Mathematics of Guitar Strings

Jerich Lee

2023-05-08

The mechanical analysis for the rotary spring mechanism begins with the free body diagram of the initial state of the guitar, as shown in Figure 12, and the final state of the guitar, as shown in Figure 13. The free body diagram gives initial length as the sum of $a+b$, where a is the length between the boundaries of the string, which the user plays, and b is the length between the boundary point of a and the guitar string tuning key.

`classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig12.png`

Figure 1: Free Body Diagram of Initial State

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig13.png

Figure 2: Free Body Diagram of Deformed State

The equation that relates frequency to the tension of a guitar string is the vibrating–string equation shown in **Equation 1**:

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho_L}} \tag{1}$$

By performing a Taylor-series expansion of Equation 1 with respect to tension, the partial derivative of frequency with respect to tension is

$$\frac{\partial f}{\partial T} = \frac{1}{4L\sqrt{T\rho_L}} \tag{2}$$

Simple algebraic manipulation of Equation 1 yields

$$\frac{f_0}{T_0} = \frac{1}{2L\sqrt{T\rho_L}} \tag{3}$$

Using a first-order Taylor expansion, the frequency after a small change in tension ΔT becomes

$$f = f_0 + \frac{\partial f}{\partial T}\,\Delta T \tag{4}$$

Accordingly, the change in frequency is

$$\Delta f = \frac{\partial f}{\partial T}\,\Delta T \tag{5}$$

Finally, substituting Equation 3 into Equation 2 shows that

$$\frac{\partial f}{\partial T} = \tfrac{1}{2}\,\frac{f_0}{T_0} \tag{6}$$

By plugging Equation (6) into Equation (5), we obtain

$$\Delta f = \frac{1}{2}\,\frac{f_0}{T_0}\,\Delta T \tag{7}$$

Re-arranging Equation (7) gives the change in tension in terms of the initial tension T_0, initial frequency f_0, and the desired frequency shift Δf:

$$\Delta T = 2T_0\,\frac{\Delta f}{f_0} \tag{8}$$

Using the definition of normal stress σ as force divided by cross-sectional area and the engineering strain ϵ as the fractional change in length,

$$\sigma = \frac{F}{A} \tag{9}$$

$$\epsilon = \frac{L - L_0}{L_0}, \tag{10}$$

and assuming linear elasticity (Hooke's law)

$$\sigma \;=\; E\,\epsilon, \tag{11}$$

we can combine (9)–(11) to obtain

$$\frac{F}{A} \;=\; E\,\frac{L - L_0}{L_0}. \tag{12}$$

The axial displacement of the string is simply the difference between the current length L and the original length L_0:

$$\delta \;=\; L - L_0. \tag{13}$$

Thus, once δ is determined from the geometric relations in the next section, Equations (8) and (12) can be used to calculate the tension increment ΔT and the corresponding axial force F required to achieve the target frequency shift.

