SMU

SINGAPORE MANAGEMENT UNIVERSITY

QF 620

Stochastic Modelling in Finance

Hung Lin Yin

Jeriel Wong Kwan Kit

Lee Qianxin

Pang Hong Yun

Wei Xiaoxin

Project — Part I (Analytical Option Formulae)

For derivation of the respective models for options valuation, please refer to the appendix Model Derivation for model valuation.

			Bachelier	Black Scholes	Black76	Displaced Diffusion
			$d1 = \frac{(S-K)}{(\sigma\sqrt{T})}$	$d1 = \frac{\left(log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)}{\sigma\sqrt{T}}$	$d1 = \frac{\left(\log\left(\frac{F}{K}\right) + \sigma^2 T\right)}{\sigma\sqrt{T}}$	$d1 = \frac{\left(\log\left(\frac{F}{K\beta - F\beta + F}\right) + \left(\beta^2\sigma^2\frac{T}{2}\right)\right)}{\beta\sigma\sqrt{T}}$
			d2 = -d1	$d2 = d1 - \sigma\sqrt{T}$	$d2 = d1 - \sigma\sqrt{T}$	$d2 = d1 - (\sigma\beta\sqrt{T})$
Call	Asset – K*Cash = Vanilla	Vanilla	$e^{-rT}((S-K)\Phi(d1) + \sigma\sqrt{T}\phi(d1))$	$S\Phi(d1) - Ke^{-rT}\phi(d2)$	$e^{-rT}(F\phi(d1) - K\phi(d2))$	$e^{-rT}\left(\frac{F}{\beta}\phi(d1) - \left(K + \left((1-\beta) * \frac{F}{\beta}\right)\right)\phi(d2)\right)$
		Digital Asset	$e^{-rT}\left(S\Phi(d1) + \sigma\sqrt{T}\phi(d1)\right)$	SΦ(d1)	$e^{-rT}(F\phi(d1))$	$e^{-rT}\left(\frac{F}{\beta}\Phi(d1) - \left((1-\beta) * \frac{F}{\beta}\right)\Phi(d2)\right)$
		Digital Cash	$e^{-rT}\phi(d1)$	$e^{-rT}\Phi$(d2)	$e^{-rT}(\phi(d2))$	$e^{-rT}\phi(d2)$
Put	K*Cash - Asset = Vanilla	Vanilla	$e^{-rT}\left((K-S)\Phi(d2) + \sigma\sqrt{T}\phi(d2)\right)$	$-S\Phi(-d1) + Ke^{-rT}\phi(-d2)$	$e^{-rT}(-F\phi(d1) + K\phi(d2))$	$e^{-rT}\left(-\frac{F}{\beta}\Phi(-d1) + \left(K + \left((1-\beta) * \frac{F}{\beta}\right)\right)\Phi(-d2)\right)$
		Digital Asset	$e^{-rT}\left(S\Phi(d2) - \sigma\sqrt{T}\phi(d2)\right)$	Sϕ(-d1)	$e^{-rT}(F\phi(-d1))$	$e^{-rT}\left(\frac{F}{\beta}\Phi(-d1) - \left((1-\beta) * \frac{F}{\beta}\right)\Phi(-d2)\right)$
		Digital Cash	$e^{-rT}\phi(d2)$	$e^{-rT}\phi(-d2)$	$e^{-rT}(\phi(-d2))$	$e^{-rT}\Phi(-d2)$

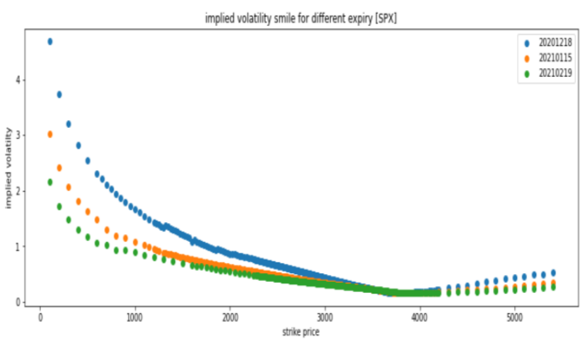
To test our formula are correct, we have provided test cases in the attached Jupyter notebook. The test cases will check that:

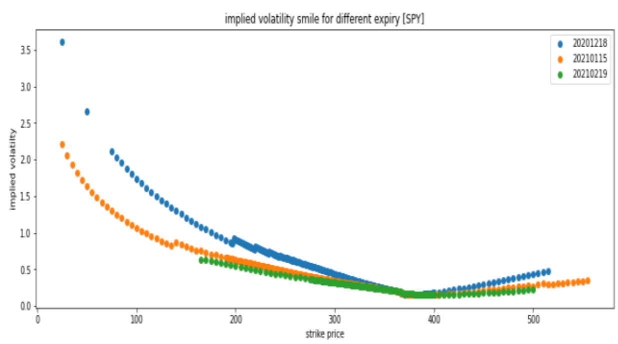
- For call options, digital asset payout – K*digital cash payout == vanilla payout
- For put options, K*digital cash payout – digital asset payout == vanilla payout

Project — Part II (Model Calibration)

On 1-Dec-2020, the S&P500 (SPX) index value was 3662.45, while the SPDR S&P500 Exchange Traded Fund (SPY) stock price was 366.02. The call and put option prices (bid & offer) over 3 maturities are provided in the spreadsheet:
• SPX options.csv
• SPY options.csv
The discount rate on this day is in the file: zero rates 20201201.csv.

