van Emde Boas Trees

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Problem definition

- Maintain a **sorted integer set** $S \subseteq \{0,\ldots,U-1\}$, for some **universe size** U, under the following operations and queries for an integer $0 \leq x < U$.

  - Min(): return the smallest element in S.
  - Max(): return the largest element in S.
  - Member(x): return "Yes" if $x \in S$, "No" otherwise.
  - Insert(x): add x to S (assuming $x \notin S$).
  - Delete(x): remove x from S (assuming $x \in S$).
  - Successor(x): return the smallest $y \in S$ such that $y > x$, or $\perp$ if no such element exists.
  - Predecessor(x): return the largest $y \in S$ such that $y < x$, or $\perp$ if no such element exists.

Problem definition

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- Notes:
  - We assume U fits in $O(1)$ memory words.
  - S is a set, hence **no duplicates are allowed**.
  - Balanced binary search trees (like AVL trees and RB trees) solve the problem in $O(\log n)$ time per op/query, where $n = |S|$. This holds for general keys.

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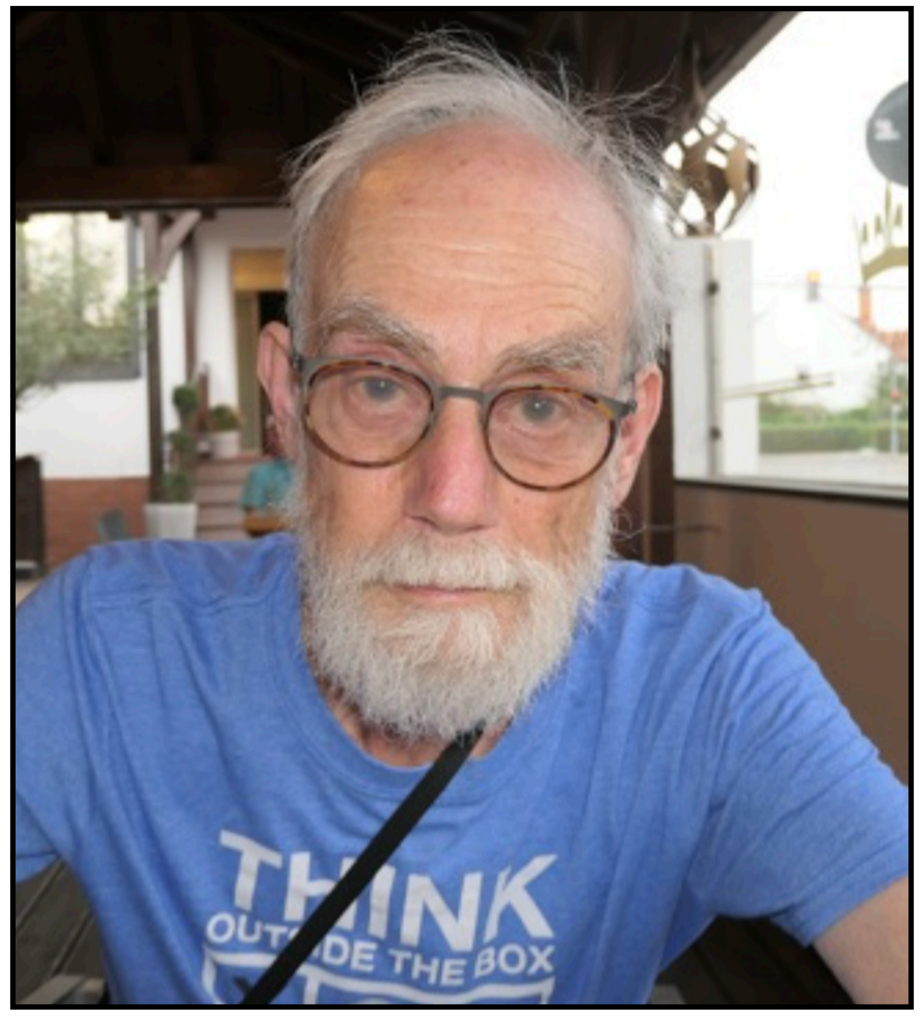
    Here, we deal with **integers from a bounded domain**.

- The solution we are going to describe was first proposed by **Peter van Emde Boas** in 1975, and later refined in 1977.



Peter van Emde Boas

Plain bitvector

- **Idea.** Use a bitvector $B[0..U-1]$ of U bits where $B[x] = 1$ iff $x \in S$.
  (Sometimes called the *characteristic* bitvector of S.)

- Example for $U = 16$ and $S = \{2,4,7,13\}$.

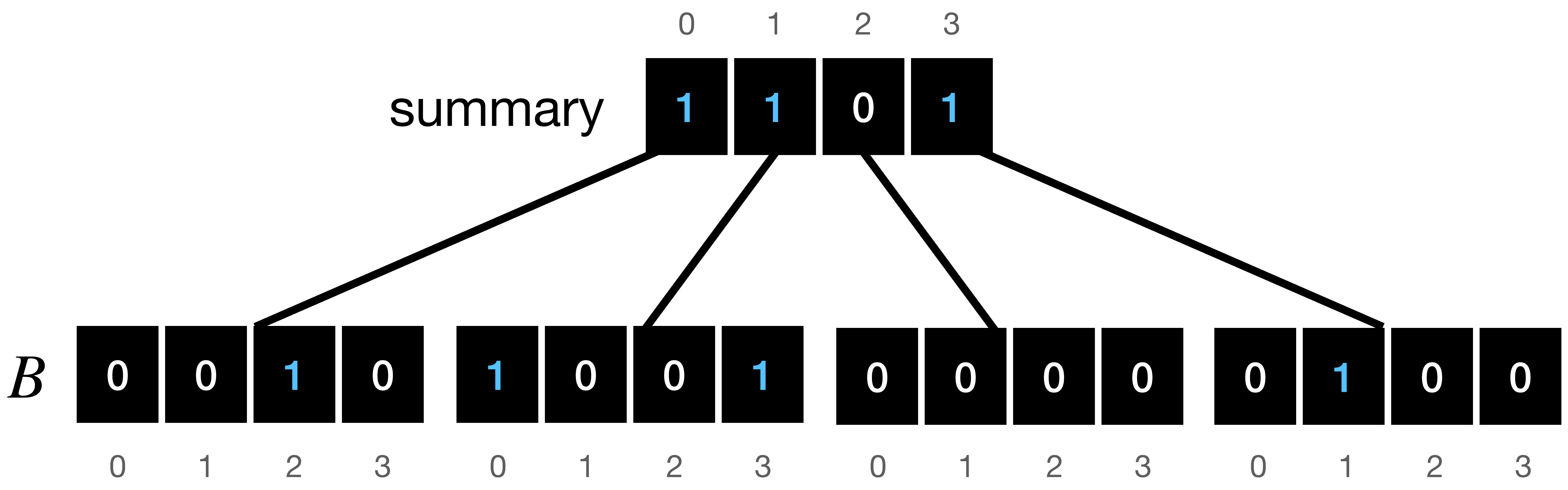
B	0	0	1	0	1	0	0	1	0	0	0	0	0	1	0	0
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

- Insert/Delete x: flip bit in position x.
- Member(x): check bit in position x.
- Runtime is $O(1)$.

- But Min/Max, Predecessor/Successor are slow: $O(U)$ time by scanning B.

Chunking

- **Idea.** Split B into chunks of R bits. Define a new bitvector summary$[0..U/R-1]$ such that summary$[i] = 1$ iff chunk i is not empty. (We assume R divides U.)
- **Intuition.** Use the summary as a **shortcut**, avoiding to scan the entire bitvector.
- In general, ops/queries operate on the summary and at most two chunks: $O(U/R + R)$ time, which is minimized for $R = \sqrt{U}$.



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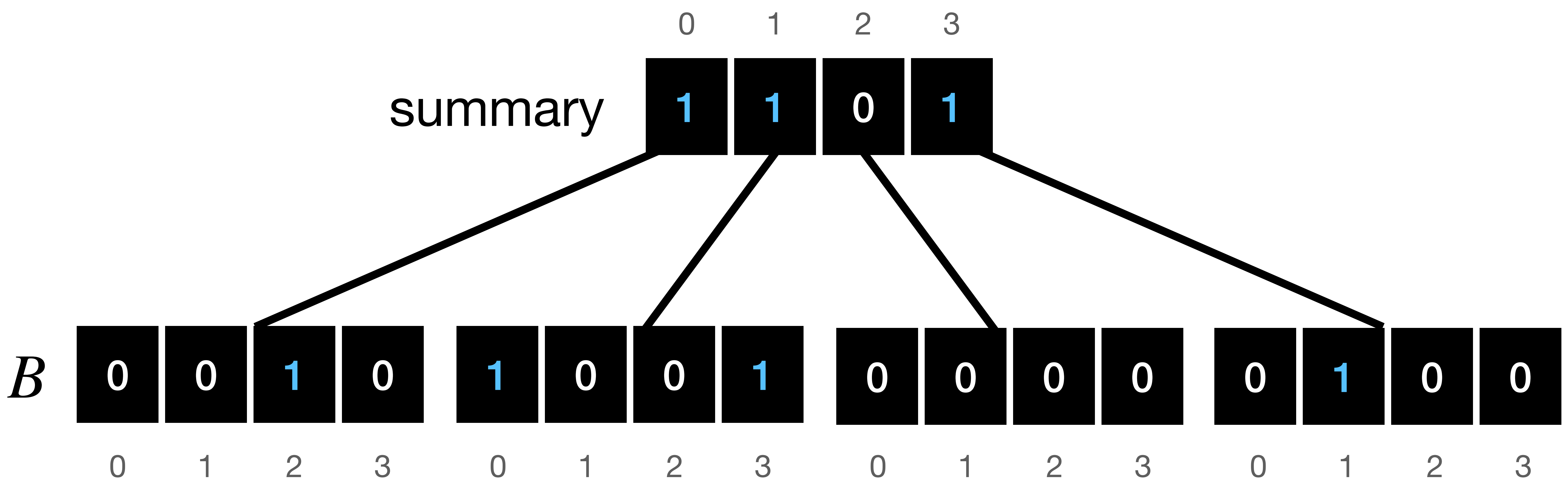

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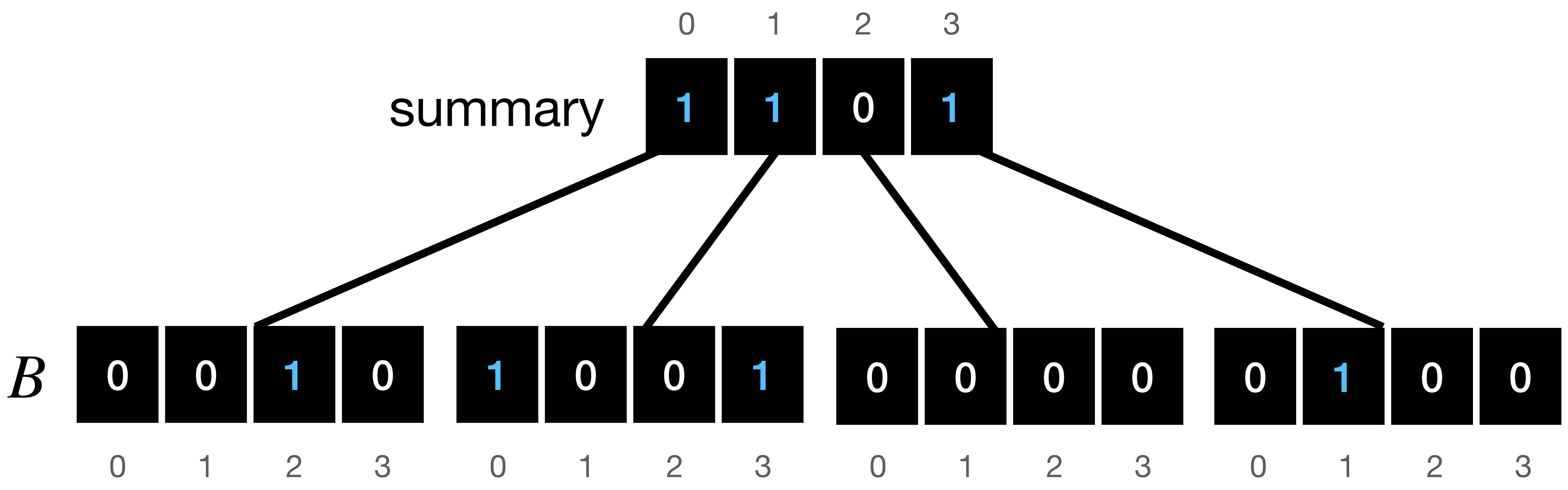

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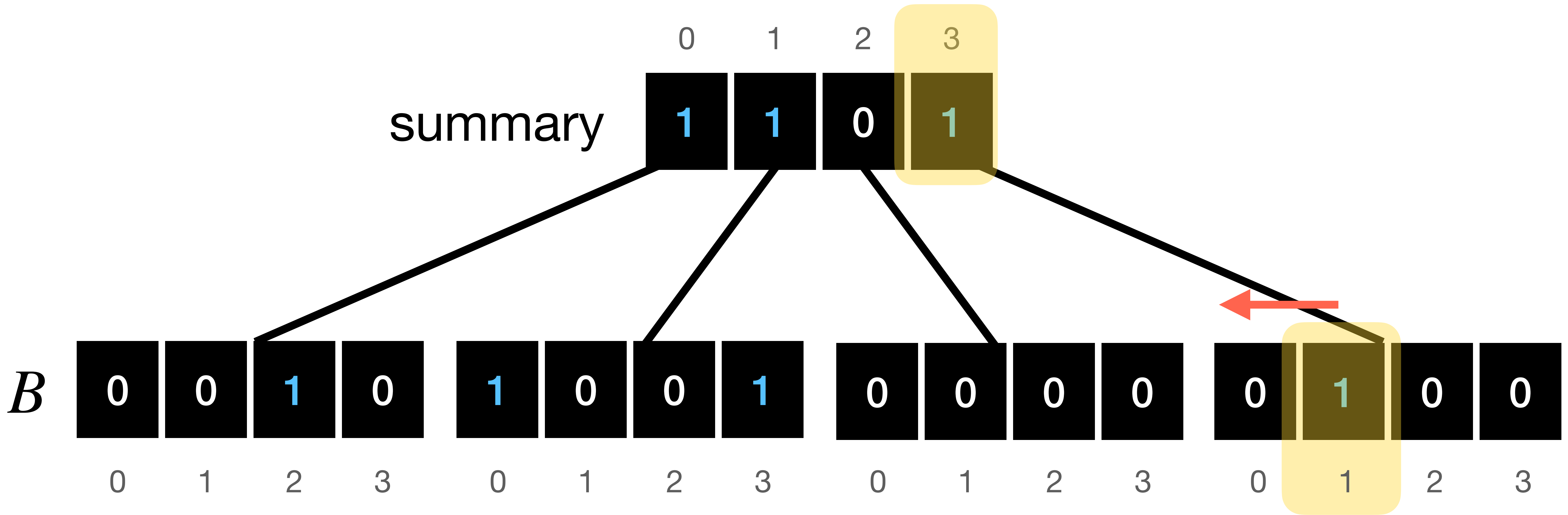

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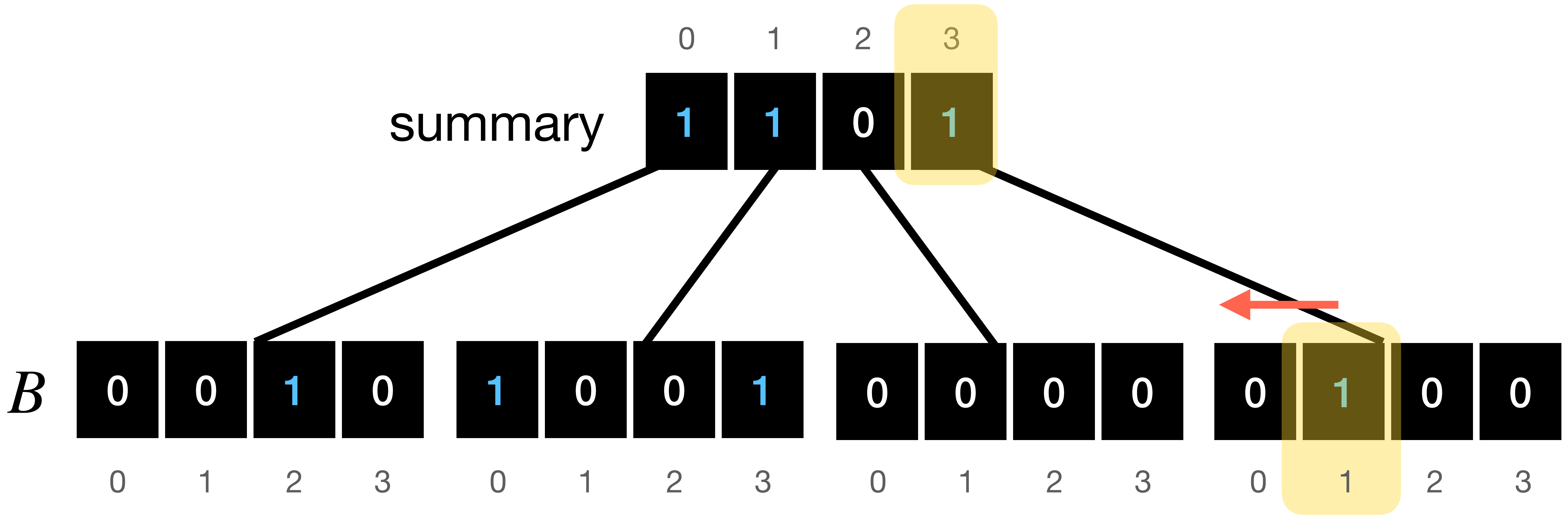

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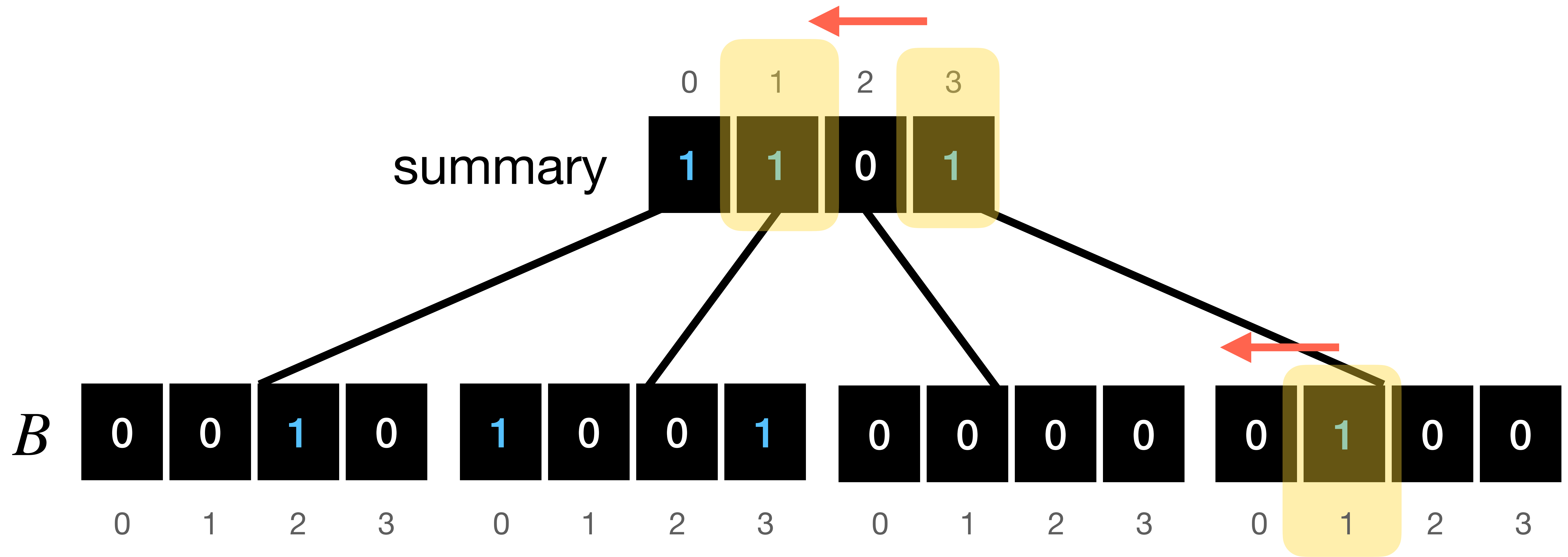

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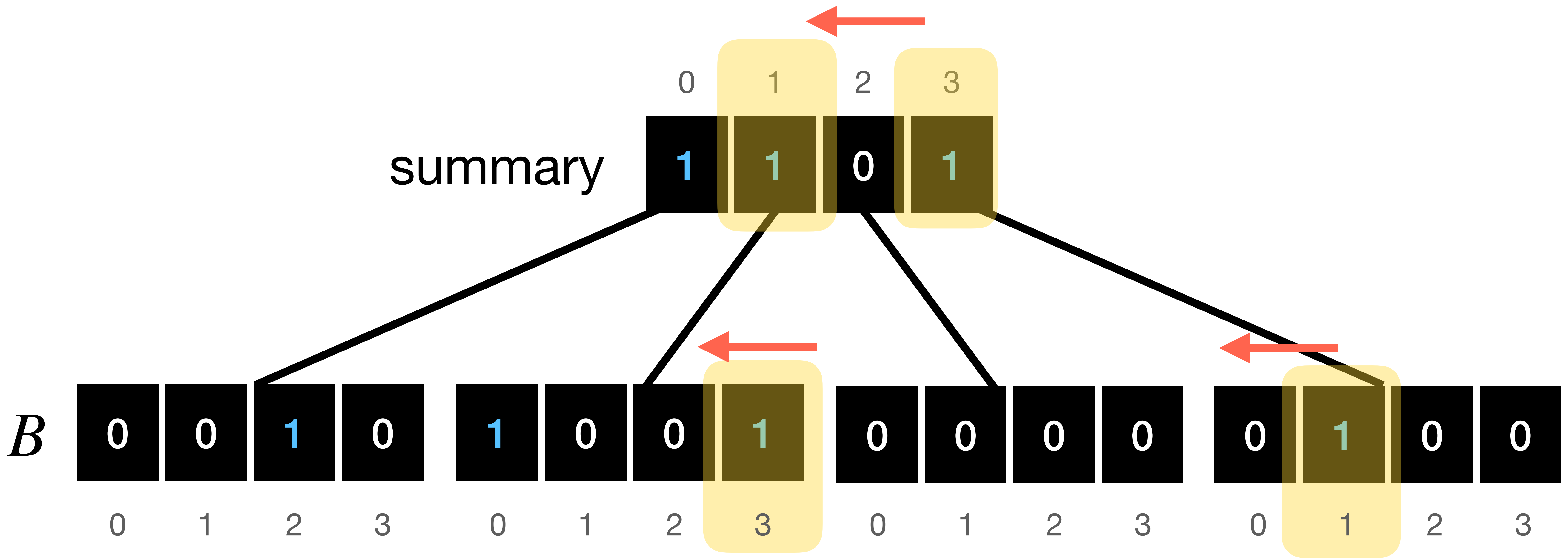

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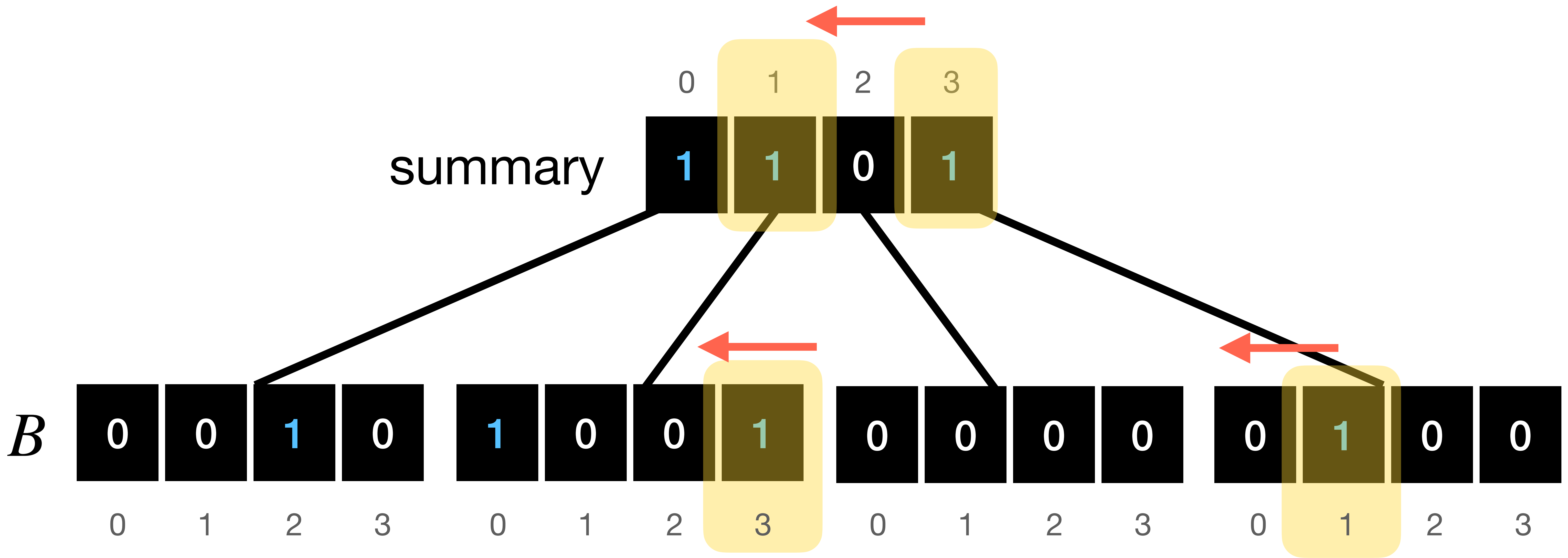

