

Limits and Sequences

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Problem 1 (9.12). • Assume all $s_n \neq 0$ and that the limit $L = \lim \left| \frac{s_{n+1}}{s_n} \right|$ exists.

- (a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint:* Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.
- (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint:* Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

Solution 1. 1. *Proof.* $\forall s_n, s_n \neq 0, \lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| = L$.

Proposition. If $L < 1$, then $\lim_{n \rightarrow \infty} s_n = 0$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies \left| \frac{s_{n+1}}{s_n} - L \right| < \epsilon$. This implies that $\frac{s_{n+1}}{s_n} < \epsilon + L$. Then we select a value a such that $L < a < 1$.

$$s_{n+1} < (\epsilon + L)s_n \quad (1)$$

$$a = \epsilon + L \quad (2)$$

To show that $|s_n| < a^{n-N}|s_N|$, we will show the following:

$$|s_n| = \left| \frac{s_n}{s_{n-1}} \cdot \frac{s_{n-1}}{s_{n-2}} \cdots \frac{s_{N+1}}{s_N} \cdot |s_N| \right| \quad (4)$$

$$|s_n| < a^{n-N}|s_N| \quad (5)$$

$\frac{1}{a^N}|s_N|$ is a constant, so we can rename it as the following:

$$C = \frac{1}{a^N}|s_N| \quad (6)$$

Then, we can say the following:

$$|s_n| < a^n \quad (7)$$

If $a < 1$, we know that $\lim_{n \rightarrow \infty} a^n = 0$. By Theorem 9.7 (b) in Ross, we know that $\lim_{n \rightarrow \infty} a^n = 0$. Therefore, by Theorem 9.2 in Ross, $\lim_{n \rightarrow \infty} |s_n| = 0$. \square

2.

Proposition. If $L > 1$, then $\lim_{n \rightarrow \infty} |s_n| = \infty$.

Proof. Let $t_n = \frac{1}{|s_n|}$. Our goal is to prove $\lim_{n \rightarrow \infty} t_n = 0$. Then:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{t_n}{t_{n+1}} \right| \end{aligned} \quad (9)$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies \left| \frac{t_n}{t_{n+1}} - L \right| < \epsilon.$$

$$\frac{t_n}{t_{n+1}} < \left(\frac{\epsilon + L}{a} \right) t_n \quad (11)$$

Applying the same process as in (a), we can say that $L < a < 1$. Then:

$$t_{n+1} < at_n \quad (12)$$

$$\frac{t_{n+1}}{t_n} < a \quad (13)$$

$$|t_n| = \left| \frac{t_n}{t_{n-1}} \cdot \frac{t_{n-1}}{t_{n-2}} \cdots \frac{t_{N+1}}{t_N} \cdot |t_N| \right| \quad (14)$$

$$C = a^{-N} |t_N| \quad (15)$$

We can say that $|t_n| < a^n C$ if $a < 1$, so $\lim_{n \rightarrow \infty} a^n = 0$. Then $\lim_{n \rightarrow \infty} t_n = 0$. Therefore:

$$\lim_{n \rightarrow \infty} \frac{1}{|s_n|} = 0 \quad (16)$$

By Theorem 9.10 in Ross, $\lim_{n \rightarrow \infty} |s_n| = \infty$. □

Problem 2 (9.14). Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n \rightarrow \infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \leq 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases} \quad (1)$$

Hint: For the $a > 1$ case, use Exercise 9.12(b).

$$\frac{1}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solution 2. 1. *Proof.* We begin with $s_n = \lim_{n \rightarrow \infty} \frac{a^n}{n^P}, |a| \leq 1$. Applying Problem 1:

$$\left| \frac{s_{n+1}}{s_n} \right| = \frac{a^{n+1}}{(n+1)^P} \cdot \frac{n^P}{a^n} \quad (17)$$

$$= a \left(\frac{n^P}{(n+1)^P} \right) \quad (18)$$

$$= a \quad (\text{since } \lim_{n \rightarrow \infty} \frac{n^P}{(n+1)^P} = 1). \quad (19)$$