There are 3 options expiry dates, 20201218, 20210215 and 20210219, in the option chains csv file given. Using mid-price of OTM options, time to expiry and interpolated rates, implied volatilities for each strike price are plotted below:

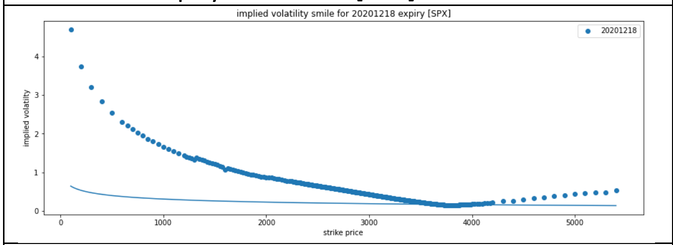
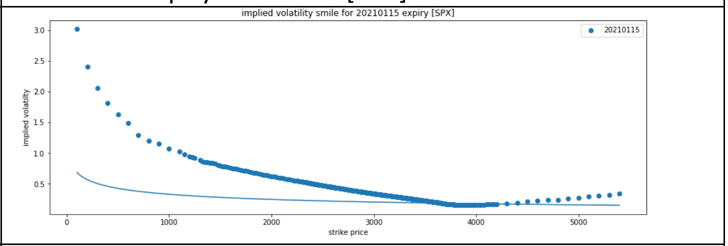
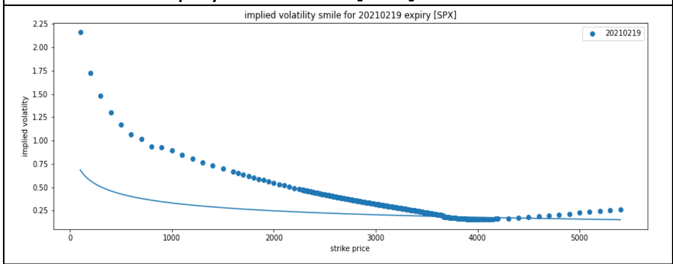
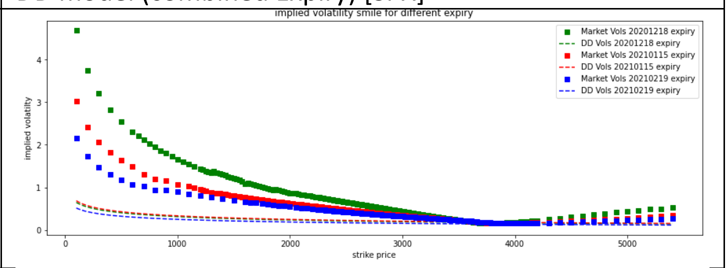




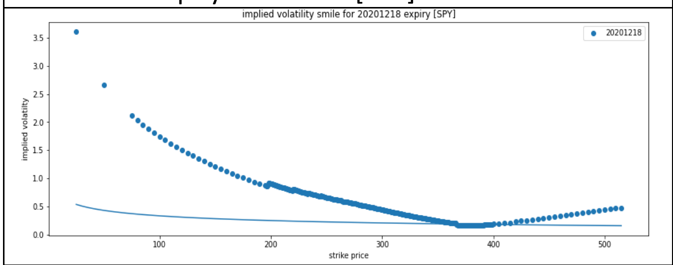
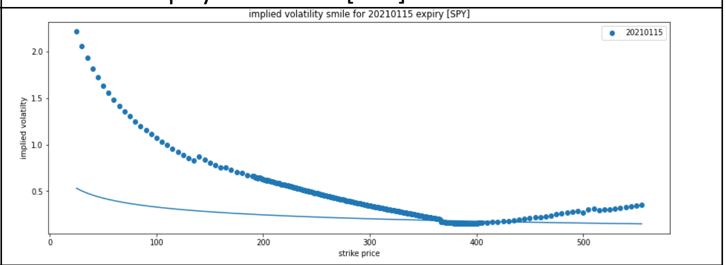
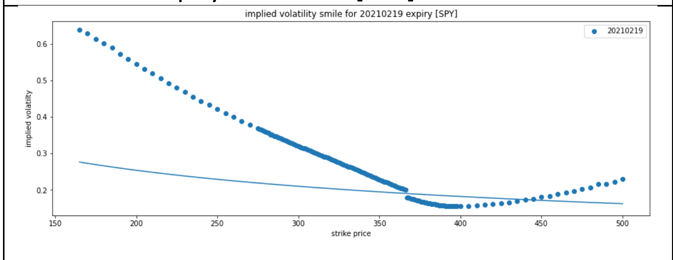
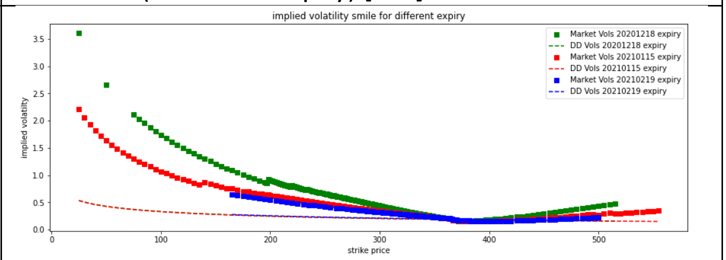
Calibrate the following models to match the option prices and plot the fitted implied volatility smile against the market data.

