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The Radiation Reaction Revisited

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THE RADIATION REACTION REVISITED

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A Brief Exploration of the Self-Force in Classical Electrodynamics

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PREFACE

Accelerating point charges emit radiation, as proclaimed in standard textbooks on classical electrodynamics. Parts of the electromagnetic fields surrounding a charged particle can disconnect themselves and irreversibly drift off to infinity, when acceleration sets in. Yet as this happens, the particle loses energy and momentum. Therefore, the emission of radiation must affect the particle's motion, similar to the way a boat can be influenced by the waves it caused in its wake. In electrodynamics, the recoil force experienced by an emitting charge is called the radiation reaction force.

The question rises why this effect is so often unaccounted for in daily usage of electromagnetic theory, which, after all, successfully manages to explain so many phenomena convincingly. The reason is twofold. First of all, the radiation reaction only gives rise to a small correction to the motion of a particle. Timescales where it would be paramount are, as of the moment, experimentally inaccessible. Secondly, the classical treatment of the problem exposes complications of a much more fundamental nature.

Undesirable theoretical features emerge upon including the radiation reaction force in the equation describing the motion of charged point particles. Either they would be able to accelerate infinitely in the absence of an external force, or they would react to such a force before it kicks in, violating causality. Surprisingly, the classical equations seem to allow tunnelling as well, even in the familiar context of a potential barrier.

It is pivotal in this matter that point charges are intrinsically quantum mechanical objects, and probing their nature pushes the boundaries of classical theory. A parallel to Russell's paradox comes to mind, which revealed inconsistencies in the set-theoretical foundations of mathematics. Indeed, also classical electromagnetism ultimately breaks down at its building blocks. Illuminating how this comes about is the central ambition of this report.

The first part of the report is dedicated to the classical attempt at describing the radiation reaction and how it influences the dynamics of a charged particle. Chapter 1 provides details concerning the modified equation of motion for particles at low speeds, as well as its immediate implications. Chapter 2 sheds light on the physical origin of the self-force affecting a charged particle. A natural transition is made to its relativistic generalisation and the related discussion regarding electromagnetic inertia. The treatment is inspired by Refs. [4, 6, 7].

In the second part, two examples are elaborated in order to illustrate how tunnelling solutions and indeterminacy can follow from a classical equation of motion (Chapter 3). The setting of a rectangular potential barrier is investigated in Section 3.1. Section 3.2 presents an overview of the numerical work done for the case of a continuous potential ramp. The discussion is based on suggestions in Ref. [2], but the calculations were performed independently and enable a cross-check of the results.

A few closing remarks follow at the end, as well as an Appendix (Part A) containing some of the longer calculations, and a list of selected references.

I wish to express my gratitude to Prof. Dr. Ben Craps and Joris Vanhoof, for their advice and helpful suggestions.

CONVENTIONS

Despite Gaussian units being widely used in books on electrodynamics, all calculations and results in this report are stated in SI units. Vectors are denoted by bold-faced lowercase letters (e. g. $\boldsymbol{r}$). Their length is written as an unbolded letter (e. g. $r = |\boldsymbol{r}|$). Unit vectors, such as $\hat{\boldsymbol{x}}$ in the x direction, are indicated with a hat. Using the convention in Ref. [4], $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$ denotes the vector connecting a source point $\boldsymbol{r}'$ to a field point $\boldsymbol{r}$. The unit vector in that direction is written as $\hat{\boldsymbol{\mathscr{r}}} = (\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|$.

Where special relativity enters the picture, the spacelike convention $(- + ++)$ is used. Proper time is referred to using τ, as opposed to coordinate time t. Dots as in $\dot{\boldsymbol{r}} = \mathrm{d}\boldsymbol{r}/\mathrm{d}\tau$ refer to proper-time derivatives.

In the description of relativistic dynamics, $x^\mu = (ct, \boldsymbol{r})$ is the position 4-vector of an event happening at spatial location $\boldsymbol{r}$ at time t in some reference frame. The ordinary velocity of an object is denoted by $\boldsymbol{v}$, whereas its proper velocity is written as $\boldsymbol{\eta} = \gamma \boldsymbol{v}$, where $\gamma = \gamma_v = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor using the object's speed. The 4-velocity is given by $\eta^\mu = (\gamma c, \boldsymbol{\eta})$. Defining $\boldsymbol{p} = m\boldsymbol{\eta}$ as the relativistic momentum of a particle with rest mass m leads to the energy-momentum 4-vector $p^\mu = m\eta^\mu = (E/c, \boldsymbol{p})$, where $E = \gamma mc^2$ is the relativistic energy. The 4-acceleration is denoted by $\alpha^\mu = \dot{\eta}^\mu$, and $K^\mu = \dot{p}^\mu$ refers to the 4-force, also called the Minkowski force.

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Part I

CLASSICAL DYNAMICS OF A CHARGED PARTICLE

1 CLASSICAL RADIATION REACTION

1.1 PRELIMINARY CONSIDERATIONS

Classical electrodynamics focuses on formulating a consistent description of electromagnetic interactions. Electric and magnetic *fields* are introduced to mediate forces between charges, and accordingly, determining these vector fields, generated by specified charge and current distributions, becomes a central objective. Maxwell's equations relate the fields and the sources, but to find generic, direct expressions for the fields, it is convenient to rewrite these equations in terms of a scalar and a vector *potential* describing the fields.

1.1.1 *Fields of a Point Charge in Motion*

For an electric point charge in motion, the Liénard-Wiechert potentials reflect the fact that electromagnetic signals travel at the finite speed of light. The current position $\boldsymbol{w}(t)$ of a moving point charge does not matter when determining its influence on a stationary charge at a field point $\boldsymbol{r}$. Instead, the signal arriving at the field point at some time t must have left the moving charge at an earlier time. This *retarded time* t_r is implicitly defined by the distance the signal has to travel in the time interval $t - t_r$: $|\boldsymbol{r} - \boldsymbol{w}(t_r)| = c(t - t_r)$.

Subsequently, from the Liénard-Wiechert potentials, one can calculate the fields of the moving point charge at an arbitrary field point $\boldsymbol{r}$ at any time t (Ref. [4]):

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^3} \left[(c^2 - v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{a}) \right] \tag{1}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{2}$$

In Eqs. 1 and 2, $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{w}(t_r)$ represents the vector which connects the particle's retarded position to the field point, the particle's velocity $\boldsymbol{v} = \dot{\boldsymbol{w}}(t_r)$ and acceleration $\boldsymbol{a} = \ddot{\boldsymbol{w}}(t_r)$ are evaluated at the retarded time, and the notation $\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$ is employed. The first term in Eq. 1 is referred to as the *velocity field*; the second one is called the *acceleration field*.

1.1.2 *Radiation from an Accelerated Point Charge*

To comply with conservation laws, one is compelled to acknowledge the electromagnetic fields as dynamical entities carrying energy, momentum and angular momentum. Parts of the field of an accelerating point charge can detach themselves and propagate to infinity in vacuum, irreversibly carrying off energy and (linear and angular) momentum. This is the defining characteristic of electromagnetic *radiation* from a localized source.

LARMOR FORMULA To see how radiation from a point charge comes about, it suffices to calculate the total power leaving a sphere surrounding the point charge and afterwards, investigate the limit of an infinite radius. The Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$ expresses the rate per unit area at which the electromagnetic fields transport energy across a closed surface surrounding the charge. Hence, at time t, integration of the Poynting vector over the surface of a sphere of radius $\mathscr{r}$ centered at the particle's retarded position $\boldsymbol{w}(t_r)$ gives the total power the point charge radiated at the unique retarded time for all points on the sphere. Since the area of the sphere goes like $\mathscr{r}^2$, it turns out that only the acceleration field in Eq. 1 contributes to the surface integral in the limit $\mathscr{r} \to \infty$, because it falls off as $1/\mathscr{r}$ and $|\boldsymbol{S}| \propto E^2$. For a point charge instantaneously at rest at the retarded time, the total radiated power P is then given by the *Larmor formula* (Ref. [4]):

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{3}$$

The general equation, accurate for a non-zero velocity $\boldsymbol{v}$ at the retarded time, can be deduced using an argument from special relativity (§ 2.2.2). Like Eq. 3, the generic formula shows that it takes an accelerating charged particle to generate electromagnetic waves.

1.1.3 *The Radiation Reaction*

As remarked in Jackson [5], problems in classical electrodynamics tend to belong to one of two categories: either one calculates the fields resulting from given charge and current distributions, or the fields are specified and one determines their impact on the motion of charged particles. To calculate the energy radiated by a point charge moving in an external field, one would typically first calculate its trajectory, neglecting possible emission of radiation when it is accelerated. Subsequently, one usually determines the radiated energy by integrating the radiated power as a function of time, effectively using the obtained particle trajectory $\boldsymbol{w}(t)$ as a given to substitute the particle's velocity and acceleration in the formula for the radiated power.

However, inaccuracies ensue from this stepwise treatment of problems combining both categories: the emission of radiation, disregarded in the first step, necessarily influences the motion of a charged particle. Indeed, when subjected to a specified force, a charged particle will accelerate less than its electrically neutral counterpart of the same mass, because it loses kinetic energy and momentum to the emitted radiation. Hence, there is some recoil force, due to the emission of radiation. It is called the *radiation reaction force*, which acts back on the charged particle and modifies its acceleration.

1.2 THE ABRAHAM-LORENTZ FORMULA

1.2.1 *Derivation from Energy Conservation*

The principle of energy conservation provides some insight into the functional form of the radiation reaction force $\boldsymbol{F}_{\text{rad}}$. To good approximation, the Larmor formula gives the radiated power P of a non-relativistic, accelerating particle. If energy is to be conserved, one would expect this quantity P to be equal to $-\boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{v}$, the energy loss rate of the particle when subjected to the radiation reaction force. However, this argument is incorrect, because the Larmor formula only considers energy which has irreversibly travelled off along acceleration fields propagating to infinity. Yet energy is stored in the velocity fields as well, and these exchange energy with the particle during its course of motion, as it speeds up or slows down. The total energy a charge loses is thus equal to the sum of the energy it transfers to its surrounding electromagnetic fields and the energy it loses to radiation. It is this total energy loss rate that is linked to the work rate of the radiation reaction force. Therefore, calling the radiation reaction a *field reaction* would be more accurate; essentially, it amounts to a back-reaction of the particle's retarded fields, upon energy and momentum transfer.

In the case of periodic motion, one can consider a time interval $[t_1, t_2]$ with identical system states at t_1 and t_2. Then the net energy lost by the particle is only due to emitted radiation, since the energy stored in the fields surrounding the particle returns to its initial value. Hence, the equality described in the previous paragraph holds on the time average:

$$\int_{t_1}^{t_2} \boldsymbol{F}_{\mathrm{rad}} \cdot \boldsymbol{v}\, \mathrm{d}t = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2\, \mathrm{d}t = -\frac{\mu_0 q^2}{6\pi c} \left[\left(\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}^2 \boldsymbol{v}}{\mathrm{d}t^2} \cdot \boldsymbol{v}\, \mathrm{d}t \right] \tag{4}$$

The boundary term vanishes, leaving $\int_{t_1}^{t_2} \left(\boldsymbol{F}_{\mathrm{rad}} - \frac{\mu_0 q^2}{6\pi c} \dot{\boldsymbol{a}} \right) \cdot \boldsymbol{v}\, \mathrm{d}t = 0$. The *Abraham-Lorentz formula* gives the simplest sufficient solution for $\boldsymbol{F}_{\mathrm{rad}}$:

$$\boldsymbol{F}_{\mathrm{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\boldsymbol{a}} = m\tau_0 \dot{\boldsymbol{a}}, \text{ where } m \text{ is the particle mass and } \tau_0 = \frac{\mu_0 q^2}{6\pi m c}. \tag{5}$$

Mathematically, of course, this derivation does not prove Eq. 5 convincingly. Moreover, it only provides some information about a special time average of the $\boldsymbol{F}_{\mathrm{rad}}$ component parallel to the particle's velocity. On the other hand, this expression for $\boldsymbol{F}_{\mathrm{rad}}$ has the virtue of consistency with energy conservation, by construction. Luckily then, better arguments exist (Sec. 2.1), confirming the Abraham-Lorentz formula in its range of validity. Indeed, because the Larmor formula was used when exploiting the principle of energy conservation, Eq. 5 is an approximate result, only reliable for small velocities.

1.2.2 *Runaway Solutions*

The involvement of $\dot{\boldsymbol{a}}$, the *jerk*, in the Abraham-Lorentz formula has embarrassing consequences. Applying Newton's second law to a particle of mass m under the influence of no external forces F_{ext} yields

$$ma = F_{\mathrm{rad}} + F_{\mathrm{ext}} = m\tau_0 \dot{a} + 0 \implies a(t) = a_0 \mathrm{e}^{t/\tau_0}. \tag{6}$$

Unless one takes the integration constant a_0 to be zero, one obtains a *runaway solution* with an acceleration spontaneously rising exponentially in the absence of external forces. This behaviour would be in grave contrast to experimental observations. Excluding runaways by setting $a_0 = 0$ seems to be the natural resolution of the problem, since a particle undergoing no external influences is not supposed to accelerate anyway, and hence, not to radiate nor experience any radiation reaction force. However, systematic runaway exclusion leads to *preacceleration* (§ 1.2.3): when an external force is about to be applied, the particle would start to accelerate a short while before the force starts to act, violating causality. It is worthwhile noting that these problems do not occur when considering the classical equation of motion for a spherical-shell particle, if its radius[1] is larger than $c\tau_0$ (Ref. [8]). Yet for point particles, the classical theory seems to break down.

1.2.3 *Acausal Preacceleration*

The radiation reaction implies a modification to Newton's second law for a charged particle subjected to an external force F_{ext}. The Abraham-Lorentz formula yields the *modified equation of one-dimensional motion*:

$$ma = F_{\mathrm{rad}} + F_{\mathrm{ext}} = m\tau_0 \dot{a} + F_{\mathrm{ext}} \implies a(t) = \tau_0 \dot{a}(t) + \frac{F_{\mathrm{ext}}(t)}{m}. \tag{7}$$

If the applied force is discontinuous, an uncharged particle would experience a discontinuous acceleration, since $a = F_{\mathrm{ext}}/m$. However, for a realistic[2] external force, the radiation reaction damps out abrupt acceleration changes, leaving a charged particle with a continuous acceleration. The proof follows upon integrating the equation of motion, in order to calculate a possible discontinuous jump

1 Incidentally, $c\tau_0 = (2/3)R_{\mathrm{class}}$, where R_{class} is the so-called *classical radius*, discussed in § 2.1.3.
2 A δ force still leads to a discontinuous acceleration $a(t)$. (§ 3.1.2)

$\Delta a = \lim_{\epsilon\to 0}(a(t_0+\epsilon) - a(t_0-\epsilon))$ at an arbitrary time t_0. It is based on the fact that the position $x(t)$ and velocity $v(t)$ of a particle must always be continuous on physical grounds: the particle is not able to teleport nor change its velocity without having some time to accelerate.

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{t_0-\epsilon}^{t_0+\epsilon} a(t)\mathrm{d}t &= \lim_{\epsilon\to 0}(v(t_0+\epsilon) - v(t_0-\epsilon)) = \Delta v = 0\\ &= \lim_{\epsilon\to 0}\int_{t_0-\epsilon}^{t_0+\epsilon}\left(\tau_0\dot{a}(t) + \frac{F_{\mathrm{ext}}(t)}{m}\right)\mathrm{d}t \qquad (8)\\ &= \tau_0\Delta a + 0 \implies \Delta a = 0 : a(t) \text{ is continuous in } t_0.\end{aligned}$$

Unless $F_{\mathrm{ext}}(t)$ is a delta function, $\int_{t_0-\epsilon}^{t_0+\epsilon} F_{\mathrm{ext}}(t)\mathrm{d}t$ tends to zero if $\epsilon \to 0$.

RECTANGULAR FORCE To investigate how a charged particle is influenced by a constant external force $F > 0$ lasting from $t = 0$ to $t = T$, one sets $F_{\mathrm{ext}}(t) = F$ for $0 < t < T$ and $F_{\mathrm{ext}}(t) = 0$ otherwise. Solutions of Eq. 7 are obtained separately for three time intervals.

In Period 1 ($t < 0$), $a_1 = \tau_0\dot{a}_1$ yields the particle acceleration $a_1(t) = A_1 e^{t/\tau_0}$; in Period 2 ($0 < t < T$), $a_2(t) = A_2 e^{t/\tau_0} + F/m$ follows from $a_2 = \tau_0\dot{a}_2 + F/m$; Period 3 ($t > T$) is analogous to Period 1, whence $a_3(t) = A_3 e^{t/\tau_0}$. The real constants A_1, A_2 and A_3 are determined by imposing the acceleration continuity as a matching condition at times $t = 0$ and $t = T$. The requirement that $a_1(0) = a_2(0)$ implies that $A_1 = B_1 + F/m$. Also, $a_2(T) = a_3(T)$ entails $A_3 = A_2 + (F/m)e^{-T/\tau_0}$.

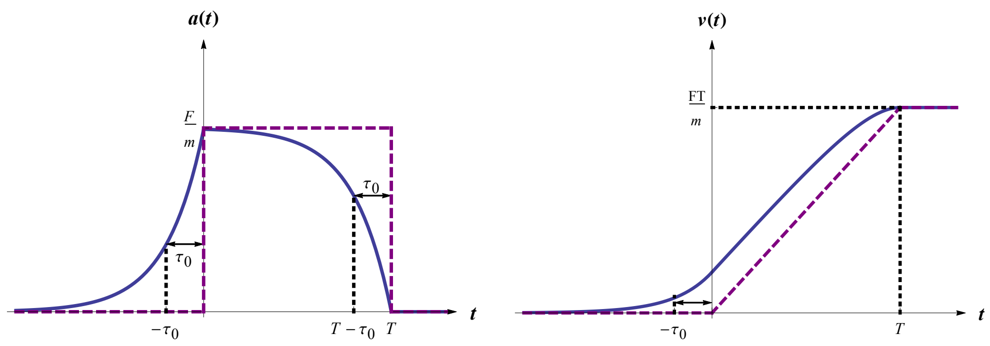

Figure 1: Particle acceleration (left) and velocity (right) due to a rectangular force. Dashed lines show the non-radiative solution.

