

x-Fast and y-Fast Tries



Giulio Ermanno Pibiri
Ca' Foscari University of Venice

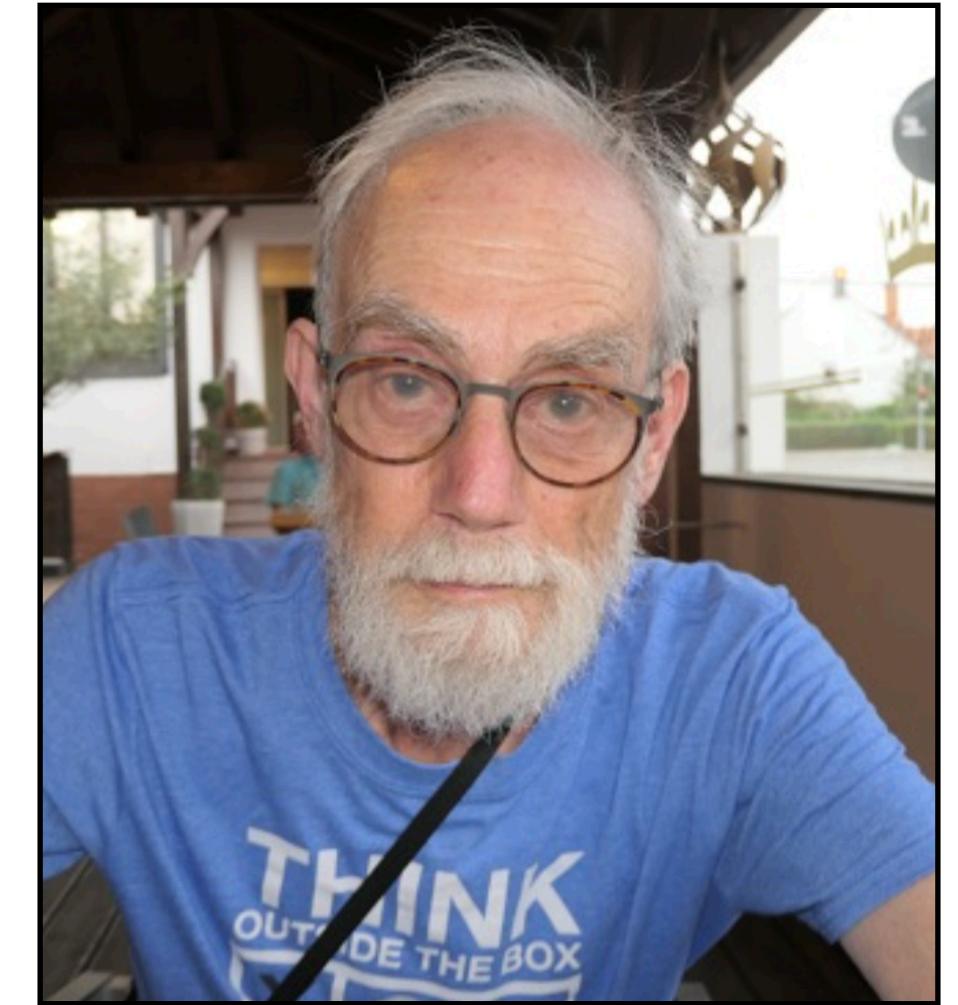
Venice, Italy, 10 February 2026

Problem definition

- Maintain a **sorted integer set** $S \subseteq \{0, \dots, U - 1\}$, for some **universe size** U , under the following operations and queries for an integer $0 \leq x < U$.
 - Min(): return the smallest element in S .
 - Max(): return the largest element in S .
 - Member(x): return “Yes” if $x \in S$, “No” otherwise.
 - Insert(x): add x to S (assuming $x \notin S$).
 - Delete(x): remove x from S (assuming $x \in S$).
 - Successor(x): return the smallest $y \in S$ such that $y > x$, or \perp if no such element exists.
 - Predecessor(x): return the largest $y \in S$ such that $y < x$, or \perp if no such element exists.
- Let n be $|S|$ in the following.

Summary from last week

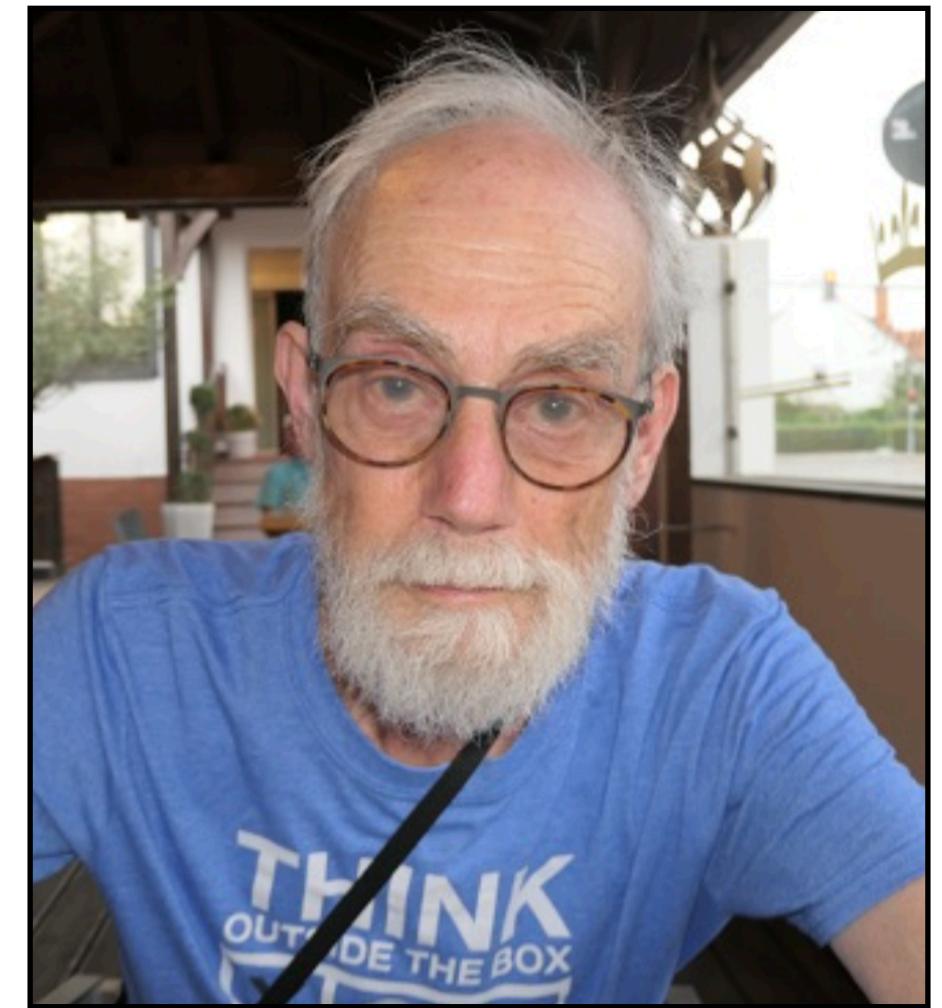
- **van Emde Boas** trees (1975-77) solve the problem in $O(\log \log U)$ worst-case time and space $\Theta(U)$.
- When $U = n^c$ for some $c \geq 1$ then $\log_2 \log_2 U = \Theta(\log \log n)$ and vEB trees are **exponentially** faster than AVL and RB trees. This is **optimal** for Successor/Predecessor.
- **Key insight.** The $\log_2 U$ -bit representation of an integer can be split recursively to speed up operations and queries.



Peter van Emde Boas

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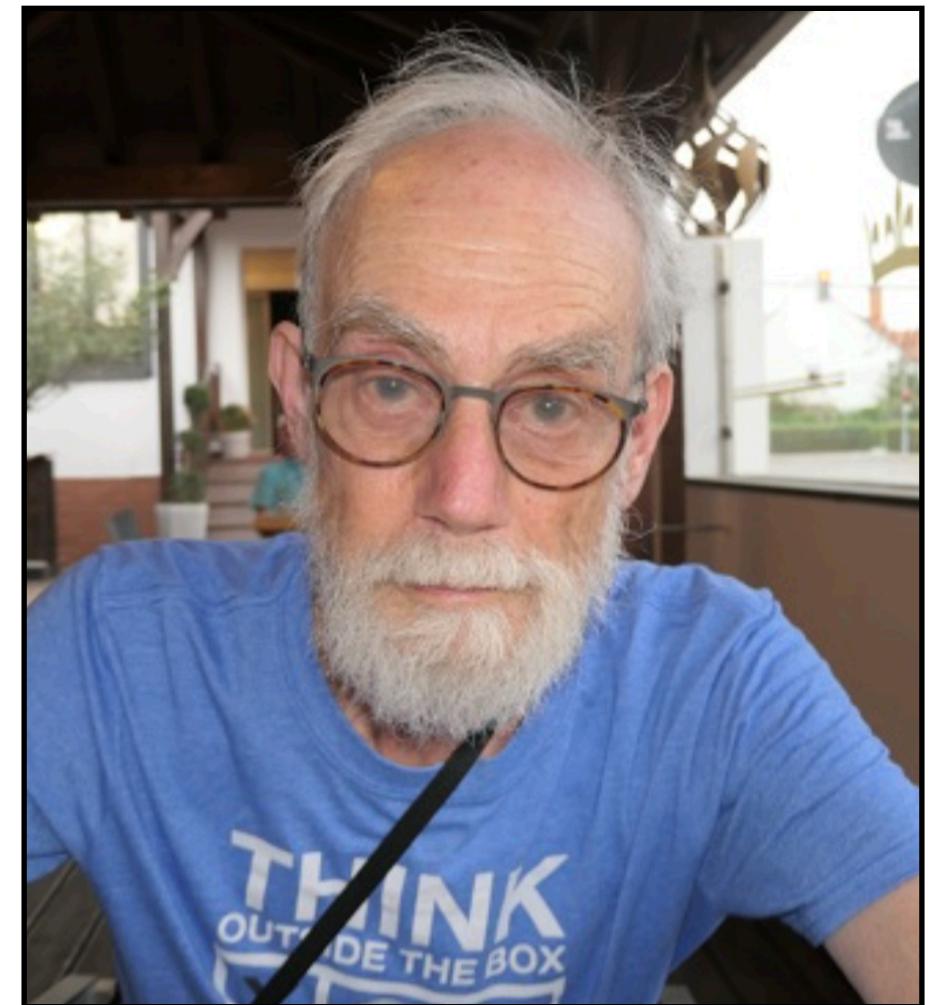
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- **Key insight.** The $\log_2 U$ -bit representation of an integer can be split recursively to speed up operations and queries.
- **Q.** Is it possible combine $O(\log \log U)$ time with space $\Theta(n)$?



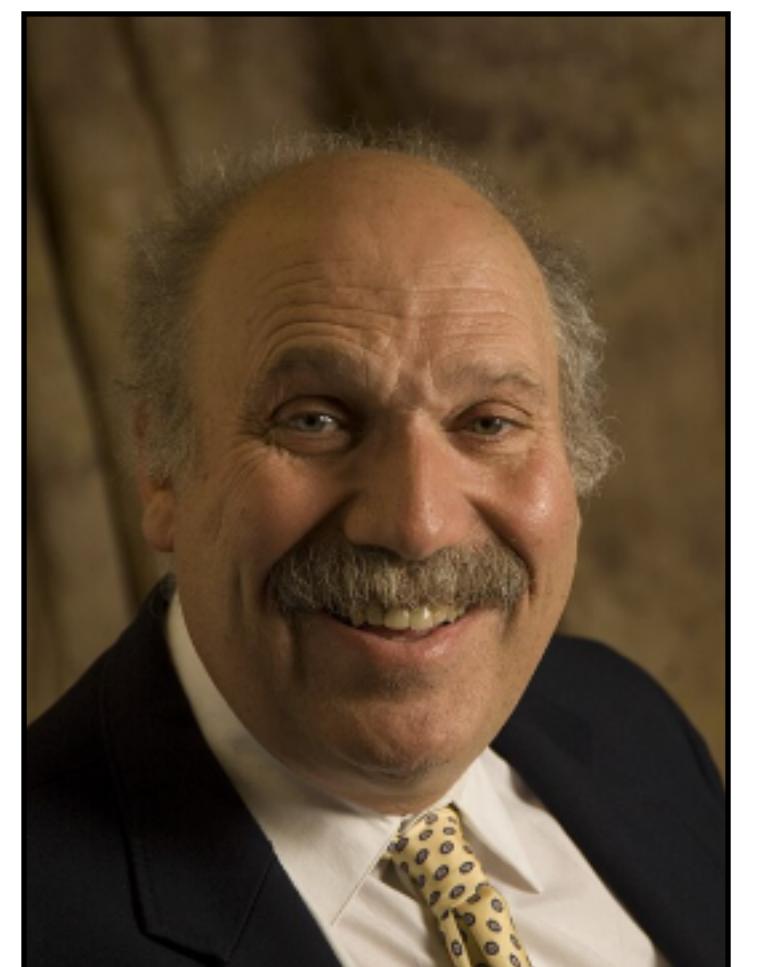
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- **Key insight.** The $\log_2 U$ -bit representation of an integer can be split recursively to speed up operations and queries.
- **Q.** Is it possible combine $O(\log \log U)$ time with space $\Theta(n)$?
- **A. Dan Willard** showed it is in 1983, with the y-fast trie.



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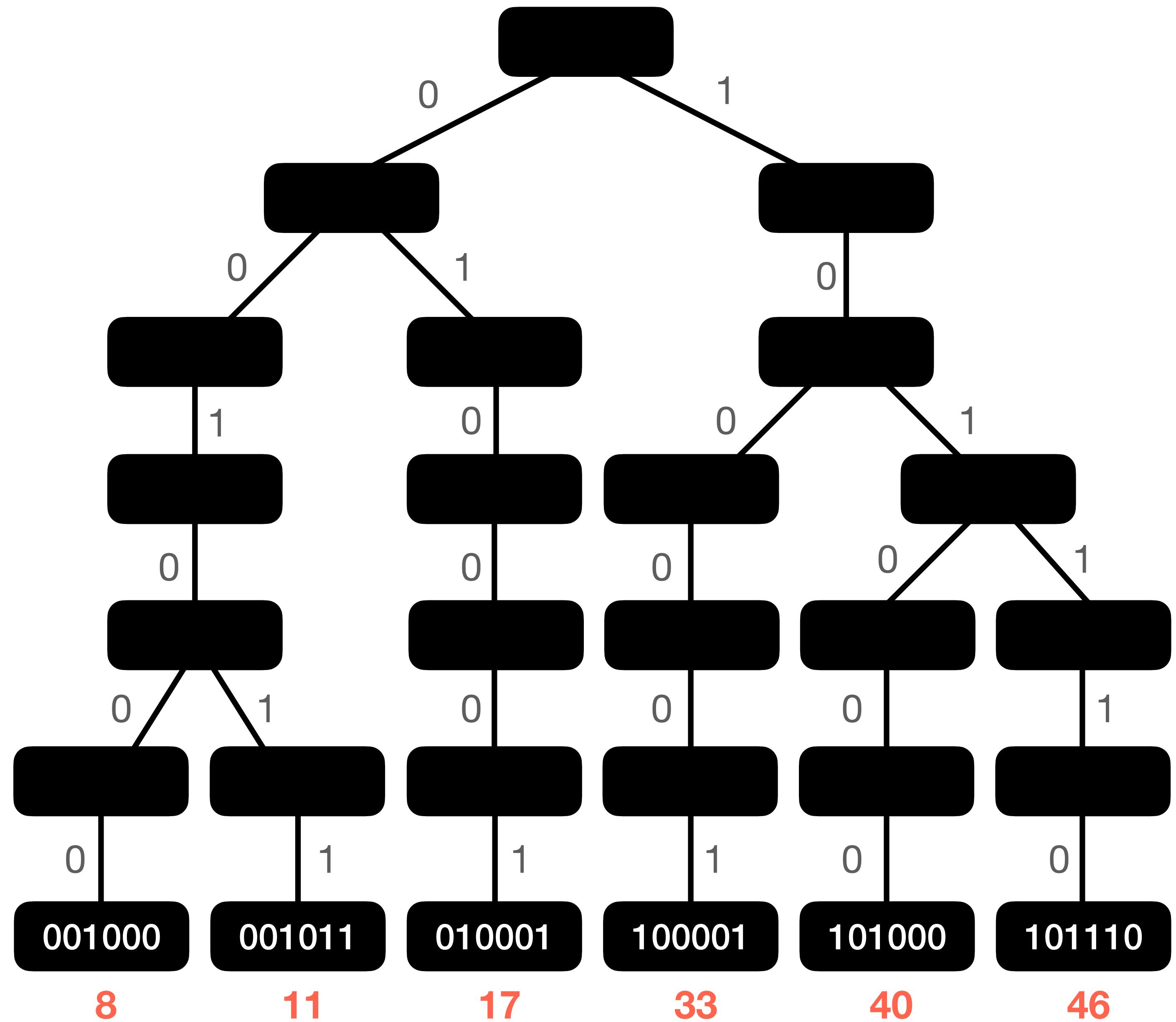


Dan Willard

Bitwise tries for integers

$$S = \{8, 11, 17, 33, 40, 46\}, U = 64$$

- **Tries** are data structures to represent a collection of strings, where common prefixes are represented once.
- Integers are binary strings of length $\log_2 U$, hence the trie has $\log_2 U$ levels.
- All operations in $O(\log U)$ time.
- For example: Successor(x) and Predecessor(x) by following the bits of x and “walking backwards” if necessary.

