TODAY: **PORTFOLIO OPTIMIZATION**

"MARKOWITZ" MODEL

↳ NOBEL PRIZE, 1990

IDEA: HAVE n ASSETS (SHARES OF n STOCKS)

LET R_i = RETURN ON ASSET i

RANDOM VARIABLE

RELATIVE CHANGE IN VALUE OVER FIXED TIME PERIOD (DAYS, WEEKS, YEARS)

CAN CODE RETURN AS A FRACTION, OR IN % PTS.

IF R_i IS A FRACTION, THEN VALUE OF ASSET i CHANGES LIKE

$$V_i \longrightarrow (1+R_i)V_i$$

EX: $R_i = .15$ +15% RETURN

$R_i = -.07$ −7% RETURN

FACT: LET x_i BE FRACTION OF INVESTMENT IN ASSET i

$$\sum_{i=1}^{m} x_i = 1 \quad , \quad x_i \geq 0$$

THEN RETURN ON PORTFOLIO IS

$$R_p = \sum_{i=1}^{m} x_i R_i$$

= WEIGHTED AVERAGE OF ASSET RETURNS

PROOF INVEST A TOTAL OF \$D. THEN INVESTMENT IN ASSET i IS $x_i D$

FINAL VALUE OF ASSET i IS

$$(1+R_i)\, x_i D$$

TOTAL VALUE OF PORTFOLIO BECOMES

$$\sum_{i=1}^{m} (1+R_i)\, x_i D$$

SO RETURN OF PORTFOLIO IS

$$\frac{\left[\cancel{D}\sum_{i=1}^{m}(1+R_i)x_i\right]-\cancel{D}}{\cancel{D}}$$

$$= \sum_{i=1}^{m}(1+R_i)x_i \quad - 1$$

$$= \underbrace{\left[\sum_{i=1}^{m}x_i\right]}_{=1} + \left[\sum_{i=1}^{m}R_i x_i\right] - 1$$

SO $R_p = \sum_{i=1}^{m} R_i x_i$ Q.E.D.

EXPECTED RETURN FOR PORTFOLIO IS THEN

$$E[R_p] = E\left[\sum_{i=1}^{m}R_i x_i\right] = \sum_{i=1}^{M} E[x_i R_i]$$

\# RV

$$= \sum_{i=1}^{m} x_i E[R_i]$$ LINEAR

NEXT, WANT

$$\text{VAR}[R_p] = \text{VAR}\left[\sum_{i=1}^{n} x_i R_i\right]$$

NOTE: VARIANCES ADD IF RVs ARE INDEPENDENT ... NOT GENERALLY THE CASE HERE.

IN GENERAL CAN USE FORMULA FOR VARIANCE OF A SUM OF RVs:

$$\text{VAR}[R_p] = \sum_{i=1}^{n} x_i^2 \,\text{VAR}[R_i] + 2 \sum_{i \neq j} \text{COV}(R_i, R_j)\, x_i x_j$$

QUADRATIC FUNCTION OF x_i

"RECALL" IF X, Y ARE RVs, THEN CORRELATION COEFFICIENT IS

$$\underset{\text{"RHO"}}{\rho} = \frac{\text{COV}(X,Y)}{\sigma_X \sigma_Y}$$

PROBLEM

FOR A GIVEN VALUE OF

$$E[R_p]$$

WOULD LIKE TO

MINIMIZE

$$VAR[R_p]$$

"EFFICIENT" PORTFOLIO

DATA CONSISTS OF EMPIRICAL ESTIMATES FOR

$$E[R_i]$$

ρ_{ij} CORRELATION BETWEEN R_i & R_j

σ_i STD DEV R_i

IN TERMS OF FUTURE, ρ_{ij} AND σ_i INFORMATION IS MORE STABLE THAN $E[R_i]$. "PAST PERFORMANCE IS NOT INDICATIVE OF FUTURE..."

PORTFOLIO PROBLEM IS
WELL-POSED

MINIMIZING CONVEX QUADRATIC
FUNCTION WITH LINEAR CONSTRAINTS.

SENSITIVITY REPORT:

- LAGRANGE MULTIPLIERS
  GIVE TANGENTS TO
  NONLINEAR CURVES

  EX: MULTIPLIER FOR
  EXPECTED VALUE
  REQUIREMENT

- REDUCED GRADIENTS ARE
  MULTIPLIERS FOR A
  TIGHT BOUND CONSTRAINT;
  USUALLY $x_i \geq 0$

  EX: FRACTION MCDONALDS

EXTENSIONS:

① REQUIRED LOWER AND/OR UPPER BOUNDS ON x_i

EX: McDONALDS $\geq .2$

② UPPER AND/OR LOWER BOUNDS ON AN x_i IF $x_i > 0$

USE BINARY VARIABLES y_i

$$l_i y_i \leq x_i \leq u_i y_i \qquad y_i \in \{0,1\}$$

USE ASSET i IN PORTFOLIO

CAN ALSO ADD CONSTRAINT LIKE

$$\sum y_i \leq K$$

TO LIMIT # OF ASSETS USED IN PORTFOLIO.