QF 605 Fixed Income Securities

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Part I (Boot Strapping Swap Curves)

Bootstrap OIS Discount Factor

In the IR Data.xlsm spreadsheet, OIS data is provided. Bootstrap the OIS discount factor $D_o(0, T)$ and plot the discount curve for $T \in [0, 30]$.

Using the 6m OIS and 1 OIS we can solve for the f_0 and f_1:

$$D_0(0,6m) \times 0.5 \times 0.25\% = D_0(0,6m) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} - 1\right]$$

$$D_0(0,1y) \times 0.3\% = D_0(0,1y) \times \left[\left(1 + \frac{f_0}{360}\right)^{180}\left(1 + \frac{f_1}{360}\right)^{180} - 1\right]$$

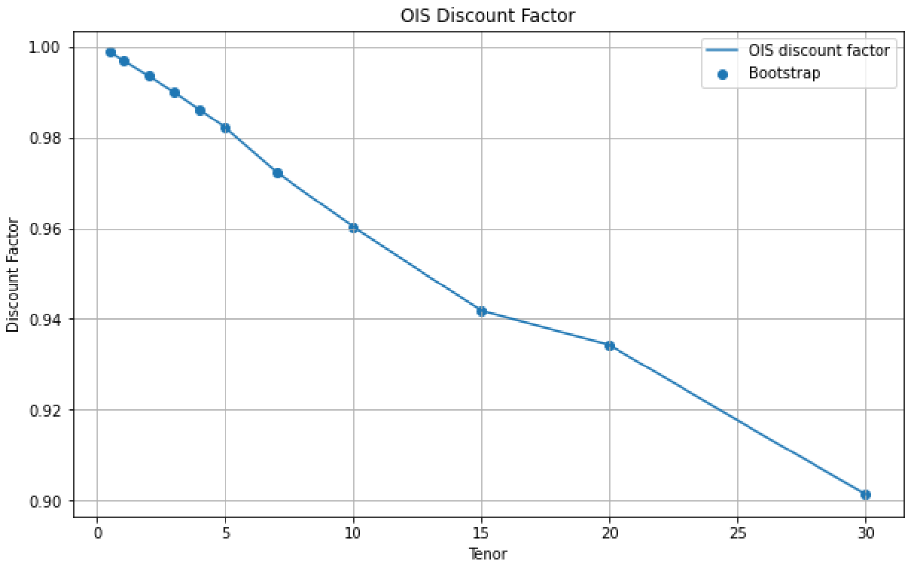
Then expressing $D_0(0,6m) = \frac{1}{\left(1+\frac{f_0}{360}\right)^{180}}$ $and\ D_0(0,1y) = \frac{1}{\left(1+\frac{f_0}{360}\right)^{180}} \times \frac{1}{\left(1+\frac{f_1}{360}\right)^{180}}$

Solve the 2y OIS with it:

$$[D_0(0,1y) + D_0(0,2y)] \times 0.325\% = D_0(0,1y) \times \left[\left(1 + \frac{f_0}{360}\right)^{180}\left(1 + \frac{f_1}{360}\right)^{180} - 1\right] + D_0(0,2y) \times \left[\left(1 + \frac{f_2}{360}\right)^{360} - 1\right]$$

Below is our result table from bootstrapping the OIS discount.

	Tenor	Product	Rate	float	OIS Discount
0	0.5	OIS	0.00250	0.002497	0.998752
1	1.0	OIS	0.00300	0.002996	0.997009
2	2.0	OIS	0.00325	0.003495	0.993531
3	3.0	OIS	0.00335	0.003545	0.990015
4	4.0	OIS	0.00350	0.003946	0.986117
5	5.0	OIS	0.00360	0.003996	0.982184
6	7.0	OIS	0.00400	0.005003	0.972406
7	10.0	OIS	0.00450	0.004163	0.960336
8	15.0	OIS	0.00500	0.003885	0.941861
9	20.0	OIS	0.00525	0.001607	0.934322
10	30.0	OIS	0.00550	0.003582	0.901444



Bootstrap LIBOR Discount Factor

Using the IRS data provided, bootstrap the LIBOR discount factor D(0, T), and plot it for T ∈ [0; 30]. Assume that the swap market is collateralized in cash and overnight interest is paid on collateral posted.

