OBTAINING ANALYTICAL PRICING FORMULA FOR VARIATION OF CALL OPTION

Risk-neutral pricing formulas can be obtained for a particular option in various ways. Two of the techniques used to accomplish this are solving the Black-Scholes equation directly, using the options associated payoff, or finding the expected payoff and discounting back to the current time. Here we are going to use the first method and solve the Black-Scholes equation using the risk neutral rate as the underlying equity's growth (diffusion) parameter.

In the absence of dividends, an option written on a stock that has a price following the SDE defined in Equation (1) has a Black-Scholes pricing equation of the form of Equation (2) (derivation omitted). Here V is the price of the option, S is the price of the underlying asset, σ is the standard deviation of the stock's return and r is the risk neutral rate which replaces the diffusion parameter μ in the SDE (making this a risk neutral price).

$$dS_t = \mu S_t dt + \sigma S_t dZ_t \tag{1}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2}$$

From this, the current price of the option can be considered as a discounted price of the future price of the option. Using the discounting factor, $e^{-r(T-t)}$, the value of the option today can be written as a discounted form of future value. This can be observed in Equation (3) where the future time is T and the current time is t.

$$V(s,t) = e^{-r(T-t)}U(s,t) \tag{3}$$

Substituting (3) into (2), Equation (4) is obtained.

$$\frac{\partial U}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0 \tag{4}$$

Since we are solving backwards in time and for simplicity, we can write $\tau = T - t$. Applying the chain rule (Equation (5)) to Equation (4), Equation (6) is obtained which is the backward Kolmogorov equation.

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\frac{\partial}{\partial \tau} \tag{5}$$

$$\frac{\partial U}{\partial \tau} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} \tag{6}$$

Writing $\xi = \log S \Leftrightarrow S = e^{\xi}$, applying the chain rule to this new variable (Equation (7) and Equation (8)) and substituting into Equation (6), Equation (9) is obtained.

$$\frac{\partial}{\partial S} = \frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi} = \frac{1}{S}\frac{\partial}{\partial \xi}$$

(7)

$$\frac{\partial^2}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial}{\partial \xi}\right) = \frac{1}{S^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right)$$

(8)

$$\frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial \xi}$$

(9)

Writing $x = \xi + \left(r - \frac{\sigma^2}{2}\right)\tau$ and $U = W(x, \tau)$, Equation (9) is simplified to Equation (10) which takes the form of the well-known one-dimensional heat equation. Here x can be represented in the original variables as $x = \log S\left(r - \frac{\sigma^2}{2}\right)(T - t)$.

$$\frac{\partial W}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 W}{\partial x^2}$$

(10)

Equation (10) can be solved using similarity reduction. Setting $W = \tau^{\alpha} f(\eta)$ where $\eta = \frac{(x - x')}{\tau^{\beta}}$, Equation (11) is obtained where $f' = \frac{\partial f}{\partial \eta}$.

$$\tau^{\alpha-1}(\alpha f - \beta \eta f') = \frac{1}{2}\sigma^2 \tau^{\alpha - 2\beta} f''$$

(11)

To reduce the dimension of this problem set $\beta = \frac{1}{2}$ which eliminates τ from the equation to give Equation (12) – a second order ODE.

$$\alpha f - \frac{1}{2}\eta f' = \frac{1}{2}\sigma^2 f''$$

(12)

We need to select α such that

$$\int_{\mathbb{R}} W \, dx = 1 \; \forall \, \tau$$

$$\Rightarrow \; \tau^{\alpha} \int_{\mathbb{R}} f\left(\frac{x - x'}{\sqrt{\tau}}\right) dx = 1 \; \forall \, \tau$$

$$\Rightarrow \; \tau^{\alpha} \int_{\mathbb{R}} f(\eta)\sqrt{\tau} \, d\eta = 1 \; \forall \, \tau$$

$$\Rightarrow \; \tau^{\alpha + \frac{1}{2}} \int_{\mathbb{R}} f(\eta) \, d\eta = 1 \; \forall \, \tau$$

We want this to hold regardless of time which implies that $\alpha = -\frac{1}{2}$.

