Lecture 2: Some local and global network properties

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Studying the characteristics of a network are like studying the characteristics of a biological organism. There isn't just one property that characterizes it but many. Some of the network properties are local, i.e., measured at or around a given node, link, or small region of a network. Other properties are global, reflecting something that applies to the whole network. Here, we go through a number of local and global properties of networks that help us analyze them.

1 Degree distribution of a network

This, perhaps simplest of all the tools in this chapter, represents one of the most deployed measurements in the current arsenal of analysis that network scientists have. The idea is that, beyond the individual degrees of nodes, it is important to know the *overall* display of node degrees present in a network and in what proportions they can be found.

Therefore, starting with the degrees of all the nodes

$$\{k_1, k_2, \ldots, k_n\} \tag{1}$$

we construct first the histogram of degrees of the network by counting the number of times (or frequency) $H(k)$ one finds a node of degree k in the network.

Example 1. *The network in the Fig. 1 has degrees:*

$$k_1 = 2, k_2 = 3, k_3 = 3, k_4 = 1, k_5 = 1$$

and therefore the degree histogram is

$$H(k = 1) = 2, H(k = 2) = 1, H(k = 3) = 2.$$

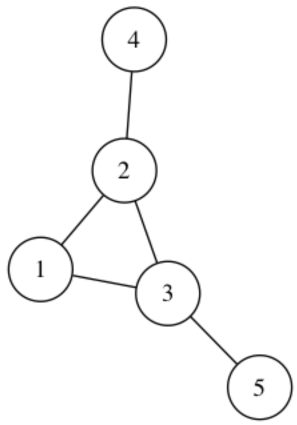

Figure 1: Example

Note that the histogram satisfies certain properties. For instance, the sum of the histogram frequencies gives us n, the number of nodes. This is clear because H is counting how many nodes in each of the possible cases. Thus

$$\sum_{k=0}^{n-1} H(k) = n. \tag{2}$$

The sum extends from $k = 0$ to $n - 1$ because these are the two limit values of k.

These equations allow to write the degree distribution of a given network as

$$\Pr(k) = \frac{H(k)}{\sum_{k=0}^{n-1} H(k)} = \frac{H(k)}{n}, \quad k = 0, \ldots, n-1. \tag{3}$$

The empirical study of the degree distribution has been responsible in large part for the new impetus that network science has acquired in the years since roughly 1999. This is because, of the surprising discovery that the Internet seemed to have a distribution in the form of a power-law that had not been empirically observed before; up to that point networks were customarily thought to have mainly Gaussian-like degree distributions.

There is another interesting set of properties that can be calculated by knowing the degree distribution. One of the simplest ones is the average degree,

$$\langle k \rangle = \sum_{k=0}^{n-1} k\Pr(k) = \frac{\sum_{k=0}^{n-1} kH(k)}{\sum_{k=0}^{n-1} H(k)} \tag{4}$$

The fact that this is the case can be seen by recalling that the average degree is also given by

$$\langle k \rangle = \frac{\sum_{i=1}^{n} k_i}{n} \tag{5}$$

but the numerator contains $H(k)$ repetitions of the a degree degree of value k, and thus adds $kH(k)$ to the sum.

Another related property is the following lemma, usually called the handshaking lemma. For an undirected network

$$\sum_{i=1}^{n} k_i = \sum_{k=0}^{n-1} kH(k) = 2m \tag{6}$$

or twice the number of links. This is easy to see in a few ways, but intuitively it means that the total sum of the degrees of a network gives a result of $2m$ because each link is identified twice when going from node to node. If on the other hand, the network is directed, one finds

$$\sum_{i=1}^{n} k_i^{(in)} = \sum_{i=1}^{n} k_i^{(out)} = m \tag{7}$$

because each link has an origin and destination nodes and thus when counting a link either from its origin or destination, we are not counting it twice.