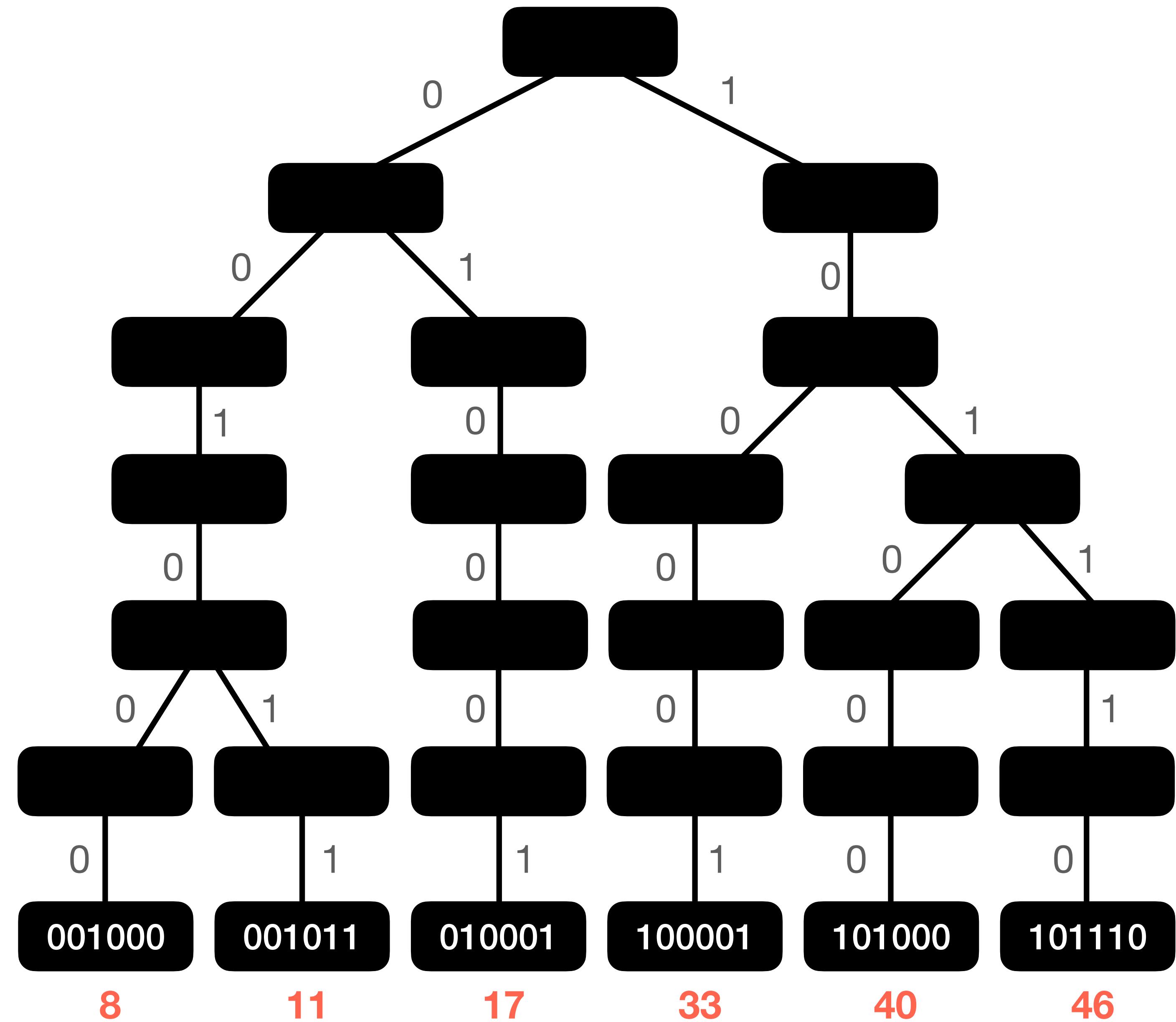


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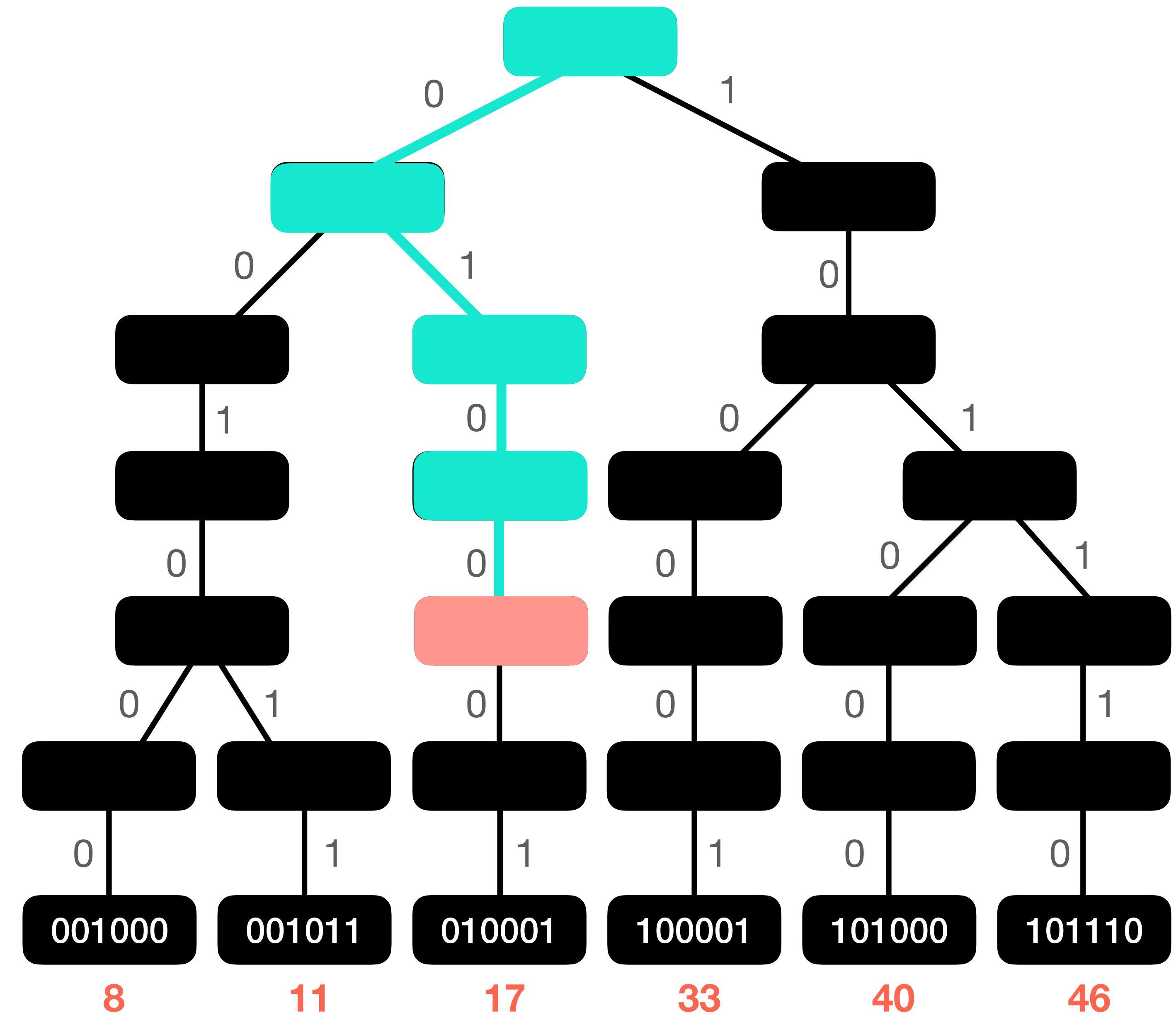


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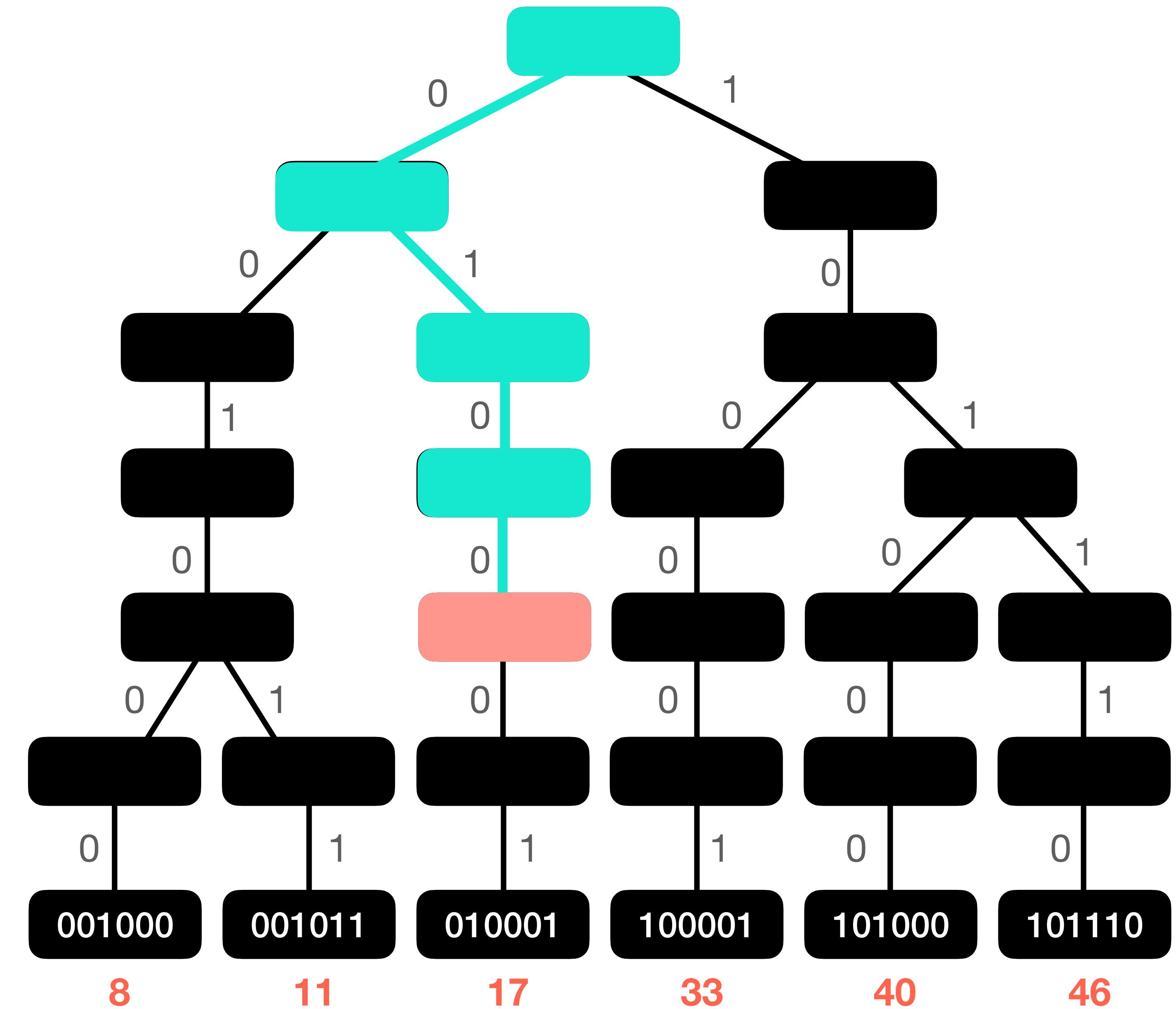
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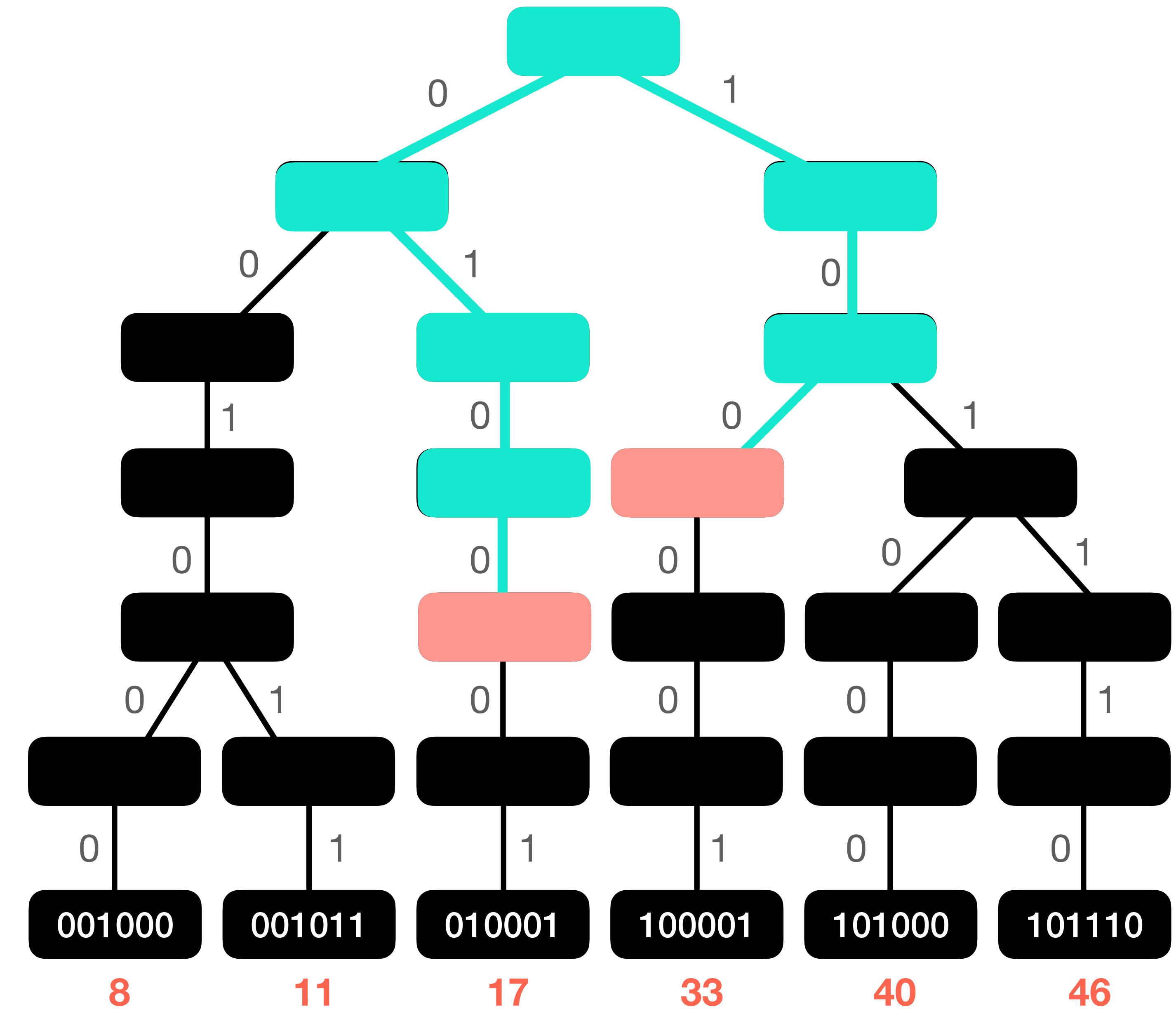
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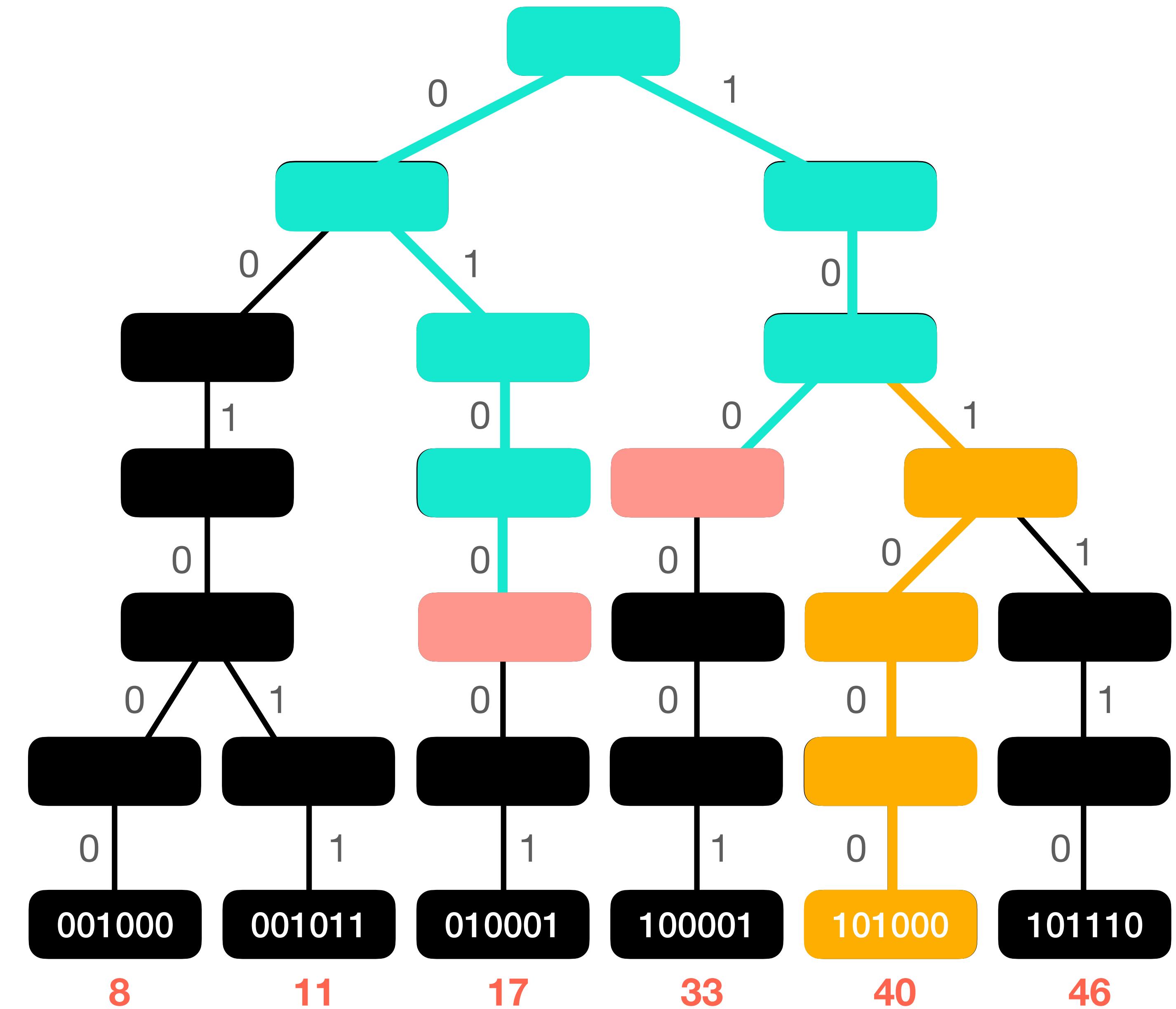
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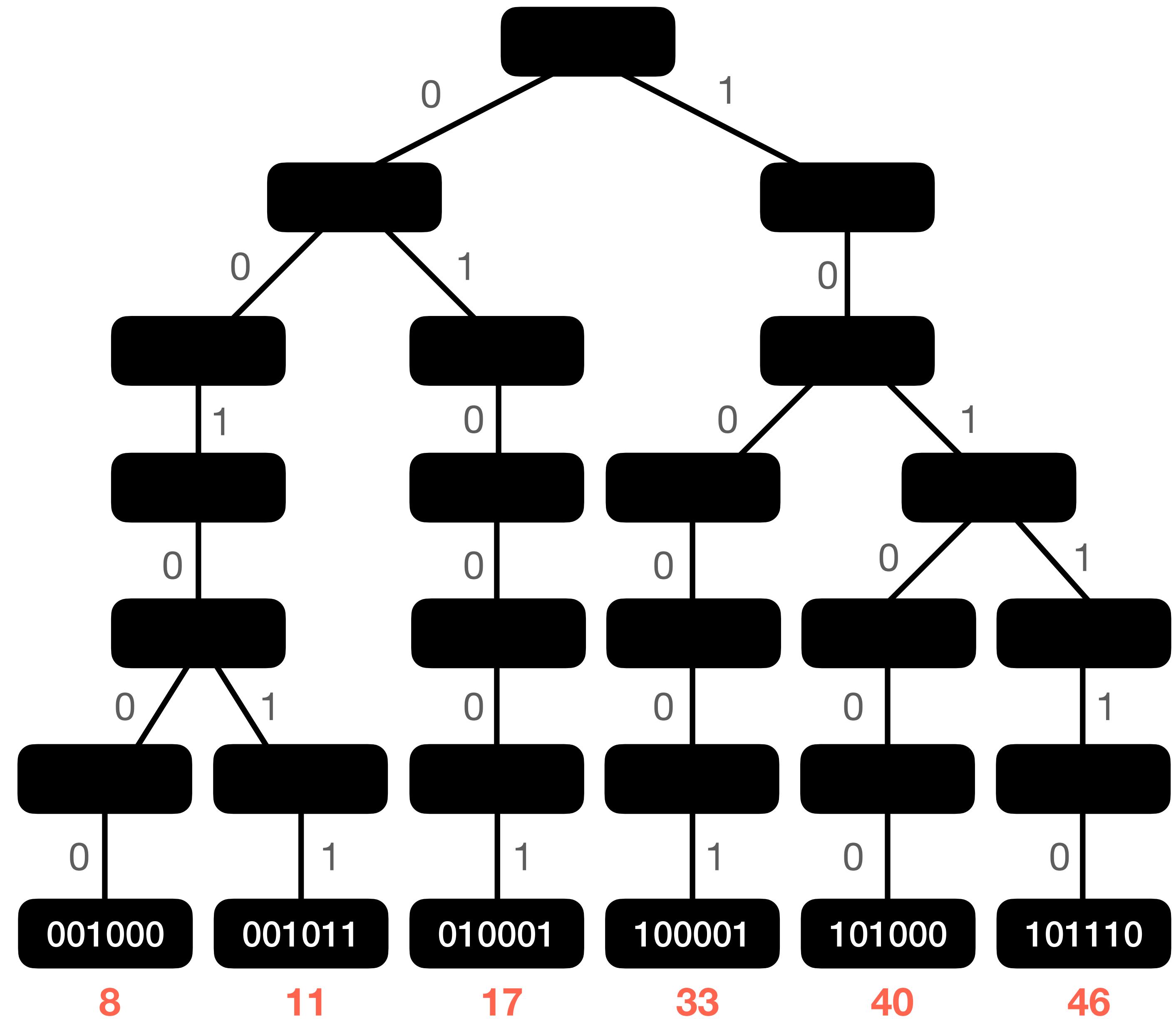
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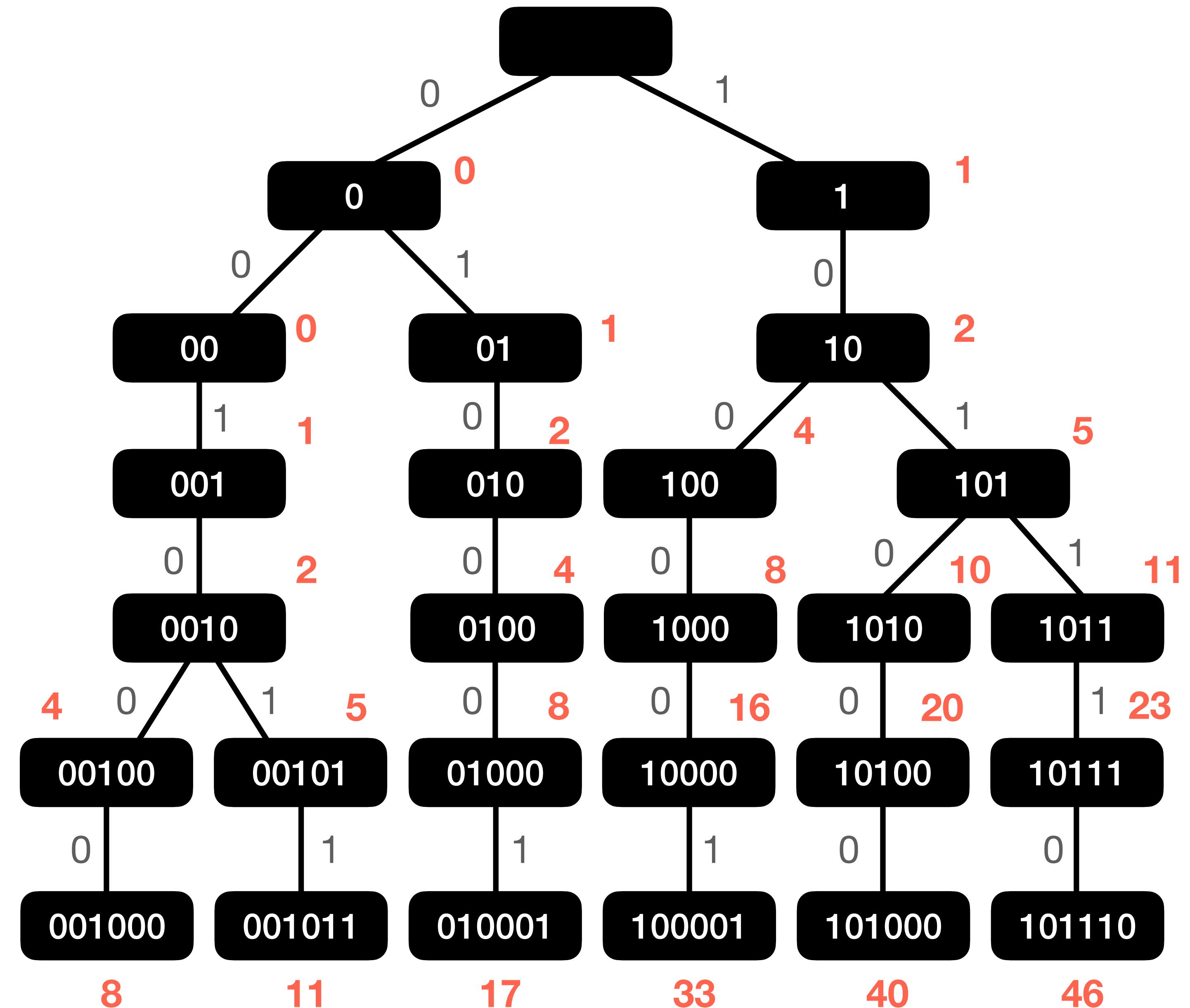
Bitwise tries for integers

