Limits and Sequences

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Problem 1 (9.12)**.** 1. Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint:* Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint:* Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

Solution 1. (a) *Proof.* Assume $s_n \neq 0$ and $L = \lim\limits_{n\to\infty} \left|\dfrac{s_{n+1}}{s_n}\right| < 1$. Pick a with $L < a < 1$. Then there is $N \in \mathbb{N}$ such that $\left|s_{n+1}\right| \leq a \left|s_n\right|$ for all $n \geq N$. Iterating,

$$|s_n| \leq a^{\,n-N} \, |s_N| \xrightarrow[n\to\infty]{} 0,$$

because $0 < a < 1$. Hence $\lim\limits_{n\to\infty} s_n = 0$. ∎

(b) *Proof.* Define $t_n = \dfrac{1}{|s_n|}$. From the ratio in part (a) we get

$$\lim_{n\to\infty} \left|\frac{t_{n+1}}{t_n}\right| = \frac{1}{L} < 1,$$

so by the result of part (a) we have $\lim\limits_{n\to\infty} t_n = 0$. Therefore $|s_n| = \dfrac{1}{t_n} \to \infty$. ∎

Problem 2 (9.14)**.** Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \leq 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases} \tag{1}$$

Hint: For the $a > 1$ case, use Exercise 9.12(b).

$\frac{1}{n^2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution 2. 1. *Proof.* We begin with $s_n = \lim_{n\to\infty} \frac{a^n}{n^P}, |a| \leq 1$. Applying Problem 1:

$$\left|\frac{s_{n+1}}{s_n}\right| = \frac{a^{n+1}}{(n+1)^P} \cdot \frac{n^P}{a^n} \tag{17}$$

$$= a\left(\frac{n^P}{(n+1)^P}\right) \tag{18}$$

$$= a \quad (\text{since } \lim_{n\to\infty} \frac{n^P}{(n+1)^P} = 1). \tag{19}$$

The $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ exists, so Problem 1 applies. According to the result from Problem 1a, if $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| < 1$, then $\lim_{n\to\infty} s_n = 0$. When $|a| \leq 1$, $\lim_{n\to\infty}\frac{a^n}{n^P} = a$. Therefore, its limit must be 0 when $|a| \leq 1$, as desired. □

2. *Proof.* We begin with $s_n = \lim_{n\to\infty} \frac{a^n}{n^P}, a > 1$. Applying Problem 1:

$$\left|\frac{s_{n+1}}{s_n}\right| = \frac{a^{n+1}}{(n+1)^P} \cdot \frac{n^P}{a^n} \tag{20}$$

$$= a\left(\frac{n^P}{(n+1)^P}\right) \tag{21}$$

$$= a \quad (\text{since } \lim_{n\to\infty} \frac{n^P}{(n+1)^P} = 1). \tag{22}$$

The $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ exists, so Problem 1 applies. According to the result from Problem 1a, if $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| > 1$, then $\lim_{n\to\infty} s_n = +\infty$. When $a > 1$, $\lim_{n\to\infty}\frac{a^n}{n^P} = a$. Therefore, its limit must be $+\infty$ when $a > 1$, as desired. □

3. *Proof.* We begin with $s_n = \lim_{n\to\infty} \frac{a^n}{n^P}, a < -1$. To show that $\lim_{n\to\infty}\frac{a^n}{n^P} = \text{DNE}$, we will show that there exists more than one limit for the sequence s_n. Let s_{n_1} be the subsequence such that n is even. Let s_{n_2} be the subsequence such that n is not even.

To show that a limit is divergent, the following must be satisfied:

$$\forall M > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies \left(\frac{a^{2n}}{(2n)^P}\right) > M \tag{23}$$

Choose N such that $N > \frac{P\ln 2 + \ln M}{2(\ln a - P)}$. Then, for all $n > N$, this implies:

$$n(2\ln a - P) > P\ln 2 + \ln M \tag{24}$$

$$2n\ln a - P\ln n > \ln(2^P M) \tag{25}$$

$$\ln\left(\frac{a^{2n}}{N^P}\right) > \ln(2^P M) \tag{26}$$

$$\frac{a^{2n}}{(2n)^P} > M \tag{27}$$

This implies that the $\lim_{n\to\infty} s_{n_1}$ is divergent. The same reasoning follows with $\lim_{n\to\infty} s_{n_2}$. We then get:

$$\lim_{n\to\infty} s_{n_1} = \lim_{n\to\infty} \frac{a^{2n}}{(2n)^P} \tag{28}$$

$$\lim_{n\to\infty} s_{n_2} = \lim_{n\to\infty} \frac{a^{2n+1}}{(2n+1)^P} \tag{29}$$