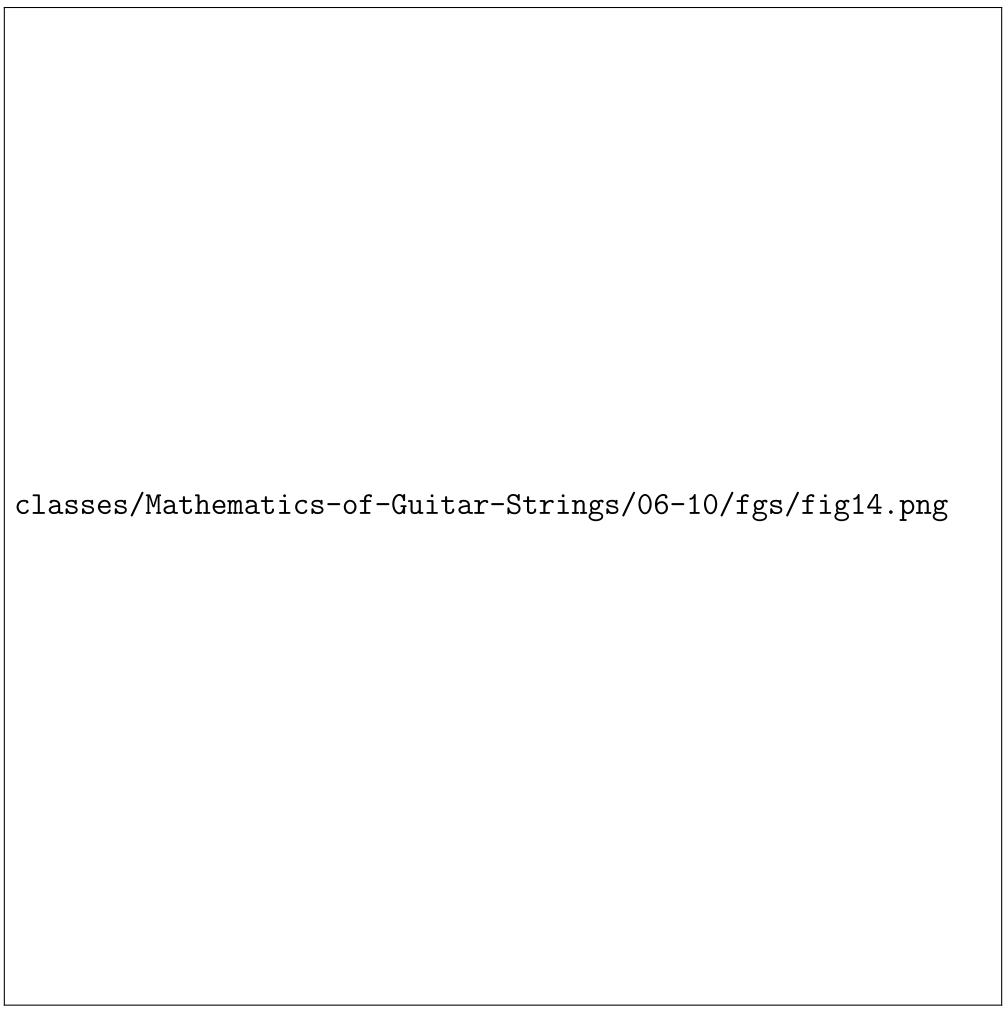

Figure 3: Free Body Diagram of Deformed Region

$$L_0 \;=\; a + b \tag{14}$$

$$L \;=\; a + 2\sqrt{\delta^2 + \left(\frac{b}{2}\right)^2} \tag{15}$$

Equation 14 and Equation 15 can be substituted into Hooke's-law relation (11) to obtain the strain of the string in terms of the vertical deflection δ and the baseline lengths a and b:

$$\frac{L - L_0}{L_0} \;=\; \frac{2\sqrt{\delta^2 + \left(\frac{b}{2}\right)^2} \;-\; b}{a + b} \tag{16}$$

Solving Equation 16 for the vertical displacement δ yields

$$\delta \;=\; \sqrt{\left(\frac{\Delta T\,(a+b)}{2(EA)_{\text{eff}}} + \frac{b}{2}\right)^2 \;-\; \left(\frac{b}{2}\right)^2} \tag{17}$$

The value of δ in Equation 17 provides the *horizontal displacement* that the user's foot must apply to the pedal so the string reaches the target frequency consistently. All quantities on the right side are either measured (a, b), experimentally characterized [T_0, $(EA)_{\text{eff}}$], or specified by the musician ($\Delta f = f - f_0$).

The force applied when the string displaces by δ is obtained from the free-body diagram in Figure 15. By resolving the component forces along the y-axis, one arrives at Equations (18) and (19).

$$\sum F_y \;=\; 0 \;=\; 2\big(T_0 + \Delta T\big)\frac{\delta}{\sqrt{\delta^2 + b^2/4}} \tag{18}$$

$$F_T \;=\; 2\big(T_0 + \Delta T\big)\,\frac{\delta}{\sqrt{\delta^2 + \left(\frac{b}{2}\right)^2}} \tag{19}$$

To solve for the vertical displacement of the string in **Equation 17** and for the applied force required in **Equation 18**, the constants a, b, f_0, Δf, T_0 and $(EA)_{\text{eff}}$ must be known. The values a and b are obtained by measuring the distance between the string boundaries and the distance from the boundary point at a to the guitar's tuning key, respectively. The initial frequency f_0 and the desired frequency change Δf are found from the difference between the initial note G♯3 and the final note A3. Finally, the initial tension T_0 and the effective axial stiffness $(EA)_{\text{eff}}$ are determined experimentally. Table 1 lists these values.

Table 1: *

Table 1. Constants for Guitar String

Constant	**Value**
a	22.50 in
b	7.374 in
f_0	207.7 Hz
T_0	29.5983 lb
Δf	12.460 Hz
$(EA)_{\text{eff}}$	4638 lb

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig15.png

Figure 4: Free Body Diagram of Forces at Deformed Region

After the constants are known, they are substituted into Equation (8), Equation (17), and Equation (18) to obtain the values of the functions ΔT, δ, and F_T, summarized in Table 2.

