Problem 1. Prove the following identities:

1. $\mathbf{S}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{S}\mathbf{u}) \otimes \mathbf{v}$

   According to the definition of the tensor product, the left hand side can be multiplied by the following:

$$\begin{aligned} \mathbf{S}(\mathbf{u} \otimes \mathbf{v})\mathbf{w} &= \mathbf{S}(\mathbf{v} \cdot \mathbf{w})\mathbf{u} \\ &= (\mathbf{v} \cdot \mathbf{w})\mathbf{S}\mathbf{u} \end{aligned} \tag{1}$$

   Multiplying by $\mathbf{w}$ on the right hand side:

$$(\mathbf{S}\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})(\mathbf{S}\mathbf{u}) \tag{2}$$

   Equation 1 and Equation 2 are equal after being both multiplied by $\mathbf{w}$. $\square$

2. $(\mathbf{u} \otimes \mathbf{v})\mathbf{S} = \mathbf{u} \otimes (\mathbf{S}^{\mathbf{T}}\mathbf{v})$

   Taking the transpose of both sides then multiplying by $\mathbf{w}$ yields:

$$\begin{aligned} ((\mathbf{u} \otimes \mathbf{v})\mathbf{S})^T\mathbf{w} &= \mathbf{S}^T(\mathbf{v} \otimes \mathbf{u}) \cdot \mathbf{w} \\ &= \mathbf{S}^T(\mathbf{u} \cdot \mathbf{w})\mathbf{v} \end{aligned}$$

   Right hand side:

$$\begin{aligned} (\mathbf{u} \otimes (\mathbf{S}^T\mathbf{v}))^T = ((\mathbf{S}^T\mathbf{v}) \otimes \mathbf{u})\mathbf{w} &= (\mathbf{u} \cdot \mathbf{w})\mathbf{S}^T\mathbf{v} \\ &= \mathbf{S}^T(\mathbf{u} \cdot \mathbf{w})\mathbf{v} \end{aligned}$$

   The LHS and RHS are equal. $\square$

3. $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d}$

   Using the identity below:

$$\mathbf{S}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{S}\mathbf{u}) \otimes \mathbf{v}$$

$$(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d}$$

   Multiplying both sides by $\mathbf{w}$:

$$(((\mathbf{b} \cdot \mathbf{c})\mathbf{a}) \otimes \mathbf{d})\mathbf{w} = (\mathbf{d} \cdot \mathbf{w})((\mathbf{b} \cdot \mathbf{c})\mathbf{a})$$

   Multiplying the RHS by $\mathbf{w}$:

$$((\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d})\mathbf{w} = (\mathbf{d} \cdot \mathbf{w})((\mathbf{b} \cdot \mathbf{c})\mathbf{a})$$

   The LHS and RHS are equal. $\square$

4. $\mathbf{R} \cdot (\mathbf{S}\mathbf{T}) = (\mathbf{S}^T\mathbf{R}) \cdot \mathbf{T} = (\mathbf{R}\mathbf{T}^T) \cdot \mathbf{S}$

   By the definition of the transpose:

$$\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^T\mathbf{v}$$

   Taking the transpose of each term:

$$\begin{aligned} &(\mathbf{R} \cdot (\mathbf{S}\mathbf{T}))^T = ((\mathbf{S}^T\mathbf{R}) \cdot \mathbf{T})^T = ((\mathbf{R}\mathbf{T}^T) \cdot \mathbf{S})^T \\ \implies &\operatorname{tr}(\mathbf{R}(\mathbf{S}\mathbf{T})^T) = \operatorname{tr}(\mathbf{S}^T\mathbf{R}\mathbf{T}^T) = \operatorname{tr}((\mathbf{R}\mathbf{T}^T)\mathbf{S}^T) \\ \implies &\operatorname{tr}(\mathbf{R}\mathbf{T}^{\mathbf{T}}\mathbf{S}^{\mathbf{T}}) = \operatorname{tr}(\mathbf{R}\mathbf{T}^{\mathbf{T}}\mathbf{S}^{\mathbf{T}}) = \operatorname{tr}(\mathbf{R}\mathbf{T}^{\mathbf{T}}\mathbf{S}^{\mathbf{T}}) \qquad (1) \end{aligned}$$

