Jeremy Tan Math 282.1

1a) Sketch the rgb components as they would appear on a monochrome monitor.

We separate the colors on the image into the three primary color components: red, green, and blue.

The monochrome monitor will display three grayscale channels, each showing the amount of red, green, blue color in the original image.

Notation: Let P_{ij} denote the color on the ith row and jth column of the image.

1st row:

Since $P_{11}, P_{12}, \dots, P_{17}$ are already grayscale on the original image, there is an equal amount of red, green, blue in each color.

Using color picker, P_{11} has rgb value: 67, 67, 67 and
P_{17} has rgb value: 243, 243, 243.

Assume that the colors in between ($P_{12}, \dots, P_{16}$) have evenly spaced out grayscale rgb values.

$$P_{12} = 67 + \frac{243-67}{6} \approx 96$$
$$P_{13} = 67 + 2\left(\frac{243-67}{6}\right) \approx 126$$
$$P_{14} = 67 + 3\left(\frac{243-67}{6}\right) \approx 155$$
$$P_{15} = 67 + 4\left(\frac{243-67}{6}\right) \approx 184$$
$$P_{16} = 67 + 5\left(\frac{243-67}{6}\right) \approx 214$$

Thus, on all three grayscale channels, the rgb values for the colors on the first row will be the same. The rgb values calculated will be used for all three channels. For example, on the red, green, blue channels, P_{12} will have rgb (96, 96, 9

2nd row:

Assume that we are using the additive color model.

P_{21} is red

P_{22} is orange = red + yellow = red + red + green

P_{23} is yellow = red + green

P_{24} is green

P_{25} is cyan = green + blue

P_{26} is blue

P_{27} is purple = blue + magenta = blue + blue + red

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1a) Assume that red has rgb(255,0,0), green has rgb(0,255,0), and blue has rgb(0,0,255). The primary colors are fully saturated. Thus, after normalizing and scaling, we have the ff. rgb values:

P_{21} rgb = (255,0,0)
P_{22} rgb = (255,128,0)
P_{23} rgb = (255,255,0)
P_{24} rgb = (0,255,0)
P_{25} rgb = (0,255,255)
P_{26} rgb = (0,0,255)
P_{27} rgb = (128,0,255)

To get the rgb color on the grayscale channel, we take the corresponding component rgb value and use the same value for red, green, and blue. For example, for the leftmost color on the second row (P_{21}), the color on the red channel will have rgb = (255,255,255), the color on the green channel will have rgb = (0,0,0), and the color on the blue channel will have rgb = (0,0,0).

For the third row, the saturation and intensity of the colors were adjusted from the rgb values of the colors from the second row using the conversion formulas in 1b. The saturation decreased in the colors in row 3 as there is more whiteness mixed with the pure color. Moreover, the intensity of the colors in row 3 are higher than the counterpart colors in row 2 because the lighter color has higher rgb color values across the red, green, and blue components

P_{31} rgb = (255,208,208)
P_{32} rgb = (255,230,200)
P_{33} rgb = (255,255,200)
P_{34} rgb = (210,255,210)
P_{35} rgb = (210,255,255)
P_{36} rgb = (210,210,255)
P_{37} rgb = (220,210,255)

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1b) Sketch the HSI components.

Similar to 1a, we will have three grayscale channels on the monochrome monitor for hue, saturation, and intensity.

Given the obtained RGB values of the colors obtained in 1a, we normalize the values and solve the HSI components.

$$H = \begin{cases} \theta & \text{if } B \leq G \\ 360° - \theta & \text{otherwise} \end{cases} \qquad S = 1 - \frac{3}{R+G+B} \cdot \min(R,G,B) \qquad I = \frac{R+G+B}{3}$$

$$\theta = \cos^{-1}\left(\frac{\frac{1}{2}((R-G)+(R-B))}{\sqrt{(R-G)^2 + (R-B)(G-B)} + \varepsilon} \right),$$ ε is added to avoid division by zero.

Note: In the table, H is normalized by dividing θ by 360°.

