02.27.2020

Lecture 12

Relationship between divergence, gradient and curl

- $\operatorname{div}(\operatorname{curl} \underset{\sim}{u}) = 0$ Divergence of curl is zero
- $\operatorname{curl}(\nabla\varphi) = \underset{\sim}{0}$ Curl of a gradient of a scalar field is the zero vector field.

Kinematics : Bodies, deformations

In continuum mechanics, a body $\mathcal{B}$ is modeled as a closed region/subset of a Euclidean point space $\mathcal{E}$ with a piecewise smooth boundary $\partial\mathcal{B}$.

What does closed mean? In last class we have seen what an open set is. We have also seen what a boundary of an open set is. For any set A, we denote

A° – largest open set within A, called the interior of A

We know $\partial(A^\circ)$ – all points outside the open set $\mathring{A}$ such that every ball centered at the points of $\partial(\mathring{A})$ has an element of $\mathring{A}$.

$\bar{A}$, called the closure of A is defined as

$$\bar{A} := \mathring{A} \cup \partial \mathring{A}$$

A set B is said to be closed if $B = \bar{B}$.

<u>Examples</u> : 1)

$$A = (0, 1] := \{x \in \mathbb{R} : 0 < x \leq 1\}.$$

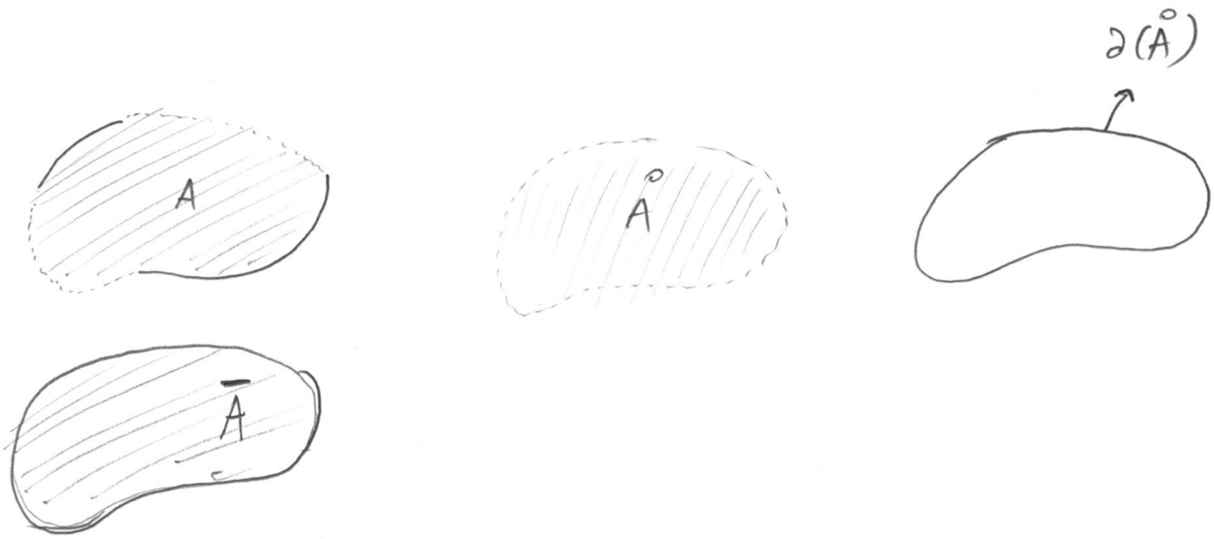
$$\mathring{A} = (0, 1) := \{x \in \mathbb{R} : 0 < x < 1\}.$$

$$\partial(\mathring{A}) = \{0, 1\}.$$

$$\bar{A} = \mathring{A} \cup \partial \mathring{A}$$

$$= [0, 1].$$

2)



<u>Note</u>: The notion of a boundary can be defined for any set, but we defined it only for open sets.

A closed region of $\mathcal{E}$ with piecewise smooth boundary is sometimes referred to as a <u>regular region</u>.