   Equation 1 are all equal. $\square$

5. $\mathbf{S} \cdot (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{S}\mathbf{v}$

   Taking the trace of the LHS:

$$\begin{aligned}
\mathbf{S} \cdot (\mathbf{u} \otimes \mathbf{v}) &= \operatorname{tr}(\mathbf{S}(\mathbf{v} \otimes \mathbf{u})) \\
&= \operatorname{tr}((\mathbf{v} \otimes \mathbf{u})\mathbf{S}) = \operatorname{tr}(\mathbf{S}\mathbf{v} \otimes \mathbf{u}) \\
&= \mathbf{S}\mathbf{v} \cdot \mathbf{u}
\end{aligned}$$

   Dot products are commutative, so the LHS is equal to the S:

$$\mathbf{u} \cdot \mathbf{S}\mathbf{v} = \mathbf{S}\mathbf{v} \cdot \mathbf{u}$$

$\square$

6. $(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$

   Taking the trace of the LHS:

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) &= \operatorname{tr}((\mathbf{a} \otimes \mathbf{b})(\mathbf{d} \otimes \mathbf{c}) \\
&= \operatorname{tr}((\mathbf{b} \cdot \mathbf{d})\mathbf{a} \otimes \mathbf{c}) \\
&= (\mathbf{b} \cdot \mathbf{d})\mathbf{a} \cdot \mathbf{c} = (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c})
\end{aligned}$$

   The LHS is equal to the RHS. $\square$

Problem 2. Prove the following identities related to the cross product of vectors:

1. Prove the following:

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u \tag{1}$$

Converting from direct to indicial:

$$u = u_i e_i$$

$$v = v_j e_j$$

$$w = w_k e_k$$

$$\begin{aligned}(u \times v) \times w &= (u \cdot w)v - (v \cdot w)u \\ &= (u_i e_i \times v_j e_j) \times w_k e_k \\ &= u_i v_j w_k (e_i \times e_j) \times e_k \\ &= u_i v_j w_k (\epsilon_{ijl} e_l) \times e_k \\ &= u_i v_j w_k \epsilon_{ijl} (e_l \times e_k) \\ &= u_i v_j w_k \epsilon_{ijl} \epsilon_{lkm} e_m \\ &= u_i v_j w_k \epsilon_{ijl} \epsilon_{kml} e_m\end{aligned}$$

Using the following identity:

$$\epsilon_{ijl}\epsilon_{kml} = \delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}$$

$$\begin{aligned}&= u_i v_j w_k (\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}) e_m \\ &= v_j (u_i w_k \delta_{ik}) \delta_{jm} e_m - u_i (v_j w_k \delta_{kj}) \delta_{im} e_m \\ &= v_j (u_i w_i) e_j - u_i (v_j w_j) e_i\end{aligned}$$

Converting back to direct notation:

$$v_j(u_i w_i)e_j - u_i(v_j w_j)e_i = (u \times v) \times w \tag{2}$$

Equation 1 and Equation 2 are equal. $\square$

2. Prove the following:

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

Converting from direct to indicial:

$$A = a_i e_i$$

$$B = b_j e_j$$

$$C = c_k e_k$$

$$\begin{aligned} A \times (B \times C) &= a_i e_i \times (b_j e_j \times c_k eK) \\ &= a_i b_j c_k e_k \times (e_j \times e_k) \\ &= a_i b_j c_k e_i \times (\epsilon_{jkl} e_l) \\ &= a_i b_j c_k \epsilon_{jkl}(e_i \times e_l) \\ &= a_i b_j c_k \epsilon_{jkl} \epsilon_{ilm} e_m \\ &= a_i b_j c_k \epsilon_{jkl} \epsilon_{mil} e_m \end{aligned}$$