Row 1:	RGB Normalized	HSI Normalized
1st col	(0.2627, 0.2627, 0.2627)	(0.25, 0, 0.2627)
2nd col	(0.3765, 0.3765, 0.3765)	(0.25, 0, 0.3765)
3rd col	(0.4941, 0.4941, 0.4941)	(0.25, 0, 0.4941)
4th col	(0.6078, 0.6078, 0.6078)	(0.25, 0, 0.6078)
5th col	(0.7216, 0.7216, 0.7216)	(0.25, 0, 0.7216)
6th col	(0.8392, 0.8392, 0.8392)	(0.25, 0, 0.8392)
7th col	(0.9529, 0.9529, 0.9529)	(0.25, 0. 0.9529)

Row 2:	RGB Normalized	HSI Normalized
1st col	(1, 0, 0)	(0, 1, 0.3333)
2nd col	(1, 0.5, 0)	(0.0833, 1, 0.5)
3rd col	(1, 1, 0)	(0.1667, 1, 0.6667)
4th col	(0, 1, 0)	(0.3333, 1, 0.3333)
5th col	(0, 1, 1)	(0.5, 1, 0.6667)
6th col	(0, 0, 1)	(0.6667, 1, 0.3333)
7th col	(0.5, 0, 1)	(0.75, 1, 0.5)

Row 3:	RGB Normalized	HSI Normalized
1st col	(1, 0.8157, 0.8157)	(0, 0.0700, 0.8771)
2nd col	(1, 0.902, 0.7843)	(0.0917, 0.1241, 0.8954)
3rd col	(1, 1, 0.7843)	(0.1667, 0.1549, 0.9281)
4th col	(0.8235, 1, 0.8235)	(0.3333, 0.0667, 0.8823)
5th col	(0.8235, 1, 1)	(0.5, 0.1250, 0.9412)
6th col	(0.8235, 0.8235, 1)	(0.6667, 0.0667, 0.8823)
7th col	(0.8627, 0.8235, 1)	(0.6692, 0.0678, 0.8834)

1. The following images are visualizations of the channels on a monochrome monitor. Canva was used to create the monochrome monitor channels. Normalized RGB and HSI values from the previous page were converted to hexadecimal by multiplying the values by 255.

Figure 1: Original Colored Image

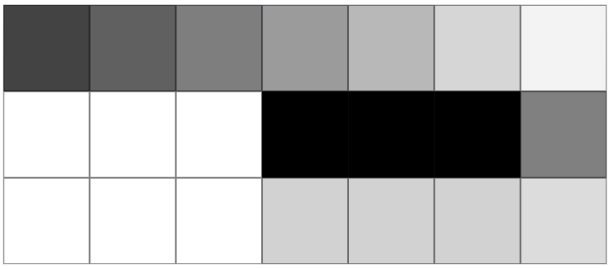

Figure 2: Red Channel as seen using a monochrome monitor (RGB)

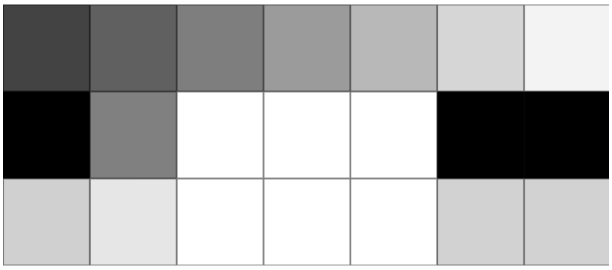

Figure 3: Green Channel as seen using a monochrome monitor (RGB)

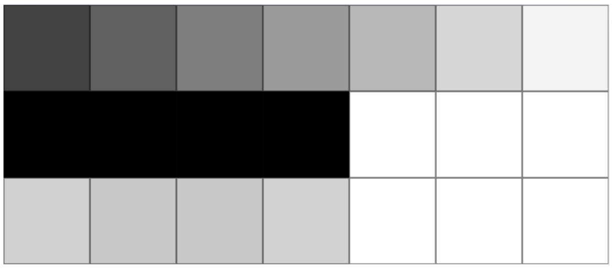

Figure 4: Blue Channel as seen using a monochrome monitor (RGB)

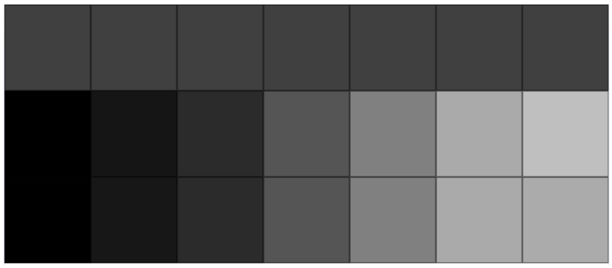

Figure 5: Hue Channel as seen using a monochrome monitor (HSI)

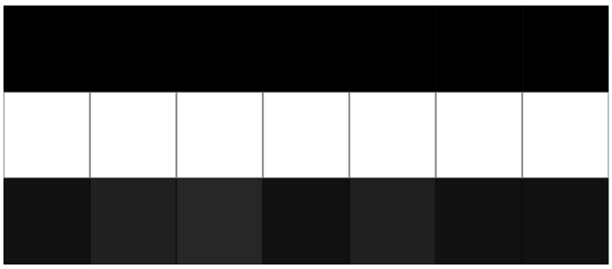

Figure 6: Saturation Channel as seen using a monochrome monitor (HSI)