A neutral particle, obeying Newton's second law, would not accelerate in the force-free Periods 1 and 3, and only speed up in Period 2, with an acceleration F/m. However, for a charged particle, the elimination of the preacceleration in Period 1 is only possible if $A_1 = 0$, meaning that $A_2 = -F/m$. Yet excluding the runaway solution in region 3 necessitates $A_3 = 0$, equivalent to $A_2 = -(F/m)e^{-T/\tau_0}$. Unless $T = 0$, which removes the external force, the requirements cannot be met simultaneously.

Since runaway solutions violate experimental evidence, one can choose to eliminate them, thus fixing the integration constants. The resulting particle velocity is obtained by integrating the acceleration and invoking continuity at $t = 0$ and $t = T$. Setting the particle's initial velocity $\lim_{t\to-\infty} v_1(t) = 0$ fully determines the integration constants. The final result is given in Sect. A.1. It is straightforward to show that the solutions are compatible with energy conservation (§ A.2.1).

The preacceleration resulting from runaway elimination jeopardizes physics' treasured tenet of causality, but only on a timescale of the order of τ_0 as illustrated in Fig. 1. For an electron, $\tau_0 \simeq 6.2664 \times 10^{-24}$ s, indicating an experimentally inaccessible timescale for the acausality to take place. Notwithstanding the lack of observational refutation, preacceleration remains an ostensibly unavoidable feature that has been called "embarrassing", "undesirable" and "philosophically repugnant" (Refs. [12, 2, 4] respectively).

DELTA FORCE Preacceleration persists when considering how a delta force like $F_{\text{ext}}(t) = k\delta(t - t_0)$ affects a charged particle, where k is a non-zero, real constant. For simplicity, one can take $t_0 = 0$ as the moment of the impulse. The acceleration of an uncharged particle subjected to that force would be given by $a(t) = (k/m)\delta(t)$. If it is at rest initially, $\lim_{t\to-\infty} v(t) = 0$ leads to $v(t) = (k/m)\int_{-\infty}^{t}\delta(t')\mathrm{d}t' = 0$ for a time $t < 0$ and $v(t) = k/m$ for $t > 0$.

For a charged particle, the solution method is the same as for a rectangular force, except that the acceleration is not continuous at $t = 0$: when substituting $F_{\text{ext}}(t)$ in Eq. 8, the integral of the second term yields k/m, whence $\Delta a = -k/m\tau_0$. This discontinuous acceleration boost between Period 1 ($t < 0$) and Period 2 ($t > 0$) serves as a matching condition. It relates the A_i coefficients in $a_i(t) = A_i \mathrm{e}^{t/\tau_0}$, where $i \in \{1, 2\}$ specifies the time interval of validity: $\Delta a = A_2 - A_1$. Two contradictory requirements arise: $A_1 = k/m\tau_0$ to exclude runaways and $A_1 = 0$ to eliminate preacceleration. Choosing to fulfil the former fixes the constants. If the particle is at rest initially, its velocity $v_1(t)$ for $t < 0$ is obtained by integrating $a_1(t)$ from $-\infty$ to t, yielding $v_1(t) = (k/m)\mathrm{e}^{t/\tau_0}$. Since $a_2(t) = 0$, the particle velocity in Period 2 is a constant: $v_2(t) = v_2(0) = k/m$. The results, consistent with energy conservation (§ A.2.2), are shown in Fig. 2.

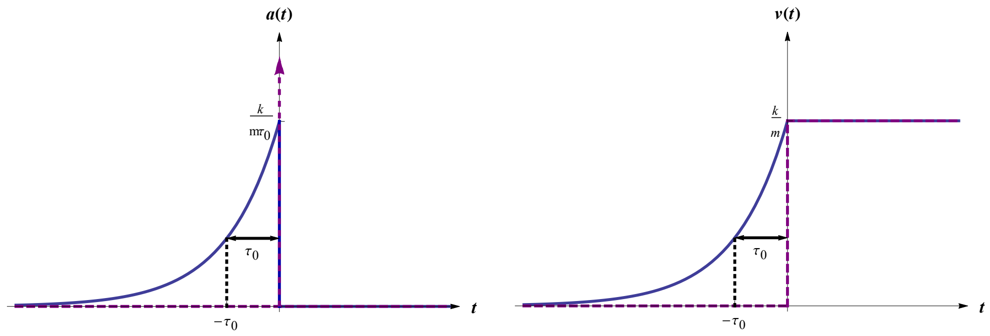

Figure 2: Particle acceleration (left) and velocity (right) due to a delta force. Dashed lines pertain to the non-radiative solution.

Incidentally, the preceding examples reveal why it is often permissible to omit the radiation reaction in calculations, as stated in § 1.1.3. The resultant error, related to neglecting the $m\tau_0\dot{a}$ term in Eq. 7, is only crucial in phenomena involving timescales of the order of τ_0, and hence usually imperceptible.

1.3 INTEGRAL FORMULATION OF THE EQUATION OF MOTION

Using Green's functions, it is possible to find an alternative formulation for the modified equation of motion (Eq. 7), which takes the form of an integral equation:

$$a(t) = \frac{1}{m\tau_0}\int_{t}^{+\infty} \mathrm{e}^{(t-t')/\tau_0} F_{\text{ext}}(t')\,\mathrm{d}t'. \tag{9}$$

This version of the modified equation of motion exhibits preacceleration even more conspicuously than Eq. 7. At some time t, the acceleration depends on the external force at all later times $t' > t$. The exponential weighting factor ensures that the sensitivity of $a(t)$ to future values of F_{ext} decreases rapidly for $t' > t + \tau_0$. As mentioned in § 1.2.3, preacceleration would only be appreciable on a timescale set by τ_0. Furthermore, the integral formulation seems to rid the equation of motion of runaways. If the external force is zero after some time T, it immediately follows from the integral equation that $a(t) = 0$ for $t > T$, since $F_{\text{ext}}(t') = 0$ for $t' \geq t > T$. The automatic runaway exclusion is due to the choice of the Green's function, which is not uniquely determined. Further elaboration on this point can be found in Sect. A.3, together with the calculations leading to Eq. 9.

2 ELECTROMAGNETIC SELF-FORCE AND RELATIVISTIC GENERALISATION

2.1 PHYSICAL ORIGIN OF THE RADIATION REACTION

2.1.1 *Electromagnetic Self-Force*

The derivation of the Abraham-Lorentz formula, based on energy conservation, provides little insight into the physical mechanism at hand. In § 1.1.3, the radiation reaction force was linked to the recoil effect due to the particle's fields acting back on it. Yet since these fields (Eqs. 1 and 2) become infinite at the location of the charge, one has to resort to a workaround to calculate the force they exert on the particle that generates them.

A possible way, suggested in Refs. [4, 6, 7], is to investigate the finite forces that infinitesimal lumps of an extended charge distribution exert on one another, and to calculate the limit as its spatial extent becomes zero. In general, the electromagnetic forces of two parts of the distribution on each other do not cancel pairwise. Adding up all the internal imbalances, one has to conclude that a charged particle exerts a net force on itself. This *electromagnetic self-force*, arising from the breakdown of Newton's third law in the context of electrodynamics, is responsible for the radiation reaction.

Considering a spherical charge distribution seems the most natural choice, but it complicates the calculations unnecessarily. The simplest extended distribution of charge, a dumbbell model (Fig. 3), suffices to illuminate the basic mechanism. As shown in the figure, the particle's charge q is split into two halves with a fixed distance d between them. At this point, the objective is to calculate the net self-force on the dumbbell. Later on, taking the limit $d \to 0$ will remove the model-dependent terms in that expression, leaving a result which is prescribed by conservation of energy alone.

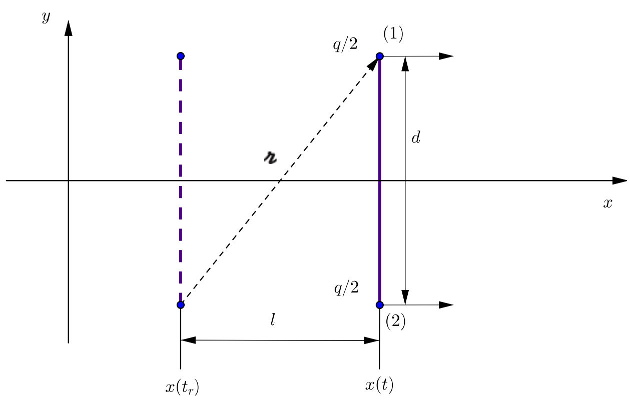

Figure 3: Dumbbell model for the radiation reaction.

In the calculations, transverse motion of the dumbbell was considered, along the x axis, with an arbitrary velocity $\boldsymbol{v}(t) = v(t)\hat{\boldsymbol{x}}$ and acceleration $\boldsymbol{a}(t) = a(t)\hat{\boldsymbol{x}}$. As shown in Fig. 3, $x(t_r)$ refers to the retarded position of the dumbbell, whereas $x(t)$ is its present position. The electric field at charge (1) due to charge (2) (and vice versa) is calculated using Eq. 1. The y components of the Lorentz forces due to the electrical fields at (1) and (2) cancel out. Similarly, the forces due to the magnetic fields at (1) and at (2) are equal in magnitude, but opposite. The end result for the net self-force on the dumbbell can be expressed as a power series in d:

$$\boldsymbol{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\gamma^3 a}{4c^2 d} + \frac{va^2\gamma^6}{c^5} + \frac{\gamma^4 \dot{a}}{3c^3} + (\dots)d + \dots\right]\hat{\boldsymbol{x}}, \tag{10}$$

where v, a, $\dot{a}$ and γ are evaluated at the present time t. In the point-charge limit, d goes to zero and all positive powers of d vanish. The derivation of Eq. 10 is relegated to Sect. A.4.

In the low-speed limit ($v \ll c$), the second term in Eq. 10 drops out and $\gamma \simeq 1$. This yields the non-relativistic approximation of the self-force, which is easier to derive: in Ref. [4], the dumbbell velocity at the retarded time is immediately set to zero, simplifying the equations. However, Eq. 10 exhibits more general validity: it does not require the particle to be instantaneously at rest at the retarded time. It should be noted that a transverse dumbbell does not experience length contraction[1], so the considered model is valid at relativistic velocities.

The self-force clearly illustrates how the action-reaction principle cannot be upheld within a particle's structure. Yet it is important to remark that Eq. 10 only yields a non-zero self-force in the case of an accelerating particle. In Ref. [6], it is noted that this fact ensures the preservation of Newton's first law in electrodynamics: no external force is needed for an object to maintain a constant velocity. This becomes clear when looking at Newton's second law for an object with mechanical mass m_0: $m_0 a = F_{\text{tot}} = F_{\text{self}} + F_{\text{ext}}$. If the self-force would be non-zero in the case of uniform motion ($a = 0$), an external force would be required to cancel it and ensure that the object does not accelerate.

In another comment, Lyle [6] points out that Newton's third law is preserved when the first law applies (i. e. in the absence of external forces, for uniform motion), but breaks down exactly when the first law is no longer valid (i. e. in the presence of external forces, for accelerated motion). Actually, the underlying cause is the fact that the electric field of a point charge moving with constant velocity points radially outwards from the present position of the charge, not from the retarded location where the electromagnetic 'message' left the charge. This special result can be obtained from Eq. 1 (Ref. [4]). It entails that the fields of a point charge in uniform motion seem to compensate for retardation effects, and owing to the mirror symmetry, the self-force contributions of the dumbbell ends cancel out perfectly.

2.1.2 *Radiation Reaction*

RELATIVISTIC DYNAMICS In the theory of special relativity, Newton's second law $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ for the net force $\boldsymbol{F}$ acting on an object with rest mass m is upheld, when using the relativistic momentum $\boldsymbol{p} = m\boldsymbol{\eta}$, where $\boldsymbol{\eta} = \gamma\boldsymbol{v}$ denotes the proper velocity, $\boldsymbol{v}$ the coordinate velocity, t the coordinate time and $\gamma = (1 - v^2/c^2)^{-1/2}$. In the case of rectilinear motion, this implies

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m v) = \gamma^3 ma\frac{v^2}{c^2} + \gamma ma = \gamma^3 ma, \tag{11}$$

$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$

where $a = dv/dt$ is the coordinate acceleration.

If the separation d goes to zero, positive powers of d disappear from the equation for the self-force acting on the dumbbell (Eq. 10). However, the first term seems to diverge in the point-charge limit, as it is proportional to $1/d$, so the self-force of an accelerated point charge is infinite. In the interest of resolving this problem, it is crucial to note that the divergent term is proportional to the acceleration a. Hence, it can be combined with the $\gamma^3 m_0 a$ term on the other side of the relativistic

1 There is no Lorentz contraction perpendicular to the direction of motion.

version of Newton's second law (Eq. 11), where $m_0/2$ is the mass of one dumbbell end.

$$\gamma^3 m_0 a = F_{\text{tot}} = F_{\text{self}} + F_{\text{ext}} \Rightarrow \left(\gamma^3 m_0 + \frac{q^2}{4\pi\epsilon_0}\frac{\gamma^3}{4c^2 d}\right) a = \frac{q^2}{4\pi\epsilon_0}\left[\frac{va^2\gamma^6}{c^5} + \frac{\gamma^4 \dot{a}}{3c^3}\right] + F_{\text{ext}} \quad (12)$$

The term with the square brackets is readily identified as the radiation reaction force: for small velocities, the first term inside is negligible, leaving half the Abraham-Lorentz force (Eq. 5).

$$F_{\text{rad}}^{\text{int}} = \frac{q^2}{4\pi\epsilon_0}\left[\frac{va^2\gamma^6}{c^5} + \frac{\gamma^4\dot{a}}{3c^3}\right] \underset{v \ll c}{\longrightarrow} \frac{q^2}{12\pi\epsilon_0 c^3}\dot{a} = \frac{\mu_0 q^2}{12\pi c}\dot{a} = \frac{1}{2}F_{\text{rad}} \quad (13)$$

The superscript in $F_{\text{rad}}^{\text{int}}$ is meant to indicate that the calculated self-force only accounts for the interaction between both ends of the dumbbell. To retrieve the Abraham-Lorentz formula, one also has to consider the force each end exerts on itself. A nice way to achieve this is to spread the total charge q evenly along a strip of length L, in the y direction (perpendicular to the direction of motion), and subsequently integrate over all pairs of infinitesimal segments ($\mathrm{d}y_1$ and $\mathrm{d}y_2$). Hence, the linear charge density is given by $\lambda = q/L$. Since each of the pairs can be considered as a dumbbell, Eq. 13 is applicable, after making the substitution $q/2 \to \lambda\, \mathrm{d}y$, which is the charge at one dumbbell end. Adding a factor $1/2$ because each interaction pair is intentionally[2] counted twice, one obtains the cumulative[3] interaction force in Eq. 14.

$$F_{\text{rad}} = \frac{1}{2}\int_0^L \frac{1}{4\pi\epsilon_0}\frac{\gamma^4}{c^3}\left(\frac{va^2\gamma^2}{c^2} + \frac{\dot{a}}{3}\right)\cdot 2\lambda \left[\int_0^L 2\lambda\, \mathrm{d}y_1\right] \mathrm{d}y_2 = \frac{\mu_0 q^2}{6\pi c}\gamma^4\left(\dot{a} + \frac{3va^2\gamma^2}{c^2}\right) \quad (14)$$

All kinematic variables in Eq. 14 are evaluated at the present time t, and it is clear that this relativistic expression reduces to the Abraham-Lorentz formula (Eq. 14) in the non-relativistic limit ($v \ll c$). Accordingly, one is compelled to conclude that the radiation reaction stems from the force an accelerating charge exerts on itself, as anticipated in § 1.1.3, or more accurately: from the net force due to interactions between different parts of the charge distribution – even in the point-charge limit, the self-force leads to the increase of a particle's resistance to acceleration (§ 2.1.3).

2.1.3 *Classical Mass Renormalisation*

The following discussion concerning electromagnetic inertia summarizes arguments from Refs. [6, 7].

SELF-FORCE-DERIVED MASS Replacing $F_{\text{rad}}^{\text{int}}$ by F_{rad} (Eq. 14) in Eq. 12 leads to the relativistic version of the modified equation of motion mentioned earlier (Eq. 7), where m is actually the *renormalised mass*, as it turns out.

$$\gamma^3 ma = F_{\text{rad}} + F_{\text{ext}}\,, \quad \text{where } m = m_0 + m_{\text{sf}} \text{ and } m_{\text{sf}} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4c^2 d} \quad (15)$$

This equation replaces Newton's second law for charged particles; the classical renormalisation procedure amounts to stating that the renormalised mass m must represent the total physical inertia of the dumbbell model for a charged particle. It has two components: the particle's *bare mass* m_0 and the *self-force-derived mass* m_{sf}. This last term clearly goes to infinity in the point-charge limit, but it no longer matters. The thorny issue is overcome by imposing that the only requirement

2 As an alternative, running the $\mathrm{d}y_1$ integral up to y_2 instead of L would avoid double-counting.

3 Inside the outer integral, all interaction forces due to charge pairs dy_1 (at position y_1) and dy_2 (at y_2) are added up for a fixed y_2 coordinate and all $0 \le y_1 \le L$.

is for the total inertial mass m to be finite, since this is the only experimentally measurable resistance to acceleration. The unobservable bare mass of a charged particle might contain some negative infinity compensating the divergent m_{sf} term.

