02.03.2020

Lecture 5

Todays topics :

1) Recap 2) order of a tensor
3) Transformation rules for vectors and tensors.
4) Transpose of a tensor.

In the last class, we defined tensors as linear transformations between vector spaces. In particular, we considered only tensors that map the Euclidean vector space to itself. We showed that for a given basis, a tensor can be represented as a matrix, and the vector Tu can be obtained by the multiplication of the matrix T_{ij} with the column matrix u_k. We then defined some special tensors such as I, O, $\underset{\sim}{u} \otimes \underset{\sim}{v}$ (where $\underset{\sim}{u}, \underset{\sim}{v}$ are vectors). We then showed that the space of all tensors denoted by Lin is a vector space. We then showed that every tensor T can be represented as

$$T = T_{ij} \, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j} .$$

Order of a tensor : The tensors we have discussed so far are called second-order tensors.

0-order tensors — scalars

1st-order " — vectors.

An n^{th}-order tensor is a linear transformation that maps a vector space to the set of $(n-1)^{th}$-order tensors which is also a vector space.

The above definition implies, a vector can be viewed as a linear transformation from the space in which it lives to 0-order tensor, i.e scalars. In other words, $\underset{\sim}{v} : V \longrightarrow \mathbb{R}$. How is a vector a linear transformation?

$$\underset{\sim}{v}(\underset{\sim}{u}) = \underset{\sim}{v} \cdot \underset{\sim}{u} \quad , \quad \underset{\sim}{u} \in V$$

is a linear transformation. Of course, this identification depends on the inner-product. By tensor, we usually mean a second-order tensor.

Transformation rules for vectors and tensors:

V - Euclidean vector space. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ denote its basis.

Recall that a vector can be represented by its components with respect to the chosen basis as

$$\underset{\sim}{v} = v_i \underset{\sim}{e}_i \quad \text{——} \quad (1)$$

<u>Note</u>: By basis, we always mean orthonormal basis.

How would the components change if we choose an alternate basis? Let v_1', v_2', v_3' denote the components of $\underset{\sim}{v}$ in a new basis $\{\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3'\}$:

$$\underset{\sim}{v} = v_i' \underset{\sim}{e}_i' . \quad \text{——} \quad (2)$$

Recall that a component, say v_i' can be obtained from $\underset{\sim}{v}$ as $v_i' = \underset{\sim}{v} \cdot \underset{\sim}{e}_i'$. But $\underset{\sim}{e}_i'$ itself can be represented with respect to basis $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ as

$$\underset{\sim}{e}_i' = \lambda_{ij} \underset{\sim}{e}_j \quad \text{——} \quad \text{these are three equations.} \quad (3)$$

What are λ_{ij}'s?

λ_{ij} is the cosine of the angle between vectors $\underset{\sim}{e}_i'$ and $\underset{\sim}{e}_j$. Since the angle between $\underset{\sim}{e}_i'$ and $\underset{\sim}{e}_j$ is the same as angle between $\underset{\sim}{e}_j$ and $\underset{\sim}{e}_i'$ it follows that

$$\underset{\sim}{e}_i = \lambda_{ji}\, \underset{\sim}{e}_j' \quad — (4)$$

From (1), (2) and (4) we have

$$\underset{\sim}{v} = v_j'\, \underset{\sim}{e}_j' = v_i\, \underset{\sim}{e}_i$$

$$= v_i\, \lambda_{ji}\, \underset{\sim}{e}_j'$$

$$\Rightarrow \boxed{v_j' = \lambda_{ji}\, v_i} \quad — (5)$$

Similarly, from (1), (2) and (3) we have

$$\underset{\sim}{v} = v_j\, \underset{\sim}{e}_j = v_i'\, \underset{\sim}{e}_i'$$

$$= v_i'\, \lambda_{ij}\, \underset{\sim}{e}_j$$

$$\Rightarrow \boxed{v_j = \lambda_{ij}\, v_i'} \quad — (6)$$

If λ_{ij}s can be collected as a matrix

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix},$$

then 5 and 6 show that

$$\boxed{\Lambda^{-1} = \Lambda^{T}.} \quad \text{--- } 7$$

or in other words $\Lambda \Lambda^T = \Lambda^T \Lambda =$ identity matrix. Such matrices are called orthogonal matrixes.

Ex: Is Λ^T also an orthogonal matrix?

Ex: Recall the definition of dot product:

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_i v_i .$$

How does an inner product appear when we choose a different basis?

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= u_i v_i \\ &= (\lambda_{ji} u'_j)(\lambda_{ki} v'_k) \end{aligned}$$

$$= \underbrace{\lambda_{ji}\lambda_{ki}}_{(\Lambda\Lambda^T)_{jk} = \delta_{jk}} u_j' v_k'$$

$$= u_k' v_k' .$$

This means the inner product between two vectors does not depend on the choice of the basis

Let us now see how tensors transform under a change of basis. Recall that any tensor can be represented as

$$T = T_{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = T_{ij}'\, \underset{\sim}{e}_i' \otimes \underset{\sim}{e}_j'$$

for a given basis' $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ and $\{\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3'\}$

From 3, we have

$$T_{ij}'\, \underset{\sim}{e}_i' \otimes \underset{\sim}{e}_j' = T_{ij}' (\lambda_{ik} \underset{\sim}{e}_k) \otimes (\lambda_{jl} \underset{\sim}{e}_l)$$

$$= T_{ij}' \lambda_{ik} \lambda_{jl}\, \underset{\sim}{e}_k \otimes \underset{\sim}{e}_l$$

Interchange dummy indices

$$\overset{i \leftrightarrow k}{\underset{j \leftrightarrow l}{=}} T_{kl}' \lambda_{ki} \lambda_{lj}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\Rightarrow \quad T_{ij} = T_{kl}' \lambda_{ki} \lambda_{lj} .$$

In matrix form, we have

$$[T] = \Lambda^T [T'] \Lambda .$$

Similarly (using 7) $[T'] = \Lambda [T] \Lambda^T$, ie.

$$T'_{ij} = \Lambda_{ik} T_{kl} \Lambda_{jl} .$$

<u>Transpose of a second-order tensor</u> : V = Euclidean vector space

S - second order tensor. The transpose of S, denoted as S^T is a unique second-order tensor that satisfies the condition

$$S\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot S^T \underset{\sim}{v} \qquad \forall \ \underset{\sim}{u}, \underset{\sim}{v} \in V$$

How does a transpose look like in indicial notation?

Recall that for any tensor T,

$$T_{ij} = T\underset{\sim}{e_j} \cdot e_i$$

$$\therefore S^T_{ij} = S^T e_j \cdot e_i = e_j \cdot S e_i = S_{ji}$$

$$\therefore \boxed{S^T_{ij} = S_{ji}}$$ $\rightarrow$ This also show transpose is unique!