For LIBOR discount factor the formula is $D_L(0{,}0.5y) = \frac{1}{1+0.5\times L(0{,}0.5y)}$ and LIBOR is $L(T_i, T_{i+1}) = \frac{D_L(0,T_i)-D_L(0,T_{i+1})}{\Delta D_L(0,T_i)}$

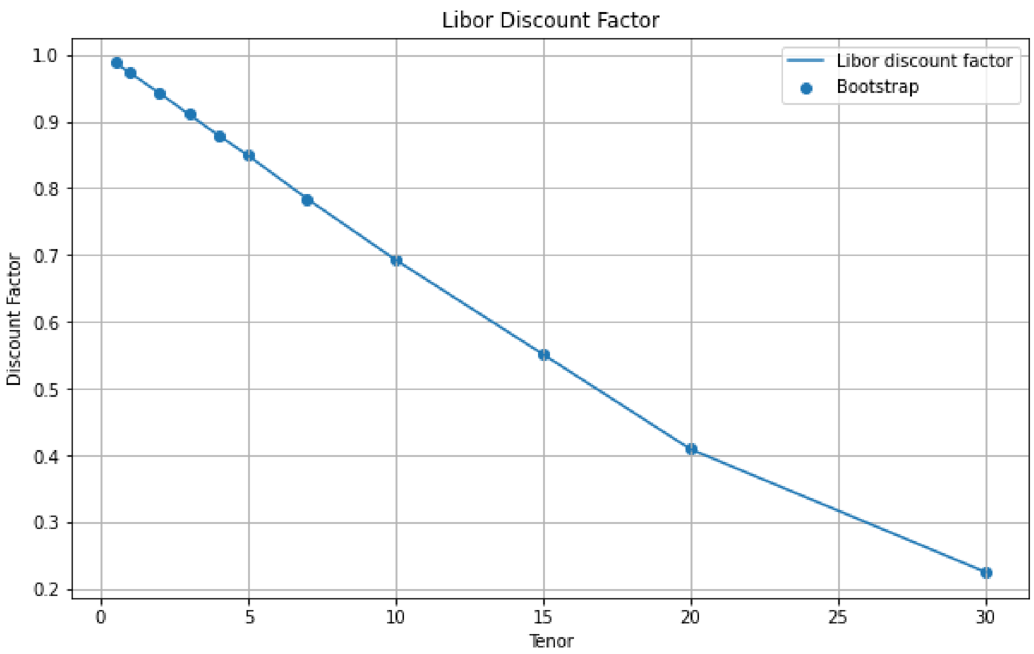
Once again solve for the Discount Factor then get the LIBOR. If there are missing IRS rate use linear interpolation on LIBOR Discount Factor.

$$PV_{fix} = PV_{float}$$

$$0.5 * [\, D_0(0{,}6m) + D_0(0{,}1y)] = 0.5 * [D_0(0{,}6m)L(0{,}6m) + D_0(0{,}1y)L(6m, 1y)]$$

The LIBOR discount curve sample data is shown as the table below.

	Tenor	Rate	float	OIS Discount	Libor Discount	Forward Libor
0	0.5	0.0025	0.002497	0.99875	0.987654	0.025
1	1	0.003	0.002996	0.99701	0.972577	0.03101
2	1.5	NaN	0.003495	0.99527	0.957378	0.03175
3	2	0.00325	0.003495	0.99353	0.942179	0.03226
4	2.5	NaN	0.003545	0.99177	0.92633	0.03422
5	3	0.00335	0.003545	0.99002	0.910482	0.03481
6	3.5	NaN	0.003946	0.98806	0.894731	0.03521
7	4	0.0035	0.003946	0.98612	0.878981	0.03584
8	4.5	NaN	0.003996	0.98415	0.863985	0.03471
9	5	0.0036	0.003996	0.98218	0.848989	0.03533
10	5.5	NaN	0.005003	0.97973	0.832796	0.03889



Forward Swap Rate Calculation

Calculate the following forward swap rates:

- 1y * 1y, 1y * 2y, 1y * 3y, 1y * 5y, 1y * 10y
- 5y * 1y, 5y * 2y, 5y * 3y, 5y * 5y, 5y * 10y
- 10y * 1y, 10y * 2y, 10y * 3y, 10y * 5y, 10y * 10y

Thus from the interpolated libor discount rate in the previous questions, we can generate Forward Swap Rates for different expiries and tenors as below:

Tenor	Forward Swap Rates
1y x 1y	0.032007
1y x 2y	0.033259
1y x 3y	0.034011
1y x 5y	0.035255
1y x 10y	0.038424

Tenor	Forward Swap Rates
5y x 1y	0.039274
5y x 2y	0.040075
5y x 3y	0.040068
5y x 5y	0.041087
5y x 10y	0.043612

Tenor	Forward Swap Rates
10y x 1y	0.042132
10y x 2y	0.043057
10y x 3y	0.044038
10y x 5y	0.04619
10y x 10y	0.053307

Part II (Swaption Calibration)

Swaption Pricing

The pricing formula for payer swaption can be derived by:

$$[P_{n+1,N}(T)(S_{n,N}(T)-K)]^+$$

Under measure $Q^{n+1,N}$, payer swaption can be derived as:

$$V_{n,N}^{\text{payer}}(0) = P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(S_{n,N}(T)-K)^+]$$

$$V_{n,N}^{payer}(0) = P_{n+1,N}(0)[S_{n,N}(0)\Phi(d_1) - K\Phi(d_2)]$$

$$d_1 = \frac{\log\frac{S_{n,N}(0)}{K} + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{n,N}\sqrt{T}$$

Given a discount factor and a normal black call option, we can price the payer swaptions. Similar to the receiver swaption, which can be derived by put option using normal black model with a discount factor.

$$\text{Payer Swap} = P_{n+1,N}(t)(S_{n,N}(t)-K)$$

Swaption implied volatilities were provided for varying tenor and expiry. However, the strikes were given in basis point differences from ATM, we need to derive forward par swap rates from Part I discount curve and use them to match the implied volatilities to different strike prices

Displace Diffusion (DD) model calibration

The DD model can be thought of as a combination of normal and lognormal stochastic price model, in which the contribution of lognormal behaviour relative to normal behaviour is determined by beta.

To calibrate the implied volatilities using DD, we tune the beta parameter. The beta parameter is tuned such that it gives swaption prices which when put into inverted Black Scholes function, gives calculated implied volatilities with least square errors from provided implied volatilities. At the same time, when optimising beta parameter, parameter sigma is found by putting ATM swaption price and beta into inverted DD function to get sigma.

One sigma and beta are used to fit the implied volatility smile for each expiry/tenor swaption

DD Model Parameters

Displaced-Diffusion Model	Expiry/Tenor	1Y	2Y	3Y	5Y	10Y
Sigma (σ)	1Y	22.5	28.72	29.78	26.07	24.47
	5Y	27.26	29.83	29.98	26.6	24.51
	10Y	28.54	29.28	29.4	26.74	24.37
Beta (β)	1Y	4.4074E-06	1.0445E-07	1.0113E-07	1.0464E-07	2.8467E-05
	5Y	1.0286E-07	4.1466E-07	3.1323E-06	4.51E-05	0.1015625
	10Y	7.3197E-07	1.783E-06	6.3704E-07	0.00012659	0.123442

SABR model calibration

In SABR model calibration, there are 4 parameters alpha, nu, rho and beta. Beta is fixed at 0.9 while alpha, rho and nu are tuned such that when used with the SABR function, they provide a set of σ_{SABR} across various strikes with least square error compared to observed black-Scholes implied volatilities.

SABR Model Parameters

	alpha				
Tenor	**1**	**2**	**3**	**5**	**10**
Expiry					
1	0.140424	0.184880	0.197018	0.177524	0.168069
5	0.166129	0.199469	0.210170	0.186872	0.168234
10	0.178110	0.196505	0.208989	0.203030	0.173160

	nu				
Tenor	**1**	**2**	**3**	**5**	**10**
Expiry					
1	2.025269	1.674793	1.436597	1.070212	0.817115
5	1.334670	1.061596	0.936514	0.686967	0.556583
10	1.008019	0.927915	0.872413	0.723678	0.599174