Another point worth noting is that it is somewhat remarkable that the bare mass and the m_{sf} contribution to the inertia combine so nicely. First of all, the fact that the divergent self-force term is proportional to the acceleration saves the point-charge limit, to some extent. Secondly, its factor γ^3, perhaps deriving from the Lorentz invariance of Maxwell's equations (Ref. [6]), ensures that the total inertial mass can be written as $\gamma^3 m$ (from Eq. 12) and varies with speed in the way special relativity (that is to say, Eq. 15) requires[4].

On a side note, Lyle [6] points out that the self-force gets a minus sign in the case of an electric dipole ($q^2 \to -q^2$) with opposite charges at either end. Hence, m_{sf} becomes negative and the dumbbell's inertia gets lowered. This is in full agreement with special relativity, since the electrostatic energy of a dipole is negative, and due to $E = mc^2$, a reduction of inertial mass is expected to ensue from a negative (binding) energy contribution in a bound-state particle.

ENERGY-DERIVED MASS In the light of special relativity, the electromagnetic mass correction would certainly be sensible. If the dumbbell is at rest, the like-charge repulsion gives the configuration the electrostatic (Coulomb) energy E_{pot} (Eq. 16). Then, from Einstein's formula for relativistic energy ($E = \gamma mc^2$), it follows that this potential energy enhances the object's inertia, as it leads to a contribution called the *energy-derived mass* m_{en}. Since an object at rest is being considered in this case, $\gamma = 1$, whence $m_{\text{en}} = E_{\text{pot}}/c^2$.

$$E_{\text{pot}} = \frac{1}{4\pi\epsilon_0}\frac{(q/2)^2}{d} \qquad \Longrightarrow \qquad m_{\text{en}} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4c^2 d} = m_{\text{sf}} \tag{16}$$

However, the equality between self-force-derived mass and energy-derived mass does not hold in general. It happens to be valid for a dumbbell in transverse motion, but for a dumbbell moving longitudinally, it has been calculated[5] that $m_{\text{sf}} = 2m_{\text{en}}$, where m_{en} equals the quantity given in Eq. 16 (Ref. [7]). Furthermore, for a uniformly charged spherical shell, one can obtain $m_{\text{sf}} = (4/3)m_{\text{en}}$ by integrating the dumbbell self-force over all interaction pairs spread over the sphere, leading to the so-called 4/3 *paradox*. A sphere at rest, with a uniformly distributed charge q on the surface, has no magnetic field, and its electrostatic energy is calculated as in Eq. 17.

$$E_{\text{pot}} = \frac{\epsilon_0}{2}\int E^2 \, \mathrm{d}V = \frac{4\pi\epsilon_0}{2}\int_R^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 r^2 \, \mathrm{d}r \Longrightarrow m_{\text{en}} = \frac{1}{c^2}\frac{q^2}{8\pi\epsilon_0 R} \tag{17}$$

If the electron could be modelled as a charged spherical shell and its mass m_e would have a purely electromagnetic origin, then its radius would have to equal $R = (1/2)R_c$, where $R_c = e^2/(4\pi\epsilon_0 m_e c^2) \simeq 2.8 \times 10^{-15}$ m is the *classical electron radius*[6]. The proportionality factor between R and R_c depends on the chosen electron model. For instance, it differs for a sphere whose volume is uniformly filled with charge. However, experiments show that the electron radius cannot exceed 10^{-18} m, indicating a failure of the assumptions; the electron is a point particle, as far as known today.

4 Lyle [6] and Feynman [3] mention that this velocity-dependence of inertial mass was discovered before the theory of special relativity was fully established.

5 This result takes into account the Lorentz contraction of the dumbbell in an external lab frame.

6 The classical electron radius can also be rewritten as $R_c = \alpha \frac{\hbar}{m_e c}$, where $\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c}$ is the fine-structure constant. Furthermore, one can notice that $c\tau_0 = (2/3)R_c$, so τ_0 is approximately the time it takes light to cross the classical electron radius.

MOMENTUM-DERIVED MASS If total momentum is to be conserved, electromagnetic fields generated by particles in a volume $\mathcal{V}$ must carry some momentum given by $\boldsymbol{p} = \epsilon_0 \int_{\mathcal{V}} \boldsymbol{E} \times \boldsymbol{B} \, \mathrm{d}V$ (Ref. [4]). For a dumbbell moving at a constant velocity, $m_{\mathrm{mom}} = p/v$ is called the *momentum-derived mass*. For dumbbells in transverse and longitudinal motion as well as for a charged sphere, it turns out that the momentum-derived mass is equal to the self-force-derived mass (Ref. [7]).

4/3 PARADOX The 4/3 factor specifically refers to the fact that $m_{\mathrm{sf}} = (4/3)m_{\mathrm{en}}$ for a charged spherical shell. In general, $m_{\mathrm{sf}} = m_{\mathrm{mom}} \geq m_{\mathrm{en}}$ is found, since it holds for a dumbbell and an arbitrary charge configuration can be decomposed into dumbbells of infinitesimal interacting charge pairs. Generalising the transverse dumbbell case, all three electromagnetic mass measures are only equal when the charge distribution is fully located in a plane perpendicular to the direction of motion. Basically, the problem boils down to the fact that the electromagnetic self-force (resp. momentum) for a dumbbell inclined at an angle, with respect to the direction of motion, is not in general parallel to the acceleration (resp. velocity). In that case, the notions of m_{sf} and m_{mom} become ambiguous and can no longer be defined as scalars. The geometry of the charge configuration with respect to the direction of motion influences the way fields due to different parts of the configuration interact with the other parts.

However, for the infinitesimal dumbbells making up a sphere, the self-force (resp. momentum) components perpendicular to the acceleration (resp. velocity) cancel out for symmetry reasons, so m_{sf} (resp. m_{mom}) is defined in a consistent manner. Another point of view is required to resolve the 4/3 paradox (Ref. [7]). Poincaré was led by the idea that the self-force and momentum calculations mentioned above do not reflect the whole picture. The considered model is incomplete: stabilizing forces of non-electromagnetic origin, called *Poincaré stresses*, have to be added to counteract the repulsive electrical forces acting on the charged sphere. Otherwise, the particle model would be unstable and fly apart due to the unbalanced forces. The mass discrepancy vanishes when the stabilizing cohesive forces are accounted for in the calculations. Yet the complexity of this resolution spoils the beauty of the examined charged-particle model, a simple sphere.

Fermi and Rohrlich suggested another procedure, as they were focused on the fact that the electromagnetic stress-energy tensor $T^{\mu\nu}$ is not divergenceless at the position of a charge. In that case, the generalised definition of the electromagnetic momentum[7] does not yield a 4-vector. After redefining the 4-momentum, it is possible to adjust the momentum-derived mass, so that $m_{\mathrm{mom}} = m_{\mathrm{en}}$. Essentially, this approach does not contradict Poincaré's method, which just makes $T^{\mu\nu}$ divergenceless by adding a non-electromagnetic part. Carefully assessing how Lorentz contraction affects the sphere – rigidity is a notoriously precarious concept in special relativity – and its self-force, it can also be shown that $m_{\mathrm{sf}} = m_{\mathrm{en}}$ (Ref. [12]).

2.2 THE LORENTZ-DIRAC EQUATION

In Sect. 1.1, it was noted that the Abraham-Lorentz formula is non-relativistic, as its derivation explicitly relies on the Larmor formula. The relativistic formula for the power radiated by a point charge allows to check whether the relativistic expression for the radiation reaction force complies with energy conservation.

7 Normally, the electromagnetic 4-momentum is obtained as the integral of the four $T^{\mu 0}$ components over all space: $p^{\mu} = \int T^{\mu 0} \mathrm{d}V$ (Ref. [7]).

2.2.1 *Preliminary Definitions*

4-ACCELERATION Following the conventions detailed on p. iv, a particle's ordinary acceleration is written as $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$. Noting that $\mathrm{d}\gamma/\mathrm{d}t = (\gamma^3/c^2)\boldsymbol{v}\cdot\boldsymbol{a}$, the particle's *4-acceleration* $\alpha^\mu = \dot{\eta}^\mu$ can be expressed as

$$\alpha^\mu = \frac{\mathrm{d}\eta^\mu}{\mathrm{d}\tau} = (\alpha^0, \boldsymbol{\alpha}) = \left(\frac{\gamma^4}{c}\boldsymbol{v}\cdot\boldsymbol{a}, \gamma^2\left[\boldsymbol{a} + \frac{\gamma^2}{c^2}\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{a})\right]\right). \tag{18}$$

Basic arithmetic leads to

$$\alpha^\mu\alpha_\mu = -\frac{\gamma^8}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})^2 + \gamma^4\left(\boldsymbol{a} + \frac{\gamma^2}{c^2}\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{a})\right)^2 = \gamma^4\left(a^2 + \frac{\gamma^2}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})^2\right). \tag{19}$$

Furthermore, from $\eta^\mu\eta_\mu = -(\gamma c)^2 + (\gamma v)^2 = -c^2$, it follows that the 4-velocity and the 4-acceleration are orthogonal:

$$\eta^\mu\alpha_\mu = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\eta^\mu\eta_\mu) = 0. \tag{20}$$

4-FORCE As noted in § 2.1.2, $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ holds for an ordinary force $\boldsymbol{F}$ acting on a particle. The *4-force*, also called the Minkowski force, is defined as

$$K^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}p^0}{\mathrm{d}\tau}, \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau}, \gamma\boldsymbol{F}\right) = (K^0, \boldsymbol{K}), \tag{21}$$

where $\mathrm{d}E/\mathrm{d}\tau$ is the proper rate of change of the particle's energy.

2.2.2 *Liénard's Generalisation of the Larmor Formula*

It can be shown that the power radiated by a charge is Lorentz invariant (Refs. [5, 10]): in all inertial frames, the charge emits the same (ordinary) power. The proof hinges on the fact that radiated electromagnetic energy transforms like the zero component of a 4-vector. Hence, the goal is to find a Lorentz invariant quantity[8] reducing to the Larmor formula (Eq. 3) in the non-relativistic limit.

Noting that $\alpha^\mu\alpha_\mu \to a^2$ when $v \to 0$, one finds the radiated power $\mathrm{d}E/\mathrm{d}t$ as a Lorentz scalar which reduces to the Larmor formula:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mu_0 q^2}{6\pi c}(\alpha^\nu\alpha_\nu) \quad \xrightarrow{v\to 0} \quad \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{22}$$

If $\boldsymbol{v}$ and $\boldsymbol{a}$ make an angle θ, it is possible to rewrite $\alpha^\nu\alpha_\nu$ as follows, starting from Eq. 19:

$$\begin{aligned}\alpha^\nu\alpha_\nu &= \gamma^6\left(a^2\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})^2\right)\\ &= \gamma^6\left(a^2 + \frac{1}{c^2}v^2a^2(\cos^2(\theta) - 1)\right) = \gamma^6\left(a^2 - \frac{|\boldsymbol{v}\times\boldsymbol{a}|^2}{c^2}\right).\end{aligned} \tag{23}$$

This is one way to find *Liénard's generalisation*, enabling one to calculate the power emitted by a moving charge:

$$P = \frac{\mu_0 q^2}{6\pi c}\gamma^6\left(a^2 - \frac{|\boldsymbol{v}\times\boldsymbol{a}|^2}{c^2}\right). \tag{24}$$

8 Jackson [5] states that the Lorentz invariant is unique, when imposing that the power can only depend on $\boldsymbol{v}$ and $\boldsymbol{a}$.

2.2.3 *Generalisation of the Abraham-Lorentz Formula*

In order to find a relativistic version of the Abraham-Lorentz formula (Eq. 5), it seems natural to consider

$$K^{\mu}_{\text{rad}} = m\tau_0 \dot{\alpha}^{\mu} = \frac{\mu_0 q^2}{6\pi c}\frac{\mathrm{d}\alpha^{\mu}}{\mathrm{d}\tau}. \tag{25}$$

This is indeed a 4-vector reducing to the Abraham-Lorentz formula in the non-relativistic case. However, it cannot represent a 4-force, since $K^{\mu} = \dot{p}^{\mu} = m\alpha^{\mu}$ entails $K^{\mu}\eta_{\mu} = 0$, using the orthogonality of α^{μ} and η_{μ} (Eq. 20). Yet for K^{μ}_{rad} as defined in Eq. 25, one finds

$$K^{\mu}_{\text{rad}}\eta_{\mu} = m\tau_0\dot{\alpha}^{\mu}\eta_{\mu} = -m\tau_0\alpha^{\mu}\alpha_{\mu}\,,\ \text{since}\ \frac{\mathrm{d}}{\mathrm{d}\tau}(\eta^{\mu}\alpha_{\mu}) = 0 \implies \dot{\alpha}^{\mu}\eta_{\mu} = -\alpha^{\mu}\alpha_{\mu}. \tag{26}$$

In general, $\alpha^{\mu}\alpha_{\mu} \neq 0$, so as it stands, Eq. 25 is incorrect. In order to make K^{μ}_{rad} orthogonal to η_{μ}, one can add a 4-vector term $X^{\mu} = (X^0, \boldsymbol{X})$ to $\dot{\alpha}^{\mu}$, ensuring that the non-relativistic limit of K^{μ}_{rad} does not change (and that K^{μ}_{rad} stays a 4-vector).

$$(\dot{\alpha}^{\mu} + X^{\mu})\,\eta_{\mu} = 0 \implies X^{\mu}\eta_{\mu} = -\dot{\alpha}^{\mu}\eta_{\mu} = \alpha^{\mu}\alpha_{\mu}. \tag{27}$$

The easiest way to let $\boldsymbol{X}$ vanish as $v \to 0$, is by making X^{μ} proportional to η^{μ}, since $\eta^{\mu} \to (c, \mathbf{0})$ in the non-relativistic limit. Trying $X^{\mu} = (\alpha^{\nu}\alpha_{\nu})(-1/c^2)\eta^{\mu}$ fulfils the requirement: using $\eta^{\mu}\eta_{\mu} = -c^2$, one verifies that $X^{\mu}\eta_{\mu} = \alpha^{\mu}\alpha_{\mu}$.

Hence, the simplest covariant generalisation for the radiation reaction 4-force is given by

$$K^{\mu}_{\text{rad}} = m\tau_0\left(\dot{\alpha}^{\mu} - \frac{1}{c^2}(\alpha^{\nu}\alpha_{\nu})\eta^{\mu}\right). \tag{28}$$

The expression has been confirmed by other methods. It was, for instance, proven by Dirac that the formula follows from energy-momentum conservation. For the case of a dumbbell in one-dimensional, transverse motion (§ 2.1.2), it has been checked that a calculation exploiting the collinearity of $\boldsymbol{v}$ and $\boldsymbol{a}$ confirms that $|\boldsymbol{K}_{\text{rad}}| = \gamma F_{\text{rad}}$, where F_{rad} is taken from Eq. 14. Furthermore, the relativistic generalisation leads to a more general equation of motion for point charges, the *Lorentz-Dirac equation*, which reduces to Eq. 7 when $v \to 0$.

$$m\alpha^{\mu}(\tau) = K^{\mu}_{\text{ext}}(\tau) + m\tau_0\left(\dot{\alpha}^{\mu}(\tau) - \frac{1}{c^2}(\alpha^{\nu}\alpha_{\nu})\eta^{\mu}(\tau)\right) \tag{29}$$

In Eq. 29, m is the renormalized mass of the particle and K^{μ}_{ext} refers to the external 4-force acting on it.

2.2.4 *Runaway Solutions*

It is possible to find runaway solutions for the Lorentz-Dirac equation, complying with the relativistic constraint on velocities. That is, solutions with an exponentially increasing *rapidity* exist, implying a boundless rise of γ and the particle's kinetic energy, and that the particle gets accelerated to light speed.

For simplicity, one-dimensional particle motion along the x direction is considered. From the 3-vector part of the Lorentz-Dirac equation (Eq. 29), one finds

$$m\boldsymbol{\alpha} = \boldsymbol{K}_{\text{ext}} + m\tau_0\left(\dot{\boldsymbol{\alpha}} - \frac{1}{c^2}(\alpha^{\nu}\alpha_{\nu})\boldsymbol{\eta}\right). \tag{30}$$

In this expression, $\boldsymbol{\eta} = (\eta, 0, 0)$, $\boldsymbol{\alpha} = (\alpha, 0, 0)$, $\dot{\boldsymbol{\alpha}} = (\dot{\alpha}, 0, 0)$ and $\boldsymbol{K}_{\text{ext}} = (K_{\text{ext}}, 0, 0)$.

If the magnitude of the ordinary velocity of the particle is v, its rapidity is defined as $\varphi(\tau) = \operatorname{arctanh}(v/c)$. In the case of one-dimensional motion, it is readily shown (Sect. A.5) that Eq. 30 can be written as

$$\dot{\varphi} = \frac{1}{mc\gamma} K_{\text{ext}} + \tau_0 \ddot{\varphi} = \frac{F_{\text{ext}}}{mc} + \tau_0 \ddot{\varphi}. \tag{31}$$

The second equality follows from Eq. 21, which implies that $K_{\text{ext}} = \gamma F_{\text{ext}}$, where F_{ext} is the ordinary force from Newton's second law.

Finally, the runaway phenomenon is recovered: even if no external force is exerted on the particle, Eq. 31 still admits rapidity solutions rising exponentially:

$$\ddot{\varphi} = \frac{1}{\tau_0} \dot{\varphi} \Rightarrow \varphi(\tau) = A + B\mathrm{e}^{\tau/\tau_0}, \text{ where } A \text{ and } B \text{ are integration constants.} \tag{32}$$

2.2.5 *Energy Conservation*

To check whether the radiation reaction 4-force K^{μ}_{rad} (Eq. 28) is compatible with energy conservation, one can inspect the power loss of the particle affected by K^{μ}_{rad} and compare it with the radiated power P, as obtained from Liénard's generalisation (Eq. 24).

