Connectedness and Uniform Continuity

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Problem 1 (22.3)**.** Prove that if E is a connected subset of a metric space (S, d), then its closure E^- is also connected.

Solution 1. *Proof.* To solve this problem, we will prove the contrapositive, i.e., if E^- is disconnected, then E is also disconnected. Because E^- is disconnected, there exist U_1, U_2 such that:

1. $E^- \subseteq U_1 \cup U_2$
2. $(E^- \cap U_1) \cap (E^- \cap U_2) = \emptyset$
3. $(E^- \cap U_1) \neq \emptyset, (E^{-1} \cap U_2) \neq \emptyset$

By the definition of closure, $E \subset E^-$. Then,

$$(E \cap U_1) \cap (E \cap U_2) \subset (E^- \cap U_1) \cap (E^- \cap U_2) \tag{1}$$

$$(E \cap U_1) \cap (E \cap U_2) = \emptyset \tag{2}$$

Therefore, the following holds true:

1. $E \subset E^- \subseteq U_1 \cup U_2 \implies E \subseteq U_1 \cup U_2$
2. $(E \cap U_1) \cap (E \cap U_2) = \emptyset$
3. $(E \cap U_1) \neq \emptyset, (E \cap U_2) \neq \emptyset$

Therefore, E is also disconnected. □

Problem 2. Prove that an intersection of convex sets in $\mathbb{R}^n$ is convex.

Solution 2. *Proof.* Suppose we have sets E, F that are both convex. We wish to find $E \cup F$ is convex. We know the following:

$$\forall x, y \in E, 0 < t < 1 \implies tx + (1-t)y \in E \tag{3}$$

$$\forall u, v \in F, 0 < t < 1 \implies tu + (1-t)v \in F \tag{4}$$

Choose $\forall a, b \in E \cup F$. Then,

$$ta + (1-t)b \in E \tag{5}$$

$$ta + (1-t)b \in F \tag{6}$$

By the definition of an intersection of a set, we know that $x \in P$ iff $x \in E_\alpha$ for every $\alpha \in A$. Let $a \in E$, and $b \in F$, and $E = E_1$, $F = E_2$, $E_1, E_2 \in E_\alpha$. $P = E \cap F$. By the above, we know that $ta + (1-t)b \in E \cap F$. Therefore, the intersection of convex sets E and F is also convex. □

Problem 3. On the metric space $\mathbb{R}^n$ (with the Euclidean metric d), denote by P_i $(1 \leq i \leq n)$ the projection onto the i-th coordinate. Specifically, $P_i : \mathbb{R}^n \to \mathbb{R}$ takes $\vec{x} = (x_1, \ldots, x_n)$ to x_i. Prove that P_i is Lipschitz.

Solution 3. *Proof.* The idea is to choose the largest value of $P_i(s)$, where $s \in \mathbb{R}^n$, and P_i is the projection function of $\mathbb{R}^n$, i.e., choose the projection with the largest magnitude. By the triangle inequality, we know that the difference between the largest projection of values $s, t \in \mathbb{R}^n$ is always less than or equal to their respective Euclidean distances. As a result, we can always bound the differences between the difference of the outputs by the difference of the inputs by some constant value k, thereby making the function P_i Lipschitz. For elements $s, t \in \mathbb{R}^n$, choose $k = 1$. Then,

$$|s_i - t_i| < (1)d(s,t) \tag{7}$$

$$= \left(\sum_{i=1}^{n} |s_i - t_i|^2 \right)^{\frac{1}{2}} \tag{8}$$

Therefore, P_i is Lipschitz. □

Problem 4. Denote by ℓ_1 the set of all absolutely convergent series: the elements of ℓ_1 are sequences $a = (a_i)_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} |a_i| < \infty$. For $a = (a_i)_{i=1}^{\infty}$ and $b = (b_i)_{i=1}^{\infty}$, define

$$d(a,b) = \sum_{i=1}^{\infty} |a_i - b_i|.$$

1. Prove that d is a metric.
2. Prove that the function $f : \ell_1 \to \mathbb{R} : (a_i) \mapsto \sum_{i=1}^{\infty} a_i$ is Lipschitz.
3. Determine whether the function $g : \mathbb{R} \to \ell_1$, taking $t \in \mathbb{R}$ to the sequence $\left(\frac{t^2}{2^i}\right)_{i=1}^{\infty}$, is uniformly continuous.

