Convergent Series

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Problem 1 (14.2(d))**.** Determine which of the following series converge. Justify your answers.

1. $\sum_{n=1}^{\infty}(\frac{n^3}{3^n})$

Solution 1. *Proof.* We are given the series:

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} \tag{1}$$

We will apply the ratio test to check for convergence. First, compute the ratio:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3}\right| \tag{2}$$

Simplifying the expression:

$$= \frac{(n+1)^3}{n^3} \times \frac{1}{3} \tag{3}$$

Now expand $(n+1)^3$:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 \tag{4}$$

Thus:

$$= \frac{n^3 + 3n^2 + 3n + 1}{n^3} \times \frac{1}{3} \tag{5}$$

Simplifying the fraction:

$$= \frac{1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}}{3} \tag{6}$$

As $n \to \infty$, the terms involving $\frac{3}{n}, \frac{3}{n^2}, \frac{1}{n^3}$ approach zero, so:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} \tag{7}$$

Since $\frac{1}{3} < 1$, the series converges by the ratio test.

□

Problem 2 (14.4(a,b)). Determine which of the following series converge. Justify your answers.

1. $\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$
2. $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

Solution 2. 1. *Proof.* We are given the following expression:

$$\frac{(n+(-1)^n)^2}{(n+1)(-1)^n)^2} \cdot \frac{(n+1)^2}{1} \tag{8}$$

Expanding the numerator and denominator:

$$= \frac{n^2 + 2n(-1)^n + 1}{n^2 + 2n(-1)^n + (-1)^{2n} + 2(-1)^n + 2} \tag{9}$$

Breaking it down further:

$$(n+(-1)^n)^2 = (n+(-1)^n)(n+(-1)^n) \tag{10}$$

Which expands to:

$$n^2 + n(-1)^n + n(-1)^n + (-1)^{2n} = n^2 + 2n(-1)^n + 1 \tag{11}$$

Thus:

$$= \frac{n^2 + 2n(-1)^n + 1}{n^2 + 2n(-1)^n + 2(-1)^n + 2} \tag{12}$$

Finally, simplifying the entire expression, we get:

$$= \frac{1 + \frac{2}{n}(-1)^n + \frac{1}{n^2}}{1 + \frac{2}{n} + \frac{2}{n}(-1)^n + \frac{2}{n^2}} = 2 \tag{13}$$

Since the limit results in a constant value, we conclude:

□

2. *Proof.* We are tasked with evaluating the series:

$$\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \tag{14}$$

To simplify, multiply both the numerator and denominator by the conjugate:

$$\left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \tag{15}$$

This simplifies to:

$$\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \tag{16}$$

Now, we observe that:

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \tag{17}$$

Thus, we can compare this to the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which is a p-series with $p = \frac{1}{2}$, and since $p \leq 1$, the series diverges. Therefore:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges.} \tag{18}$$

Since $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ is bounded by a divergent series, by the comparison test:

$$\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \text{ also diverges.} \tag{19}$$

□

Problem 3 (14.5(a,b,c))**.** Suppose $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, where A and B are real numbers. Use limit theorems from section 9 to quickly prove the following.

1. $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$
2. $\sum_{n=1}^{\infty} ka_n = kA$ for $k \in \mathbb{R}$

3. Is $\sum_{n=1}^{\infty} a_n b_n = AB$ a reasonable conjecture? Discuss.

Solution 3. We are tasked with evaluating the following properties of series.

1. *Proof.* Given $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, we want to show:

$$\sum_{n=1}^{\infty}(a_n + b_n) = A + B \tag{20}$$

Let a_n^* and b_n^* be the partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. Then, we have:

$$\lim_{n\to\infty} a_n^* + \lim_{n\to\infty} b_n^* = \lim_{n\to\infty} (a_n^* + b_n^*) \quad \checkmark \tag{21}$$

$\square$

2. *Proof.*

For a constant $k \in \mathbb{R}$, we want to show:

$$\sum_{n=1}^{\infty} k a_n = kA \tag{22}$$

The limit of the partial sums satisfies:

$$\lim_{n\to\infty} (k a_n^*) = k \lim_{n\to\infty} a_n^* = kA \quad \checkmark \tag{23}$$

$\square$

3. *Proof.* The series $\sum a_n b_n$ converges if and only if a_n and b_n converge absolutely. $\square$

Problem 4 (14.6(a)). 1. Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges and b_n is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges. *Hint*: Use Theorem 14.4.