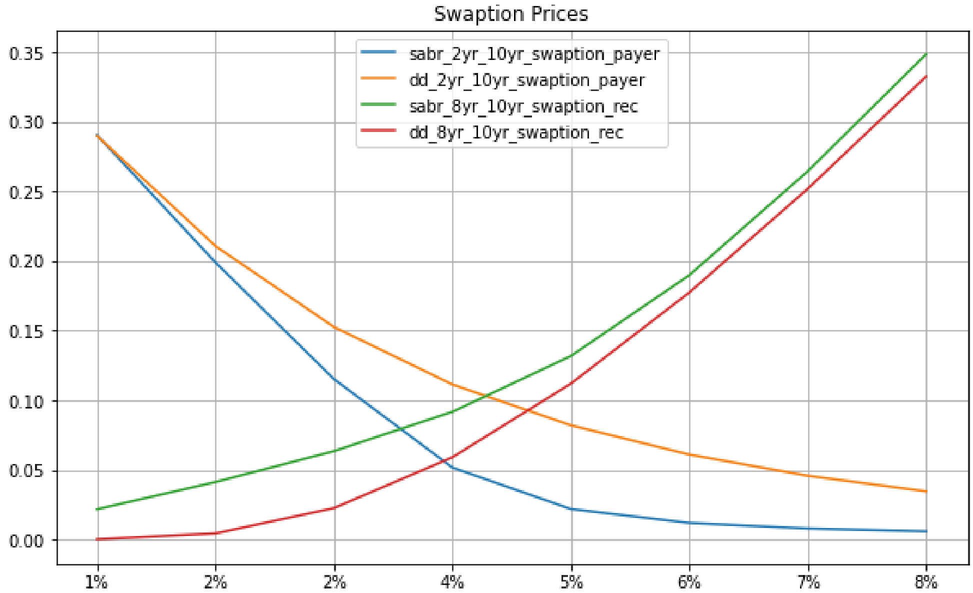
	rho				
Tenor	**1**	**2**	**3**	**5**	**10**
Expiry					
1	-0.627410	-0.524940	-0.483092	-0.411590	-0.237841
5	-0.582356	-0.546637	-0.549103	-0.487206	-0.358822
10	-0.547401	-0.547800	-0.555302	-0.567969	-0.478303

Price following swaption using calibrated DD and SABR model:

Under DD model, the swaption price can be derived as:

$$V_{n,N}(0) = P_{n+1,N}(0)\text{Black76}\left(\frac{S_{n,N}(0)}{\beta},\ K + \frac{1-\beta}{\beta}S_{n,N}(0),\ \sigma\beta,\ T\right)$$

Using the swaption price by the above formula under DD model to substitute in the function of Black-Scholes implied volatility function, volatility under different strikes is calculated as below. The following figure also shows that the payer swaption payoff is similar to the put option, on the other hand, receiver swaption is similar to the call option.



To price the swaption using calibrated models, we need to know SABR model parameters (alpha, rho and nu) and DD parameters (sigma and beta) for corresponding expiry and tenor. This can be found using linear interpolation of model parameters calibrated on swaptions with nearest expiry/tenor

Next, the forward swap rate needed as input to the models were derived from Libor discount curve from question 1.

	Payer 2y x 10y		Receiver 8y x 10y	
K	DD	SABR	DD	SABR
1%	0.290247	0.290420	0.003351	0.021640
2%	0.211959	0.198932	0.023303	0.041212
3%	0.154614	0.115147	0.060661	0.063411
4%	0.113649	0.051584	0.111361	0.091675
5%	0.084313	0.021784	0.171884	0.131749
6%	0.063099	0.011773	0.239674	0.189821
7%	0.047588	0.007771	0.312919	0.264489
8%	0.036126	0.005759	0.390325	0.348659

Part III (Convexity Correction)

Forward Swap Rate Calculation for given Libor Discounting Factor

The Par Swap Rate for the $[T_n - T_N]$ swap as R $S_{n,N}$:

$$S_{n,N}(t) = \frac{D_n(t) - D_N(t)}{\sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)}$$

Present Value of 2 CMS Receiver Leg

The present value of a CMS leg is a summation of regular interval of CMS rates discounted to the present, as shown in this formula:

$$CMS\ leg\ PV = \sum_{i=0.5}^{N} D_0(0,T_i) * Delta * CMS\left(S_{n,N}(T_i)\right)$$

Where expected value of CMS($S_{n,N}$ (T)) is given by:

$$E^T[S_{n,N}(T)] = g(F) + \frac{1}{D_0(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]$$

And IRR settled swaption (Vrec and Vpay) are calculated as such:

$$V_{n,N}(0) = D_0(0,T_n) * IRR\left(S_{n,N}(0)\right) * Black\ 76\left(S_{n,N}(0), K, \sigma_{sabr}, T\right)$$

According to the swap market model, there is no drift term, so F in the integrand limits is simply the par swap rate calculated from spot Libor discount curve in question 1. In our case, for calculating the PV of CMS10y with semi-annual payment, we will need the par swap rate of 0.5y x 10y, 1y x 10y, 1.5y x 10y ... 10y x 10y calculated from spot Libor discount curve
Also, we got SABR parameters for these different swap rate expiries using cubic spline interpolation of alpha, rho and nu from calibrated swaption of nearest expiry and same tenor.