By the way, connecting these results with the average degree, one finds in the undirected case that

$$\langle k \rangle = \frac{2m}{n}. \tag{8}$$

2 Path lengths and their distributions

Next, we look at paths in a network. They are given by sequences of links that can be visited in consecutive order when travelling in a network. To travel in a network, let us imagine we begin the trip in nodes i. From this node, one can travel along any of the links connected to i to one of its neighbors, say j. At j, one can again travel to a neighbor of j (which could be i again) along the links connected to j. Trips like this constitute network paths, and can be represented in several ways. If a path involves ℓ links, then it is a path of length ℓ. In terms of representations, we could write it down in any of the following forms:

$$(i, h_1), (h_1, h_2) \ldots (h_{\ell-1}, j) \tag{9}$$

$$i \to h_1 \to h_2 \to \ldots j. \tag{10}$$

These are two possible valid presentations of a path of length ℓ between origin node i and destination node j.

General network paths can revisit the same nodes or links, which makes them inefficient. Therefore, in many circumstances we have to restrict the kinds of paths we study even further by focusing on *simple* paths, those that do not intersect with themselves. One special kind of simple path, the *shortest path* is one of the key network paths.

To test whether a network path is present in a network, we can use link indicator products. Thus, the path indicated above between i and j as a link indicator product is represented by

$$a_{i,h_1} \times a_{h_1,h_2} \times \cdots \times a_{h_{\ell-1},j} = 1. \tag{11}$$

Note that if a path is not present, then its link indicator product is simply 0 instead of 1. This happens because for a path to be present, all the link indicators of the links that belong to the path must be equal to 1.

There is an efficient way to write down *all* the paths in a network when we do not care about the path backtracking or intersecting itself (simple path is not required). This is achieved by matrix multiplication of the adjacency matrix with itself. Note that, for instance, the square of the adjacency matrix is given by

$$\mathbf{A}^2_{ij} = \sum_{h_1=0}^{n-1} a_{i,h} a_{h,j}. \tag{12}$$

Note that the definition of matrix multiplication automatically gives us products of link indicators with the correct format for a path, i.e., with consecutive links. Unfortunately as indicated above, these paths do not exclude crossing over themselves, for instance.

In fact, it is important to state the following:

$$\#\text{paths of length } \ell \text{ between } i \text{ and } j = \mathbf{A}^{\ell}_{i,j} \tag{13}$$

or in other words the matrix element i, j of the matrix $\mathbf{A}^{\ell}$. The value of $\mathbf{A}^{\ell}_{ij}$ when it is greater than 0 is often greater than 1, This is because typically there is more than 1 path between i, j of length ℓ provided this length is greater (or even equal) than the shortest path between i, j (see below).

3 Structural connectivity

There are pairs of nodes for which no value of ℓ leads to $\mathbf{A}^{\ell}_{i,j} > 0$. Such node pairs are said to be structurally disconnected. This means that the nodes belong to different *clusters* of the network. Structural connectivity is a fundamental global property of networks.

We can extend the notion of structural connectivity from node pairs to entire groups of nodes. In this case, a structurally connected group of nodes is called a cluster. When a network is structurally disconnected into, say, C different clusters, each cluster has a number of the total n nodes. Thus, if we label the number of nodes of cluster c as n_c, we have

$$n = \sum_{c=1}^{C} n_c. \tag{14}$$

Structural connectivity has the features of an equality relation (formally an equivalence class). Thus, if i and j are structurally connected, and then j and h are structurally connected, then i and h are structurally connected. If, on the other hand, i and j are structurally disconnected, but j and h are structurally connected, then i and h are structurally disconnected.

4 Shortest paths and their histograms

There are a few things to say about shortest paths. First, note that shortest paths are paths between a pair of nodes i, j such that no other path of less links can be found between those nodes.

The shortest path between i and j can be found by determining the smallest value $s(i,j) = s_{ij}$ for which $\mathbf{A}_{i,j}^{s_{ij}} > 0$. What this means is that for $\ell < s_{ij}$ the paths are not long enough to make a connection between i, j. Note that the value of $\mathbf{A}_{i,j}^{s_{ij}}$ can be greater than 1 because, as explained above, there may be more than one path of length s_{ij} between i and j.

Another important quantity associated with shortest paths is their histogram over a network. There a number of technical details that need to be kept in mind when talking about histograms of shortest paths. This is because shortest paths are relevant to pairs of nodes, which leads to some important details.

First, when studying the shortest paths in a network, we can do so by studying several sampling methods of pairs of nodes. We could sample:

1. all pairs,
2. a fixed origin and all other destinations,
3. a selection of all pairs, or

4. from a fixed origin to all other destinations and then loop through the origin (this is done for memory purposes).