1 Displaced-diffusion (DD) model

1.1 for SPX

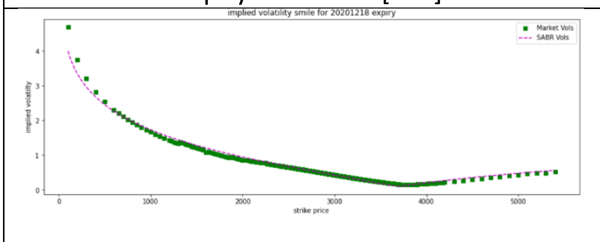
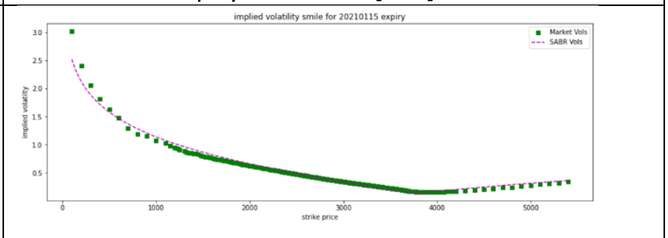
DD model Expiry 20201218 [SPX]	DD model Expiry 20210115 [SPX]
	
DD model Expiry 20210219 [SPX]	**DD model (combined Expiry) [SPX]**
	

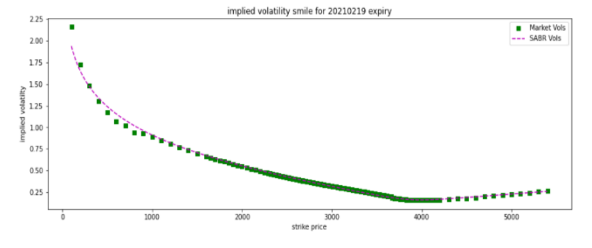
1.2 for SPY

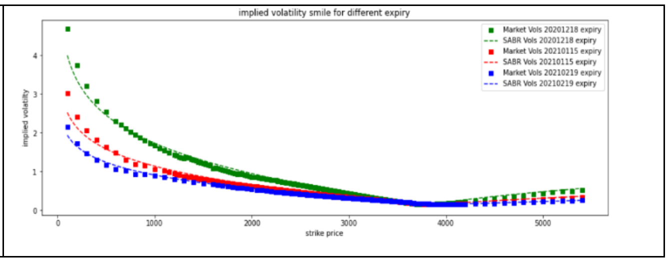
DD model Expiry 20201218 [SPY]	DD model Expiry 20210115 [SPY]
	
DD model Expiry 20210219 [SPY]	**DD model (combined Expiry) [SPY]**
	

2. SABR model (fix $\beta = 0.7$)

2.1 for SPX

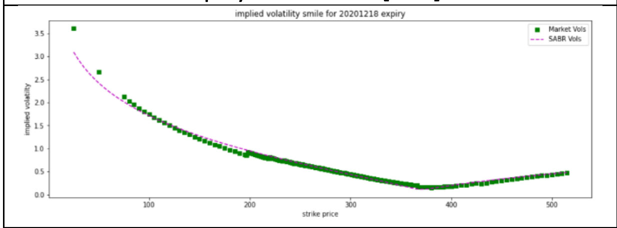
SABR model Expiry 20201218 [SPX]	SABR model Expiry 20210115 [SPX]
	
SABR model Expiry 20210219 [SPX]	**SABR model (combined Expiry) [SPX]**



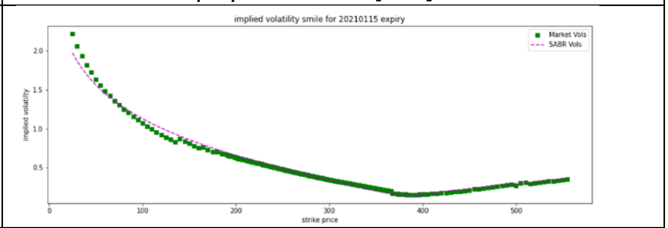


2.2 for SPY

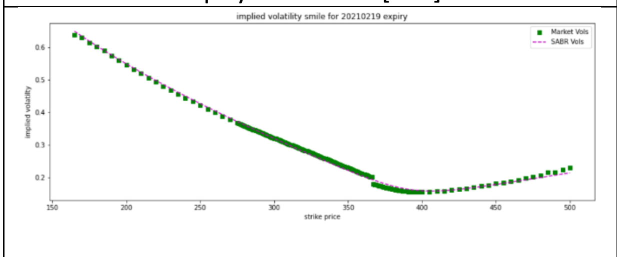
SABR model Expiry 20201218 [SPY]



