Continuity

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Problem 1 (18.7)**.** Prove $xe^x = 2$ for some x in $(0, 1)$.

Solution 1. *Proof.* Our goal is to show that $f = x$ and $g = e^x$ are both continuous, then invoke Theorem 17.4 part iii): Let f and g be real-valued functions that are continuous at x_0 in $\mathbb{R}$. Then:

$$fg \text{ is continuous at } x_0 \tag{1}$$

To show that $f = x$ is continuous, we can show the following:

$$|x_n - x| < \varepsilon \tag{2}$$

Choosing $\delta = \epsilon$, we get:

$$|x_n - x| < \delta \implies |f(x_n) - f(x)| < \epsilon \tag{3}$$

We wish to prove that the function $f(x) = e^x$ is continuous.

Let $\epsilon > 0$. We need to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|e^x - e^{x_0}| < \epsilon$.

Starting with the expression:

$$|e^x - e^{x_0}| = e^{x_0}|e^{x-x_0} - 1| \tag{4}$$

Now, we use the elementary inequality:

$$e^y \geq 1 + y \tag{5}$$

For $y = -y$, we get:

$$e^{-y} \geq 1 - y \tag{6}$$

which implies:

$$\frac{1}{1-y} \geq e^y \quad \text{for } y < 1 \tag{7}$$

Hence:

$$|e^y - 1| \leq \max\left\{|y|, \left|\frac{y}{1-y}\right|\right\} \quad \text{for } y < 1 \tag{8}$$

Thus, taking $y = x - x_0$, we have:

$$|e^y - 1| \leq \max\left\{|y|, \left|\frac{y}{1-y}\right|\right\} \tag{9}$$

Now, choose δ small enough such that $|y| = |x - x_0| < \delta$ satisfies:

$$\max\left\{|y|, \left|\frac{y}{1-y}\right|\right\} < e^{-x_0}\epsilon \tag{10}$$

and $|y| < 1$.

Therefore, we can conclude that e^x is continuous at x_0. Invoking Theorem 17.4 iii), we

show that xe^x is a continuous function. To show that there exists x such that $h = xe^x = 2$, we can show that pick two points in $(0, 1)$: $x_1 = 0.01$, and $x_2 = 0.99$, and substitute these into h, getting $h(x_1) = 0.01$ and $f(x_2) = 2.66$. $h(x_1) < 2 < h(x_2)$, and because h is continuous on $(0, 1)$, we can invoke the IVT, which proves that there exists x such that $xe^x = 2$.

□

Problem 2 (21.2)**.** Consider $f : S \to S^*$ where S, d and S^*, d^* are metric spaces. Show that f is continuous at $s_0 \in S$ if and only if for every open set U in S^* containing $f(s_0)$, there is an open set V in S containing s_0 such that $f(V) \subseteq U$.

Solution 2. *Proof.* $\implies$: The goal is to show that every point in V has a neighborhood, i.e., is open. Because U is open, we know that there exists $r = \varepsilon$ for each point $y \in U$. Because f is continuous, we also know that there is a corresponding $\delta > 0$ such that $s \in B_\delta(x) = V$, so this implies that there exists an open ball V around each point s in S. Because $d(s, s_0) < \delta \implies d(f(s), f(s)) < \varepsilon$, and $\delta = V$, $\varepsilon \subseteq U$, as there may be other points not in δ that map into U, this implies $f(V) \subseteq U$.

$\impliedby$: We know that V is open for every U in S^*. Choose $\varepsilon \in U$ *s.t.* $\varepsilon > 0$. Then choose $\delta \in V$ *s.t.* $\delta > 0$. By implication, we know that δ is open. Then, we can state that $d(s, s_0) < \delta \implies d(f(s), f(s_0)) < \varepsilon$, which is the definition of continuity at a point. □

Problem 3 (21.3)**.** Let (S, d) be a metric space and choose $s_0 \in S$. Show $f(s) = d(s, s_0)$ defines a uniformly continuous real-valued function f on S.

