Uniform Convergence

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Throughout this homework, we assume that (reverse) trigonometric and exponential functions (such as exp, sin, cos, tan, or arctan) are continuous. Other common calculus facts about these functions can also be used.

Problem 1 (20.16)**.** Suppose the limits $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exist.

(a) Show if $f_1(x) \le f_2(x)$ for all x in some interval (a, b), then $L_1 \le L_2$.

(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all x in some interval (a, b). Can you conclude $L_1 < L_2$?

Solution 1. 1. *Proof.* $\forall \varepsilon > 0, \exists \delta > 0\ s.t.\ |f_1(x) - L_1| < \varepsilon_1$ whenever $x \in (a-\delta, a+\delta) \cap S$. Let S be $(a, b), b > a$. $\forall \varepsilon > 0, \exists \delta_2 > 0\ s.t.\ |f_2(x) - L_2| < \varepsilon_2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$L_2 - \varepsilon < f_2(x) < L_2 + \varepsilon \tag{1}$$

$$L_1 - \varepsilon < f_1(x) < L_1 + \varepsilon \tag{2}$$

$$L_1 - \frac{\varepsilon}{2} < f_1(x) \le f_2(x) < L_2 + \frac{\varepsilon}{2} \tag{3}$$

$$\forall \varepsilon, L_1 < L_2 + \varepsilon \implies \tag{4}$$

$$L_1 \le L_2 \tag{5}$$

□

2. *Proof.* No, due to the proof of part a of this problem. We can say that $L_1 = L2$, which satisfies $L_1 < L_2 + \varepsilon$ but not $L_1 < L_2$. □

Problem 2 (20.17)**.** Show that if $\lim_{x\to a^+} f_1(x) = \lim_{x\to a^+} f_3(x) = L$ and if $f_1(x) \le f_2(x) \le f_3(x)$ for all x in some interval (a, b), then $\lim_{x\to a^+} f_2(x) = L$. This is called the squeeze lemma. *Warning*: This is not immediate from Exercise 20.16(a), because we are not assuming $\lim_{x\to a^+} f_2(x)$ exists; this must be proved.

Solution 2. *Proof.* $\lim_{x\to n^+} f_1(x) = \lim_{x\to n^+} f_3(x) = L$, if $\forall x \in (a,b), f_1(x) \le f_2(x) \le f_3(x) \implies \lim_{x\to a^+} f_2(x) = L$.

$$\text{Let } x \in (a-\delta, a+\delta) \cap S,\ S = (a,b)\ s.t.\ b > a. \tag{6}$$

$$\forall \varepsilon > 0, \exists \delta > 0\ s.t.\ |f_1(x) - L| < \varepsilon \tag{7}$$

$$|f_3(x) - L| < \varepsilon \tag{8}$$

$$L - \varepsilon < f_1(x) \le f_2(x) \le f_3(x) < \varepsilon + L \tag{9}$$

$$L - \varepsilon < f_2(x) < \varepsilon + L \tag{10}$$

$$|f_2(x) - L| < \varepsilon \implies \tag{11}$$

$$\lim_{x\to a^+} f_2(x) = L \tag{12}$$

□

Problem 3 (23.4c)**.** For $n = 0, 1, 2, 3, \ldots$, let $a_n = \left[\frac{4+2(-1)^n}{5}\right]^n$.

(a) Find $\limsup (a_n)^{1/n}$, $\liminf (a_n)^{1/n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.

(b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Explain briefly.

(c) Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.

Solution 3. 1. (a)

$$\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = \limsup_{n\to\infty} \left(\left[\frac{4+2(-1)^n}{5}\right]^n\right)^{\frac{1}{n}} \tag{13}$$

$$= \max\left\{\limsup_{k\to\infty} \left(\frac{6}{5}\right)^{\frac{2k}{2k}}, \limsup_{k\to\infty} \left(\frac{2}{5}\right)^{\frac{2k+1}{2k+1}}\right\} \tag{14}$$

$$= \frac{6}{5} \tag{15}$$

(b)

$$\liminf_{n\to\infty} (a_n)^{\frac{1}{n}} = \min\left\{\liminf_{k\to\infty} \left(\frac{6}{5}\right)^{\frac{2k}{2k}}, \liminf_{k\to\infty} \left(\frac{2}{5}\right)^{\frac{2k+1}{2k+1}}\right\} \tag{16}$$

$$= \frac{2}{5} \tag{17}$$

(c)