Table 2: *

Table 2. Values of Interest for Guitar String

Function	**Value**
ΔT	3.55 lb
δ	0.29 in
F_T	5.21 lb

Using these solved values, the required change in string length $\delta = 0.29$ in produces the pitch A3, and the force required to achieve this displacement is $F_T = 5.21$ lb. Because the player will apply the load via a foot pedal rather than directly at the head-stock, a static analysis of the pedal is needed. Of particular interest are (i) the angular displacement needed to produce δ and (ii) the force required at the pedal. Figures 16 and 17 illustrate the pedal mechanism. A stop pin fixes the pedal at an exact angular position, ensuring consistent notes. The design task is therefore to determine the pin height h (see the free-body diagram in Figure 18) as a function of the initial and final frequencies.

The constant dimensions of the foot pedal are listed in Table 3. The green line d represents the string segment that runs from pin C to pin A. An external force is applied at point D, causing the pedal to rotate about pin B. This rotation lengthens d; our goal is to choose the pedal's angular displacement so that d increases by precisely the vertical string deflection δ required to reach the target frequency A3. That same rotation fixes the pin height h that will hold the pedal at the desired angle when the player presses down, because the reaction forces at the base keep the angle stable. Determining h therefore begins with computing the length d via the Law of Cosines (Figure 19).

Table 3: *

Table 3. Constants for Foot Pedal

Constant	**Value**
c	2.50 in
e	3.10 in
f	3.40 in
θ	60°

The Law of Cosines gives the original length d of the green segment:

$$d^2 = e^2 + c^2 - 2ec\cos\theta \tag{20}$$

$$d = \sqrt{e^2 + c^2 - 2ec\cos\theta} \tag{21}$$

Once d is known, the new pedal angle γ is obtained with a second application of the Law of Cosines in Equations 22 and 23 (see the free-body diagram in Figure 20). Here γ is the post-actuation angle $\angle ABC$ that produces a total string displacement of $d + \delta$ inches.

$$(d + \delta)^2 = c^2 + e^2 - 2ce\cos\gamma \tag{22}$$

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig16.png

Figure 5: Side Angle View of Pedal

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig17.png

Figure 6: Side Angle View of Foot Pedal Base

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig18.png

Figure 7: Free Body Diagram of Initial State of Foot Pedal

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig19.png

Figure 8: Free Body Diagram of Initial State of Triangle ABC

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig20.png

Figure 9: Free Body Diagram of Actuated State of Triangle ABC

$$\gamma \ = \ \arccos\left(\frac{c^2 + e^2 - (d+\delta)^2}{2ce}\right) \tag{23}$$

After γ is known, its complement $\alpha = \angle DBE$ (see Figure 21) is

$$\alpha \ = \ 90^\circ - \gamma \tag{24}$$

The angle α is then used to determine the nail height h in Equation 21.

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig21.png

Figure 10: Free Body Diagram of Actuated State of Foot Pedal

$$\tan\alpha \ = \ \frac{h}{f} \tag{24}$$

$$h \ = \ f\ \tan\alpha \tag{25}$$

The constant dimensions of the foot pedal are substituted into Equation (21), Equation (23), and Equation (25) to obtain the quantities listed in Table 4.

Table 4: *

Table 4. Values of Interest for Foot Pedal

Function	Value
d	2.85 in
α	22.82°
h	1.43 in

Table 4 shows the exact length h required for the nail at the end of the pedal to ensure the final string frequency is 220.16 Hz (note A3). To determine the force applied to the pedal to achieve this frequency, the angle β must be obtained from the Law of Cosines, as illustrated in Figure 22 and expressed in Equation 27.

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig22.png

Figure 11: Free Body Diagram of Triangle ABC

$$e^2 = c^2 + (d+\delta)^2 - 2c(d+\delta)\cos\beta \tag{26}$$

$$\beta = \arccos\left(\frac{c^2 + (d+\delta)^2 - e^2}{2c(d+\delta)}\right) \tag{27}$$

The force that must be applied at the end of the pedal to realize this geometry is obtained from the free-body diagram in Figure 23 and is expressed in Equation 28 and Equation 29.

classes/Mathematics-of-Guitar-Strings/06-10/fgs/fig23.png

Figure 12: Free Body Diagram of Forces on Pedal

$$\sum M_B = 0 \; = \; cF_T \sin\beta \; - \; f\, F_F \tag{28}$$

$$F_F \; = \; \frac{cF_T \sin\beta}{f} \tag{29}$$

Table 5 summarizes the solved values of β and the pedal force F_F. The required force on the pedal to lower the string to note A3 is $F_F = 3.49\,\text{lb}$.

Table 5: *

Table 5. Values of Interest for Forces on Foot Pedal

Function	**Value**
β	65.56°
F_F	3.49 lb

The force applied at the pedal is therefore smaller than the direct force needed at the head-stock: the pedal requires only 3.49 lb whereas the string tension change demands $F_T = 5.21$ lb. The resulting mechanical advantage—because the pedal acts as a class-I lever—is

$$MA = \frac{F_T}{F_F} = 1.49 \tag{30}$$