Solution 3. *Proof.* We to show the definition of continuity:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_Y(f(p), f(q)) < \epsilon \tag{11}$$
$$\forall p, q \in X \text{ for which } d_X(p, q) < \delta. \tag{12}$$

We know that:

$$f(s_0) = d(s_0, s_0) = 0 \tag{13}$$

$$f(p) = d(p, s_0) \tag{14}$$

$$f(q) = d(q, s_0) \tag{15}$$

$$d_Y(f(p), f(q)) \leq d(f(p)) + d(f(q)) \tag{16}$$

$$\tag{17}$$

We want to show using the triangle inequality:

$$d(f(p), f(q)) \leq d(f(p), f(s_0)) + d(f(q), f(s_0)) < \varepsilon \tag{18}$$

From Eqn 13, 14, and 15, we can say:

$$d(f(p), f(q)) \leq d(p, s_0) + d(q, s_0). \tag{19}$$

Choosing p, q such that $d(p, s_0) < \frac{\varepsilon}{2}$ and $d(q, s_0) < \varepsilon$, we can pick $\delta = \varepsilon$ and use Eqn 14 and 15 to show:

$$d(f(p), f(q)) \leq d(f(p)) + d(f(q)) \tag{20}$$

$$\leq d(p, s_0) + d(q, s_0) < \varepsilon \tag{21}$$

□

Problem 4 (21.4)**.** Consider $f : S \to \mathbb{R}$ where (S, d) is a metric space. Show the following are equivalent:

1. f is continuous;
2. $f^{-1}((a, b))$ is open in S for all $a < b$;
3. $f^{-1}((a, b))$ is open in S for all rational $a < b$.

Solution 4. *Proof.* $1 \implies 2$: We know that any open interval (a, b) in R^1 is complete, i.e., every subsequence converges to a limit point contained in $\mathbb{R}$. Therefore, (a, b) is an open set in $\mathbb{R}$. We know from Problem 2 that if f is continuous, then every open set in the range corresponds to an open set in the domain. (a, b) is open, so $f^{-1}(a, b)$ is also open. □

Proof. $1 \implies 3$: By $1 \implies 2$, we know that if $a, b \in \mathbb{Q}$, then (a, b) is open by the denseness of the rationals, i.e., between every real there exists a rational. □

Proof. $3 \implies 1$: We achieve this by taking the converse of $1 \implies 3$, which exists by the bijection of $1 \implies 2$. □

Proof. $2 \implies 1$: This exists by the bijection of $1 \implies 2$. □

Proof. 3 $\implies$ 2: This is always true by the denseness of the rationals. $\square$

Proof. 2 $\implies$ 3: This is always true, as $\mathbb{Q} \subset \mathbb{R}$ $\square$

Problem 5 (21.5)**.** Let E be a noncompact subset of $\mathbb{R}^k$.

1. Show there is an unbounded continuous real-valued function on E. *Hint*: Either E is unbounded or else its closure E^- contains $\mathbf{x}_0 \notin E$. In the latter case, use $\frac{1}{g}$ where $g(\mathbf{x}) = d(\mathbf{x}, \mathbf{x}_0)$.
2. Show there is a bounded continuous real-valued function on E that does not assume its maximum on E.

Solution 5.
1. (a) Suppose that E is bounded, and x_0 is not a point of E. To show that there exists a continuous unbounded real-valued function on E, consider the function:

$$f(x) = \frac{1}{x - x_0} \tag{22}$$

This function is continuous on E.

   (b) Suppose that E is unbounded. Then, $f(x) = x$ is unbounded and is a continuous real-valued function on E.

2. Suppose that E is bounded. Then the following:

$$g(x) = \frac{1}{1 + (x - x_0)^2} \tag{23}$$

$g(x)$ is bounded, as $0 < g(x) < 1$ for all x. $g(x)$ has no maximal element, as x_0 is not a member of E.

Problem 6 (21.10(d))**.** Explain why there are no continuous functions mapping $[0, 1]$ onto $(0, 1)$ or $\mathbb{R}$.

Solution 6. We know that $[0, 1]$ is compact according to the according to the Heine-Borel Theorem, as it is closed and bounded. According to Theorem 21.4, if E is compact, then $f(E)$ is compact. $(0, 1)$ and $\mathbb{R}$ are not compact, so this contradicts Theorem 21.4..

Problem 7. Is it true that any bounded continuous function on $\mathbb{R}$ is uniformly continuous?

Solution 7. No. Choose $f = \frac{1}{x - x_0}$ with $x \in (0, x_0)$. Uniform continuity states that $\forall \varepsilon, \exists \delta$ *s.t.* $d(p, q) < \delta \implies d(f(p), f(q))$ for all p, q. Choose an arbitrary ε. Then, there exists a corresponding δ such that $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$. But as we take the same δ about x closer and closer to x_0, $d(f(x), f(x_n))$ will grow larger and larger, eventually exceeding our chosen ε. Therefore, there exists no constant δ that satisfies our chosen ε. Therefore, our bounded, continuous function f is not uniformly continuous.