$\boxed{\text{A body is modeled as a regular region in } \mathcal{E}.}$

When the body is moved or deformed it occupies different closed regions of $\mathcal{E}$. We would like to describe its deformation. In order to do so, we choose a particular region occupied by $\mathcal{B}$ and call it the reference configuration.

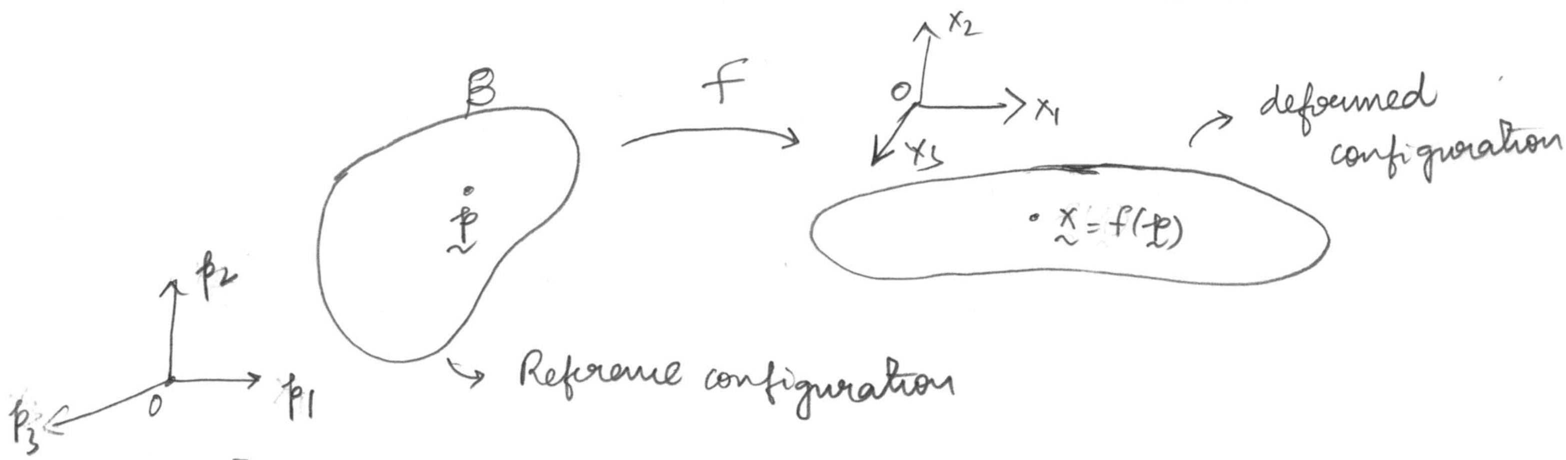
We can then describe the motion of $\mathcal{B}$ relative to the reference configuration using a mapping

$$f : \mathcal{B} \longrightarrow \mathcal{E}$$

$$f(\underset{\sim}{p}) = \underset{\sim}{x}.$$

The points in $\mathcal{B}$ are called material points.

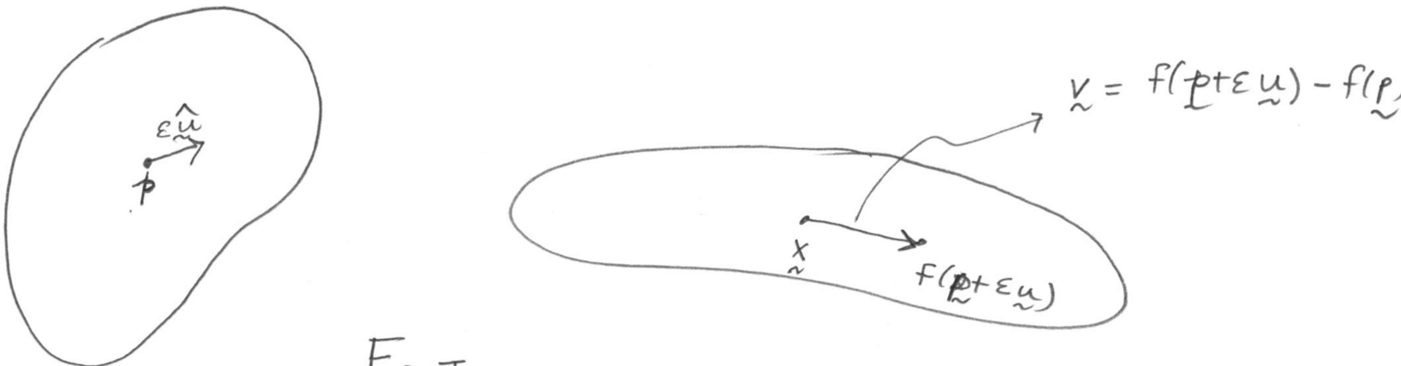
Any bounded subregion of $\mathcal{B}$ are called parts.