Using above formulas and SABR parameters interpolation the PV of CMS legs are as follows:

PV of Receiver CMS 10Y (Semi-Annual)	PV of Receiver CMS2y (Quarterly)
0.207	0.452

CMS Rate

Tenor	CMS Rates
1y x 1y	0.031878
1y x 2y	0.033101
1y x 3y	0.033829
1y x 5y	0.035005
1y x 10y	0.037895

Tenor	CMS Rates
5y x 1y	0.039079
5y x 2y	0.039858
5y x 3y	0.039855
5y x 5y	0.040806
5y x 10y	0.043011

Tenor	CMS Rates
10y x 1y	0.041907
10y x 2y	0.042804
10y x 3y	0.04374
10y x 5y	0.045741
10y x 10y	0.051786

Difference (Forward – CMS)

Tenor	Forward Swap - CMS
1y x 1y	0.000837
1y x 2y	0.00125
1y x 3y	0.001002
1y x 5y	0.00042
1y x 10y	0.000581

Tenor	Forward Swap - CMS
5y x 1y	0.016965
5y x 2y	0.018227
5y x 3y	0.017253
5y x 5y	0.003191
5y x 10y	0.003019

Tenor	Forward Swap - CMS
10y x 1y	0.028625
10y x 2y	0.043472
10y x 3y	0.05454
10y x 5y	0.045159
10y x 10y	0.048487

Discuss the effect of maturity and tenor on convexity correction

Forward Swap Rate is always greater than the CMS rate, which is due to convexity adjustment. As the time to expiry of the option increases, the difference gap also increases. The CMS rate can be replicated by the formula below:

$$\underset{\text{A}}{\boxed{\underbrace{\mathbb{E}^T\left[S_{n,N}(T)\right]}_{\text{CMS Rate}}}} = \underset{\text{B}}{\boxed{g(F)}} + \underset{\text{C}}{\boxed{\frac{1}{D(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]}}$$

In above equation, term A is the CMS rate, which is larger than term B (swap rate) by the convexity term C. Convexity correction term C increases with time to expiry because V(K) in the term is a function of Black model option price which increases with time to expiry. Thus for the same tenor larger time to maturity contains higher convexity correction.

Part IV (Decompounded Options)

A decompounded option pays the following at time T = 5y:

$$\mathrm{CMS10y}^{1/P} - 0.04^{1/q}$$

where p = 4 and q = 2. Use static replication to value the PV of this payoff.

Part A = 0.21777799185220725

Suppose the Payoff is now

$$\left(CMS10y^{1/P} - 0.04^{1/q}\right)^{+}$$

Use static replication to value the PV of this payoff.

Part B = 0.14298798751111846

For Part A, forward rate F is the 5y x 10y CMS par swap rate. Discount factor is derived from the 5y OIS rate. $h(K)$ is defined as $\frac{g(K)}{IRR(K)}$, where $g(K)$ is the payoff and $IRR(K)$ is the IRR-Settled Swaption discount rate. $g(K)$ and its second and third derivatives are

$$g(K) = K^{\frac{1}{4}} - 0.04^{\frac{1}{2}}, g'(K) = \left(\frac{1}{4}\right)K^{-\frac{3}{4}}, \ g''(K) = \left(-\frac{3}{16}\right)K^{-\frac{7}{4}}$$

For Part B, to have a positive payoff, CMS $10y^{\frac{1}{p}} - 0.04^{\frac{1}{q}}$> 0, the lower bound for the option should be 0.0016. As it's a CMS caplet, only the integrand of the payer swaption needs to be considered.

$$D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^{\infty} h''(K)V^{pay}(K)dK$$

using static replication, where

$$h''(K) = \frac{-\text{IRR}''(K)\cdot K - 2\cdot\text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2\cdot\text{IRR}'(K)^2\cdot K}{\text{IRR}(K)^3}.$$

Valuation for CMS

SABR variables from previous questions:

Sabr_alpha	Sabr_beta	Sabr_rho	Sabr_nu
0.168234	0.9	-0.358823	0.556587

Swap rate and discount factor extract from previous questions as follow

Discount Factor	Forward Swap Rate
0.98218411973321	0.043243

V $\left(CMS10y^{1/P} - 0.04^{1/q}\right)$ = 0.23694977260270478

Valuation for CMS Cap

$$\text{CMS Caplet} = V^{pay}(F)h'(F) + \int_F^{\infty} h''(K)V^{pay}(K)dK$$

using static replication, where

$$h'(K) = \frac{\text{IRR}(K) - \text{IRR}'(K)\cdot(K-F)}{\text{IRR}(K)^2}$$

$$h''(K) = \frac{-\text{IRR}''(K)(K-F) - 2\cdot\text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2\cdot\text{IRR}'(K)^2\cdot(K-F)}{\text{IRR}(K)^3}$$

Submit variables into integration function:

Sabr_alpha	Sabr_beta	Sabr_rho	Sabr_nu
0.168234	0.9	-0.358823	0.556587

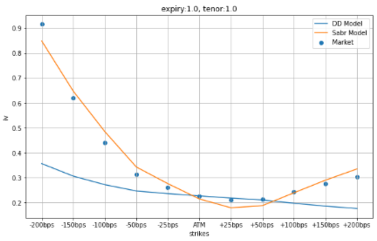
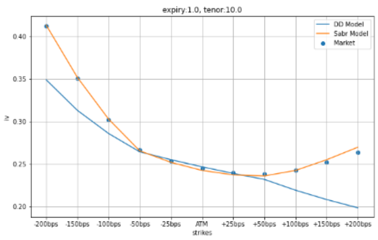
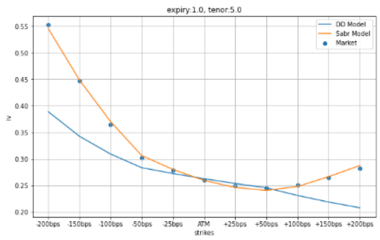
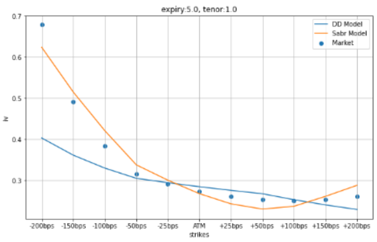
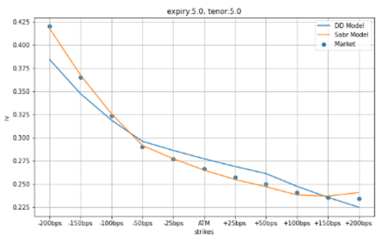
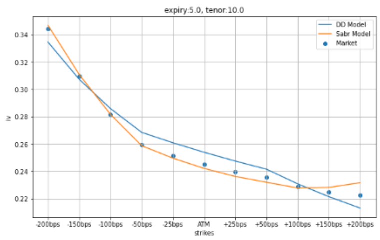
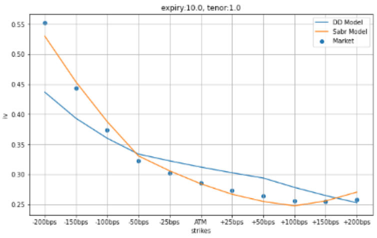
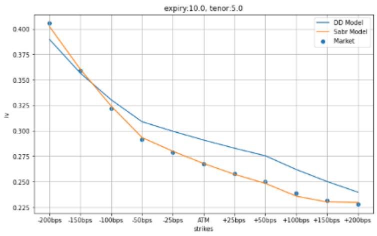
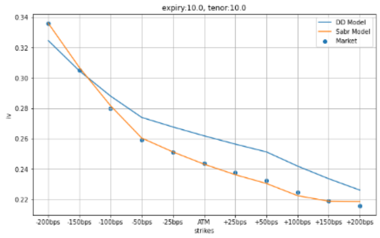
Swap rate and discount factor extract from previous questions as follow

