# Convergent Series

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**Problem 1** (14.2(d))**.** Determine which of the following series converge. Justify your answers.

1. $\sum_{n=1}^{\infty}(\frac{n^3}{3^n})$

**Solution 1.** *Proof.* We are given the series:

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} \tag{1}$$

We will apply the ratio test to check for convergence. First, compute the ratio:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3}\right| \tag{2}$$

Simplifying the expression:

$$= \frac{(n+1)^3}{n^3} \times \frac{1}{3} \tag{3}$$

Now expand $(n+1)^3$:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 \tag{4}$$

Thus:

$$= \frac{n^3 + 3n^2 + 3n + 1}{n^3} \times \frac{1}{3} \tag{5}$$

Simplifying the fraction:

$$= \frac{1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}}{3} \tag{6}$$

As $n \to \infty$, the terms involving $\frac{3}{n}, \frac{3}{n^2}, \frac{1}{n^3}$ approach zero, so:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} \tag{7}$$

Since $\frac{1}{3} < 1$, the series converges by the ratio test.

□

**Problem 2** (14.4(a,b)). Determine which of the following series converge. Justify your answers.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\big(n + (-1)^n\big)^2}$

2. $\displaystyle\sum_{n=1}^{\infty} \big(\sqrt{n+1} - \sqrt{n}\big)$

**Solution 2.** 1. *Proof.* For even $n$ we have $n + (-1)^n = n - 1$, and for odd $n$ we have $n + (-1)^n = n + 1$. In either case

$$(n-1)^2 \;\le\; \big(n + (-1)^n\big)^2 \;\le\; (n+1)^2.$$

Hence

$$\frac{1}{(n+1)^2} \;\le\; \frac{1}{\big(n + (-1)^n\big)^2} \;\le\; \frac{1}{(n-1)^2} \quad (n \ge 2).$$

Because the $p$-series $\sum n^{-2}$ converges and the middle expression is sandwiched between two constant multiples of $n^{-2}$, the comparison test shows the given series converges. □

2. *Proof.* Write one term and telescope:

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since $\sqrt{n+1} + \sqrt{n} \ge 2\sqrt{n}$, each term satisfies

$$0 < \frac{1}{\sqrt{n+1} + \sqrt{n}} \le \frac{1}{2\sqrt{n}},$$

and the comparison with the divergent $p$-series $\sum n^{-1/2}$ shows that the series diverges.

□

**Problem 3** (14.5(a,b,c)). Suppose $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, where $A$ and $B$ are real numbers. Use limit theorems from section 9 to quickly prove the following.

1. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

2. $\sum_{n=1}^{\infty} k a_n = kA$ for $k \in \mathbb{R}$

3. Is $\sum_{n=1}^{\infty} a_n b_n = AB$ a reasonable conjecture? Discuss.

**Solution 3.** We are asked to verify some basic algebra of convergent series.

1. *Proof.* Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. Because $\displaystyle\sum_{k=1}^{\infty} a_k = A$ and $\displaystyle\sum_{k=1}^{\infty} b_k = B$, we have $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$. Then

$$\sum_{k=1}^{n}(a_k + b_k) = A_n + B_n \quad \Longrightarrow \quad \lim_{n\to\infty}(A_n + B_n) = A + B,$$

so $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = A + B$. □

2. *Proof.* Fix a constant $k \in \mathbb{R}$. The $n$-th partial sum of $\displaystyle\sum_{k=1}^{\infty} ka_n$ is $kA_n$, and

$$\lim_{n\to\infty} kA_n \;=\; k \lim_{n\to\infty} A_n \;=\; kA,$$

proving $\displaystyle\sum_{n=1}^{\infty} ka_n = kA$. □

3. *Proof.* In general the series $\sum a_n b_n$ **does not** equal $AB$. A simple counter-example: take $a_n = b_n = (-1)^{n+1}$. Then $A = B = 1 - 1 + 1 - 1 + \dots$ converges conditionally by Cesàro summation to $\frac{1}{2}$, but $a_n b_n = 1$ for every $n$, so $\sum a_n b_n$ diverges to $+\infty$. The correct statement is that $\sum a_n b_n$ converges (and the naive identity holds) **only if** the series of absolute values $\sum |a_n|$ and $\sum |b_n|$ both converge. □

**Problem 4** (14.6(a))**.** 1. Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges and $b_n$ is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges. *Hint*: Use Theorem 14.4.