Since the zero component of K^{μ}_{rad} is related to the emitted power, it is instructive to write its expression explicitly. Using Eqs. 18 and 23 yields the following result:

$$\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau} = K^{0}_{\text{rad}} = m\tau_0 \left(\gamma \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\gamma^4}{c} \boldsymbol{v} \cdot \boldsymbol{a} \right) - \frac{1}{c^2} \gamma^6 \left(a^2 - \left| \frac{\boldsymbol{v} \times \boldsymbol{a}}{c} \right|^2 \right) \gamma c \right). \tag{33}$$

It follows that the time-average energy change of the particle equals (minus) the energy lost to radiation, when integrating over a time interval between two identical system states. For in that case, the derivative term vanishes from the integral, leading to

$$\begin{aligned} \int_{t_1}^{t_2} \frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t &= \int_{t_1}^{t_2} \frac{c}{\gamma} K^{0}_{\text{rad}}\,\mathrm{d}t = m\tau_0 \left(\left[\gamma^4 \boldsymbol{v} \cdot \boldsymbol{a} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \gamma^6 \left(a^2 - \left| \frac{\boldsymbol{v} \times \boldsymbol{a}}{c} \right|^2 \right) \mathrm{d}t \right) \\ &= -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} \gamma^6 \left(a^2 - \frac{|\boldsymbol{v} \times \boldsymbol{a}|^2}{c^2} \right) \mathrm{d}t = -\int_{t_1}^{t_2} P\,\mathrm{d}t. \end{aligned} \tag{34}$$

The term integrating to zero is minus the (coordinate) time derivative of the *Schott energy* $E_S = -m\tau_0\gamma^4 \boldsymbol{v} \cdot \boldsymbol{a}$. Hence, the (ordinary) power $\mathrm{d}E/\mathrm{d}t$ due to the radiation reaction force equals $-\mathrm{d}E_S/\mathrm{d}t - P$. The energy reversibly transferred from the charge to its bound fields (e. g. the velocity field in Eq. 1) is quantified by E_S, whereas P (from Liénard's formula) refers to the power irreversibly radiated to infinity.

Part II

CONSEQUENCES OF THE MODIFIED EQUATION OF MOTION

3 CLASSICAL TUNNELLING

A remarkable consequence of taking into account the effect of the radiation reaction on the motion of a charged point particle, is the emergence of so-called *tunnelling* solutions, recently described by Carati et al. [1] and Denef et al. [2]. Considering the traditional quantum mechanical context of a charged particle impinging on a one-dimensional potential barrier, it is straightforward to show that the modified classical equation of motion allows for the particle to pass through even in cases where its kinetic energy is lower than the step height (Sect. 3.1). Another peculiar feature of the considered equation of motion is the generic non-uniqueness of its physical solutions. To study the impact of the sharpness of the discontinuous potential step, analogous calculations can be made to investigate the setting of a more physical, continuous potential ramp (Sect. 3.2).

3.1 THE POTENTIAL BARRIER

3.1.1 *Preliminary Definitions*

To correctly examine the behaviour of a non-relativistic point particle with charge q and renormalised mass m, incident on the left-hand side of a one-dimensional potential energy barrier (Fig. 4), one attempts to solve the modified equation of motion (§ 1.2.3):

$\tau_0 = \frac{\mu_0 q^2}{6\pi mc}$

$$a(t) = \tau_0 \dot{a}(t) + \frac{F}{m} \tag{35}$$

The potential barrier is described by

$\theta(x)$ denotes the unit step function and is related to the Dirac delta function: $\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = \delta(x)$.

$$V(x) = V_0(\theta(x) - \theta(x-L)) = \begin{cases} V_0 & \text{if } 0 < x < L \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Hence, the associated external force, acting upon the particle, is determined as

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x} = V_0(-\delta(x) + \delta(x-L)). \tag{37}$$

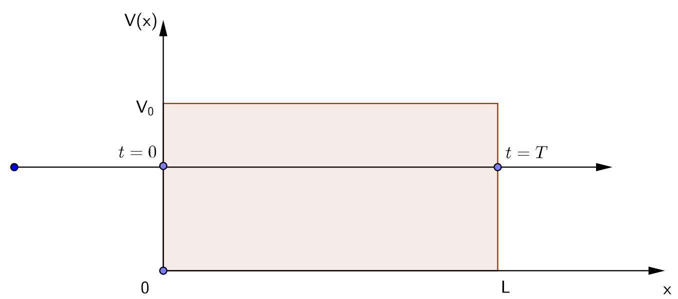

Figure 4: Potential barrier of height $V_0 > 0$ and transmitted particle.

For a transmitted particle, the localized potential barrier defines the following three regions in space, in which Eq. 35 shall be solved separately. Setting $t = 0$ for the moment when the particle passes $x = 0$ and letting T denote the time it spends traversing the barrier, one observes that each spatial region can be linked to the time interval in which the particle can be found inside it.

Region 1: $x < 0 \leftrightarrow t < 0$

Region 2: $0 < x < L \leftrightarrow 0 < t < T$

Region 3: $x > L \leftrightarrow t > T$

In each of these regions, the force equals zero, since its only non-zero points are $x = 0$ and $x = L$. Therefore, it follows from the equation of motion that

If $F = 0$, one finds $a(t) = \tau_0 \dot{a}(t)$.

$$a_i(t) = A_i \mathrm{e}^{t/\tau_0}, \tag{38}$$

where $a_i(t)$ denotes the acceleration in Region i (with $i \in \{1,2,3\}$) and A_i is a constant to be determined by matching conditions. Integrating these solutions yields equations – each only applicable in its respective region – for the velocity and the position of the particle in rectilinear motion:

$$v_i(t) = \tau_0 A_i \mathrm{e}^{t/\tau_0} + B_i \tag{39}$$

$$x_i(t) = \tau_0^2 A_i \mathrm{e}^{t/\tau_0} + B_i t + C_i \tag{40}$$

In Eqs. 39 and 40, B_i and C_i are integration constants.

3.1.2 *Acceleration Discontinuity*

On physical grounds, the position $x(t)$ and velocity $v(t)$ of a particle must always be continuous. However, since delta forces are involved, a calculation similar to the one in § 1.2.3 shows that the particle's acceleration $a(t)$ is discontinuous at the edges of the potential barrier. Letting $x^{-1}(0)$ and $x^{-1}(L)$ denote the moments the particle passes an edge, one finds that the particle's acceleration increases abruptly with $\Delta a(x = 0)$ when the particle encounters the barrier and subsequently drops with $\Delta a(x = L)$ when leaving. The calculations (§ A.6.1) lead to

$$\Delta a(x = 0) = \frac{+V_0}{m\tau_0 v(x^{-1}(0))} \quad \text{and} \quad \Delta a(x = L) = \frac{-V_0}{m\tau_0 v(x^{-1}(L))}. \tag{41}$$

3.1.3 *Matching Conditions*

THE ASYMPTOTIC CONDITION Writing the equation of motion as a third-order equation of the position, i. e. as $\ddot{x}(t) = \tau_0 \dddot{x}(t) + \frac{F}{m}$, it becomes clear that specifying the initial position, velocity and acceleration is mathematically sufficient to find a unique solution, that is, to determine the constants defined in § 3.1.1. However, in general this solution is an experimentally unobserved *runaway solution*. The only way to avoid such unbounded, exponentially increasing acceleration and velocity (Eqs. 38 and 39) in the force-free Region 3, is to set $A_3 = 0$. This *asymptotic condition* implies that the particle does not accelerate beyond the barrier, and replaces the initial-value specification of the acceleration. The essentially mathematical question at hand then becomes whether acceptable and, if possible, unique physical solutions can still be found (Ref. [2]). Further elaboration upon this question follows in § 3.1.6 and Sect. 3.2.

BOUNDARY CONDITIONS If v_f denotes the particle's final velocity as $t \to \infty$ (implying $B_3 = v_f$) and the Region 3 runaway is excluded, one obtains C_3 from $L = x_3(t) = v_f T + C_3$. The other integration constants are obtained by use of the (dis)continuity conditions in $x = 0$ and $x = L$. The exact expressions are listed in § A.6.2.

BARRIER RELATIONS Using the calculated expressions for the constants, one finds an equation for the particle position in Region 2:

$$\begin{aligned} x_2(t) &= \tau_0^2 A_2 e^{t/\tau_0} + B_2 t + C_2 \\ &= \frac{V_0}{mv_f}(\tau_0(e^{(t-T)/\tau_0} - 1) + T - t) + v_f(t - T) + L \end{aligned} \tag{42}$$

A relation between the barrier length and the crossing time (Eq. 43) follows from the condition that $x_2(0) = 0$.

$$L = v_f T - \frac{V_0}{mv_f}(\tau_0 e^{-T/\tau_0} + T - \tau_0) \tag{43}$$

If the particle's initial velocity is denoted by v_i, it is straightforward to see that $v_i = \lim_{t\to-\infty} v_1(t) = \lim_{t\to-\infty}(\tau_0 A_1 e^{t/\tau_0} + B_1) = B_1$. Furthermore, $v_1(0) = v_2(0)$ implies that $v_i = v(0) - \tau_0 A_1$. Noting that the particle velocity at $t = 0$ is given by Eq. 44, one can work out the expression for A_1 (Eq. 135).

$$v(0) = v_2(0) = \tau_0 A_2 e^{0/\tau_0} + B_2 = \frac{V_0}{mv_f}(e^{-T/\tau_0} - 1) + v_f \tag{44}$$

By extension, one is now able to formulate the following relation between the initial and final velocities (Eq. 45).

$$v_i = v(0) - \tau_0 A_1 = v_f - \frac{V_0}{mv_f}\left(1 - \left(\frac{V_0}{mv_f^2}(e^{-T/\tau_0} - 1) + 1\right)^{-1}\right) \tag{45}$$

Eqs. 43 and 45, also mentioned in [2, 4], turn out to be of great practical use. Indeed, if the barrier length and height are fixed, it is possible to extract the crossing time from Eq. 43, given a certain final velocity, by numerically solving the equation for T. Subsequently, the initial velocity can be obtained from Eq. 45. An analogous procedure will be used in Sect. 3.2, in order to make a plot of the initial velocity as a function of the final velocity, in keeping with the practice of solving the matching conditions backwards in time. Although this retrogressive solution strategy might seem counterintuitive, it is a natural consequence of applying the asymptotic condition in the most effective way.

3.1.4 *Tunnelling Solutions*

A specific example of a tunnelling solution can be obtained by letting the final kinetic energy equal half the barrier height, implying that $mv_f^2 = V_0$. Substituting this value for V_0 into Eq. 43, gives the following result.

$$L = v_f T - \frac{mv_f^2}{mv_f}(\tau_0 e^{-T/\tau_0} + T - \tau_0) = v_f \tau_0 (1 - e^{-T/\tau_0}) \tag{46}$$

Hence, $e^{-T/\tau_0} - 1 = \frac{-L}{v_f \tau_0}$, so Eq. 45 reduces to

$$v_i = v_f - \frac{mv_f^2}{mv_f}\left(1 - \left(\frac{mv_f^2}{mv_f^2}\left(\frac{-L}{v_f T}\right) + 1\right)^{-1}\right) = \frac{v_f}{1 - \frac{L}{v_f \tau_0}}. \tag{47}$$

If one chooses a barrier length $L = \frac{v_f \tau_0}{4}$, the initial velocity is found to be $v_i = \frac{4}{3} v_f$. Thus, despite the particle's initial kinetic energy $\frac{1}{2} m v_i^2 = \frac{8}{9} m v_f^2 = \frac{8}{9} V_0$ being a fraction lower than the potential barrier height V_0, it tunnelled through.

According to classical mechanics, this is not possible, since the total particle energy is given by the sum of its kinetic and potential energy. The kinetic energy is non-negative by definition, whereas the latter is zero before and positive inside the barrier. Hence, conservation of energy dictates that it is impossible for the particle to pass a potential barrier higher than its initial total energy, equal to its initial kinetic energy. A tunnelling particle would have a negative kinetic energy inside the barrier region.

3.1.5 *Interpreting Classical Tunnelling*

The remarkable tunnelling phenomenon gives rise to a plethora of questions, the most obvious one perhaps concerning the physical mechanism behind it. How is it possible that a completely classical framework allows such tunnelling solutions without any quantum mechanical considerations whatsoever entering the argument?

First of all, it is of crucial importance to investigate the role played by the length of the barrier. Indeed, the given example could strike one as artificial, in the sense that the details, i. e. the chosen barrier parameters, are very specific. Be that as it may, it is straightforward to generalize the example. Taking the final kinetic energy $\frac{1}{2}mv_f^2 = \frac{1}{2}V_0$ as in § 3.1.4, a tunnelling solution is retrieved if the initial kinetic energy is lower than the barrier height: $\frac{1}{2}mv_i^2 < V_0$. Using Eq. 47, this condition becomes equivalent to

$$\frac{1}{2}m\left(\frac{v_f}{1-\frac{L}{v_f\tau_0}}\right)^2 < mv_f^2 \quad \Longleftrightarrow \quad \left(\frac{L}{v_f\tau_0}\right)^2 - 2\frac{L}{v_f\tau_0} + \frac{1}{2} > 0. \tag{48}$$

The obtained inequality brings about two cases: $0 < \frac{L}{v_f\tau_0} < 1 - \frac{\sqrt{2}}{2}$ and $\frac{L}{v_f\tau_0} > 1 + \frac{\sqrt{2}}{2}$. The latter however entails a negative initial velocity, when substituted into Eq. 47, a conclusion contrary to the assumption that the particle is incident on the left-hand side of the barrier. This leaves only the former case, indicating that tunnelling with the specified final velocity occurs whenever the barrier length is small compared to $v_f\tau_0$, a measure for the distance travelled by the particle in a time interval τ_0:

$$L < \left(1 - \frac{\sqrt{2}}{2}\right) v_f\tau_0 \simeq 0.29\, v_f\tau_0. \tag{49}$$

In other words, if the potential barrier is so narrow that the particle can cross it on a timescale of order τ_0, it tunnels through the classically forbidden region. Bringing into account the typical order of magnitude of τ_0, it is clear that the theoretical phenomenon only manifests itself on small scales. A rough calculation shows that for a non-relativistic electron with final velocity $v_f = 0.1\,c$, the order of magnitude of the involved action can be estimated by

For an electron: $\tau_0 \simeq 6.27 \times 10^{-24}$ s, *and* $\lambda_c \simeq 2.43 \times 10^{-12}$ m.

$$V_0 T \lesssim V_0\tau_0 = m_e v_f^2 \tau_0 \sim 10^{-30}\, 10^{15}\, 10^{-23}\ \mathrm{J\,s} < \hbar. \tag{50}$$

This indicates that quantum mechanics inevitably must enter the picture and that an adequate treatment should rely on quantum electrodynamics, unfortunately beyond the scope of current report, rather than classical electromagnetism. Even though the latter is clearly not the right framework for probing the issue, it is nevertheless somewhat surprising that a classical formula allows tunnelling solutions mathematically.

In the end, however, one should bear in mind the words of Rohrlich [9] about the restrictions of a physical theory, classical electrodynamics in particular. Reminiscent of the logical principle *ex falso quodlibet*, senseless results might ensue from applying a theory beyond its range of validity. Where quantum mechanics prevails, i. e. at length scales below or of the order of the Compton wavelength $\lambda_c = h/mc$ of the particle, it is perhaps not abnormal that a classical equation of motion breaks down. For the example above, one finds $\tau_0 < \hbar/(m_e v_f^2)$ from the order-of-magnitude calculation for the action, whence

$$L < 0.29\, v_f \tau_0 < 0.29 \frac{\hbar}{m_e v_f} = 0.29 \frac{h/(2\pi)}{m_e (c/10)} < \lambda_c. \tag{51}$$

This, as well as the fact that $\tau_0 c \sim 10^{-23}\, 10^8$ m$< \lambda_c$, shows that the involved length scale is below the electron's Compton wavelength, as expected, implying that the *classical tunnelling* phenomenon theoretically occurs in the quantum regime. The apparent contradiction in terms is resolved by noting that, whereas its manifestation only takes place on the quantum scale, the tunnelling is classical in the sense that it follows from purely classical considerations about the radiation emitted by an accelerating particle.

BOUND MOMENTUM In order to investigate the classical mechanism behind this phenomenon, Denef et al. [2] refer to the behaviour of the *bound momentum*, defined in Eq. 52. It is claimed that this definition allows for the bound momentum to have a consistent interpretation as the total combined momentum of an electrically charged point particle and its bound electromagnetic field, as established by Teitelboim [11].

$$\wp = m(v - \tau_0 a) \tag{52}$$

One can note that the time derivative of the bound momentum reproduces the modified equation of motion when equated to an external force F (Eq. 35):

$$\dot{\wp} = ma - m\tau_0 \dot{a} = F. \tag{53}$$

This is in effect a modified version of Newton's second law, which takes the radiation reaction into account for accelerating charged particles. In the absence of acceleration, the bound momentum reduces to the usual particle momentum $p = mv$.

For the case of the potential barrier, writing $\wp_i$ for the bound momentum in Region i, one obtains that it has a constant value in each region:

$$\wp_i = m((\tau_0 A_i e^{t/\tau_0} + B_i) - \tau_0 (A_i e^{t/\tau_0})) = mB_i. \tag{54}$$

Hence, using Eq. 136, under the barrier its value[1] is $\wp_2 = m\left(v_f - \frac{V_0}{mv_f}\right)$. In Region 3, one finds $\wp_3 = mv_f$ for the bound momentum, since the asymptotic condition entails that the particle's acceleration ceases upon entering this spatial region. Furthermore, Eq. 54 implies that $F = \dot{\wp} = 0$, in each region, as noted in § 3.1.1. From this, one can now conclude that the bound momentum is a conserved quantity in each region, despite being able to make discontinuous jumps at the edges of the barrier.