From this Equation (12) becomes

$$-f - \eta f' = \sigma^2 f''$$

Which can be written as

$$\sigma^2 f'' + (f\eta)' = 0$$

And can be integrated to obtain

$$\sigma^2 f' + f\eta = A$$

In the limit of $\eta \to \infty$, $f' \to 0$ and $f \to 0 \Rightarrow A = 0$ (by definition of the Kolmogorov equation). Integrating as a separable ODE it follows that

$$f(\eta) = C \exp\left(-\frac{\eta^2}{2\sigma^2}\right)$$

Where C is a normalising coefficient such that $C\int_{\mathbb{R}} \exp\left(-\frac{\eta^2}{2\sigma^2}\right) d\eta = 1$. After manipulation, it follows that $C = \frac{1}{\sqrt{2\pi}\sigma}$. From this f is fully defined as

$$f(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\eta^2}{2\sigma^2}\right)$$

From this it can be concluded that the fundamental solution takes form

$$W_f = \frac{1}{\sqrt{2\pi\tau}\sigma} \exp\left(-\frac{(x - x')^2}{2\sigma^2\tau}\right)$$

Which is the probability density function for a normal random variable x with mean of x' and standard deviation of $\sqrt{\tau}\sigma$. This allows us to find the solution to the BSE at different values of x'.

Introducing the payoff at time $t = T$ $(\tau = 0)$

$$V(S,T) = Payoff(S)$$

Recall $x = \xi + \left(r - \frac{\sigma^2}{2}\right)\tau$, so at maturity $\tau = 0 \Longrightarrow x = \xi = \log S \Longrightarrow S = e^x$

$$\therefore W(x, 0) = Payoff(e^x)$$

$$\Rightarrow W(x,\tau) = \int_{\mathbb{R}} W_f(x,\tau; x') Payoff(e^{x'}) dx' \tag{13}$$

Recall also

$$x' = \log S' \Rightarrow dx' = \frac{dS'}{S'}$$

$$e^{x'} = S'$$

$$\Rightarrow Payoff\left(e^{x'}\right)dx' = Payoff(S')\frac{dS'}{S'}$$

Substituting this (and the original variables) into Equation (13) we obtain

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} Payoff(S')\frac{dS'}{S'}$$

Substituting the payoff for this European Call option, $Payoff(S') = \max(\ln(S') - \ln(K), 0)$, into this we obtain the following

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{K}^{\infty} e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} (\log(S') - \log(K))\frac{dS'}{S'}$$

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{K}^{\infty} e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} (\log(S'))\frac{dS'}{S'}$$
$$- \log(K)\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{K}^{\infty} e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} \frac{dS'}{S'}$$

Substituting the variable $x' = \log S' \Rightarrow \log\frac{1}{S'} = -x'$

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log K}^{\infty} e^{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\, x'dx'$$
$$- \log(K)\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log K}^{\infty} e^{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} dx' \tag{14}$$

Observing the second term in equation (14)

$$- \log(K)\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log K}^{\infty} e^{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} dx'$$

We can use the substitution

$$u = \frac{-x' + \log(S) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

$$du = \frac{-1}{\sigma\sqrt{T-t}}dx' \Rightarrow dx' = -\sigma\sqrt{T-t}\, du$$

With a limit change of

$$\int_0^{\infty} = \int_{\frac{\log\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}}^{-\infty}$$

Substituting these we can express this term in the following way

$$- \log(K)\frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}u^2} du$$

Where

$$d = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

This can be expressed in terms of the cumulative distribution function for a normal distribution $N(x)$ obtaining

$$\log(K)\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log K}^{\infty} e^{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}dx' = \log(K)\,e^{-r(T-t)}N(d)$$

(15)

Focusing on the first term in equation (14)

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log K}^{\infty} e^{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,x'dx'$$

We can make the substitutions

$$u = \frac{-x' + \log(S) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

$$du = \frac{-1}{\sigma\sqrt{T-t}}dx' \Rightarrow dx' = -\sigma\sqrt{T-t}\,du$$

$$x' = \log S + \left(r - \frac{\sigma^2}{2}\right)(T-t) - \left(\sigma\sqrt{T-t}\right)u$$

The limits change in the same way as in the previous term evaluated. Substituting this into the first term we obtain

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{d}\left(\log(S) + \left(r - \frac{\sigma^2}{2}\right)(T-t) - \left(\sigma\sqrt{T-t}\right)u\right)\left(\sigma\sqrt{T-t}\right)e^{-\frac{1}{2}u^2}du$$

Which can be evaluated as two integrals

$$\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{d}\left(\log(S) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)e^{-\frac{1}{2}u^2}du - \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{d}\left(\left(\sigma\sqrt{T-t}\right)u\right)e^{-\frac{1}{2}u^2}du$$

(16)