Solution 4. 1.

2. To show that d is a metric, we need to show that d satisfies the three criteria of a metric space:

(a) Non-degeneracy:

$$d(x, y) = 0 \iff x = y \tag{9}$$

$$d(a, b) = \sum_{i=1}^{\infty} |a_i - b_i| \tag{10}$$

Proof. $\implies$:

$$\sum_{i=1}^{\infty} |a_i - b_i| = \sum_{i=1}^{\infty} |0| = 0 \tag{11}$$

$\impliedby$:

$$\sum_{i=1}^{\infty} |a_i - b_i| = 0 \implies |a_i - b_i| = 0 \tag{12}$$

$$a_i = b_i = 0 \tag{13}$$

□

(b) Symmetry:

$$\forall x, y \in \ell_1, d(x, y) = d(y, x) \tag{14}$$

Proof.

$$|a_i - b_i| = |b_i - a_i| \tag{15}$$

□

(c) Triangle Inequality:

$$\forall x, y, z \in \ell_1, d(x, y) + d(y, z) \geq d(x, z) \tag{16}$$

Proof.

$$\sum_{i=1}^{\infty} |a_i - b_i| + \sum_{i=1}^{\infty} |b_i - c_i| \geq \sum_{i=1}^{\infty} |a_i - c_i| \tag{17}$$

$$\sum_{i=1}^{\infty} a_i = A, \sum_{i=1}^{\infty} b_i = B, \sum_{i=1}^{\infty} c_i = C \tag{18}$$

$$|A - B| + |B - C| \geq |A - C| \tag{19}$$

$$|A - B| + |B - C| \geq |(A - B) + (B - C)| \tag{20}$$

Where Equation 18 is proven in Homework 5 Problem 3.1, and Equation 20 is proven by the triangle inequality. □

3.

$$f : \ell_1 \to \mathbb{R} : (a_i) \mapsto \sum_{i=1}^{\infty} a_i \tag{21}$$

$$s, t \in \ell_1, d^*(f(s), f(t)) \leq kd(s, t) \tag{22}$$

Proof. We wish to find:

$$\sum_{i=1}^{\infty} s_i - \sum_{i=1}^{\infty} t_i \leq k \sum_{i=1}^{\infty} |a_i - b_i| \tag{23}$$

$$|S| - |T| \leq k\,|S - T| \tag{24}$$

Choose $k = 1$. Then:

$$|(S - T) + T| \leq |S - T| + |T| \tag{25}$$

$$|S| \leq |S - T| + |T| \tag{26}$$

By the triangle inequality established in Problem 4.1, at $k = 1$, f is Lipschitz. □

4. To prove that $g : \mathbb{R} \to \ell_1$ is uniformly continuous, we will use the definition, i.e.,

$$\forall \varepsilon > 0, \exists \delta \text{ } s.t. \text{ } x, y \in S \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \tag{27}$$

We will determine δ by the following discussion:

$$f(x) - f(y) = \left(\frac{x^2}{2^n} - \frac{y^2}{2^n} \right) = \tag{28}$$

$$\left(\frac{x^2 - y^2}{2^n} \right) = \left(\frac{(x + y)(x - y)}{2^n} \right) \tag{29}$$

$$\left(\frac{(x + y)(x - y)}{2^n} \right) < \varepsilon \tag{30}$$

$$(x - y) < \frac{\varepsilon \cdot 2^n}{(x + y)} \tag{31}$$

$$\tag{32}$$

Choose $\delta = \frac{\varepsilon \cdot 2^n}{(x+y)}$. Then, $|x-y| < \delta$ implies:

$$|f(x) - f(y)| = \left| \left(\frac{x^2}{2^n} - \frac{y^2}{2^n} \right) \right| = \tag{33}$$

$$\left(\frac{x^2 - y^2}{2^n} \right) = \left(\frac{(x+y)(x-y)}{2^n} \right) < \left(\frac{(x+y)\delta}{2^n} \right) \tag{34}$$

$$\frac{(x+y) \cdot \varepsilon \cdot 2^n}{(x+y) \cdot 2^n} = \varepsilon \tag{35}$$

$$\tag{36}$$

Therefore,

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon \tag{37}$$

Then g is uniformly continuous.