Lecture 4

Recap: Last class we discussed

vector spaces,
dimension of a vector space
basis.

Then we introduced an additional structure called the inner product.

A useful identity about the inner product:

$$\boxed{\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \leq \|\underset{\sim}{u}\| \, \|\underset{\sim}{v}\|,} \quad \text{— (1)}$$

where $\|u\| := \sqrt{\langle u, u \rangle}$ is called the norm of $\underset{\sim}{u}$.
Equation (1) is called the Cauchy-Schwartz inequality (see note on Piazza for a proof).

Clarification: The word Euclidean space in some books ~~is some books~~ is used to refer to the Euclidean vector space, while in others it refers to the Euclidean point space. We will avoid this ambiguity by always referring explicitly to either "Euclidean VS" or "Euclidean point space".

We then discussed affine spaces, with the Euclidean point space as an example.

Tensor: A tensor is a linear transformation from one vector space to another. It is represented as

$$T : V \longrightarrow W \quad , \text{ where}$$

V, W are vector spaces.

What does "linear transformation" mean?

$$\underset{\sim}{T}(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha \underset{\sim}{T}\underset{\sim}{u} + \beta \underset{\sim}{T}\underset{\sim}{v} \qquad \forall\ \alpha, \beta \in \mathbb{R};\ \underset{\sim}{u}, \underset{\sim}{v} \in V$$

Because of linear transformation property ($T\underset{\sim}{u} = \alpha_i T\underset{\sim}{e}_i$, where $\underset{\sim}{u} = \alpha_i \underset{\sim}{e}_i$), T is completely described by its action on the basis, i.e. $\{T\underset{\sim}{e}_1, T\underset{\sim}{e}_2, T\underset{\sim}{e}_3\}$.

Typically, we deal with situations where $W = V$. Therefore, T is completely described by

$$T_{ij} := \langle T\underset{\sim}{e}_j, \underset{\sim}{e}_i \rangle .$$

Therefore, T can be represented as a matrix. But this representation depends on the choice of the basis. We will come back to the choice of basis later. For now, we will always assume we have a fixed orthonormal basis

What does $T\underset{\sim}{u}$ look like in indicial notation?

$T\underset{\sim}{u}$ is a vector, and for a given basis, let its components be $\alpha_1, \alpha_2, \alpha_3$:

$$T\underset{\sim}{u} = \alpha_1 \underset{\sim}{e}_1 + \alpha_2 \underset{\sim}{e}_2 + \alpha_3 \underset{\sim}{e}_3 .$$

Each component can be obtained as

$$\alpha_i = T\underset{\sim}{u} \cdot \underset{\sim}{e}_i .$$

Since $\underset{\sim}{u}$ can be represented as $\underset{\sim}{u} = u_j \underset{\sim}{e}_j$,

$$\alpha_i = T(u_j \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i$$

$$= u_j \, T\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i$$

$$= u_j \, T_{ij} \quad (\text{By the definition of } T_{ij})$$

Therefore, the ith component of $T\underset{\sim}{u}$ is given by $T_{ij} u_j$. In other words, $T\underset{\sim}{u}$ may be viewed as the product of a matrix and a vector!

We will now put together some facts resulting from the linearity property of a tensor.

1) A second-order tensor maps the zero vector into itself

$$T\underset{\sim}{0} = \underset{\sim}{0}$$

Two second-order tensors S and T are equal if

$$S\underset{\sim}{u} = T\underset{\sim}{u} \quad \forall \ \underset{\sim}{u} \in V.$$

The set of all second-order tensors is denoted by Lin.

2) Let $S, T \in$ Lin. Then their sum $S+T$ defined as

$$(S+T)u = Su + Tu,$$

is a second-order tensor. Similary the entities αT (where $\alpha \in \mathbb{R}$), O and I defined as

$$(\alpha T)\underset{\sim}{u} := \alpha(T\underset{\sim}{u}),$$

$$O\underset{\sim}{u} := \underset{\sim}{0},$$

$$I\underset{\sim}{u} := \underset{\sim}{u}$$

are all second-order tensors.

③ For any $T \in Lin$, $\lambda \in \mathbb{R}$

(i) $0T = \mathbf{0}$, (iv) $\mathbf{0} + T = T + \mathbf{0} = T$

(ii) $\lambda \mathbf{0} = \mathbf{0}$ (v) $T + (-T) = -T + T = \mathbf{0}$,

(iii) $1T = T$ where $-T := (-1)T$.

The subtraction of two second-order tensors is defined as $S - T := S + (-1)T$.

Theorem : The set of all second-order tensors is a real vector space.

Let us now look at a special kind of a second order tensor that is constructed from a pair of vectors. Given two vectors $\underset{\sim}{u}, \underset{\sim}{v} \in V$, the tensor $\underset{\sim}{u} \otimes \underset{\sim}{v}$ defined as

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})\, \underset{\sim}{w} := (\underset{\sim}{v} \cdot \underset{\sim}{w})\, \underset{\sim}{u},$$

is called a dyadic product of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

(convince yourself that it is indeed a tensor.)

Ex: Show that $\underset{\sim}{u} \otimes \underset{\sim}{v}$ is a tensor

Ex: The tensor product is homogeneous,

(i) $(\alpha \underset{\sim}{u}) \otimes (\beta \underset{\sim}{v}) = (\alpha \cdot \beta)(u \otimes v)$,

and distributive with respect to addition

(ii) $\underset{\sim}{u} \otimes (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \otimes \underset{\sim}{v} + \underset{\sim}{u} \otimes \underset{\sim}{w}$

$(\underset{\sim}{u} + \underset{\sim}{v}) \otimes \underset{\sim}{w} = \underset{\sim}{u} \otimes \underset{\sim}{w} + \underset{\sim}{v} \otimes \underset{\sim}{w}$

Ex: $u \otimes v \neq v \otimes w$ . . — Not commutative

Recall that for a given basis, a tensor T is completely represented by $T\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i$. We also showed that the space of tensors itself formed a vector space. Can we construct a basis for Lin? Before we do that, we have the following theorem

Theorem: Any second-order tensor T that maps a Euclidean vector space to itself, has a representation as

$$T = T_{ij} \, \underset{\sim}{e}_i \otimes e_j ,$$

where $T_{ij} = Te_j \cdot e_i$.

Proof. It is enough to show that

$$T\underset{\sim}{u} = (T_{ij}\, e_i \otimes e_j)\, \underset{\sim}{u}$$

We know that $T\underset{\sim}{u}$ is given by

$$(T\underset{\sim}{u})_i = T_{ij}\, u_j \quad \text{——} \ (*)$$

On the other hand

$\boxed{\text{By the definition of } u \otimes v}$

$$(T_{ij}\, e_i \otimes e_j)\, \underset{\sim}{u} = T_{ij}\, (\underset{\sim}{e_j} \cdot \underset{\sim}{u})\, \underset{\sim}{e_i}$$

$$= T_{ij}\, u_j\, \underset{\sim}{e_i}.$$

$\Rightarrow$ its ith component is $T_{ij} u_j$, which is identical to (*).