Rearranging and evaluating Equation (16) we obtain

$$e^{-r(T-t)}\left\{\left(\log(S) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)N(d) + \frac{\left(\sigma\sqrt{T-t}\right)e^{-\frac{d^2}{2}}}{\sqrt{2\pi}}\right\}$$

From this we can conclude the first term takes the form

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log K}^{\infty} e^{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\, x'dx'$$

$$= e^{-r(T-t)}\left\{\left(\log S + \left(r-\frac{\sigma^2}{2}\right)(T-t)\right)N(d) + \frac{\left(\sigma\sqrt{T-t}\right)e^{-\frac{d^2}{2}}}{\sqrt{2\pi}}\right\} \tag{17}$$

Substituting (15) and (17) into (14) we obtain

$$V(S,t) = e^{-r(T-t)}\left\{\left(\log(S) + \left(r-\frac{\sigma^2}{2}\right)(T-t)\right)N(d) + \frac{\left(\sigma\sqrt{T-t}\right)e^{-\frac{d^2}{2}}}{\sqrt{2\pi}}\right\} - e^{-r(T-t)}\,N(d)\log(K)$$

Rearranging, we obtain the analytical risk-neutral pricing formula for this option

$$V(S,t) = e^{-r(T-t)}\left\{\left(\log\left(\frac{S}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)(T-t)\right)N(d) + \frac{\sigma}{\sqrt{2\pi}}\left(\sqrt{T-t}\right)e^{-\frac{d^2}{2}}\right\}$$

Where

$$\begin{cases} d = \dfrac{\log\left(\frac{S}{K}\right) + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \\ N(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\,du \end{cases}$$

VERIFYING THE ANALYTICAL PRICE WITH MONTE CARLO PRICING

In Monte Carlo based pricing, the aim is to simulate the equities price using Equation (1) with the risk-neutral rate replacing μ obtaining the following

$$dS_t = rS_t dt + \sigma S_t dZ_t$$

Payoffs are then calculated for each path, summed, averaged and discounted back to the current time.

This can be simulated under this SDE in multiple ways.

Method 1: Discretise the equation using first order Euler approximations for the terms in the SDE above then set $dZ_t = \sqrt{dt}\emptyset$ where $\emptyset \sim N(0,1)$ by GBM.

Applying this we obtain

$$S_{t+1} - S_t = rS_t dt + \sigma S_t\sqrt{dt}\emptyset$$

$$\Rightarrow S_{t+1} = S_t\left(1 + r\,dt + \sigma\sqrt{dt}\emptyset\right)$$

Where we know S_0 and dt is the time step in respect to the time frame you are simulating over.

Method 2: Transform the SDE using Itô's lemma and obtain the discrete equation

$$S_{t+1} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\sqrt{dt}\emptyset\right)$$

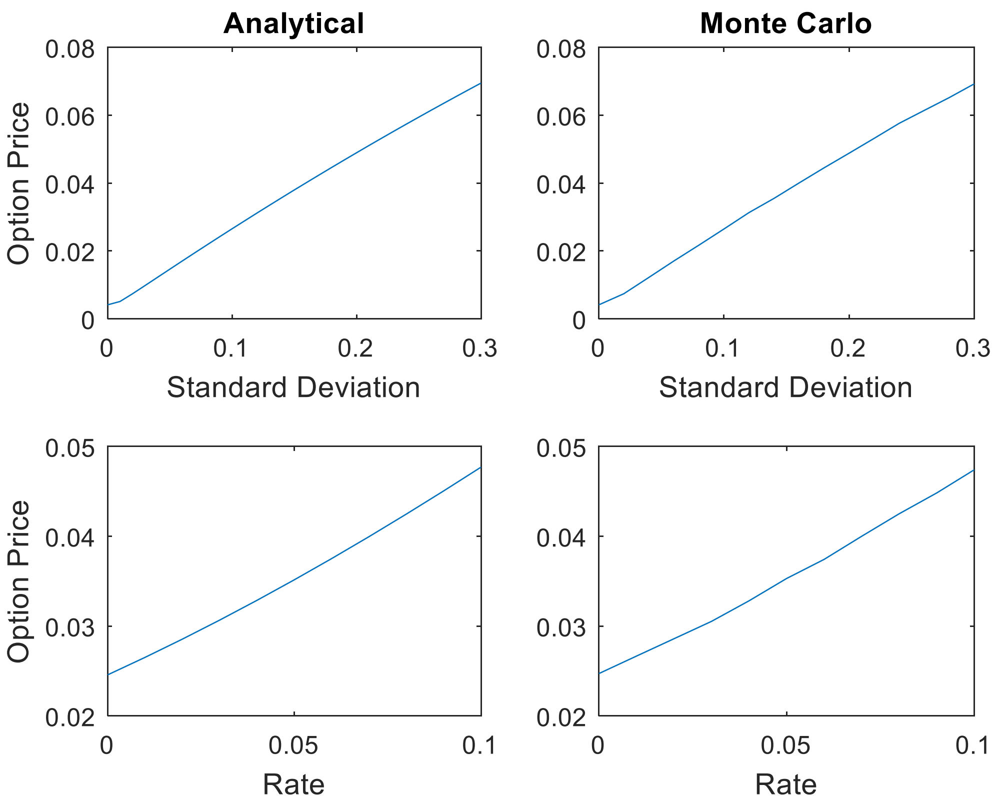
And simulate in the same way as method 1.