Taking a closer look, one notices that the bound momentum suddenly decreases when the particle enters Region 2, due to the positive acceleration boost Δa and the fact that the velocity is continuous at $x = 0$:

$$\Delta\wp(x=0) = m(\Delta v(x=0) - \tau_0 \Delta a(x=0)) = -m\tau_0 \frac{V_0}{m\tau_0 v(0)} < 0. \tag{55}$$

There are two distinct possibilities, then: either the particle never makes the opposite side of the barrier and turns back at some point inside Region 2, or it manages

1 The specific example chosen in § 3.1.4, where $V_0 = mv_f^2$, gives the rather peculiar result $\wp_2 = 0$.

to get across the barrier. In the first case, the calculations based on the premise that the particle gets to Region 3 do not hold, but it is clear that this can only happen if the velocity turns negative, necessitating a deceleration in Region 2. The other case is more interesting: as the calculations in § 3.1.3 do not involve the assumption of tunnelling, one can observe that the equations for the velocity (Eq. 56) and the acceleration (Eq. 57) in Region 2 hold regardless of whether the particle has enough energy to pass the barrier classically. Since, by definition, the velocity of an incoming particle is positive at $t = 0$, and because Eq. 56 increases as a function of time, the velocity in Region 2 stays positive, as one would expect for a tunnelling particle. However, just like the acceleration, it increases as a function of time.

Since calculations were made for a particle managing to cross the barrier, $v_f > 0$ follows by assumption.

$$v_2(t) = \frac{V_0}{mv_f}(\mathrm{e}^{(t-T)/\tau_0} - 1) + v_f > 0 \tag{56}$$

$$a_2(t) = \frac{V_0}{m\tau_0 v_f}\mathrm{e}^{(t-T)/\tau_0} > 0 \tag{57}$$

This implies that the physically appealing, intuitive image of a decelerating particle, barely making it to Region 3, is simply wrong. If the particle manages to surmount the potential barrier, it underwent a positive acceleration and had an increasing velocity. Whether it did or did not have an initial kinetic energy higher than the barrier height, does not matter: if the particle eventually ended up behind the barrier, it never[2] decelerated in Region 2. The only difference would be that a tunnelling particle necessarily has to traverse the barrier in a time of order τ_0, as suggested by Eq. 49.

In conclusion, the main dichotomy can be traced back to the behaviour of the particle's acceleration in Region 2. Either the particle starts to decelerate during its passage under the barrier, in which case it will be reflected, or it continues to accelerate throughout Region 2, in the course of time making it to Region 3. When the particle leaves the barrier, entering Region 3, the bound momentum gets a positive boost, owing to the sudden acceleration drop $\Delta a(x = L) < 0$. In this way, a particle that manages to traverse the barrier always ends up with a positive bound momentum mv_f in Region 3, even though its value $m[v_f - V_0/(mv_f)]$ under the barrier can be positive or negative. In other words, in Region 2 the bound momentum can be directed along or opposite to the positive velocity. However, in Region 3, the acceleration ceases, so the bound momentum ends up being aligned to the velocity, as one would expect from the usual definition of linear momentum. Furthermore, an analogous remark can be made for a reflected particle. After turning back to Region 1, when all acceleration ceases, its bound momentum is negative, just like its final velocity. Accordingly, the alignment of bound momentum and velocity always turns out to be restored in the end.

It is important to note that the final velocity can be smaller than the initial velocity, in spite of the acceleration in Region 2. This can be inferred from Eq. 47, where the denominator is less than unity. The reason behind this fact is basically the *pre-deceleration* which the particle is subjected to before entering the potential barrier. To understand this, it suffices to picture the encounter between the particle and the rectangular barrier as a sequence of events, seen from the viewpoint of the particle. From Eq. 37 it follows that at $t = 0$, and $t = T$, the particle is confronted with an infinite force. As calculated in § 1.2.3, if runaway solutions are excluded, the particle will undergo an acceleration or deceleration starting a time of order τ_0 before effectively encountering the force. Paying attention to the signs in the equivalent expression $F(t) = V_0(-\delta(t) + \delta(t - T))$, one is led to surmise that the

2 This point is stated somewhat confusingly in Ref. [2], where it is suggested that if the velocity is directed opposite to the bound momentum, the former turns in a time of order τ_0, to be in the same direction as the latter. For small barrier lengths, the particle would have crossed the barrier by that time. Following the argument presented here, it becomes clear that a transmitted particle cannot have a decreasing, let alone turning, velocity. Instead, it is subjected to an acceleration.

particle decelerates during a short time before entering the barrier and experiences a brief surge of acceleration just before leaving the barrier. Following the argument above, involving Eq. 57, the latter is certainly the case. It is straightforward to demonstrate the former as well, using the example from § 3.1.4, where $V_0 = mv_f^2$. It leads to the following expression for the particle acceleration in Region 1, obtained from Eqs. 44 and 135.

$$\begin{aligned} a_1(t) &= \frac{mv_f^2}{m\tau_0}\left(\frac{e^{-T/\tau_0}}{v_f} - \left(\frac{mv_f^2}{mv_f}(e^{-T/\tau_0} - 1) + v_f\right)^{-1}\right) e^{t/\tau_0} \\ &= \frac{v_f}{\tau_0}(e^{-T/\tau_0} - e^{T/\tau_0})e^{t/\tau_0} \end{aligned} \tag{58}$$

Since $e^{-T/\tau_0} - e^{T/\tau_0}$ is a negative factor, $a_1(t)$ is a decreasing function which reaches negative values over its domain, which consists of all time parameters $t < 0$, by definition. The deceleration becomes noticeable at some time near $t = -\tau_0$, after which it would attain its maximum value at $t = 0$. At that point, the particle acceleration would receive a positive boost $\Delta a(x = 0)$ and subsequently become positive and increase (Eq. 57). To sum up, the pre-deceleration causes the particle velocity to decline rapidly, after which it increases in Region 2, up to the value of the final velocity, which can be lower than the initial velocity. It can be noted that the principle of energy conservation is not violated: since an accelerating particle emits radiation (and hence, a pre-accelerating particle as well), the irreversibly lost kinetic energy has simply been radiated.

NON-RELATIVISTIC APPROXIMATION One might be inclined to attribute the emergence of tunnelling solutions to the fact that the equation of motion concerned (Eq. 35) is based on the non-relativistic Abraham-Lorentz formula for the radiation reaction force on a charged particle. For the sake of completeness, it can be mentioned that this is not the case. In Ref. [2], it is asserted that tunnelling solutions of the relativistic equation of motion (Eq. 29) can be constructed, despite being more complex.

3.1.6 *Non-Uniqueness of Solutions*

From a mathematical point of view, one would expect that the specification of the initial particle position, velocity and acceleration allows for a unique solution of the third-order equation of motion. In order to exclude runaway solutions, the asymptotic condition usually replaces the initial-value specification of the acceleration, as motivated in § 3.1.3. However, this turns out to jeopardize the uniqueness of the obtained solutions in a serious way.

A specific example can quickly be found for a one-dimensional potential barrier between $x = 0$ and $x = L$. If one specifies the initial velocity of the particle as zero and its initial position x_0, in the infinite past, inside the barrier, it can immediately be confirmed that $x(t) = x_0$ constitutes a stationary solution to the equation of motion (Eq. 35), since its derivatives are zero and $F = 0$ inside the barrier.

Since $F = 0$, $a(t) = \tau_0 \dot{a}(t)$.

One might, however, also try to solve the equation for a situation in which the particle somehow passes $x = L$ at some time $t = T$. This context also provides a valid solution for the equation. To demonstrate this, it suffices to divide the area into two regions, the barrier region (Region 1) and the space behind the barrier (Region 2). As in § 3.1.1, each region is linked to the time interval which the particle spends inside it.

Region 1: $0 < x < L \leftrightarrow t < T$

Region 2: $x > L \leftrightarrow t > T$

A solution for the equation of motion is sought in both regions. The particle's acceleration, velocity and position have the same functional form as described in Eqs. 38, 39 and 40. As dictated by the asymptotic condition, $a_2(t) = 0$ and $v_2(t) = v_f$ in the second region, leading to $x_2(t) = v_f t + C_2$, where $C_2 = x_2(T) - v_f T = L - v_f T$. Hence, one finds $x_2(t) = L + v_f(t - T)$ for $t > T$. In the first region, one can invoke the acceleration discontinuity at the rightmost barrier edge to determine A_1.

$$a_2(T) - a_1(T) = \Delta a(x = L): \quad 0 - A_1 e^{T/\tau_0} = \frac{-V_0}{m\tau_0 v(T)} \tag{59}$$

Furthermore, the fact that the particle has no initial velocity implies that $B_1 = 0$, since

$$0 = v_i = \lim_{t\to-\infty} v_1(t) = \lim_{t\to-\infty} (\tau_0 A_1 e^{t/\tau_0} + B_1) = B_1. \tag{60}$$

The asymptotic initial value of the position subsequently fixes C_1:

$$x_0 = \lim_{t\to-\infty} x_1(t) = \lim_{t\to-\infty} (\tau_0^2 A_1 e^{t/\tau_0} + B_1 t + C_1) = C_1. \tag{61}$$

It then follows, substituting $v(T) = v_2(T) = v_f$, that the expression below, valid for $t < T$, forms the Region 1 part of this solution for the chosen initial-value problem.

$$x_1(t) = \tau_0^2 \frac{V_0}{m\tau_0 v_f} e^{(t-T)/\tau_0} + x_0 \tag{62}$$

Therefore, the given initial data do not determine the physical solution of the equation of motion in a unique way. Nevertheless, the existence of the aforementioned stationary solution remains an isolated[3] case. It arises because the initial velocity was specified to be zero; any other value would disallow it. At this point, it comes to mind that this feature of the equation of motion might be attributable to the asymptotic nature of the described initial data. It turns out that this is not the case. In Ref. [2], it is stated that the non-uniqueness of physical (hence, non-runaway) solutions is persistent, even when the given boundary conditions are the particle velocity and position at some finite time.

As a matter of fact, the non-uniqueness problem is not at all limited to the special example mentioned. When looking at a particle with initial velocity v_i in the non-radiative case, it follows from the conservation of energy that the particle will be reflected on a potential barrier, with final velocity $v_f = -v_i$, if its initial kinetic energy is lower than the barrier height V_0. In the other case, if $\frac{1}{2}mv_i^2 > V_0$, the particle will be transmitted and its final velocity will be $v_f = v_i$. In this manner, the particle's initial velocity of course determines its initial kinetic energy (which distinguishes between reflection and transmission), and a fortiori fixes the final velocity uniquely. This clear-cut correspondence between v_i and v_f cannot be maintained in the classical description of a radiating charged particle, as is shown in Sect. 3.2 for the case of a linear potential ramp: several final velocities are found to be possible when specifying initial velocities in a certain range.

However, an even more troublesome feature of the potential barrier is the blatant non-existence of non-runaway solutions for a range of small incident velocities, which was noted in Ref. [13]. From the physical point of view, it is not acceptable that some incident velocities, for which the particle is expected to be reflected, do not give rise to a physical solution. This raises further questions about the validity of the equation of motion. The cause of this undesired feature can be

3 The fact that the stated example is more generic than the one given in Ref. [2] does not alter this point.

traced back to the rectangular shape of the potential. Its idealized, sharp edges render a non-analytic potential function. Luckily, for a more physical, continuous potential ramp, the problematic range vanishes and a solution can be found for each initial velocity, as is proven in Sect. 3.2. Thus, a mitigating factor can be found for the fact that some incident velocities do not allow a solution of the equation of motion in the case of a steep potential barrier. Nevertheless, the outcome remains unsatisfactory in a conceptual way of thinking, in spite of being saved for realistic potentials.

3.2 THE POTENTIAL RAMP

3.2.1 *Preliminary Definitions*

In § 3.1.6, it was mentioned that a certain range of initial velocities of an incident particle does not admit physical solutions to the modified equation of motion, for the case of a rectangular potential barrier. This behaviour is profoundly unphysical. On that account, the question rises whether the entire tunnelling phenomenon and the non-uniqueness of physical solutions are not linked to this breakdown, hence limited to potentials with discontinuous features. In order to check into this point thoroughly, it is possible to examine the case of scattering at a continuous potential ramp.

As before, a point particle of charge q and mass m is considered. In view of compliance with the conventions in Ref. [2], it is considered to be incident on the right-hand[4] side of a linearly rising, one-dimensional potential energy step (Fig. 5). Once again, a solution for the equation of motion (Eq. 35) is desired. A description of the potential ramp is given by Eq. 63, where V_0 denotes its step height and $\epsilon > 0$ the slope width.

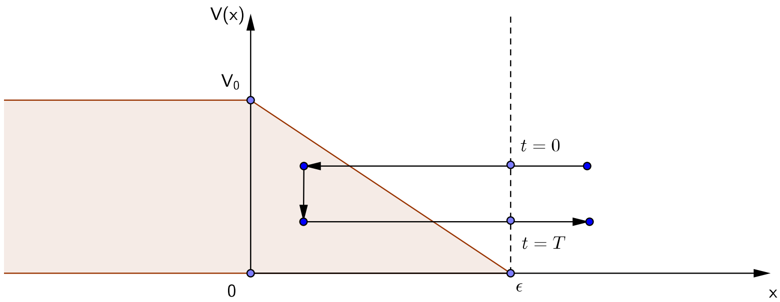

Figure 5: Potential ramp of height $V_0 > 0$ and reflected particle (Case I).

$$V(x) = \begin{cases} V_0 & \text{if } x < 0 \\ V_0(1 - \frac{x}{\epsilon}) & \text{if } 0 < x < \epsilon \\ 0 & \text{if } x > \epsilon \end{cases} \tag{63}$$

Consequently, the associated external force is determined by Eq. 64, as a function of the position of the particle.

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x} = \begin{cases} \frac{V_0}{\epsilon} & \text{if } 0 < x < \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

Since this force is not a delta function, a glance at the calculation in § 3.1.2 confirms the usual condition that the particle's acceleration is a continuous function of time.

4 Except for Case III, the particle is taken to have a negative incident velocity.

The potential ramp naturally defines three regions in space, in which the equation of motion has to be solved separately. It is a convenient choice to define $t = 0$ as the moment when the particle passes $x = \epsilon$ and enters Region 2.

Region 1: $x > \epsilon$

Region 2: $0 < x < \epsilon$

Region 3: $x < 0$

The equation of motion can be written as

$$a_i(t) = \begin{cases} \tau_0 \dot{a}_i(t) & \text{in Regions 1 and 3 } (i \in \{1,3\}) \\ \tau_0 \dot{a}_i(t) + \frac{V_0}{\epsilon m} & \text{in Region 2 } (i = 2). \end{cases} \tag{65}$$

Hence, the acceleration will have the general form

$$a_i(t) = \begin{cases} A_i \mathrm{e}^{t/\tau_0} & \text{in Regions 1 and 3 } (i \in \{1,3\}) \\ \tau_0 A_i \mathrm{e}^{t/\tau_0} + \frac{V_0}{\epsilon m} & \text{in Region 2 } (i = 2). \end{cases} \tag{66}$$

At this point, it becomes possible to distinguish between four physically different situations. Adopting the conventions in Ref. [2], Cases I and II correspond to reflections of the incident particle, the only difference is the location of the particle's turning point. In Case I, the particle turns back in the area of the slope (Region 2), whereas in Case II, it manages to penetrate the plateau and the turning point is located in Region 3. Case III will refer to a particle with a positive incident velocity, which means that it originates from the plateau on the left-hand side. The possibility of a particle, coming from the right-hand side, that has enough energy to overcome the potential ramp and emerge with a negative final velocity will be considered as Case IV. Details concerning the four cases are given in Sect. A.7, together with the solutions for a non-radiating particle.

3.2.2 *Relating Initial and Final Velocities*

In order to compare the solutions of the modified equation of motion and the non-radiative solutions for the problem of scattering on a potential ramp, it is instructive to construct a plot relating the initial and final particle velocities. From the solution method of the different cases, relying on the asymptotic condition to exclude runaway solutions and applying the boundary conditions backwards in time, it becomes clear that it is a natural choice to assign the horizontal axis to the final particle velocity v_f. Indeed, each of the treated cases involves one or more time parameters linked to the time the particle spends in each region, and one can find a nonlinear equation or system of equations which enables the determination of the relevant time parameter(s) from a given value of v_f. The corresponding value of the particle's initial velocity v_i follows from an expression relating v_i and v_f, which depends on the obtained time parameter(s).

In practice, it is not a trivial numerical[5] task to solve the nonlinear equations for the time parameters, taking into account additional boundary conditions such as $T > 0$ in Cases I and IV, or $0 < T_1 < T_2 < T$ in Case II. Furthermore, aside from some physical intuition and basic requirements such as $v_f > 0$ in Cases I, II and III, a priori one has no knowledge of the exact range of v_f values for which solutions exist.

5 To solve a system of n equations of the form $f_i(x_1, x_2, \ldots, x_n) = 0$ for $i \in \{1, 2, \ldots, n\}$, one can search for the values of the parameters $x_1, x_2, \ldots, x_n$ that agree with the boundary conditions and simultaneously minimise the expression $\sum_{i=1}^{n} f_i^2$, not letting its value exceed an infinitesimal tolerance ε. This way, one takes into account that the solutions found numerically for such complex equations are hardly ever exact.

