Basic Adaptive Control

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1 Theoretical Formulation

The fundamental purpose of an adaptive controller is to design a control law such that it can deal with parametric uncertainties. For example, given some dynamics:

$$\dot{x} = f(t, x, u)$$

, we can design a control law such that $\tau = -kr + V_r$, where V_r is the adaptive component of the control law that deals with the some unknown parameter in the mathematical model.

1.1 Example

Consider an Euler-Lagrange system:

$$m\ddot{q} + mglsin(q) = \tau \tag{1}$$

where $q, \dot{q} \in \mathbb{R}$ are the states, $m \in \mathbb{R}$ is the mass of the system, $l \in \mathbb{R}$ is a constant length, $g \in \mathbb{R}$ is gravity, and $\tau \in \mathbb{R}$ is the control input. The objective here is to control our system such that the error signal converges to 0 as time goes to ∞. We now define our error signal to be:

$$e = q_d - q \tag{2}$$

. The derivative of the error signal can be also expressed as:

$$\dot{e} = \dot{q_d} - \dot{q} \tag{3}$$

$$\ddot{e} = \ddot{q}_d - \ddot{q} \tag{4}$$

To analyse a non-linear system, we have to formulate a Lyapunov function. In this case, our Lyapunov candidate function is expressed as:

$$V = \frac{1}{2}mr^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{5}$$

where

$$r = \dot{e} + \alpha e \tag{6}$$

, is known as the filter tracking error term, and Γ is a positive definite matrix. $\tilde{\theta}$ is the difference between the true value of an unknown parameter and its estimated value. $\tilde{\theta}$ can be expressed as:

$$\tilde{\theta} = \theta - \hat{\theta} \tag{7}$$

. Taking the derivative of Eq. 7, we get

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}} \tag{8}$$

since θ is just a constant value. From Eq. 5, we can see that the Lyapunov function is going to be positive definite. We now take the derivative of the function, which then gives us:

$$\dot{V} = rm\dot{r} + \tilde{\theta}\Gamma^{-1}\dot{\tilde{\theta}} \tag{9}$$

. With $\dot{r} = \dot{e} + \alpha\ddot{e}$, then

$$\dot{r} = \alpha\dot{e} + \ddot{q}_d - \ddot{q} \tag{10}$$

Substituting Eq. (10) into Eq. (9), we then get

$$\dot{V} = rm(\alpha\dot{e} + \ddot{q}_d - \ddot{q}) + \tilde{\theta}\Gamma^{-1}\dot{\tilde{\theta}} \tag{11}$$

$$= r(m\alpha\dot{e} + m\ddot{q}_d + mglsin(q) - \tau) + \tilde{\theta}\Gamma^{-1}\dot{\tilde{\theta}} \tag{12}$$

. At this point, we want to design τ so that we can remove the unwanted terms in Eq. (12). One of the common forms that produce at worst a negative semidefinite Lyapunov derivative can be expressed as:

$$\dot{V} = -kr^2 \tag{13}$$

. In this case, the uncertain term of our system is its mass and length. Therefore, we must find a way to estimate it using θ. We can express all the linearly parameterised mass in Eq. (12) to become:

$$m\alpha\dot{e} + m\ddot{q}_d + mglsin(q) = \begin{bmatrix} q_d + \alpha e & gsin(q) \end{bmatrix} \begin{bmatrix} m \\ ml \end{bmatrix} \tag{14}$$

$$= \mathbf{Y}\theta \tag{15}$$

. Now, we substitute Eq. (15) back into Eq. (12), which then gives us:

$$\dot{V} = r\mathbf{Y}\theta - \tau r - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} \tag{16}$$

. Using $\tau = kr + V_r$, we can then define V_r to be:

$$V_r = \mathbf{Y}\hat{\theta} \tag{17}$$

. Then, Eq. (16) becomes:

$$\dot{V} = r\mathbf{Y}(\theta - \hat{\theta}) - kr^2 - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} \tag{18}$$

$$= r\mathbf{Y}\tilde{\theta} - kr^2 - \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} \tag{19}$$