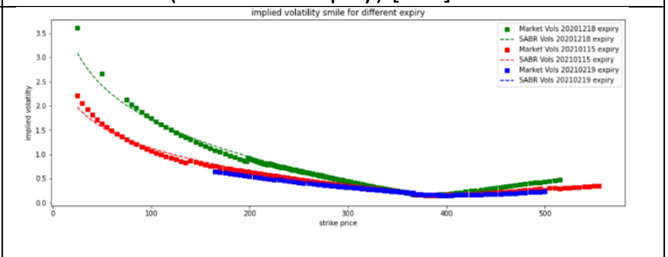
SABR model Expiry 20210115 [SPY]



SABR model Expiry 20210219 [SPY]



SABR model (combined Expiry) [SPY]



Report the model parameters:

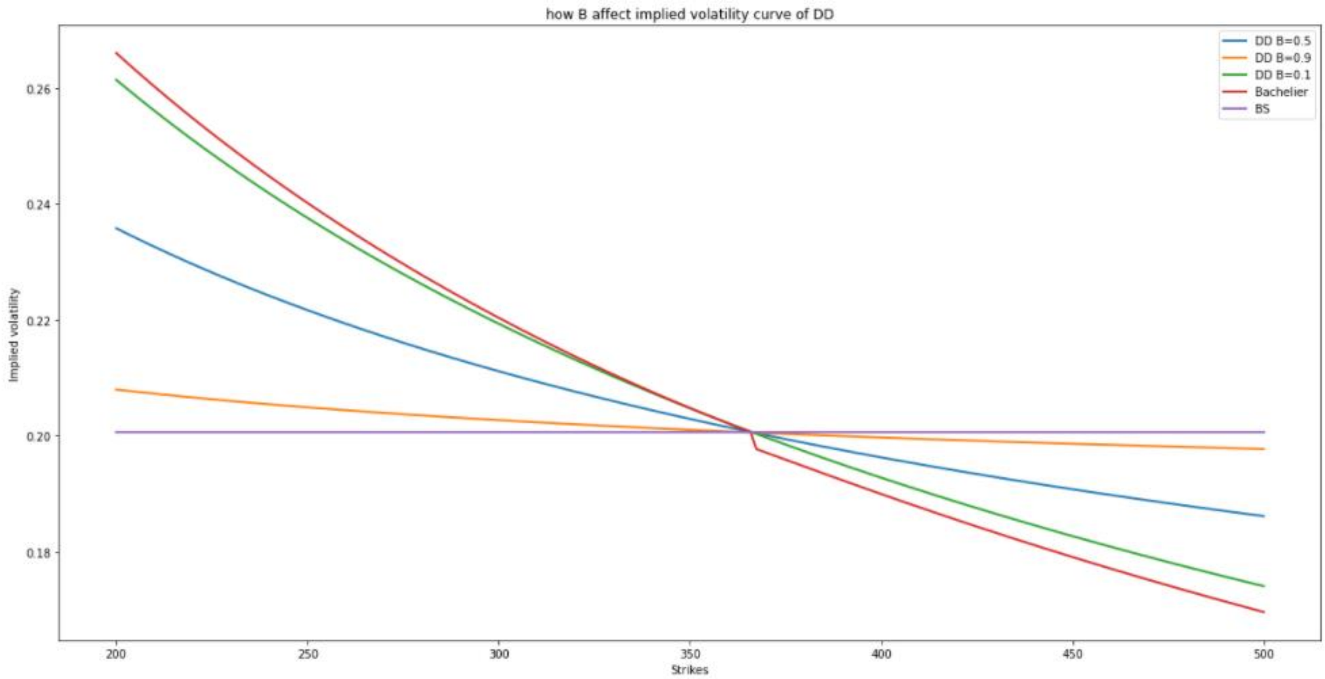
1. 1 σ, β
2. 2 α, ρ, ν

		SPX			SPY		
Model	params	Ex 20201218	Ex 20210115	Ex 20210219	Ex 20201218	Ex 20210115	Ex 20210219
Displaced-diffusion model	σ *	0.174	0.185	0.141	0.201	0.197	0.200
	β	0.000	0.000	0.000	0.000	0.000	0.000
SABR model (fix β = 0.7)	α	1.212	1.817	2.140	0.665	0.908	1.121
	ρ	-0.301	-0.404	-0.575	-0.412	-0.489	-0.633
	ν	5.460	2.790	1.842	5.250	2.729	1.742

