Limits and Sequences Continued

Jerich Lee

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Problem 1 (11.5)**.** *Let* (q_n) *be an enumeration of all the rationals in the interval* $(0, 1]$*.*

1. *Give the set of subsequential limits for* (q_n)*.*
2. *Give the values of* $\limsup q_n$ *and* $\liminf q_n$*.*

Solution 1. 1. *The set of subsequential limits for* (q_n) *is* $q \in Q : 0 < q \leq 1$*.*

2. (a) $\limsup q_n = 1$
   (b) $\liminf q_n = 0$

Problem 2 (11.11)**.** *Let* S *be a bounded set. Prove there is an increasing sequence* (s_n) *of points in* S *such that* $\lim_{n\to\infty} s_n = \sup S$*. Compare Exercise 10.7. Note: If* $\sup S$ *is in* S*, it's sufficient to define* $s_n = \sup S$ *for all* n*.*

Solution 2. $u_n = \sup s_n$*, and* $s_n \leq \sup s_n$*. If* S *is bounded, then there exists a real number* a *such that* $s_n \leq a$*. By Theorem 11.4 in Ross, there exists a monotonic subsequence for bounded set* S*. We shall prove that this sequence is increasing.*

Proof. Because S is bounded, we know that there exists $\sup S$ by the Axiom of Completeness. We also know that monotone bounded sequences converge by Theorem 10.2 in Ross. Therefore, we can say:

$$\sup S - s_n < 1 \tag{1}$$

Where 1 is chosen arbitrarily. We can then choose a subsequence of S such that it satisfies the following conditions:

$$\sup S - s_n = \min\left(\sup S - s_{n-1}, \frac{1}{n}\right) \tag{2}$$

Choosing such s_n enforces that the sequence is increasing, and is getting closer and closer to the supremum. With the the values selected, we now have a subsequence of S such that $n_1 < n_2 < \ldots n_k$, $s_{n_1} < s_{n_2} < \ldots s_{n_k}$, and $\lim S_{n_k} = \sup S$. □

Problem 3 (12.3 (d,e))**.** *Let* (s_n) *and* (t_n) *be the following sequences that repeat in cycles of four:*

1. $(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$

2. $(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \dots)$

Find:

1. $\limsup(s_n + t_n)$

2. $\limsup s_n + \limsup t_n$

Solution 3. *1.* $\limsup(s_n + t_n) = 3$

2. $\limsup(s_n) + \limsup(t_n) = 4$

Problem 4 (12.4)**.** *Show* $\lim_{N\to\infty} \sup_{n>N}(s_n + t_n) \leq \lim_{N\to\infty} \sup_{n>N} s_n + \lim_{N\to\infty} \sup_{n>N} t_n$ *for bounded sequences* (s_n) *and* (t_n)*. Hint: First show:*

1. $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$.

Then apply Exercise 9.9 (c).
Suppose there exists N_0 *such that* $s_n \leq t_n$ *for all* $n > N_0$
Exercise 9.9 (c):

1. Prove that if $\lim s_n$ *and* $\lim t_n$ *exist, then* $\lim s_n \leq \lim t_n$*.*

Solution 4. *Proof.* We want to find the following:

$$\limsup_{N\to\infty} s_N + t_N \leq (\limsup_{N\to\infty} s_n) + (\limsup_{N\to\infty} t_n) \tag{3}$$

We know that the following holds true, as this is the definition of $\limsup$.

$$\limsup_{N\to\infty} s_N = \limsup_{M\to\infty} \{s_N | N \geq M\} \tag{4}$$

We also know that any element of a sequence s_n is always less than or equal to the supremum of s_n due to the Axiom of Completeness. Then, we can state the following.