Depending on which of these methods is used, there are certain values that $\sum_{i,j} H(s_{ij})$ will take. Without trying to list all of them, one can rather simply justify that when $H(s_{ij})$ is constructed for all pairs of nodes, we have that

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} H(s_{i,j}) \geq \sum_{c=1}^{C} \binom{n_c}{2}, \tag{15}$$

where the right hand side is the count of all node pairs among each of the structurally connected clusters that constitute the network. The inequality comes from the possibility that $A_{i,j}^{s_{i,j}} > 1$. The last equation basically *counts* pairs of structurally connected nodes in a network that has C clusters with sizes $\{n_c\}_c$.

4.1 The small world property

It turns out that if we look at the typical or average shortest path lengths in a network, oftentimes we find that it is related to the number of network nodes by a logarithmic relation of the form

$$\langle s \rangle \sim \log n. \tag{16}$$

This relation is usually called the small-world effect.

The small-world effect is an idea that has been present in social science since the early part of the 20th century. However, it was S. Milgram who in the 1960s devised an experiment to try and find this effect. The details of the research are interesting, but the key consequence of the experiment was that it led to the notion of *six degrees of separation* representing that two nodes are typically distant from each other by only about 6 links (in general the number is different, but still a small number).

We are not going to prove this result now, but only mention that it relies on the tree-like nature of networks. By the way, there are other "size" classes in networks including

1. large world networks, and
2. ultra-small worlds.

Ask me about the details if you're keen.

5 Centrality and Betweenness

There is a general notion is networks called centrality, and it is the idea that certains nodes play central roles in a network. However, the details of that centrality are associated with the way in which we conceive of a node as being central.

Of the many kinds of centrality measures on a network, two are particularly famous: eigenvector centrality and betweeness centrality. The first of these two (eigenvector) assigns to each node a value x_i that represents a type of score of importance in the network. It is a quantity that basically is supposed to satisfy the following relation with the x_j of its neighbors

$$x_i = \beta \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n \tag{17}$$

Thus, if a node i is connected to a node j, and the later has a large value x_j then some of this should help increase x_i. In other words, it is a measure of importance of a node due to two factors, the number of connections of the node, and to what other nodes the node is connected. Since the β indicated in the equation has to be consistent for all nodes, then we end up with the vector equation

$$\lambda \mathbf{x} = \mathbf{A}\mathbf{x}, \tag{18}$$

where $\lambda = 1/\beta$. This equation represents an eigenvalue problem for $\mathbf{A}$ with eigenvalue λ. The eigenvector $\mathbf{x}$ associated with λ provides the centrality values for the nodes when λ is the largest eigenvalue.

To be clear, Eq. 18 provides an equation to solve for the values of x_i so that they satisfy the condition imposed by Eq. 17 (which is in fact the same as the condition in Eq. 18 which is the matrix form of the conditions all put together). Thus, solving the eigenvalue problem in Eq. 18 is the required step to get the problem solved. The values of x_i that come out are important in a comparative sense: if $x_i > x_j$ then i is more central than j; the actual values x_i, x_j are not as important.

The second centrality is called betweenness. For a given node i, its betweenness b_i corresponds to the number of shortest paths between all other node pairs in the network that visit i. There are different convention on how to count: i) some include the origin and destination nodes, while others do not, ii) if there are multiple shortest paths between node pairs, in some conventions each path is counted as one path whereas others count it as a fraction such that the sum of all shortest paths counts to 1. Once these details are established, one has to look at all pairs of nodes in a network, and count for each node how many of those paths between node pairs pass through the node.

The notion that stems from this is that a node with high betweenness is a node that is likely to be exposed to a lot of information travelling in a network. It is a node that also is likely to have a disproportionate effect to communication if it decides to stops acting as an intermediary.

6 Practicalities for shortest paths measures

Without going into too much detail, it is important to remind ourselves that calculating shortest paths is not usually done mathematically (unless for small example graphs) but rather computationally. Shortest path finding is not particularly demanding computationally when done from one origin and maybe all other nodes as destinations, but it becomes demanding when one tries to do this for all node pairs.