When $a < -1, n \in \mathbb{N}$, a^{2n} is always positive and a^{2n+1} is always negative. We then get:

$$\lim_{n\to\infty} s_{n_1} = \lim_{n\to\infty} \frac{a^{2n}}{(2n)^P} = +\infty \tag{31}$$

$$\lim_{n\to\infty} s_{n_2} = \lim_{n\to\infty} \frac{a^{2n+1}}{(2n+1)^P} = -\infty \tag{32}$$

There are two subsequences of s_n with two distinct limits, so by Theorem 11.8 iii) in Ross, the limit of s_n with $a < -1$ does not exist. □

Problem 3 (10.6)**.** (a) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Solution 3. 1. *Proof.* Let s_n be a sequence such that $\forall n \in \mathbb{N}, |s_{n+1} - s_n| < 2^{-n}$. To show that a sequence is Cauchy, we must satisfy the following:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n, m > N \implies |s_n - s_m| < \epsilon. \tag{3}$$

We will show this by showing that:

$$\forall \epsilon, \exists N_0 \in \mathbb{N} \text{ s.t. } n > N_0 \implies |2^{-n} - 0| < \epsilon. \tag{4}$$

Then we can bound $s_n < \epsilon$ for all $n, m \in \mathbb{N}$, thereby showing s_n is Cauchy. We will solve for N in the expression:

$$\frac{1}{2^N} < \epsilon \tag{34}$$

$$2^N > \frac{1}{\epsilon} \tag{35}$$

$$N > \log_2\left(\frac{1}{\epsilon}\right) \tag{36}$$

Then, we can choose $N = \log_2\left(\frac{1}{\epsilon}\right)$. Then, $\forall n > N$,

$$|s_{n+1} - s_n| < 2^{-n} \tag{37}$$

$$|s_m - s_n| < 2^{-n} < \epsilon \tag{38}$$

$$|s_m - s_n| < \epsilon \tag{39}$$

Therefore, the sequence s_n is Cauchy, as required. □

2. *Proof.* We can follow a similar line of reasoning from the previous question. We want $\forall n \in \mathbb{N}, s_n = |s_{n+1} - s_n| < \frac{1}{n}$. So we will prove that:

$$\forall \epsilon > 0, \exists N \text{ s.t. } m, n > N \implies |s_{n+1} - s_n| < \epsilon. \tag{5}$$

$$|s_{n+1} - s_n| < \frac{1}{n} \tag{40}$$

$$\left|\frac{1}{n} - 0\right| < \epsilon \tag{41}$$

To determine N, we will use algebra as follows:

$$\frac{1}{N} < \epsilon \tag{42}$$

$$\frac{1}{\epsilon} < N \tag{43}$$

We will choose $N = \frac{1}{\epsilon}$. Then, $n > N$ implies:

$$\frac{1}{n} < \epsilon \tag{44}$$

And hence,

$$\left|\frac{1}{n} - 0\right| < \epsilon \tag{45}$$

$$|s_m - s_n| < \frac{1}{n} < \epsilon \tag{46}$$

$$|s_m - s_n| < \epsilon \tag{47}$$

Therefore, the sequence satisfies the Cauchy criterion. □

Problem 4 (10.8)**.** Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

Solution 4. *Proof.* Suppose by contradiction, that

$$\frac{1}{n+1}(s_1 + s_2 + \cdots + s_n + s_{n+1}) < \frac{1}{n}(s_1 + s_2 + \cdots + s_n). \tag{6}$$

Then,

$$\sum_{i=1}^{n+1} s_i < \frac{n+1}{n} \sum_{i=1}^{n} s_i \tag{48}$$

$$\sum_{i=1}^{n+1} s_i < \left(1 + \frac{1}{n}\right) \sum_{i=1}^{n} s_i \tag{49}$$

$$s_{n+1} < \frac{1}{n} \sum_{i=1}^{n} s_i \tag{50}$$

$$ns_{n+1} \leq \sum_{i=1}^{n} s_n \tag{51}$$

$$s_{n+1} < ns_n \tag{52}$$

Contradiction. □

Problem 5 (10.10)**.** Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

(c) Show (s_n) is a decreasing sequence.

(d) Show $\lim s_n$ exists and find $\lim s_n$.