*Sigma is implied vol of DD model at ATM, calculated by interpolating OTM put and call DD volatility nearest to ATM

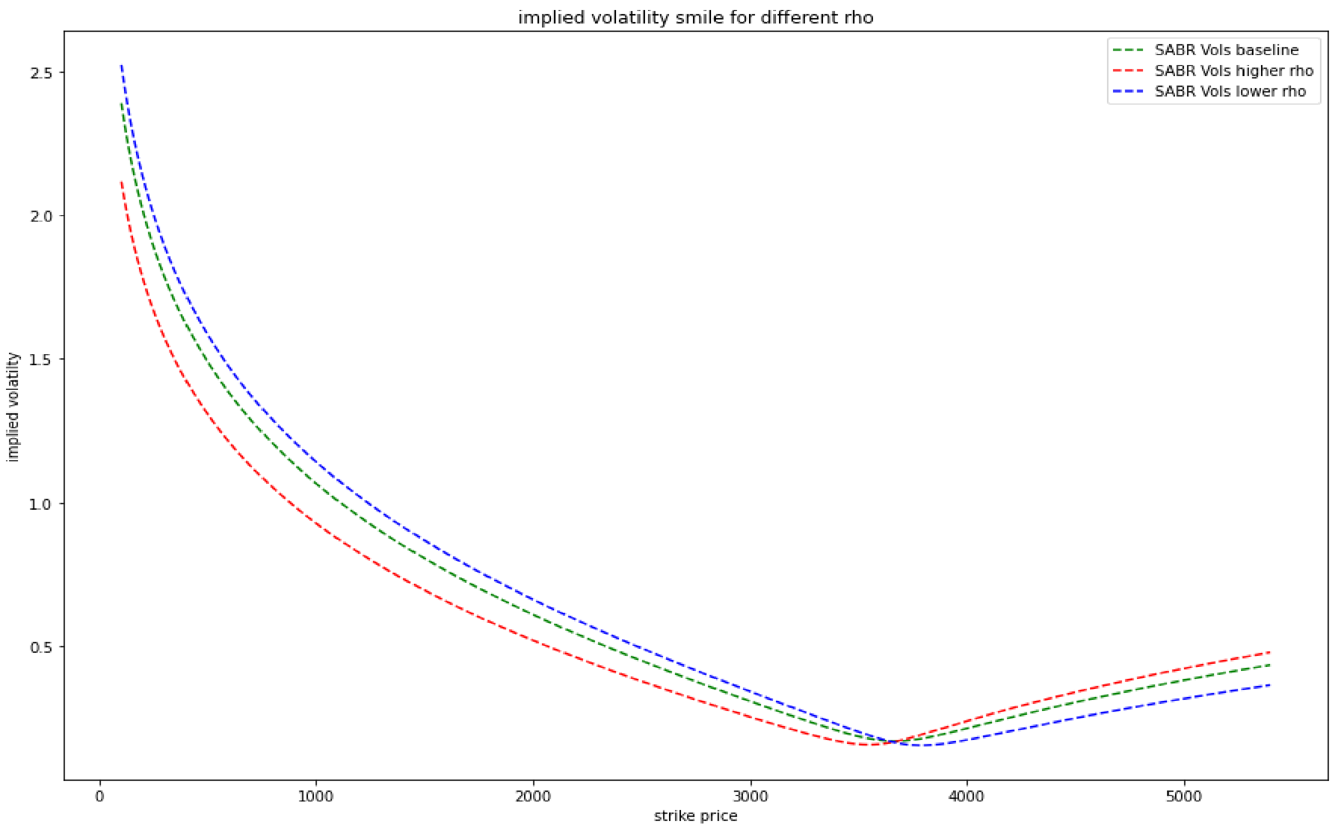
How does change β in the displaced-diffusion model affect the shape of the implied volatility smile.

- Suppose β = 0.5, then the DD volatility curve (blue line) will lie somewhere between that of Bachelier (red line) and Black-Scholes (purple line).
- Increasing β (orange line, where β = 0.9) will cause the DD volatility smile to be flatter and closer to that of Black-Scholes (purple line). The stochastic process has more multiplicative behavior than arithmetic behavior.
- Lowering β (green line, where β = 0.1) will cause the DD volatility smile to be more convex and closer to that of Bachelier (red line). The stochastic process has more arithmetic behavior than multiplicative behavior