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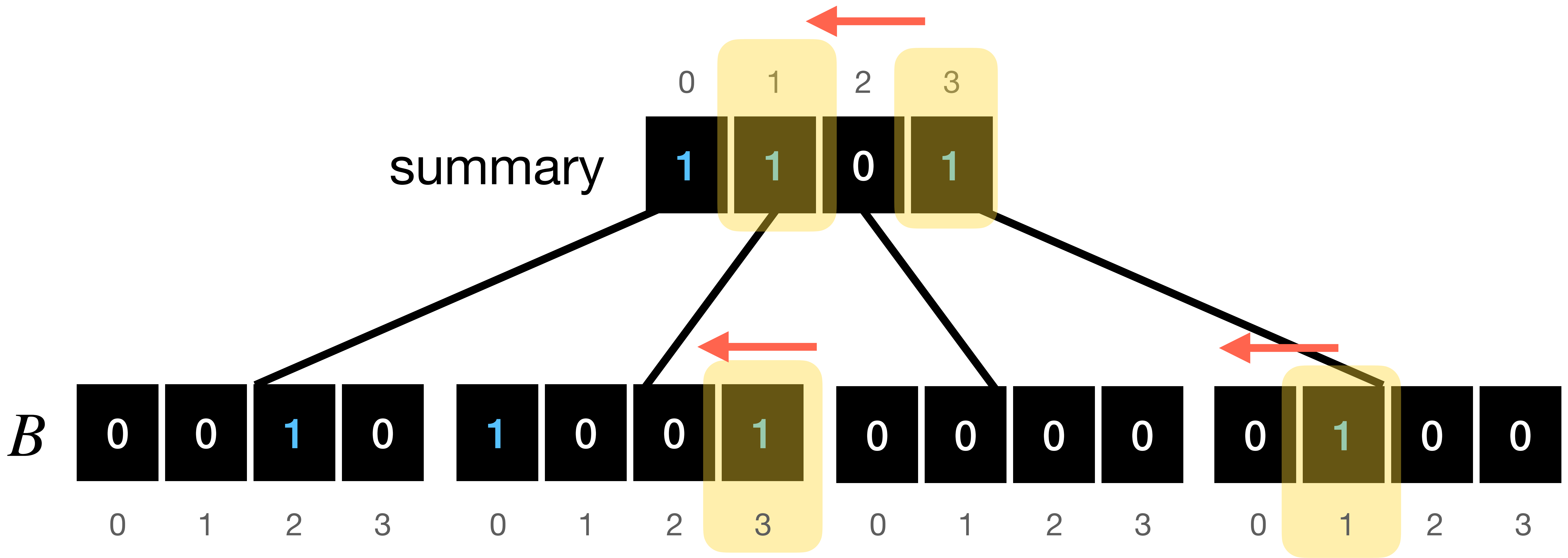

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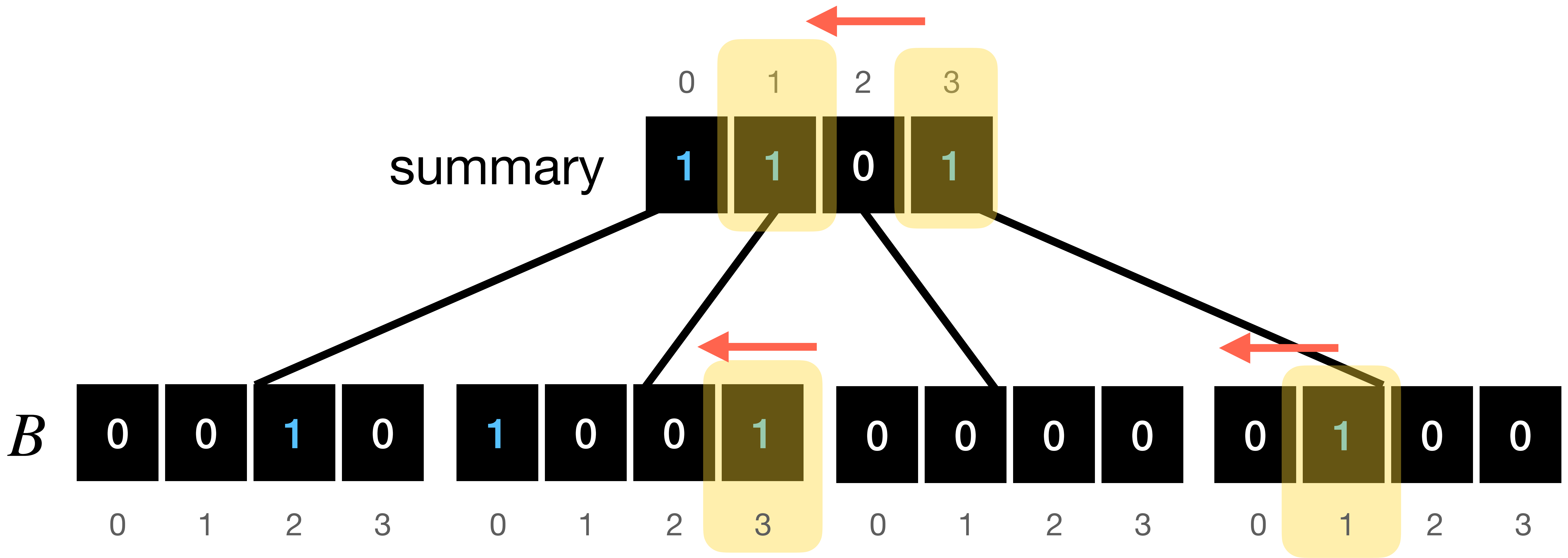

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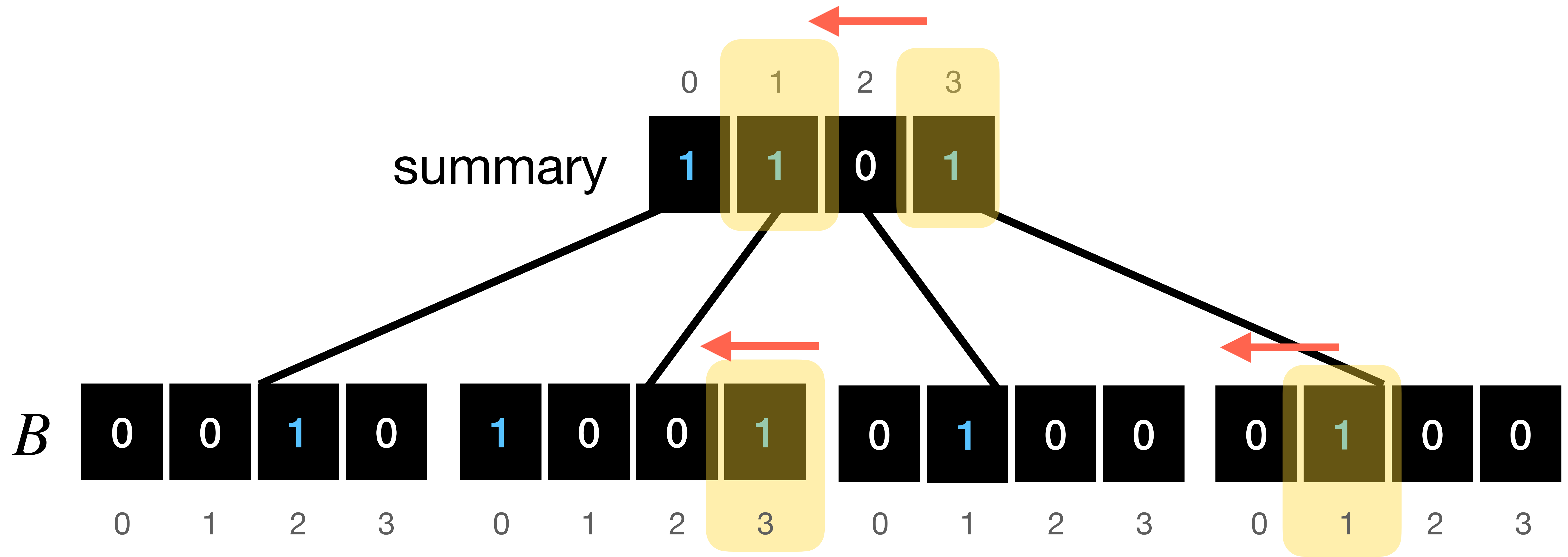

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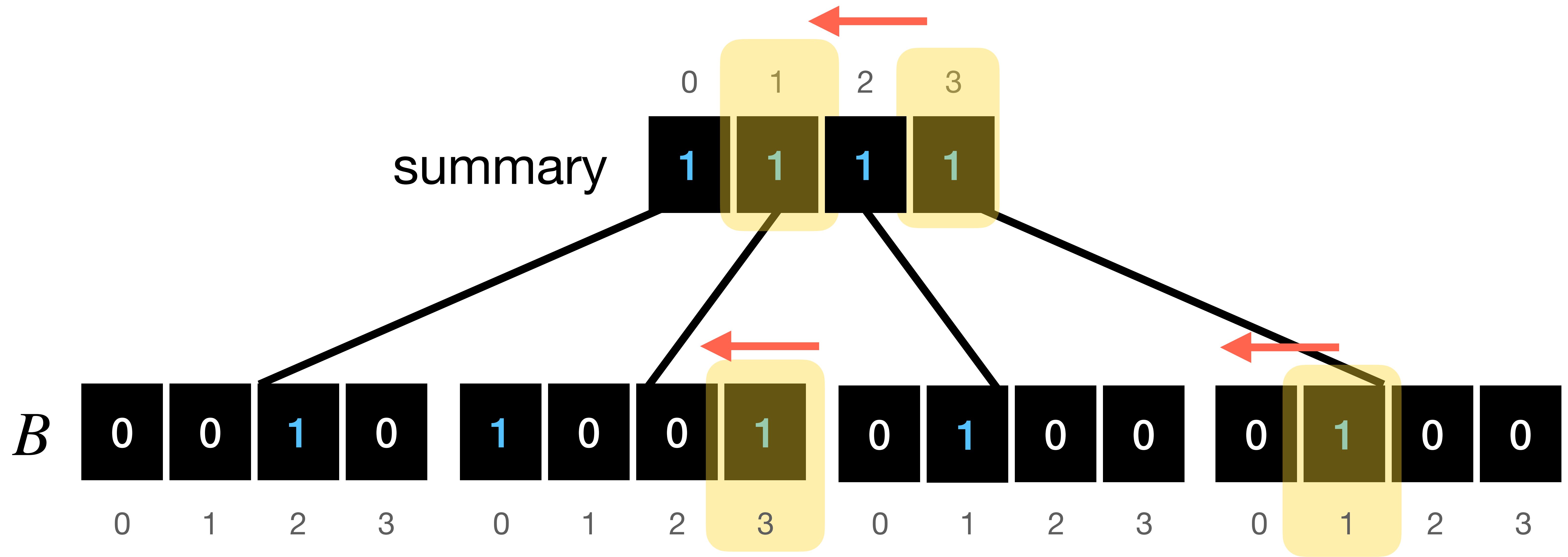

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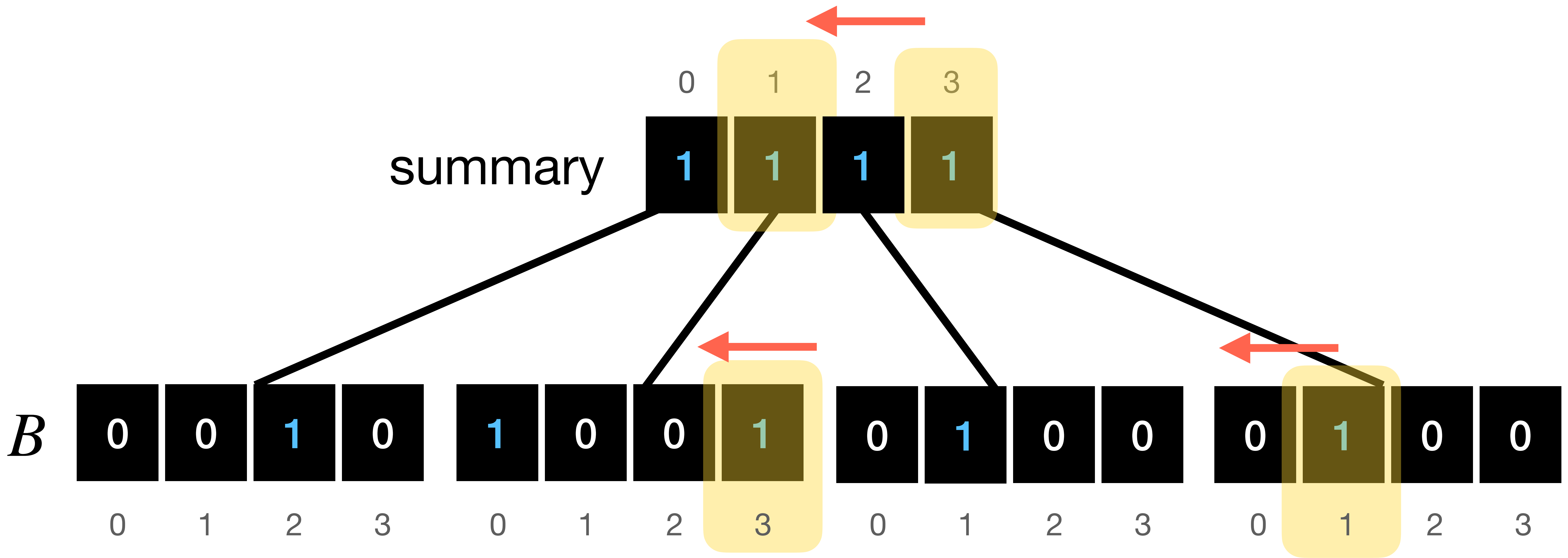

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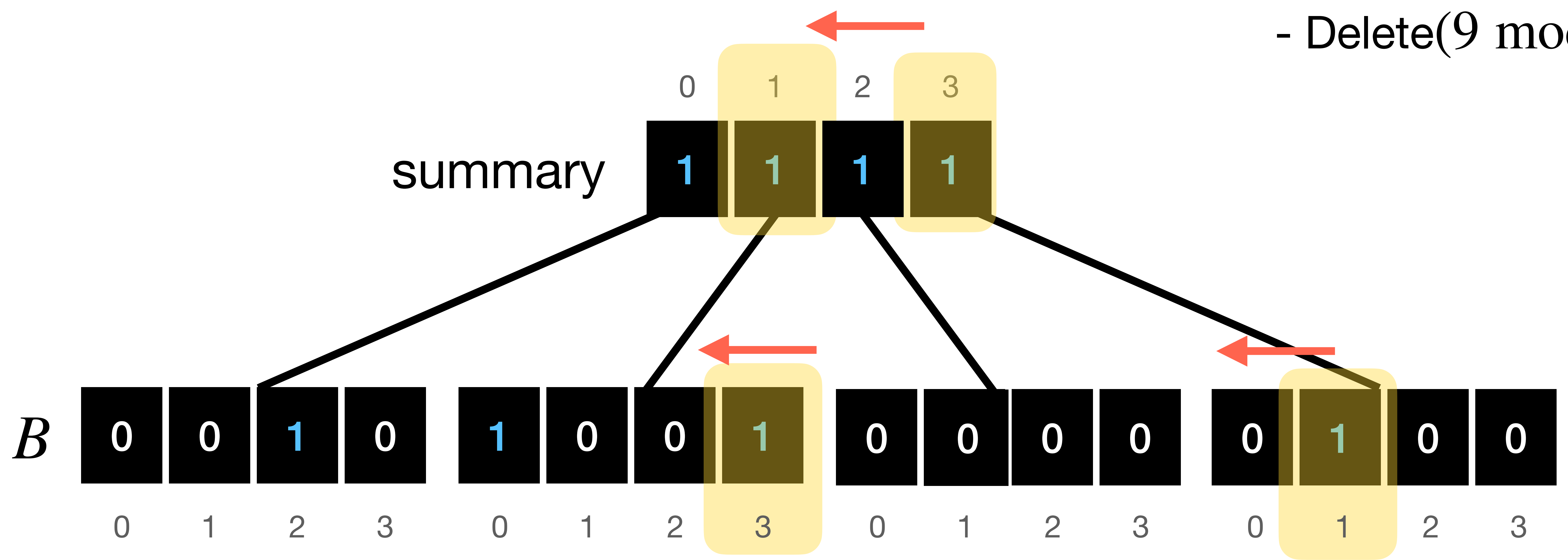

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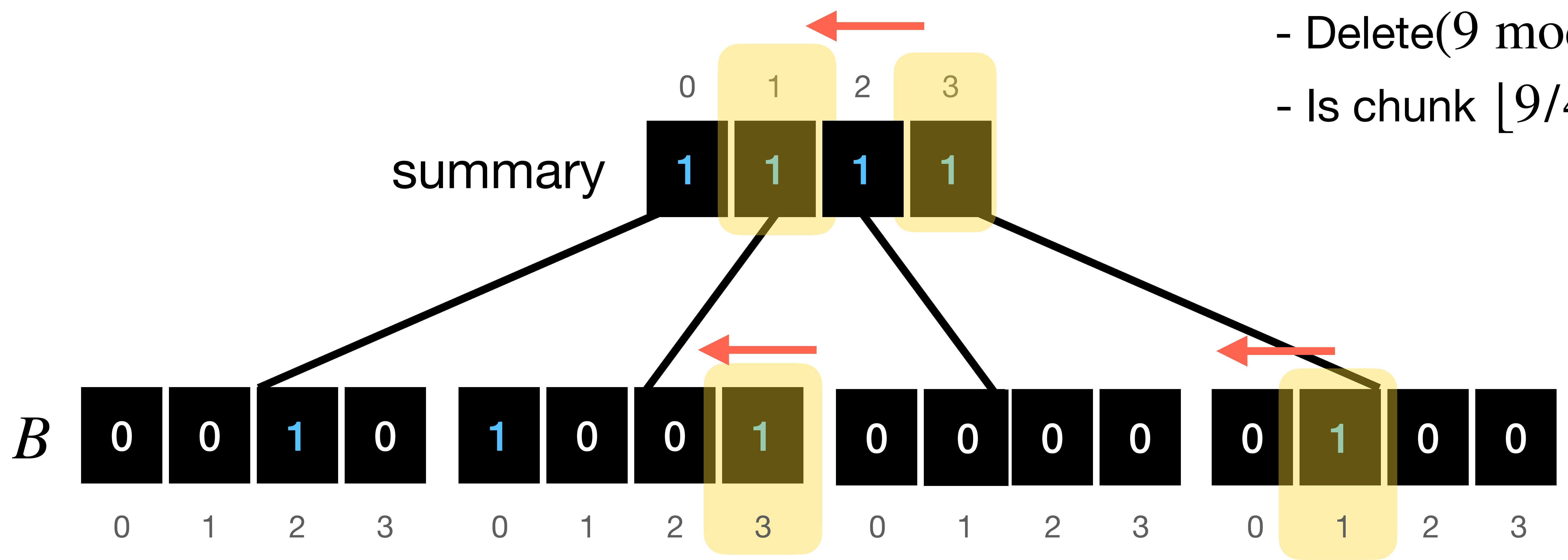

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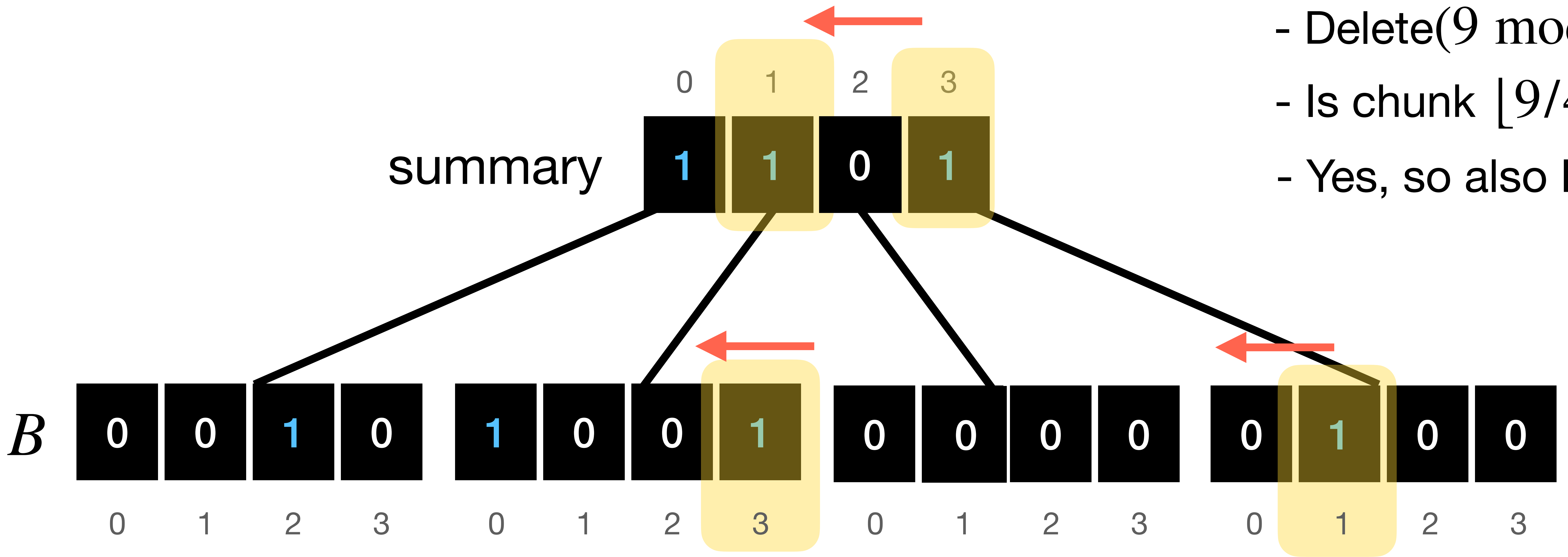

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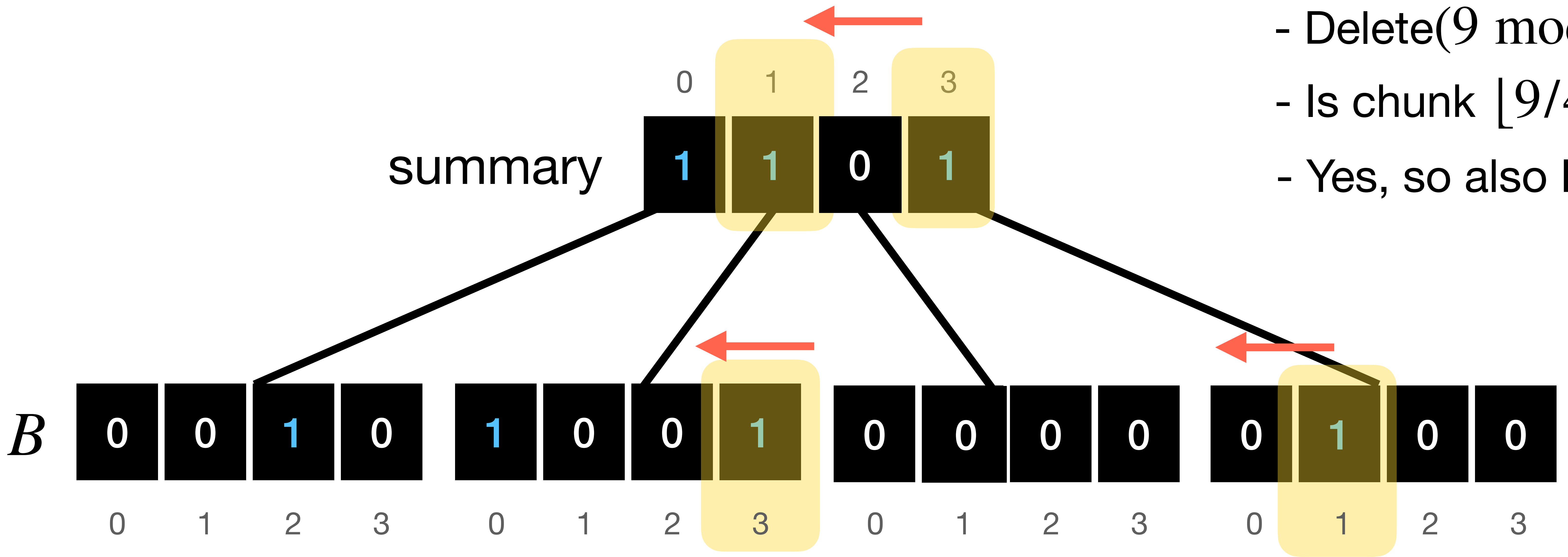
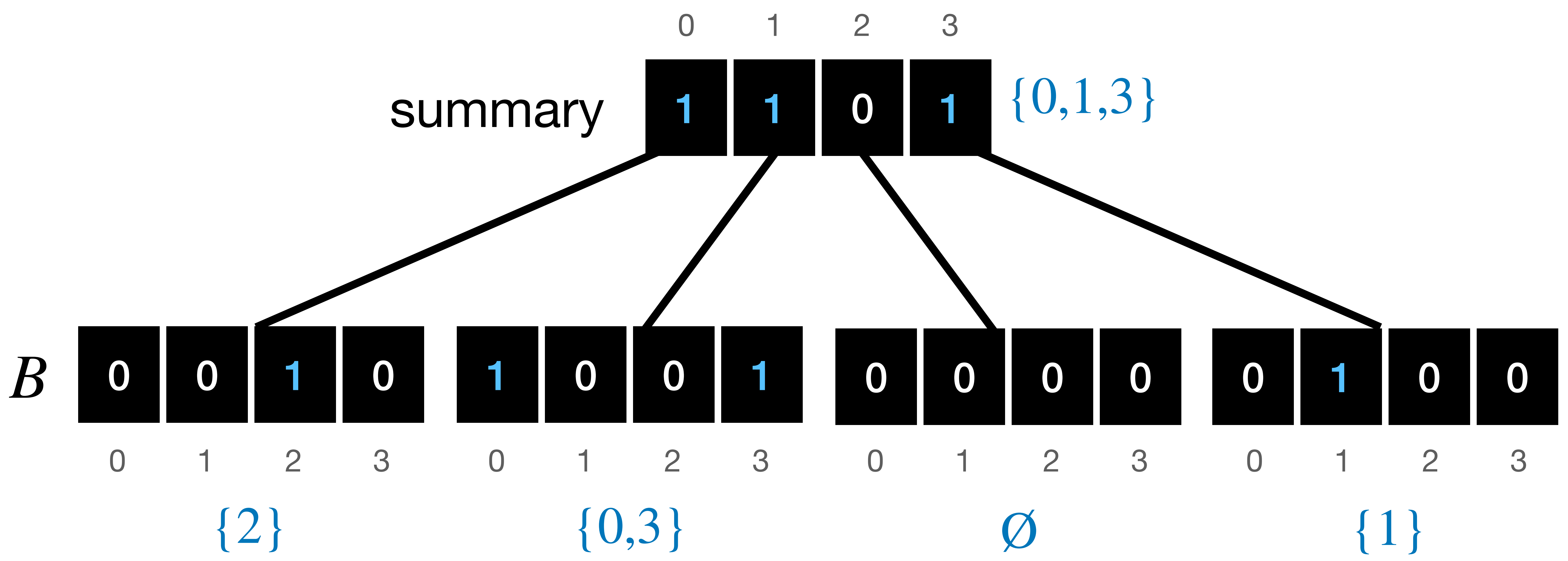


In general, we recurse **twice**: on the summary **and** on a chunk.