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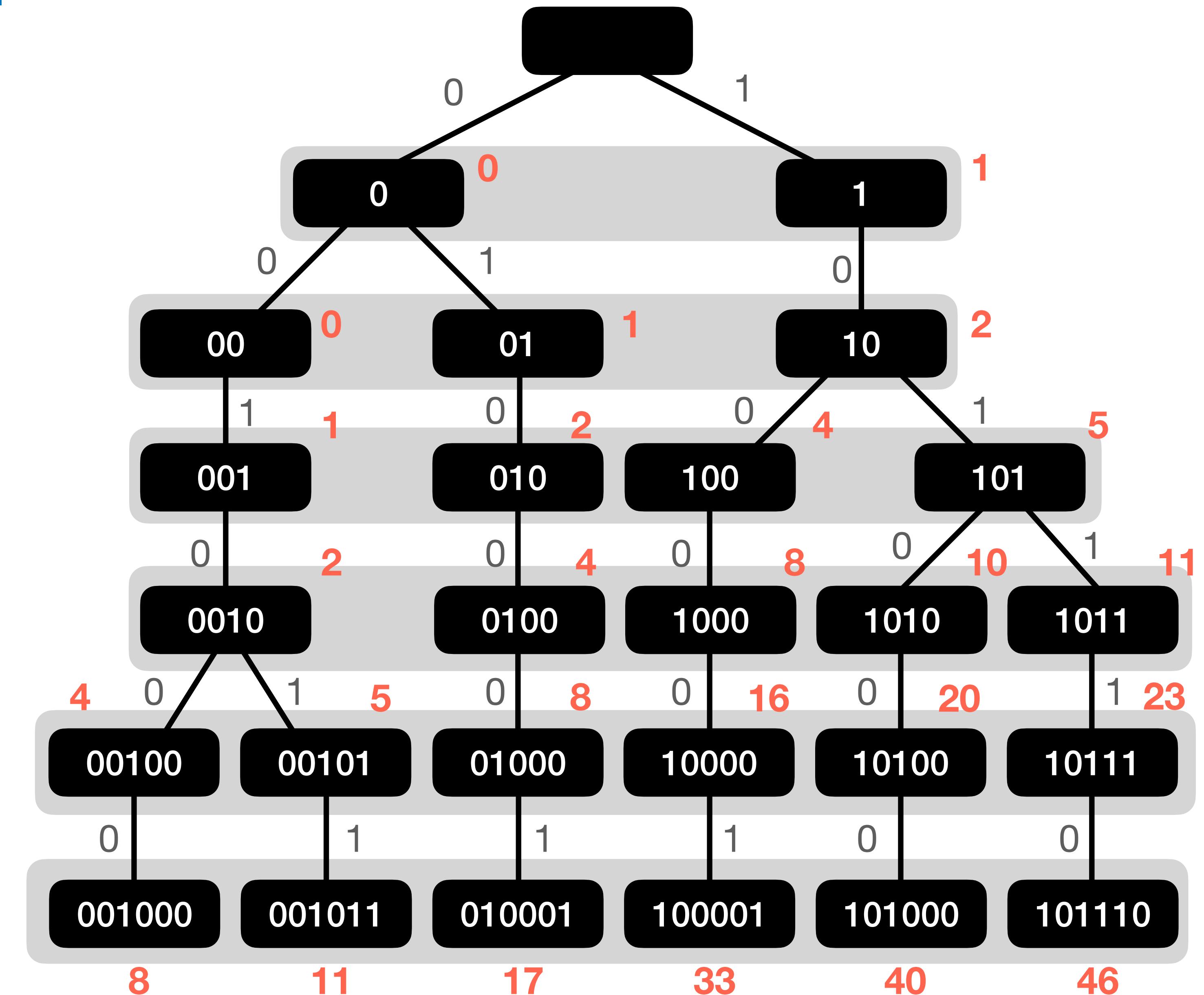
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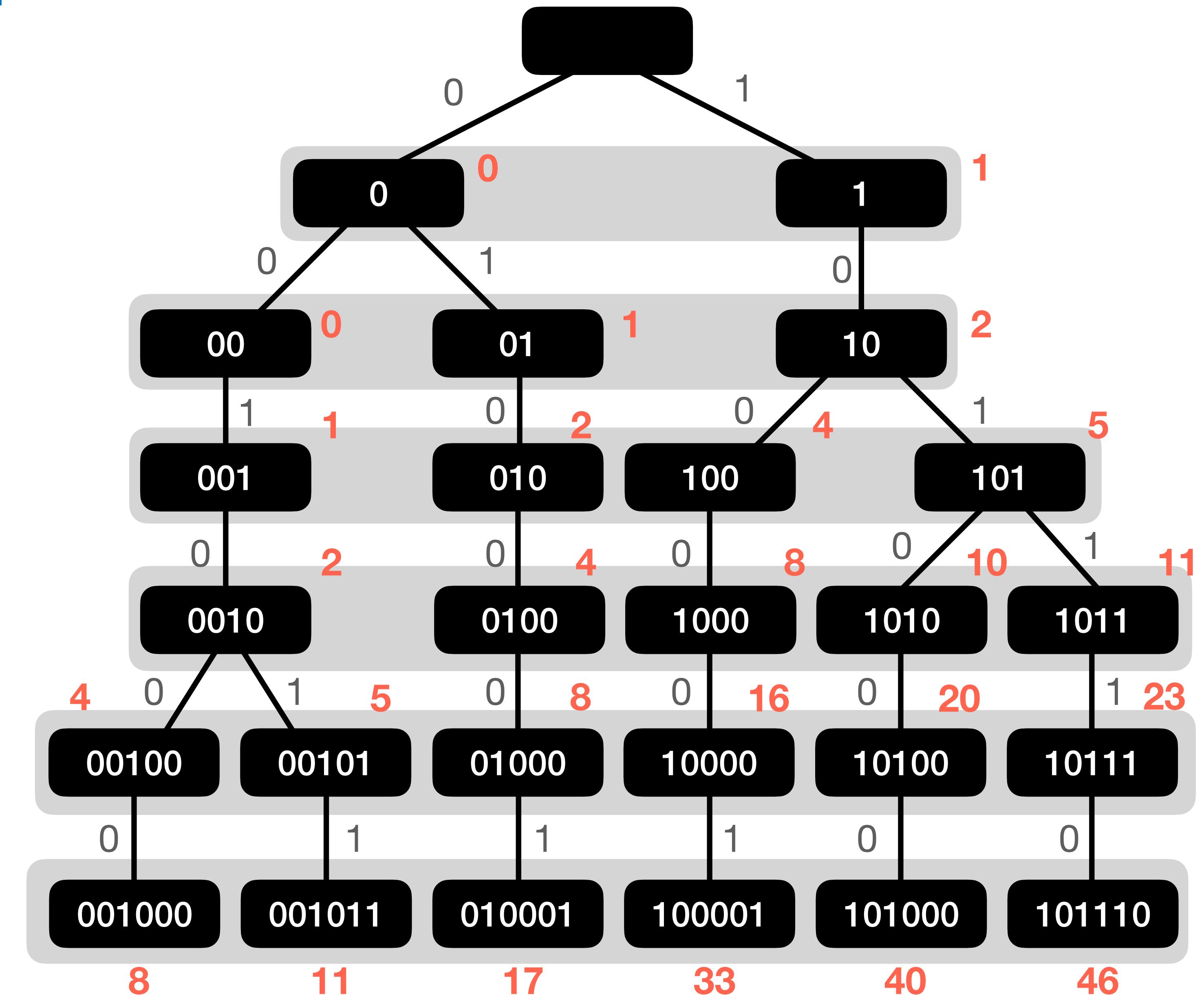
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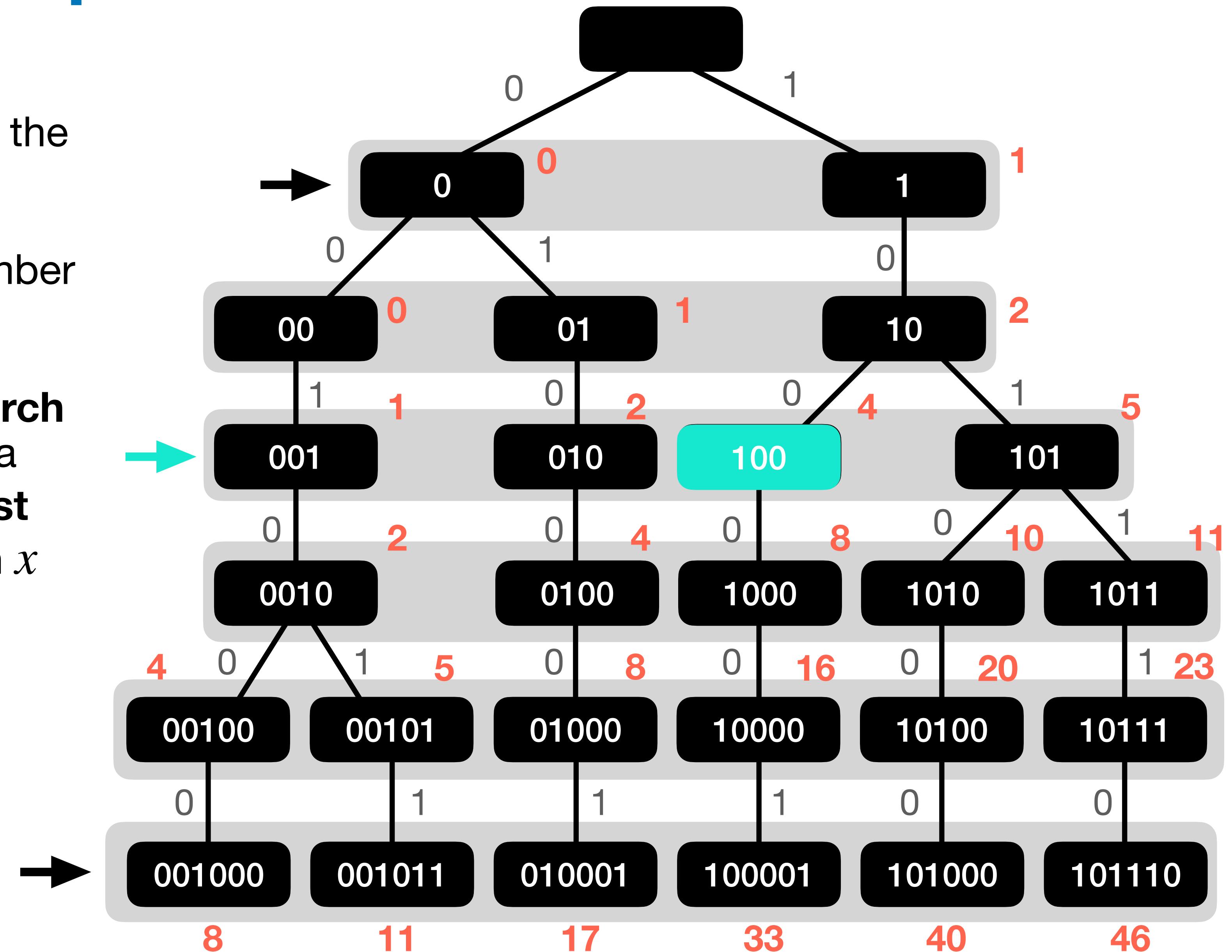
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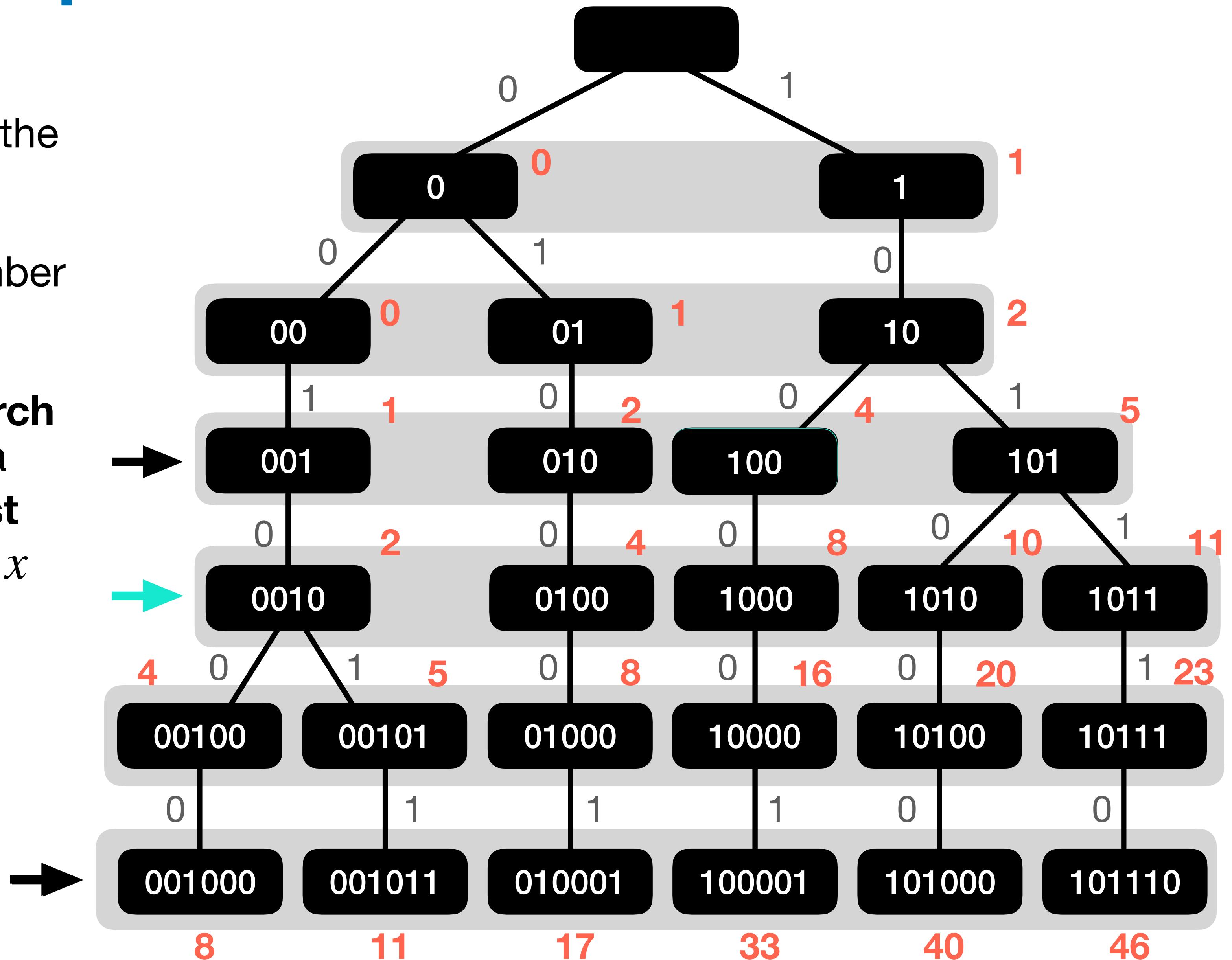
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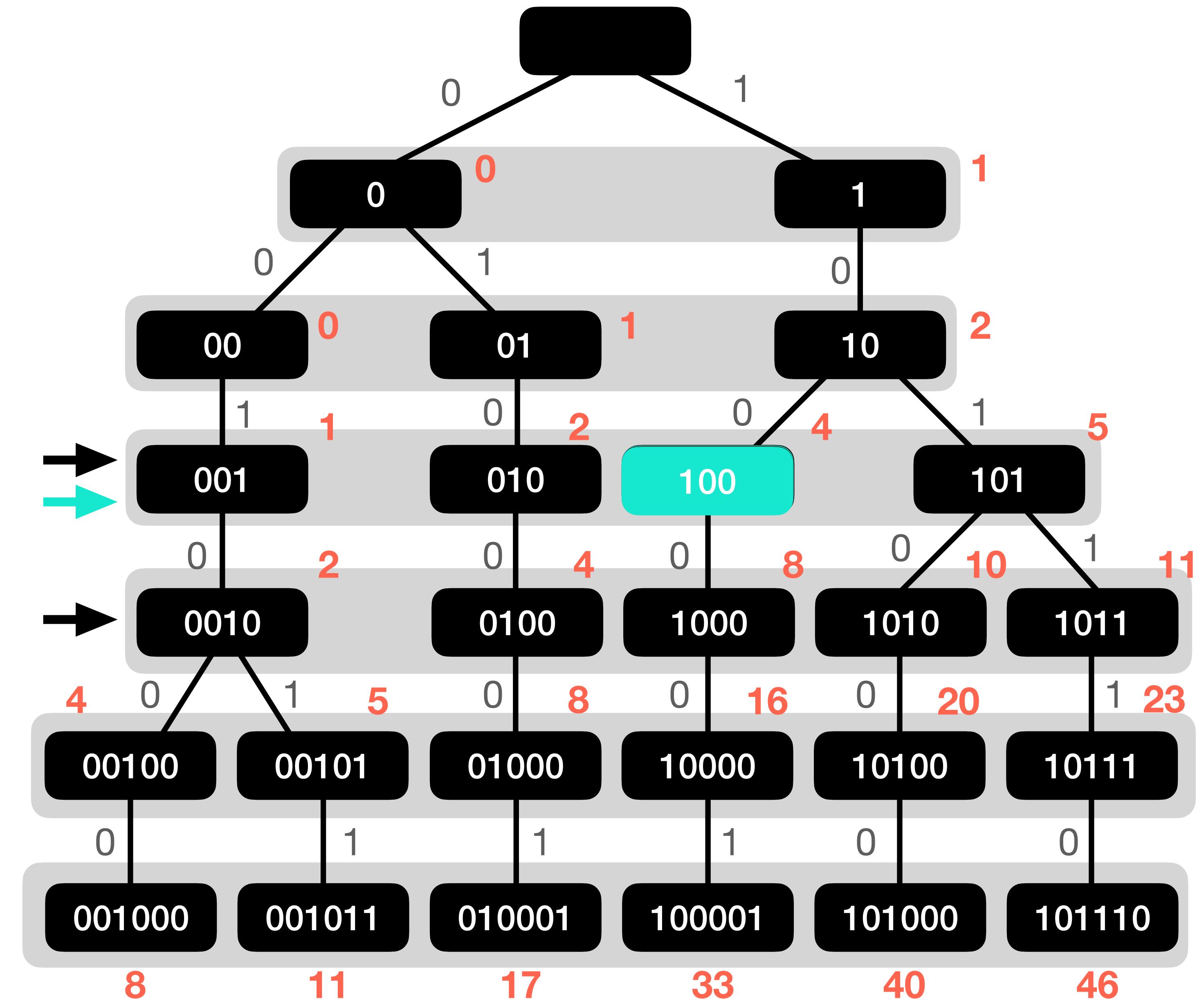
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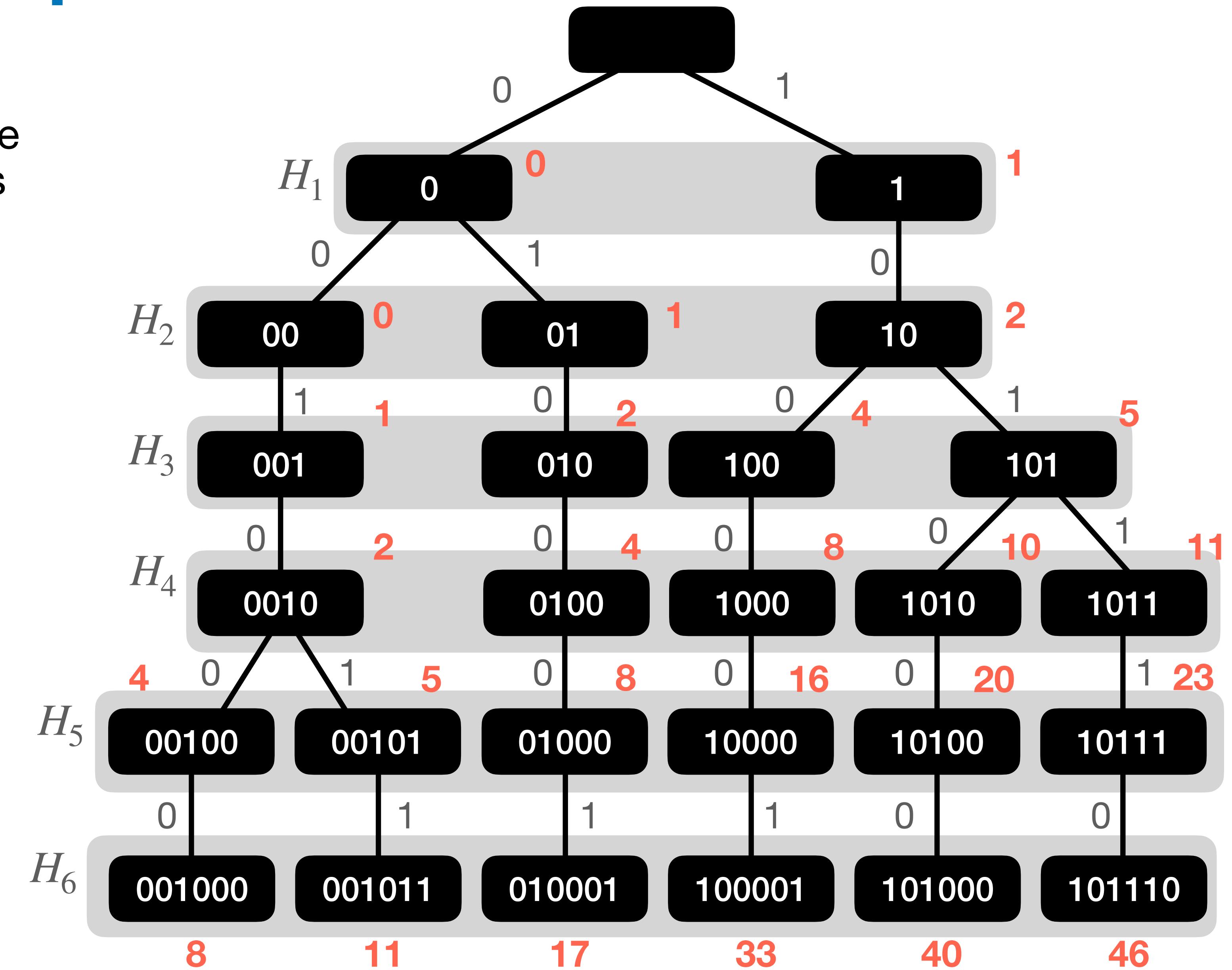
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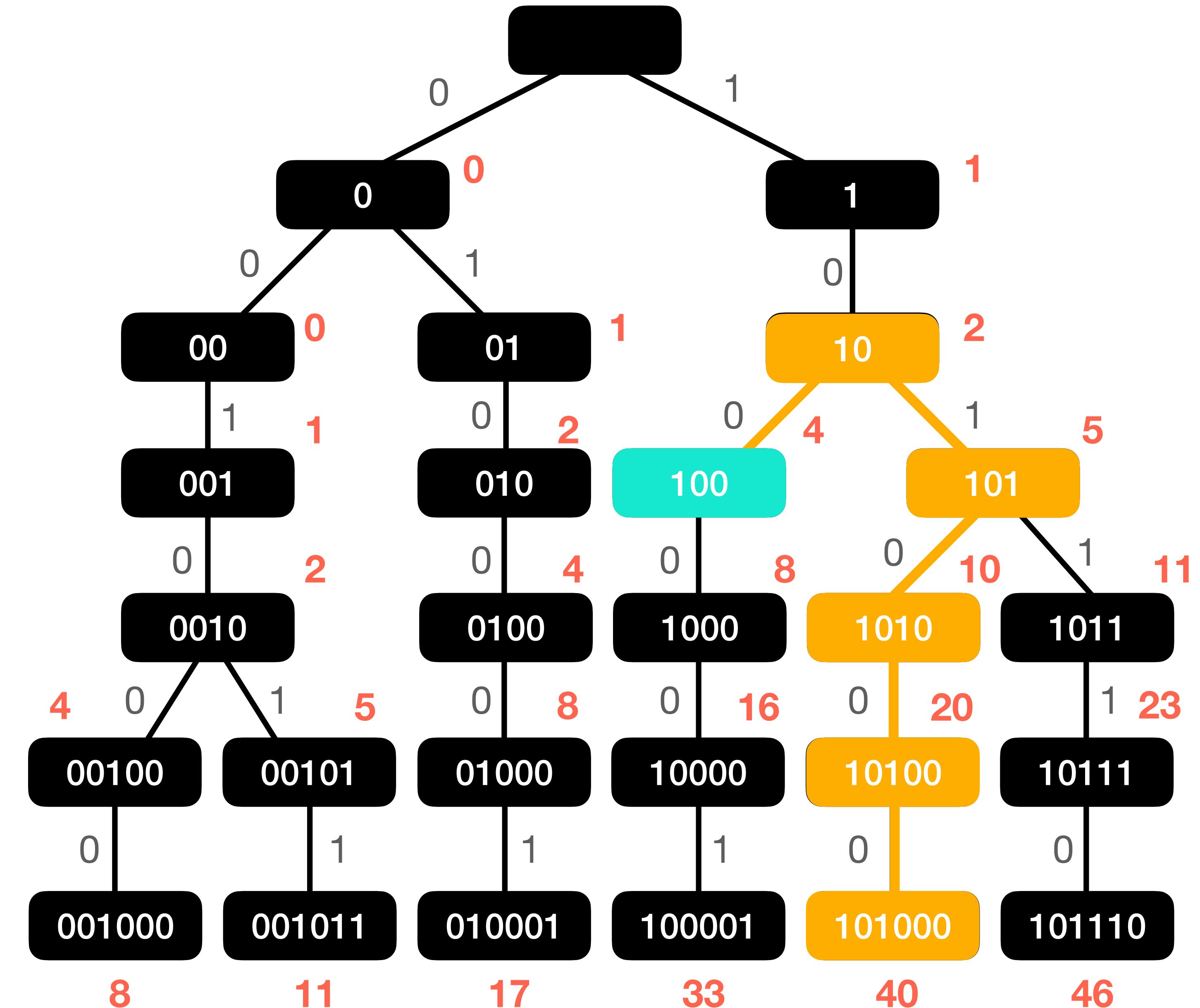
- Binary search needs to determine whether a given node (a prefix) is present at a level.
- So, all the nodes at level i are inserted in a **hash table** H_i with $O(1)$ time per lookup.
- The LCP can now be determined in $O(\log \log U)$.
- **Bonus.** $\text{Member}(x)$ now runs in $O(1)$ by probing the bottom-level hash table.