Solution 5. 1. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$. Then:

(a) $s_1 = 1$
(b) $s_2 = \frac{1}{3}(s_1 + 1) = \frac{1}{3}(2) = \frac{2}{3}$
(c) $s_3 = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9}$
(d) $s_4 = \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27}$

2. *Proof.* We will use induction. The base case is as follows:

$$s_1 = 1 \tag{53}$$

The induction hypothesis is:

$$\forall n \geq 1, \frac{1}{2} < s_{n+1} < s_n < 1 \tag{54}$$

The inductive step is as follows:

$$\frac{1}{3}(s_{n+1} + 1) < s_{n+1} \tag{55}$$

$$\frac{s_{n+1}}{3} + \frac{1}{3} < s_{n+1} \tag{56}$$

$$\frac{1}{3} < \frac{2s_{n+1}}{3} \tag{57}$$

$$\frac{1}{2} < s_{n+1} \tag{58}$$

To finish the proof, we need $\frac{1}{3}(s_{n+1}+1) > \frac{1}{2}$:

$$\frac{s_{n+1}}{3} - \frac{1}{6} > 0 \tag{59}$$

$$\frac{1}{6}(2s_{n+1} - 1) > 0 \tag{60}$$

$$s_{n+1} > \frac{1}{2} \tag{61}$$

□

3. *Proof.* To prove that $\forall n \geq 1, s_1 = 1, s_n = \frac{1}{3}(s_n + 1)$ is decreasing, we will show that $\forall n \geq 1, s_{n+1} \leq s_n$:

$$\frac{1}{3}(s_{n+1} + 1) < s_n \tag{62}$$

$$\frac{s_n}{3} + \frac{1}{3} < s_n \tag{63}$$

$$\frac{1}{3} < \frac{2s_n}{3} \tag{64}$$

$$\frac{1}{2} < s_n \tag{65}$$

The last line was proved by induction in part (b) of this problem. So s_n is decreasing.

□

4. *Proof.* To show that $\lim_{n\to\infty} s_n$ exists, we can state that because s_n is decreasing, it is monotone. Because $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$, the sequence is also bounded. Therefore, by Theorem 10.2 in Ross, s_n converges and must have a limit.

   Let $\epsilon > 0$, $S = \{s_n : n \in \mathbb{N}\}$, and $u = \inf s_n$. $u + \epsilon$ is not a lower bound of S, so $\exists N$ s.t. $s_N < u + \epsilon$ for all $n \geq N$.

   Thus, $u \leq s_n < u + \epsilon \implies |s_n - u| < \epsilon$. From part (b), we proved $\frac{1}{2} < s_{n+1} < s_n < 1$. So $u = \inf s_n = \frac{1}{2}$. Therefore, by Theorem 10.2 from Ross,

$$\lim_{n\to\infty} s_n = u = \frac{1}{2}. \tag{7}$$

□

Problem 6 (10.12)**.** Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

(c) Use induction to show $t_n = \frac{n+1}{2n}$.

(d) Repeat part (b).

Solution 6. 1. *Proof.* To show that t_n is decreasing, we will show that $t_{n+1} < t_n$ for all $n \in \mathbb{N}$ s.t. $n \geq 1$:

$$\left[1 - \frac{1}{(n+1)^2}\right] \cdot t_n < t_n \tag{66}$$

$$\left[1 - \frac{1}{(n+1)^2}\right] < 1 \tag{67}$$

$$-\frac{1}{(n+1)^2} < 0 \tag{68}$$

$$0 < \frac{1}{(n+1)^2} \tag{69}$$

The last line is always true for $n \in \mathbb{N}$, so the $\lim_{n\to\infty} t_n$ exists. We have shown that t_n is monotone, and therefore has a limit. □

2. I think the limit is $\frac{1}{2}$.

3. *Proof.* We will use induction. For the base case, $n = 1$:

$$t_n = \frac{n+1}{2n} \tag{70}$$

$$t_1 = \frac{2}{2} = 1 \tag{71}$$

The inductive hypothesis is as follows:

$$t_n = \frac{n+1}{2n} \quad \text{for } n \geq 1 \tag{72}$$

The inductive step:

$$t_{n+1} = \frac{(n+1)+1}{2(n+1)} \tag{74}$$

$$= \frac{n+2}{2(n+1)} \tag{75}$$

$$= \left[1 - \frac{1}{(n+1)^2}\right] t_n \tag{76}$$

$$= \left[1 - \frac{1}{(n+1)^2}\right] \left(\frac{n+1}{2n}\right) \tag{77}$$

$$= \frac{(n+1)^2(n+1)}{2n(n+1)^2} - \frac{n+1}{2n(n+1)^2} \tag{78}$$

$$= \frac{n+2}{2(n+1)} \tag{79}$$

Thus, the proof is complete. □

4. *Proof.*

$$\lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+1/n}{2} \tag{80}$$

$$= \frac{1}{2} \tag{81}$$

Equation (80) is established from Theorem 9.10 in Ross. □