Betweenness centrality, in particular, does require a respectable amount of computation. The complexity of the algorithm scales roughly as n^2.

The main algorithm to determine shortest paths from one origin to all other nodes is the *breadth-first-search* algorithm. It is a greedy algorithm, and functions extremely well.

7 Motifs

There is another important concept in network analysis relating to the presence of particular shapes. Let us take, for instance, the network in Fig. 2. This is a network in which we can distinguish two triangles embedded in the structure. The fact that a given network has many, or few, examples of a particular shape is taken in many situations to be an informative feature. As another example, social networks are well known for having a large number of triangles, whereas random networks like Erdős-Rényi ones have a small number of triangles.

The interest in finding particular shapes in a network is not limited to triangular shapes, but it can be applied to many different such shapes. A network shape is usually called a *motif* and corresponds to a connected network subgraph with node labels stripped off, that one can find in numerous places in a network.

To systematically analyze a network for its motifs, there have been several approaches, such as those by Alon at al. In social science, the identification of triangles and an associated property called clustering have been

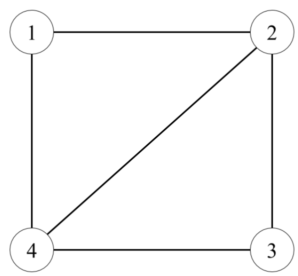

Figure 2: Example network in which two triangular motifs can be seen.

very important.

To expand a bit on the notions of motifs, we can mention a few generalities and a few concrete examples. In terms of generalities, a motif can be represented by a link indicator product appropriately chosen to the shape of interest. For instance, for a triangular motif, we can write the following link indicator product to represent it:

$$\text{triangular motifs link product: } a_{i,h}a_{h,j}a_{j,i}, \quad i \neq j \neq h = 1, \ldots, n. \tag{19}$$

Every time that one finds a combination of i, j, h in a specific network G for which the product is 1, it means that these three nodes make up a triangle. Thus, to represent a motif, one can draw on a link indicator product that acts as a *template* for the shape. Note, for instance, that the indicator product above requires that the initial and final nodes are the same node i, and that the nodes are not all the same.

There are some generic useful numbers when it comes to motifs. One is the number of nodes r, and the other the number of links λ. For instance, a triangle motif has $r = 3$ and $\lambda = 3$.

Another useful motif, specially for social scientists, is called a v-shape (or a 2-path). It is a motif basically equivalent to a path of length 2 in a network. The indicator product for it is given by

$$\text{v-shape motif link product: } a_{i,h}a_{h,j}, \quad i \neq j \neq h = 1, \ldots, n. \tag{20}$$

This link indicator has $r = 3$ and $\lambda = 2$. It has, as the name suggests, the shape of a 'v'.

Note an important generic feature of motifs: it is typical for larger motifs to contain in them smaller ones, i.e. small motifs are embedded into larger

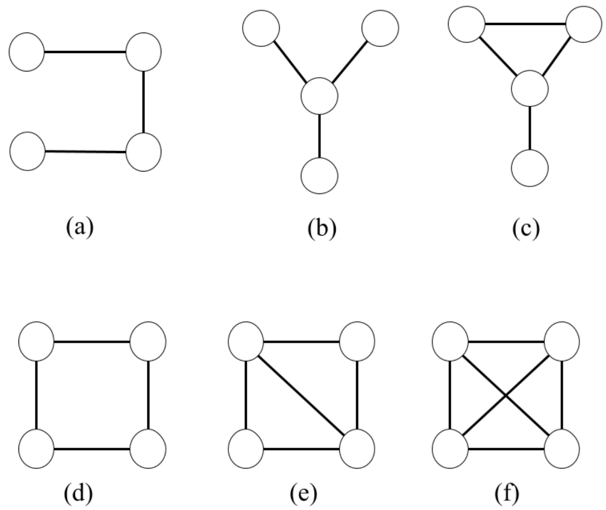

Figure 3: Possible undirected $r = 4$ motifs.

motifs. This is a feature that comes back again and again when it comes to counting motifs, a popular data analysis technique. When counting motifs it is important to understand as much as possible about the structure of the embedded counting. To illustrate, note that a triangle contains inside it 3 v-shapes: for triangle (i, j, h), there is a v-shape between i, j, h, another between h, i, j, and another between j, h, i. In other words, the vertex of the v-shape can be any one of the three nodes of the triangle.

