***Problem 1*:**

Whether certain mice are black or brown depends on a pair of genes, each of which is either *B* or *b*. If both members of the pair are alike, the mouse is said to be homozygous, and if they are different it is said to be heterozygous. Homozygous *bb* mice are brown; heterozygous mice and homozygous *BB* mice are black. The offspring of a pair of mice have two such genes, one from each parent. If the parent is heterozygous, the inherited gene is equally likely to be *B* or *b*.

a. Suppose two heterozygous mice mate and the offspring is black. We will call this mouse Boff (for black offspring). What is the probability that Boff is homozygous *BB*?

b. Now suppose Boff is mated with a brown mouse, resulting in seven offspring, all of which are black. Apply Bayes theorem to find the posterior probability that Boff is homozygous *BB*, given that Boff has produced seven black offspring when mated with a brown mouse.

c. Show that the result to Part b is the same if you redo the previous calculation, processing the seven observations sequentially, using the posterior probability from each observation as the prior probability for the next observation.

***Solution*:**

Part a.

If two heterozygous mice mate, there are four possibilities for the offspring: BB, Bb, bB, and bb (where the first letter is the allele from the mother and the second is the allele from the father). Each of these has equal probability. Homozygous BB mice and heterozygous mice are black; homozygous bb mice are brown. Therefore, the probability that Boff is homozygous BB is ¼, and the probability that he is black is ¾.

After observing that Boff is black, we apply Bayes Rule:

$$P(BB|\text{black}) = \frac{P(\text{black}|BB)P(BB)}{P(\text{black})} = \frac{P(\text{black}|BB)P(BB)}{P(BB)+P(bB)+P(Bb)} = \frac{1\times\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Therefore, the probability that Boff is black is 1/3.

Part b.

There are two possibilities for Boff's genetic makeup given that he is black: homozygous (probability 1/3) or heterozygous (probability 2/3). If Boff is homozygous, then all Boff's offspring will be black, with probability 100%. If Boff is heterozygous and mated with a brown mouse (that is, a homozygous brown mouse), then the offspring will be heterozygous Bb with probability ½, and homozygous bb with probability ½. In the first case, the offspring will be black; in the second case, they will be brown.

Therefore, if Boff has 7 babies with the homozygous bb mouse, then the babies will all be black if Boff is homozygous BB, and the number of black offspring will have a Binomial(7,½) distribution if Boff is heterozygous. That is:

P(7 black offspring | BB) =1
P(7 black offspring | Bb) = $(½)^7$ = 1/128

Applying Bayes rule:

$$P(BB|7\text{ black offspring}) = \frac{P(7\text{ black offspring}|BB)P(BB|\text{black})}{P(7\text{ black offspring}|\text{BB})P(BB|\text{black}) + P(7\text{ black offspring}|Bb)P(Bb|\text{black})}$$
$$= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{128} \times \frac{2}{3}} = 0.9846$$

Therefore, the probability that Boff is BB given that he has 7 black offspring when mated with a brown mouse is 0.9846.

Part c.

We begin with a prior probability of 1/3 that Boff is BB. At each trial, we use the prior probability from the previous trial, and apply the evidence of an additional black baby:

$$P(\text{BB}\,|\,1\text{ black offspring}) = \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3}} = \frac{1}{2}$$
$$P(\text{BB}\,|\,2\text{ black offspring}) = \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} = \frac{2}{3}$$
$$P(\text{BB}\,|\,3\text{ black offspring}) = \frac{1 \times \frac{2}{3}}{1 \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3}} = \frac{4}{5}$$
$$P(\text{BB}\,|\,4\text{ black offspring}) = \frac{1 \times \frac{4}{5}}{1 \times \frac{4}{5} + \frac{1}{2} \times \frac{1}{5}} = \frac{8}{9}$$
$$P(\text{BB}\,|\,5\text{ black offspring}) = \frac{1 \times \frac{8}{9}}{1 \times \frac{8}{9} + \frac{1}{2} \times \frac{1}{9}} = \frac{16}{17}$$
$$P(\text{BB}\,|\,6\text{ black offspring}) = \frac{1 \times \frac{16}{17}}{1 \times \frac{16}{17} + \frac{1}{2} \times \frac{1}{17}} = \frac{32}{33}$$
$$P(\text{BB}\,|\,7\text{ black offspring}) = \frac{1 \times \frac{32}{33}}{1 \times \frac{32}{33} + \frac{1}{2} \times \frac{1}{33}} = \frac{64}{65} = 0.9846$$

Therefore, we get the same result of 0.9846 = 16/17 whether we process the observations one by one or all at once.

***Problem 2*:**

In an experiment, subjects were given the choice between two gambles:

Gamble 1:

A: $2500 with probability 0.33 B: $2400 with certainty

$2400 with probability 0.66
$0 with probability 0.01

Suppose that a person is an expected utility maximizer. Set the utility scale so that u($0) = 0 and u($2500) = 1. person is an expected utility maximizer. Set the utility scale so that u($0) = 0 and u($2500) = 1. Whether a utility maximizing person would choose Option A or Option B depends on the person's utility for $2400. For what values of u($2400) would a rational person choose Option A? For what values would a rational person choose Option B?

Gamble 2:

C:	$2500 with probability 0.33 $0 with probability 0.67	D:	$2400 with probability 0.34 $0 with probability 0.66

For what values of x would a person choose Option C? For what values would a person choose Option D? Explain why no expected utility maximizer would prefer B and C.

This problem is a version of the famous *Allais paradox*, named after the prominent critic of subjective expected utility theory who first presented it. Many people's choices violate subjective expected utility theory. For example, Kahneman and Tversky found that 82% of subjects preferred B over A, and 83% preferred C over D. For more information see http://en.wikipedia.org/wiki/Allais_paradox.

***Solution*:**

Define x = u($2400), the utility of $2400.

For A versus B:

Expected utility of A is 0.33* (1) + 0.66*(x) + 0.1*(0)
Expected utility of B is x

Setting them equal and solving for x tells us the value of x for which an expected utility maximizer would be indifferent between the two options

$0.33 * (1) + 0.66 * (x) + 0.1 * (0) = x$

x = 0.33/0.34

If x < 33/34. then an expected utility maximizer would choose option A. If x >33/34, option B would be chosen.

For C versus D:

Expected utility of C is 0.33* (1) + 0.67*(0)
Expected utility of D is 0.34*(x)

Setting them equal and solving for x tells us the value of x for which an expected utility maximizer would be indifferent between the two options

$0.33*(1) = 0.34*(x)$

$x = 0.33/0.34$

If $x < 33/34$, then an expected utility maximizer would choose option C. If $x > 33/34$, option D an expected utility maximizer would choose option D.

Why no utility maximizer would prefer B and C:

A utility maximizer would pick B if $x>33/34$, and would pick C if $x<33/34$. These regions do not overlap. By definition, an expected utility maximizer has a consistent utility value for a given payout regardless of the probability structure. Therefore, no utility maximizer would prefer B and C. A utility maximizer would be *indifferent* among all four of these gambles if $x = 33/34$. But no utility maximizer would strictly prefer B over A, and C over D.