$$N > M \implies s_N \leq \sup \{s_N | N \geq M\} \tag{5}$$

Therefore, the following also holds true from above:

$$s_N + t_N \leq \sup \{s_N | N > M\} + \sup \{t_N | N > M\} \tag{6}$$

The RHS of the above is a constant, so the following is true:

$$(s_N + t_N \leq C) \tag{7}$$

If any value of $s_n + t_n$ is always less than the RHS of above, then we can say the the supremum of $s_n + t_n$ is also less than or equal to the RHS of above:

$$\sup \{s_N + t_N | N > M\} \leq \sup \{s_N | N > M\} + \sup \{t_N | N > M\} \tag{8}$$

We have now established $\sup(s_n + t_n) \leq \sup s_n + \sup t_n$. Now, we can use a familiar limit theorem to prove our original hypothesis.

$$\limsup_{N\to\infty} s_N + t_N \tag{9}$$

We know that the above is equal to the following from the definition of lim sup. Then, we can take the limit of both sides. From Exercise 9.9c, we know that if if given N_0 *s.t.* $s_n \leq t_n$ for all $n > N_0$ $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ exist, then $\lim_{n\to\infty} \leq \lim_{n\to\infty} t_n$

$$= \limsup_{M\to\infty} \{s_N + t_N | N > M\} \leq \lim_{M\to\infty} (\sup \{s_N | N > M\} + \sup \{t_N | N > M\}) \tag{10}$$

From a known limit theorem, we know that $\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$:

$$= \limsup_{M\to\infty} \{s_N | N > M\} + \limsup_{M\to\infty} \{t_N | N > M\} \tag{11}$$

$$\limsup_{N\to\infty} (s_N + t_N) \leq \limsup_{N\to\infty} s_N + \limsup_{N\to\infty} t_N \tag{12}$$

□

Problem 5 (12.12)**.** *Let* (s_n) *be a sequence of nonnegative numbers, and for each* n *define* $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

1. *Show:*

$$\lim_{N\to\infty} \inf_{n>N} s_n \leq \lim_{N\to\infty} \inf_{n>N} \sigma_n \leq \lim_{N\to\infty} \sup_{n>N} \sigma_n \leq \lim_{N\to\infty} \sup_{n>N} s_n \tag{13}$$

   Hint: for the last inequality, show first that $M > N$ *implies:*

$$\sup \{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_n) + \sup \{s_n : n > N\} \tag{14}$$

2. *Show that if* $\lim_{n\to\infty} s_n$ *exists, then* $\lim_{n\to\infty} \sigma_n$ *exists and* $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} s_n$.

3. *Give an example where* $\lim_{n\to\infty} \sigma_n$ *exists, but* $\lim_{n\to\infty} s_n$ *does not exist.*

Solution 5. 1. *Proof.*

$$\liminf_{N\to\infty} s_N \leq \liminf_{N\to\infty} \sigma_N \leq \limsup_{N\to\infty} \sigma_N \leq \limsup_{N\to\infty} s_N \tag{15}$$

$$\lim_{n\to\infty} s_N = s \implies s = \liminf_{N\to\infty} s_N \leq \liminf_{N\to\infty} \sigma_N \leq \limsup_{N\to\infty} \sigma_N \leq \limsup_{N\to\infty} s_N = s \tag{16}$$

By the Squeeze Theorem,

$$\implies \liminf_{N\to\infty} \sigma_N = \limsup_{n\to\infty} \sigma_N = s \tag{17}$$

$$\implies \lim_{N\to\infty} \sigma_N = s \tag{18}$$

We wish to find the third inequality as shown below, as the second inequality is obvious, and the first inequality can be proved similarly to the third inequality.

$$\limsup_{N\to\infty} \sigma_N \leq \limsup_{N\to\infty} s_N \tag{19}$$

We will fix L. Then, if $M > L$, we can consider $\sup\{\sigma_N|N > M\}$

$$\sigma_N = \frac{s_1 + \ldots s_N}{N} \tag{20}$$

$$= \frac{s_1 + \ldots + s_N}{N} + \frac{s_{L+1} + \ldots + s_N}{N} \tag{21}$$

We know that there are $N - (L + 1) + 1 = N - L$ total terms in the denominator of the second term of the RHS above. Therefore, we can state the following:

$$\sigma_N \leq \frac{s_1 + \ldots s_L}{N} + \underbrace{\frac{N - L}{N}}_{\leq 1} \sup\{s_N|N > L\} \tag{22}$$

$$\leq \frac{s_1 + \ldots + s_L}{M} + \sup\{s_N|N > L\} \tag{23}$$

$$\sup\{\sigma_N|N > M\} \leq \frac{s_1 + \ldots + s_L}{M} + \sup\{s_N|N > L\} \tag{24}$$