Deformation gradient $\quad F := \nabla f \quad \longleftrightarrow \quad F_{iJ} = f_{i,J}$

F is a tensor field. $F(\underset{\sim}{p})$ is a tensor that maps the vector space attached to $\underset{\sim}{p}$ to the vector space attached to $f(\underset{\sim}{p})$. Recall the coordinate-free definition of F:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\| f(\underset{\sim}{p} + \varepsilon \underset{\sim}{\hat{u}}) - f(\underset{\sim}{p}) - F(\underset{\sim}{p})(\varepsilon \underset{\sim}{\hat{u}}) \right\| = 0$$

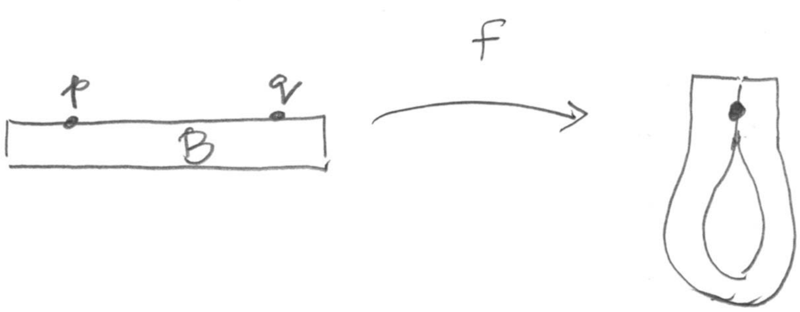


F_{iJ}

index corresponding to reference config (i)

represents index corresponding to the deformed configuration (J)

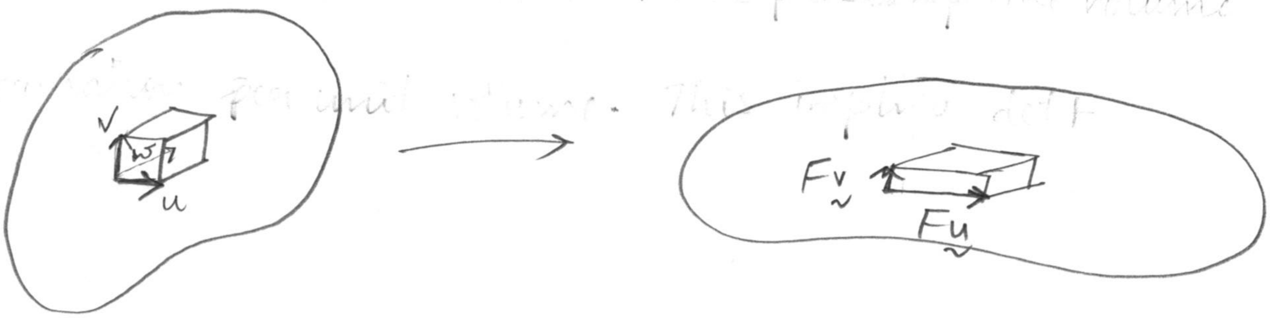
Since a function (by definition) cannot map a point $\underset{\sim}{p}$ to two points, our theory for deformation cannot describe <u>tearing, cracking, cleaving, splitting, fracture</u> etc. We make an additional assumptions on deformation: 1) we want every point $\underset{\sim}{p} \in B$ to be mapped to a unique point. In other words, if $\underset{\sim}{p} \neq \underset{\sim}{q}$ then $f(\underset{\sim}{p}) \neq f(\underset{\sim}{q})$.