. Now, the last and obvious step is to equate $r\mathbf{Y}\tilde{\theta} = \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}}$. As of this moment, we have explicitly discussed if every term is known or not. Considering that we can measure what our state q is, and we know what trajectory we want to track, i.e. what our $q_d(t)$ is, we only have to define $\dot{\hat{\theta}}$. To remove the remaining unwanted terms, we define $\dot{\hat{\theta}}$ to be:

$$\dot{\hat{\theta}} = \Gamma\mathbf{Y}r \tag{20}$$

. Then, $\dot{V}$ finally becomes:

$$\dot{V} = r\mathbf{Y}\tilde{\theta} - kr^2 + \tilde{\theta}\Gamma^{-1}\dot{\hat{\theta}} \tag{21}$$

$$= r\mathbf{Y}\tilde{\theta} - kr^2 - \tilde{\theta}\mathbf{Y}r \tag{22}$$

. Since $r\mathbf{Y}\tilde{\theta}$ and $\tilde{\theta}\mathbf{Y}r$ are both scalars, they then cancel each other out, hence giving us:

$$\dot{V} = -kr^2 \tag{23}$$

Now, recalling that

$$\tau = kr + \mathbf{Y}\hat{\theta}$$

, we know that $\hat{\theta}$ is a special term because it is not a fixed value. We call $\hat{\theta}$ the adaptive law, which constantly updates by using

$$\int_0^T \dot{\hat{\theta}} dt$$

. It is important to note that the adaptive law does not necessarily converge to the true value of the unknown parameters. However, we do know that $\dot{V}$ is negative semidefinite while V is positive definite, which fulfils the stability criterion of a non-linear system.

We find that our adaptive controller can track a sinusoidal trajectory pretty well. Fig. 1, 2, 3 shows the results of a Python simulation of this particular adaptive controller.

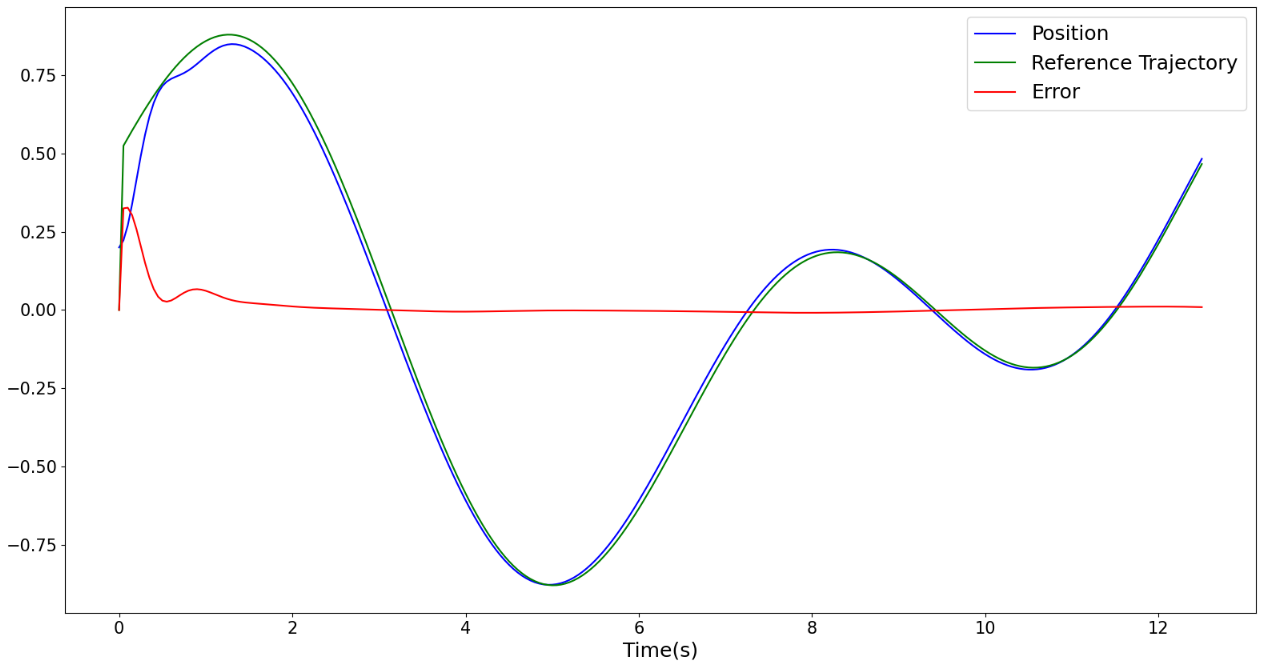

Figure 1: Tracking performance of states.

