Jeremiah Lübke
108015230366

Statistical Mechanics
Homework 8

June 24, 2019

H8.1

Consider the hyperrelativistic fermi gas with $\epsilon_p = p\,c$.

(a) In order to compute the grand potential $\Omega = -k\,T \log Z$ consider at first the partition function for fermions:

$$Z = \prod_\nu \delta_{\nu\nu} \left(1 + e^{\beta(\mu - \epsilon_\nu)}\right)$$

$$\begin{aligned}
\log Z &= \sum_\nu \delta_{\nu\nu} \log\left(1 + e^{\beta(\mu-\epsilon_\nu)}\right) \\
&= \underbrace{\sum_s}_{=g_s} \int \mathrm{d}^3p \, \frac{V}{(2\pi\hbar)^3} \log\left(1 + e^{\beta(\mu - pc)}\right) \\
&= \frac{g_s\,V}{2\pi^2\hbar^3} \int_0^\infty \mathrm{d}p \, \underbrace{p^2}_{u'} \underbrace{\log\left(1 + e^{\beta(\mu-pc)}\right)}_{v} \\
&= \frac{g_s\,V}{2\pi^2\hbar^3} \frac{\beta\,c}{3} \int_0^\infty \frac{p^3\,\mathrm{d}p}{e^{\beta(pc-\mu)} + 1}
\end{aligned}$$

Continue with the integral $I = \int_0^\infty \frac{p^3\,\mathrm{d}p}{e^{\beta(pc-\mu)}+1}$. Substituting $z = \beta(p\,c - \mu)$ yields:

$$\begin{aligned}
(c\beta)^4\,I &= \int_{-\beta\mu}^\infty \frac{(z+\beta\mu)^3}{e^z+1}\,\mathrm{d}z \\
&= \int_0^\infty \frac{(\beta\mu+z)^3}{e^z+1}\,\mathrm{d}z + \int_0^{\beta\mu} \frac{(\beta\mu-z)^3}{e^{-z}+1}\,\mathrm{d}z \quad \text{with} \quad \frac{1}{e^{-z}+1} = 1 - \frac{1}{e^z+1} \quad \text{and} \quad \beta\mu \gg 1 \\
&= \int_0^{\beta\mu} (\beta\mu - z)^3\,\mathrm{d}z + \int_0^\infty \frac{(\beta\mu+z)^3 - (\beta\mu-z)^3}{e^z+1}\,\mathrm{d}z
\end{aligned}$$

The first integral is easily evaluated:

$$\int_0^{\beta\mu} (\beta\mu - z)^3\,\mathrm{d}z = \int_0^{\beta\mu} w^3\,\mathrm{d}w = \frac{(\beta\mu)^4}{4}$$

And for the second, one works out the numerator of the fraction

$$(\beta\mu+z)^3 - (\beta\mu-z)^3 = 6\,(\beta\mu)^2\,z + 2\,z^3$$

and uses

$$\int_0^\infty \frac{z^{a-1}}{e^z+1}\,\mathrm{d}z = (1 - 2^{1-a})\,\Gamma(a)\,\zeta(a)$$

$$\implies \int_0^\infty \frac{z\,\mathrm{d}z}{e^z+1} = \frac{\pi^2}{12}, \quad \int_0^\infty \frac{z^3\,\mathrm{d}z}{e^z+1} = \frac{7\pi^4}{120}$$

Therewith:

$$
\begin{aligned}
(c\beta)^4\, I &= \frac{(\beta\mu)^4}{4} + \frac{(\beta\mu\pi)^2}{2} + \frac{7\pi^4}{60} \\
\implies \log Z &= \frac{g_s V \beta\mu^4}{24\pi^2\hbar^3c^3}\left(1 + \frac{2\pi^2}{(\beta\mu)^2} + \frac{7}{15}\left(\frac{\pi}{\beta\mu}\right)^4\right) \\
\implies & \boxed{\Omega = -k\,T\log Z = -\frac{g_s V\mu^4}{24\pi^2\hbar^3c^3}\left(1 + \frac{2(\pi kT)^2}{\mu^2} + \mathcal{O}\left(T^4\right)\right)}
\end{aligned}
$$

(b) For the thermal equation of state, integrate the energy per state with respect to the state number:

$$
\begin{aligned}
E &= \int\limits_{pc<\mu} \epsilon\,\mathrm{d}N = \frac{g_s V c}{(2\pi\hbar)^3} 4\pi \underbrace{\int\limits_0^{\mu/c} \mathrm{d}p\, p^3}_{=\frac{\mu^4}{4c^3}} = \frac{g_s V\mu^4}{8\pi^2\hbar^3c^3} \\
\implies \Omega &= -p\,V = -\frac{g_s V\mu^4}{24\pi^2\hbar^3c^3} = -\frac{1}{3}E \\
\implies & \boxed{3\,p\,V = E}
\end{aligned}
$$

(c) In order to find the heat capacity, one needs to take the first correction into account (in the following: $C = C_V \approx C_p$, since $C_V - C_p \sim \mathcal{O}(T^3)$):

$$
\begin{aligned}
S &= -\frac{\partial\Omega}{\partial T} = \frac{g_s V\mu^2k^2}{6\hbar^3c^3}T \\
& \boxed{C = T\frac{\partial S}{\partial T} = \frac{g_s V\mu^2k^2}{6\hbar^3c^3}T}
\end{aligned}
$$