

# The Fundamental Theorem of Calculus

Jerich Lee

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**Problem 1** (26.6). Let  $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$  and  $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$  for  $x \in \mathbb{R}$ .

(a) Prove  $s' = c$  and  $c' = -s$ .

(b) Prove  $(s^2 + c^2)' = 0$ .

(c) Prove  $s^2 + c^2 = 1$ .

Actually,  $s(x) = \sin x$  and  $c(x) = \cos x$ , but you do **not** need these facts.

**Solution 1.** (a) (a) *Proof.* For  $x \in \mathbb{R}$ :

$$s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (1)$$

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (2)$$

$$\lim_{t \rightarrow x} \phi(t) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \quad (3)$$

$$s(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1)!} \quad (4)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (5)$$

$$c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n)}}{(2n)!} \quad (6)$$

$$s'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} \quad (7)$$

$$= \frac{(-1)^n x^{2n}}{(2n)!} \quad (8)$$

$$= c(x) \quad (9)$$

□

(b) *Proof.*

$$c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (10)$$

$$c'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} \quad (11)$$

$$= \frac{(-1)^n x^{2n-1}}{(2n-1)!} \quad (12)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} \quad (13)$$

$$= -s(x) \quad (14)$$

□

(b) *Proof.*

$$(s^2 + c^2)' = 0 \quad (15)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad (16)$$

$$= \sum_{k=0}^{\infty} \left( \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^k x^{2k}}{(2k)!} \right) \quad (17)$$

$$= 0 \quad (18)$$

□

(c) todo

**Problem 2** (33.3). A function  $f$  on  $[a, b]$  is called a *step function* if there exists a partition

$$P = \{a = u_0 < u_1 < \cdots < u_m = b\} \quad (19)$$

of  $[a, b]$  —not  $P = \{a = u_0 < u_1 < \cdots < c_m = b\}$ , as stated in the textbook— such that  $f$  is constant on each interval  $(u_{j-1}, u_j)$ , say  $f(x) = c_j$  for  $x$  in  $(u_{j-1}, u_j)$ .

(a) Show that a step function  $f$  is integrable and evaluate  $\int_a^b f$ .

**Solution 2.** *Proof.* If  $f$  is constant on every interval, then  $M_i = m_i$ . Then,  $f$  is monotone and bounded on  $(u_{j-1}, u_j)$ , (in fact it is constant from  $(u_{j-1}, u_j)$ ), so therefore it is uniformly continuous, which implies that there exists a continuous extension to the closed set  $[u_{j-1}, u_j]$ . Invoking Theorem (3.38) from Ross (Piecewise Monotone), we show that  $f \in \mathcal{R}$ . □

**Problem 3** (33.7). Let  $f$  be a bounded function on  $[a, b]$ , so that there exists  $B > 0$  such that  $|f(x)| \leq B$  for all  $x \in [a, b]$ .

(a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions  $P$  of  $[a, b]$ . *hint:*  $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$ .

(b) Show that if  $f$  is integrable on  $[a, b]$ , then  $f^2$  also is integrable on  $[a, b]$ .

**Solution 3.** *Proof.*  $f$  is a bounded function on  $[a, b]$ , so there exists  $B > 0$  s.t.  $|f(x)| \leq B$  for all  $x \in [a, b]$ .

$$u(f^2, p) = \sum_{i=1}^{\infty} M_i^2 \Delta x_i \quad (20)$$

$$= M_i^2 = \sup f(x)^2 \quad (21)$$

$$= \sum_{i=1}^n (M_i^2 - m_i^2) \Delta x_i \quad (22)$$

$$= \sum_{i=1}^n (M_i + m_i) (M_i - m_i) \Delta x_i \quad (23)$$

$$\leq \sum_{i=1}^n 2B (M_i - m_i) \Delta x_i \quad (24)$$

$$= 2B \sum_{i=1}^{\infty} (M_i - m_i) \Delta x_i \quad (25)$$

$$= 2B (U(f, p) - L(f, p)) \quad (26)$$

□

**Problem 4** (34.2). Calculate

$$(a) \lim_{h \rightarrow 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt.$$

**Solution 4.** *Proof.*

$$\int_3^{3+h} e^{t^2} dt \quad (27)$$

$$= \lim_{h \rightarrow 0} \frac{F(3+h) - F(3)}{h} \quad (28)$$

$$(29)$$

By FTC I,

$$F'(3) = e^9 \quad (30)$$

□

**Problem 5** (34.5). Let  $f$  be a continuous function on  $\mathbb{R}$  and define

$$F(x) = \int_{x-1}^{x+1} f(t) dt \quad \text{for } x \in \mathbb{R}. \quad (31)$$

Show  $F$  is differentiable on  $\mathbb{R}$  and compute  $F'$ .

**Solution 5.** *Proof.*

$$F(x) = \int_x^0 f(t) dt + \int_0^{x+1} f(t) dt \quad (32)$$

$$= - \int_0^{x-1} f(t) dt + \int_0^{x+1} f(t) dt \quad (33)$$

$$(34)$$

By FTC II, we get:

$$F'(x_0) = f(x_0 + 1) - f(x_0 - 1) \quad (35)$$

□

**Problem 6. A. [Bonus problem]** Suppose  $f$  is a continuous non-negative function on  $[a, b]$ , with

$$M = \max_{x \in [a, b]} f(x). \quad (36)$$

For  $n \in \mathbb{N}$ , let

$$M_n = \left( \int_a^b f^n dt \right)^{1/n}. \quad (37)$$

Prove that  $\lim M_n = M$ .

**Solution 6.** *Proof.*

$$\left( \int_a^b f^n \right)^{\frac{1}{n}} \leq ((b-a)(M^n))^{\frac{1}{n}} \quad (38)$$

$$= \lim_{n \rightarrow \infty} \underbrace{(b-a)^{\frac{1}{n}}}_1 M = M \quad (39)$$

□