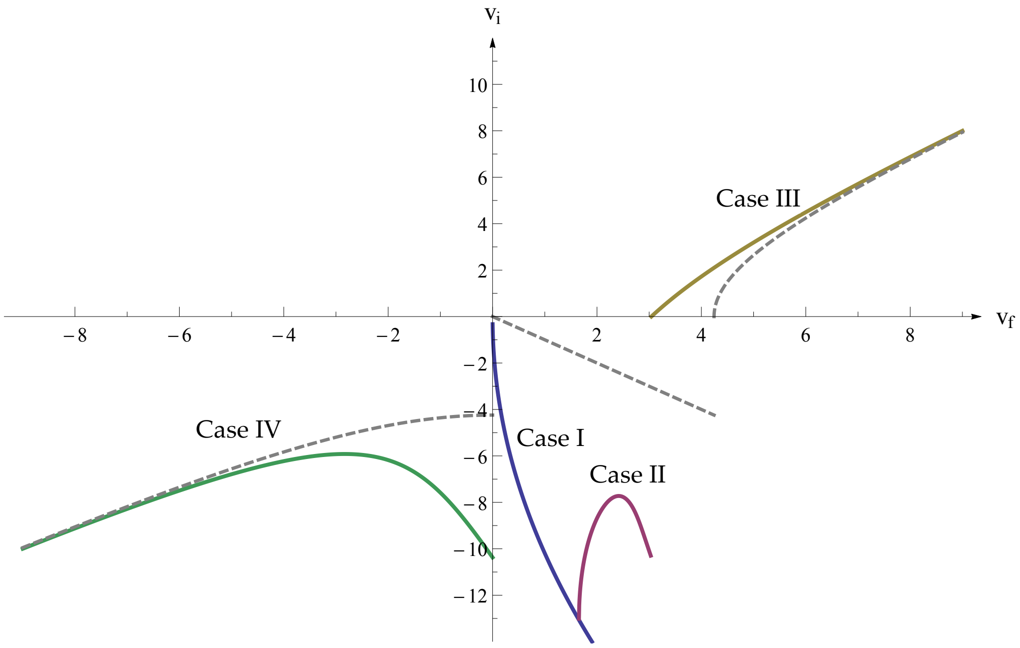

Figure 6: Plot of the initial velocity v_i versus the final velocity v_f, for the solution of the modified equation of motion in the context of a potential ramp with parameters $V_0 = 9$ and $\epsilon = 0.5$. The dashed lines pertain to the classical solution neglecting the radiation reaction.

Continuing in units where the particle mass m and the characteristic time τ_0 equal unity, the case-specific relations between the initial and final particle velocities yield the full lines in Fig. 6 for a potential ramp characterised by a slope width $\epsilon = 0.5$ and a step height $V_0 = 9$. The dashed lines show the behaviour of a non-radiating particle, obtained in § A.7.5. The combined curves for Cases I and II reduce to a straight line in the non-radiative case, as noted above, as well as in the limit of a very gentle slope ($\epsilon \to \infty$). This was observed in Ref. [2] and can be understood by noticing that a gentle slope does not give rise to strong variations in the particle acceleration[6], rendering a negligible $\dot{a}$ term in the modified equation of motion (Eq. 65). There is a small overlap of v_f values, where Case I and Case II solutions appear to coexist, meaning that different initial velocities can result in the same final velocity. This point was not noted in Ref. [2], despite following from the same numerical methodology leading to an otherwise identical plot[7]. The numerical solutions for the time parameter in the tail of the Case I curve fulfil their defining equation (Eq. 143) excellently[8], leaving them beyond doubt.

If the radiation reaction is neglected or negligible ($\epsilon \to \infty$), reflection only takes place for initial velocities up to $\sqrt{2V_0}$ in absolute value, beyond which transmission occurs (dashed Case IV curve). In other words, an initial kinetic energy exceeding V_0 is required to overcome the barrier in the non-radiative case, whereas this threshold increases to $2V_0$ if the radiation energy loss is not negligible. Indeed, from the plot one can infer that $v_i \lesssim -\sqrt{4V_0} = -6$ is required if the slope is steep. Since the high potential plateau has an infinite length, tunnelling solutions for Case IV are not present. However, for a sloped barrier with finite length, calculations analogous to those in Sect. 3.1 can show their existence for quantum-scale barrier lengths (Ref. [2]).

6 In all of the considered cases, the A_i coefficient of the particle acceleration (Eq. 66) is proportional to $V_0/(\epsilon m)$ and vanishes as the slope width tends to infinity.

7 However, Ref. [2] provides no details about their solution methods and calculations for the different cases under consideration.

8 Typically, their substitution in the minimised expression yields a value of the order of 10^{-30}, way below the allowed tolerance.

Furthermore, it is visible in Fig. 6 that the final velocity in the non-radiative Case III exceeds $\sqrt{2V_0}$, a situation approached in the case of a gentle slope. However, a steeper slope entails larger acceleration and energy loss to radiation, so that the final velocity (exceeding $\sqrt{V_0}$) is lower: the full Case III curve is located to the left of the dashed one. It is not possible that particles with a final velocity exceeding $\sqrt{V_0}$ have undergone a reflection, they must have come over from the plateau on the left. This transition between Cases II and III occurs smoothly; no forbidden final velocities emerge.

3.2.3 *Non-Uniqueness of Solutions*

The most striking feature emerging from Fig. 6 is probably the non-uniqueness of physical solutions of the modified equation of motion, related to the discussion in § 3.1.6. The classical bijective correspondence between a given initial velocity value and the resulting final velocity, is completely lost when the radiation reaction is taken into account, even for a continuous potential. Essentially, this indicates the failure of the asymptotic condition to replace the specification of the initial particle acceleration.

In the most extreme case for the relatively steep potential ramp considered here, a total of five possibilities arises. On the one hand, a particle incident on the right-hand side of the ramp with an initial velocity $v_i = -9$, can be transmitted with a low final velocity, indicating significant energy loss while trying to overcome the ramp, or it could end up travelling on and hardly losing energy, since the curve for Case IV intersects the horizontal $v_i = -9$ line twice. As it can be shown that the vertical intercept of the Case IV curve moves downward for steeper ramps, one sees how this non-uniqueness persists in the case of a potential barrier, as mentioned in § 3.1.6. On the other hand, the particle could be reflected and have any of three allowed final velocities. This clearly shows an unexpected indeterminacy arising from classical electrodynamics.

Yet it is reassuring that, given any initial velocity, non-runaway solutions exist for any realistic potential ramp with $\epsilon \neq 0$, even with a steep slope like this one. In Ref. [2], it is noted that the branch pertaining to Case I moves towards the vertical axis if the slope width tends to zero. This clarifies the unphysical vanishing of solutions for an interval of small incident velocities in the limiting case where $\epsilon = 0$, which does not occur as long as the potential is not discontinuous, and the involved force is not infinite. In addition, it can be remarked that Denef et al. [2] have confirmed that the non-uniqueness persists for smoother, analytic potentials, and also when relying on the relativistic equation of motion including the radiation reaction.

Even though the discrete ambiguity in the particle's final velocity is reminiscent of quantum mechanical scattering, the similarity is only superficial. There are no classical analogues for the quantum mechanical reflection and transmission coefficients, and a comment in Ref. [1] asserts that, quantitatively, several orders of magnitude separate the obtained classical energy range where the indeterminacy occurs and its quantum mechanical analogue for an electron. In view of this point, a quantum electrodynamical treatment of the scattering problem considering the radiation reaction would clearly be illuminating.

CLOSING REMARKS

The fact that accelerating charges emit electromagnetic radiation is found to influence their dynamics by adding a radiation reaction force to Newton's second law. The Abraham-Lorentz formula, which can be derived invoking energy and momentum conservation, describes this recoil force for particles at low speeds. The modified equation of motion depends on the acceleration as well as the time derivative of the acceleration. Its implications contradict common sense: it entails that a charge will either exhibit an acausal response to an external force in the future (i. e. preacceleration), or accelerate without bound after the force ceases to act (i. e. runaway solutions). The Lorentz-Dirac equation, the relativistic description of point-charge motion, admits runaway solutions too, where a particle's rapidity is allowed to increase exponentially.

A careful assessment showed that the radiation reaction actually ensues from internal electromagnetic force imbalances within the structure of an accelerating charged particle. In a sense, charged particles feel their own electromagnetic fields. However, an extended particle model is required for a classical examination of the self-interaction, since the fields of a point charge are infinite at its own location. Electrons and other elementary particles, though, are not known to possess internal structure.

Therefore, it is expedient to consider a simple extended charge distribution and investigate its zero-size limit. Upon expanding the self-force on a charge dumbbell as a power series in its size parameter, one notices a term which diverges in the point-charge limit. Luckily, it happens to have exactly the right form to be absorbed into a factor representing the total inertia of a point charge. In this classical mass renormalisation process, the electromagnetic inertia contribution is called the self-force-derived mass. Subtleties involving the geometry of the configuration and the definitions of electromagnetic energy and momentum yield apparent paradoxes between different kinds of electromagnetic mass.

When solving the modified equation of motion for point charges impinging on a rectangular potential barrier, a class of tunnelling solutions was found, which allow a charge to go through the barrier even when its kinetic energy is below the step height. However, classical tunnelling is only expected to take place on scales where quantum mechanical effects prevail. Another peculiar feature is the absence of solutions for a range of incident velocities, which can be traced back to the unphysical sharpness of a rectangular barrier. The problem vanishes when considering a continuous potential ramp, as illustrated in a plot relating initial and final particle velocities. Yet the general non-uniqueness of the solutions does not disappear: given the initial particle velocity, the condition excluding the unobserved runaway solutions fails to determine the final velocity uniquely.

The intricate issues concerning self-force and electromagnetic mass touch on the fundamental concepts underlying the theory. Together with the tunnelling phenomenon, they indicate complications that appear unresolvable within the framework that brings them about. Classical electrodynamics eventually breaks down when one ventures beyond its domain of validity.

Part III

ATTACHMENTS

APPENDIX

A

A.1 ACAUSAL PREACCELERATION: RECTANGULAR FORCE

The final result for the particle velocities in the three time intervals, defined in § 1.2.3, can be written as

$$v_1(t) = \frac{F\tau_0}{m}(1 - e^{-T/\tau_0})e^{t/\tau_0} \tag{67}$$

$$v_2(t) = \frac{F}{m}(t + \tau_0(1 - e^{(t-T)/\tau_0})) \tag{68}$$

$$v_3(t) = \frac{FT}{m}. \tag{69}$$

A.2 ENERGY CONSERVATION

A.2.1 *Rectangular Force*

The objective of the following calculations is to show that the sum of the total work W_{ext} done on a particle by the external force $F_{\text{ext}}(t) = F(\theta(t) - \theta(t-T))$ and the radiated energy $W_{\text{rad}} < 0$ equals the final kinetic kinetic energy E^f_{kin}. Using the results from § 1.2.3, one can find

$$\begin{aligned} W_{\text{ext}} &= \int F_{\text{ext}}\,dx = \int F_{\text{ext}}\frac{dx}{dt}\,dt \\ &= \int_0^T F v_2(t)\,dt = \frac{F^2}{m}\int_0^T (t + \tau_0(1 - e^{(t-T)/\tau_0}))\,dt \\ &= \frac{F^2}{m}\left(\frac{T^2}{2} + \tau_0 T - \tau_0^2 + \tau_0^2 e^{-T/\tau_0}\right) \end{aligned} \tag{70}$$

$$E^f_{\text{kin}} = \frac{1}{2}mv_3^2 = \frac{1}{2}\frac{F^2T^2}{m} \tag{71}$$

$$\begin{aligned} W_{\text{rad}} &= \int F_{\text{rad}}\,dx = \int_{-\infty}^{+\infty} m\tau_0 \dot{a} v\,dt \\ &= m\tau_0\left(\int_{-\infty}^0 \dot{a}_1 v_1\,dt + \int_0^T \dot{a}_2 v_2\,dt + \int_T^{+\infty} \dot{a}_3 v_3\,dt\right) \\ &= m\tau_0 \int_{-\infty}^0 \left(\frac{F}{m}(1 - e^{-T/\tau_0})\right)^2 \frac{1}{\tau_0} e^{t/\tau_0}\tau_0 e^{t/\tau_0}\,dt \\ &\quad + m\tau_0 \int_0^T \left(-\frac{F}{m\tau_0}\right) e^{(t-T)/\tau_0}\frac{F}{m}(t + \tau_0 - \tau_0 e^{(t-T)/\tau_0})\,dt + 0 \\ &= m\tau_0\left(\frac{F^2\tau_0}{2m^2}(1 - e^{-T/\tau_0})^2 - \frac{F^2}{m^2}\left(T - \frac{\tau_0}{2}(1 - e^{-2T/\tau_0})\right)\right) \\ &= \frac{F^2\tau_0}{m}(\tau_0 - T - \tau_0 e^{-T/\tau_0}). \end{aligned} \tag{72}$$

Consequently, $W_{\text{ext}} + W_{\text{rad}} = E^f_{\text{kin}}$, which shows that energy is conserved.

A.2.2 *Delta Force*

Calculations analogous to those in § A.2.1 shows that

$$W_{\text{ext}} = kv(0) = \frac{k^2}{m} \tag{73}$$

$$E^f_{\text{kin}} = \frac{k^2}{2m} \tag{74}$$

$$W_{\text{rad}} = -\frac{k^2}{2m} \tag{75}$$

Again, energy is conserved: $W_{\text{ext}} + W_{\text{rad}} = E^f_{\text{kin}}$.

A.2.3 *Radiation Reaction Force on a Transverse Dumbbell*

The radiation reaction force for a dumbbell in one-dimensional, transverse motion (§ 2.1.2) was found to be given by Eq. 14. It is straightforward to show that this result is compatible with energy conservation. From Liénard's formula (Eq. 24), using the fact that $\boldsymbol{v} \times \boldsymbol{a} = \boldsymbol{0}$, the radiated power is

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}. \tag{76}$$

When integrating by parts over a time interval between two identical system states, one finds

$$\int_{t_1}^{t_2} \gamma^4 \dot{a} v \, \mathrm{d}t = \left[\gamma^4 va\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} a \frac{\mathrm{d}}{\mathrm{d}t}(\gamma^4 v) \, \mathrm{d}t = -\int_{t_1}^{t_2} a \left(\gamma^4 a + 4\gamma^3 \left(\frac{va}{c^2}\gamma^3\right) v\right) \, \mathrm{d}t \tag{77}$$

Hence, the time-averaged energy loss due to the radiation reaction force equals the irreversible loss of energy to radiation.

$$\begin{aligned}
\int_{t_1}^{t_2} F_{\text{rad}} v \, \mathrm{d}t &= \frac{\mu_0 q^2 a^2}{6\pi c} \int_{t_1}^{t_2} \gamma^4 \left(\dot{a} + \frac{3va^2\gamma^2}{c^2}\right) v \, \mathrm{d}t \\
&= \frac{\mu_0 q^2 a^2}{6\pi c} \int_{t_1}^{t_2} \left(-a\left(\gamma^4 a + 4\gamma^6 \frac{v^2 a}{c^2}\right) + \frac{3v^2a^2\gamma^6}{c^2}\right) \mathrm{d}t \\
&= \frac{\mu_0 q^2 a^2}{6\pi c} \int_{t_1}^{t_2} \left(-\gamma^6 a^2 \left(1 + 3\frac{v^2}{c^2}\right) + \frac{3v^2a^2\gamma^6}{c^2}\right) \mathrm{d}t \\
&= -\int_{t_1}^{t_2} P \, \mathrm{d}t
\end{aligned} \tag{78}$$

A.3 INTEGRAL FORMULATION OF THE EQUATION OF MOTION

The goal of the following calculations is to derive an alternative formulation for the modified equation of motion, as indicated in Sect. 1.3. Rewriting Eq. 7 as in Eq. 79 defines a linear operator $(\mathrm{d}/\mathrm{d}t - 1/\tau_0)$, and a source term $Q(t)$ which, in a sense, makes the differential equation inhomogeneous.

$$m\tau_0 \dot{a} - ma = -F_{\text{ext}} \implies \left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{\tau_0}\right) a(t) = -\frac{1}{m\tau_0} F_{\text{ext}}(t) = Q(t) \tag{79}$$

The objective is to find a Green's function $G(t)$, such that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{\tau_0}\right) G(t) = \delta(t). \tag{80}$$

When the Green's function is found, the acceleration can be obtained as the convolution of the Green's function and the source term.

$$a(t) = \int_{-\infty}^{+\infty} G(t-t')Q(t')\,\mathrm{d}t' \tag{81}$$

This is readily checked by substituting Eq. 81 into Eq. 79 and using the fact that t and t' are independent, whence $\mathrm{d}Q(t')/\mathrm{d}t = (\mathrm{d}Q(t')/\mathrm{d}t')(\mathrm{d}t'/\mathrm{d}t) = 0$.

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{\tau_0}\right)\int_{-\infty}^{+\infty} G(t-t')Q(t')\,\mathrm{d}t' &= \int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}t}G(t-t')\right)Q(t')\,\mathrm{d}t' \\ &\quad - \frac{1}{\tau_0}\int_{-\infty}^{+\infty} G(t-t')Q(t')\,\mathrm{d}t'\end{aligned} \tag{82}$$

Now, the definition of $G(t)$ entails

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t-t') = \frac{\mathrm{d}G(t-t')}{\mathrm{d}(t-t')}\frac{\mathrm{d}(t-t')}{\mathrm{d}t} = \delta(t-t') + \frac{1}{\tau_0}G(t-t'). \tag{83}$$

Hence, Eq. 82 shows that the acceleration given in Eq. 81 solves the equation of motion (Eq. 79).

