CSI-674 Jericho McLeod
Assignment 8 G00986513

Problem 1:
The conditional normal distributions for Thetas are given by the mean μ and the standard deviation τ. As functions of the other parameters, these are:

$$\tau^* = (1/\tau^2 + n\rho)^{-1/2}$$

$$\mu^* = \frac{\mu/\tau^2 + \rho\sum_i x_i}{1/\tau^2 + n\rho}$$

While the conditional gamma distributions for Rhos are given by the shape and scale parameters α and β. As functions of the other parameters, these are:

$$\alpha^* = \alpha + n/2$$

$$\beta^* = (\beta^{-1} + \frac{1}{2}\sum_i (x_i - \theta)^2)^{-1}$$

Solving for these using the priors gives the following conditional distributions, where the first two parameters are for a normal distribution, and the following two for a gamma distribution:

	Surface	**Bottom**
$\boldsymbol{\mu}$	4.837	5.845
$\boldsymbol{\tau}$	0.198	0.314
$\boldsymbol{\alpha}$	9.5	9.5
$\boldsymbol{\beta}$	0.054	0.072

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Code used for Problem 1:

```
wolf.surface=c(3.74, 4.61, 4.00, 4.67, 4.87, 5.12, 4.52, 5.29, 5.74, 5.48)
wolf.bottom=c(5.44, 6.88, 5.37, 5.44, 5.03, 6.48, 3.89, 5.85, 6.85, 7.16)

s_n <- length(wolf.surface)
b_n <- length(wolf.bottom)

s_xbar <- mean(wolf.surface)
b_xbar <- mean(wolf.bottom)

#Theta Parameters
mu0 <- 6
tau0 <- 1.5

#Rho parameters
alpha0 <- 4.5
beta0 <- 0.19

# Standard deviations of input data
b_sigma0 <- sd(wolf.bottom)
s_sigma0 <- sd(wolf.surface)

# Describing the precision as the inverse of the variance
b_rho0 <- 1/b_sigma0^2
s_rho0 <- 1/s_sigma0^2
b_lambda0 <- 1/tau0^2
s_lambda0 <- 1/tau0^2

# Conditional Distribution for Theta:
s_mu1 <- ((mu0/tau0^2)+(sum(wolf.surface)/s_sigma0)) /
   ((1/tau0^2)+(length(wolf.surface)/s_sigma0))
b_mu1 <- ((mu0/tau0^2)+(sum(wolf.bottom)/b_sigma0)) /
   ((1/tau0^2)+(length(wolf.bottom)/b_sigma0))
s_tau1 <- (1/tau0^2 + length(wolf.surface)*s_rho0)^-.5
b_tau1 <- (1/tau0^2 + length(wolf.bottom)*b_rho0)^-.5

# Conditional Distribution for Rho
b_var_theta <- sum((b_n-b_mu1)^2)
s_var_theta <- sum((s_n-s_mu1)^2)
b_alpha1 <- alpha0 + b_n/2
s_alpha1 <- alpha0 + s_n/2
b_beta1 <- 1/(1/beta0 + b_var_theta/2)
s_beta1 <- 1/(1/beta0 + s_var_theta/2)
```

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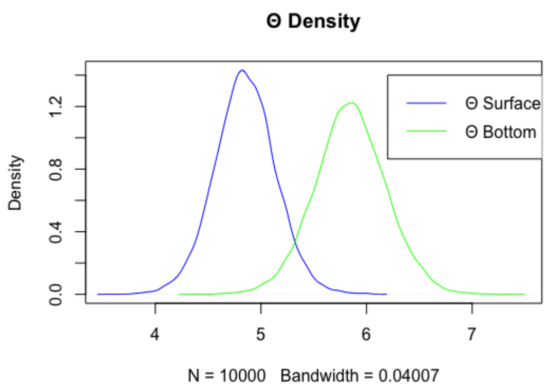
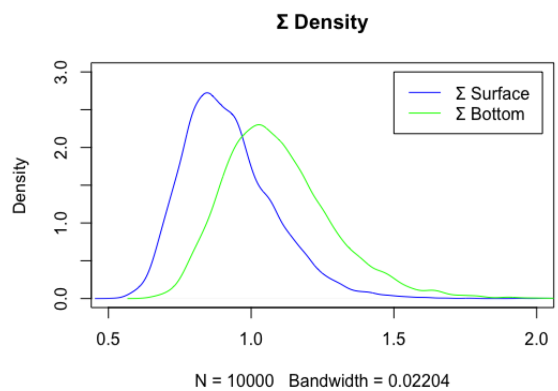
Problem 2:

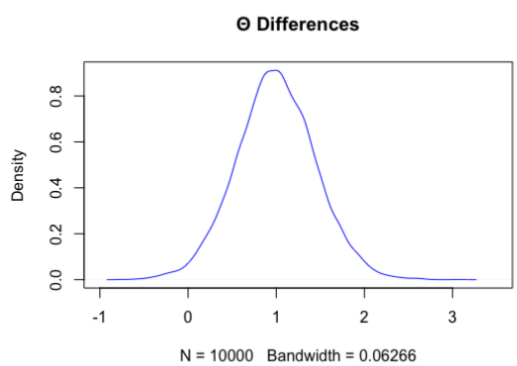
Drawing 10,000 Gibbs Samples from these distributions iteratively and taking the 90% credible interval provides the following quantile data:

	0.05	**0.95**
$\boldsymbol{\Theta_S}$	4.376	5.324
$\boldsymbol{\Theta_B}$	5.292	6.410
$\boldsymbol{\Sigma_S}$	0.699	1.212
$\boldsymbol{\Sigma_B}$	0.828	1.450
$\boldsymbol{\Theta_B - \Theta_S}$	0.255	1.738

Drawing density plots to spot check these values yields the following:







The quantiles look reasonable given these, and haven't diverged unexpectedly far from the initial priors. We see that the 90% credible intervals for Thetas have very little overlap, while those for Sigmas are significantly overlapped. This is reflected in the density of the differences; it is largely distributed around 1, with only a small portion less than 0, indicating that in most trials Theta Bottom > Theta Surface.

Code used for Problem 2:

```
numSim <- 10000

# Initial guess for precision
s_rGprev<-rgamma(1,shape=s_alpha1,scale=s_beta1)
b_rGprev<-rgamma(1,shape=b_alpha1,scale=b_beta1)

for (k in 1:numSim) {
  # reduces to xbar when tau = Inf
  s_muG = (mu0/tau0^2 + s_n*s_rGprev*s_xbar) /
    (1/tau0^2 + s_n*s_rGprev)
  b_muG = (mu0/tau0^2 + b_n*b_rGprev*b_xbar) /
    (1/tau0^2 + b_n*b_rGprev)

  # reduces to sigmaGibbs/sqrt(n) when tau = Inf
  s_tauG = 1/sqrt(1/tau0^2 + s_n*s_rGprev)
  b_tauG = 1/sqrt(1/tau0^2 + b_n*b_rGprev)

  # simulate new value for theta
  s_thetaGibbs[k] <- rnorm(1,mean=s_muG,sd=s_tauG)
  b_thetaGibbs[k] <- rnorm(1,mean=b_muG,sd=b_tauG)

  # update scale given current theta
  s_betaG<-1/(1/beta0 + 0.5*sum((wolf.surface-s_thetaGibbs[k])^2))
  b_betaG<-1/(1/beta0 + 0.5*sum((wolf.bottom-b_thetaGibbs[k])^2))

  # sample new value for rho
  s_rhoGibbs[k]<-rgamma(1,shape=s_alpha1,scale=s_betaG)
  b_rhoGibbs[k]<-rgamma(1,shape=b_alpha1,scale=b_betaG)

  # calculate new value of sigma
  s_sigmaGibbs[k]<-1/sqrt(s_rhoGibbs[k])
  b_sigmaGibbs[k]<-1/sqrt(b_rhoGibbs[k])

  # previous value of rho
  s_rGprev = s_rhoGibbs[k]
  b_rGprev = b_rhoGibbs[k]
}

# Calculate Quantiles
s_thetaGibbs_Q <- quantile(s_thetaGibbs,c(0.05,0.95))
b_thetaGibbs_Q <- quantile(b_thetaGibbs,c(0.05,0.95))
s_sigmaGibbs_Q <- quantile(1/sqrt(s_rhoGibbs),c(0.05,0.95))
b_sigmaGibbs_Q <- quantile(1/sqrt(b_rhoGibbs),c(0.05,0.95))
theta_diff <- b_thetaGibbs-s_thetaGibbs
theta_diff_Q <- quantile(theta_diff,c(0.05,0.95))

# print quantiles for report out
s_thetaGibbs_Q
b_thetaGibbs_Q
s_sigmaGibbs_Q
b_sigmaGibbs_Q
theta_diff_Q
```