Towards van Emde Boas trees — Recursion

- Now that we have a solution that runs in $O(\sqrt{U})$ time for a universe size U, we can use it for both the summary and the chunks, recursively, over a universe size $\sqrt{U}$.



- We let high(x) $:= \lfloor x/\sqrt{U} \rfloor$ and low(x) $:= x \bmod \sqrt{U}$.

x	high(x)	low(x)
2	0	2
4	1	0
7	1	3
13	3	1

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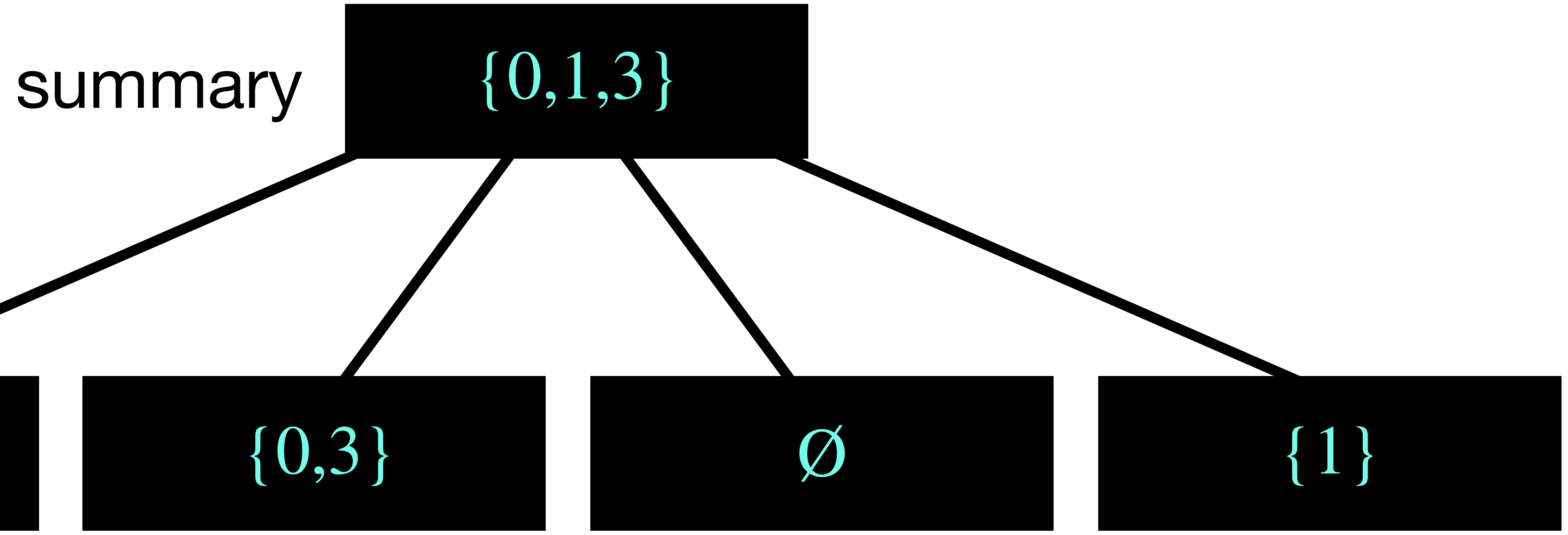

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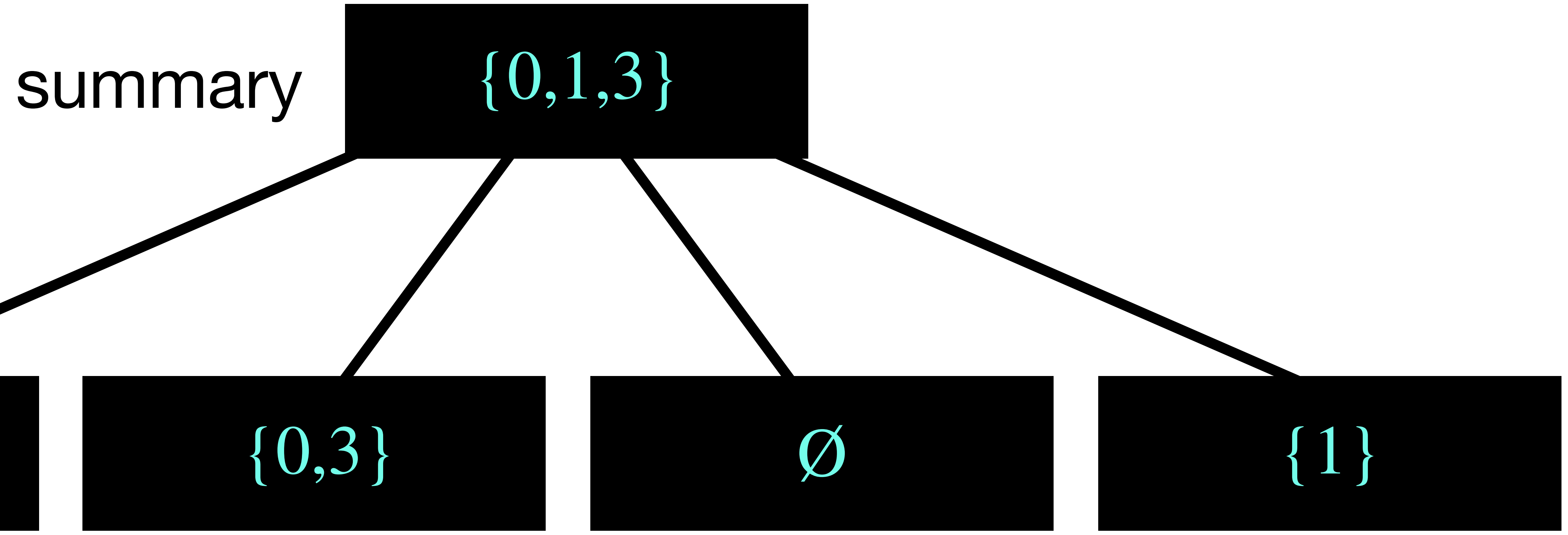

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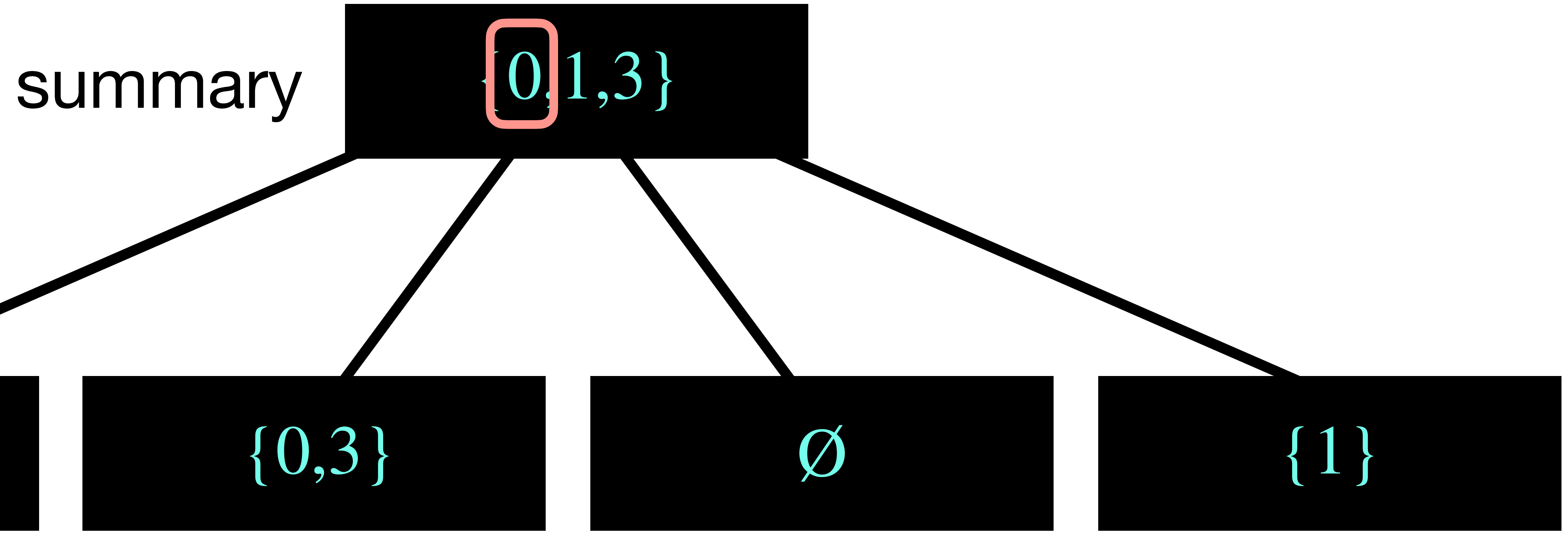

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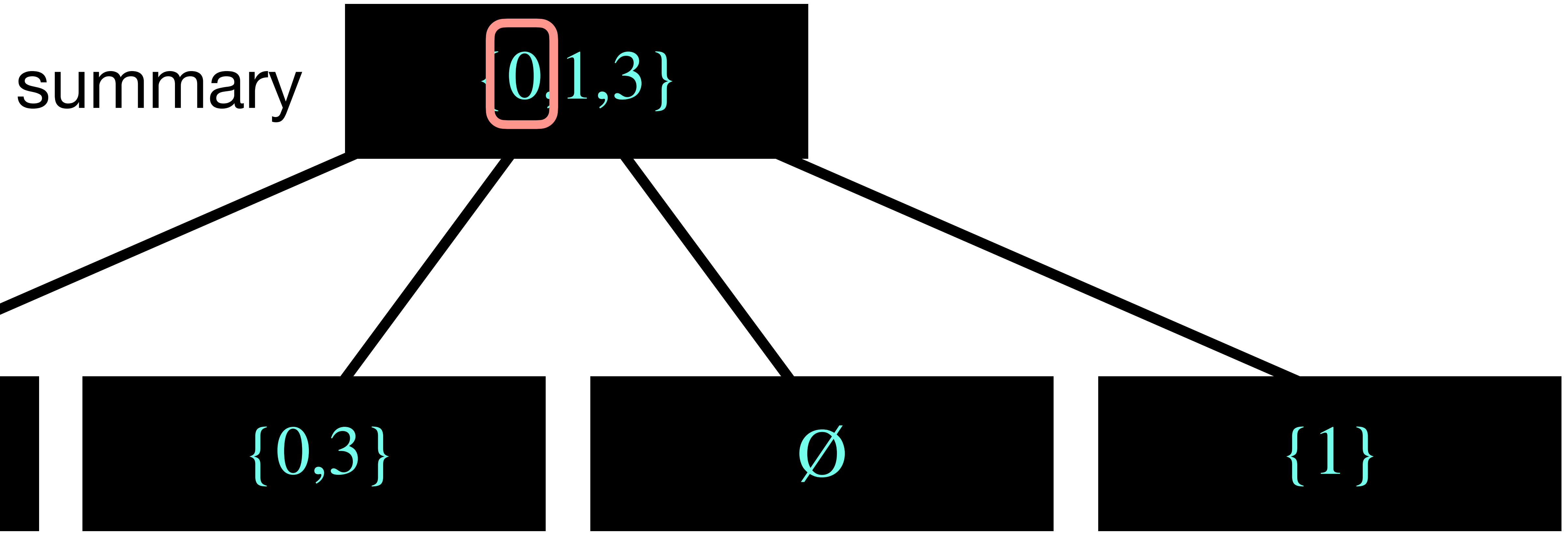

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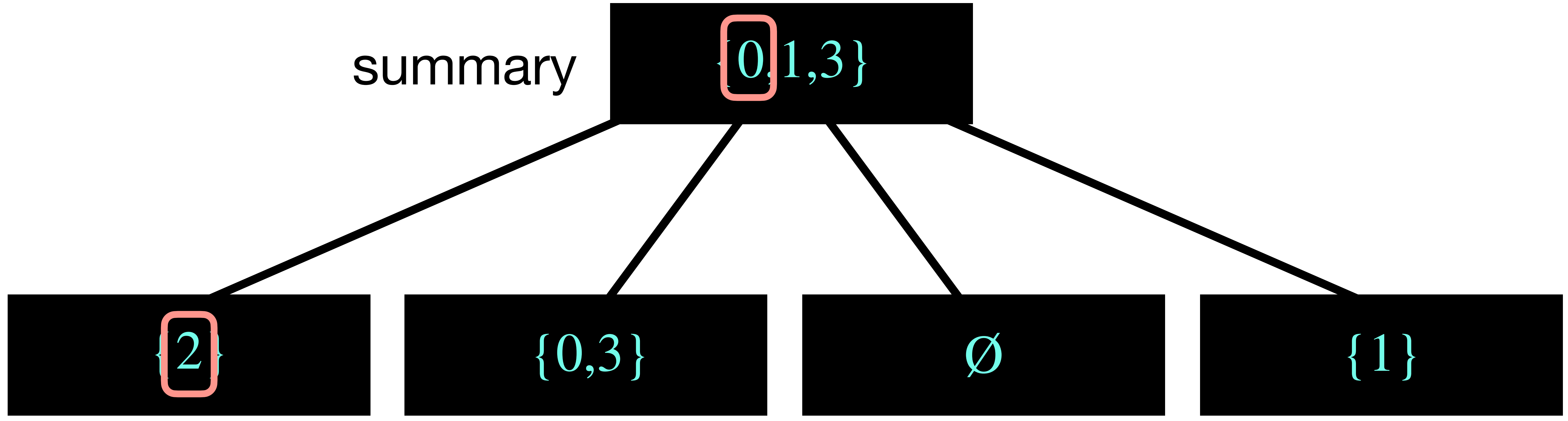

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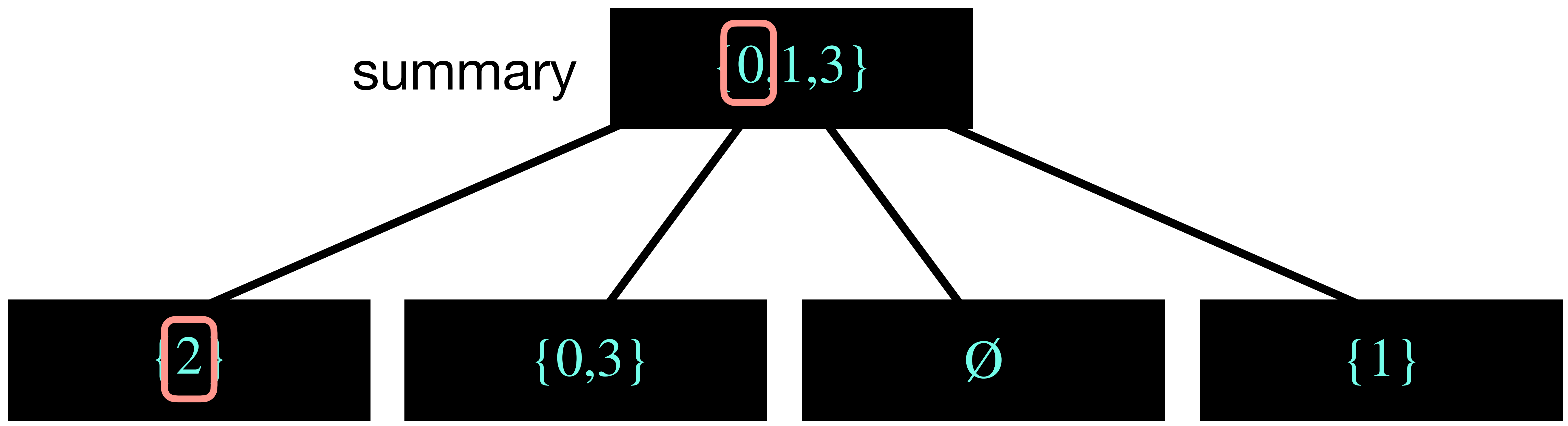

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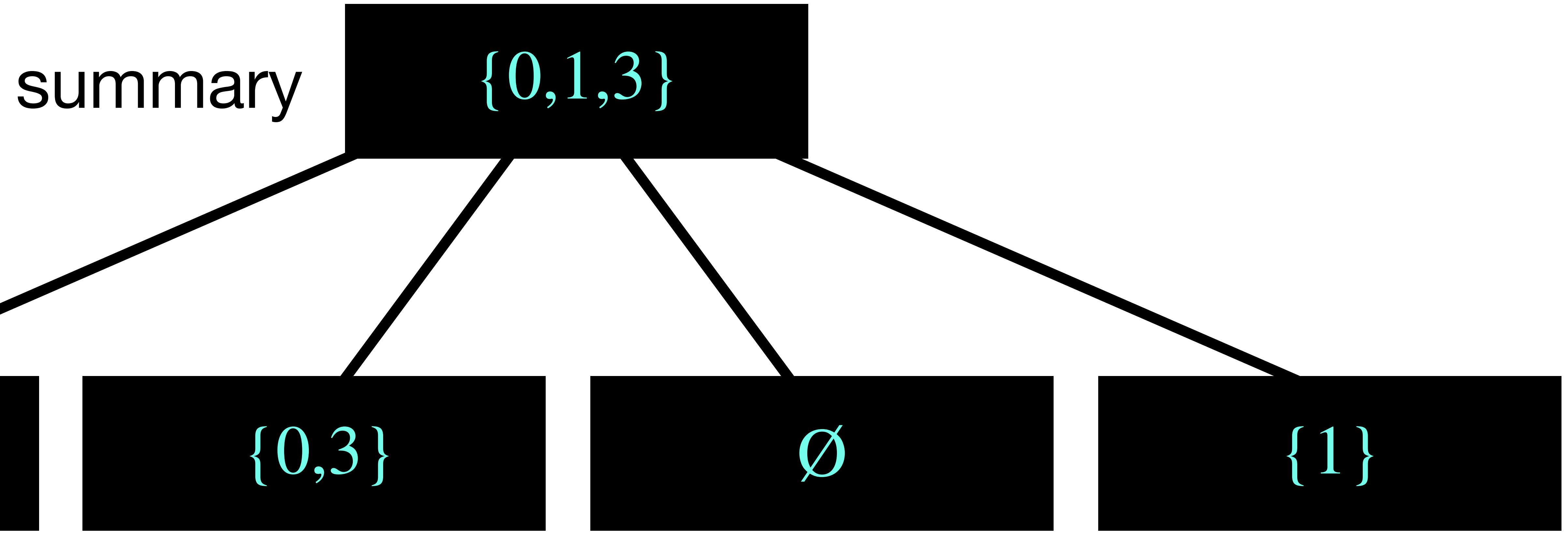

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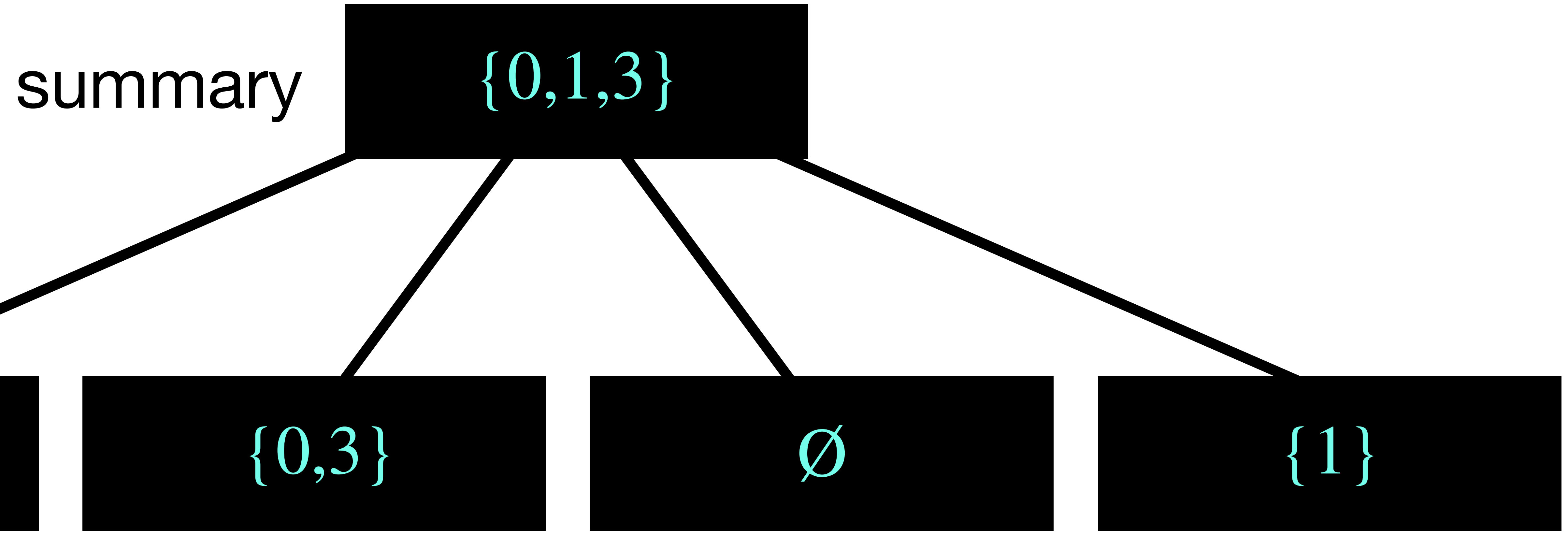

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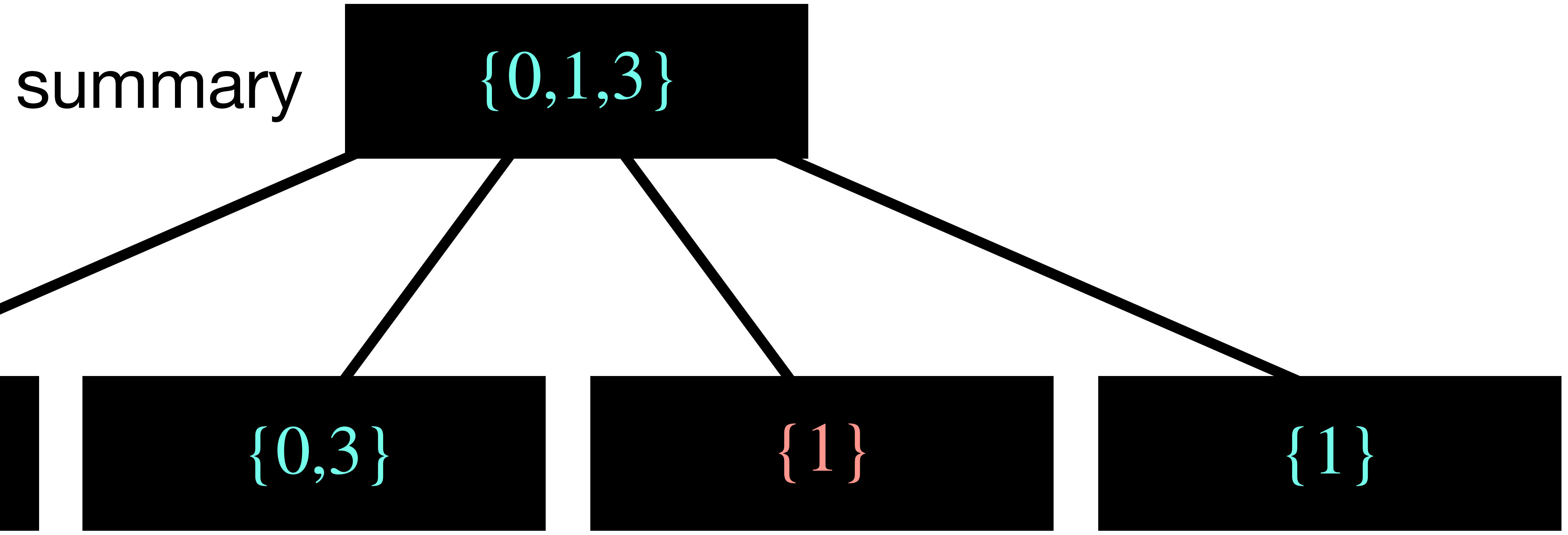