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{\tau_0}\right)\int_{-\infty}^{+\infty} G(t-t')Q(t')\,\mathrm{d}t' = \int_{-\infty}^{+\infty}\delta(t-t')Q(t')\,\mathrm{d}t' = Q(t) \tag{84}$$

In order to solve Eq. 80, one can take its Fourier transform and in that way, switch from a differential equation to an algebraic one. Writing $g(\omega)$ for the Fourier transform of $G(t)$, one has

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega. \tag{85}$$

Exploiting the requirement in Eq. 80 leads to

$$\delta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\frac{\partial}{\partial t} - \frac{1}{\tau_0}\right)\mathrm{e}^{i\omega t}g(\omega)\,\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(i\omega - \frac{1}{\tau_0}\right)\mathrm{e}^{i\omega t}g(\omega)\,\mathrm{d}\omega. \tag{86}$$

Fourier's inversion theorem prescribes

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \iff F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t. \tag{87}$$

Applying it to $f(t) = \delta(t)$, one finds $F(\omega) = 1/\sqrt{2\pi}$ and

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{i\omega t}\,\mathrm{d}\omega. \tag{88}$$

Combining the terms in Eqs. 86 and 88 into one integral, equal to zero, and by Plancherel's theorem, one can find the following necessary and sufficient condition, leading to an expression for $G(t)$:

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{i\omega - \frac{1}{\tau_0}} \quad \overset{\text{Eq. 85}}{\Longrightarrow} \quad G(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{i\omega t}}{i\omega - \frac{1}{\tau_0}}\,\mathrm{d}\omega. \tag{89}$$

Writing $\omega_0 = -i/\tau_0$ for the zero of the denominator (Eq. 89), one observes that the integral is equal to

$$G(t) = \frac{-i}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{i\omega t}}{\omega - \omega_0}\,\mathrm{d}\omega. \tag{90}$$

Cauchy's integral formula enables one to evaluate this integral along the real axis. For a holomorphic function $f : U \to \mathbb{C}$ on an open subset U of the complex plane and a point z_0 inside a simple, closed contour $\mathcal{C}$ lying within U, the formula states that

$$\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\,\mathrm{d}z = 2\pi i f(z_0). \tag{91}$$

Two cases are to be considered. If $t > 0$, the contour containing the real axis is closed above, by adding a semicircle at infinity. The numerator in Eq. 90, $\mathrm{e}^{i\omega t}$ tends to zero as the imaginary part of ω goes to $+\infty$, and it can be shown that the semicircle does not contribute to the contour integral, leaving only the value of the integral along the real axis. However, the chosen contour is located in the upper half-plane, and does not contain the singularity ω_0. In this case, when the integrand is analytic at all points on and interior to the contour, the Cauchy-Goursat theorem implies that $G(t) = 0$ if $t > 0$. For $t < 0$, the situation is inverted: now, the contour is closed below, again by a semicircle at infinity, since $\mathrm{e}^{i\omega t}$ goes to zero as the imaginary part $\mathrm{Im}(\omega)$ tends to $-\infty$. This contour encloses ω_0, the singularity located on the negative part of the imaginary axis. Hence, adding a minus sign to correct for the clockwise orientation of the chosen integration path, Cauchy's formula yields

$$G(t) = -\frac{i}{2\pi}(-2\pi i)\left[\mathrm{e}^{i\omega t}\right]_{\omega=\omega_0} = -\mathrm{e}^{t/\tau_0} \quad \text{if } t < 0. \tag{92}$$

Finally, putting the Green's function back into Eq. 81, one retrieves the integral formulation of the modified equation of motion:

$$a(t) = 0 - \int_t^{+\infty} \mathrm{e}^{(t-t')/\tau_0} Q(t')\,\mathrm{d}t' = \frac{1}{m\tau_0}\int_t^{+\infty} \mathrm{e}^{(t-t')/\tau_0} F_{\text{ext}}(t')\,\mathrm{d}t' \tag{93}$$

In order to calculate the integral, it is noted that $G(t - t')$ is non-zero only when $t - t' < 0$, or equivalently, for $t' > t$.

Applying Eq. 93 to the examples in § 1.2.3 yields identical results. Using the notation introduced in that section, the calculations using Eq. 93 go as follows.

RECTANGULAR FORCE For $t < 0$:

$$\begin{aligned} a_1(t) &= \frac{1}{m\tau_0}\left(\int_t^0 \mathrm{e}^{(t-t')/\tau_0}\cdot 0\,\mathrm{d}t' + \int_0^T \mathrm{e}^{(t-t')/\tau_0} F\,\mathrm{d}t' + \int_T^{+\infty} \mathrm{e}^{(t-t')/\tau_0}\cdot 0\,\mathrm{d}t'\right) \\ &= \frac{F}{m}(1 - \mathrm{e}^{-T/\tau_0})\mathrm{e}^{t/\tau_0}. \end{aligned} \tag{94}$$

For $0 < t < T$:

$$\begin{aligned} a_2(t) &= \frac{1}{m\tau_0}\left(\int_t^T \mathrm{e}^{(t-t')/\tau_0} F\,\mathrm{d}t' + \int_T^{+\infty} \mathrm{e}^{(t-t')/\tau_0}\cdot 0\,\mathrm{d}t'\right) \\ &= \frac{F}{m}(1 - \mathrm{e}^{(t-T)/\tau_0}). \end{aligned} \tag{95}$$

For $t > T$:

$$a_3(t) = \frac{1}{m\tau_0}\int_t^{+\infty} \mathrm{e}^{(t-t')/\tau_0}\cdot 0\,\mathrm{d}t' = 0. \tag{96}$$

DELTA FORCE For $t < 0$:

$$a_1(t) = \frac{1}{m\tau_0}\int_t^{+\infty} \mathrm{e}^{(t-t')/\tau_0}\cdot k\delta(t')\,\mathrm{d}t' = \frac{k}{m\tau_0}\mathrm{e}^{t/\tau_0}. \tag{97}$$

For $t > 0$:

$$a_2(t) = \frac{1}{m\tau_0}\int_t^{+\infty} e^{(t-t')/\tau_0} \cdot k\delta(t')\,dt' = 0. \tag{98}$$

However, it is notable that the integral formulation appears to have rid the equation of motion of runaways. If the external force is zero after some time T, it immediately follows from the integral equation that $a(t) = 0$ for $t > T$, since $F_{\text{ext}}(t') = 0$ for $t' \geq t > T$. The automatic runaway exclusion is due to the choice of the Green's function, which is determined up to a solution $G_0(t)$ of the homogeneous version of Eq. 80, where A is an integration constant.

$$\left(\frac{d}{dt} - \frac{1}{\tau_0}\right) G_0(t) = 0 \quad \Longrightarrow \quad G_0(t) = Ae^{t/\tau_0} \tag{99}$$

Changing $G \to G + G_0$ in the calculation of Eq. 93 adds a term which restores runaways, unless the external force vanishes completely:

$$a(t) = \frac{1}{m\tau_0}\int_t^{+\infty} e^{(t-t')/\tau_0} F_{\text{ext}}(t')\,dt' - \frac{A}{m\tau_0}\int_{-\infty}^{+\infty} e^{(t-t')/\tau_0} F_{\text{ext}}(t')\,dt'. \tag{100}$$

Either way, the modified equation of motion becomes nonlocal in time.

A.4 ELECTROMAGNETIC SELF-FORCE ON A TRANSVERSE DUMBBELL

In the following calculation, transverse motion of the dumbbell is considered, along the x axis, with an arbitrary velocity $\boldsymbol{v}(t) = v(t)\hat{\boldsymbol{x}}$ and acceleration $\boldsymbol{a}(t) = a(t)\hat{\boldsymbol{x}}$. As shown in Fig. 7, $x(t_r)$ refers to the retarded position of the dumbbell, whereas $x(t)$ is its present position. The electric field $\boldsymbol{E}_1$ at charge (1) due to charge (2) is calculated using Eq. 1, with $\boldsymbol{\mathscr{r}} = l\hat{\boldsymbol{x}} + d\hat{\boldsymbol{y}}$ (whence $\mathscr{r} = \sqrt{l^2 + d^2}$) denoting the vector from the retarded position of dumbbell end (2) to the present position of end (1). With $\boldsymbol{v}$, $\boldsymbol{u}$ and $\boldsymbol{a}$ evaluated at the retarded time $t_r = t - \mathscr{r}/c$, it is straightforward to show the following equalities.

$$\boldsymbol{u} = \left(\frac{cl}{\mathscr{r}} - v\right)\hat{\boldsymbol{x}} + \frac{cd}{\mathscr{r}}\hat{\boldsymbol{y}}\,; \quad \boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \boldsymbol{v} = c\mathscr{r} - lv\,; \quad \boldsymbol{\mathscr{r}} \cdot \boldsymbol{a} = la \tag{101}$$

$$\begin{aligned} \boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{a}) &= \boldsymbol{u}(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}) \\ &= \left[la\left(\frac{cl}{\mathscr{r}} - v\right) - a(c\mathscr{r} - lv)\right]\hat{\boldsymbol{x}} + \left[la\frac{cd}{\mathscr{r}}\right]\hat{\boldsymbol{y}} \end{aligned} \tag{102}$$

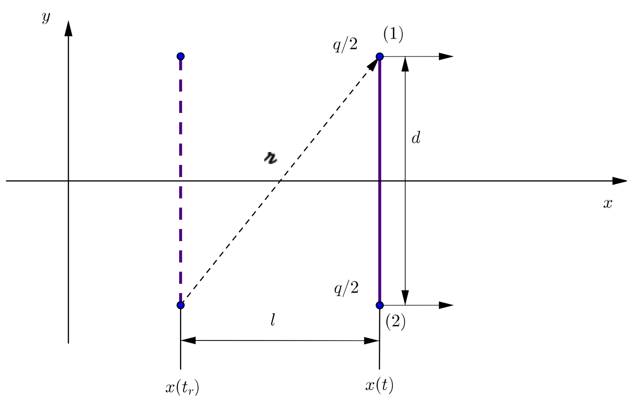

Figure 7: Dumbbell model for the radiation reaction.

In the calculation of the electric field $\boldsymbol{E}_2$ at (2) due to (1), $\boldsymbol{\mathscr{r}} = l\hat{\boldsymbol{x}} - d\hat{\boldsymbol{y}}$ and the y component of $\boldsymbol{u}$ gets a minus sign, but $\mathscr{r}$ and the other expressions in Eq. 101

stay the same. Hence, the y component of $\boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{a})$ gets a minus sign as well, and from Eq. 1 it follows that the y components of the Lorentz forces due to the electrical fields at (1) and (2) cancel out. Similarly, the forces due to the magnetic fields $\boldsymbol{B}_1$ at (1) and $\boldsymbol{B}_2$ at (2) are equal in magnitude, but opposite. This can be deduced from Eq. 2: since the second term in $\boldsymbol{E}$ (Eq. 1) disappears in the cross product, a term proportional to $\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{u}$ is left. The scalar front factor is identical for $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, but $\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{u} = [(l/\mathscr{r})(cd/\mathscr{r}) - (d/\mathscr{r})(cl/\mathscr{r} - v)]\,\hat{\boldsymbol{z}}$ when calculating $\boldsymbol{B}_1$ and the exact opposite for $\boldsymbol{B}_2$, so the magnetic forces cancel.

Since only the net force on the dumbbell is of interest here, it suffices to work out $[\boldsymbol{E}_1]_x$, the x component of $\boldsymbol{E}_1$, which equals $[\boldsymbol{E}_2]_x$ for symmetry reasons.

$$\begin{aligned}
[\boldsymbol{E}_1]_x &= \frac{q/2}{4\pi\epsilon_0} \frac{\mathscr{r}}{(c\mathscr{r} - lv)^3} \left[(c^2 - v^2 + la) \left(\frac{cl}{\mathscr{r}} - v \right) - (c\mathscr{r} - lv)a \right] \\
&= \frac{q}{8\pi\epsilon_0} \frac{1}{(c\mathscr{r} - lv)^3} \left[(c^2 - v^2)(cl - v\mathscr{r}) - cd^2 a \right]
\end{aligned} \tag{103}$$

It follows that the total self-force is given by Eq. 104.

$$\begin{aligned}
\boldsymbol{F}_{\text{self}} &= \frac{q}{2}(\boldsymbol{E}_1 + \boldsymbol{v} \times \boldsymbol{B}_1 + \boldsymbol{E}_2 + \boldsymbol{v} \times \boldsymbol{B}_2) = \frac{q}{2}([\boldsymbol{E}_1]_x + [\boldsymbol{E}_2]_x)\hat{\boldsymbol{x}} = q\,[\boldsymbol{E}_1]_x\,\hat{\boldsymbol{x}} \\
&= \frac{q^2}{8\pi\epsilon_0} \frac{1}{(c\mathscr{r} - lv)^3} \left[(c^2 - v^2)(cl - v\mathscr{r}) - cd^2 a \right]
\end{aligned} \tag{104}$$

This result is exact, but it is useful to calculate its expansion in powers of the particle size d in order to investigate the point-charge limit. In that limit, $d \to 0$ and all positive powers of d vanish.

The expansion of the dumbbell position as a Taylor series around the retarded time, yields Eqs. 105 and 106, where $T = t - t_r$.

$$x(t) = x(t_r) + \dot{x}(t_r)T + \frac{1}{2}\ddot{x}(t_r)T^2 + \frac{1}{3!}\dddot{x}(t_r)T^3 + \ldots \tag{105}$$

$$l = x(t) - x(t_r) = v(t_r)T + \frac{1}{2}a(t_r)T^2 + \frac{1}{6}\dot{a}(t_r)T^3 + \ldots \tag{106}$$

The definition of the retarded time implies that $T^2 = (l^2 + d^2)/c^2$, whence

$$\begin{aligned}
c^2T^2 = d^2 + l^2 &= d^2 + \left(vT + \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \ldots \right)^2 \\
&= d^2 + v^2T^2 + vaT^3 + \left(\frac{1}{4}a^2 + \frac{1}{3}v\dot{a} \right) T^4 + \ldots \quad .
\end{aligned} \tag{107}$$

Writing $\gamma = \left(1 - v(t_r)^2/c^2\right)^{-1/2}$, one obtains Eq. 108, which determines d as a function of T.

$$\frac{c^2T^2}{\gamma^2} = c^2T^2 \left(1 - \frac{v^2}{c^2} \right) = d^2 + vaT^3 + \left(\frac{1}{4}a^2 + \frac{1}{3}v\dot{a} \right) T^4 + \ldots \quad . \tag{108}$$

By reverting the series expressing d in terms of T, one can obtain T as a function of d. This task is accomplished by substituting

$$T = \frac{\gamma d}{c}(1 + C_1 d + C_2 d^2 + C_3 d^3 + \ldots) \tag{109}$$

into both sides of Eq. 108, which allows for the determination of the C_i constants when comparing like powers of d. The result can be written as

$$T = \frac{\gamma d}{c} \left(1 + \left[\frac{1}{2} va \frac{\gamma^3}{c^3} \right] d + \left[\frac{1}{2} \frac{\gamma^4}{c^4} \left(a^2\gamma^2 \left(\frac{v^2}{c^2} + \frac{1}{4} \right) + \frac{1}{3} v\dot{a} \right) \right] d^2 + \ldots \right). \tag{110}$$

Substituting this power series for T in Eq. 106 leads to

$$l = \left[\frac{v\gamma}{c}\right] d + \left[\frac{a\gamma^4}{2c^2}\right] d^2 + \left[\frac{\gamma^5}{2c^5}\left(\frac{5}{4}va^2\gamma^2 + \frac{1}{3}c^2\dot{a}\right)\right] d^3 + \dots \quad . \tag{111}$$

Using the fact that $\mathscr{r} = cT$, one can expand the following expressions, which appear in Eq. 104.

$$cl - v\mathscr{r} = \left[\frac{a\gamma^2}{2c}\right] d^2 + \left[\frac{\gamma^3}{2c^2}\left(\frac{\gamma^2 va^2}{c^2} + \frac{1}{3}\dot{a}\right)\right] d^3 + \dots \tag{112}$$

$$c\mathscr{r} - lv = \left[\frac{c}{\gamma}\right] d + \left[\frac{a^2\gamma^5}{8c^3}\right] d^3 + \dots \tag{113}$$

$$(c\mathscr{r} - lv)^{-3} = \left(\frac{\gamma}{cd}\right)^3 \left(1 - \frac{3a^2\gamma^6}{8c^4} d^2 + \dots\right) \tag{114}$$

Combining all these expansions, Eq. 104 eventually yields

$$\boldsymbol{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\gamma^3 a}{4c^2 d} + \frac{\gamma^4}{4c^3}\left(\frac{\gamma^2 va^2}{c^2} + \frac{1}{3}\dot{a}\right) + (\dots)d + \dots\right]\hat{\boldsymbol{x}}. \tag{115}$$

Essentially the same result is obtained by Lyle [6, p. 109], in a slightly less transparent way and with a small typographical error in the γ^4 term.

A.5 RELATIVISTIC RUNAWAY SOLUTIONS

Rapidity is defined as $\varphi = \operatorname{arctanh}(v/c)$, where v is the magnitude of the particle's velocity. In the case of one-dimensional motion, it is not hard to show that this implies the following equalities, noting that $\varphi = \varphi(\tau)$:

$$v = c\tanh(\varphi) \qquad \gamma = \cosh(\varphi) \qquad \eta = \gamma v = c\sinh(\varphi) \tag{116}$$

$$\alpha = c\cosh(\varphi)\dot{\varphi} \qquad \dot{\alpha} = c(\sinh(\varphi)\dot{\varphi}^2 + \cosh(\varphi)\ddot{\varphi}) \tag{117}$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{c}{\gamma}\frac{\dot{\varphi}}{\cosh^2(\varphi)} = \frac{c}{\cosh^3(\varphi)}\dot{\varphi}. \tag{118}$$

Using Eq. 18, one can find

$$\alpha^0 = \frac{\gamma^4}{c} va = \gamma^4 c \frac{\tanh(\varphi)}{\cosh^3(\varphi)}\dot{\varphi} = c\sinh(\varphi)\dot{\varphi} \tag{119}$$

$$\alpha^\nu \alpha_\nu = -(\alpha^0)^2 + \alpha^2 = -c^2\sinh^2(\varphi)\dot{\varphi}^2 + c^2\cosh^2(\varphi)\dot{\varphi}^2 = c^2\dot{\varphi}^2 \tag{120}$$

With these results, the equation of motion (Eq. 30) becomes

$$mc\cosh(\varphi)\dot{\varphi} = K_{\text{ext}} + m\tau_0\left(c(\sinh(\varphi)\dot{\varphi}^2 + \cosh(\varphi)\ddot{\varphi}) - \frac{1}{c^2}(c^2\dot{\varphi}^2)c\sinh(\varphi)\right). \tag{121}$$

From Eq. 21, it can be noted that $K_{\text{ext}} = \gamma F_{\text{ext}}$, where F_{ext} is the ordinary force from Newton's second law. Hence, upon dividing Eq. 121 by $mc\cosh(\varphi) = mc\gamma$, which is non-zero, one finds

$$\dot{\varphi} = \frac{1}{mc\gamma}K_{\text{ext}} + \tau_0\ddot{\varphi} = \frac{F_{\text{ext}}}{mc} + \tau_0\ddot{\varphi}. \tag{122}$$

Finally, the runaway phenomenon is recovered: even if no external force is exerted on the particle, Eq. 122 still admits rapidity solutions rising exponentially:

$$\ddot{\varphi} = \frac{1}{\tau_0}\dot{\varphi} \Rightarrow \varphi(\tau) = A + Be^{\tau/\tau_0}, \text{ where } A \text{ and } B \text{ are integration constants.} \tag{123}$$

A.6 THE POTENTIAL BARRIER

A.6.1 *Acceleration Discontinuity*

It is possible to show that the particle's acceleration $a(t)$ is discontinuous at the edges of a potential barrier (§ 3.1.2). Indeed, setting $t_0 = x^{-1}(0)$ or $t_0 = x^{-1}(L)$, the moment the particle passes one of the edges, one can calculate the acceleration boost $\Delta a = \lim_{\epsilon\to 0}(a(t_0+\epsilon) - a(t_0-\epsilon))$ as follows, making use of the equation of motion (Eq. 35) and the continuity of $v(t)$.