Successor ?

- The LCP can be determined in $O(\log \log U)$.
- From the node corresponding to the LCP we then walk backwards until we find a branching node with a 1 on the right and take the leftmost path from there.
- Still $O(\log U)$ time in the worst case.

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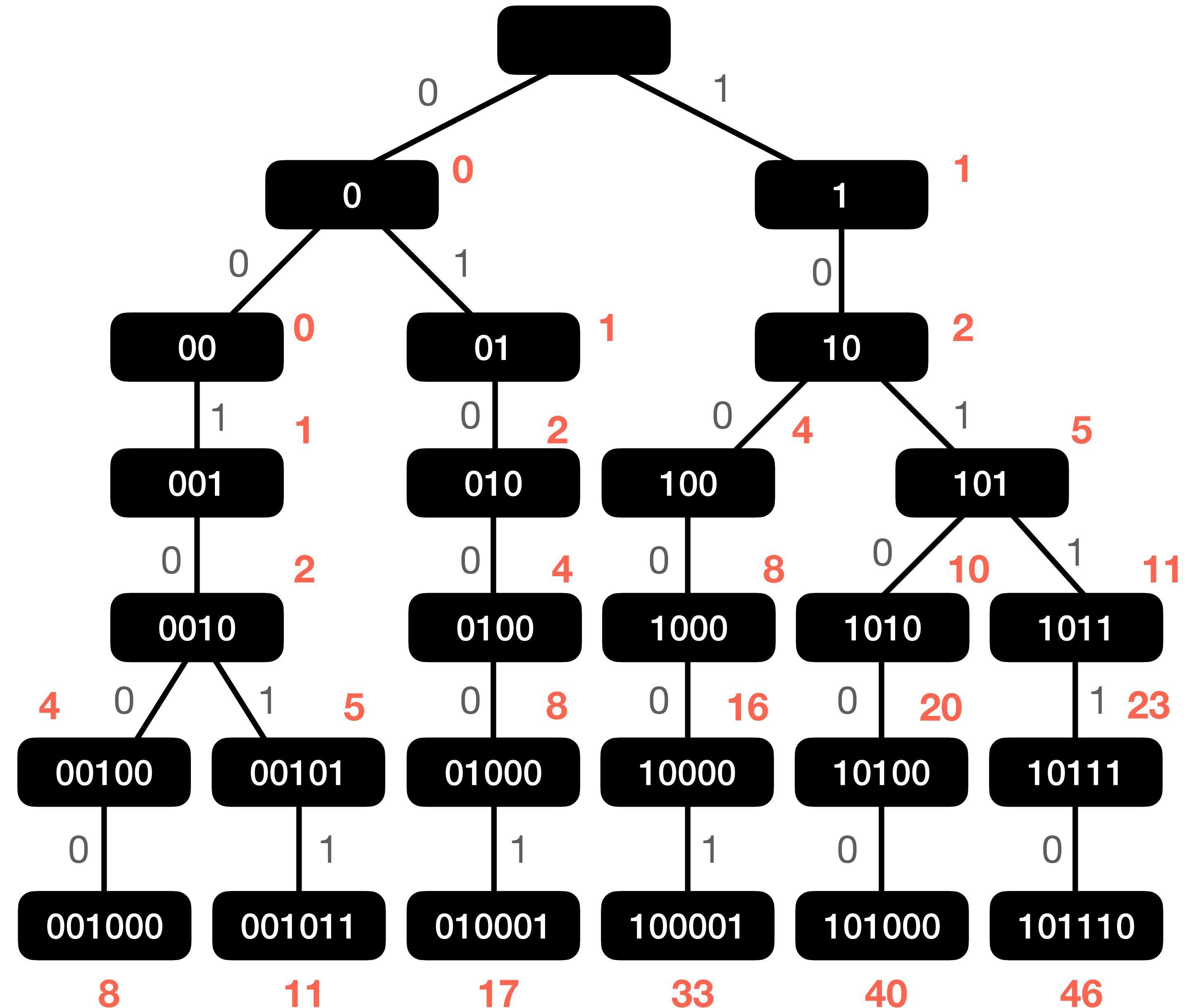


The x -fast trie

- **Claim.** If the binary search stops in a node ν , then either ν is a leaf or it has **one child** only.

(If that were not the case, a longer match could have been found.)

- **Idea.** From ν , skip to a leaf.



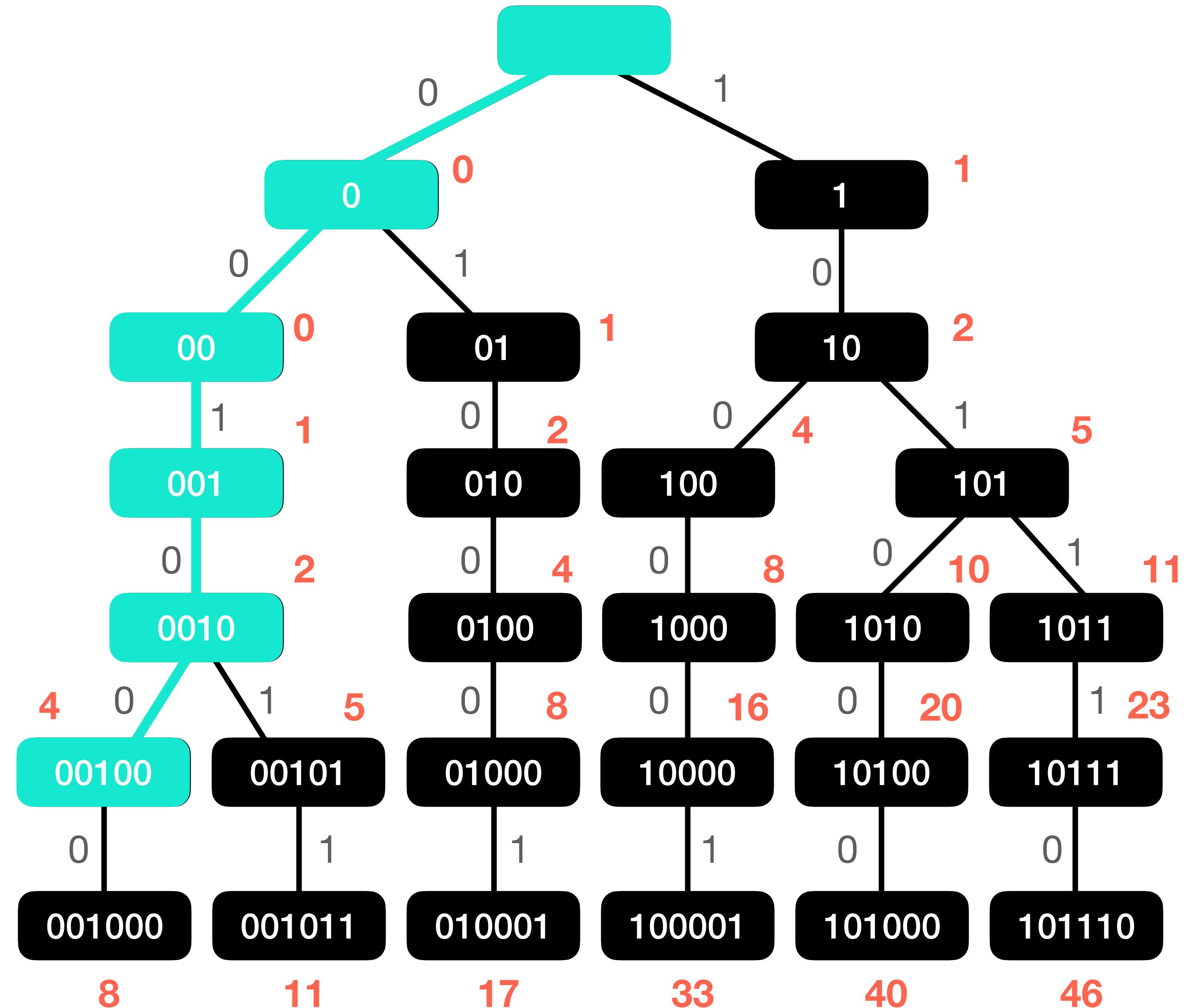
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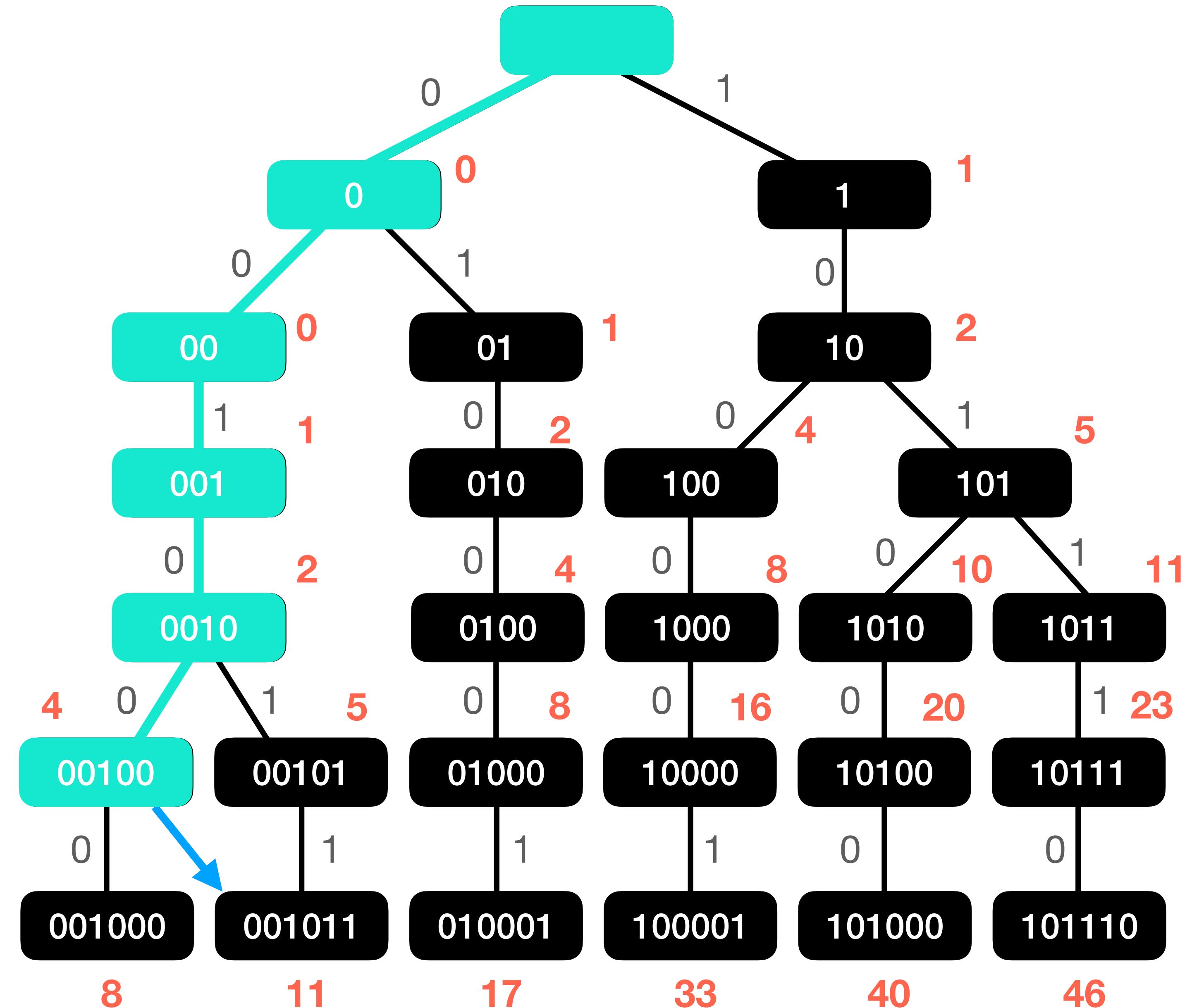
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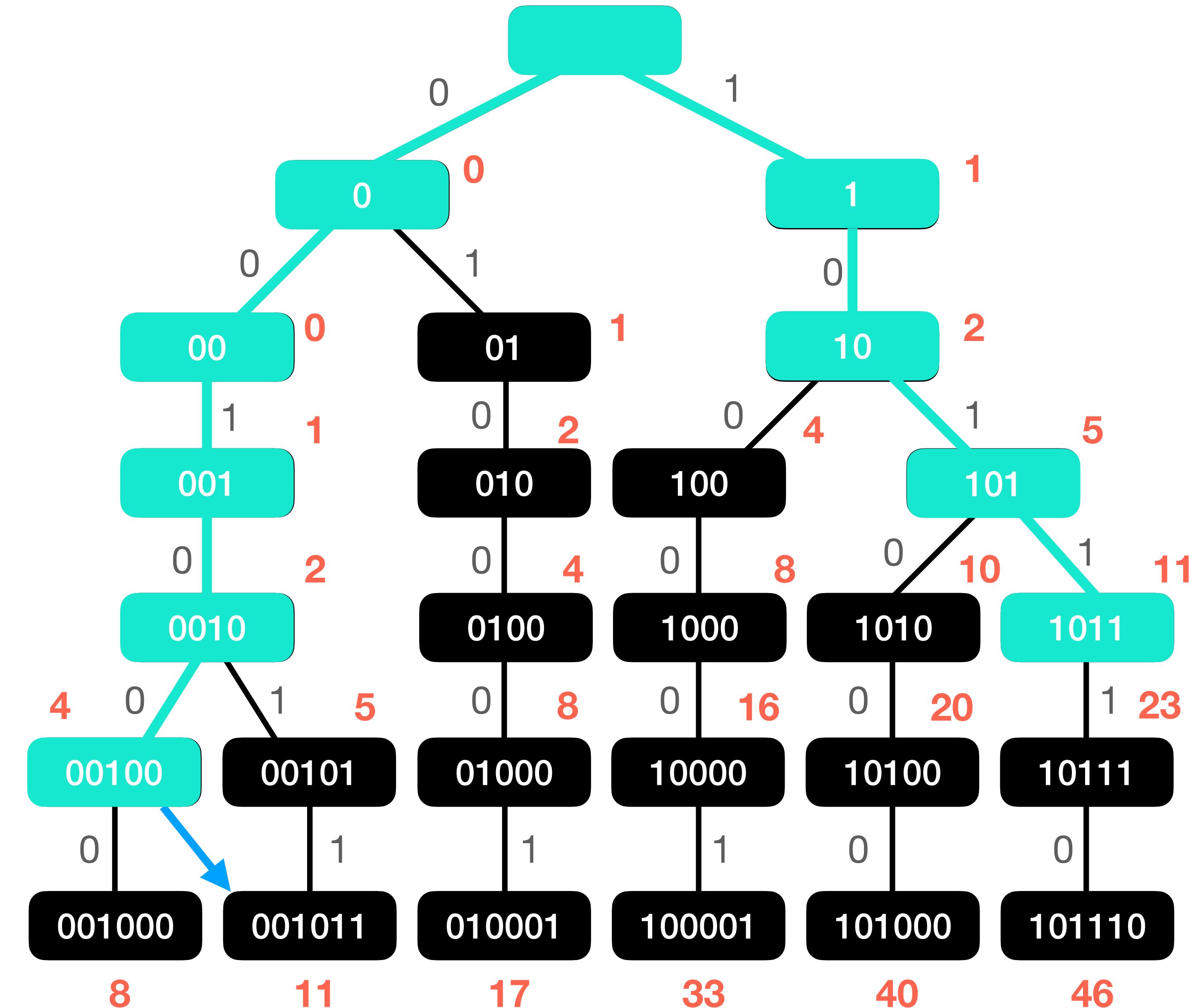
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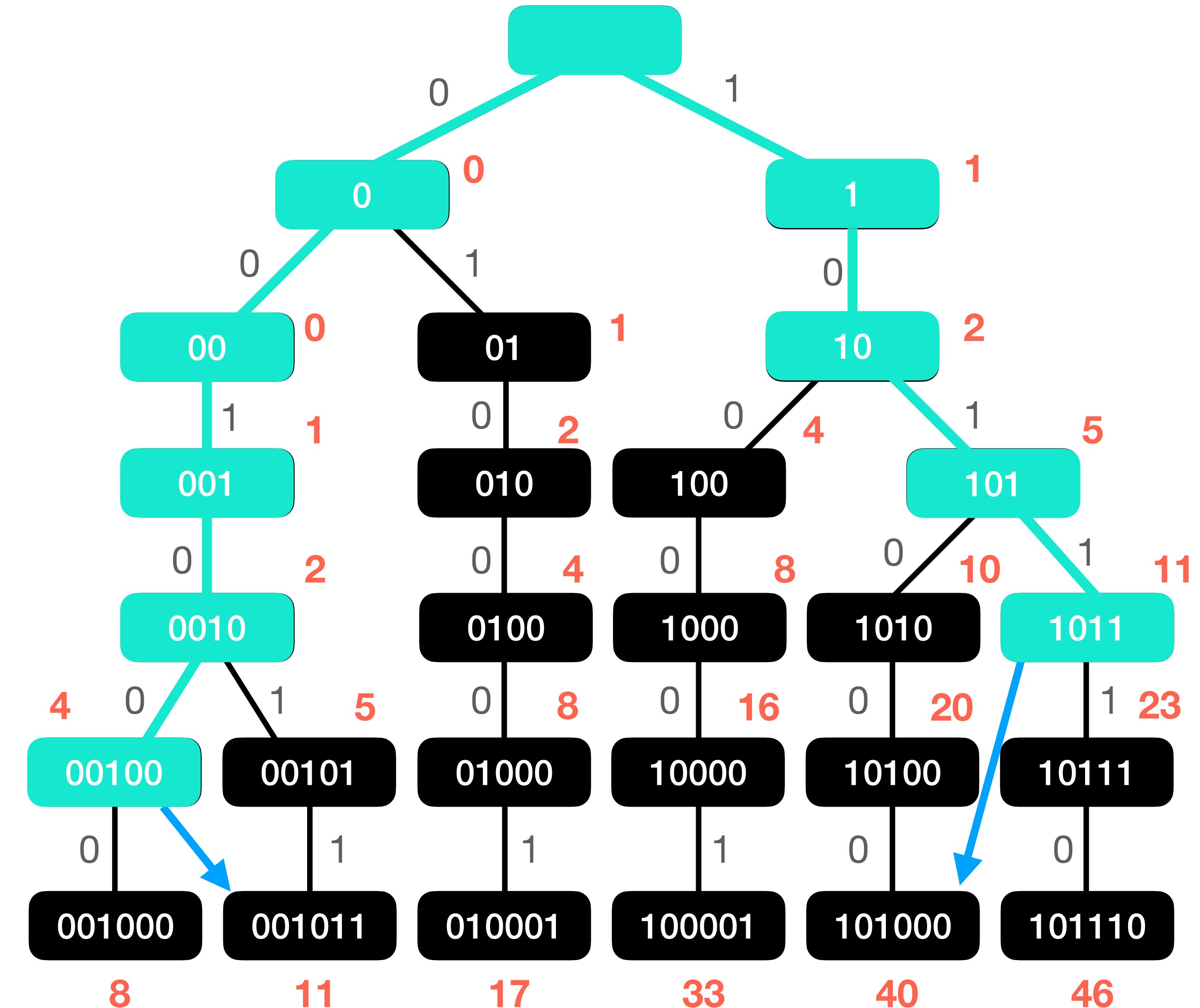
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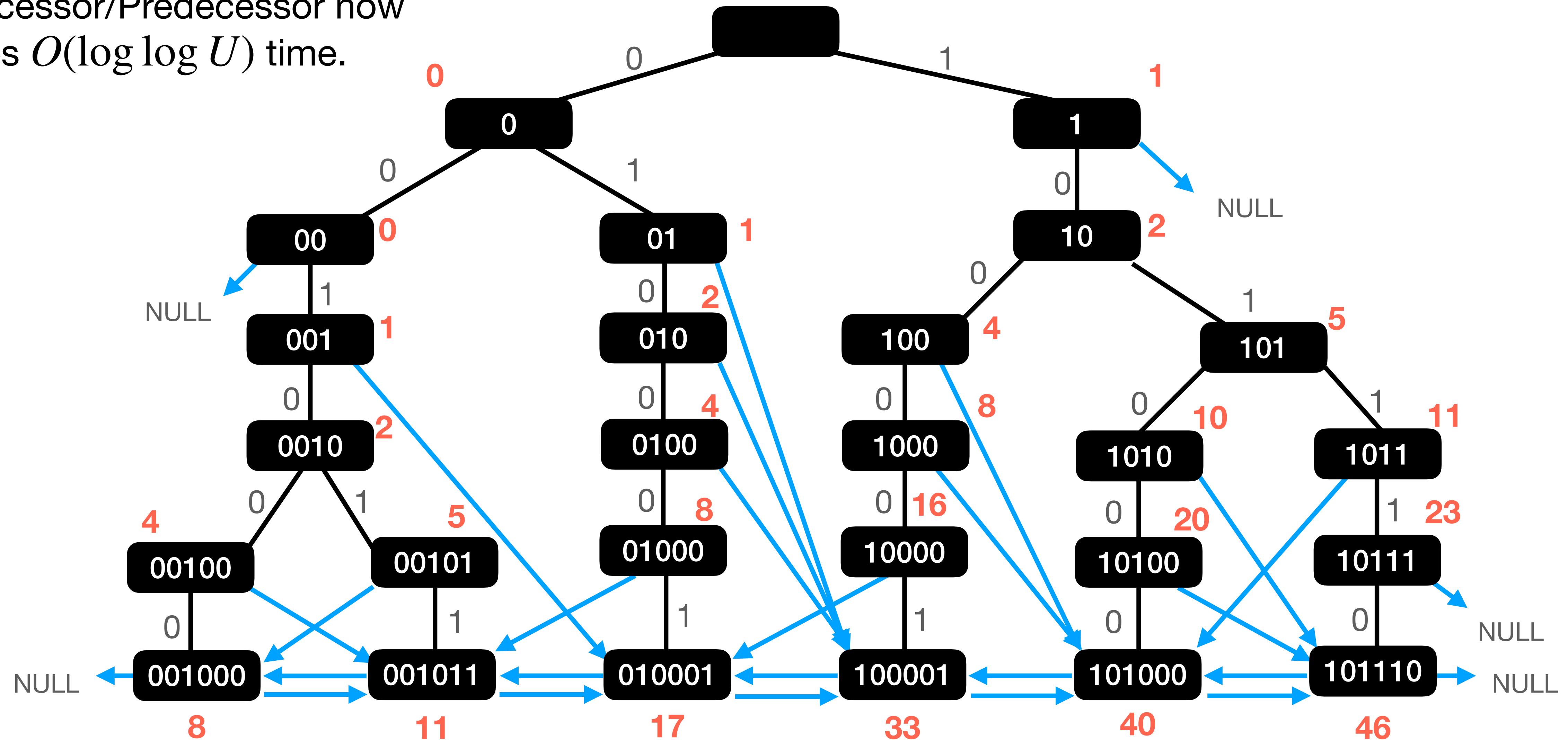