Self penetration is ruled out.

Such a map is called a one-to-one map.

2) $\det F > 0$.



From HW3, you know $\det F = \dfrac{|F\mathbf{u}\cdot(F\mathbf{v}\times F\mathbf{w})|}{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}$

$\therefore$ $\det F > 0$ implies positive volumes do not collapse to zero volumes.

<u>Definition</u>: A <u>deformation</u> of $\mathcal{B}$ is a <u>smooth</u>, one-to-one map f that maps $\mathcal{B}$ to a closed region in $\mathcal{E}$, and satisfies

$$\det F > 0,$$

where $F = \nabla f$.

A deformation is called <u>translation</u> if $f(\mathbf{p}) = \mathbf{p} + \mathbf{u}$, $\mathbf{u}$ is a constant vector (as opposed to being a field)

A deformation is called <u>homogeneous</u> if the deformation gradient is constant.

Examples:

1) $\underset{\sim}{x} = \underset{\sim}{p} + \underset{\sim}{u}$, constant u — translation

2) $\underset{\sim}{x} = Q\underset{\sim}{p}$, constant $Q \in \text{Orth}$ — rotation

3) $\underset{\sim}{x} = Q\underset{\sim}{p} + \underset{\sim}{u}$ — rotation + translation.

4) $\underset{\sim}{x} = F\underset{\sim}{p}$ where $F = I + e_1 \otimes e_2$ — shear.

A property that follows from the definition of deformation is

$$f(\overset{\circ}{B}) = \overset{\circ}{(f(B))} \quad \text{— interior maps to interior}$$

$$f(\partial B) = \partial f(B) \quad \text{— boundary maps to boundary}$$

Concept of strain

We would like to define a quantity that characterizes the "stretch" of material for any given deformation. Our intuition tells us that the first three examples of deformation given above do not result in material stretching.

Let $f: \mathcal{B} \to \mathcal{E}$ be a deformation, and

$F: \mathcal{B} \to \underbrace{Lin^+}$ be the resulting deformation gradient

Linear transformation with positive determinant.

Perform a polar decomposition of $F(p)$:

$$F(\underset{\sim}{p}) = R(\underset{\sim}{p})\, U(\underset{\sim}{p})$$

$$= V(\underset{\sim}{p})\, R(\underset{\sim}{p})$$

for each $\underset{\sim}{p} \in \mathcal{B}$ resulting in tensor fields.

U - right stretch tensor field

V - left " " "

R - rotation field.

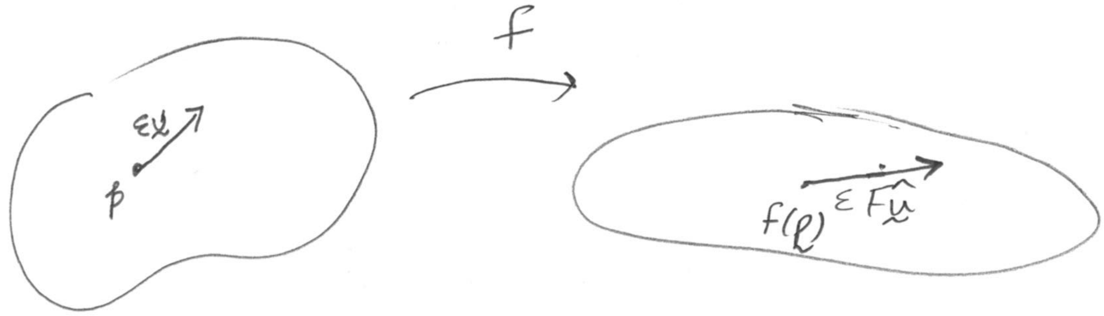
Geometric significance of F, R, U, V

We have already seen that $F(p)$ maps a vector in the vector space attached to p to a vector in a vector space attached to $F(p)$. But what does this physically mean?