Using the $\epsilon - \delta$ relation:

$$= a_i b_j c_k (\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km}) e_m$$

$$= b_j (a_i c_k \delta_{ki}) e_m - c_k (a_i b_j \delta_{ji}) \delta_{km} e_m$$

$$= b_j (A \cdot C) e_j - c_k (A \cdot B) e_k$$

$$= B(A \cdot C) - C(A \cdot B)$$

$\square$

3. Prove the following:

$$\det S = \frac{Su \cdot (Sv \times Sw)}{u \cdot (v \times w)}$$

Converting from direct notation to indicial notation:

$$Su = S_{ij} u_j e_i$$

$$Sv = S_{km} v_m e_k$$

$$Sw = S_{ol} w_l e_o$$

$$u = u_j e_j$$

$$v = v_m e_m$$

$$w = w_l e_l$$

We can convert the numerator to the following:

$$\begin{aligned}
Su \cdot (Sv \times Sw) &= (S_{ij} u_j e_i) \cdot (S_{km} v_m e_k \times S_{ol} w_l e_o) \\
&= (S_{ij} u_j e_i) \cdot (S_{km} v_m S_{ol} w_l (e_k \times e_o)) \\
&= (S_{ij} u_j e_i) \cdot (S_{km} v_m S_{ol} w_l e_p \epsilon_{kop}) \\
&= \epsilon_{kop} S_{ij} u_j S_{km} v_m S_{ol} w_l (e_l \cdot e_p) \\
&= \epsilon_{kop} S_{ij} u_j S_{km} v_m S_{ol} w_l \delta_{ip} \\
&= \epsilon_{koi} S_{ij} S_{km} S_{ol} u_j v_m w_l \qquad (1)
\end{aligned}$$

Converting the denominator

$$\begin{aligned} u \cdot (v \times w) &= (u_j e_j) \cdot (v_m e_m \times w_l e_l) \\ &= (u_j e_j) \cdot (v_m w_l (e_m \times e_l)) \\ &= (u_j e_j) \cdot (v_m w_l e_p \epsilon_{mlp}) \\ &= \epsilon_{mlp} u_j v_m w_l \delta_{jp} \\ &= \epsilon_{mlj} u_j v_m w_l \qquad (2) \end{aligned}$$

Taking the ratio of Equation 1 and Equation 2:

$$\begin{aligned} \det S &= \frac{\epsilon_{koi} S_{ij} S_{km} S_{ol} \cancel{u_j v_m w_l}}{\epsilon_{mlj} \cancel{u_j v_m w_l}} \\ &= \frac{\epsilon_{koi} S_{ij} S_{km} S_{ol}}{\epsilon_{jml}} \end{aligned}$$

We know from the following identity:

$$\epsilon_{jml} \det S = \epsilon_{iko} S_{ij} S_{km} S_{ol}$$

Dividing the above by ϵ_{jml}:

$$\det S = \frac{\epsilon_{iko} S_{ij} S_{km} S_{ol}}{\epsilon_{jml}}$$

Equation 3 and Equation 4 are equal. $\square$

Problem 3. 1.

$$\Lambda = \begin{bmatrix} -0.5 & -0.5 & 0.707 \\ 0.707 & -0.707 & 0 \\ 0.5 & 0.5 & 0.707 \end{bmatrix}$$

2.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} -2.207 \\ -0.707 \\ 0.793 \end{bmatrix} \cdot \begin{bmatrix} e_1' & e_2' & e_3' \end{bmatrix}$$

3.

$$\mathbf{T} = \begin{bmatrix} -0.5 & -1 & -0.707 \\ 0.707 & -1.414 & 0 \\ 0.5 & 1 & -0.707 \end{bmatrix} \cdot \begin{bmatrix} e_1' & 0 & 0 \\ 0 & e_2' & 0 \\ 0 & 0 & e_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The determinant and trace are -2 and 2 respectively.

Problem 4.

$$n = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-}{1}\sqrt{6} \\ \frac{-}{1}\sqrt{6} & \frac{1}{\sqrt{6}} & 0 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{I} + (\sin(\theta))\mathbf{W} + (1 + \cos(\theta))\mathbf{W}^2$$

$$\mathbf{R} = \begin{bmatrix} 0.88 & -0.38 & 0.24 \\ 0.43 & 0.88 & -.159 \\ -0.159 & 0.24 & 0.955 \end{bmatrix}$$

$$\mathbf{Rn} = \begin{bmatrix} 0.40 \\ 0.40 \\ 0.81 \end{bmatrix}$$

It is inferred that $\mathbf{Rn}$ does not change.