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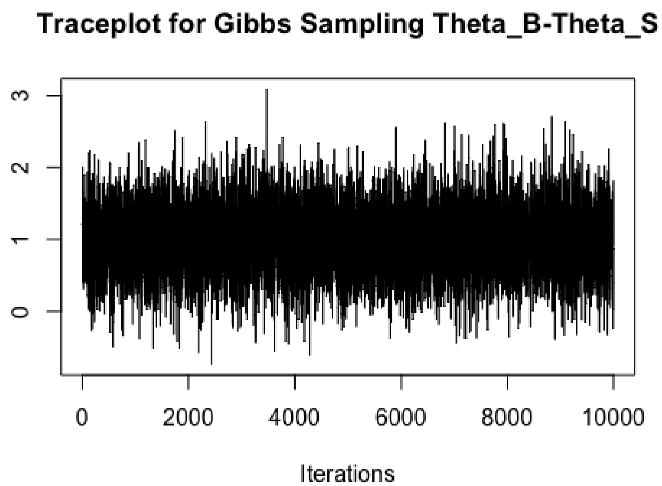
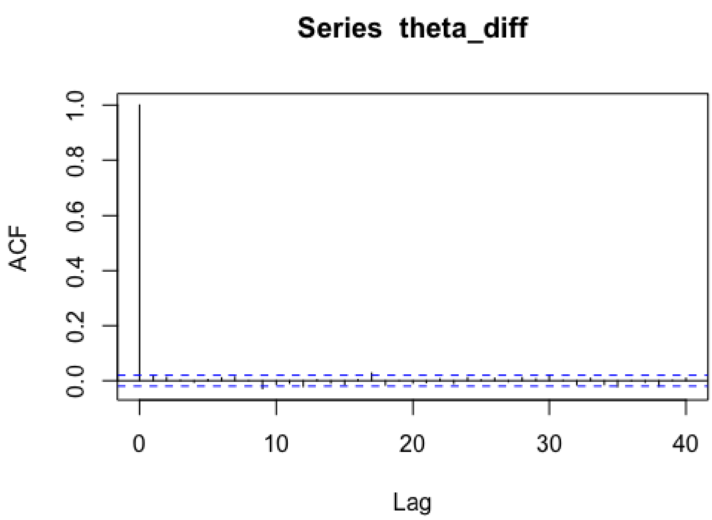
```
# Plots to verify
plot(density(s_thetaGibbs),col='blue',type='l',xlim=c(3.5,7.5),main='Θ
Density')
lines(density(b_thetaGibbs),col="green")
legend(6.2,1.4,c("Θ Surface ","Θ Bottom"),col=c("blue","green"),lty=c(1,1))

plot(density(1/sqrt(s_rhoGibbs)),col='blue',type='l',xlim=c(0.5,2),ylim=c(0,3
),main='Σ Density')
lines(density(1/sqrt(b_rhoGibbs)),col="green")
legend(1.5,3,c("Σ Surface ","Σ Bottom"),col=c("blue","green"),lty=c(1,1))

plot(density(theta_diff),col='blue',type='l',xlim=c(-1,3.5),main='Θ
Differences')
```

Problem 3:

Checking for the simulation getting stuck in a local minima or maxima yields the chart below. It does not indicate any issues..