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{t_0-\epsilon}^{t_0+\epsilon} a(t)\mathrm{d}t &= \lim_{\epsilon\to 0}\left(v(t_0+\epsilon) - v(t_0-\epsilon)\right) = \Delta v = 0 \\
&= \lim_{\epsilon\to 0}\int_{x(t_0-\epsilon)}^{x(t_0+\epsilon)}\left(\tau_0\dot{a}(t) + \frac{F}{m}\right)\frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x \\
&= \tau_0 \cdot \lim_{\epsilon\to 0}(a(t_0+\epsilon) - a(t_0-\epsilon)) + \\
&\qquad \lim_{\epsilon\to 0}\frac{1}{m}\int_{x(t_0-\epsilon)}^{x(t_0+\epsilon)}\frac{V_0}{v}(-\delta(x) + \delta(x-L))\mathrm{d}x \\
&= \tau_0\Delta a \mp \frac{V_0}{mv(t_0)} \\
&\qquad (-\text{ for } t_0 = x^{-1}(0), + \text{ for } t_0 = x^{-1}(L))
\end{aligned}$$

This calculation entails

$$\Delta a(x=0) = \frac{+V_0}{m\tau_0 v(x^{-1}(0))} \quad \text{and} \quad \Delta a(x=L) = \frac{-V_0}{m\tau_0 v(x^{-1}(L))}, \tag{124}$$

implying that the particle's acceleration abruptly increases when the particle encounters the barrier and subsequently drops when leaving.

A.6.2 *Boundary Conditions*

If v_f denotes the particle's final velocity as $t \to \infty$ (implying $B_3 = v_f$) and the Region 3 runaway is excluded, one obtains C_3 from $L = x_3(t) = v_f T + C_3$, whence the solutions for Region 3 below.

$$a_3(t) = 0 \tag{125}$$

$$v_3(t) = v_f \tag{126}$$

$$x_3(t) = v_f(t-T) + L \tag{127}$$

Invoking the (dis)continuity conditions in $x = 0$ and $x = L$ gives the following equations.

$$a_2(0) - a_1(0) = \Delta a(x=0): \quad A_2 - A_1 = \frac{V_0}{m\tau_0 v(0)} \tag{128}$$

$$a_3(T) - a_2(T) = \Delta a(x=L): \quad 0 - A_2 e^{T/\tau_0} = \frac{-V_0}{m\tau_0 v(T)} \tag{129}$$

$$v_1(0) = v_2(0): \quad \tau_0 A_1 + B_1 = \tau_0 A_2 + B_2 \tag{130}$$

$$v_2(T) = v_3(T): \quad \tau_0 A_2 e^{T/\tau_0} + B_2 = v_f \tag{131}$$

$$x_1(0) = x_2(0) = 0: \quad \tau_0^2 A_1 + C_1 = \tau_0^2 A_2 + C_2 \tag{132}$$

$$x_2(T) = x_3(T) = L: \quad \tau_0^2 A_2 e^{T/\tau_0} + B_2 T + C_2 = L \tag{133}$$

The constants are determined by solving the equations backwards in time, in the following order:

$$129 \to 128 \to 131 \to 130 \to 133 \to 132.$$

$$A_2 = \frac{V_0}{m\tau_0 v_f} \mathrm{e}^{-T/\tau_0} \tag{134}$$

$$A_1 = A_2 - \frac{V_0}{m\tau_0 v(0)} = \frac{V_0}{m\tau_0}\left(\frac{\mathrm{e}^{-T/\tau_0}}{v_f} - \frac{1}{v(0)}\right) \tag{135}$$

$$B_2 = v_f - \tau_0 A_2 \mathrm{e}^{T/\tau_0} = v_f - \frac{V_0}{m v_f} \tag{136}$$

$$B_1 = \tau_0(A_2 - A_1) + B_2 = v_f + \frac{V_0}{m}\left(\frac{1}{v(0)} - \frac{1}{v_f}\right) \tag{137}$$

$$C_2 = L - \tau_0^2 A_2 \mathrm{e}^{T/\tau_0} - B_2 T = L - v_f T + \frac{V_0}{m v_f}(T - \tau_0) \tag{138}$$

$$C_1 = \tau_0^2(A_2 - A_1) + C_2 = L - v_f T + \frac{V_0}{m}\left(\frac{\tau_0}{v(0)} - \frac{T - \tau_0}{v_f}\right) \tag{139}$$

A.7 THE POTENTIAL RAMP

Preliminary definitions are given in Sect. 3.2.

A.7.1 *Case I*

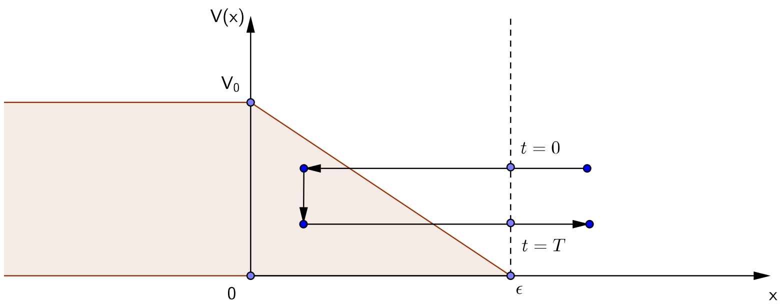

Figure 8: Potential ramp of height $V_0 > 0$ and reflected particle (Case I).

For Case I, one can infer from Fig. 8 that the particle's initial velocity v_i is negative and its final velocity v_f is positive. Its turning point is located inside Region 2, and it is useful to define T as the total time the particle spends in this region, or, equivalently, $t = T > 0$ as the moment the particle re-enters Region 1. Making the connection between the time and the particle's location and direction, one can observe that two solutions will be needed in Region 1: one for $t < 0$, when the particle is incoming and one for $t > T$, after the particle's reflection. They will be referred to as $a_1^{\text{in}}(t)$ and $a_1^{\text{out}}(t)$, respectively. These solutions in Region 1 are separate, due to the time gap in the particle's presence in this region. Solving the third-order form of the equation of motion, one obtains

$$x_1^{\text{in}}(t) = \tau_0^2 A_1^{\text{in}} \mathrm{e}^{t/\tau_0} + B_1^{\text{in}} t + C_1^{\text{in}} \tag{140}$$

$$x_2(t) = \tau_0^2 A_2 \mathrm{e}^{t/\tau_0} + \frac{V_0}{\epsilon m}\frac{t^2}{2} + B_2 t + C_2 \tag{141}$$

$$x_1^{\text{out}}(t) = \tau_0^2 A_1^{\text{out}} \mathrm{e}^{t/\tau_0} + B_1^{\text{out}} t + C_1^{\text{out}}. \tag{142}$$

Invoking the asymptotic condition, one immediately obtains that $A_1^{\text{out}} = 0$ and $B_1^{\text{out}} = v_f$. In a procedure analogous to the one described in § 3.1.3, now mainly relying on the continuity of the position, velocity and acceleration functions, the boundary conditions at $t = 0$ and $t = T$ serve to determine the other constants. Comparing two equivalent expressions for C_2 yields the condition expressed in Eq. 143, relating the slope width and the total time the particle spends in Region 2.

$$\frac{V_0}{\epsilon m}\left(\tau_0^2 e^{-T/\tau_0} - \tau_0^2 - \frac{1}{2}T^2 + \tau_0 T\right) + v_f T = 0 \tag{143}$$

Furthermore, a relation between the initial and final particle velocities can be deduced from the fact that $v_i = \lim_{t\to-\infty} v_1^{\text{in}}(t) = B_1^{\text{in}}$:

$$v_i = v_f - \frac{V_0}{\epsilon m}T. \tag{144}$$

These two equations (Eqs. 143 and 144) make it possible to construct a (v_f, v_i)-plot. For fixed values of the potential plateau height and the slope width, one can solve Eq. 143 numerically for the time parameter T, given a certain positive value of the final velocity. Following from the definition of T, this solution is required to be greater than zero. After plugging in the obtained value of T and the given final velocity, Eq. 144 yields the particle's initial velocity.

A.7.2 *Case II*

As in Case I, the initial particle velocity v_i is negative, whereas the final velocity v_f is positive. The fact that the particle turns back at some point in Region 3 (Fig. 9) complicates the solution of this case, when compared to Case I. If $t = 0$ is taken to be the moment when the particle enters Region 2, the context suggests the definition of three time parameters $T_1 < T_2 < T$: at $t = T_1$, the particle enters Region 3, in order to re-enter Region 2 subsequently at $t = T_2$, and at last, leave the middle region for Region 1 at $t = T$.

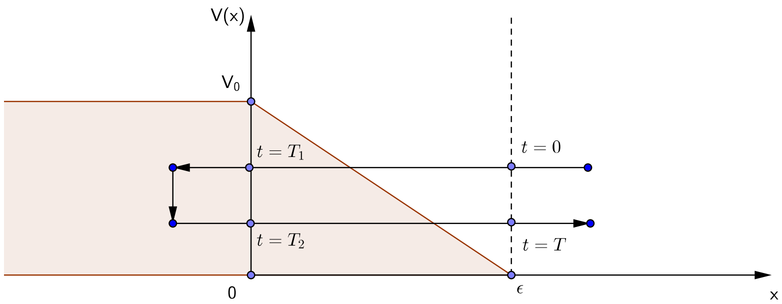

Figure 9: Potential ramp of height $V_0 > 0$ and reflected particle (Case II).

Furthermore, two solutions of the equation of motion are required in Region 1 (for the temporally separated intervals $t < 0$ and $t > T$), as well as in Region 2 (for the time intervals $0 < t < T_1$ and $T_2 < t < T$). Like in Case 1, one can add a superscript to show the difference between the solution in a region before and after the particle's reflection. After invoking the asymptotic condition and the continuity conditions at the regional boundaries, one obtains equivalent expressions for the constants C_2^{in}, C_3 and C_2^{out}, which lead to a system of three nonlinear equations.

$$\begin{aligned}\epsilon - \tau_0^2 \frac{V_0}{\epsilon m}(e^{-T_2/\tau_0} + e^{-T/\tau_0} - e^{-T_1/\tau_0})(e^{T_1/\tau_0} - 1) = \\ \frac{V_0}{\epsilon m}\left(\frac{1}{2}T_1^2 - \tau_0 T_1 - T_1 T_2 + T_1 T\right) - v_f T_1\end{aligned} \tag{145}$$

$$\tau_0^2 \frac{V_0}{\epsilon m}(e^{-T_2/\tau_0} - e^{-T/\tau_0})(e^{T_1/\tau_0} - e^{T_2/\tau_0}) = \\ -\left(\frac{V_0}{\epsilon m}(T_2 - T) + v_f\right)(T_2 - T_1) \tag{146}$$

$$\tau_0^2 \frac{V_0}{\epsilon m}(e^{(T_2-T)/\tau_0} - 1) + \frac{1}{2}\frac{V_0}{\epsilon m}(T^2 - T_2^2) = \\ \epsilon + \left(v_f + \frac{V_0}{\epsilon m}(\tau_0 - T)\right)(T_2 - T) \tag{147}$$

It is possible, though not entirely trivial, to solve this system numerically and get the allowed values of the three time parameters for a given final velocity v_f. In addition, one can find an equation (Eq. 148) relating the initial and final velocities by using the fact that $v_i = B_1^{\text{in}}$.

$$v_i = \frac{V_0}{\epsilon m}(T_2 - T - T_1) + v_f \tag{148}$$

A.7.3 *Case III*

Case III corresponds to the intuitive case of a particle originating from the plateau on the left-hand side, traversing the potential ramp with a positive velocity (Fig. 10). If one keeps the convention that $t = T$ refers to the moment when the particle passes $x = 0$ and that it crosses $x = \epsilon$ at $t = 0$, Case III will lead to a negative time parameter T. After excluding the runaway solutions in Region 1 and imposing the continuity of the particle acceleration, velocity and position as a function of time, one can find expressions for the various constants in the regional solutions of the equation of motion. The numerical determination of the time parameter follows from a comparison of two expressions for the constant C_2 (Eq. 149), obtained using the boundary condition at $t = T$.

$$\epsilon + \tau_0^2 \frac{V_0}{\epsilon m} = \frac{V_0}{\epsilon m}\left(\tau_0^2 e^{T/\tau_0} - \frac{1}{2}T^2 - \tau_0 T\right) - v_f T \tag{149}$$

The relation between the initial and final velocities (Eq. 150) is a consequence of the equality $v_i = \lim_{t\to-\infty} v_3(t) = B_3$.

$$v_i = v_f + \frac{V_0}{\epsilon m} T \tag{150}$$

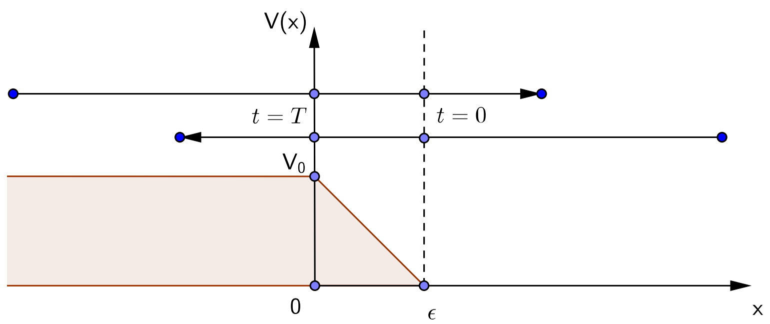

Figure 10: Potential ramp of height $V_0 > 0$ and traversing particle (Case III on top and Case IV below).

A.7.4 *Case IV*

Case IV is very similar to Case III, in that both represent the motion of a particle which doesn't turn back. In Case IV, however, the particle is incident on the right-hand side of the potential ramp, which it passes without being reflected. Hence, the particle velocity is negative. Maintaining the same definition of the time parameter T as in Case III, its value will be positive now in the context of Case IV. Applying the asymptotic condition in order to remove runaways in Region 3 and invoking the matching conditions at the boundaries of the regions lead to Eq. 151, obtained from two equivalent formulas for C_2.

$$\epsilon = \frac{V_0}{\epsilon m}\left(-\tau_0^2 e^{-T/\tau_0} + \tau_0^2 + \frac{1}{2}T^2 - \tau_0 T\right) - v_f T \tag{151}$$

Since $v_i = B_1$ holds for the initial velocity, one can find Eq. 152, relating the initial and final velocities.

$$v_i = v_f - \frac{V_0}{\epsilon m} T \tag{152}$$

A.7.5 *Classical Non-Radiative Solutions*

If one neglects the radiation reaction and regards the scattering problem from a classical viewpoint, several cases occur as well. A particle impinging on the potential ramp from the right ($v_i < 0$) will undergo elastic scattering if its initial kinetic energy $\frac{1}{2}mv_i^2$ is below the potential step height V_0. The principle of energy conservation dictates the equality of the particle's initial and final kinetic energy, so that $v_i = -v_f$ for the reflected particle. Furthermore, from $\frac{1}{2}mv_f^2 = \frac{1}{2}mv_i^2 < V_0$, it follows that v_f can take any value between 0 and $\sqrt{2V_0/m}$, for a reflected particle. This classical reflection regime splits into Cases I and II, when the radiation reaction is taken into account.

To obey the law of energy conservation, a non-radiating particle originating from the left of the plateau ($v_i > 0$) must have an initial kinetic energy larger than V_0. Hence, the particle's initial velocity has got to exceed $\sqrt{2V_0/m}$. Equating the particle's total initial energy $\frac{1}{2}mv_i^2 + V_0$ and its final energy $\frac{1}{2}mv_f^2$ yields the initial velocity $v_i = (v_f^2 - 2V_0/m)^{1/2}$ in the non-radiative solution of Case III. In addition, one notices that the final velocity can take any value greater than $\sqrt{2V_0/m}$, given an appropriate initial velocity.

In the context of Case IV, a non-radiating particle entering the potential ramp region from the right-hand side ($v_i < 0$) and continuing its motion without being reflected ($v_f < 0$), must have an initial kinetic energy exceeding V_0 to overcome the potential ramp. Accordingly, its initial velocity is required to be below $-\sqrt{2V_0/m}$. Invoking the energy conservation principle ($\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 + V_0$), one can find the following relation between the initial and final particle velocities: $v_i = -(v_f^2 + 2V_0/m)^{1/2}$.

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