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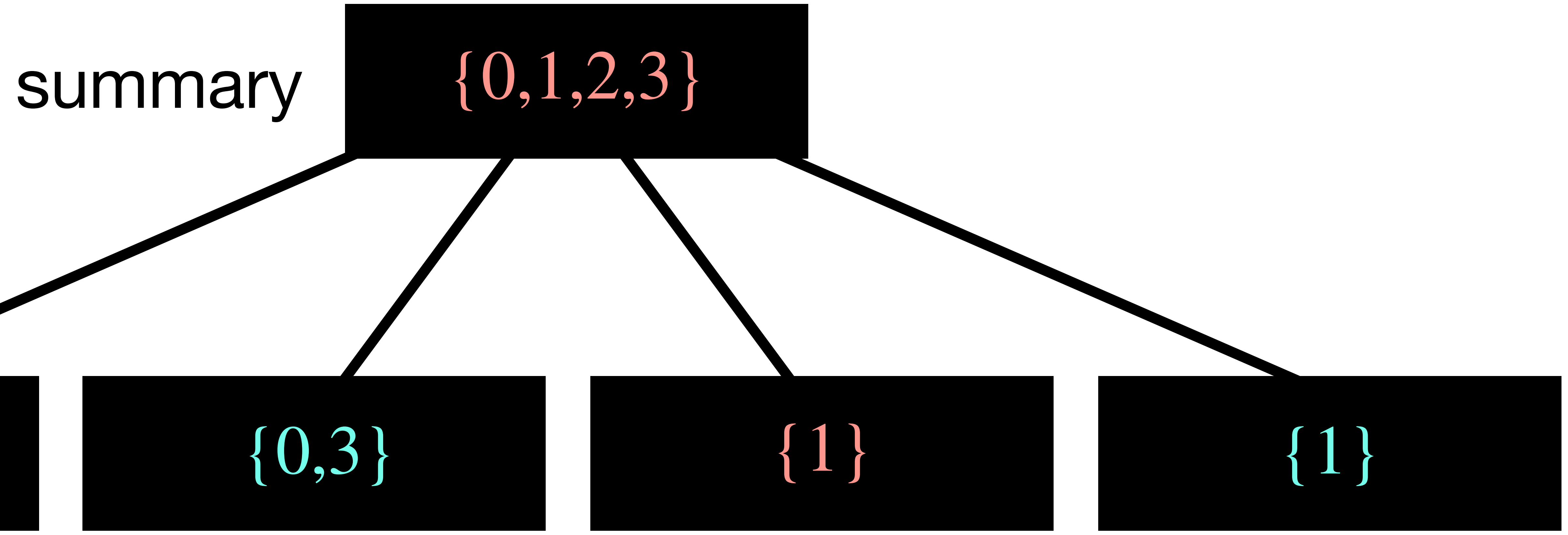

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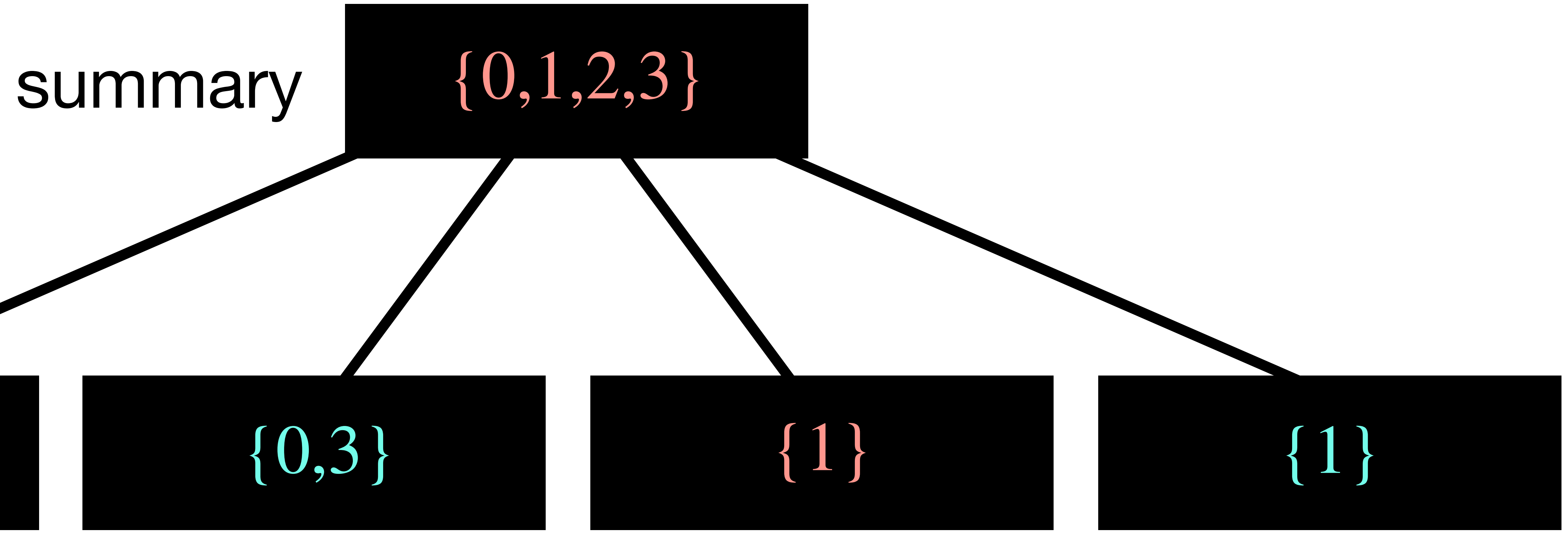

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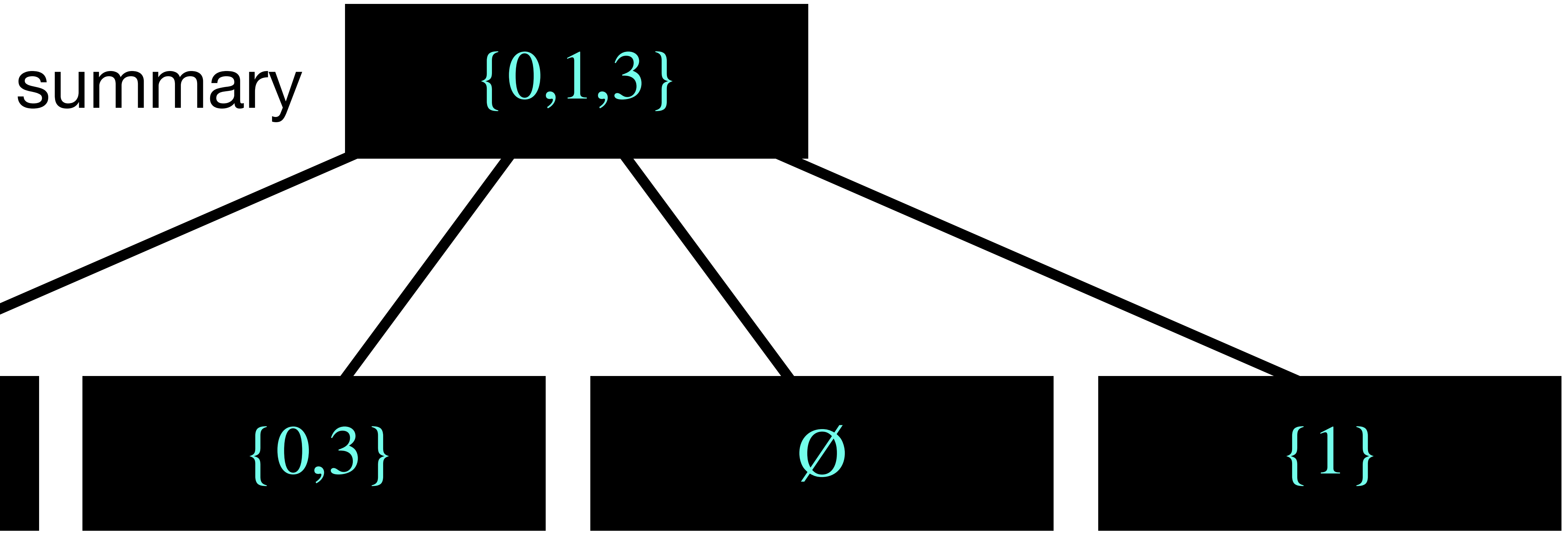

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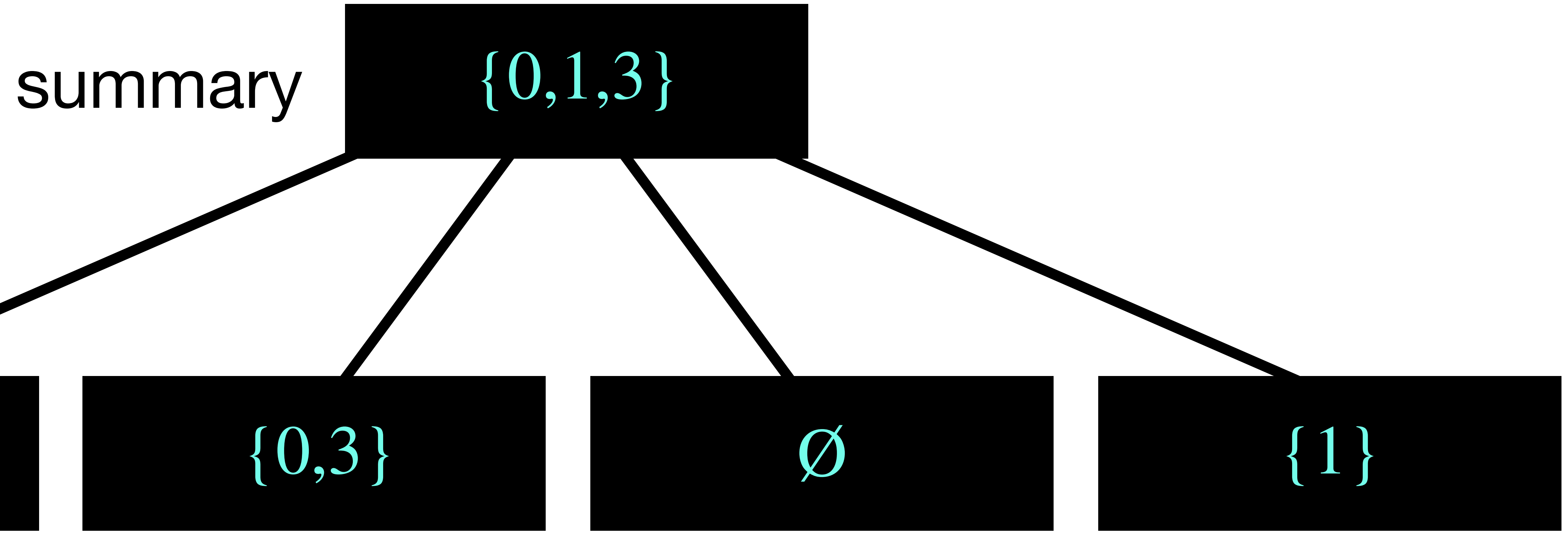

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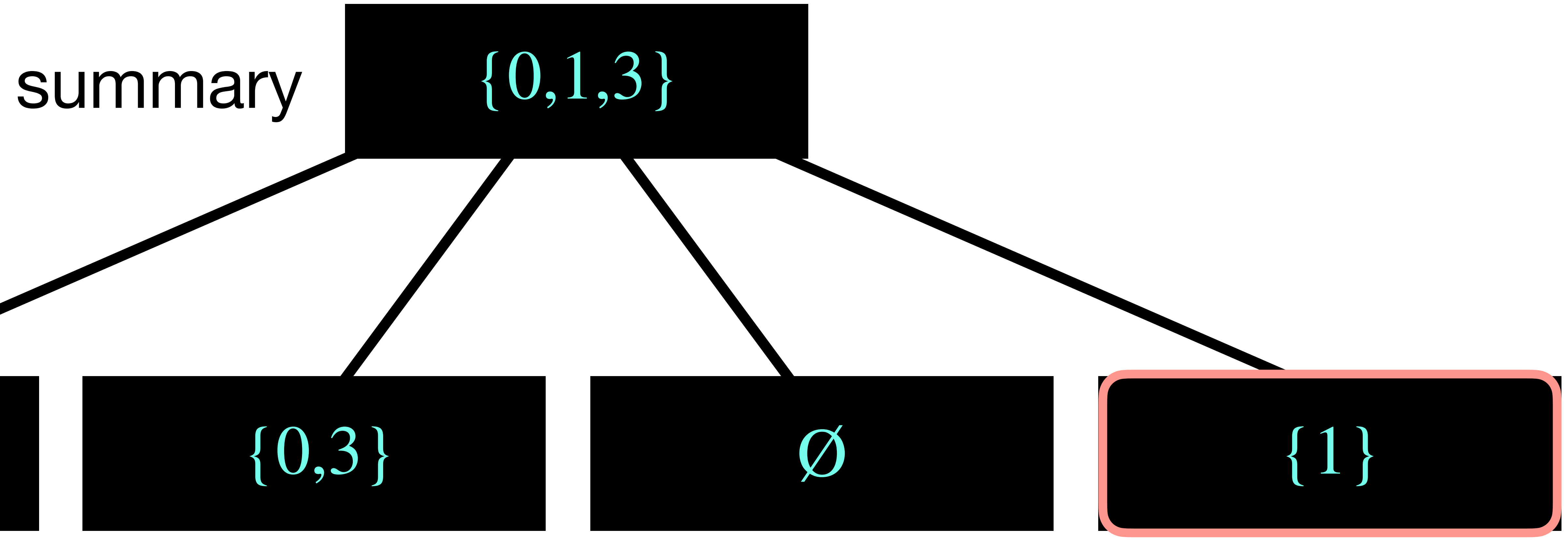

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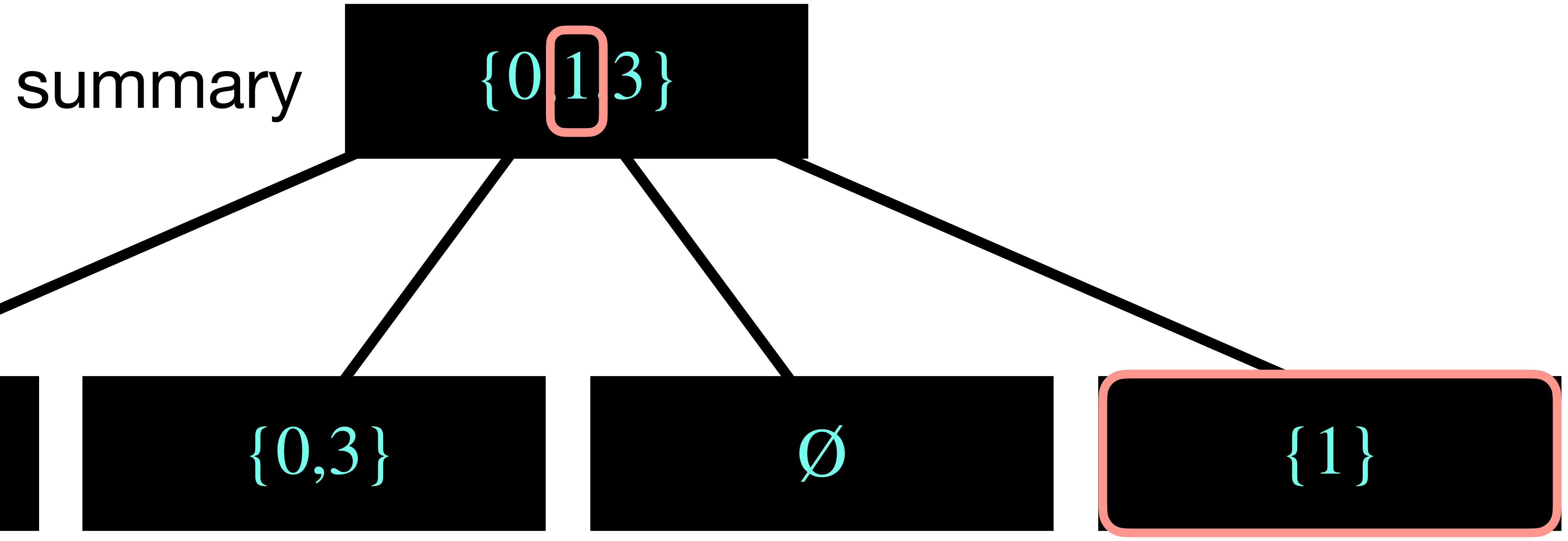

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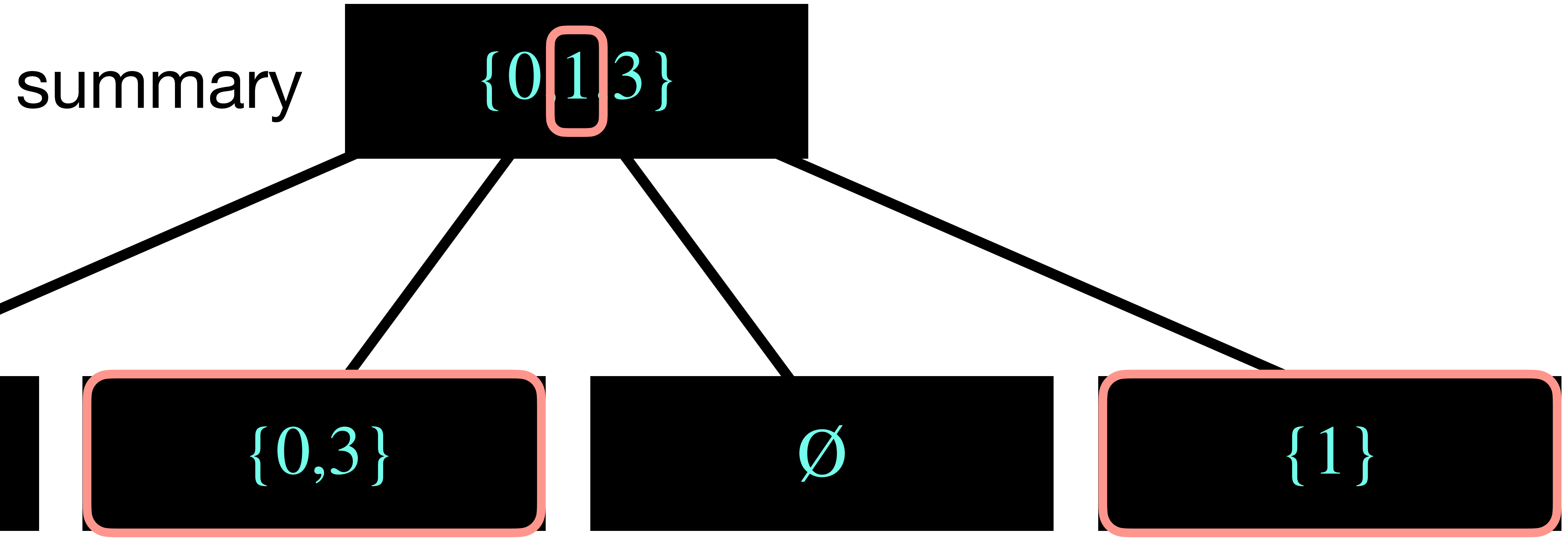

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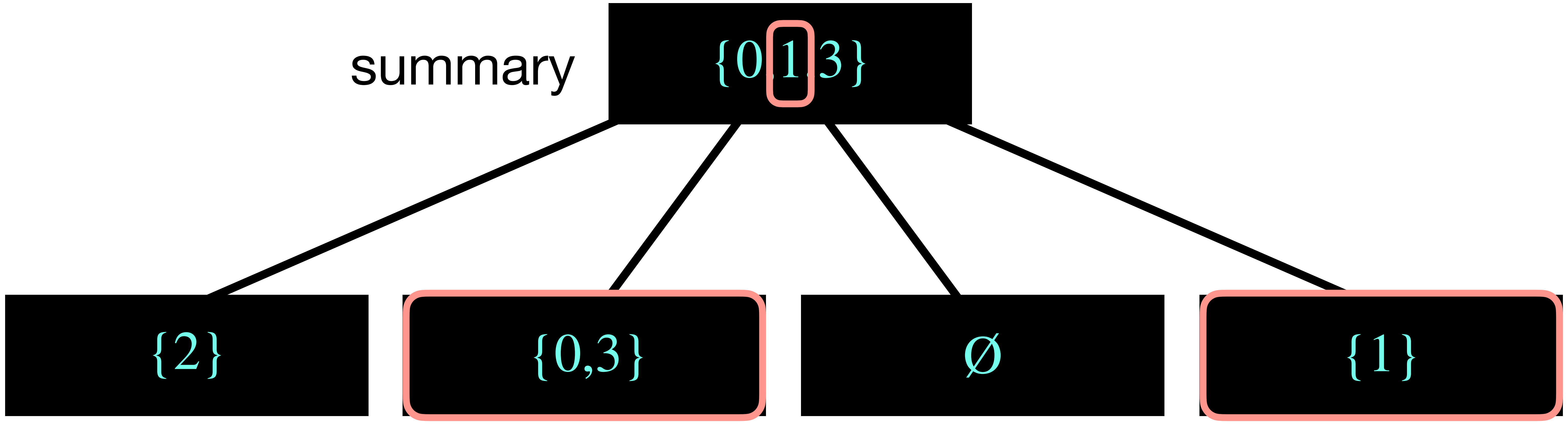

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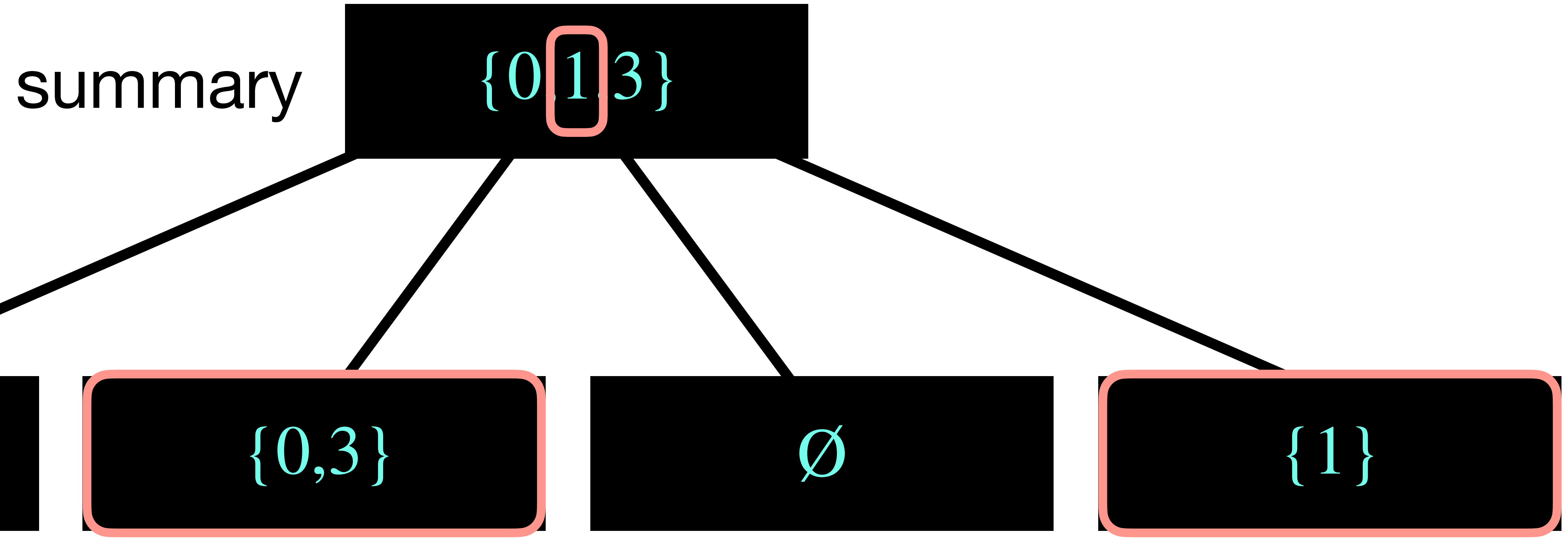
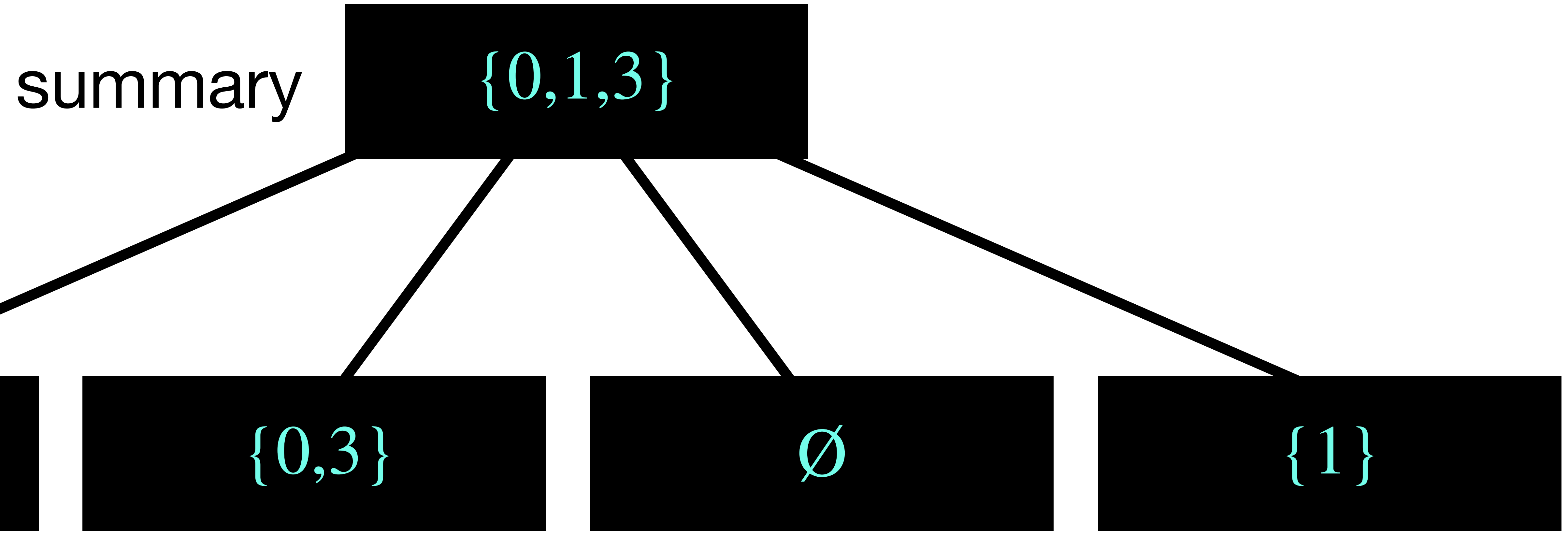


Towards van Emde Boas trees

- Let’s see how efficient this data structure is.