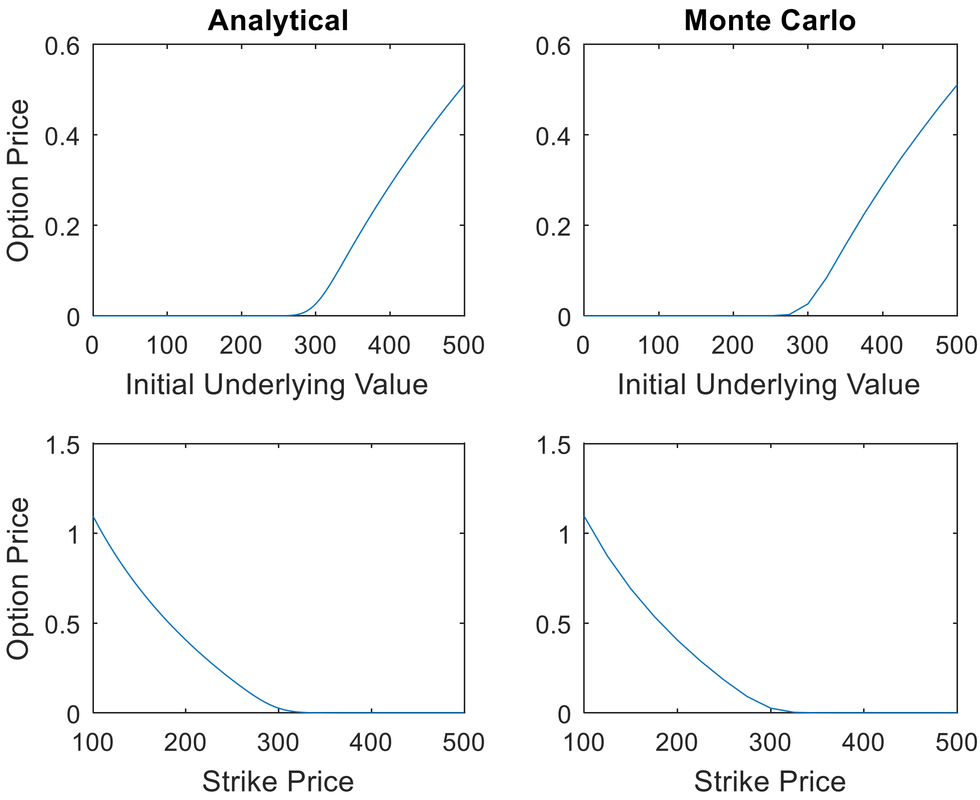
Link to GitHub repository: https://github.com/jerivington/LogCallPrice