We know that the RHS above is independent of N, so we take the limit of both sides above. Doing so results in the following:

$$\limsup_{N\to\infty} \sup \sigma_N = \limsup_{M\to\infty} \sigma_N\{N|N > M\} \tag{25}$$

$$\leq \lim_{M\to\infty} (\underbrace{\frac{s_1 + \ldots + s_L}{M}}_{\text{goes to 0 as } M \to infinity} + \sup\{s_N|N > L\}) \tag{26}$$

$$\limsup_{N\to\infty} \sigma_N \leq \sup\{s_N|N > L\} \tag{27}$$

□

2. *Proof.*

$$|s_n - L| < \varepsilon \tag{28}$$

$$\sigma_n = \frac{1}{n}\Sigma_{i=1}^{n} s_n \tag{29}$$

$$\lim \frac{1}{n}\Sigma_{i=1}^{n} s_n \tag{30}$$

$$n > N \implies \Sigma_{i=1}^{n} s_n \tag{31}$$

$$\implies |ns_n - nL| \tag{32}$$

$$\frac{1}{n}|n(s_n - L)| < \varepsilon \tag{33}$$

$$= |(s_n - L)| < \varepsilon \tag{34}$$

$$= |s_n - L| < \varepsilon \tag{35}$$

$$= \lim \sigma_n = L = \lim s_n \tag{36}$$

□

3. *Proof.* Suppose $s_n = (-1)^n$. Then $\lim_{n\to\infty} s_n$ does **not** exist, because the subsequence of even indices converges to 1 while the subsequence of odd indices converges to -1. However

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n}(-1)^k = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and in both cases $\sigma_n \to 0$. Thus $\lim_{n\to\infty} \sigma_n = 0$ even though $\lim_{n\to\infty} s_n$ does not exist. □

Problem 6 (12.13)**.** *Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many s_n are $\geq a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \lim_{N\to\infty} \inf_{n>N} s_n$ and $\inf B = \lim_{N\to\infty} \sup_{n>N} s_n$.*

Solution 6. *From Theorem 11.7, there exists a monotonic subsequence whose limit is $\limsup s_n$ and $\liminf s_n$. So $\limsup s_n$ and $\liminf s_n$ exist. From Theorem 10.1, we know that these limits converge because s_n is bounded. Our goal is to show that $\limsup s_n = \inf B$, and $\liminf s_n = \sup A$, for sets $A = \{s_n : s_n < a\}$, and $B = \{s_n : s_n > b\}$, where A, B both have finite cardinality.*

Proof. Choose N_1 *s.t.* $n > N_1 \implies \{n : |s_n - a| < \varepsilon\}$ is infinite. By Theorem 11.2, a is a subsequential limit. Choose N_2 *s.t.* $n > N_2 \implies \{n : |s_n - b| < \varepsilon\}$ is infinite. By 11.2, b is also a subsequential limit. To prove that $a = \sup A$, choose $N = \min\{d(s_N, a)\}$. Then

$N = n + 1 \implies s_{n+1} > a$, so s_n is not the least upper bound of A. So $a = \sup A$. A similar argument follows for $\inf B = b$. We know that $a < b$, or else B would be infinite, contradictory to B being finite in the given. Let $S = \{a, b\}$. Then, by Theorem 11.8, $\inf S = \liminf s_n, \sup S = \limsup s_n$. This implies $\sup A = \liminf s_n, \inf B = \limsup s_n$. $\square$