The $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right|$ exists, so Problem 1 applies. According to the result from Problem 1a, if $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| < 1$, then $\lim_{n \rightarrow \infty} s_n = 0$. When $|a| \leq 1$, $\lim_{n \rightarrow \infty} \frac{a^n}{n^P} = a$. Therefore, its limit must be 0 when $|a| \leq 1$, as desired. \square

2. *Proof.* We begin with $s_n = \lim_{n \rightarrow \infty} \frac{a^n}{n^P}, a > 1$. Applying Problem 1:

$$\left| \frac{s_{n+1}}{s_n} \right| = \frac{a^{n+1}}{(n+1)^P} \cdot \frac{n^P}{a^n} \quad (20)$$

$$= a \left(\frac{n^P}{(n+1)^P} \right) \quad (21)$$

$$= a \quad (\text{since } \lim_{n \rightarrow \infty} \frac{n^P}{(n+1)^P} = 1). \quad (22)$$

The $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right|$ exists, so Problem 1 applies. According to the result from Problem 1a, if $\lim_{n \rightarrow \infty} \left| \frac{s_{n+1}}{s_n} \right| > 1$, then $\lim_{n \rightarrow \infty} s_n = +\infty$. When $a > 1$, $\lim_{n \rightarrow \infty} \frac{a^n}{n^P} = a$. Therefore, its limit must be $+\infty$ when $a > 1$, as desired. \square

3. *Proof.* We begin with $s_n = \lim_{n \rightarrow \infty} \frac{a^n}{n^P}, a < -1$. To show that $\lim_{n \rightarrow \infty} \frac{a^n}{n^P} = \text{DNE}$, we will show that there exists more than one limit for the sequence s_n . Let s_{n_1} be the subsequence such that n is even. Let s_{n_2} be the subsequence such that n is not even. To show that a limit is divergent, the following must be satisfied:

$$\forall M > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies \left(\frac{a^{2n}}{(2n)^P} \right) > M \quad (23)$$

Choose N such that $N > \frac{P \ln 2 + \ln M}{2(\ln a - P)}$. Then, for all $n > N$, this implies:

$$n(2 \ln a - P) > P \ln 2 + \ln M \quad (24)$$

$$2n \ln a - P \ln n > \ln(2^P M) \quad (25)$$

$$\ln \left(\frac{a^{2n}}{N^P} \right) > \ln(2^P M) \quad (26)$$

$$\frac{a^{2n}}{(2n)^P} > M \quad (27)$$

This implies that the $\lim_{n \rightarrow \infty} s_{n_1}$ is divergent. The same reasoning follows with $\lim_{n \rightarrow \infty} s_{n_2}$. We then get:

$$\lim_{n \rightarrow \infty} s_{n_1} = \lim_{n \rightarrow \infty} \frac{a^{2n}}{(2n)^P} \quad (28)$$

$$\lim_{n \rightarrow \infty} s_{n_2} = \lim_{n \rightarrow \infty} \frac{a^{2n+1}}{(2n+1)^P} \quad (29)$$

When $a < -1$, $n \in \mathbb{N}$, a^{2n} is always positive and a^{2n+1} is always negative. We then get:

$$\lim_{n \rightarrow \infty} s_{n_1} = \lim_{n \rightarrow \infty} \frac{a^{2n}}{(2n)^P} = +\infty \quad (31)$$

$$\lim_{n \rightarrow \infty} s_{n_2} = \lim_{n \rightarrow \infty} \frac{a^{2n+1}}{(2n+1)^P} = -\infty \quad (32)$$

There are two subsequences of s_n with two distinct limits, so by Theorem 11.8 iii) in Ross, the limit of s_n with $a < -1$ does not exist. \square

Problem 3 (10.6). (a) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}. \quad (2)$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Solution 3. 1. *Proof.* Let s_n be a sequence such that $\forall n \in \mathbb{N}, |s_{n+1} - s_n| < 2^{-n}$. To show that a sequence is Cauchy, we must satisfy the following:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n, m > N \implies |s_n - s_m| < \epsilon. \quad (3)$$

We will show this by showing that:

$$\forall \epsilon, \exists N_0 \in \mathbb{N} \text{ s.t. } n > N_0 \implies |2^{-n} - 0| < \epsilon. \quad (4)$$