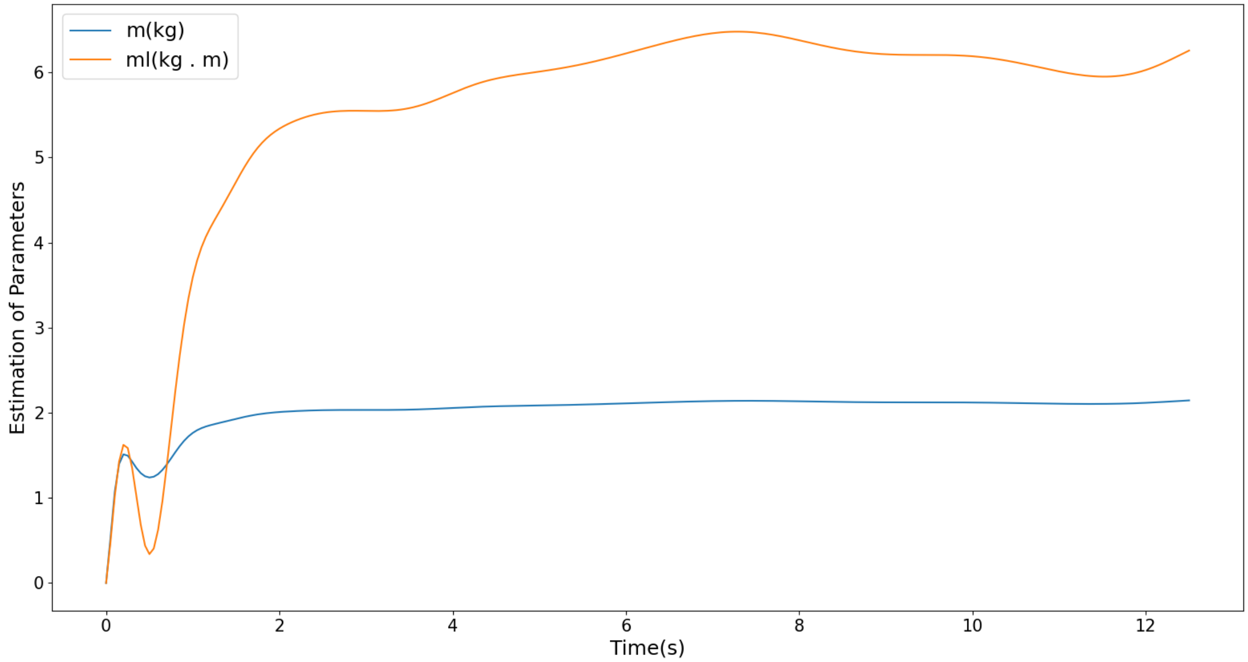

Figure 2: Estimation of unknown parameters.

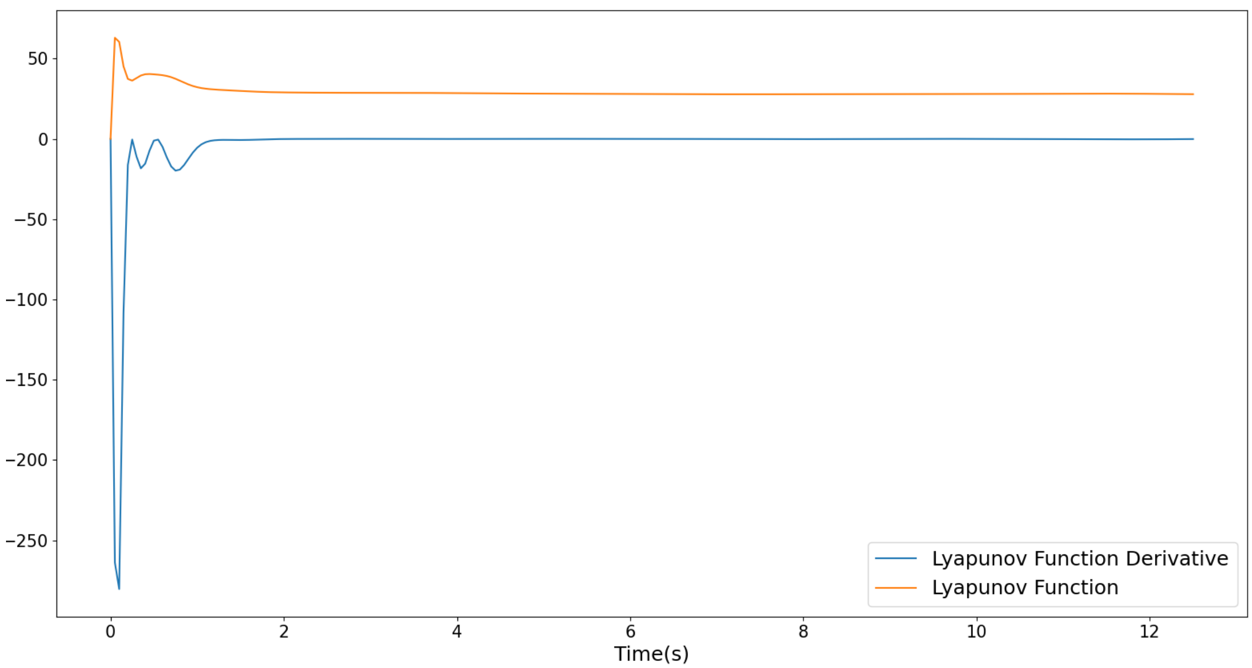

Figure 3: Lyapunov function and its derivative.