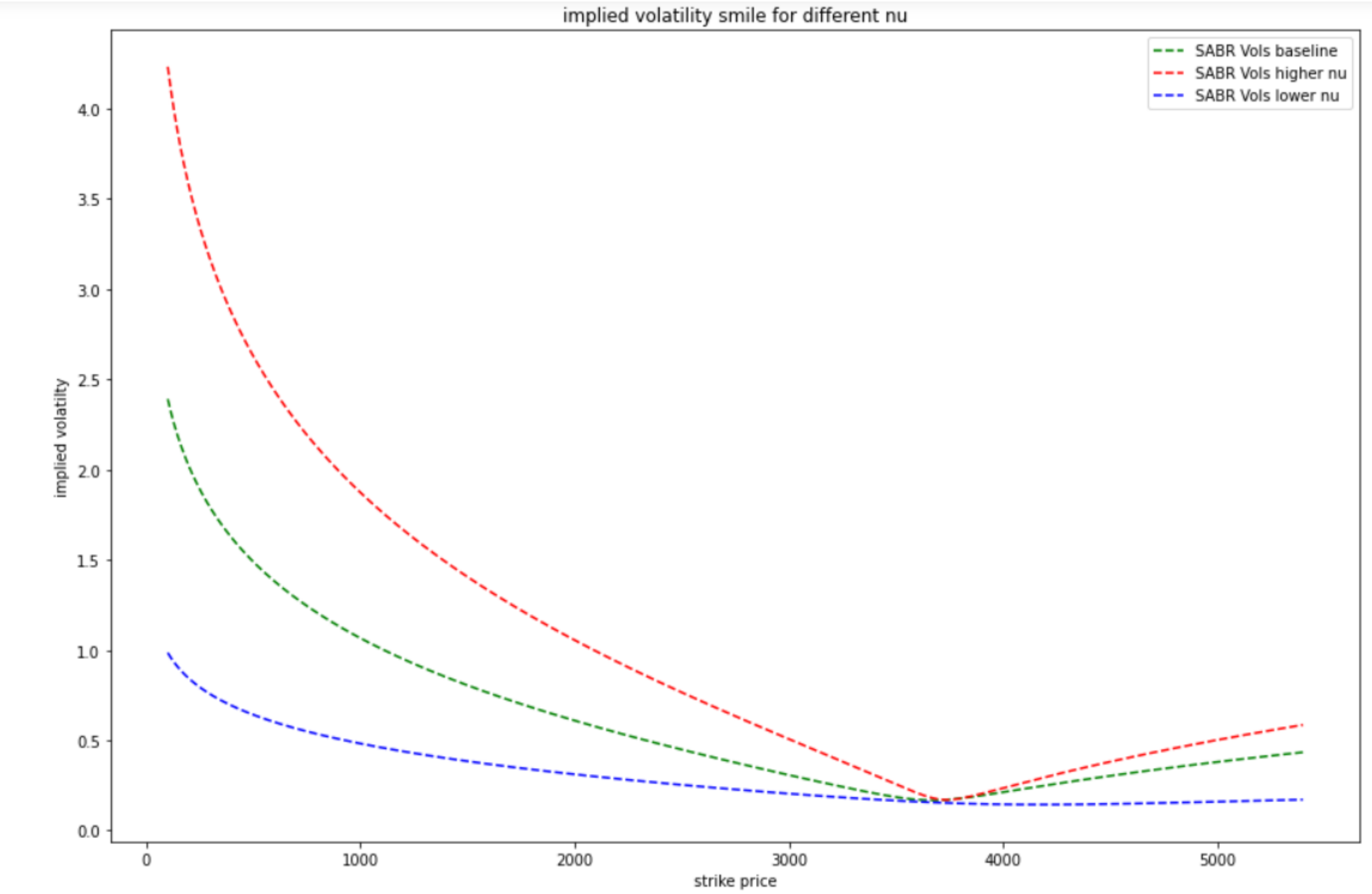


How does change in ρ, ν in the SABR model affect the shape of the implied volatility smile.

- We use the SPX options chain that expires in Jan 20210115 but modify ρ to be 0 and plot its volatility smile, and suppose that this is the baseline curve (green line).
- Decreasing ρ will make the volatility smile to be steeper and higher for Put OTM and less steep/lower for Call OTM (blue line) compared to baseline curve.
- Increasing ρ has the opposite effect. It will make the volatility smile to be steeper and higher for Call OTM and less steep/lower for Put OTM (red line) compared to baseline curve.



- We use the SPX options chain that expires in Feb 20210219 with ν = 2.790 and plot its volatility smile as the baseline
- Increasing ν, for example by 2 in the plot (red line), causes both put OTM and call OTM part of the curve to be steeper than baseline
- Decreasing ν, for example by 2 in the plot (blue line), causes both put OTM and call OTM part of the curve to be less steep than baseline



Project — Part III (Static Replication)

Assumption:

1. SPX Jan expiry data as illustration.
2. The implied volatility is linearly interpolated with OTM call and put option nearest to underlying price
3. The risk-free rate is linearly interpolated.