Solution 4. If b_n is bounded, then $\forall n, \exists M \in \ s.t. \ |s_n| \leq M$.

$$\sum_{n=1}^{\infty} a_n b_n \leq \sum_{n=1}^{\infty} a_n M \tag{24}$$

By Problem 3.2 in this document, we can state:

$$\sum_{n=1}^{\infty} a_n M = AM \tag{25}$$

$$\left|\sum_{n=1}^{\infty} a_n\right| = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty} a_k\right) \tag{26}$$

$$\left|\sum_{k=1}^{\infty} a_k - S\right| < \frac{\varepsilon}{M} \tag{27}$$

Problem 5 (17.4)**.** Prove the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$. *Hint:* Apply Example 5 in section 8.

Solution 5. *Proof.* We will utilize the definition of continuity of a function at a point for this proof. To assist us in our proof, we can use Example 5 in section 8.

$$x = \lim_{n\to\infty} x_n \tag{28}$$

Invoking Example 5 in section 8, we obtain:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sqrt{x_n} = \sqrt{x} \tag{29}$$

□

Problem 6 (17.9(c,d))**.** Prove each of the following functions is continuous at x_0 by verifying the $\epsilon - \delta$ property of Theorem 17.2 .

1. $f(x) = x \sin\left(\frac{1}{x}\right), x_0 = 0$ for $x \neq 0$ and $f(0) = 0$, $x_0 = 0$
2. $g(x) = x^3$, x_0 arbitrary. *Hint*: $x^3 - x_0^3 = (x - x_0)(x^2 + x_0 x + x_0^2)$

Solution 6. 1. *Proof.*

$$f(x) = x \sin\left(\frac{1}{x}\right) \tag{30}$$

$$x \sin\left(\frac{1}{x}\right) < \varepsilon \tag{31}$$

We know that the value of $f(x)$ will always be less than or equal to x, as the value of $\sin(x)$ is bounded from $[-1, 1]$. Thus,

$$|f(x) - f(0)| = |f(x)| \leq x < \varepsilon \tag{32}$$

Setting $\delta = \varepsilon$:

$$|x - 0| < \delta \implies |x - 0| < \varepsilon \tag{33}$$

$$\implies |f(x) - f(0)| < \varepsilon \tag{34}$$

□

2. *Proof.* For all ε, we want to find δ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. We state:

$$\left|x^3 - x_0^3\right| = |x - x_0| \left|x^2 + x_0 x + x_0^2\right| < \varepsilon \tag{35}$$

$$|x| < |x_0| + 1 \tag{36}$$

$$\left|x^2 + x_0 x + x_0^2\right| \leq \left|x^2\right| + |x_0 x| + \left|x_0^2\right| \tag{37}$$

$$< (|x_0| + 1)^2 + \left|x_0^2\right| + |x_0(|x_0| + 1)| \tag{38}$$

Solving for $|x - x_0|$:

$$|x - x_0| < \frac{\varepsilon}{(|x_0| + 1)^2 + |x_0^2| + |x_0(|x_0| + 1)|} \tag{39}$$

Setting $\delta = \min\left\{1, \frac{\varepsilon}{(|x_0|+1)^2 + \left|x_0^2\right| + |x_0(|x_0|+1)|}\right\}$:

$$|x - x_0| < \delta \implies |f(x) - f(0)| < \varepsilon \tag{40}$$

□

Problem 7 (17.10(b))**.** Prove the following functions are discontinuous at the indicated points. You may use either Def 17.1 or the $\varepsilon - \delta$ property in Theorem 17.2.

1. g(x)=sin $\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = 0, x_0 = 0$.

Solution 7. *Proof.* Our goal is to find $x_n \to 0$ such that $g(x_n) \not\to g(0) = 0$. It suffices to use the definition of continuity at a function at a point by finding a sequence x_n converging to 0 such that $f(x_n)$ does not converge to $g(0) = 0$.

$$\frac{1}{x} = 2\pi n + \frac{\pi}{2} \tag{41}$$

$$x_n = \frac{1}{2\pi n + \frac{\pi}{2}} \tag{42}$$

$$\lim_{n\to\infty} x_n = 0 \tag{43}$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \tag{44}$$

$$0 \neq 1 \tag{45}$$

□