Discount Factor	Forward Swap Rate
0.98218411973321	0.043243

V ($(CMS\ 10y^{1/4} - 0.04^{1/2})^+$)= 0.2591629795667644

Appendix Part I – Model Calibration

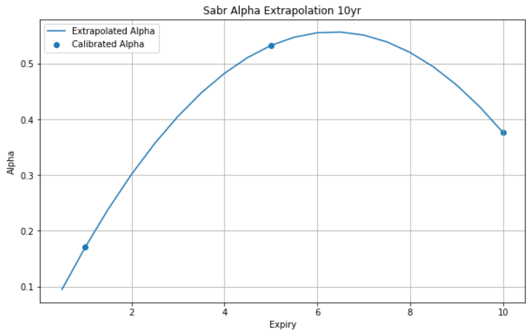
Comparison for Displaced-Diffusion Model and SABR Model

ExpiryXTenor: 1Y x 1Y	ExpiryXTenor: 1Y x 5Y	ExpiryXTenor: 1Y x 10Y
		
ExpiryXTenor: 5Y x 1Y	ExpiryXTenor: 5Y x 5Y	ExpiryXTenor: 5Y x 10Y
		
ExpiryXTenor: 10Y x 1Y	ExpiryXTenor: 10Y x 5Y	ExpiryXTenor: 10Y x 10Y
		

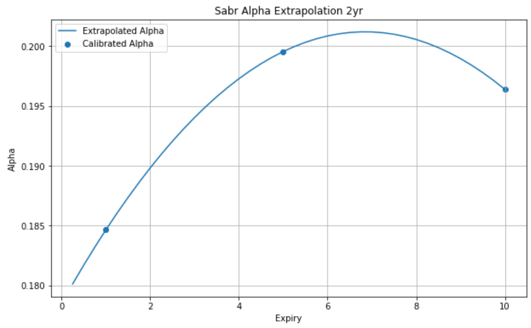
Appendix Part II – Convexity Correction

Polynomial Curve Fitting and Extrapolation For SABR Parameters

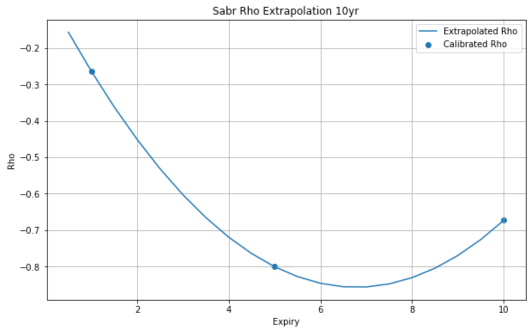
Alpha For 10Yr CMS



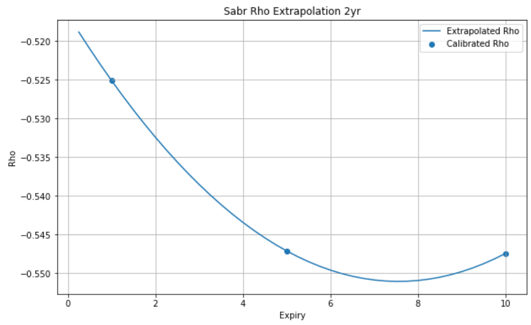
Alpha For 2Yr CMS



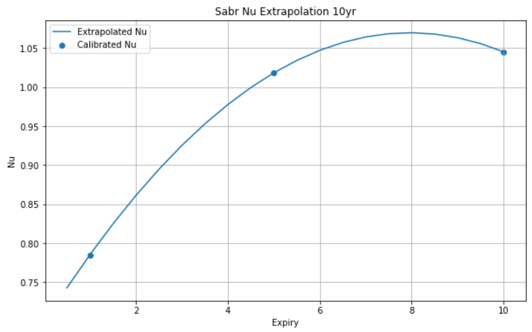