Skip pointers

- Let v be a node at level i , with **one child**.
 - If the child is labelled with **0**: v skips to the leaf Successor($\left[[v]_2 \cdot 1.0^{\log_2 U-i-1} \right]_{10}$).
 - If the child is labelled with **1**: v skips to the leaf Predecessor($\left[[v]_2 \cdot 0.0^{\log_2 U-i-1} \right]_{10}$).
- Also each leaf has a pointer to the prev/next leaf.

The x -fast trie

- Successor/Predecessor now takes $O(\log \log U)$ time.

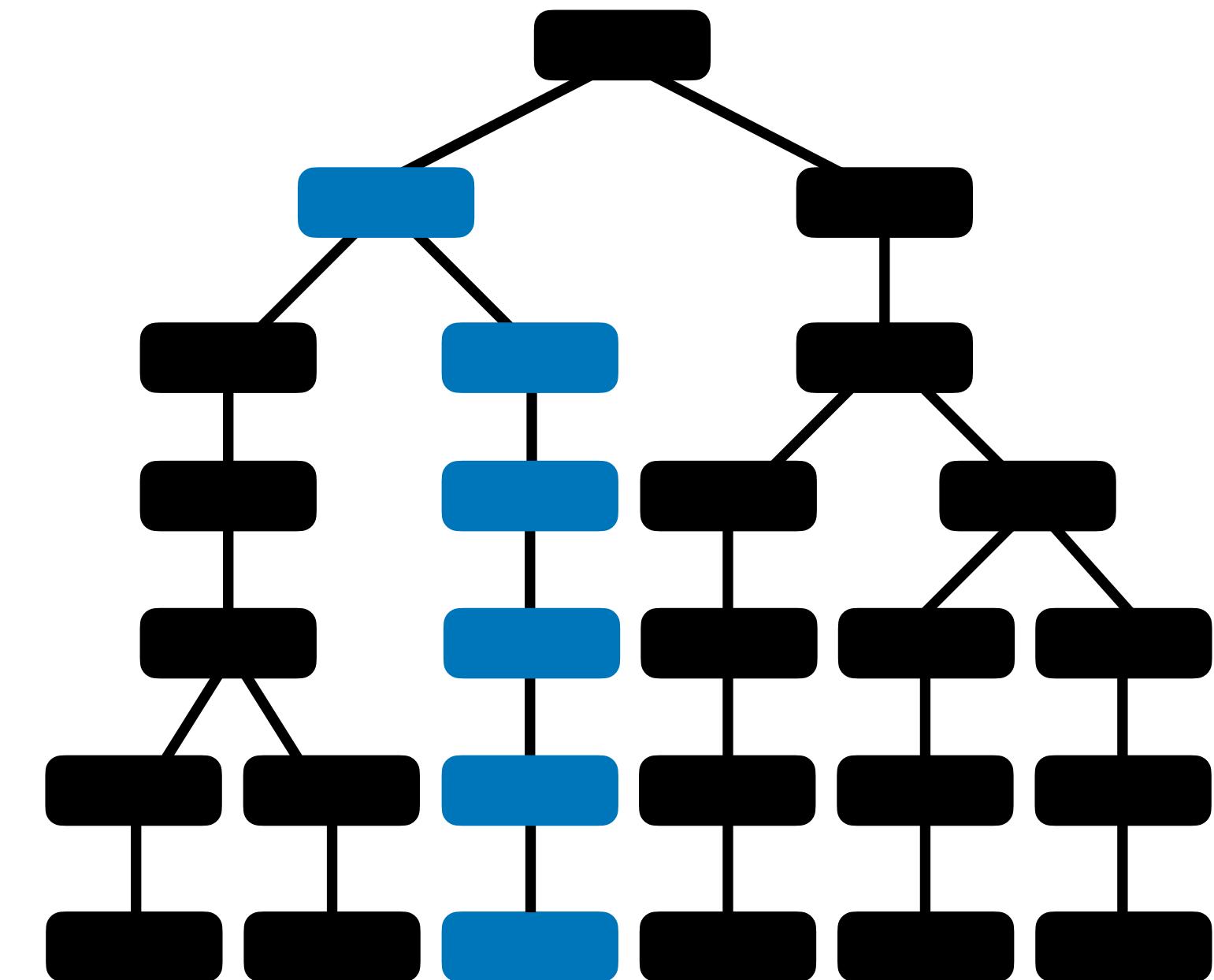


The x -fast trie

- Min/Max in $O(1)$ by caching these values.
- Member in $O(1)$.
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- Also, $\text{Subset}(\ell, r) := \{x \in S \mid \ell < x < r\}$ can be returned in time $O(\log \log U + |\text{Subset}(\ell, r)|)$: scan the linked-list from $\text{Successor}(\ell)$ to $\text{Predecessor}(r)$. (Range query.)
- **Q.** Space?

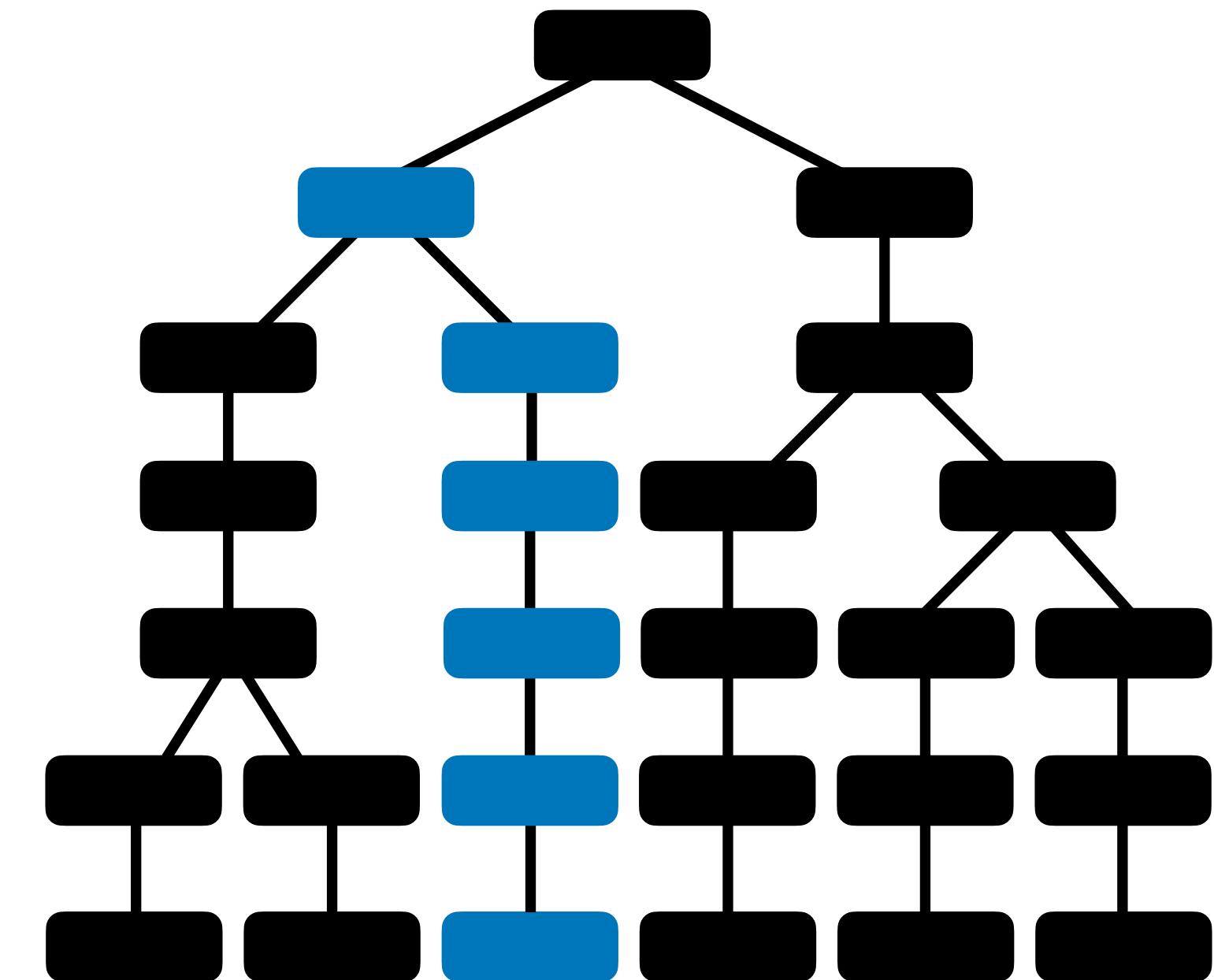
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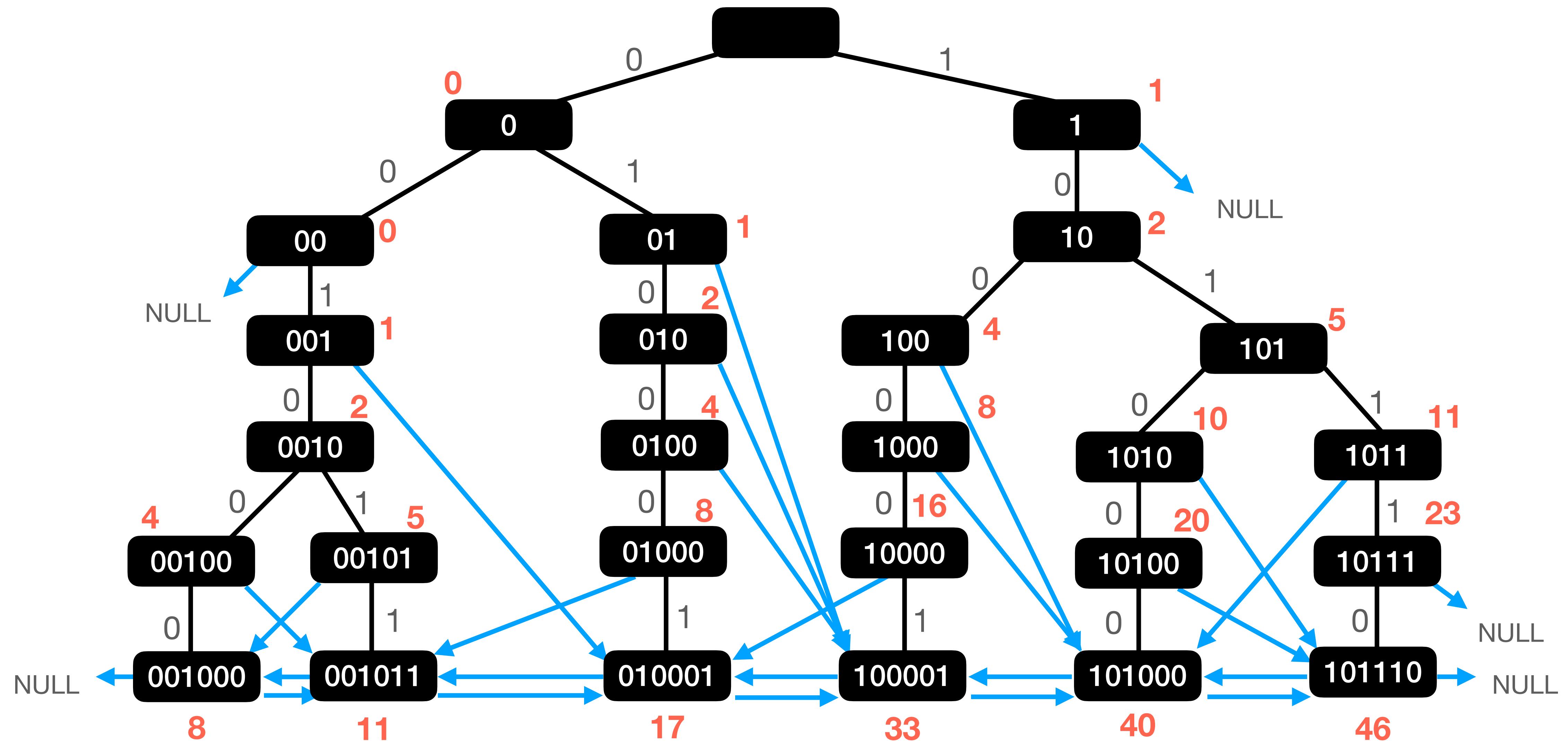
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- **Q.** Insert/Delete?