	Black-Scholes	Bachelier	Static-Replication of European Payoff (SABR Calibrated)
${S_T}^{1/3} + 1.5 * \log(S_T) + 10$	37.72398	37.72214	37.71947
$\sigma_{MF}^2 T = \mathrm{E}\left[\int \sigma^2 dt\right]$	0.00422	0.00423	0.00634
σ	0.18491	680.49473	alpha = 1.817, beta = 0.7, rho = -0.404, nu = 2.790

Project — Part IV (Dynamic Hedging)

We suppose the following parameters: S0 = \$100, σ = 0.2, r = 5%, T = 1/12 year and K = \$100. Applying Black-Scholes model to simulate the stock price over 1 month period, we explore how a short ATM call option position can be dynamically hedged using different frequency of adjusting position in underlying stock and bond.

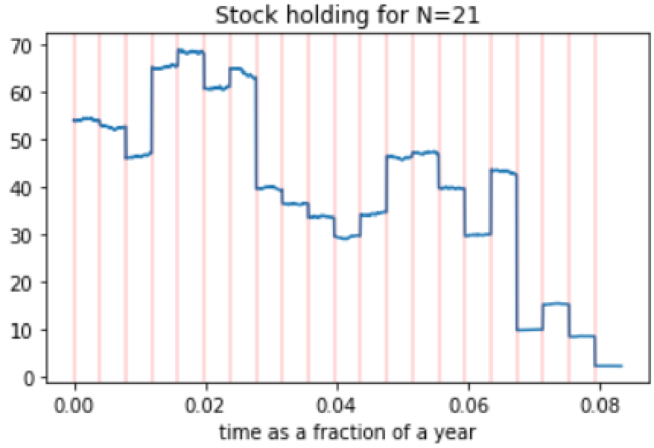
Assume there are 21 trading days over 1 month and we hedge N times during the life of the call options which expires in a month. 2 scenarios were considered:

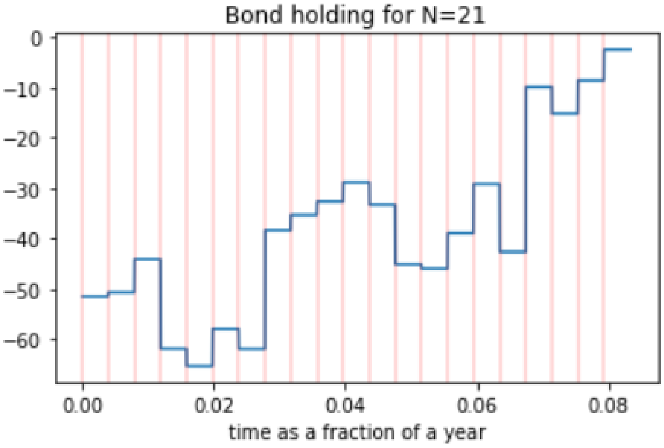
- N = 21 (hedge once every day)
- N = 84 (hedge 4 times every day)

In the simulated Brownian motion, we assumed there is total steps of 1764 (1764 = 21 * 84, lowest common multiple of 21 and 84 was chosen for easier calculation)

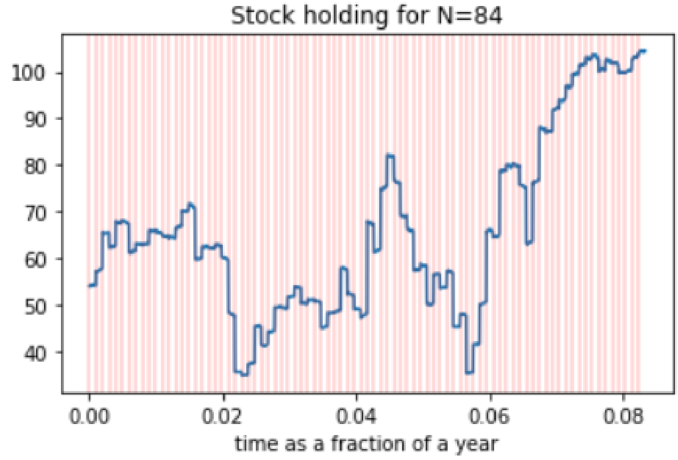
- Hedging starts at step 0 and occurs N times
- There was no hedging on the last trading day (for N = 21) and no hedging in the last quarter of last trading day (for N=84). Hence when option expires, there is a hedging error between replicated portfolio and final call option payoff

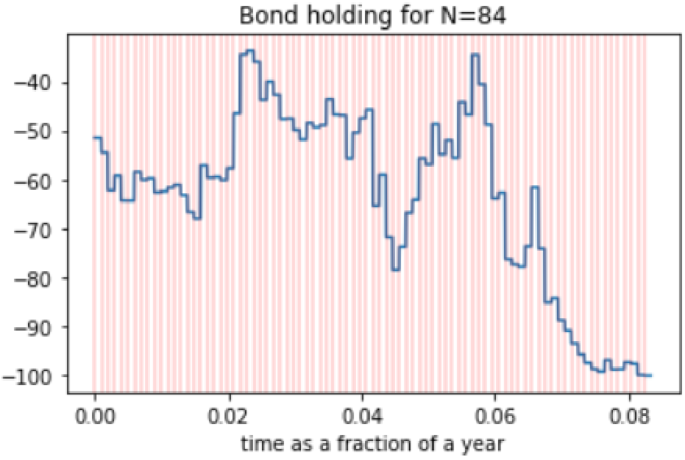
Below is the plot of Stock/Bond portfolio value across 1 month for N = 21. Red vertical line denotes the day when portfolios are hedged. This corresponds to huge jump in stock/bond position to replicate the payoff since last adjustment.