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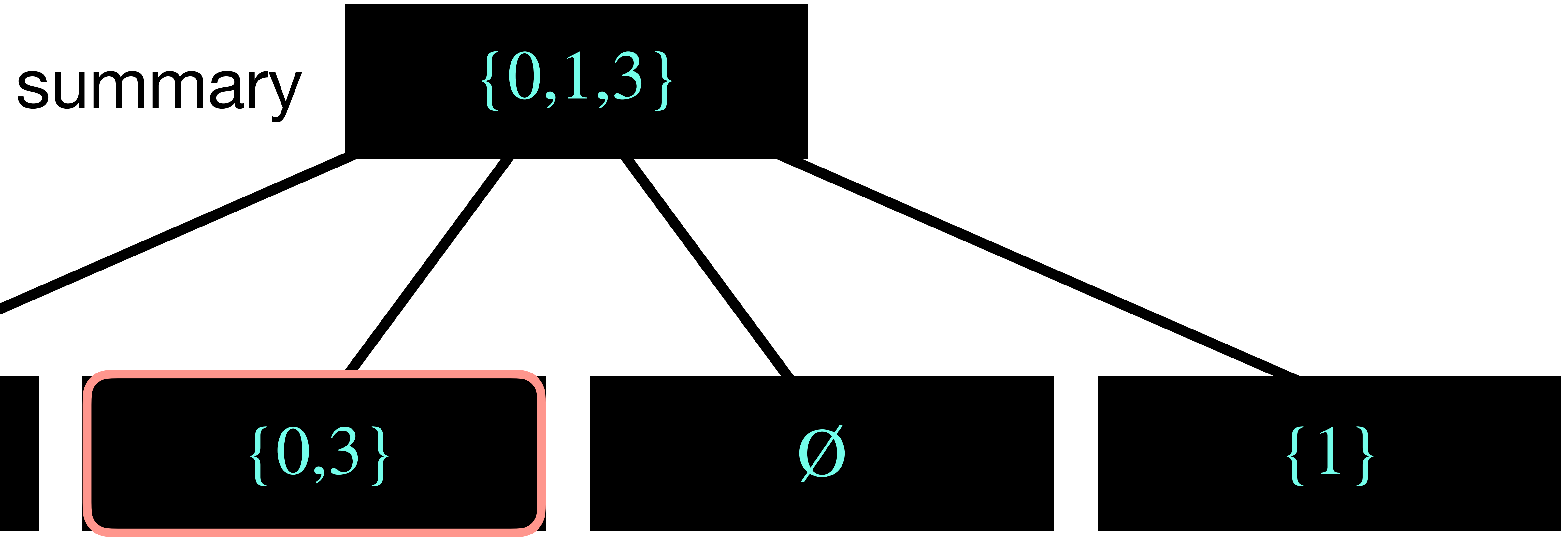
Member(7):

- Answer Member$(7 \bmod 4)$ from chunk $\lfloor 7/4 \rfloor$.



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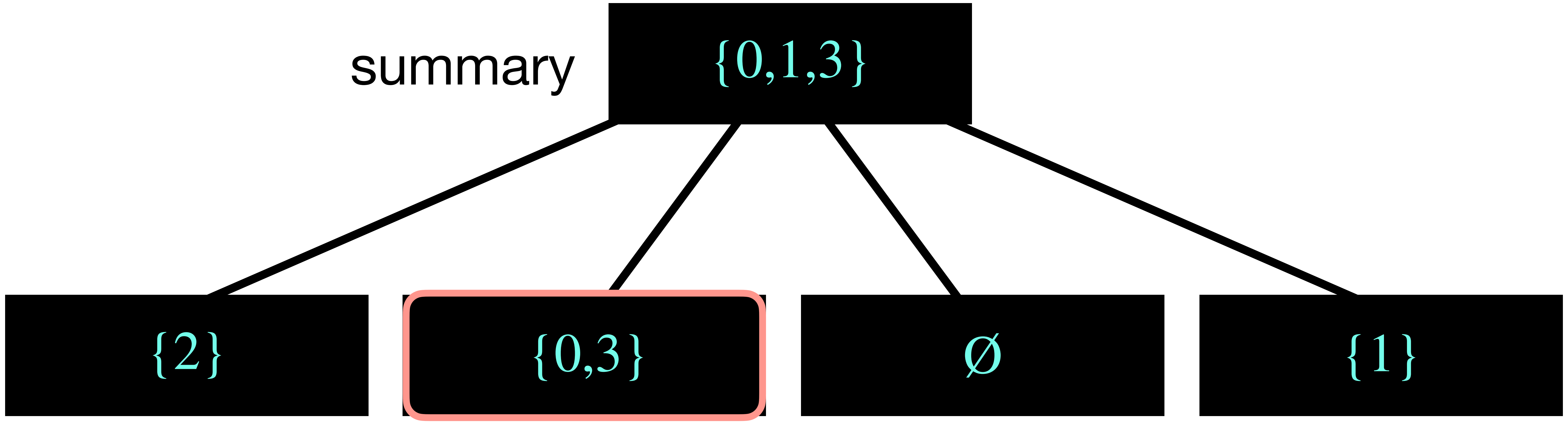

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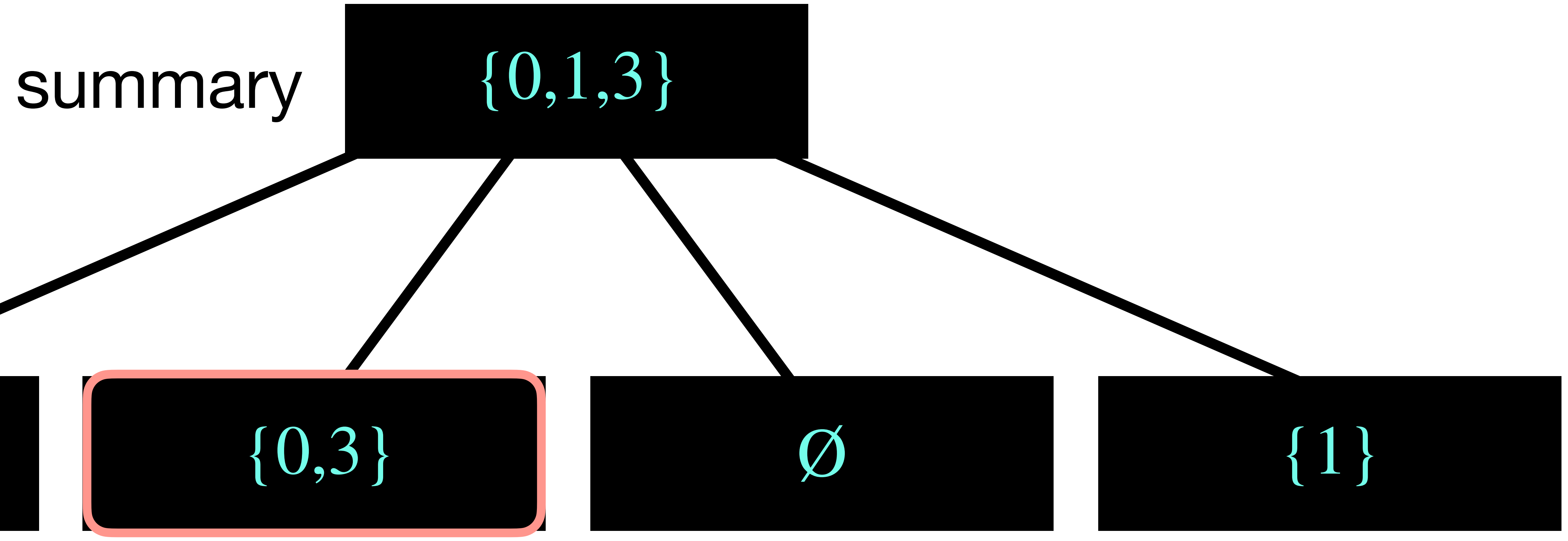
Letting $w = \log_2 U$ and $R(w) = T(2^w)$, we have
$R(w) \leq \Theta(1) + R(w/2) \to R(w) = O(\log w)$.

Hence $T(U) = O(\log \log U)$.



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- **Our goal**: $O(\log \log U)$ time for everything.
- **Q.** Where is the problem?

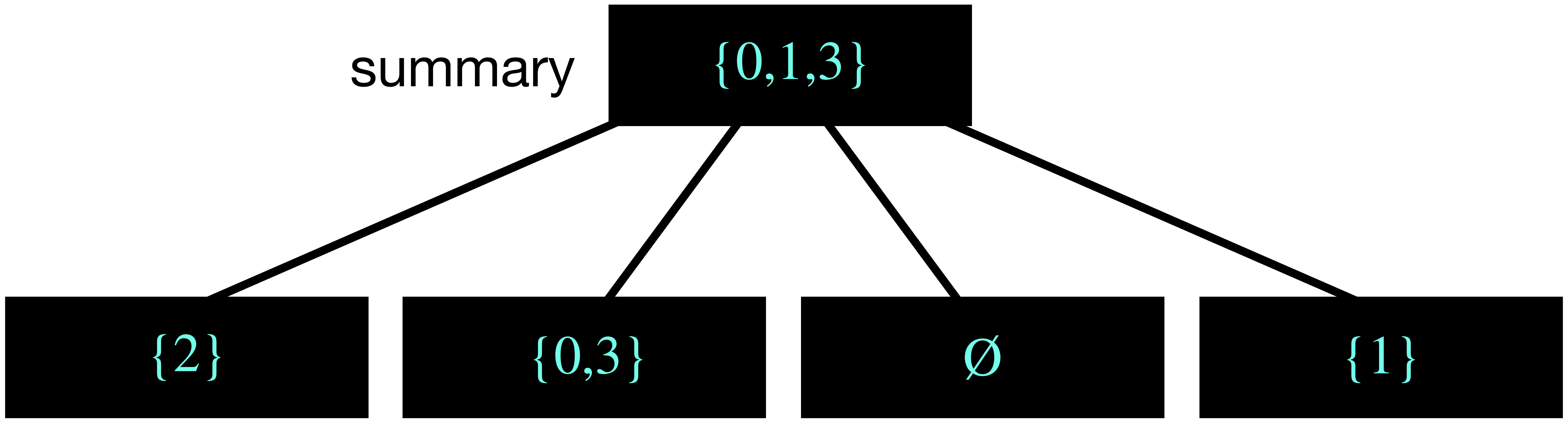
- **A.** We recurse **twice** (or we do not spend $O(1)$ per level):

$$T(U) \leq \Theta(1) + 2T(\sqrt{U}) \rightarrow O(\log U)$$

whereas $$T(U) \leq \Theta(1) + T(\sqrt{U}) \rightarrow O(\log \log U)$$

Towards van Emde Boas trees — min/max out

- Take min and max elements out from every child and summary, without representing them recursively.
- **Intuition.** Now Min() and Max(), as well as determining if a tree is empty, Insert/Delete into/from an empty tree, become constant-time ops/queries.
  With this change, we can **always** get the nice recurrence $T(U) \leq \Theta(1) + T(\sqrt{U})$, $T(2) = \Theta(1)$.



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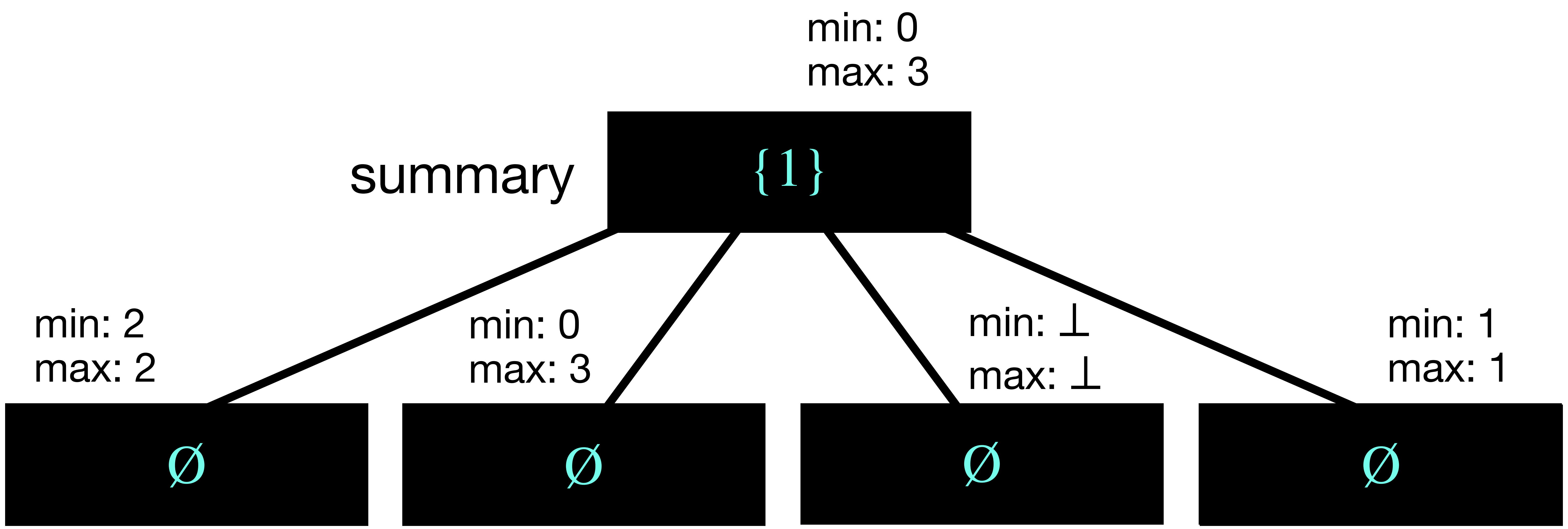

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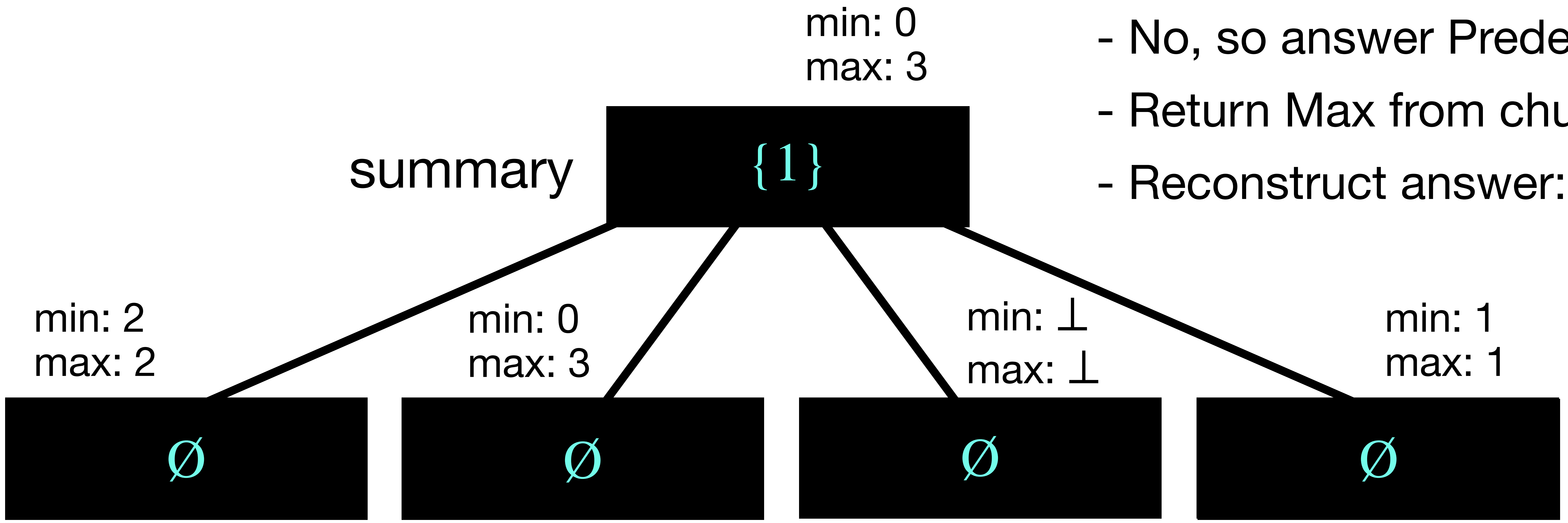

Predecessor(13):

- Retrieve Min from chunk $\lfloor 13/4 \rfloor$. Now in $\Theta(1)$.
- Is Min larger than $13 \bmod 4$?
- No, so answer Predecessor$(\lfloor 13/4 \rfloor) = 1$ on summary.
- Return Max from chunk 1, which is 3. Now in $\Theta(1)$.
- Reconstruct answer: $3 + 1 \times 4 = 7$.

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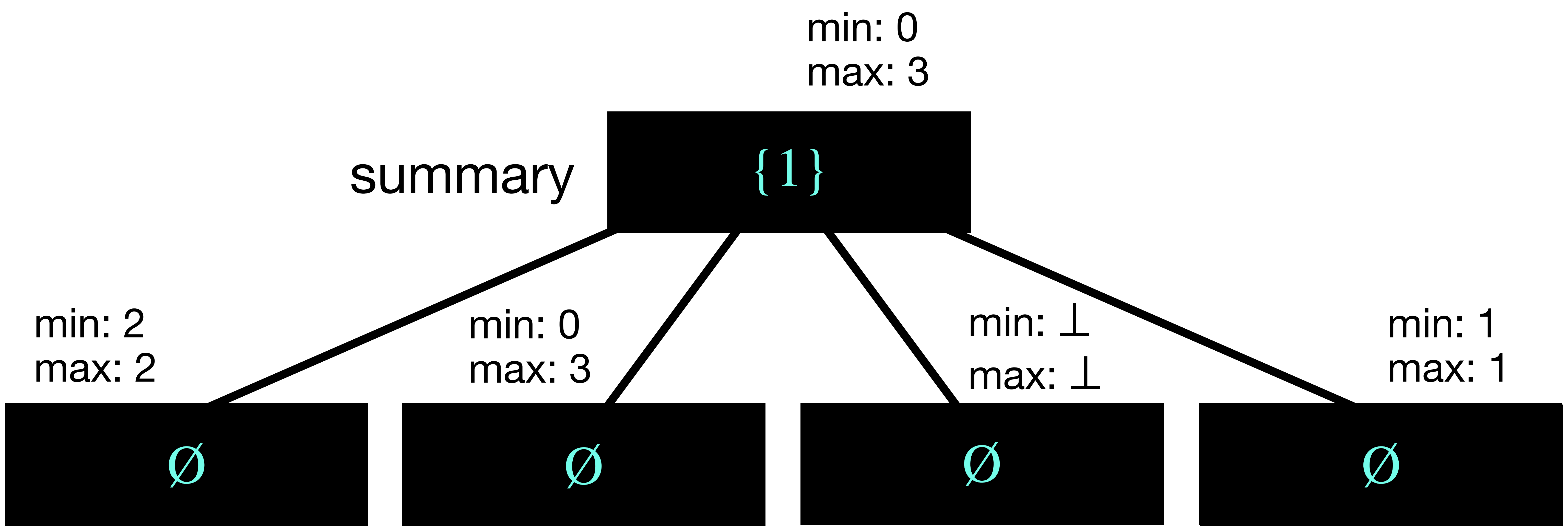

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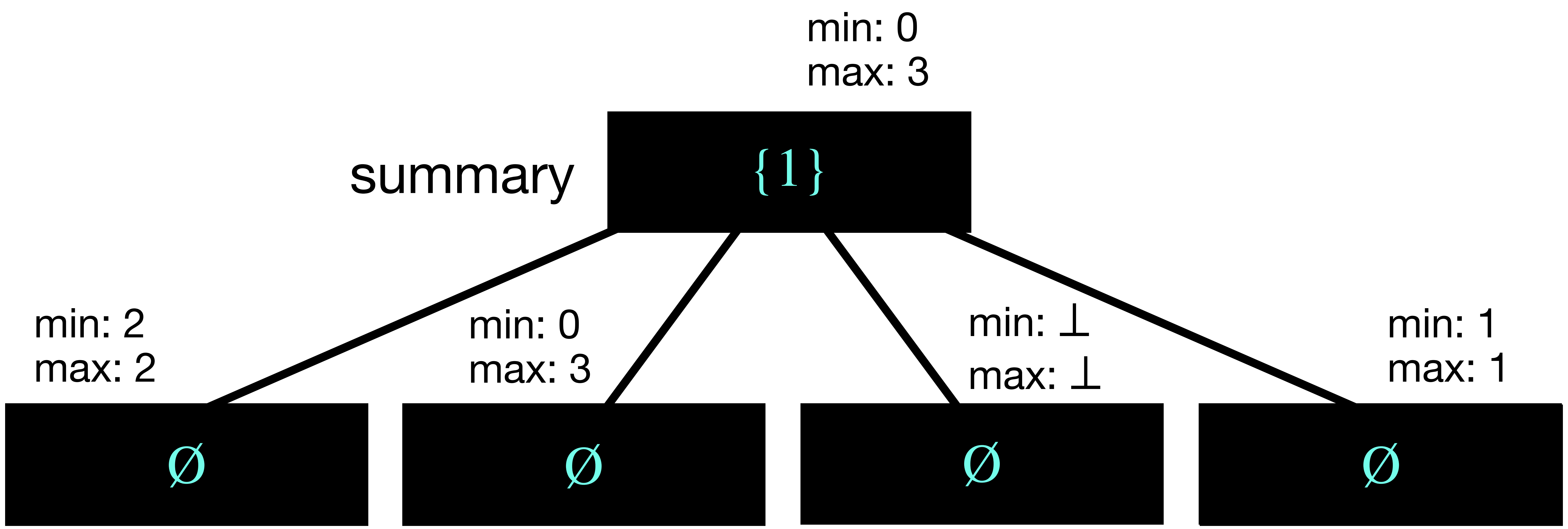

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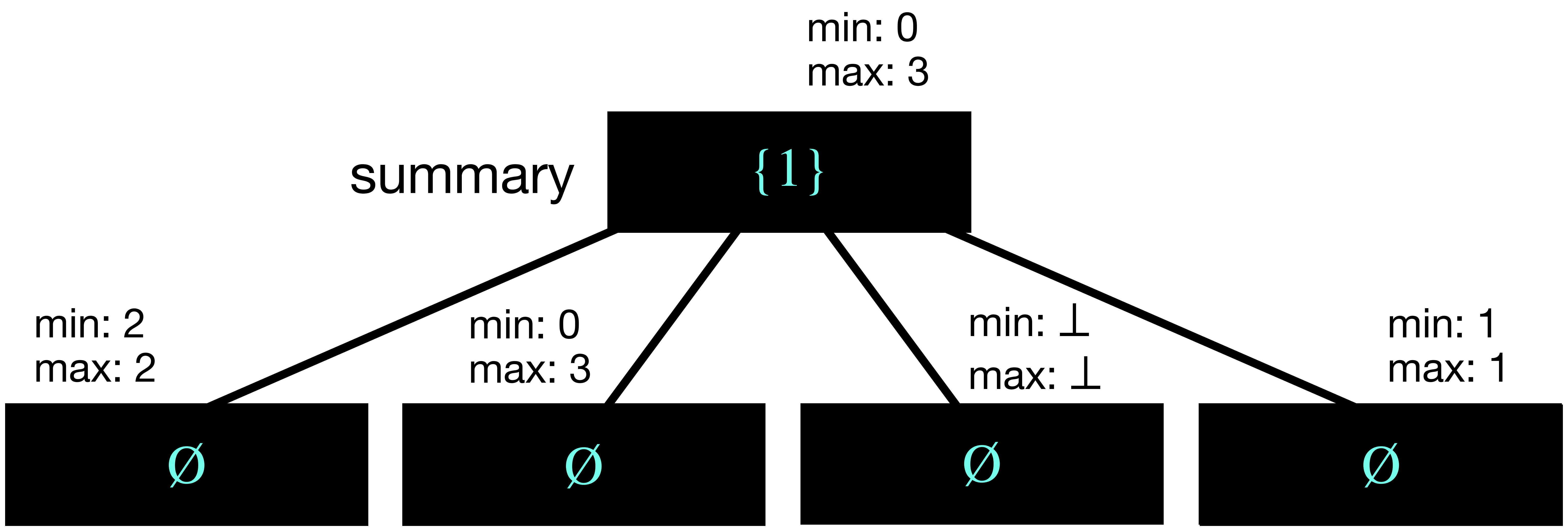
- Is chunk $\lfloor 9/4 \rfloor$ empty? Yes. Now in $\Theta(1)$.