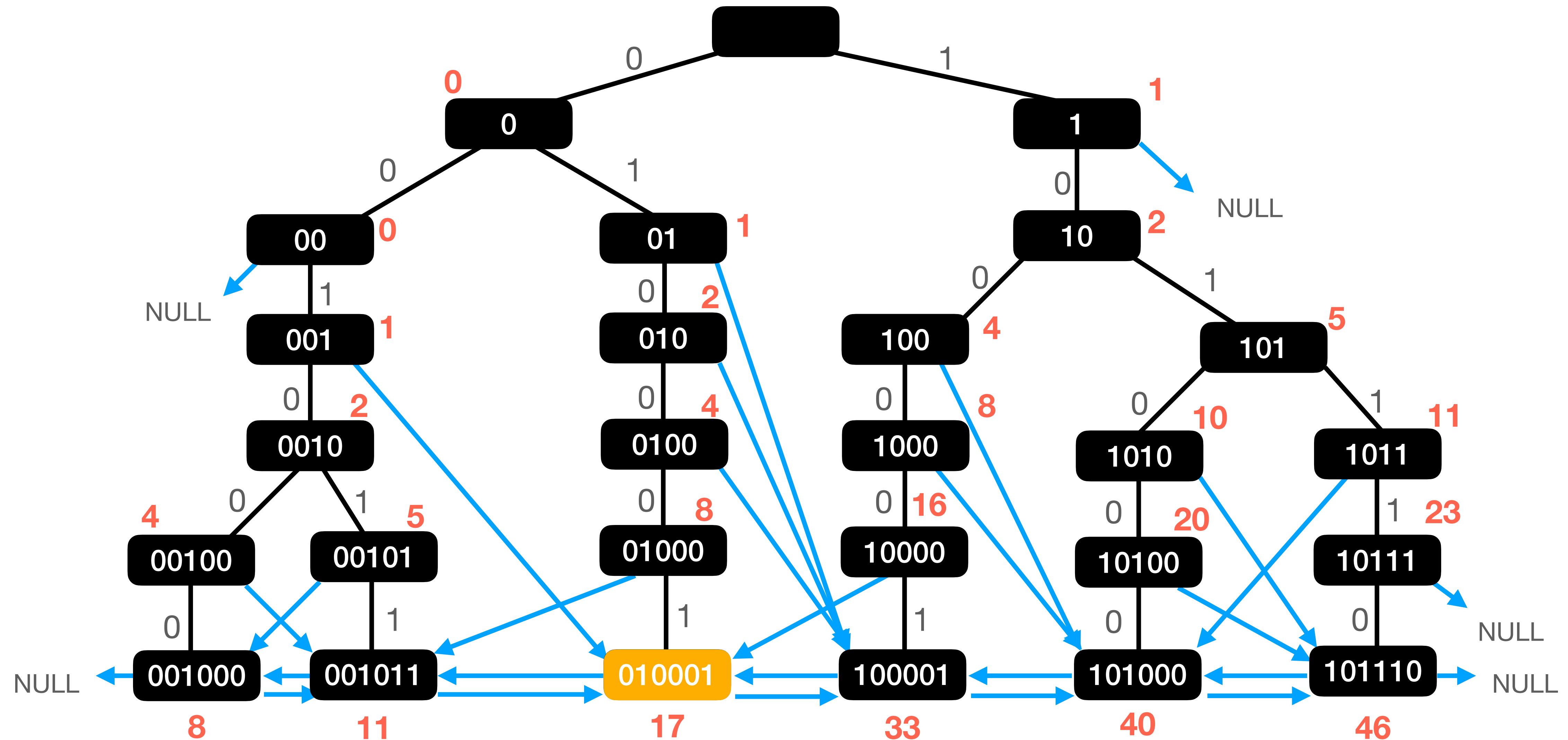
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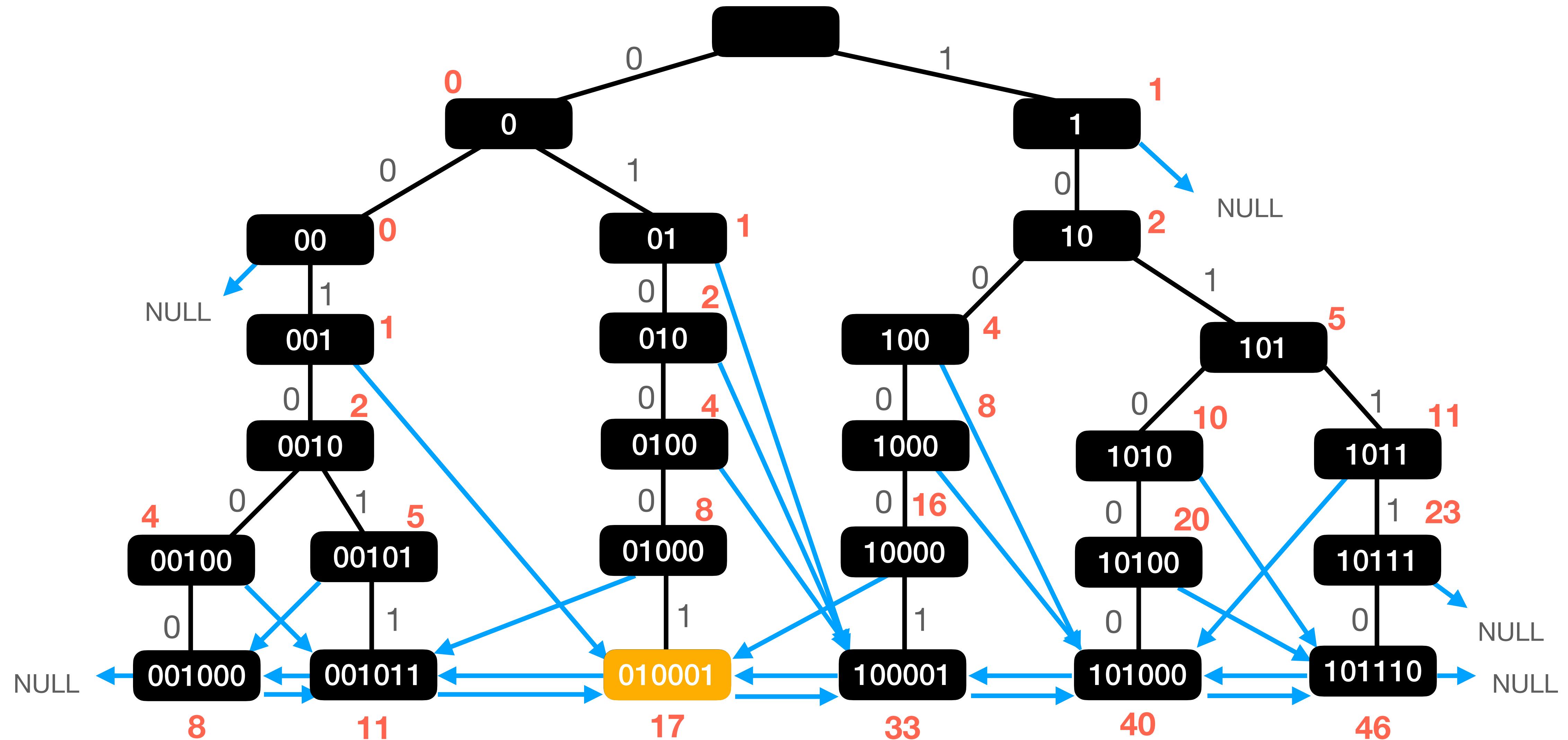
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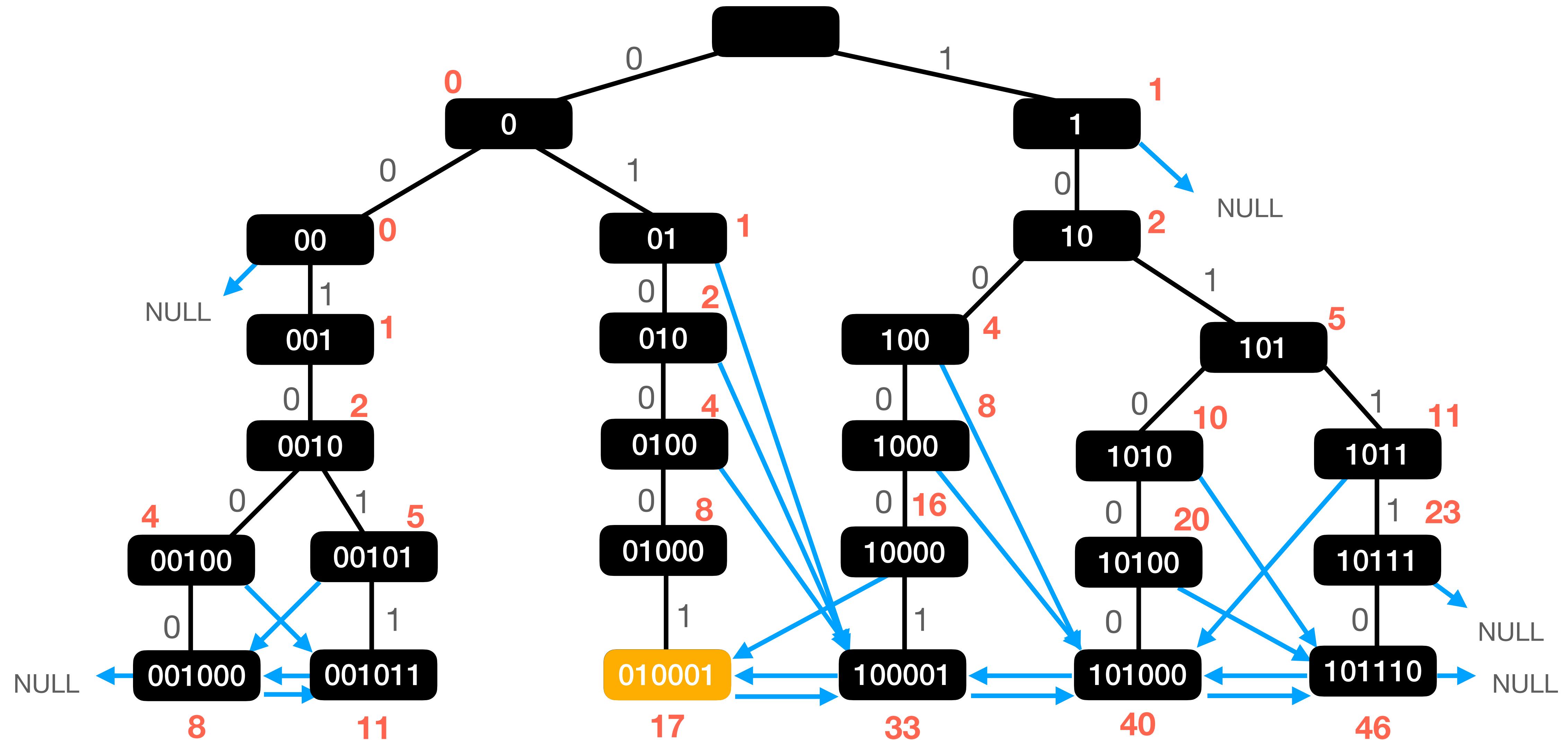
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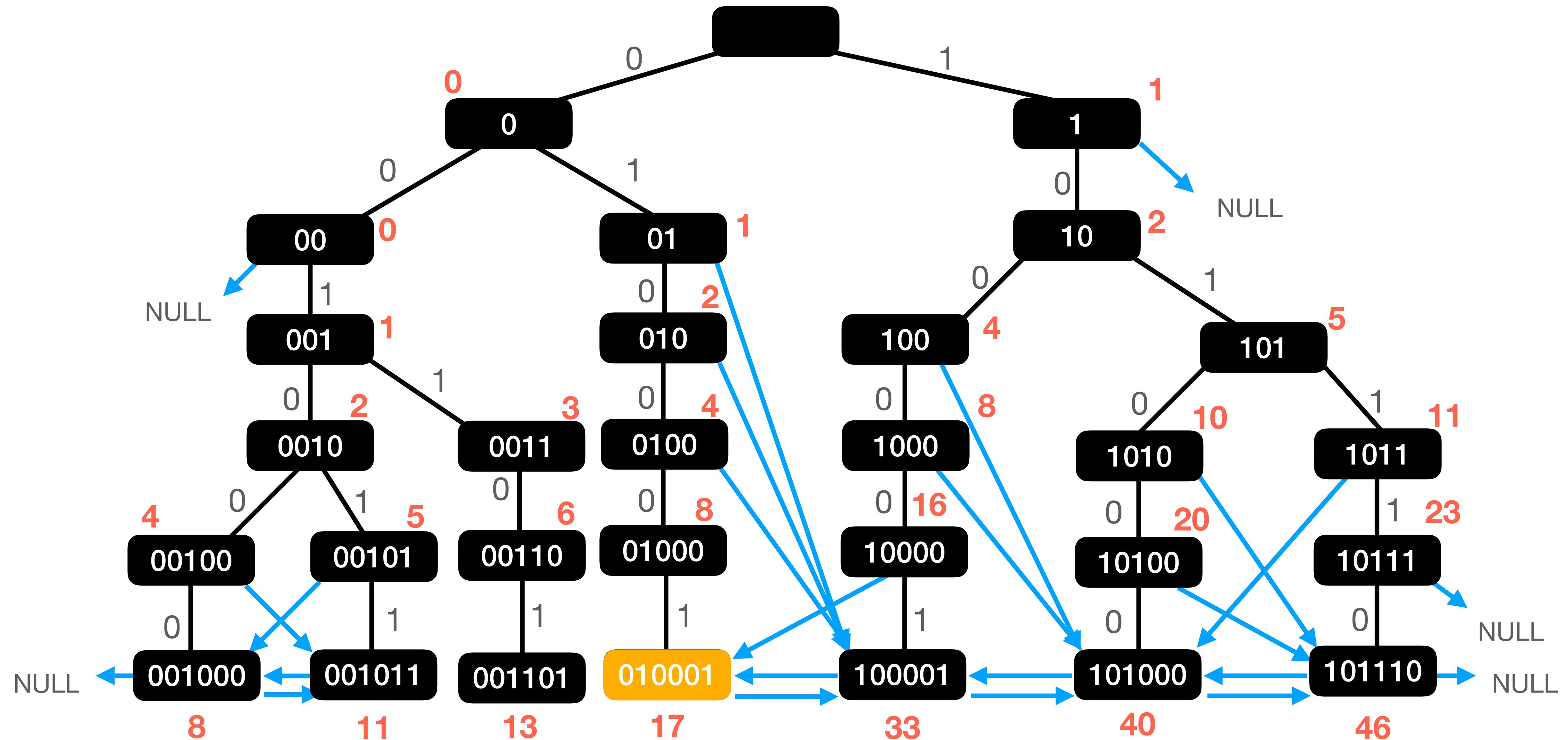
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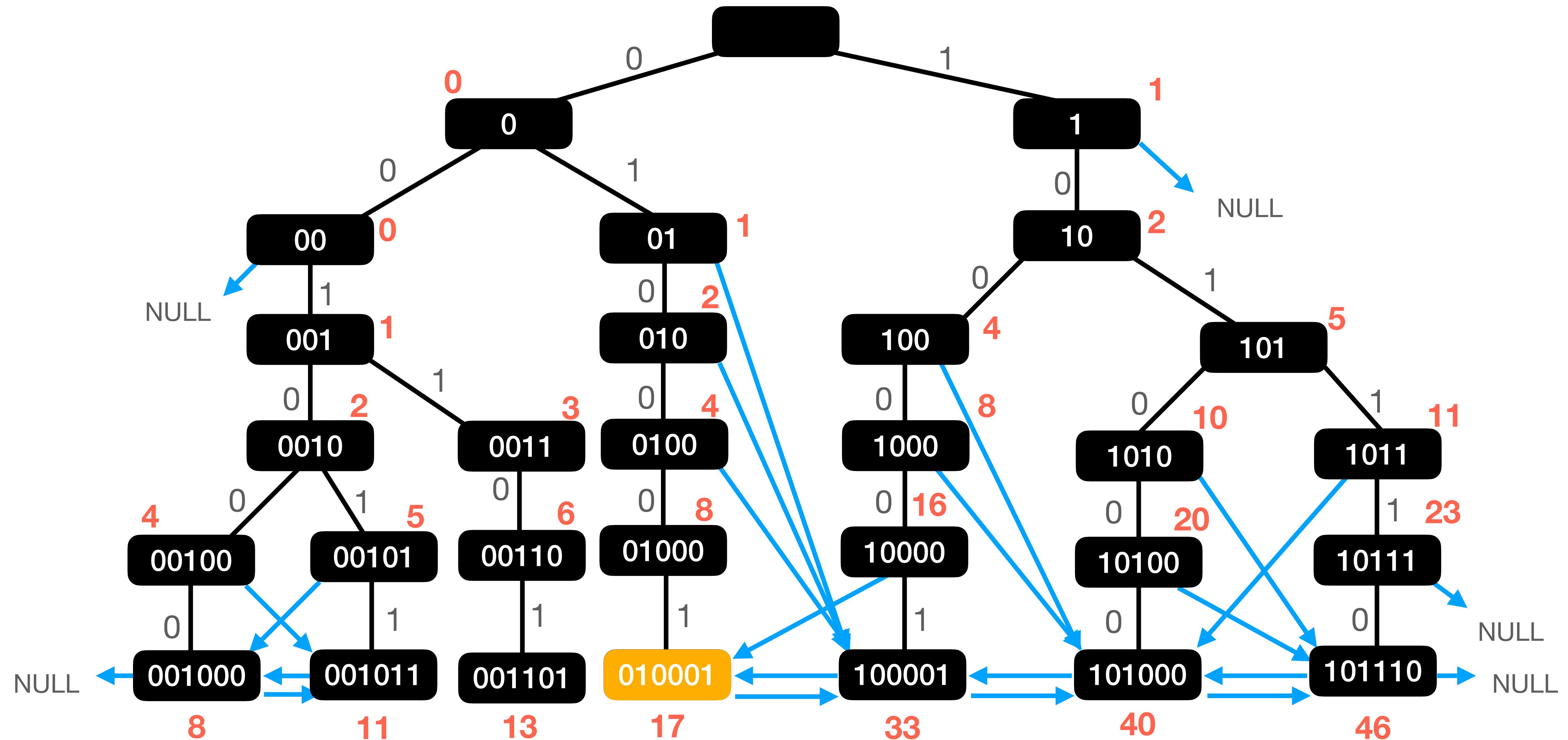
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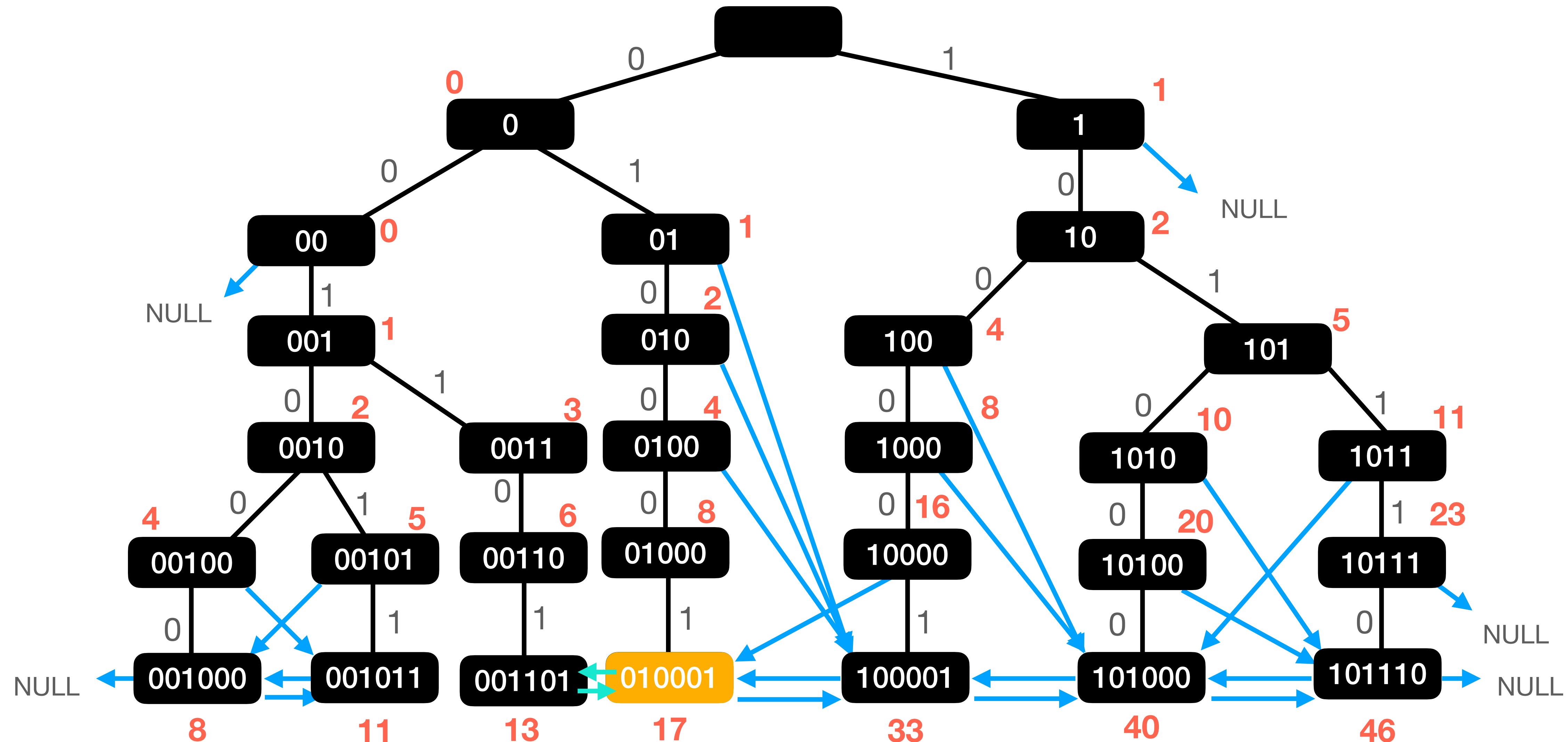
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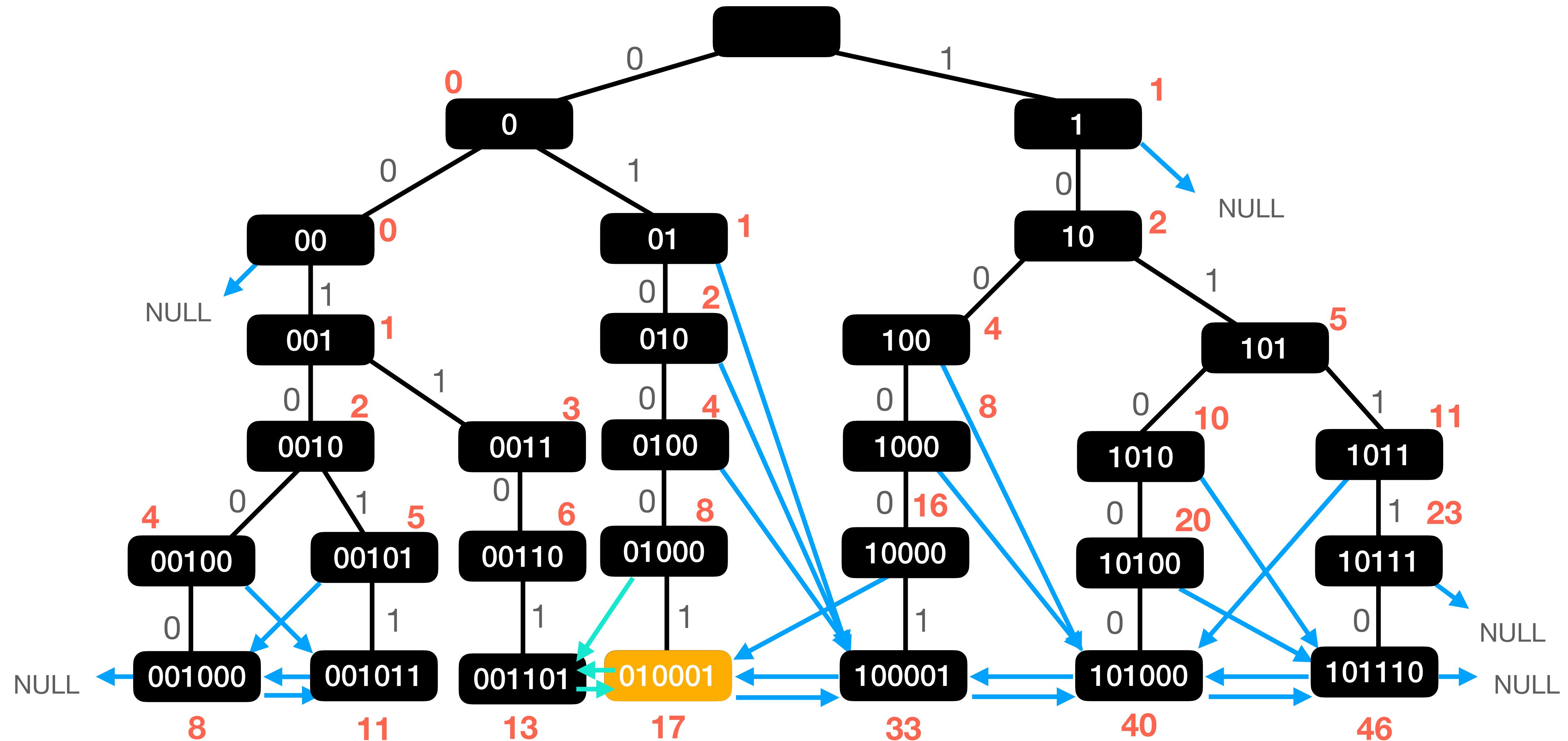
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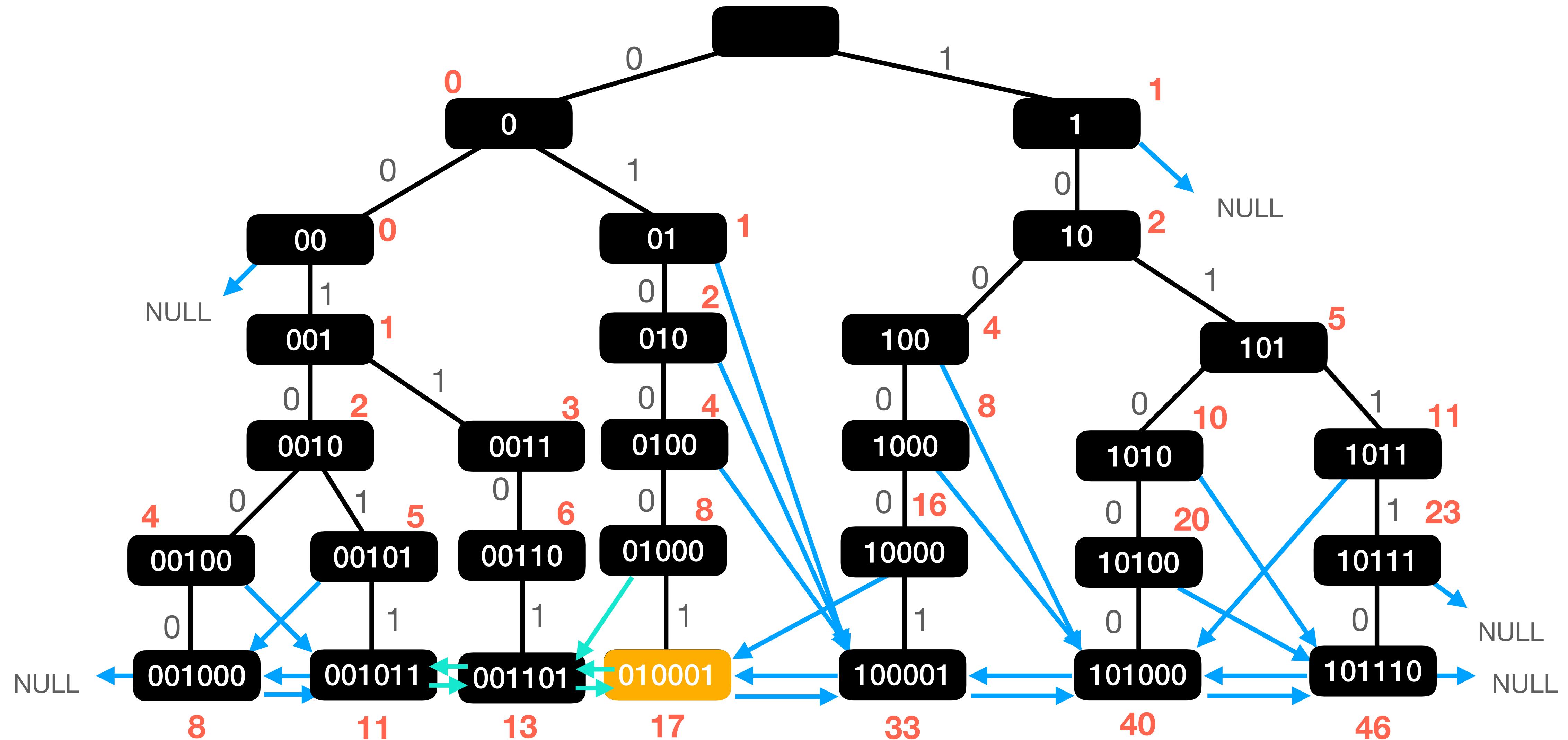
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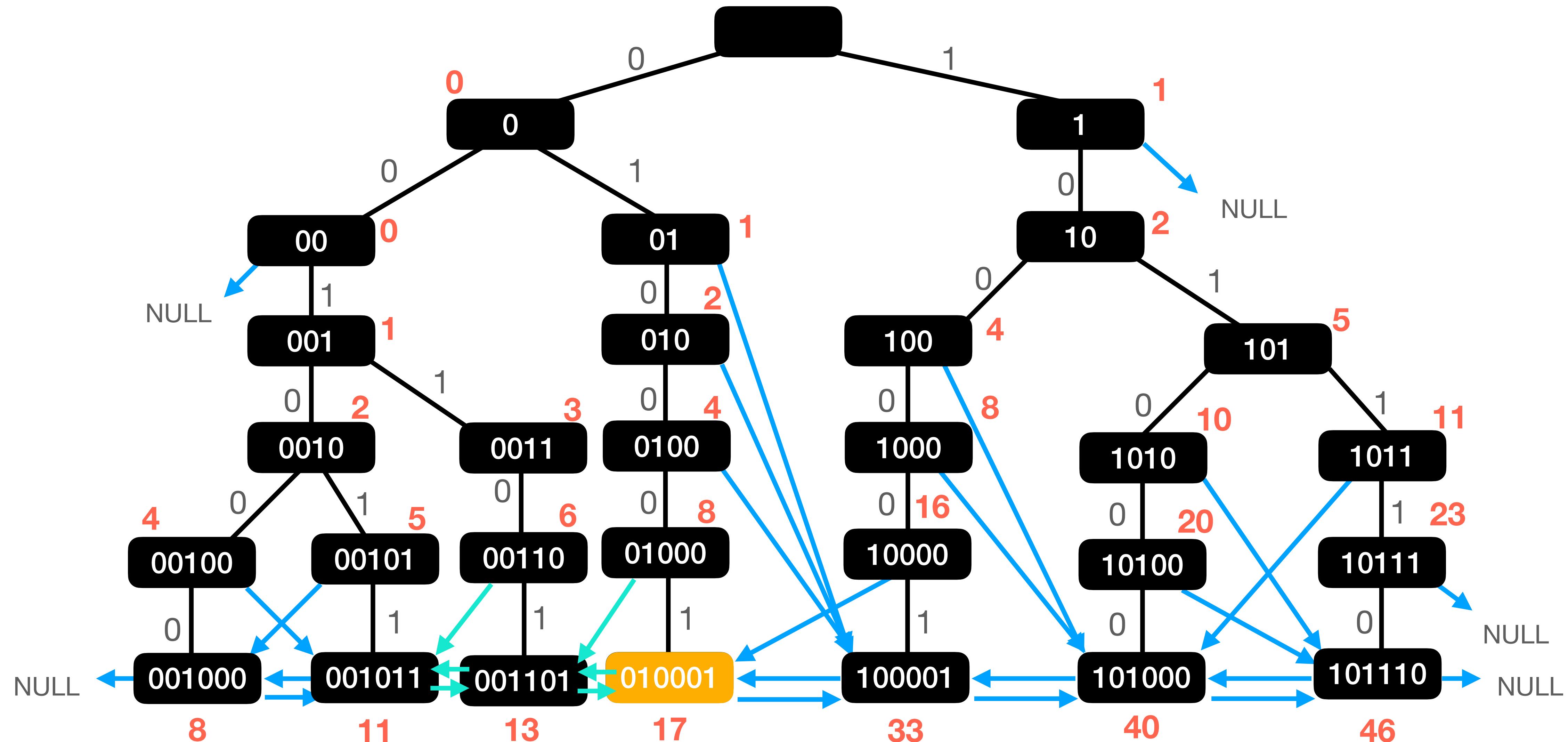
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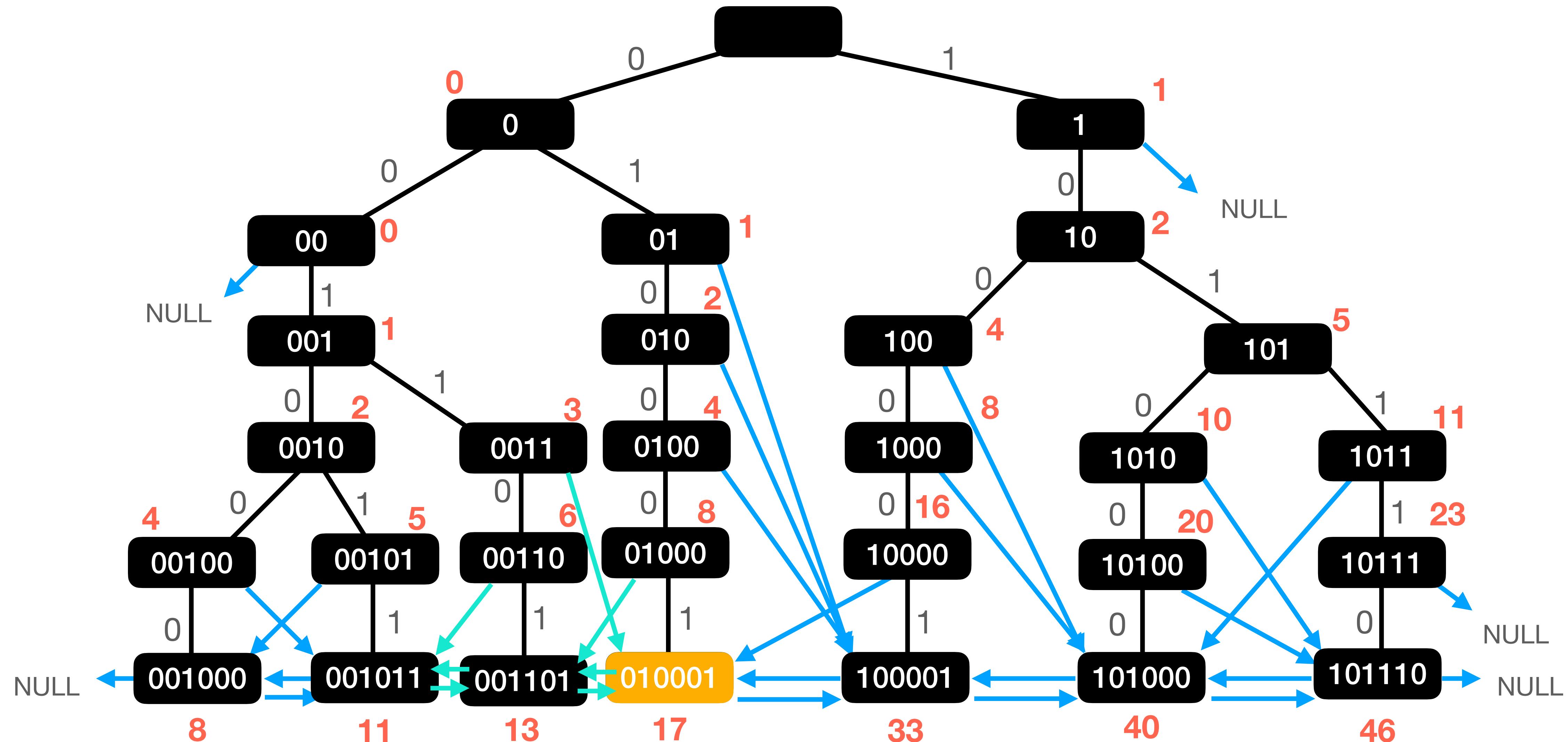
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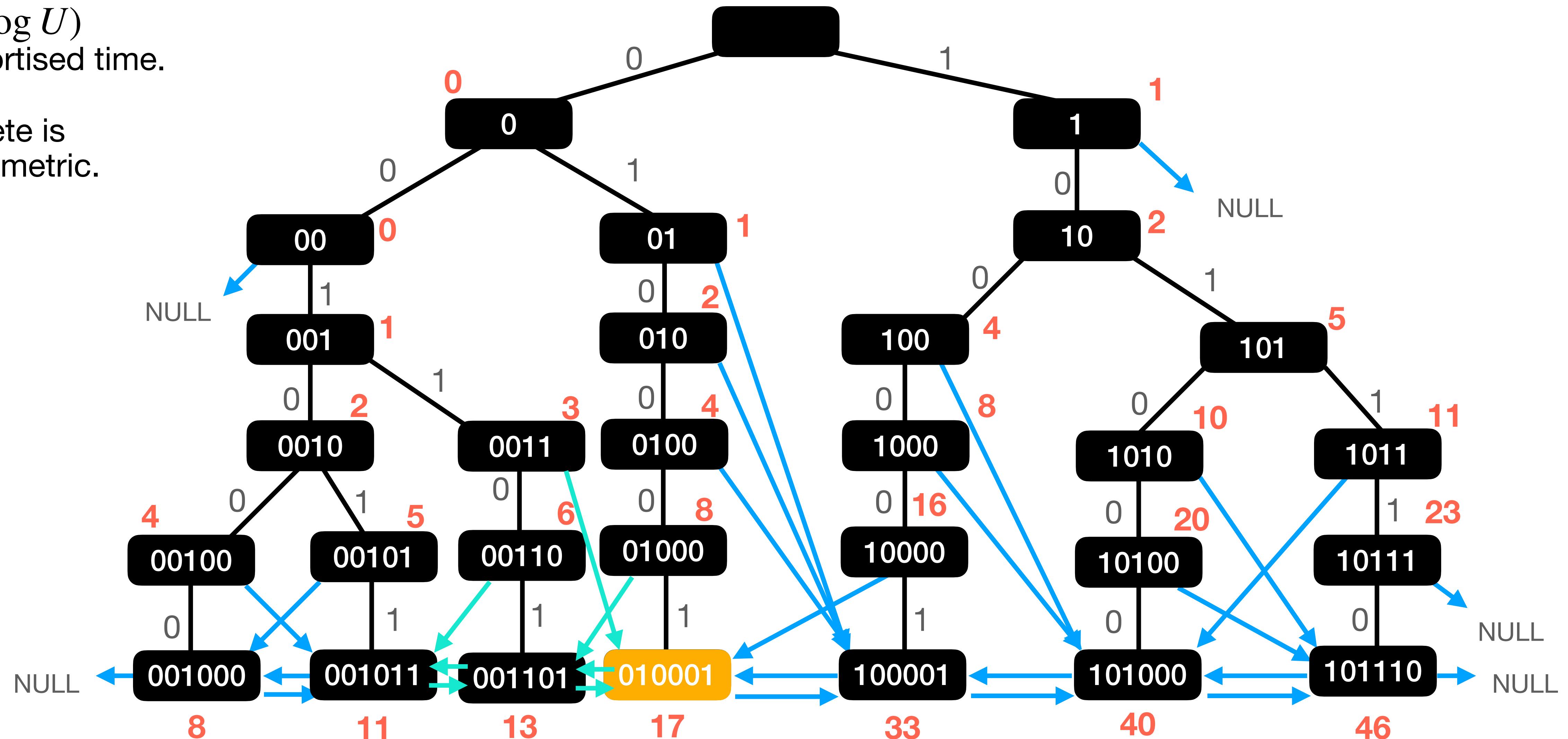
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- $O(\log U)$ amortised time.
- Delete is symmetric.