**Solution 4.** If $b_n$ is bounded, then $\forall n, \exists M \in$ $s.t.$ $|s_n| \leq M$.

$$\sum_{n=1}^{\infty} a_n b_n \leq \sum_{n=1}^{\infty} a_n M \tag{8}$$

By Problem 3.2 in this document, we can state:

$$\sum_{n=1}^{\infty} a_n M = AM \tag{9}$$

$$\left|\sum_{n=1}^{\infty} a_n\right| = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty} a_k\right) \tag{10}$$

$$\left|\sum_{k=1}^{\infty} a_k - S\right| < \frac{\varepsilon}{M} \tag{11}$$

**Problem 5** (17.4)**.** Prove the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$. *Hint:* Apply Example 5 in section 8.

**Solution 5.** *Proof.* We will utilize the definition of continuity of a function at a point for this proof. To assist us in our proof, we can use Example 5 in section 8.

$$x = \lim_{n \to \infty} x_n \tag{12}$$

Invoking Example 5 in section 8, we obtain:

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \sqrt{x_n} = \sqrt{x} \tag{13}$$

□

**Problem 6** (17.9(c,d))**.** Prove each of the following functions is continuous at $x_0$ by verifying the $\epsilon - \delta$ property of Theorem 17.2 .

1. $f(x) = x \sin\left(\frac{1}{x}\right), x_0 = 0$ for $x \neq 0$ and $f(0) = 0$, $x_0 = 0$
2. $g(x) = x^3$, $x_0$ arbitrary. *Hint*: $x^3 - x_0^3 = (x - x_0)(x^2 + x_0 x + x_0^2)$

**Solution 6.** 1. *Proof.*

$$f(x) = x \sin\left(\frac{1}{x}\right) \tag{14}$$

$$x \sin\left(\frac{1}{x}\right) < \varepsilon \tag{15}$$

We know that the value of $f(x)$ will always be less than or equal to $x$, as the value of $\sin(x)$ is bounded from $[-1, 1]$. Thus,

$$|f(x) - f(0)| = |f(x)| \leq x < \varepsilon \tag{16}$$

Setting $\delta = \varepsilon$:

$$|x - 0| < \delta \implies |x - 0| < \varepsilon \tag{17}$$

$$\implies |f(x) - f(0)| < \varepsilon \tag{18}$$

□

2. *Proof.* For all $\varepsilon$, we want to find $\delta$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. We state:

$$\left|x^3 - x_0^3\right| = |x - x_0| \left|x^2 + x_0 x + x_0^2\right| < \varepsilon \tag{19}$$

$$|x| < |x_0| + 1 \tag{20}$$

$$\left|x^2 + x_0 x + x_0^2\right| \leq \left|x^2\right| + |x_0 x| + \left|x_0^2\right| \tag{21}$$

$$< (|x_0| + 1)^2 + \left|x_0^2\right| + |x_0(|x_0| + 1)| \tag{22}$$

Solving for $|x - x_0|$:

$$|x - x_0| < \frac{\varepsilon}{(|x_0| + 1)^2 + |x_0^2| + |x_0(|x_0| + 1)|} \tag{23}$$

Setting $\delta = \min\left\{1, \frac{\varepsilon}{(|x_0|+1)^2+\left|x_0^2\right|+|x_0(|x_0|+1)|}\right\}$:

$$|x - x_0| < \delta \implies |f(x) - f(0)| < \varepsilon \tag{24}$$

□

**Problem 7** (17.10(b))**.** Prove the following functions are discontinuous at the indicated points. You may use either Def 17.1 or the $\varepsilon - \delta$ property in Theorem 17.2.

1. g(x)=sin $\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = 0, x_0 = 0$.

**Solution 7.** *Proof.* Our goal is to find $x_n \to 0$ such that $g(x_n) \not\to g(0) = 0$. It suffices to use the definition of continuity at a function at a point by finding a sequence $x_n$ converging to 0 such that $f(x_n)$ does not converge to $g(0) = 0$.

$$\frac{1}{x} = 2\pi n + \frac{\pi}{2} \tag{25}$$

$$x_n = \frac{1}{2\pi n + \frac{\pi}{2}} \tag{26}$$

$$\lim_{n\to\infty} x_n = 0 \tag{27}$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \tag{28}$$

$$0 \neq 1 \tag{29}$$

□