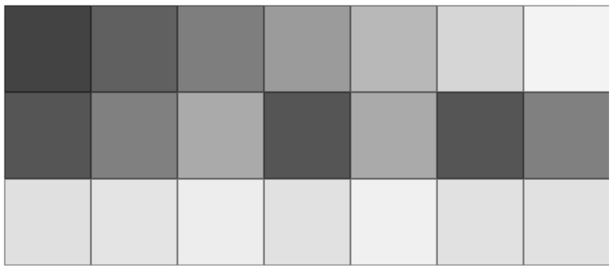

Figure 7: Intensity Channel as seen using a monochrome monitor (HSI)

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2a) Let c be a constant. Show that $\mathcal{F}\{e^{-ct^2}\} = e^{-w^2/(4c)}\sqrt{\pi/c}$ where $w = 2\pi\mu$.

By the def'n of Fourier Transform.

$$\mathcal{F}\{e^{-ct^2}\} = \int_{-\infty}^{\infty} e^{-ct^2} e^{-i2\pi\mu t}\,dt$$

Let $w = 2\pi\mu$.

$$= \int_{-\infty}^{\infty} e^{-ct^2} e^{-iwt}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-\left[ct^2 + iwt + \left(\frac{iw}{2\sqrt{c}}\right)^2 - \left(\frac{iw}{2\sqrt{c}}\right)^2\right]}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(\sqrt{c}t + \frac{iw}{2\sqrt{c}}\right)^2 + \frac{w^2}{4c}\right]}\,dt$$

$$= e^{-\frac{w^2}{4c}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{c}t + \frac{iw}{2\sqrt{c}}\right)^2}\,dt$$

Let $u = \sqrt{c}\,t + \frac{iw}{2\sqrt{c}}$

$du = \sqrt{c}\,dt \Rightarrow \frac{1}{\sqrt{c}}du = dt$

$$= \frac{1}{\sqrt{c}} e^{-\frac{w^2}{4c}} \int_{-\infty}^{\infty} e^{-u^2}\,du \quad (\star)$$

Show that $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$.

Let $I = \int_{-\infty}^{\infty} e^{-x^2}dx$.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\right)dy$$

Convert to polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$.

Area differential: $dA = dx\,dy$.

Get the Jacobian determinant for the transformation.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Thus, $dA = r\,dr\,d\theta$ so $dx\,dy = r\,dr\,d\theta$.

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

Let $u = r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$$= \int_0^{2\pi}\left(\int_0^{\infty} \frac{1}{2}e^{-u}du\right)d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{2}e^{-u}\right)\Big|_0^{\infty} d\theta = \int_0^{2\pi}\frac{1}{2}d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \pi.$$ Thus, $I = \sqrt{\pi}$.

Thus, from $(\star)$,

$$\mathcal{F}\{e^{-ct^2}\} = \sqrt{\frac{\pi}{c}}\, e^{-\frac{w^2}{4c}}.$$

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2b) $\mathcal{F}\{\cos(2\pi\mu_0 t)\} = \int_{-\infty}^{\infty} \cos(2\pi\mu_0 t)\, e^{-i2\pi\mu t}\, dt$ by def'n of Fourier Transform

Using Euler's Formula, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$.

$$\cos(2\pi\mu_0 t) = \frac{e^{i2\pi\mu_0 t}+e^{-i2\pi\mu_0 t}}{2}.$$

$$\mathcal{F}\{\cos(2\pi\mu_0 t)\} = \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i2\pi\mu_0 t}+e^{-i2\pi\mu_0 t}\right)e^{-i2\pi\mu t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i2\pi\mu_0 t}e^{-i2\pi\mu t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i2\pi\mu_0 t}e^{-i2\pi\mu_0 t}\,dt$$

$$= \frac{1}{2}\mathcal{F}\{e^{i2\pi\mu_0 t}\} + \frac{1}{2}\mathcal{F}\{e^{-i2\pi\mu_0 t}\}. \quad (\star)$$

Find $\mathcal{F}\{e^{i2\pi a t}\}$.

Compute the Fourier transform of impulse at $t=t_0$.

$\mathcal{F}(\delta(t-t_0)) = \int_{-\infty}^{\infty}\delta(t-t_0)e^{-i2\pi\mu t}\,dt = e^{-i2\pi\mu t_0}$ by sifting property

By Symmetry property, $\mathcal{F}(e^{-i2\pi\mu t_0}) = \delta(-\mu - t_0)$.

Let $t_0 = -a$.

$\mathcal{F}(e^{-i2\pi t(-a)}) = \mathcal{F}(e^{i2\pi a t}) = \delta(-\mu+a) = \delta(\mu - a)$.