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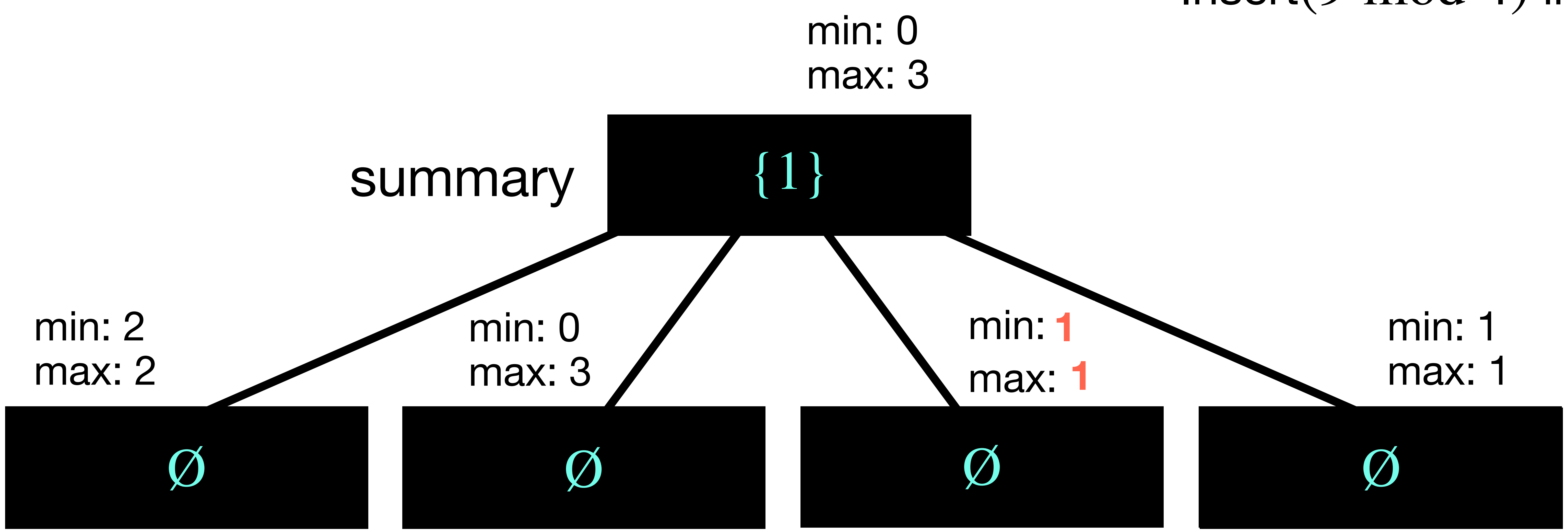

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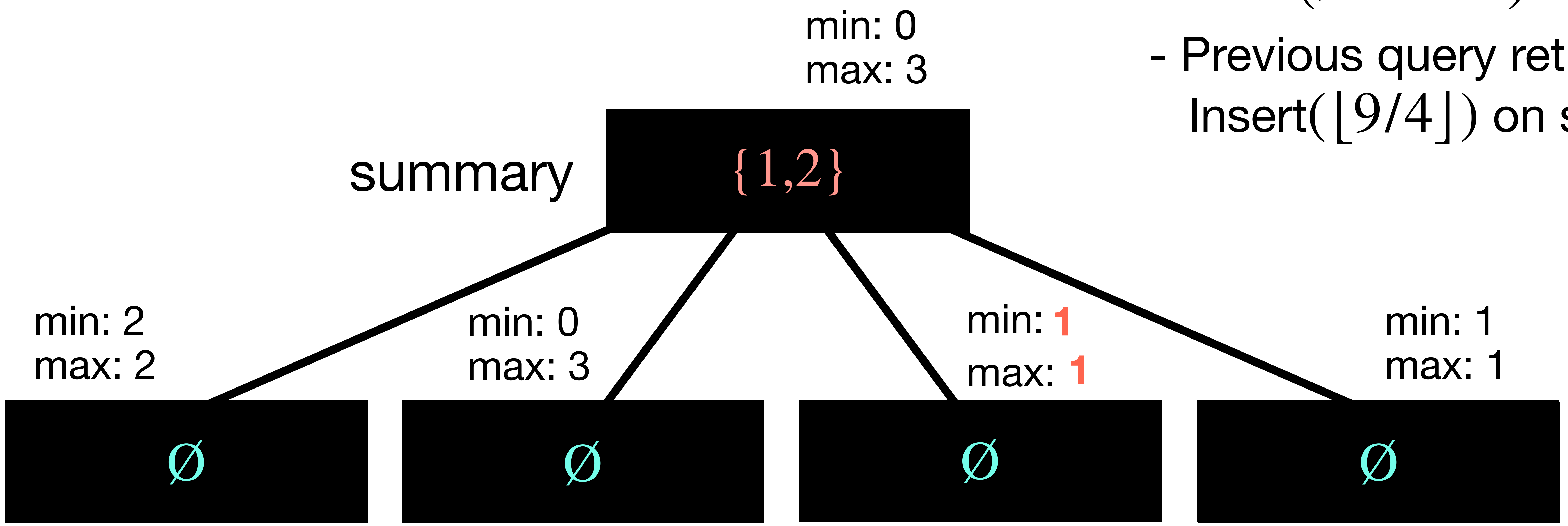
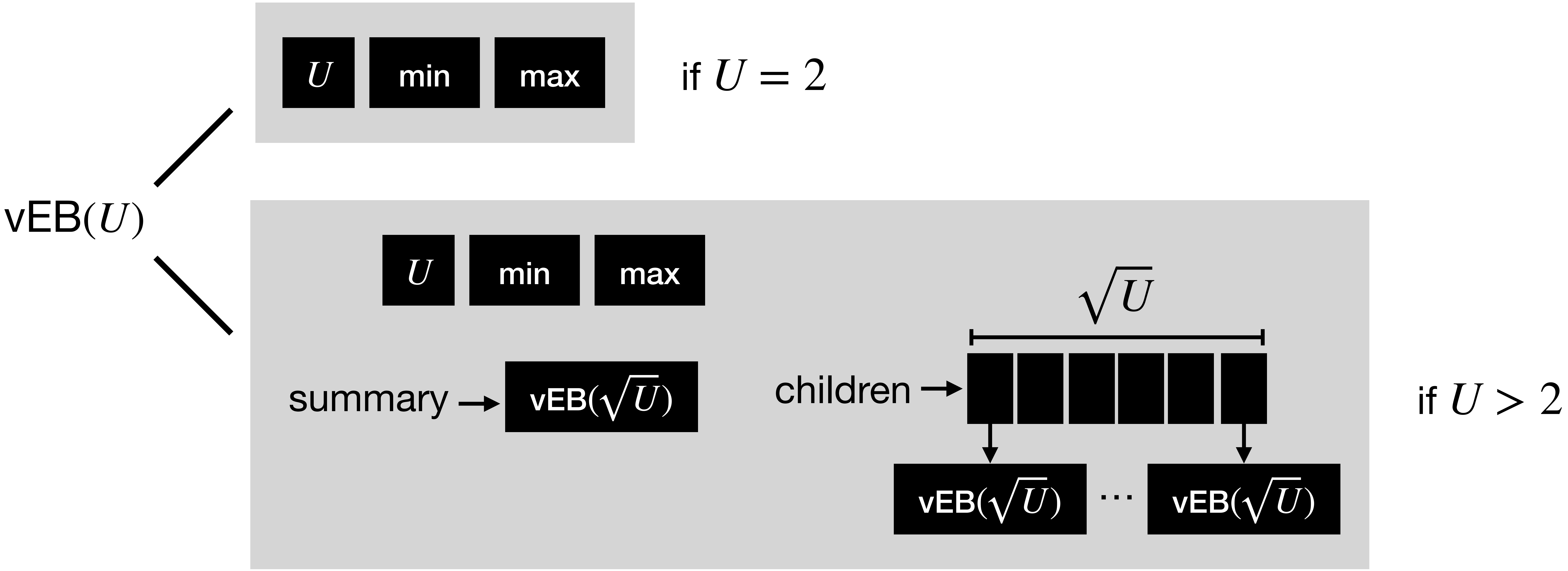
- Is chunk $\lfloor 9/4 \rfloor$ empty? Yes. Now in $\Theta(1)$.
- Insert$(9 \bmod 4)$ in chunk $\lfloor 9/4 \rfloor$. Now in $\Theta(1)$.
- Previous query returned "Yes", so Insert$(\lfloor 9/4 \rfloor)$ on summary.



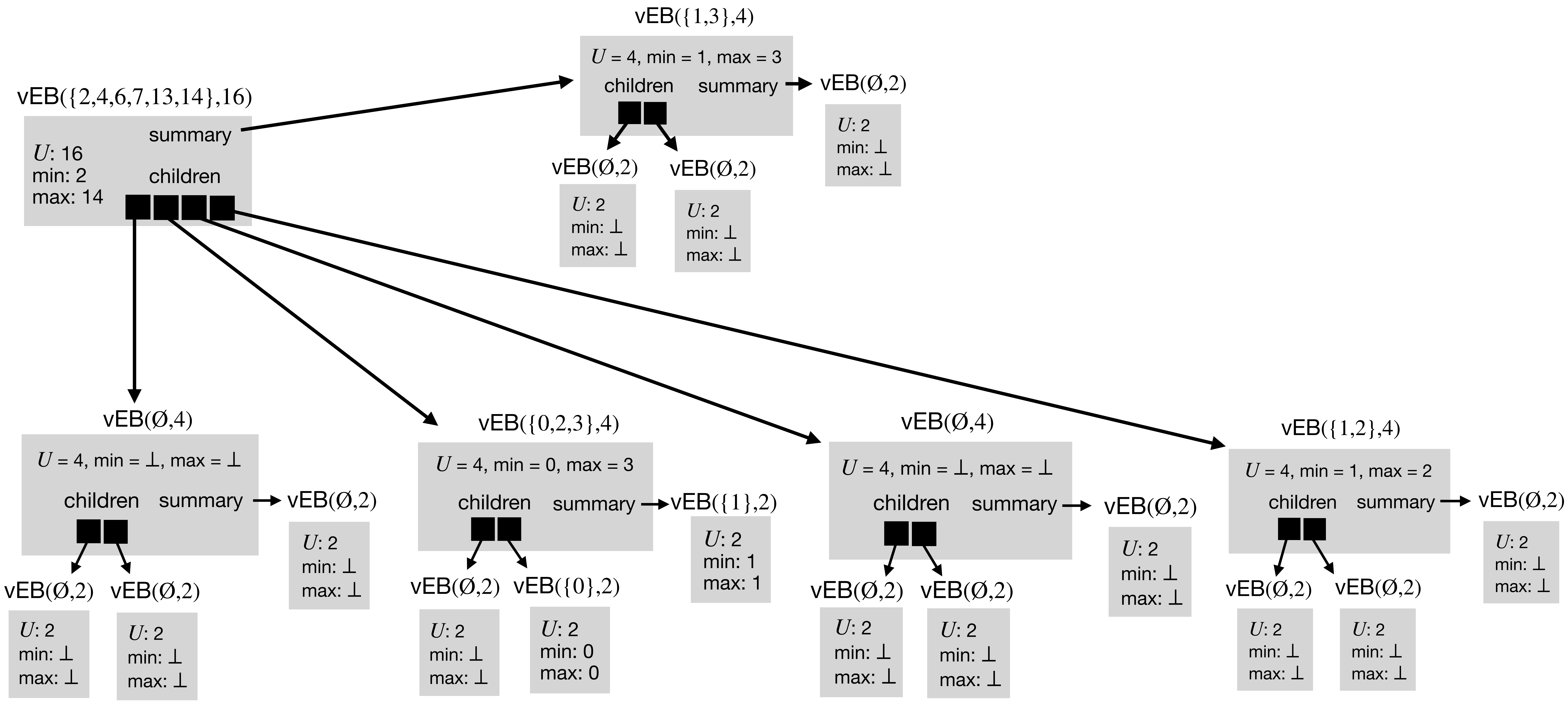
van Emde Boas trees — Rec. definition (graphical)

- A van Emde Boas tree for a universe U, vEB(U), is:



Important note: the min/max elements are not recursively represented in the children.

van Emde Boas trees — Example



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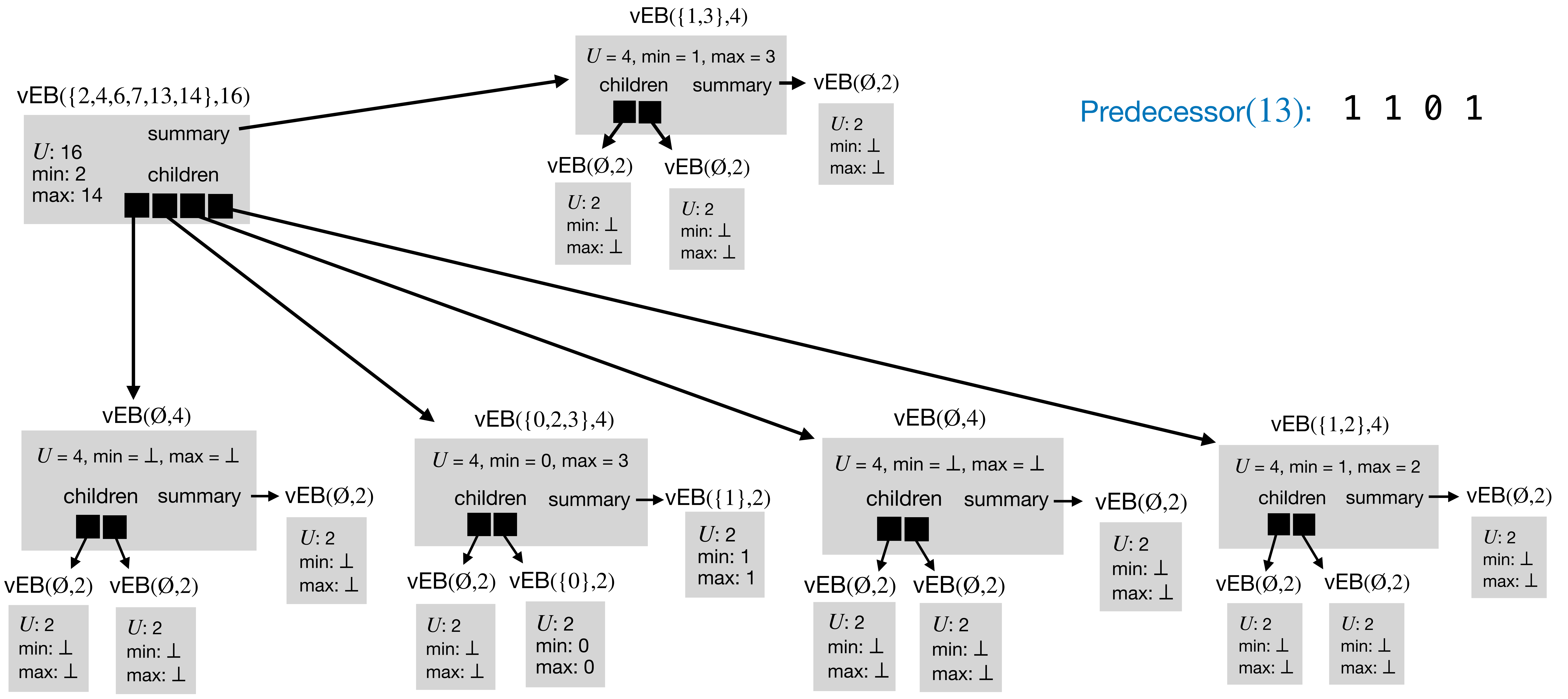
Predecessor(13): 1 1 0 1



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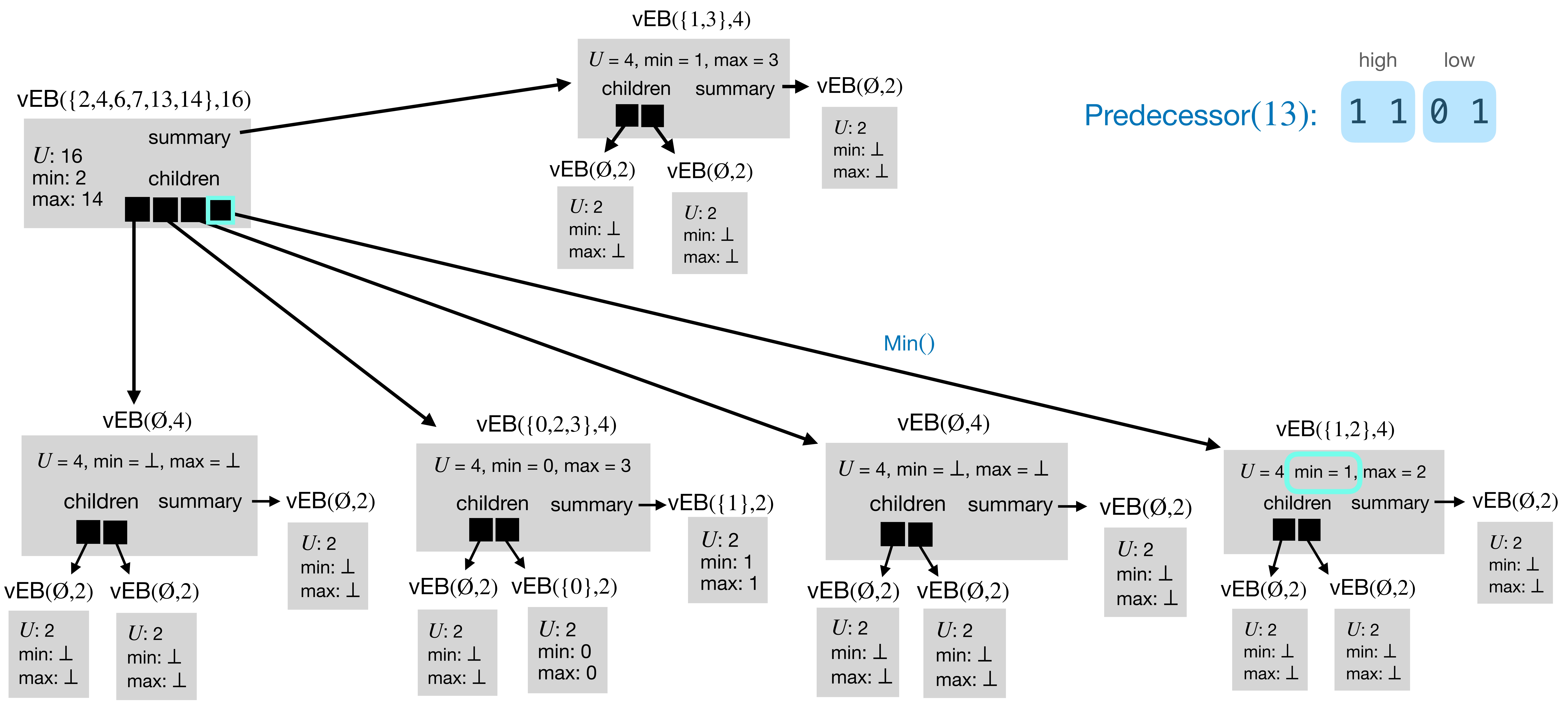

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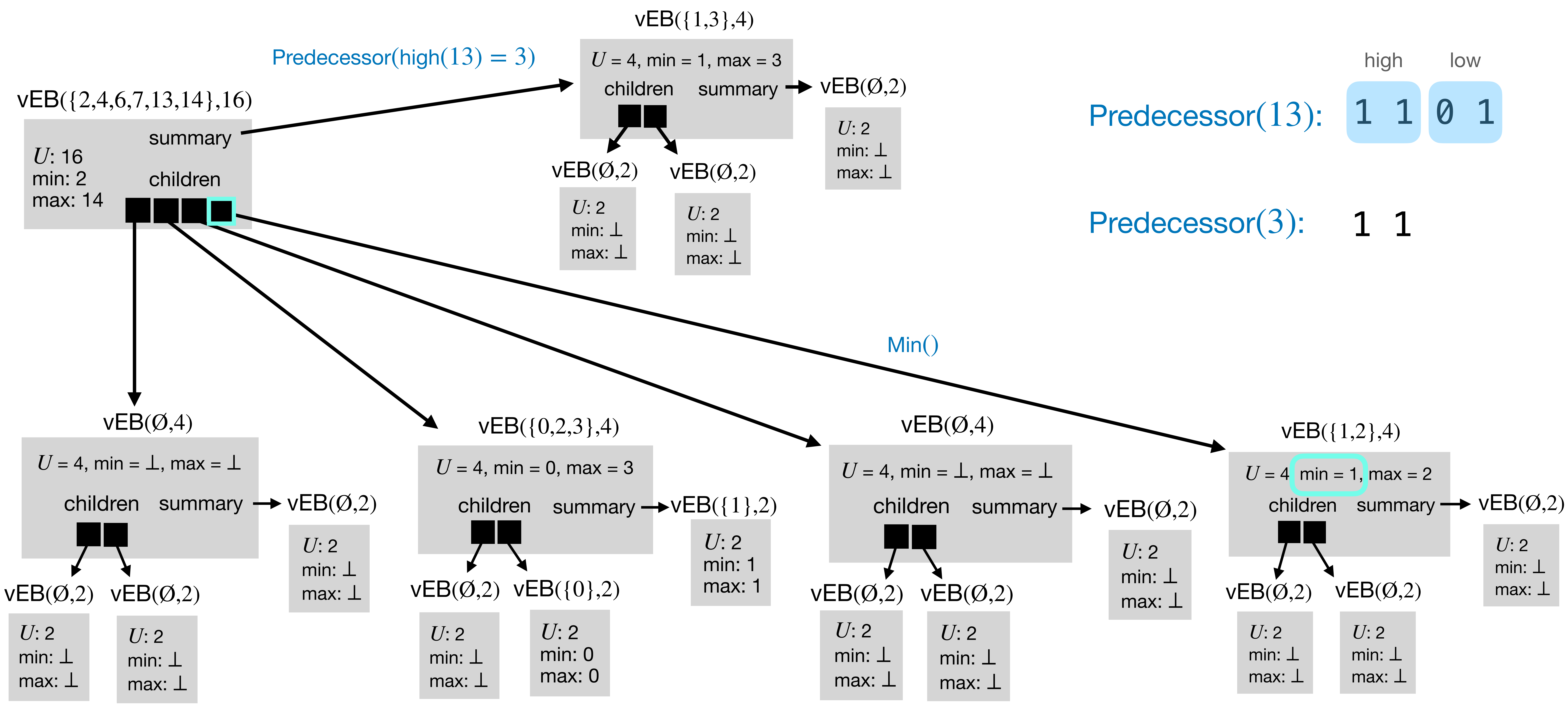