There are many motifs one can look at, basically all the connected networks that can be constructed. Thus, when we try to look for motifs with increasing r, the number of possibilities begins to get large very quickly. With $r = 3$, we presented the only two cases. But for $r = 4$, the different motifs grows to 6 (see Fig. 3).

Finally, there is a family of motifs that is interesting to study because it illustrates mathematically many of the features mentioned above. This is the family of so-called *star motifs*. These are motifs that are simply a central node surrounded by a number of other nodes. A star motif of size r has one central node, and then $r - 1$ spokes coming out of that central node. The spokes are the $\lambda = r - 1$ links of the motif. But this motif is exactly like looking at a node with degree $k = r - 1$, and therefore a node in a network with degree $k - 1$ constitutes one start motif of size r.

The embedding of motifs is also very easy to illustrate in star motifs. A star motif of size r has inside it

$$r' \text{ size start motifs inside } r \text{ size star motifs} = \binom{r-1}{r'-1}. \tag{21}$$

This is because for $r' < r$, each of the $r' - 1$ spokes of the smaller motif have $r - 1$ spokes to choose from the larger motif. Now, to determine how many star motifs of size r a network has, we must count in the following way. First, nodes with degree $k = r - 1$ have a start motif of size r, and therefore, they contribute $H(k = r - 1)$ such motifs where here H is the degree histogram of the network. However, due to embedding, each node with degree $k = r$ which constitutes a start motif of size $r + 1$ contributes $\binom{r}{r-1} = r$ extra stars of size r. Now, if there are $H(k = r)$ such nodes, then the grand total is now $H(k = r - 1) + r \times H(k = r)$. This continues until we have reached the maximum degree, and therefore, if we label $\mathbf{M}$ the count of motifs

$$\mathbf{M}(r \text{ size star motifs}) = \sum_{k=r-1}^{n-1} \binom{k}{r-1} H(k). \tag{22}$$

8 Clustering

Having looked at the triangular and v-shape motifs, we can now introduce the notion of clustering (or transitivity) in networks. This idea captures the known feature of social networks that they tend to have more triangles than would be expected by chance.

Clustering has two versions, global and local, but both share the same notional idea. *Clustering is a ratio of the number of realized triangles with the number of triangle opportunities, or*

$$\text{Clustering} = \frac{\text{realized triangles}}{\text{triangle opportunities}} \tag{23}$$

Soon, we describe what is meant by opportunities in the local and global versions.

Below we explore in detail an illustration of clustering stemming from a research paper published in 2011 by Ugander who, with internal data from Facebook, determined that users have on average a 14% chance that those that they know as Facebook friends are also Facebook friends with each other. Clustering concepts will help us analyze this situation and see why it is noteworthy.

First, let us introduce the key definitions of clustering.

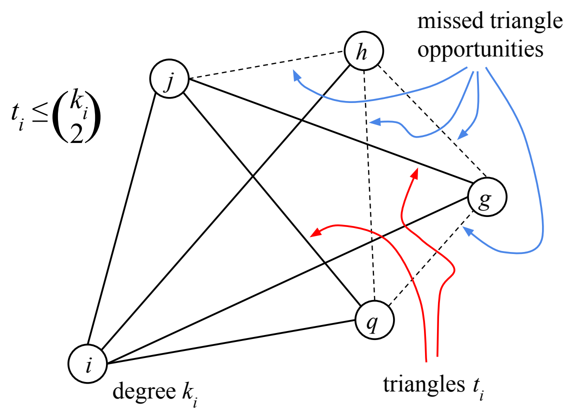

Figure 4: Node i of degree k_i. The number $\binom{k_i}{2}$ counts all the v-shapes with i as their vertex node. The possible number of triangles t_i involving i are always less than or equal to $\binom{k_i}{2}$. Local clustering for this node, where $k_i = 4$ and $\binom{4}{2} = 6$ is equal to $c_i = 2/6 = 1/3$.