From $(\star)$, $\mathcal{F}\{\cos(2\pi\mu_0 t)\} = \frac{1}{2}\delta(\mu-\mu_0) + \frac{1}{2}\delta(\mu+\mu_0)$.

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3. Properties of Radon Transform.

Notation: Let $R_{(f)}$ be the Radon Transform of $f(x,y)$.

a) Linearity. Let $a, b \in \mathbb{R}$.

$$R_{(af+bh)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (af+bh)(x,y)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[af(x,y) + b\,h(x,y)\right]\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} af(x,y)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} bh(x,y)$$

$$\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

$$= aR_{(f)} + bR_{(h)}$$

b) Translation Property.

Find the Radon transform of $f(x-x_0, y-y_0)$. Let $h(x,y) = f(x-x_0, y-y_0)$.

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x_0, y-y_0)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

Let $x' = x - x_0$, $y' = y - y_0$

$dx' = dx$, $dy' = dy$.

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,\delta(x'\cos\theta + x_0\cos\theta + y'\sin\theta + y_0\sin\theta - \rho)\,dx'\,dy'$$

Let $\rho' = \rho - x_0\cos\theta - y_0\sin\theta$.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,\delta(x'\cos\theta + y'\sin\theta - \rho')\,dx'\,dy'$$

$$R_{(h)} = g(\rho',\theta) = g(\rho - x_0\cos\theta - y_0\sin\theta, \theta) = R_{(f)}(\rho',\theta).$$

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3c) Convolution Property.

Let $h(x,y) = (f \star g)(x,y)$.

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

By the definition of convolution in 2-D,

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)\,g(x-s,y-t)\,ds\,dt\right)\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$$

Changing the order of integration.

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)\,g(x-s,y-t)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy\,ds\,dt$$

By the Translation property, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-s,y-t)\,\delta(x\cos\theta + y\sin\theta - \rho)\,dx\,dy$
$= R_{(g)}(\rho - s\cos\theta - t\sin\theta, \theta)$.

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)\,R_{(g)}(\rho - s\cos\theta - t\sin\theta, \theta)\,ds\,dt$$

Remember the sifting property, $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$.

Let $t = \rho - \rho_1$ and $t_0 = \rho - s\cos\theta - t\sin\theta$.

$$R_{(g)}(\rho - s\cos\theta - t\sin\theta, \theta) = \int_{-\infty}^{\infty} R_{(g)}(\rho - \rho_1, \theta)\,\delta(\rho_1 - s\cos\theta - t\sin\theta)\,d\rho_1$$

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)\,\delta(\rho_1 - s\cos\theta - t\sin\theta)\,R_{(g)}(\rho - \rho_1, \theta)\,ds\,dt\,d\rho_1$$

Since δ is an even function,

$$R_{(h)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)\,\delta(s\cos\theta + t\sin\theta - \rho_1)\,R_{(g)}(\rho - \rho_1, \theta)\,ds\,dt\,d\rho_1$$

$$= \int_{-\infty}^{\infty} R_{(f)}(\rho, \theta)\,R_{(g)}(\rho - \rho_1, \theta)\,d\rho_1$$

$$R_{(h)} = R_{(f)} \star R_{(g)}$$

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4. Optimization Problem

Minimize $\|Qf\|^2$ subject to $Hf=g$.

Function: $J(f,\lambda) = \|Hf-g\|^2 + \lambda\|Qf\|^2$.

Find f s.t. $\frac{\partial J}{\partial f} = 0$.

$$\frac{\partial J}{\partial f} = \frac{\partial}{\partial f}\left((Hf-g)^T(Hf-g) + \lambda(Qf)^T(Qf)\right)$$

$$= \frac{\partial}{\partial f}\left((f^TH^T-g^T)(Hf-g) + \lambda(f^TQ^T)(Qf)\right)$$

$$= \frac{\partial}{\partial f}\left(f^TH^THf - g^THf - f^TH^Tg + g^Tg\right) + \frac{\partial}{\partial f}\left(\lambda f^TQ^TQf\right)$$

Since g^THf and f^TH^Tg are scalars, $(g^THf)^T = g^THf = f^TH^Tg$.

$$\frac{\partial J}{\partial f} = 2H^THf - 2H^Tg + 2\lambda Q^TQf$$

$$0 = 2H^THf - 2H^Tg + 2\lambda Q^TQf$$

$$2H^Tg = 2(H^THf + \lambda Q^TQf)$$

$$H^Tg = (H^TH + \lambda Q^TQ)f$$

$$f = (H^TH + \lambda Q^TQ)^{-1}H^Tg.$$