Consider an arbitrarily small material fiber emanating

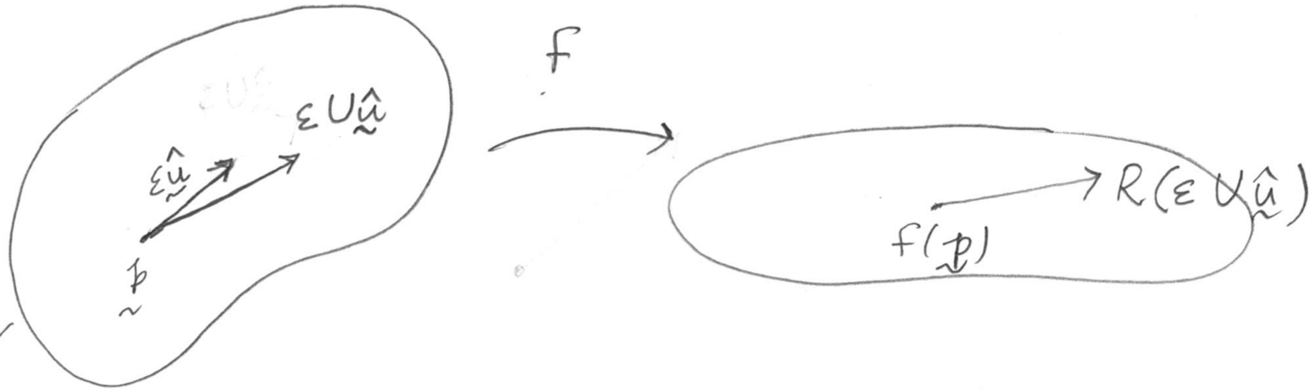
from point p in the direction of the unit normal $\hat{u}$.
Then

$$F(\varepsilon \hat{u}) = \varepsilon F \hat{u}.$$



F deforms $\varepsilon \hat{u}$ to $\varepsilon F \hat{u}$.

Caution : This has to be interpreted in an infinitesimal sense!



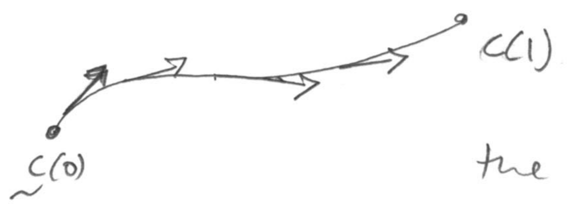
$U(p)$ maps $\varepsilon \hat{u}$ to $\varepsilon U \hat{u}$ in the same vector space, and $R(p)$ maps $\varepsilon U \hat{u}$ to $F \hat{u}$.

If $\hat{u}$ were an eigenvector of U, then $U\hat{u}$ is parallel to $\hat{u}$ and stretched/contracted relative to $\hat{u}$.

Let us now see how a deformation acts on a curve in terms of its length. But first, what is a curve?

A curve $\underset{\sim}{c}$ in $\mathcal{E}$ is a smooth map $c : [0,1] \to \mathcal{E}$ with $\dot{\underset{\sim}{c}} \neq \underset{\sim}{0}$ (note $\dot{\underset{\sim}{c}}(t)$ is a vector in the vector space attached to $\underset{\sim}{c}(t)$, and it is tangent to the curve)

$$\text{length}(\underset{\sim}{c}) := \int_0^1 \underbrace{|\dot{\underset{\sim}{c}}(t)|}_{\text{norm of } \dot{\underset{\sim}{c}}(t)} dt \qquad \text{— (1)}$$



Although the above definition depends on the parametrization of $\underset{\sim}{c}$, it can be shown that it is independent of parametrization.