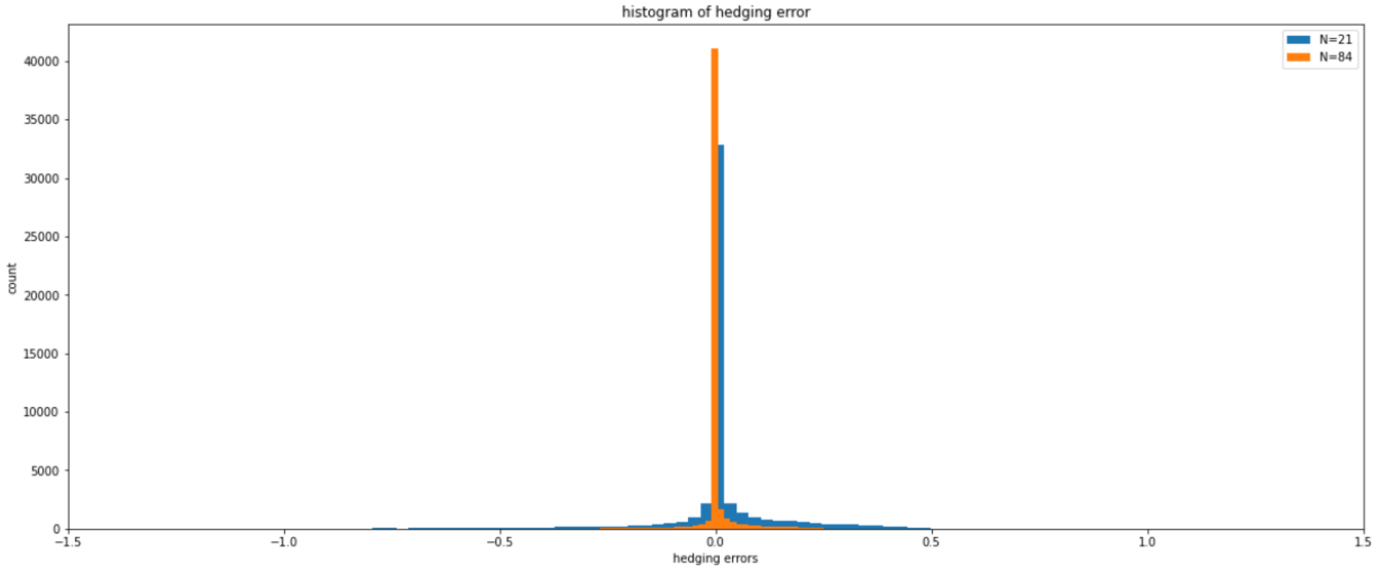
When replication increases from 21 to 84 times, as seen by the larger number of red vertical lines below, there is smaller degree of step up or step down to replicate the payoff since last adjustment, since it is dynamically hedged more often and are following the stock price and option payoff more closely





Therefore, at the end of option lifetime in a month, it is more likely that a replicated portfolio that is hedged 84 times will have less hedging error than one that is hedged only 21 times

We ran 50000 paths in the simulation and plotted the histogram of the hedging error for N = 21 and N = 84 as follows:



Indeed, for replicated portfolios with N=84, hedging errors are more concentrated at 0, compared to that with N=84. Also, histogram of hedging errors of replicated portfolios with N=21 has fatter tails, indicating higher likelihood of large hedging error when hedging frequency decreases.

To check that the hedging errors are calculated correctly, we get 50 000 runs of final call option payout (final stock price – strike price) and compare against 50 000 runs of replicated position subtracted by the hedging errors:

```
#call payoff from 50000 runs
call_payoff = blackscholespath[:, -1] - 100
call_payoff = np.where(call_payoff < 0, 0, call_payoff )
call_payoff
```

executed in 48ms, finished 23:17:26 2021-11-20

```
array([0.        , 0.        , 1.18193246, ..., 9.79459951, 7.78585948,
       0.        ])
```

```
#50000 runs of hedged portfolio with the hedging errors removed at options expiry
hedged_portfolios_84[-1] - (stockhedge_errors_84[-1] + bondhedge_errors_84[-1])
```

executed in 17ms, finished 23:17:29 2021-11-20

```
array([0.        , 0.        , 1.18193246, ..., 9.79459951, 7.78585948,
       0.        ])
```

From above, we showed that replicated position matched final call payout if hedging errors were removed. Hence hedging errors are calculated correctly.

Appendix

1. Model Derivation for model valuation.