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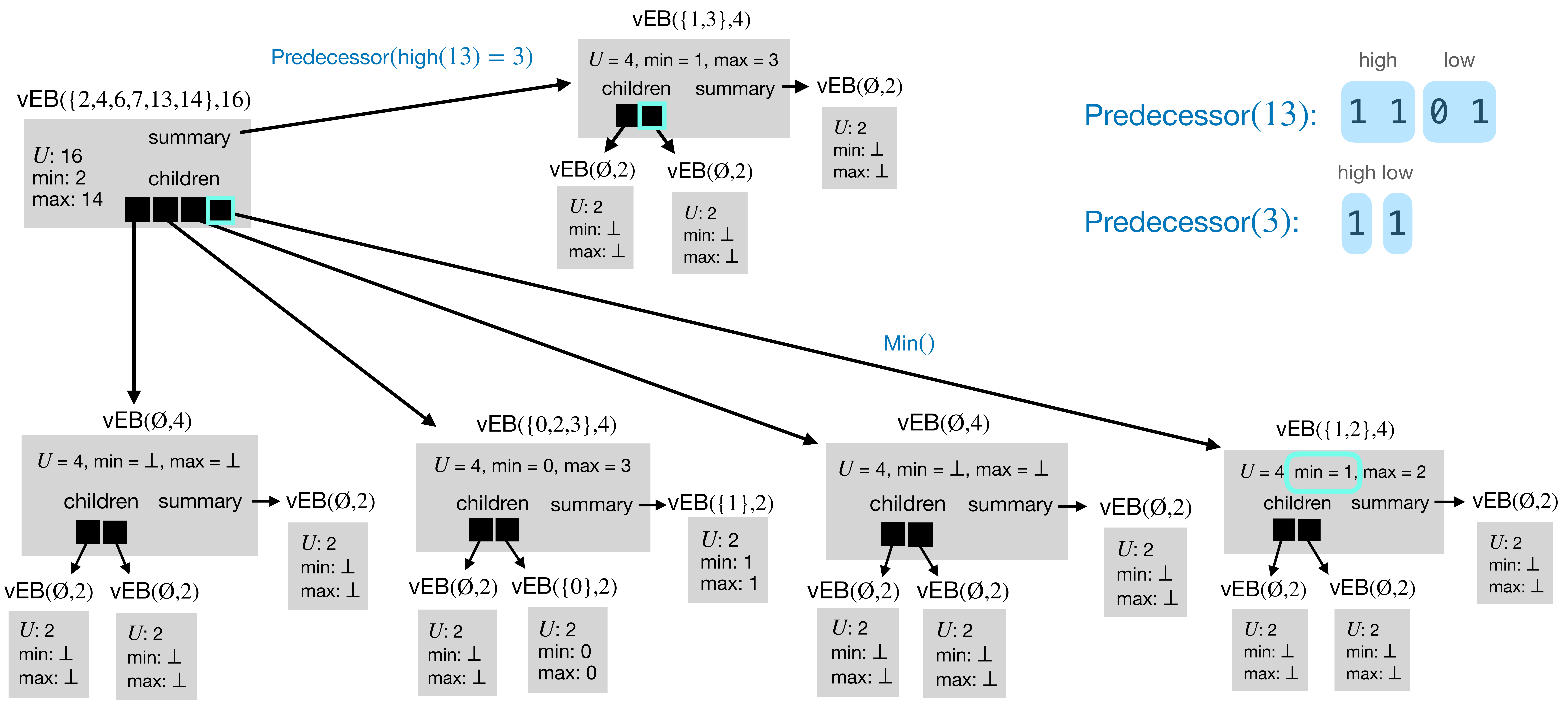

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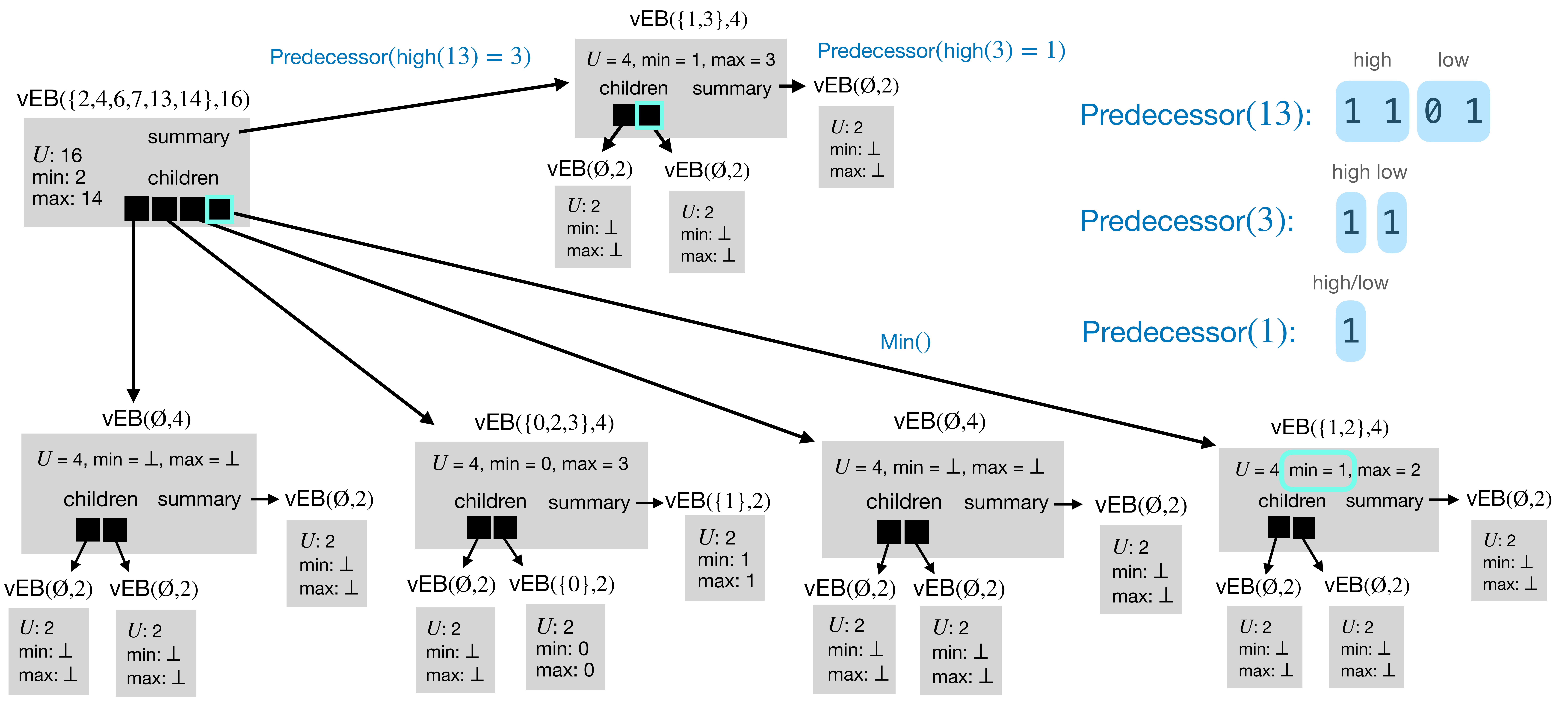

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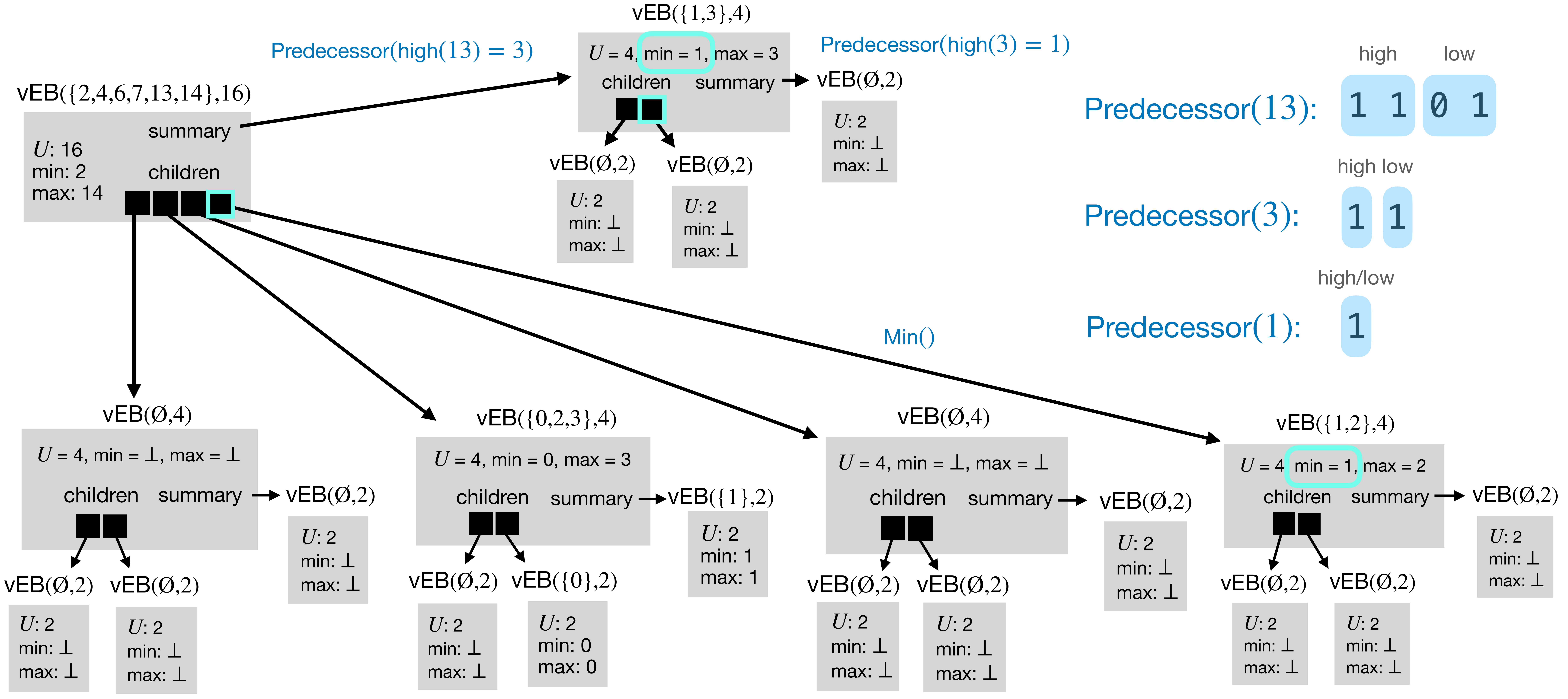

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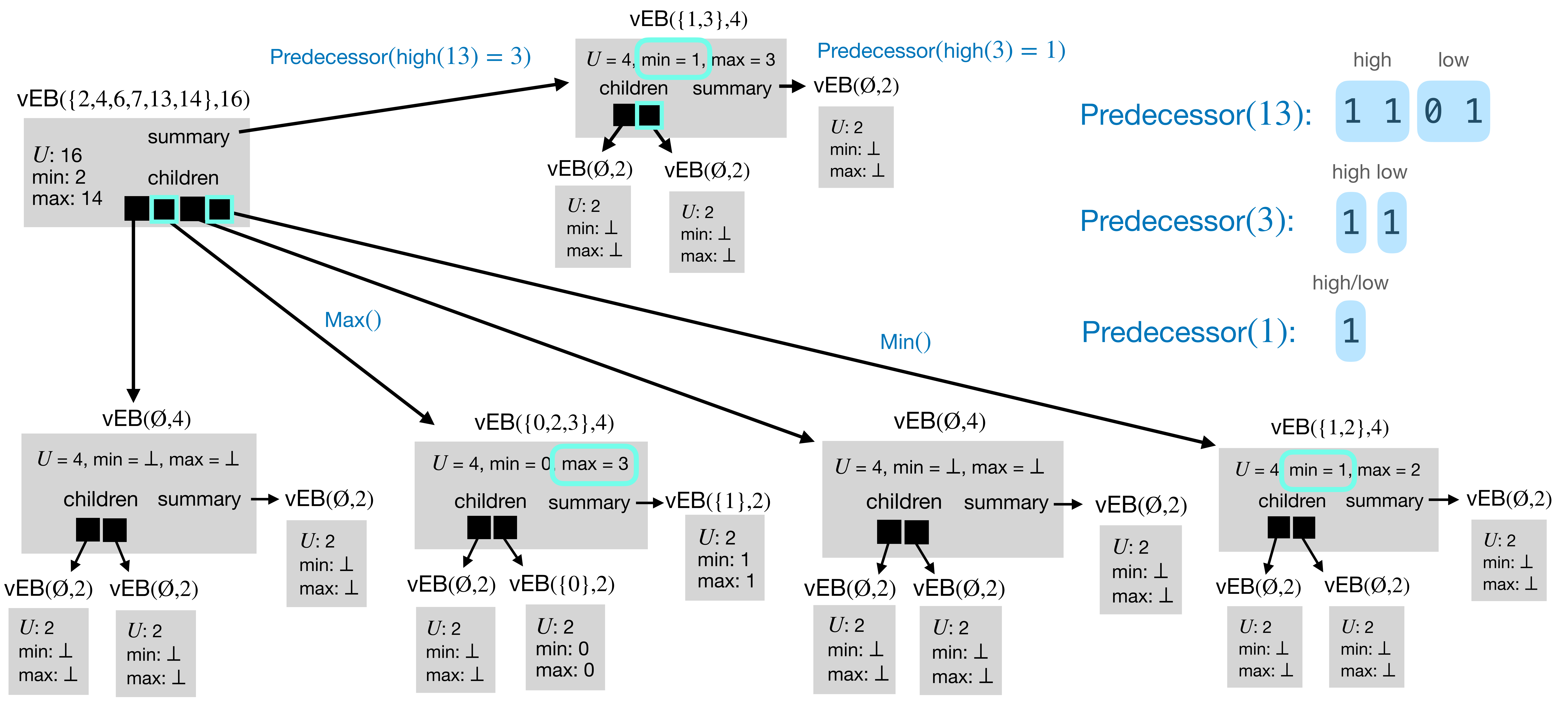
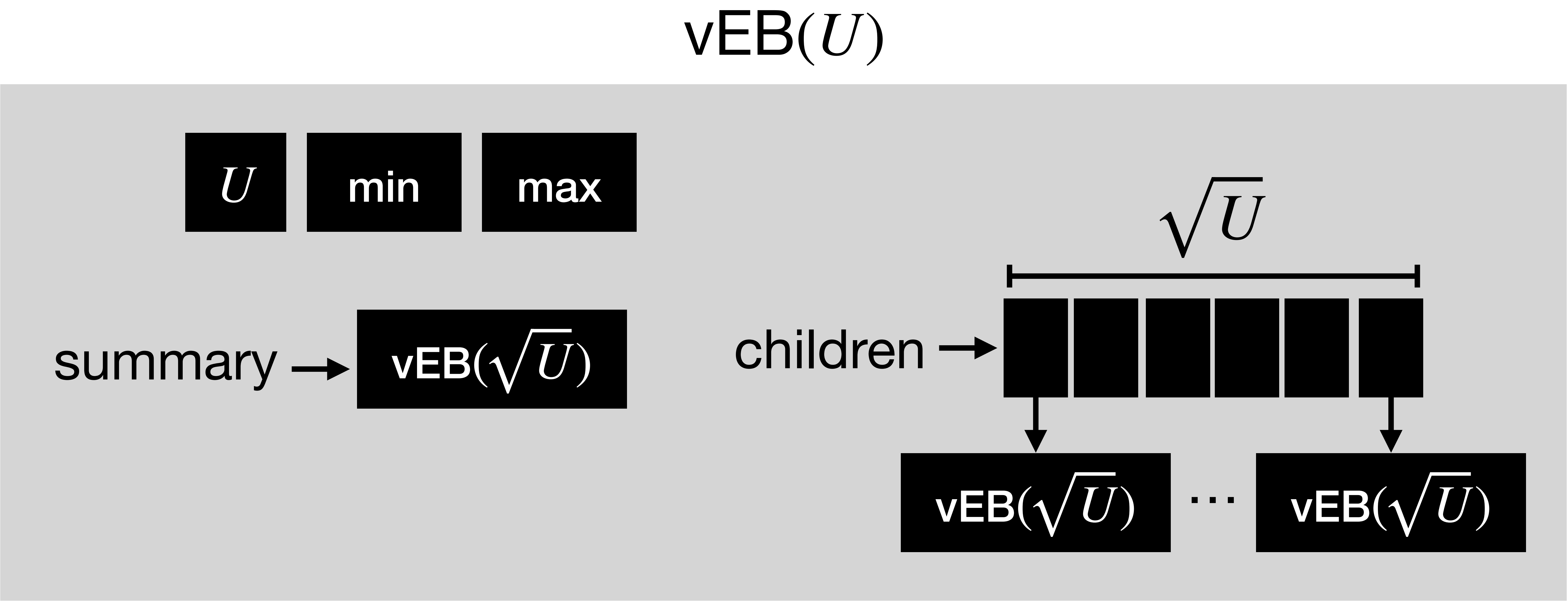


van Emde Boas trees — Space

- Recall what a vEB(U) stores.



- $S(U) = \underbrace{\Theta(1)}_{U,\ \text{min, max}} + \underbrace{\Theta(\sqrt{U})}_{\text{pointers}} + \underbrace{\sqrt{U} \cdot S(\sqrt{U})}_{\text{children}} + \underbrace{S(\sqrt{U})}_{\text{summary}}$, and $S(2) = \Theta(1)$.

van Emde Boas trees — Space

- $S(U) \simeq \Theta(\sqrt{U}) + \sqrt{U} \cdot S(\sqrt{U})$, and $S(2) = \Theta(1)$.
- Letting $w = \log_2 U$, we have:

by induction

truncated **Kempner** number, $\approx 0.81642.$

$$S(2^w) = \Theta(2^{w/2}) + 2^{w/2} S(2^{w/2}) =$$
$$= \Theta(2^{w/2}) + \Theta(2^{3w/4}) + 2^{3w/4} S(2^{w/4}) =$$
$$= \Theta(2^{w/2}) + \Theta(2^{3w/4}) + \Theta(2^{7w/8}) + 2^{7w/8} S(2^{w/8}) = \dots$$
$$= \Theta\left(\sum_{i=1}^{\log_2 w} 2^{\frac{2^i - 1}{2^i} w}\right) = \Theta\left(2^w \cdot \sum_{i=1}^{\log_2 w} 2^{-w/2^i}\right) = \Theta\left(2^w \cdot \boxed{\sum_{j=0}^{\log_2 w - 1} 2^{-2^j}}\right) = \Theta(2^w) = \Theta(U)\,.$$

by linearity

change variable $j = \log_2 w - i$

Summary

- The vEB tree maintains a sorted integer set, whose elements are less than a known quantity U, in worst-case time $O(\log\log U)$ and space $\Theta(U)$. (It can be built in $O(U)$ time: solution to the recurrence $T(U) = T(\sqrt{U}) + \sqrt{U} \cdot T(\sqrt{U})$, with $T(2) = \Theta(1)$.)

- **Key insight.** The $\log_2 U$-bit representation of an integer can be split recursively to speed up operations and queries.

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- For example, if $U = O(2^n)$ then $\log_2 \log_2 U = O(\log n)$ and vEB trees are **not** asymptotically faster than AVL nor RB trees.

- However, if $U = n^c$ for some $c \geq 1$ then $\log_2 \log_2 U = \Theta(\log\log n)$ and vEB trees are **exponentially** faster than AVL and RB trees.

Optimal predecessor search

- In 2006, Pătrașcu and Thorup published a landmark result

  Time-Space Trade-Offs for Predecessor Search, STOC 2006

  showing that **vEB is optimal for predecessor search** for polynomial universes ($U = n^c, c \geq 1$).

- Intuition: if the set is dense (n is close to U), it is faster to search over the binary representation of the integers directly, like vEB. If, instead, the set is sparse, we should just search the keys.

Mihai Pătrașcu

Mikkel Thorup

Caveats and issues

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- **Issue: Space usage $\Theta(U)$.**

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- Next week (10 Feb) spoiler:
  ***y*-fast tries** combines $O(\log \log U)$ worst-case query time with $\Theta(n)$ space.
  (Insert/Delete is $O(\log \log U)$ amortised.)

References

- P. van Emde Boas, *Preserving order in a forest in less than logarithmic time*, FOCS, 75-84, 1975.

- P. van Emde Boas; R. Kaas; E. Zijlstra, *Design and implementation of an efficient priority queue*, Math. Syst. Theory, 99–127, 1977.

- P. van Emde Boas, *Preserving order in a forest in less than logarithmic time and linear space*, Inf. Process. Lett. 6, 80–82, 1977.

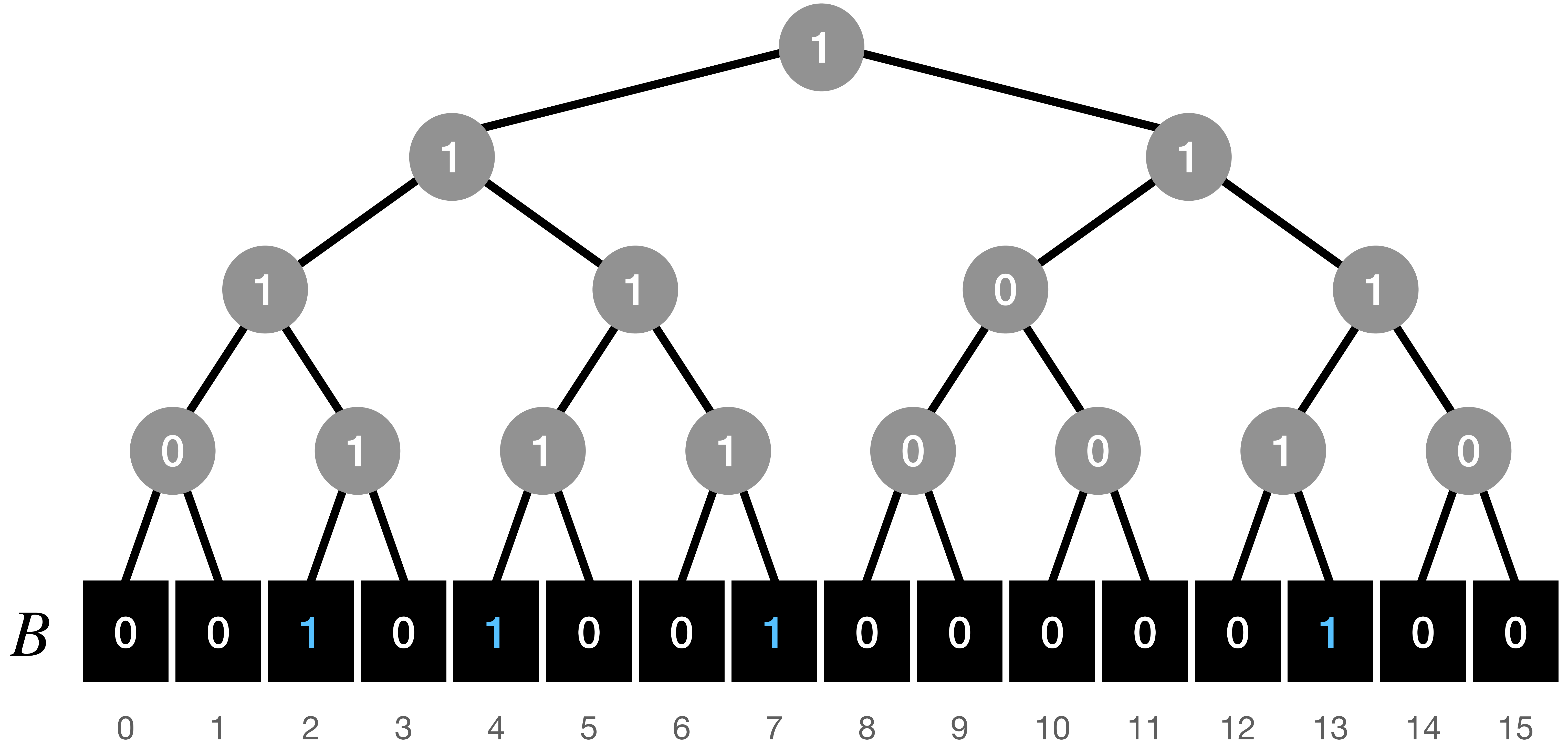
- M. Pătraşcu, and M. Thorup. *Time-space trade-offs for predecessor search*, STOC, 2006.

Bonus slides

Imposing a binary tree

- **Idea.** Impose a complete binary tree on top of B. The leaves of the tree correspond to the bits of B; an internal node stores the logical OR between the bits of its children.

- **Intuition.** Use the tree to **avoid scanning long runs on zeros** upon Predecessor/Successor.