8.1 Local clustering

A node $i \in V(G)$ with degree k_i, part of an undirected network G, has a local clustering coefficient c_i given by

$$c_i = \frac{t_i}{\binom{k_i}{2}} = \frac{2t_i}{k_i(k_i-1)}, \tag{24}$$

where t_i is the number of triangles that involve node i, and $\binom{k_i}{2}$ is the number of v-shapes that have i as their vertex node. Figure 4 illustrates this definition. In other words, what counts as an opportunity for a triangle is a v-shape.

It is easy to see the triangles that have been realized in around the node in Fig. 4. To understand why $\binom{k_i}{2}$ represents the number of v-shapes, note that what is required for a v-shape is that 3 nodes satisfy $a_{ji}a_{ih} = 1$ where i is given and j and h are free to be chosen anywhere else in the network. Since i has degree k_i it means that there are only k_i link indicators involving i and another k_i nodes that are $= 1$. Taking two such indicators at a time, we can create $\binom{k_i}{2}$ unique pairs of these link indicators, hence given $\binom{k_i}{2}$ v-shapes. What happens if k_i is 1 or 0? And to triangles?

Coefficient c_i effectively becomes the probability that the v-shapes that visit node i become a triangle. This is because the maximum number of triangles t_i around i is $\binom{k_i}{2}$, i.e., when all the v-shapes become closed. Let us now look at Facebook.

Example 2. *A study by Ugander et al. with internal data from Facebook revealed that on average 14% of pairs of Facebook friends of a Facebook user are also Facebook friends. In other words, 14% of local v-shapes in Facebook are closed into triangles. Is this a surprising result? Even by random chance, if a network does not have a special tendency to form triangles, sometimes links just happen to line up to form them.*

For instance, we can estimate the likelihood of local triangles (the local clustering of nodes) for a network with the same n and m as Facebook but put together randomly (basically we decide at random with some probability p if each link indicator a_{ij} is 0 or 1). According to Ugander et al., $n = 721 \times 10^6$ and $m = 68.7 \times 10^9$ for Facebook circa 2011. The average degree of both Facebook and any network (random or otherwise) with the same n and m is given by

$$\langle k \rangle = \frac{2m}{n} = \frac{2 \times 6.87 \times 10^{10}}{7.21 \times 10^8} \approx 1.91 \times 10^2$$

or about 191 friendships per user, on average. Also, the density of both Facebook and any other network (random or otherwise) with the same n and m is given by

(Facebook density 2011:)

$$\rho = \frac{2 \times 6.87 \times 10^{10}}{7.21 \times 10^8 (7.21 \times 10^8 - 1)} \approx 2.64 \times 10^{-6}.$$

This density ρ is the same as the probability p that two randomly chosen nodes are connected.

Now, based on the definition for local clustering, a node i with degree k_i and clustering c_i has $t_i = c_i \binom{k_i}{2}$ triangles connected to it, where $\binom{k_i}{2}$ is the actual number of v-shapes that use i as their vertex node. In the context of Facebook, the study by Ugander and colleagues found that on average over the entire network, two friends of a given user are themselves friends about 14% of the time. This is equivalent to saying that 14% of the ***possible*** *triangles around a node are indeed realized or, using clustering, $c_i \approx 0.14$. This means that each node has, on average*

$$\langle t \rangle_{\textit{Facebook 2017}} \approx 0.14 \binom{191}{2} \approx 2540$$

triangles visiting it! This seems like a very large number of triangles for a person to be involved in. However, keep in mind the degree for a person

(191), which means that there are $\binom{191}{2} = 18,145$ *chances, in the form of v-shapes, for a triangle to be closed.*

On the other hand, in the fully random network, the probability that any of the $\binom{191}{2}$ *pairs of friends around a node (forming v-shapes) is part of a triangle is given by* ρ *because that is the probability that the third link needed to connect the two ends of the v-shape is present. Therefore, in the random network version of Facebook, the prediction for the number of triangles around a node is*

$$\langle t \rangle_{random\ network} = \rho \times \binom{191}{2} \approx 2.64 \times 10^{-6} \times \frac{191 \times 190}{2} \approx 0.048$$

In other words, a single node has, on average much less than 1 triangle connecting to it. Comparing the observed versus the predicted numbers of triangles, we find a massive difference

$$\frac{\langle t \rangle_{Facebook\ 2017}}{\langle t \rangle_{random\ network}} \approx \frac{2540}{0.048} \approx 20.9 \times 2540 \approx 53,000.$$