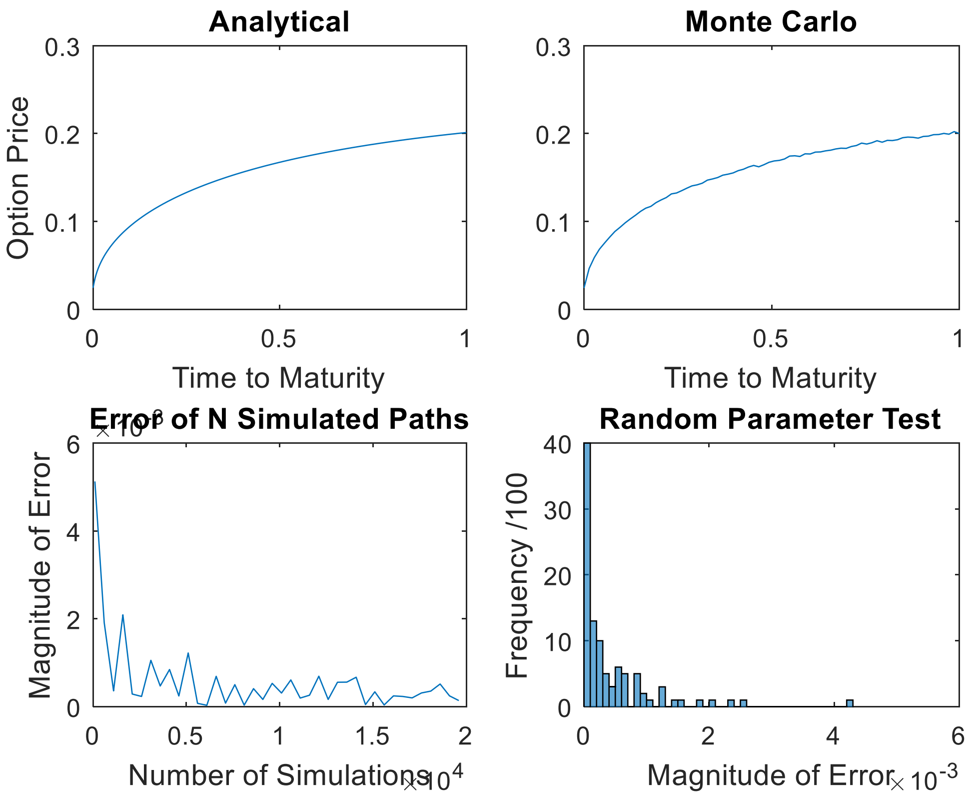
Executing 'Verification' in GitHub repository, the output below was obtained which compares the analytical price and the Monte Carlo simulated price for the option of interest using method 1. Each parameter was incremented for both the analytical and simulated prices and plotted next to each other. Unless incremented, the parameters are: $S = 300, K = 300, r = 0.01, maturity = \frac{150}{365}$ and $\sigma = 0.1$ for 100000 simulated equity paths. As we can see both the analytical and simulated prices follow the same behaviour for the parameter incriminations.

The script also tests the magnitude of error of the Monte Carlo method. The behaviour of this can be observed in the output where, for N increasing, the magnitude of error tends to zero.

The accuracy of the Monte Carlo simulation can also be tested by finding the magnitude of error between the analytical price and Monte Carlo generated price for random selection of parameters. In the final plot random parameters were selected for $S, K, r, maturity$ and σ for 100 different instances. A histogram of frequency of the error magnitude between the analytical and simulated price at each instance can be observed here.




Analytical
Monte Carlo
Option Price
0.6
0.4
0.2
0
0 100 200 300 400 500
Initial Underlying Value
Initial Underlying Value
Option Price
1.5
1
0.5
0
100 200 300 400 500
Strike Price
Strike Price


Analytical
Monte Carlo
Option Price
0.3
0.2
0.1
0
0 0.5 1
Time to Maturity
Time to Maturity
Error of N Simulated Paths
Random Parameter Test
Magnitude of Error
0 2 4 6
0 0.5 1 1.5 2
Number of Simulations
×10^4
Frequency /100
0 10 20 30 40
0 2 4 6
Magnitude of Error
× 10^-3

CONCLUSION

In conclusion, the analytical price for an option price with the payoff structure $\text{payoff} = \max(\ln(S) - \ln(K))$, $V(S,t)$, was obtained using risk-neutral pricing and takes the following form

$$V(S,t) = e^{-r(T-t)} \left\{ \left(\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t) \right) N(d) + \frac{\sigma}{\sqrt{2\pi}} \left(\sqrt{T-t}\right) e^{-\frac{d^2}{2}} \right\}$$

Where

$$\begin{cases} d = \dfrac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2 (T-t)}} \\ N(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du \end{cases}$$

The analytical price was then tested against a Monte Carlo generated price for the same option by: incrementing each parameter of the analytical price and Monte Carlo price, testing the convergence of the Monte Carlo price to the analytical price for increasing N, testing the magnitude of error of the Monte Carlo price for a random sample of variables and plotting the tests performed.

Both methods of pricing displayed the same behaviour in the parameter increment test, the Monte Carlo price converged to the analytical price as N increased with an error of $\sim 10^{-4}$ for $N = 20000$ and all errors were less than 4.3×10^{-3} in the random parameter test.