Rho For 10Yr CMS



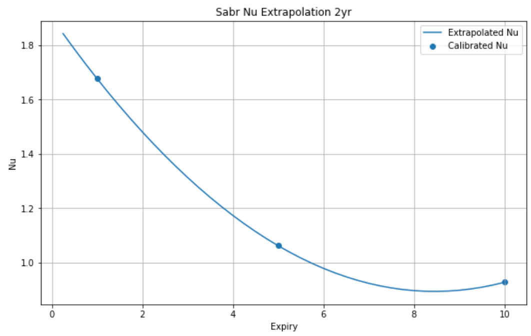
Rho For 2Yr CMS



Nu For 10Yr CMS



Nu For 2Yr CMS



Appendix Part III – Convexity Correction

Comparison for CMS

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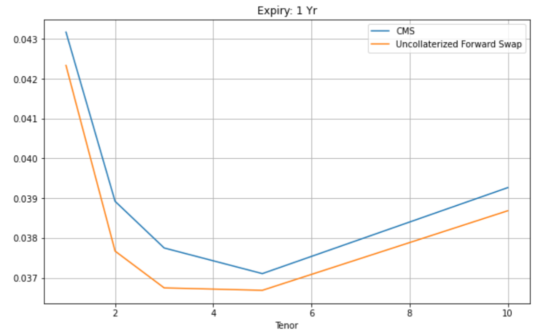
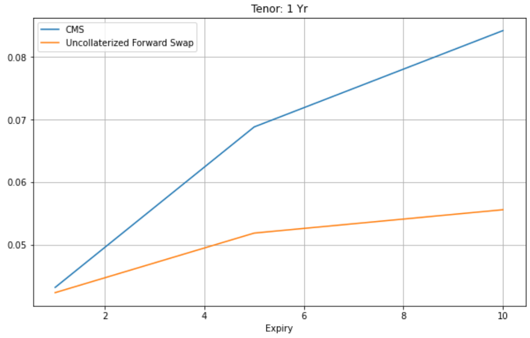
<tr><td>
</td><td>
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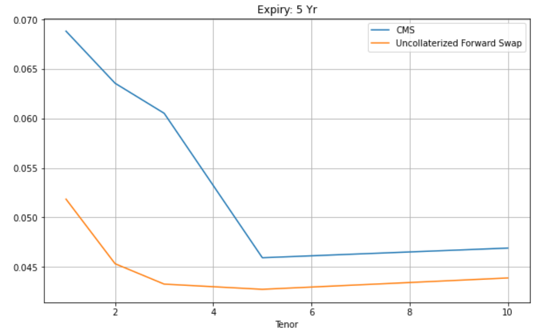
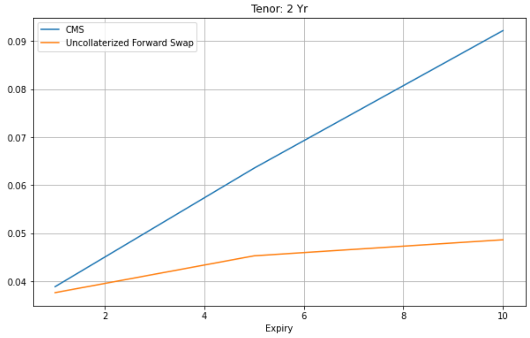
<tr><td>
</td><td>
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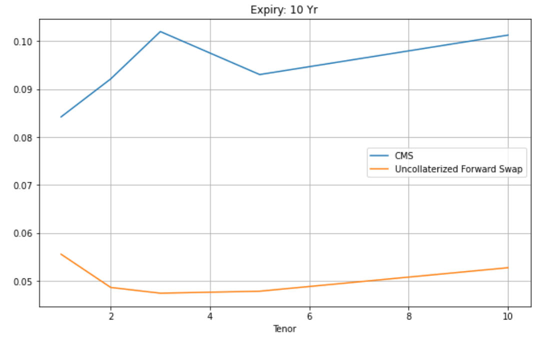
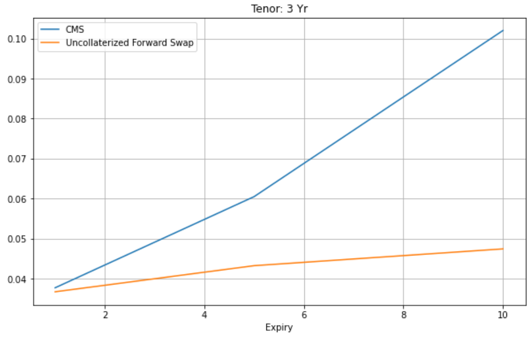
<tr><td>
</td><td>
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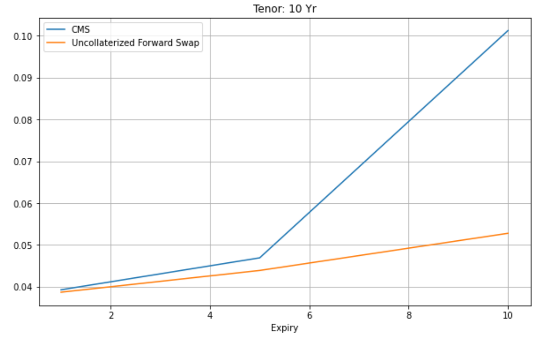
<tr><td>
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