This is a remarkable result that says that the observed number of triangles in Facebook is more than 53,000 thousand times larger than the number of triangles expected in a random network (with no social or any other structure) of the same number of nodes n *and links* m *as Facebook! The social information contained in Facebook is* ***very important****. Thus, we* ***should*** *be surprise when we see the local clustering of Facebook.*

A gratifying but slightly inefficient result from the study of paths in networks tells us that t_i can be determined from the powers of the adjacency matrix. In particular, since we know that a triangle is a path of length 3, then for an undirected network,

$$t_i = \frac{\mathbf{A}^3_{i,i}}{2}, \tag{25}$$

that is, the diagonal elements of $\mathbf{A}^3$ are equal to twice the number of triangles visiting each of the nodes in G. The double count comes from the the fact that one can travel the same triangle in two directions in an unidrected network. With this result, we have a formula for c_i. However, this is hardly ever used in practice as it is computationally inefficient. The most common approach is to computationally count local triangles and plug that number into the definition for c_i.

8.2 Global clustering (transitivity)

When we move to global clustering, our definition needs to be revised with a bit of care.

First of all, triangles are no longer counted around a node, but rather they are looked at globally. This means, in practice, that *who* owns a triangle changes. To be clear, the triangles that we deal with in local and global clustering are *the same triangles* in the network. The difference is that in local clustering, we then mind which nodes of G are involved in each specific triangle, whereas in global clustering we count all the triangles against the entire network.

Second, the opportunities to close a triangle also come from v-shapes. Interestingly, since counting v-shapes in a unique way (so that we do not overcount) on a network has to be done node by node with the v-shape matching its vertex to the given node, then local and global countings of v-shapes are equivalent. This means that for an entire network G, the total number $\Theta(G)$ of v-shapes is given by

$$\#\text{ v-shapes for network } G = \Theta(G) = \sum_{i \in V(G)} \binom{k_i}{2}. \tag{26}$$

By the way, we can prove from this last expression that $\Theta(G)$ is also equal to the sum of all unique paths of length 2 (except those that start and end in the same node) in network G. Therefore, if we assume that the nodes are labelled $1, 2, \ldots, n$

$$\Theta(G) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{A}^2_{i,j}. \tag{27}$$

Now, to complete the definition for global clustering (or transitivity) C, we have to think about what it means for each v-shape to realize its potential to close a triangle. There are two perspective. Most naturally, imagine a given realized triangle. It is formed from 3 partially overlapping v-shapes. If we go back to link indicator products, we have that

$$a_{ij}a_{jh}a_{hi} = 1 \Leftrightarrow \begin{cases} a_{ij}a_{jh} = 1 & \text{and,} \\ a_{ij}a_{hi} = a_{ji}a_{hi} = 1 & \text{and,} \\ a_{jh}a_{hi} = 1. \end{cases} \tag{28}$$

Therefore, for each closed triangle, there are *three* v-shapes. This means that in order to have a ratio for C in which the numerator and denominator have the same “units”, we must define

$$C = \frac{3T(G)}{\Theta(G)}, \tag{29}$$

where $T(G)$ is the number of global triangles in G. This number can in fact be determined from the local triangles as

$$T(G) = \frac{\sum_i t_i}{3}. \tag{30}$$

That is, because of the fact that a global triangle is claimed by three nodes in G, then adding all local triangles together gives us 3 times $T(G)$.

As a final example, let us use these concepts to calculate C for the network in Fig. 2.

Example 3. *The network has 2 global triangles, one among 1,2,4 and the other among 2,3,4. In terms of v-shapes, the nodes with degree 2 have a single v-shape, whereas the nodes with degree 3 have* $\binom{3}{2} = 3$ *v-shapes attached to them. This means that*

$$T(G) = 2, \quad \Theta(G) = 2 \times 1 + 2 \times 3 = 8 \quad \Rightarrow \quad C = \frac{3 \times 2}{8} = \frac{3}{4}.$$

You are invited to try this out using the various methods indicated above. Particularly for a small network like this, you can make use of $\mathbf{A}$*. However, other computational methods are perfectly valid.*