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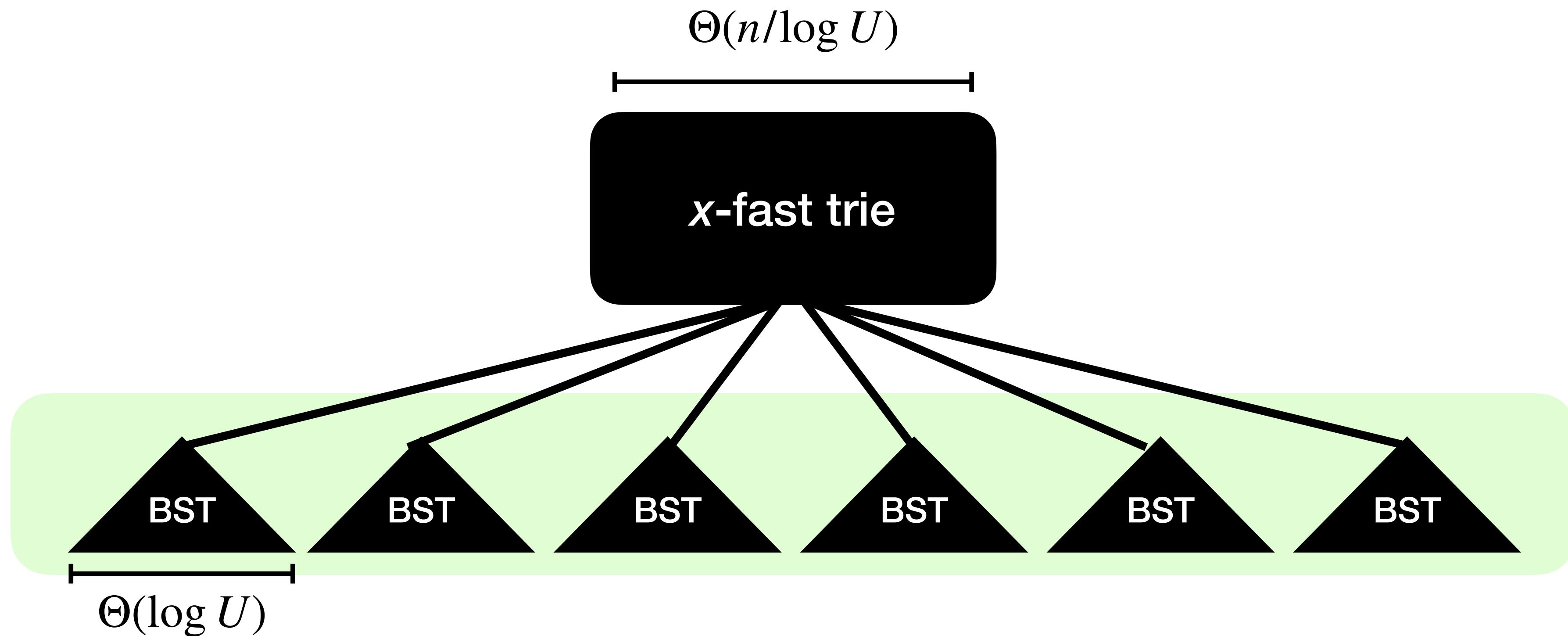
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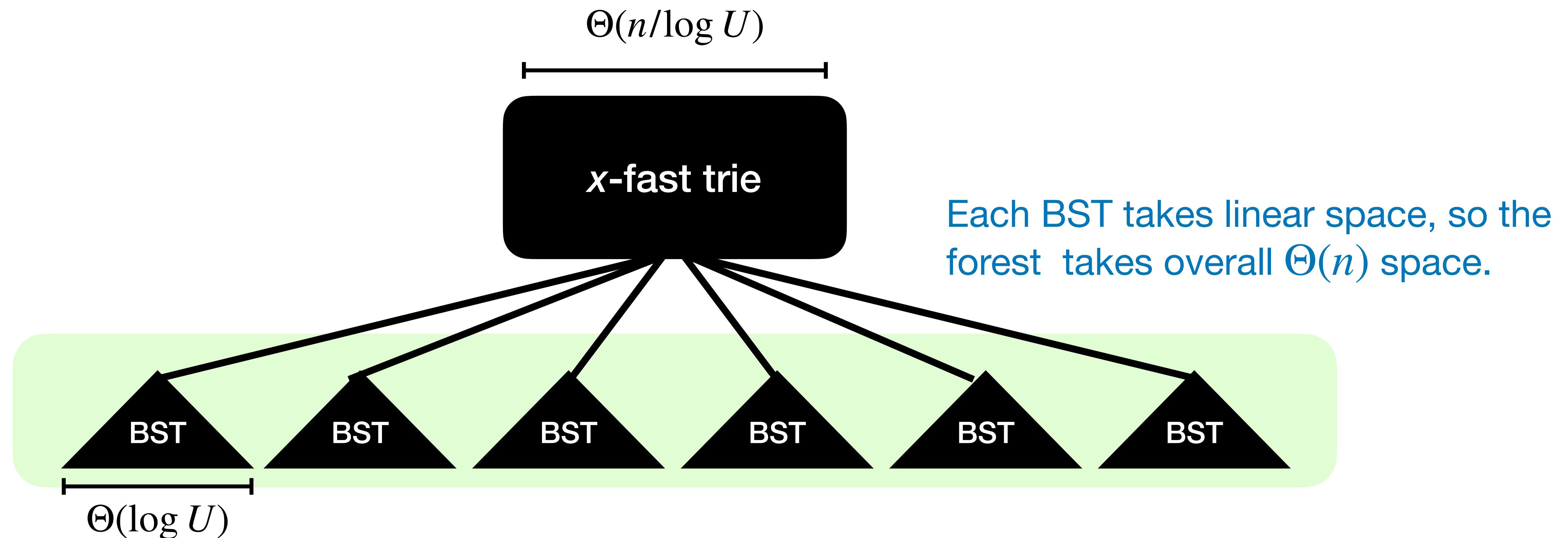
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- **Idea.** Chunking again! Split the set of keys into (mini) chunks of size $\Theta(\log U)$ and store each chunk in a balanced BST. Choose one representative per chunk. The set of $\Theta(n/\log U)$ representatives is stored with a *x*-fast trie.
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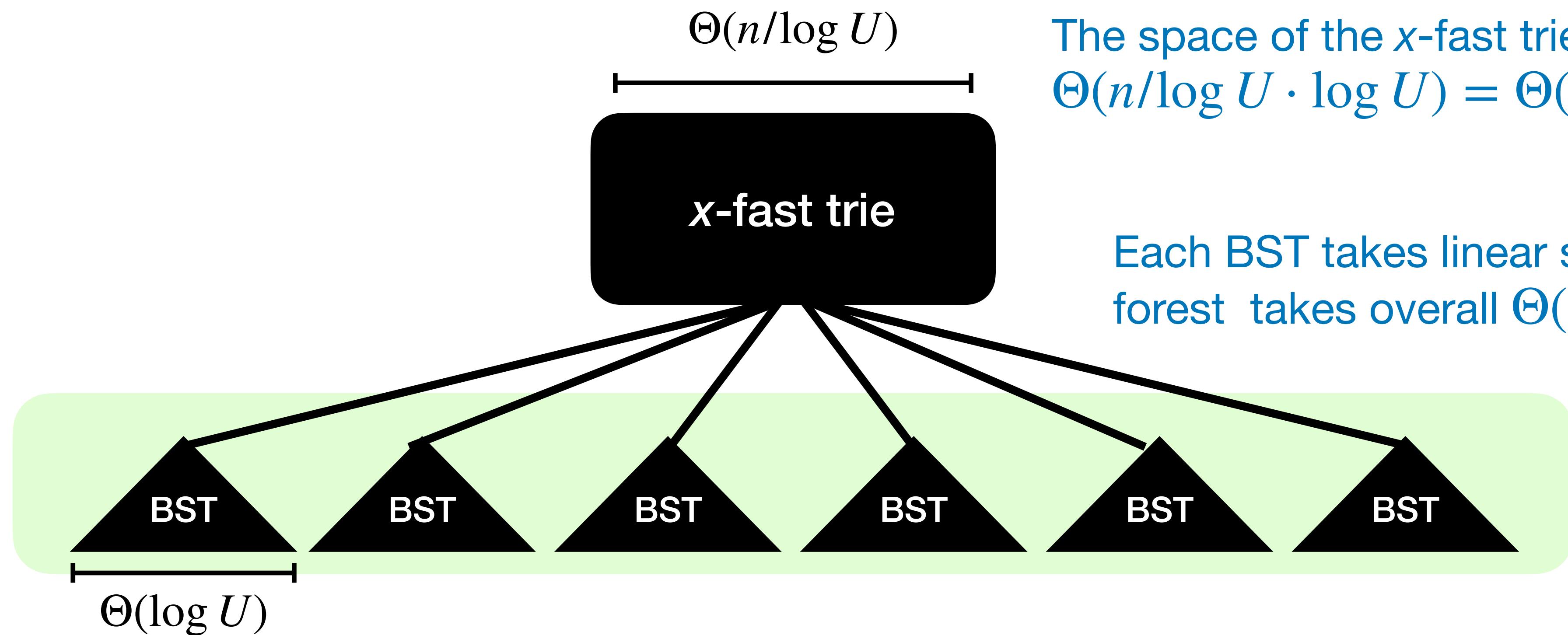
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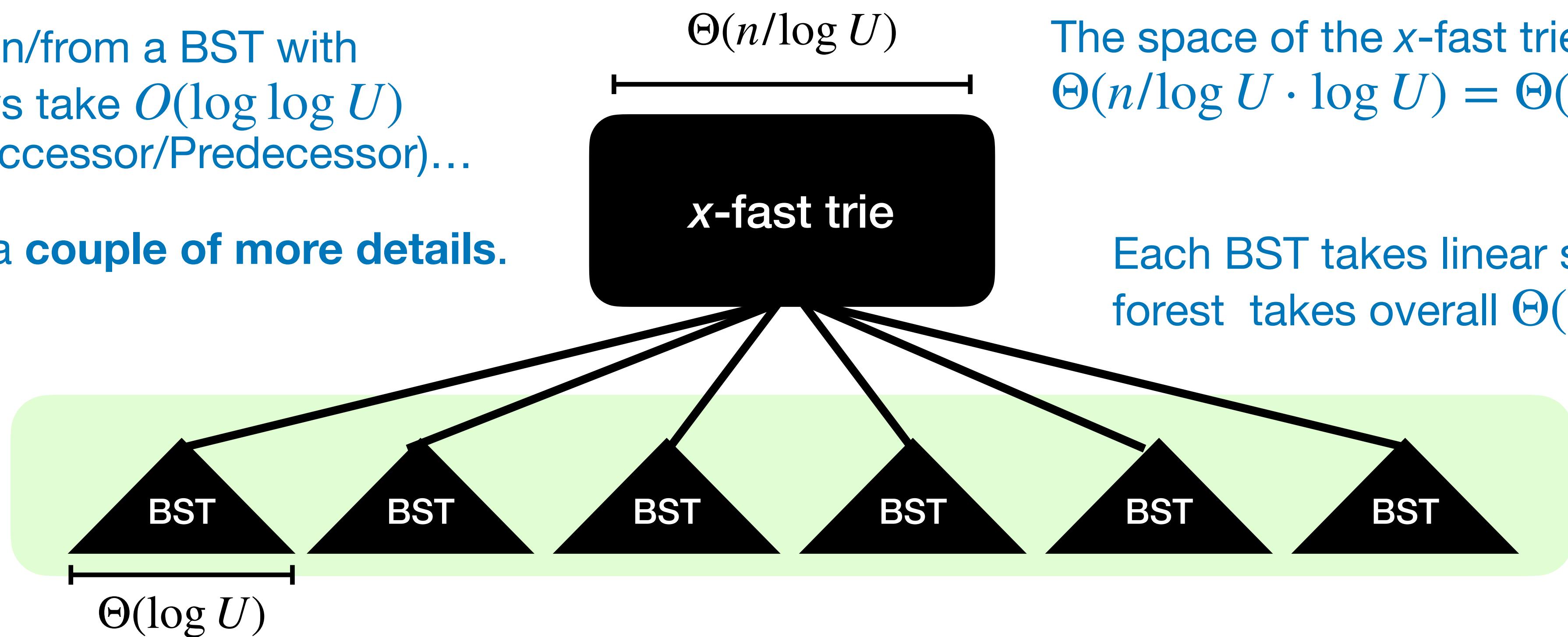