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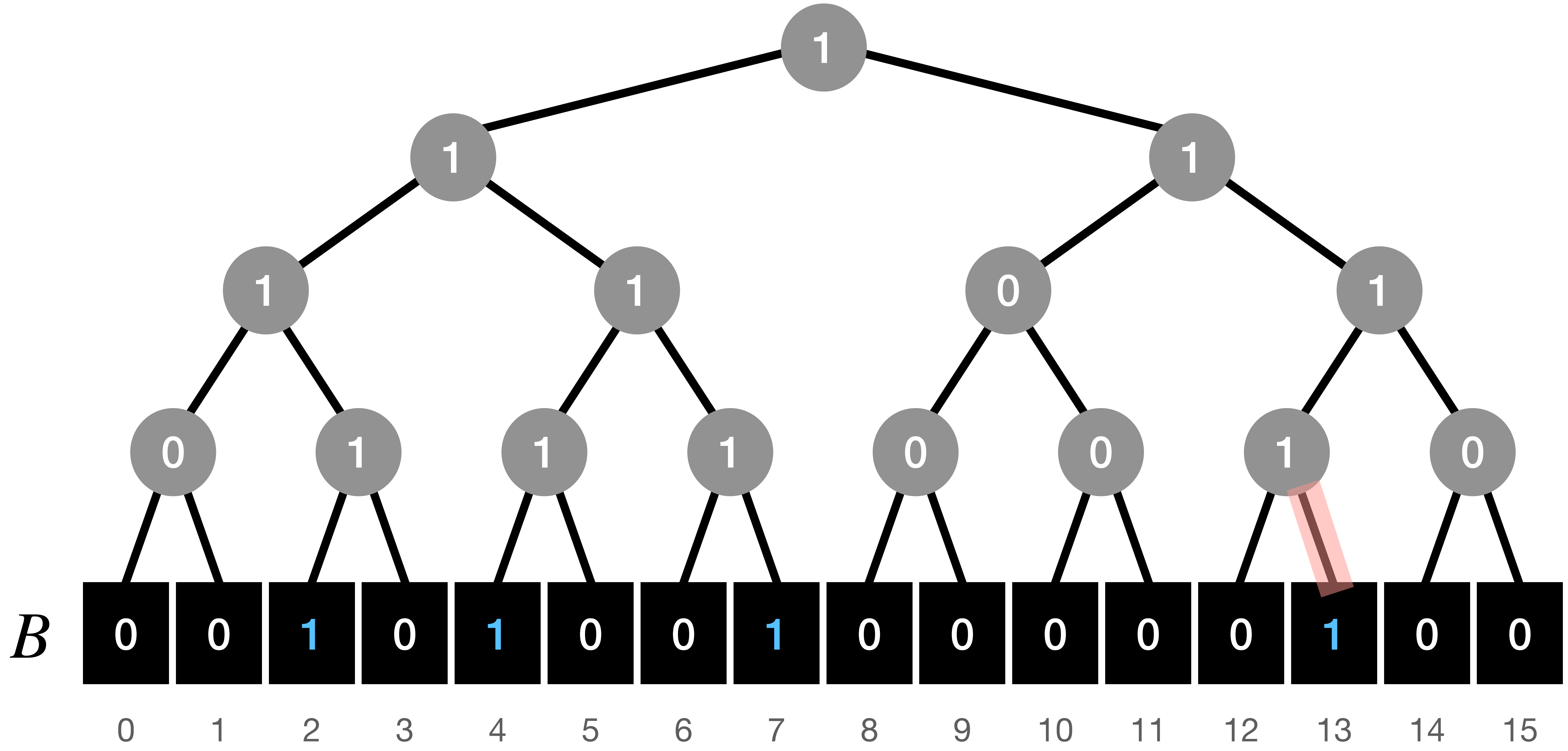

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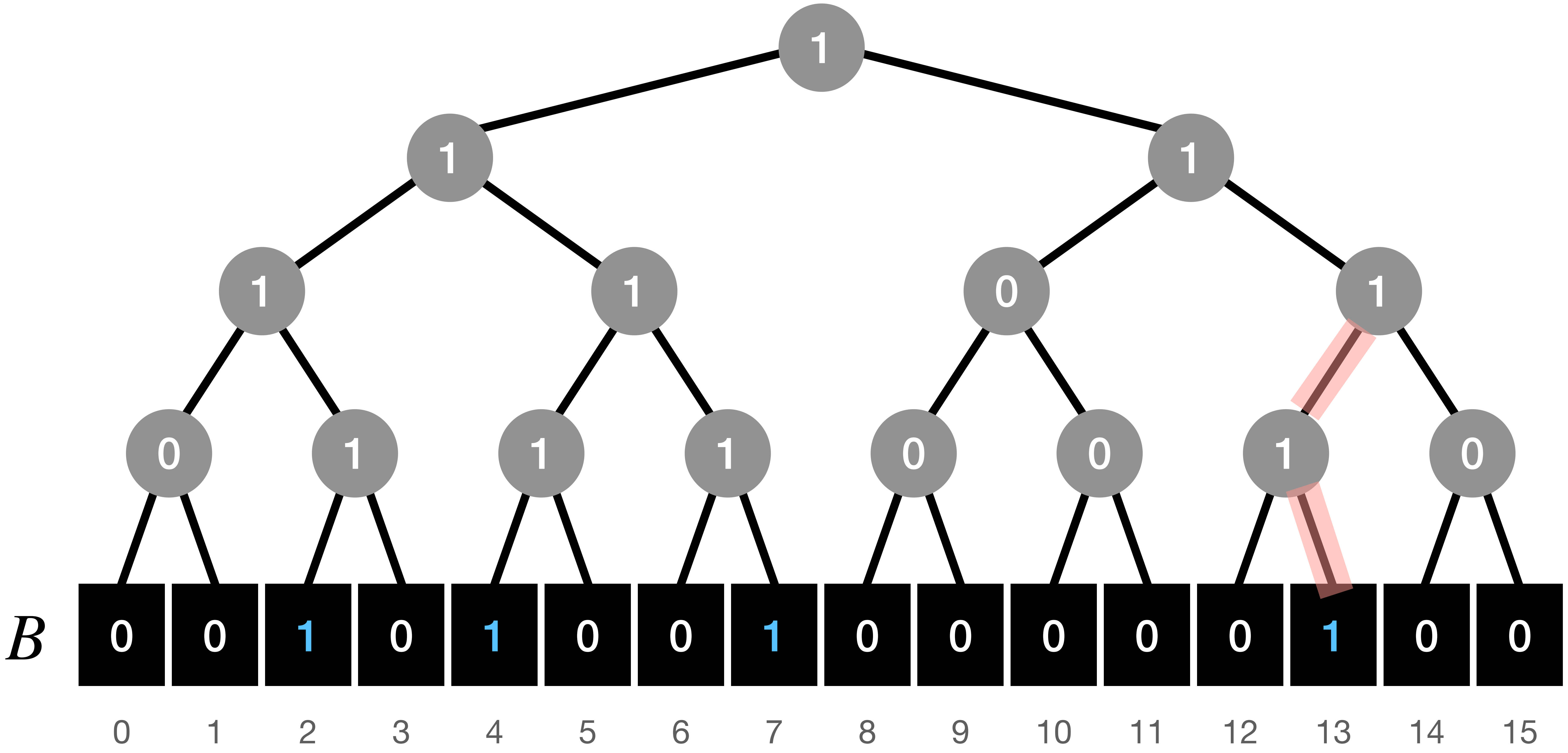

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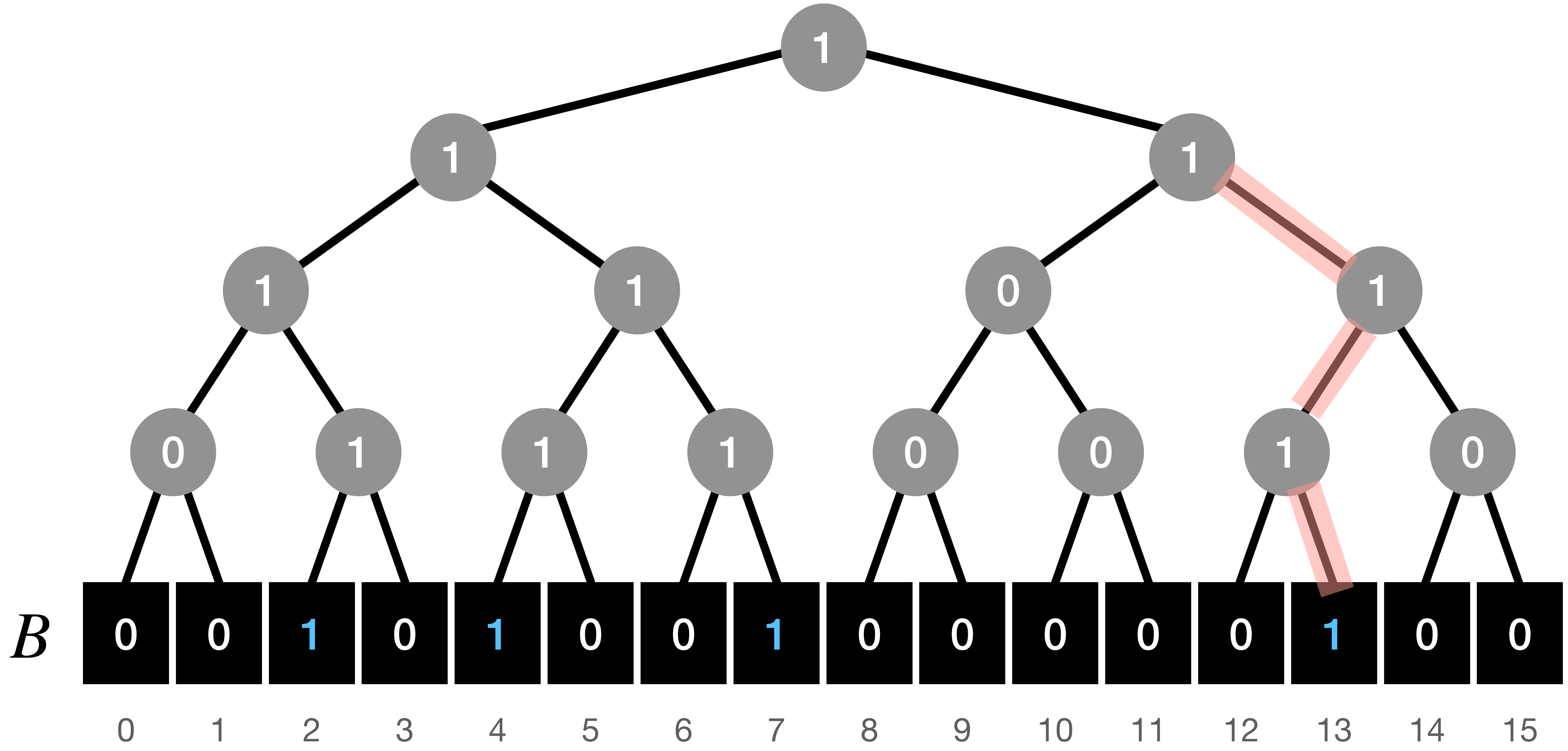

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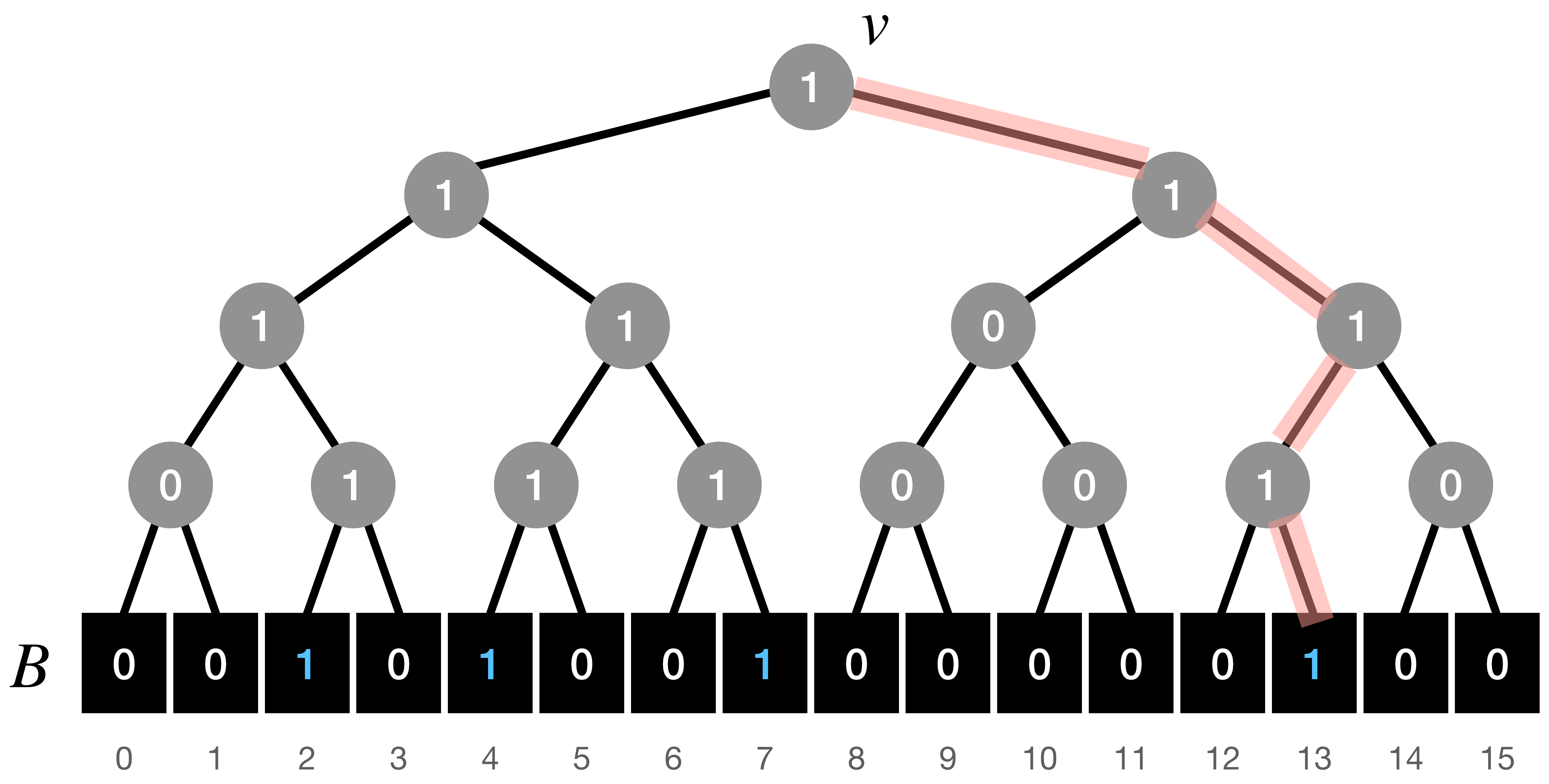

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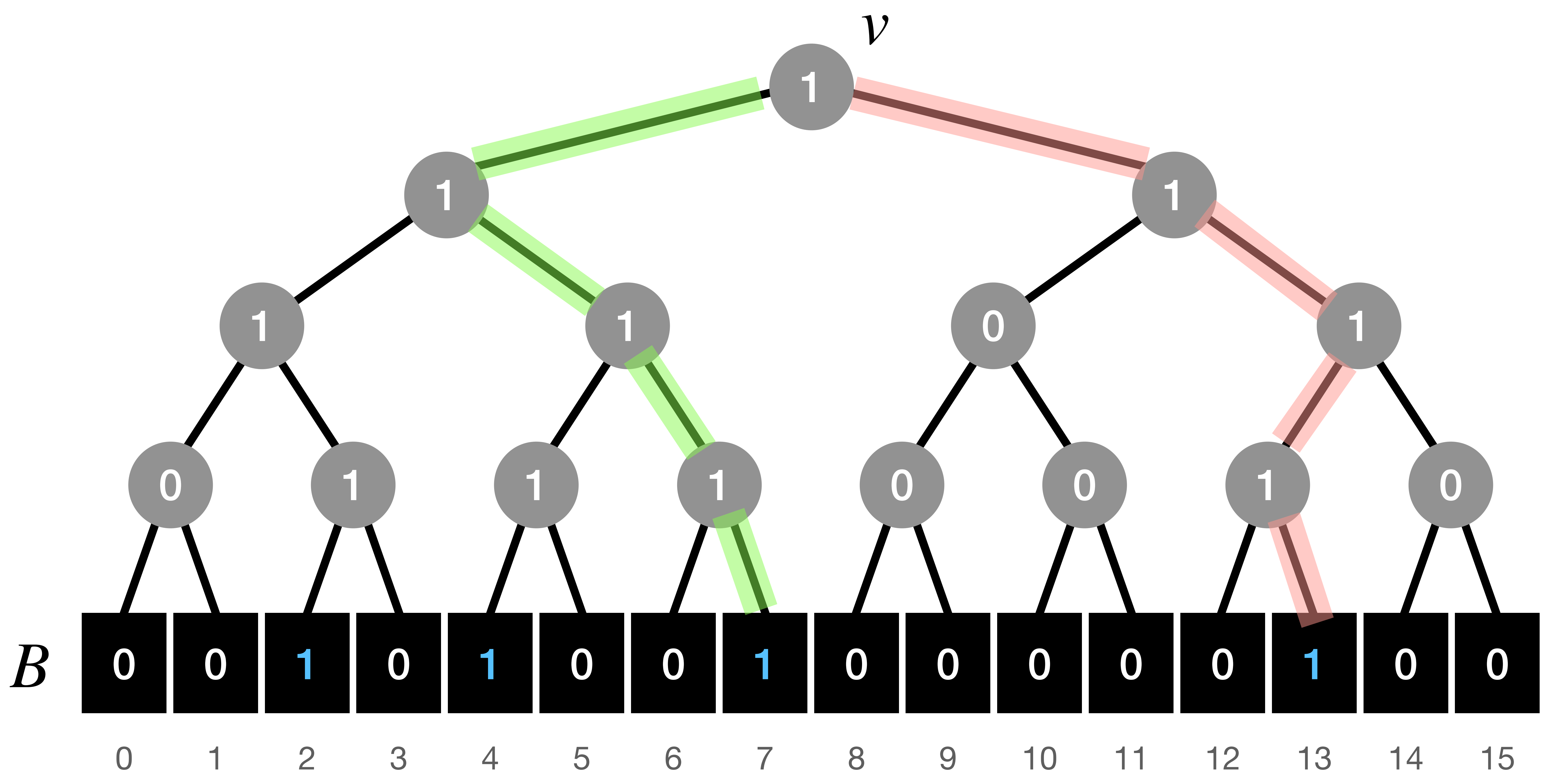

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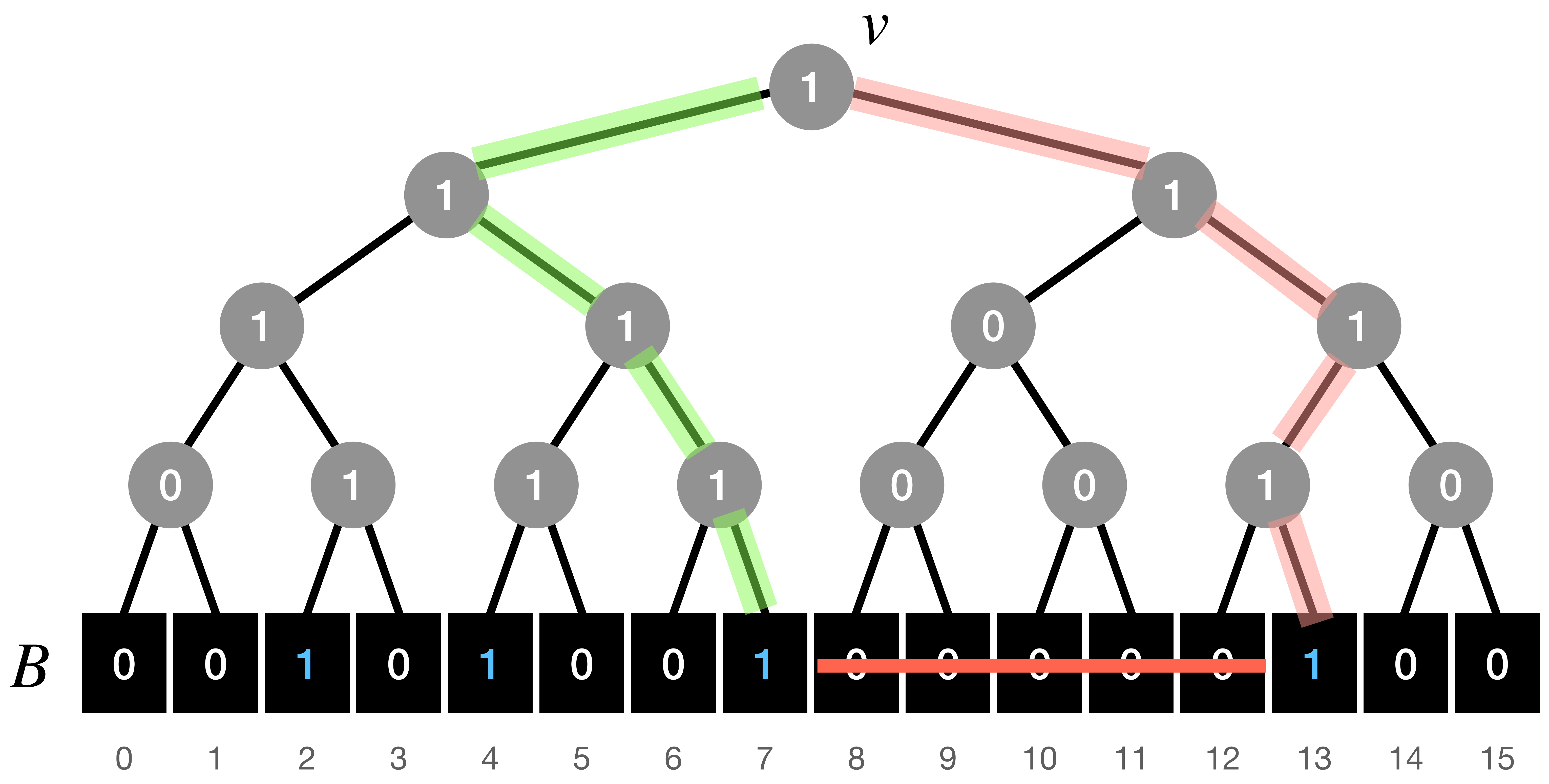

- Predecessor(x): from $B[x]$, navigate up in the tree until we enter a node v from the **right** that has a 1 in its **left** child. Then return the **max** of the subtree rooted in v . left. (If v is the root and the bit in v . left is 0, return ⊥.)

Imposing a binary tree

- **Idea.** Impose a complete binary tree on top of B. The leaves of the tree correspond to the bits of B; an internal node stores the logical OR between the bits of its children.

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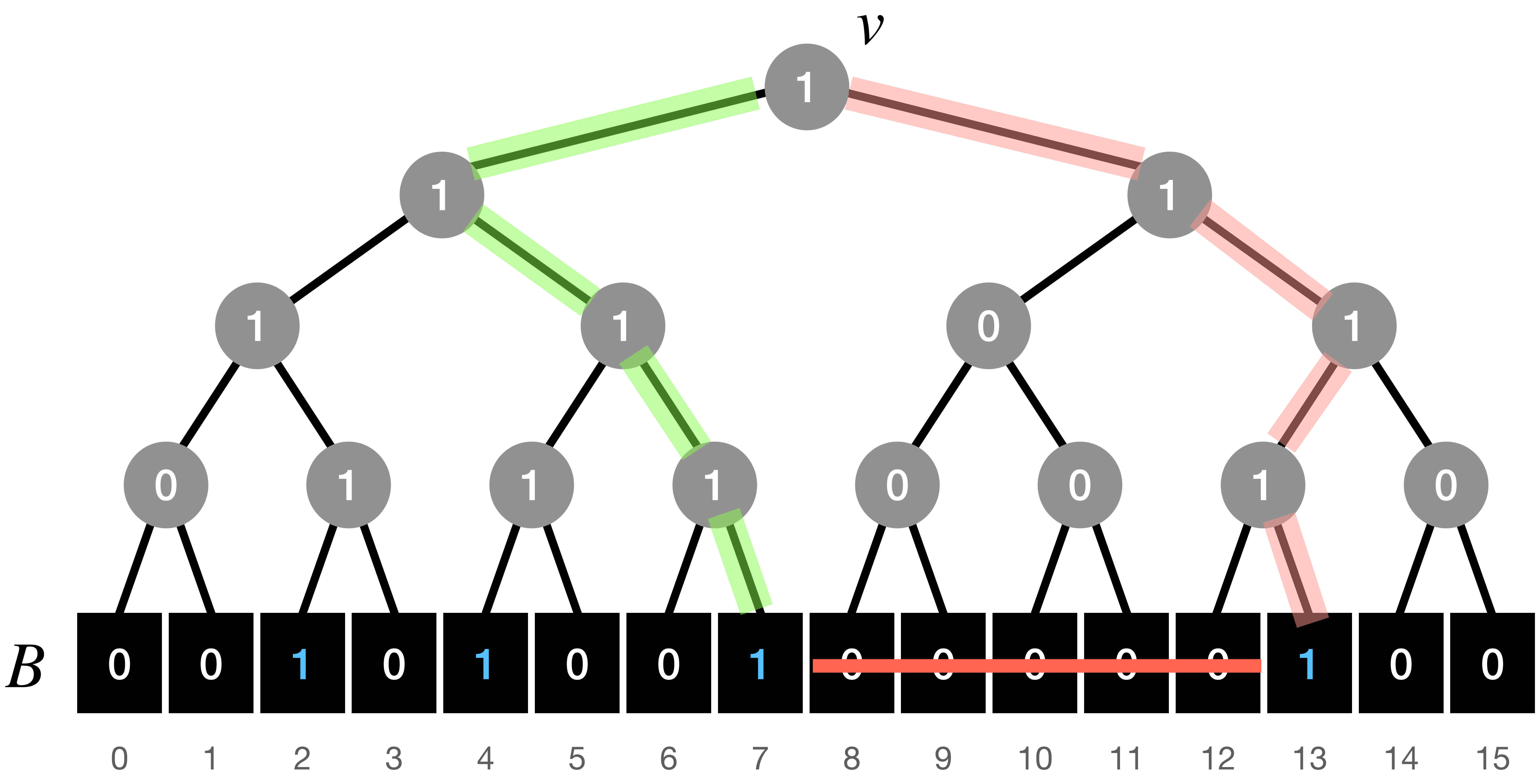

- Predecessor(x): from $B[x]$, navigate up in the tree until we enter a node v from the **right** that has a 1 in its **left** child. Then return the **max** of the subtree rooted in v.left. (If v is the root and the bit in v.left is 0, return $\perp$.)

The height of the tree is $\log_2 U$ so all ops/queries run in $O(\log U)$.