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\begin{cases}\frac{a_{2k+1}}{a_{2k}} = \frac{\left(\frac{2}{5}\right)^{2k+1}}{\left(\frac{6}{5}\right)^{2k}} & \text{if } n = 2k \\ \frac{a_{2k+2}}{a_{2k+1}} = \frac{\left(\frac{6}{5}\right)^{2k+2}}{\left(\frac{2}{5}\right)^{2k+1}} & \text{if } n = 2k+1\end{cases} \tag{18}$$

$$= \max\{0, \infty\} \tag{19}$$

$$= \infty \tag{20}$$

(d)

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \liminf_{n\to\infty}\begin{cases}\frac{a_{2k+1}}{a_{2k}} = \frac{\left(\frac{2}{5}\right)^{2k+1}}{\left(\frac{6}{5}\right)^{2k}} & \text{if } n = 2k \\ \frac{a_{2k+2}}{a_{2k+1}} = \frac{\left(\frac{6}{5}\right)^{2k+2}}{\left(\frac{2}{5}\right)^{2k+1}} & \text{if } n = 2k+1\end{cases} \tag{21}$$

$$= \min\{0, \infty\} \tag{22}$$

$$= 0 \tag{23}$$

2. These series do not converge; these do not pass the criteria of the Ratio and Root tests, i.e., either $\alpha = \limsup_{n\to\infty}|a_n|^{\frac{1}{n}} > 1$ or $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ imply that the series $\sum_{n=1}^{\infty} a_n$ does not converge.

3.

$$\sum_{i=1}^{\infty} a_n x^n \tag{24}$$

$$\beta = \limsup_{n\to\infty}|a_n|^{\frac{1}{n}} \tag{25}$$

$$R = \frac{1}{\beta} \tag{26}$$

$$\beta = \frac{5}{6} \tag{27}$$

$$R = \frac{5}{6} \tag{28}$$

$$|x| < \frac{5}{6} \tag{29}$$

The above converges for $(-\frac{5}{6}, \frac{5}{6})$.

Problem 4 (23.5b)**.** Consider a power series $\sum a_n x^n$ with radius of convergence R.

(b) Prove that if $\limsup |a_n| > 0$, then $R \leq 1$.

Solution 4. *Proof.*

$$\beta = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \tag{30}$$

$$c = \limsup_{n\to\infty} |a_n| > 0 \tag{31}$$

$$\lim_{n\to\infty} c^{\frac{1}{n}} = 1, \text{ by Theorem 9.7 in Ross} \tag{32}$$

$$|\beta - 1| < \varepsilon = 1 - \varepsilon < \beta < \varepsilon + 1 \tag{33}$$

$$\implies \beta \geq 1 \tag{34}$$

$$1 \geq \frac{1}{\beta} = R \tag{35}$$

$$1 \geq R \tag{36}$$

□

Problem 5 (24.10a)**.** (a) Prove that if $f_n \to f$ uniformly on a set S, and if $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.

Solution 5. *Proof.* $\forall \varepsilon > 0 \exists N_1$ *s.t.* $N_1(\varepsilon) \in \mathbb{N}$ *s.t.* $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N_1, \forall x \in S$, $\forall \varepsilon > 0 \exists N_2$ *s.t.* $N_2(\varepsilon) \in \mathbb{N}$ *s.t.* $|g_n(x) - g(x)| < \varepsilon$ for $n \geq N_2, \forall x \in S$. Then,

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \tag{37}$$

$$g(x) - \varepsilon < g_n(x) < g(x) + \varepsilon \tag{38}$$

$$f_n(x) - f(x) + g_n(x) - g(x) < \varepsilon \tag{39}$$

$$-f_n(x) - g_n(x) < \varepsilon - g(x) - f(x) \tag{40}$$

$$f_n(x) + g_n(x) > -\varepsilon + g(x) + f(x) \tag{41}$$

$$\implies |f_n(x) + g_n(x)| < \varepsilon \tag{42}$$

□

Problem 6 (24.11)**.** Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}$.

(a) Observe $f_n \to f$ uniformly on $\mathbb{R}$ [obvious!] and $g_n \to g$ uniformly on $\mathbb{R}$ [almost obvious].

(b) Observe the sequence $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. Compare Exercise 24.2.

Solution 6. 1. (a) $f_n \to f$ is Lipschitz at $k = 1$, $x - y \leq k|x - y|$, so $f_n \to f$ is uniformly continuous. We know that:

(b)

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| = 0 \tag{43}$$

$$= \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\frac{1}{n} - 0\right|, \lim_{n\to\infty} \frac{1}{n} = 0 \tag{44}$$

$$\implies \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{1}{n} = 0 \tag{45}$$

Therefore, $g_n \to g$ is uniformly convergent.

