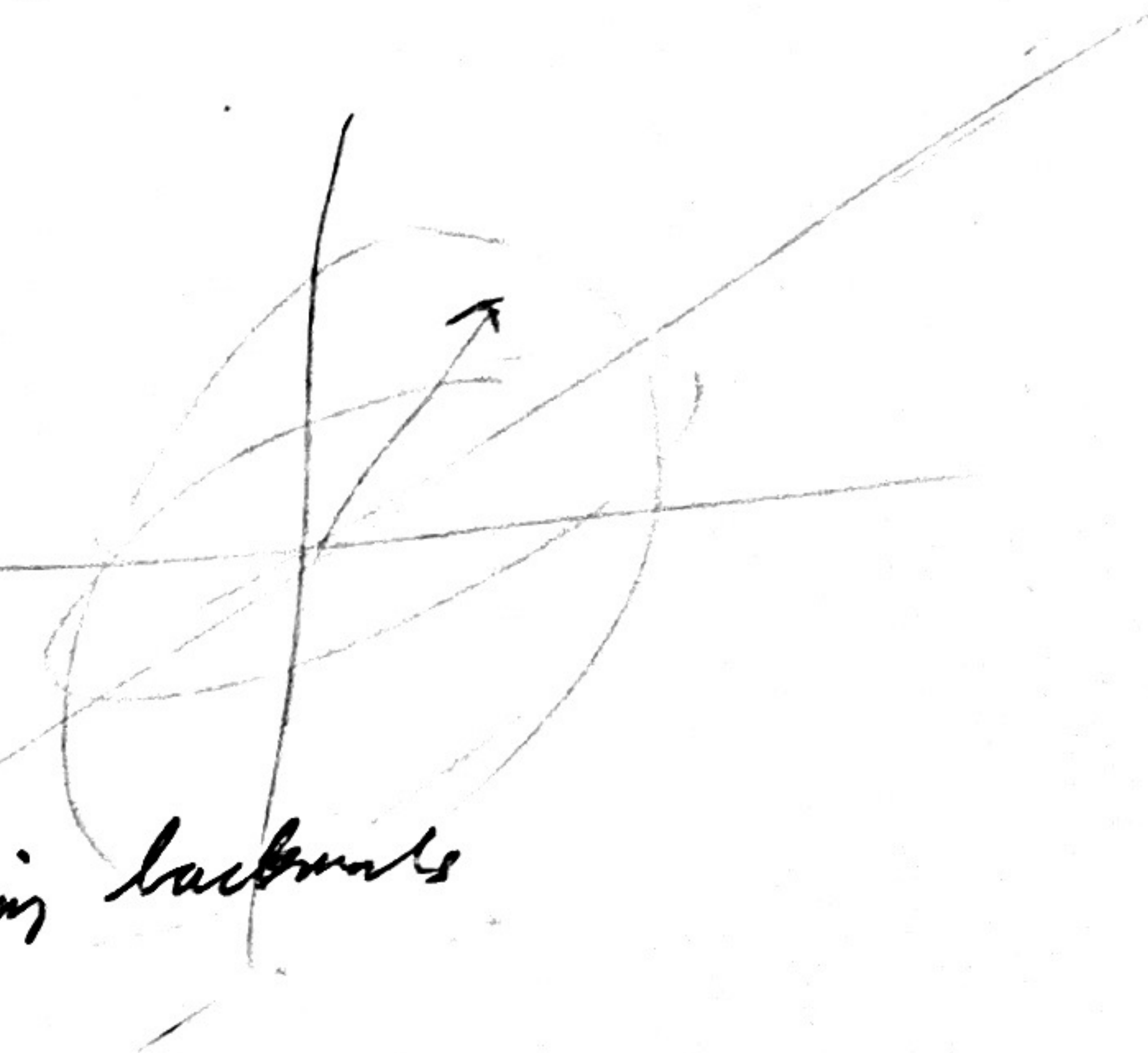
Assume $Q\underset{\sim}{u} \cdot Q\underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{v} \quad \forall \underset{\sim}{u}, \underset{\sim}{v} \in V$

$\Rightarrow \underset{\sim}{u} \cdot Q^T Q \underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{v}$

$\underset{\sim}{u} \left(Q^T Q \underset{\sim}{v} - \underset{\sim}{v} \right) = 0$

$\Rightarrow Q^T Q \underset{\sim}{v} - \underset{\sim}{v} = 0 \quad \forall \underset{\sim}{v} \in V$

$\Rightarrow (Q^T Q - I)\underset{\sim}{v} = 0 \; \forall \underset{\sim}{v} \in V$



Given a unit vector $\underset{\sim}{n}$ and an angle of rotation, can we construct a $Q \in \mathrm{Orth}^+$? and going backwards

Cross Product:

$(\underset{\sim}{u} \times \underset{\sim}{v})_i = \varepsilon_{ijk} u_j v_k$

$\hookrightarrow \varepsilon_{123} u_2 v_3 + \varepsilon_{132} u_3 v_2$

$= u_2 v_3 - u_3 v_2$

1) $\underset{\sim}{v} \times \underset{\sim}{u} = -\underset{\sim}{u} \times \underset{\sim}{v}$ (Anti-Commutative)

2) $\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \times v + \underset{\sim}{u} \times \underset{\sim}{w}$ (Distributive w.r.t. addition)