<u>Theorem</u>: Given any curve c in $\mathcal{B}$,

$$\text{length}(\underbrace{f \circ c}_{:= f(c)}) = \int_0^1 |U(\underset{\sim}{c}(t))\, \dot{\underset{\sim}{c}}(t)|\, dt$$

<u>Proof</u>: By definition given in ①, we have

$$\text{length}(f\circ c) = \int_0^1 \left| \dot{\overline{f(\underset{\sim}{c}(t))}} \right| dt$$

$$= \int_0^1 |\nabla F \, \dot{\underset{\sim}{c}}(t)| \, dt$$

(Note: $f \circ c(t) = f(c_1(t), c_2(t), c_3(t)) \Rightarrow \dot{\overline{f\circ c}} = \dfrac{\partial F}{\partial x_i} \dot{c}_i(t)$
$= \nabla F \, \dot{\underset{\sim}{c}}(t)$)

$$= \int_0^1 |R U \, \dot{\underset{\sim}{c}}(t)| \, dt$$ — Using polar decomposition $F = RU$.

$$= \int_0^1 \underbrace{|U \, \dot{\underset{\sim}{c}}(t)|}\, dt .$$

At every point of the curve, the infintesimal segment of length $|\dot{\underset{\sim}{c}}(t)|\,dt$ is streched to $|U\dot{\underset{\sim}{c}}(t)|\,dt$.

□

$$\boxed{\begin{array}{l} C := U^2 = F^T F \text{ - right Cauchy-Green deformation tensor} \\ B := V^2 = F F^T \text{ left Cauchy-Green deformation} \quad '' \end{array}}$$

$\boxed{E := \frac{1}{2}(U^2 - I)}$ — Lagrangian strain tensor.

We know U describes the stretch of material fibers.

What do C, B and E describe?

Example [uniform stretch]:

$$f_1(\underset{\sim}{p}) = \alpha_1 p_1 \; ; \quad f_2(\underset{\sim}{p}) = \alpha_2 p_2 \; ; \quad f_3(\underset{\sim}{p}) = \alpha_3 p_3$$

Deformation gradient:

$$F(\underset{\sim}{p}) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$$C(\underset{\sim}{p}) = \begin{bmatrix} \alpha_1^2 & 0 & 0 \\ 0 & \alpha_2^2 & 0 \\ 0 & 0 & \alpha_3^2 \end{bmatrix} \; ; \quad U(\underset{\sim}{p}) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$$E(\underset{\sim}{p}) = \begin{bmatrix} \frac{\alpha_1^2 - 1}{2} & 0 & 0 \\ 0 & \frac{\alpha_2^2 - 1}{2} & 0 \\ 0 & 0 & \frac{\alpha_3^2 - 1}{2} \end{bmatrix}.$$

U_{11} describes stretch in 1-direction.

E_{11} describes strain " " .

If $\alpha_i = 1$, i.e no deformation stretch $= 1$, strain $= 0$, or in other words $E \equiv 0, \quad C \equiv I.$

Example:

$$f(\underset{\sim}{p}) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$F(p) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C(p) = F^T F = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E(p) = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case what are the stretches, i.e what are the eigenvalues of $\mathbf{U}$? Eigenvalue - eigenvector pairs of C are given by:

$$\lambda_1^C = 1 - \gamma\beta^-, \quad \lambda_2^C = 1 + \gamma\beta^+, \quad \lambda_3^C = 1$$

where $\beta^{\pm} = \frac{1}{2}\left(\sqrt{4+\gamma^2} \pm \gamma\right) \geq 1,$

$$\hat{\underset{\sim}{u}}_1 = \frac{1}{\sqrt{1+(\beta^+)^2}} \begin{bmatrix} -\beta^+ \\ 1 \\ 0 \end{bmatrix},$$

$$\hat{\underset{\sim}{U}}_2 = \frac{1}{\sqrt{1+(\beta^-)^2}} \begin{bmatrix} \beta^- \\ 1 \\ 0 \end{bmatrix},$$

$$\hat{\underset{\sim}{U}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Recall that U and C have same eigenvectors while

$$\lambda_i^U = \sqrt{\lambda_i^C}.$$