2. *Proof.*

$$(f_n g_n) \underbrace{\not\to}_{UC} fg \tag{46}$$

$$\lim_{n\to\infty} \sup_{x\in S} |f_n(x)g_n(x) - f(x)g(x)| = 0 \tag{47}$$

$$\lim_{n\to\infty} \sup_{x\in S} \left|\frac{x}{n} - 0\right| = 0 \tag{48}$$

$$fg(x) = \frac{1}{n} \tag{49}$$

$$f_n = \frac{x}{n} \tag{50}$$

$$\sup_{x\in S} \left|\frac{x}{n}\right| = \infty \tag{51}$$

$$\lim_{n\to\infty} \frac{\infty}{n} = \infty \tag{52}$$

$$\tag{53}$$

Therefore, $f_n g_n$ is not uniformly convergent. □

Problem 7 (24.14)**.** Let $f_n(x) = \frac{nx}{1+n^2x^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$.

(a) Show $f_n \to f$ pointwise on $\mathbb{R}$.

(b) Does $f_n \to f$ uniformly on $[0, 1]$? Justify.

(c) Does $f_n \to f$ uniformly on $[1, \infty)$? Justify.

Solution 7. 1. Choose $N > n = \frac{1}{\varepsilon x}$. Then,

$$\frac{1}{\varepsilon x} \tag{54}$$

$$\frac{1}{nx} < \varepsilon \tag{55}$$

$$\left|\frac{1}{nx} - 0\right| < \varepsilon \implies \forall x \in S, \lim_{n\to\infty} \frac{1}{nx} \tag{56}$$

$$\frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} < \varepsilon \tag{57}$$

$$\frac{nx}{1+n^2x^2} < \varepsilon \tag{58}$$

$$\left|\frac{nx}{1+n^2x^2} - 0\right| < \varepsilon \tag{59}$$

2. We wish to find: $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n - f| = 0$:

Proof.

$$\lim_{n\to\infty} \sup_{x\mathbb{R}} \left|\frac{nx}{1+n^2x^2} - 0\right| = 0 \tag{60}$$

$$\sup_{x\in\mathbb{R}} \frac{nx}{1+n^2x^2} \tag{61}$$

$$\frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} \tag{62}$$

$$\sup \frac{nx}{1+n^2x^2} \leq \sup \frac{nx}{n^2x^2} = \sup \frac{1}{nx} = 0 \tag{63}$$

$$\sup \frac{nx}{1+n^2x^2} \leq 0 \tag{64}$$

$$\implies \sup \frac{nx}{1+n^2x^2} = 0 \tag{65}$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\frac{nx}{1+n^2x^2} - 0\right| = 0 \tag{66}$$

$\square$

3. Yes, due to the proof above.

Problem 8 (25.5)**.** Let (f_n) be a sequence of bounded functions on a set S, and suppose $f_n \to f$ uniformly on S. Prove f is a bounded function on S.

Solution 8. *Proof.* Because $f_n \to f$ is uniformly convergent, by Theorem 24.4, f_n is uniformly Cauchy. Then, there exists some N *s.t.* $m, n \geq N \implies$

$$\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - g_n(x)| = 0 \tag{67}$$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \tag{68}$$

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2} \tag{69}$$

$$\tag{70}$$

Eqn 68 implies that f is bounded. □

Problem 9 (25.9a)**.** (a) Let $0 < a < 1$. Show the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $\frac{1}{1-x}$.

Solution 9. *Proof.* Let $M_k = |a_k|\, b^k$, where $a_k = 1$ and $b = a$ in the context of this problem. Then, $\limsup_{n\to\infty} M_k^{\frac{1}{k}} = a$, and $R = 1$. Then, $a \cdot 1 < 1$, so $\sum_{k=1}^{\infty} M_k < \infty$. Applying the Weierstrass M-test, we show that $\sum_{k=1}^{\infty} x^k$ converges uniformly on S. Using the result in section 14.2 in Example 1, we show that $\sum_{k=1}^{\infty} x^k = \frac{1}{1-r}$. Therefore, $\sum_{k=1}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $\frac{1}{1-x}$. □

Problem 10 (25.9b (Bonus))**.** (b) Does the series $\sum_{n=0}^{\infty} x^n$ converge uniformly on $(-1, 1)$ to $\frac{1}{1-x}$? Explain.

Solution 10. This convergence is not uniform because $f(x) = \frac{1}{1-x}$ is not bounded from $(-1, 1)$. This is because the interval $(-1, 1)$ is not compact, hence f is not bounded. This violates the result from Problem 8 of this homework.