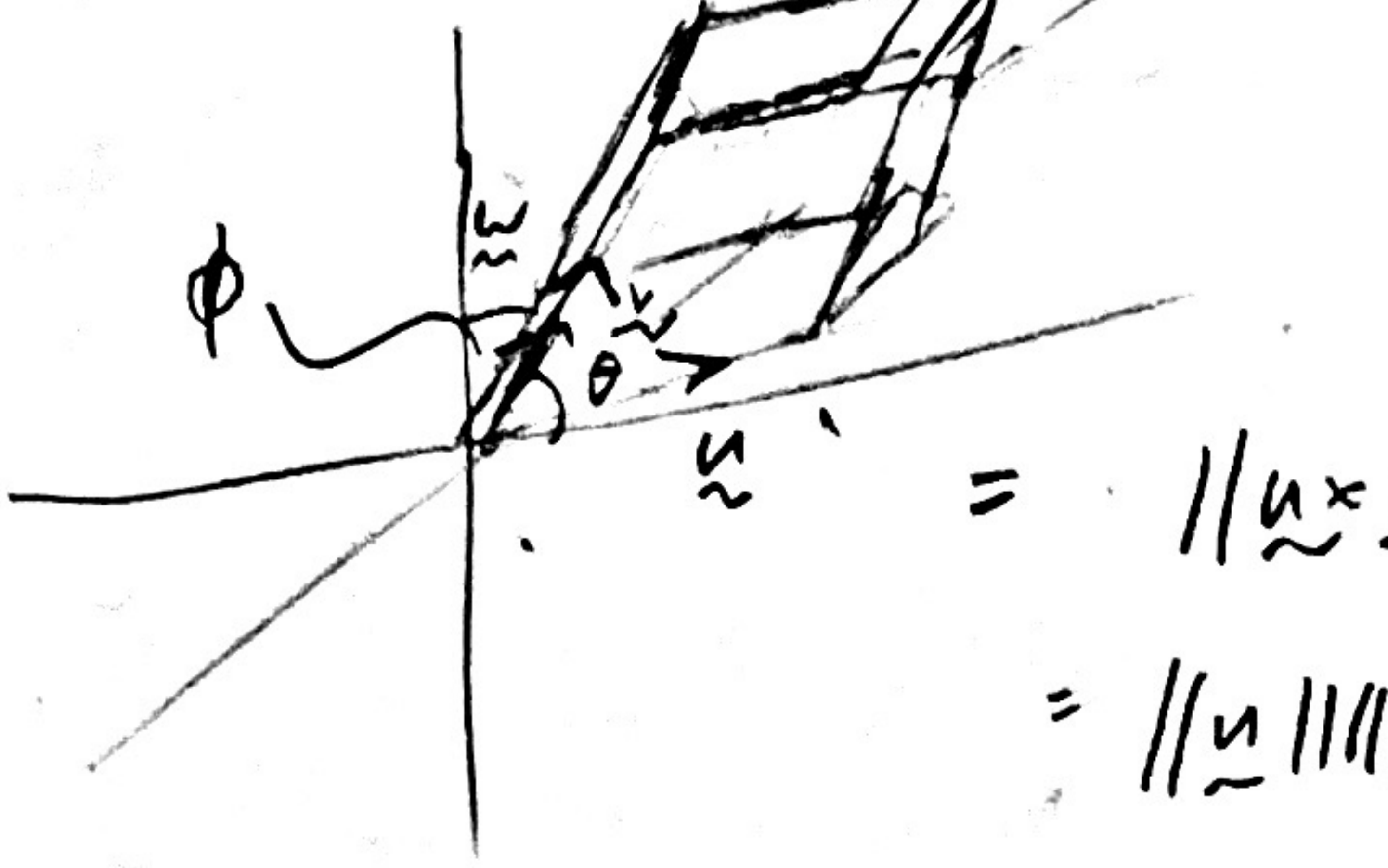
3) $(\lambda \underset{\sim}{u}) \times \underset{\sim}{v} = \lambda (\underset{\sim}{u} \times \underset{\sim}{v})$

4) $(\underset{\sim}{u} \times \underset{\sim}{v}) \times \underset{\sim}{w} = (\underset{\sim}{u} \cdot \underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{v} \cdot \underset{\sim}{w})\underset{\sim}{u}$

5) $\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w})$

$(u \times v)\cdot w = \varepsilon_{ijk} u_j v_k w_i$

$$= \det \begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$



$$= \|u \times v\| \, \|w\| \cos\phi$$

$$= \|u\| \, \|v\| \sin\theta \, \|w\| \cos\phi$$

Show that. $u \times v = 0 \iff u = \alpha v$

" " . $u \times v \perp$ to u & v

$(u \times v)\cdot v = 0$

$(u \times v)\cdot w = (w \times u)\cdot v = (v \times w)\cdot u$

Box product

Relationship between cross product and skew symmetric tensors

$W = -W^T$

<u>Theorem</u>: $W \in \text{Skw}$. There exists a unique vector $w \in V$ s.t.

$$Wu = w \times u \quad \forall u \in V$$

$W = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & & 0 \end{bmatrix}$, $w = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Given a skew tensor, there are items vectors ~~that~~ in V that when crossed, becomes a skew-tensor

$w_i = -\frac{1}{2}\varepsilon_{ijk} W_{jk}$ $\qquad Wu = (w \times u) = v$

Conversely, for a given $w \in V$, $\exists!$

$W \in \text{Skw}$ given by

$W_{ij} = -\varepsilon_{ijk} w_k$