Checking for autocorrelation in (Theta Bottom – Theta Surface) yields the chart below. In this case, there is very little autocorrelation among samples drawn, indicating chain convergence.



The effective sample size for the differences in Thetas is 9,684. This indicates that it is a robust sample size, but perhaps not so large that we are wasting computational efforts.

Code used for Problem 3:

```
library(coda)
traceplot(as.mcmc(theta_diff), main="Traceplot for Gibbs Sampling Theta_B-
Theta_S")
acf(theta_diff)
effectiveSize(theta_diff)
```

Problem 4:

Repeating from Problem 3, the lack of autocorrelation indicates the chains have converged in the Gibbs Sampling simulation. The traceplot indicates there were no issues being stuck in localized areas of the space, and the effective sample size remains good for the difference between thetas.

As previously show, we just obtained the following values:

	0.05	**0.95**
$\boldsymbol{\Theta_S}$	4.376	5.324
$\boldsymbol{\Theta_B}$	5.292	6.410
$\boldsymbol{\Sigma_S}$	0.699	1.212
$\boldsymbol{\Sigma_B}$	0.828	1.450
$\boldsymbol{\Theta_B - \Theta_S}$	0.255	1.738

From this, we can comfortably say that the mean bottom pollution of HCB in Wolf River is higher than the mean HCB pollution at the surface.

To compare with prior results from assignment six, let us remind ourselves of what we obtained previously:

	0.05	**0.95**
$\boldsymbol{\Theta_S}$	4.44	5.16
$\boldsymbol{\Theta_B}$	5.26	6.43
$\boldsymbol{P_S}$	0.930	4.67
$\boldsymbol{P_B}$	0.355	1.83

Converting the current data to Rho from Sigma yields:

	0.05 (6)	**0.05 (8)**	**0.95 (6)**	**0.95 (8)**
$\boldsymbol{\Theta_S}$	4.44	4.38	5.16	5.32
$\boldsymbol{\Theta_B}$	5.26	5.29	6.43	6.41
$\boldsymbol{P_S}$	0.930	0.681	4.67	2.048
$\boldsymbol{P_B}$	0.355	0.476	1.83	1.459

Converting in the opposite direction to match the previous data to the current output:

	0.05 (6)	**0.05 (8)**	**0.95 (6)**	**0.95 (8)**
$\boldsymbol{\Theta_S}$	4.44	4.376	5.16	5.324
$\boldsymbol{\Theta_B}$	5.26	5.292	6.43	6.410
$\boldsymbol{\Sigma_S}$	0.458	0.699	1.034	1.212
$\boldsymbol{\Sigma_B}$	0.734	0.828	1.678	1.450

The Gibbs Sampling MCMC has nearly the same means, while the standard deviations are a bit higher, or the precisions are a bit lower. It is interesting that the distribution for the Surface means are a bit higher with Gibbs Sampling, while the Bottom mean distribution is a bit narrower. The standard deviation for the Surface is shifted somewhat upwards compared to the prior study, but is a slightly narrower distribution overall as well. Comparatively, the bottom standard deviation is noticeably narrower.

These changes stem from the difference in sampling; the original data was pulled directly from a posterior distribution, while the current version is sampled from a conditional distribution that is not constant. The joint distribution from the original study had no mechanism for influence given changes in mean or standard deviation, while the Gibbs Sampling method is precisely this; draws from the gamma distribution for precision influence the draws from the normal distribution for means, and vice versa, across iterations.