Then we can bound $s_n < \epsilon$ for all $n, m \in \mathbb{N}$, thereby showing s_n is Cauchy. We will solve for N in the expression:

$$\frac{1}{2^N} < \epsilon \quad (34)$$

$$2^N > \frac{1}{\epsilon} \quad (35)$$

$$N > \log_2 \left(\frac{1}{\epsilon} \right) \quad (36)$$

Then, we can choose $N = \log_2 \left(\frac{1}{\epsilon} \right)$. Then, $\forall n > N$,

$$|s_{n+1} - s_n| < 2^{-n} \quad (37)$$

$$|s_m - s_n| < 2^{-n} < \epsilon \quad (38)$$

$$|s_m - s_n| < \epsilon \quad (39)$$

Therefore, the sequence s_n is Cauchy, as required. \square

2. *Proof.* We can follow a similar line of reasoning from the previous question. We want $\forall n \in \mathbb{N}, s_n = |s_{n+1} - s_n| < \frac{1}{n}$. So we will prove that:

$$\forall \epsilon > 0, \exists N \text{ s.t. } m, n > N \implies |s_{n+1} - s_n| < \epsilon. \quad (5)$$

$$|s_{n+1} - s_n| < \frac{1}{n} \quad (40)$$

$$\left| \frac{1}{n} - 0 \right| < \epsilon \quad (41)$$

To determine N , we will use algebra as follows:

$$\frac{1}{N} < \epsilon \quad (42)$$

$$\frac{1}{\epsilon} < N \quad (43)$$

We will choose $N = \frac{1}{\epsilon}$. Then, $n > N$ implies:

$$\frac{1}{n} < \epsilon \quad (44)$$

And hence,

$$\left| \frac{1}{n} - 0 \right| < \epsilon \quad (45)$$

$$|s_m - s_n| < \frac{1}{n} < \epsilon \quad (46)$$

$$|s_m - s_n| < \epsilon \quad (47)$$

Therefore, the sequence satisfies the Cauchy criterion. \square

Problem 4 (10.8). Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

Solution 4. *Proof.* Suppose by contradiction, that

$$\frac{1}{n+1}(s_1 + s_2 + \cdots + s_n + s_{n+1}) < \frac{1}{n}(s_1 + s_2 + \cdots + s_n). \quad (6)$$

Then,

$$\sum_{i=1}^{n+1} s_i < \frac{n+1}{n} \sum_{i=1}^n s_i \quad (48)$$

$$\sum_{i=1}^{n+1} s_i < \left(1 + \frac{1}{n}\right) \sum_{i=1}^n s_i \quad (49)$$

$$s_{n+1} < \frac{1}{n} \sum_{i=1}^n s_i \quad (50)$$

$$ns_{n+1} \leq \sum_{i=1}^n s_i \quad (51)$$

$$s_{n+1} < ns_n \quad (52)$$

Contradiction. □

Problem 5 (10.10). Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

- (a) Find s_2 , s_3 and s_4 .
- (b) Use induction to show $s_n > \frac{1}{2}$ for all n .
- (c) Show (s_n) is a decreasing sequence.
- (d) Show $\lim s_n$ exists and find $\lim s_n$.

Solution 5. 1. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$. Then:

(a) $s_1 = 1$

(b) $s_2 = \frac{1}{3}(s_1 + 1) = \frac{1}{3}(2) = \frac{2}{3}$

(c) $s_3 = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9}$

(d) $s_4 = \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27}$

2. *Proof.* We will use induction. The base case is as follows:

$$s_1 = 1 \quad (53)$$

The induction hypothesis is:

$$\forall n \geq 1, \frac{1}{2} < s_{n+1} < s_n < 1 \quad (54)$$

The inductive step is as follows:

$$\frac{1}{3}(s_{n+1} + 1) < s_{n+1} \quad (55)$$

$$\frac{s_{n+1}}{3} + \frac{1}{3} < s_{n+1} \quad (56)$$

$$\frac{1}{3} < \frac{2s_{n+1}}{3} \quad (57)$$

$$\frac{1}{2} < s_{n+1} \quad (58)$$

To finish the proof, we need $\frac{1}{3}(s_{n+1} + 1) > \frac{1}{2}$:

$$\frac{s_{n+1}}{3} - \frac{1}{6} > 0 \quad (59)$$

$$\frac{1}{6}(2s_{n+1} - 1) > 0 \quad (60)$$

$$s_{n+1} > \frac{1}{2} \quad (61)$$

□

3. *Proof.* To prove that $\forall n \geq 1, s_1 = 1, s_n = \frac{1}{3}(s_n + 1)$ is decreasing, we will show that $\forall n \geq 1, s_{n+1} \leq s_n$:

$$\frac{1}{3}(s_{n+1} + 1) < s_n \quad (62)$$

$$\frac{s_n}{3} + \frac{1}{3} < s_n \quad (63)$$

$$\frac{1}{3} < \frac{2s_n}{3} \quad (64)$$

$$\frac{1}{2} < s_n \quad (65)$$

The last line was proved by induction in part (b) of this problem. So s_n is decreasing. □

4. *Proof.* To show that $\lim_{n \rightarrow \infty} s_n$ exists, we can state that because s_n is decreasing, it is monotone. Because $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$, the sequence is also bounded. Therefore, by Theorem 10.2 in Ross, s_n converges and must have a limit.

Let $\epsilon > 0$, $S = \{s_n : n \in \mathbb{N}\}$, and $u = \inf s_n$. $u + \epsilon$ is not a lower bound of S , so $\exists N$ s.t. $s_N < u + \epsilon$ for all $n \geq N$.

Thus, $u \leq s_n < u + \epsilon \implies |s_n - u| < \epsilon$. From part (b), we proved $\frac{1}{2} < s_{n+1} < s_n < 1$. So $u = \inf s_n = \frac{1}{2}$. Therefore, by Theorem 10.2 from Ross,

$$\lim_{n \rightarrow \infty} s_n = u = \frac{1}{2}. \quad (7)$$

□

Problem 6 (10.12). Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] \cdot t_n$ for $n \geq 1$.

- (a) Show $\lim t_n$ exists.
- (b) What do you think $\lim t_n$ is?
- (c) Use induction to show $t_n = \frac{n+1}{2n}$.
- (d) Repeat part (b).

Solution 6. 1. *Proof.* To show that t_n is decreasing, we will show that $t_{n+1} < t_n$ for all $n \in \mathbb{N}$ s.t. $n \geq 1$:

$$\left[1 - \frac{1}{(n+1)^2}\right] \cdot t_n < t_n \quad (66)$$

$$\left[1 - \frac{1}{(n+1)^2}\right] < 1 \quad (67)$$

$$-\frac{1}{(n+1)^2} < 0 \quad (68)$$

$$0 < \frac{1}{(n+1)^2} \quad (69)$$

The last line is always true for $n \in \mathbb{N}$, so the $\lim_{n \rightarrow \infty} t_n$ exists. We have shown that t_n is monotone, and therefore has a limit. \square

2. I think the limit is $\frac{1}{2}$.

3. *Proof.* We will use induction. For the base case, $n = 1$:

$$t_n = \frac{n+1}{2n} \quad (70)$$

$$t_1 = \frac{2}{2} = 1 \quad (71)$$

The inductive hypothesis is as follows:

$$t_n = \frac{n+1}{2n} \quad \text{for } n \geq 1 \quad (72)$$

The inductive step:

$$t_{n+1} = \frac{(n+1)+1}{2(n+1)} \quad (74)$$

$$= \frac{n+2}{2(n+1)} \quad (75)$$

$$= \left[1 - \frac{1}{(n+1)^2}\right] t_n \quad (76)$$

$$= \left[1 - \frac{1}{(n+1)^2}\right] \left(\frac{n+1}{2n}\right) \quad (77)$$

$$= \frac{(n+1)^2(n+1)}{2n(n+1)^2} - \frac{n+1}{2n(n+1)^2} \quad (78)$$

$$= \frac{n+2}{2(n+1)} \quad (79)$$

Thus, the proof is complete. \square

4. *Proof.*

$$\lim_{n \rightarrow \infty} \frac{n+1}{2n} = \lim_{n \rightarrow \infty} \frac{1+1/n}{2} \tag{80}$$

$$= \frac{1}{2} \tag{81}$$

Equation (80) is established from Theorem 9.10 in Ross. □