The y-fast trie

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Insert/Delete in/from a BST with $\Theta(\log U)$ keys take $O(\log \log U)$ (as well as Successor/Predecessor)...

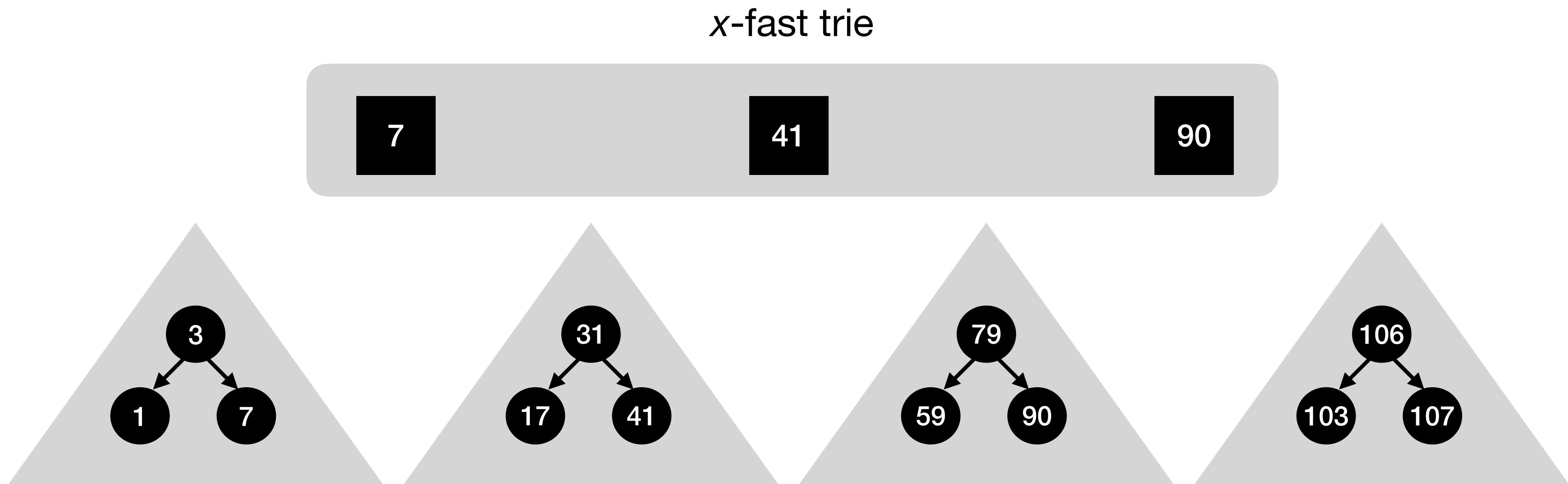
But we need a **couple of more details**.



The space of the *x*-fast trie is $\Theta(n/\log U \cdot \log U) = \Theta(n)$.

Each BST takes linear space, so the forest takes overall $\Theta(n)$ space.

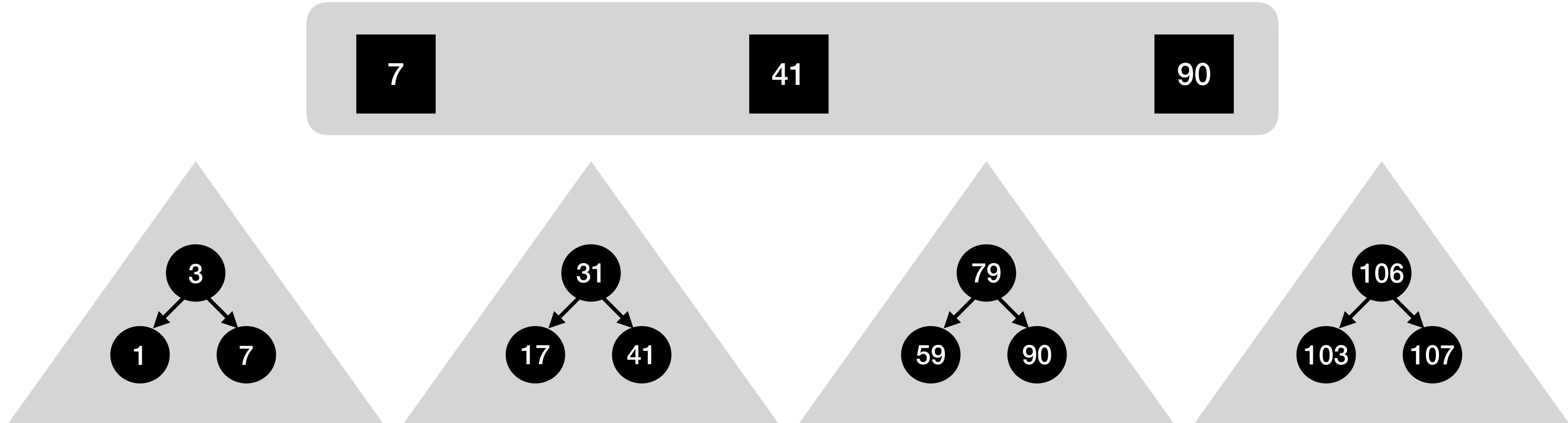
The **y**-fast trie – Successor and Predecessor



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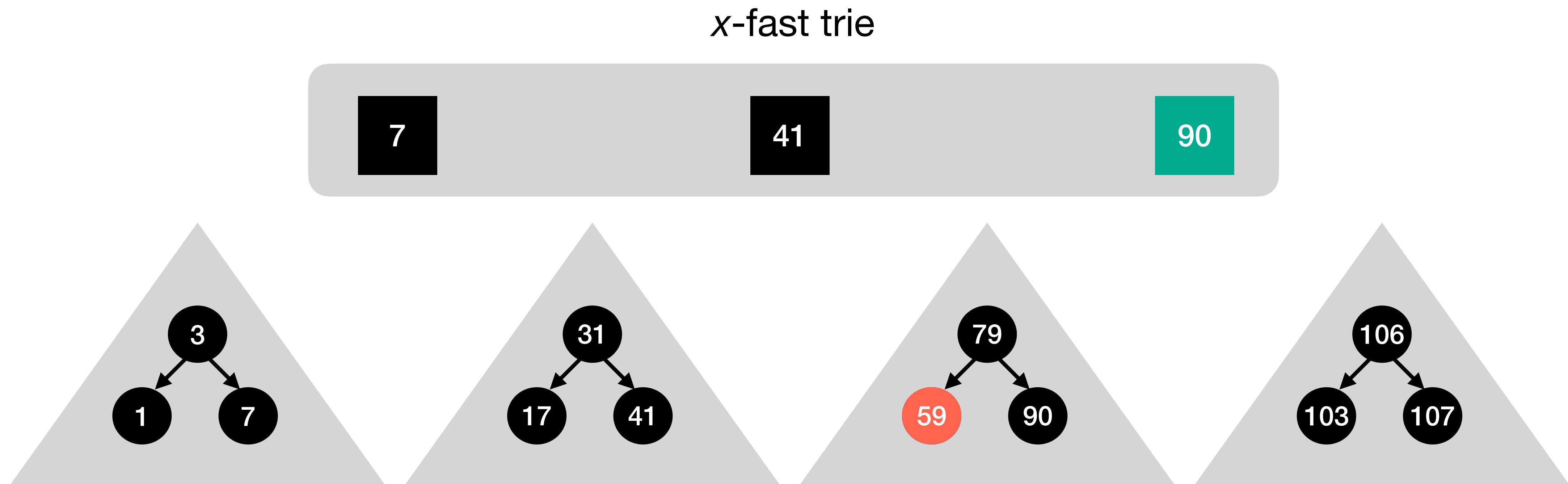
Successor(50)

x-fast trie



The **y**-fast trie – Successor and Predecessor

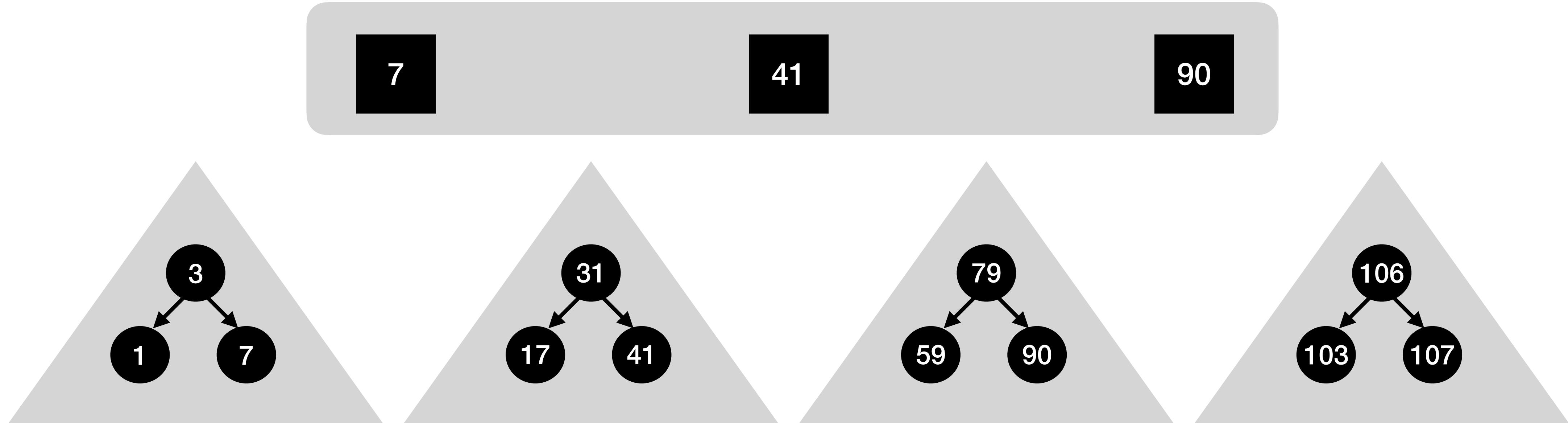
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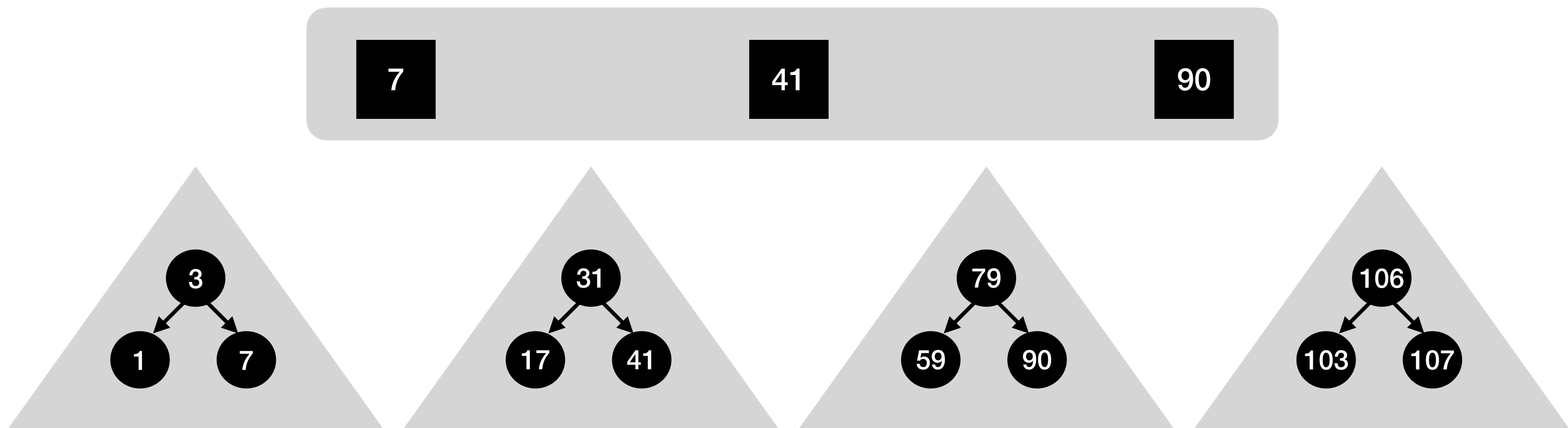


The **y**-fast trie – Successor and Predecessor

Successor(50)

Successor(101)

x-fast trie

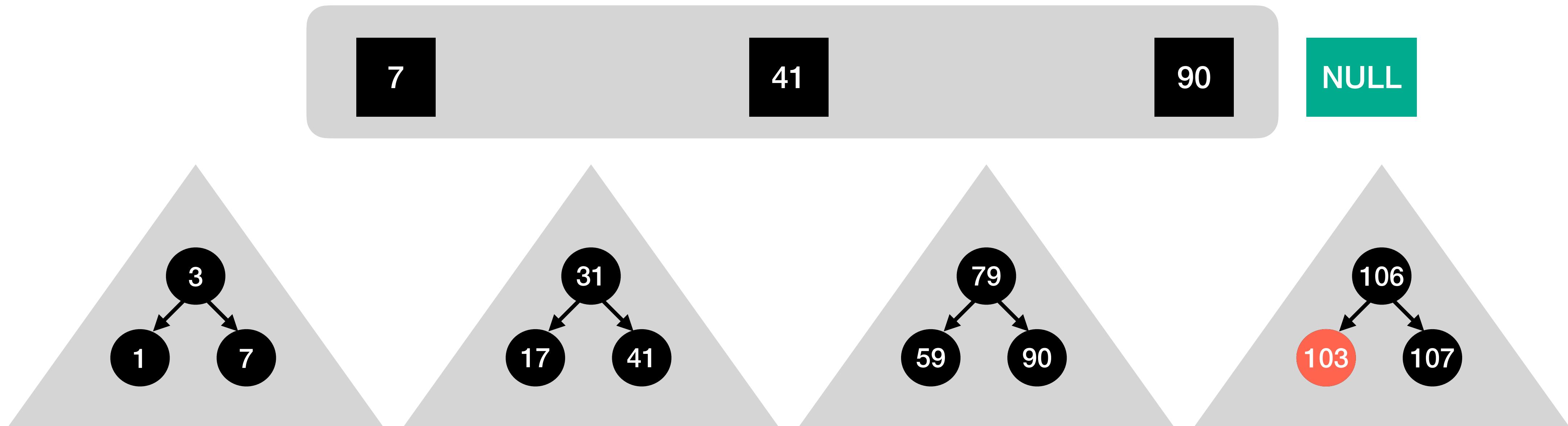


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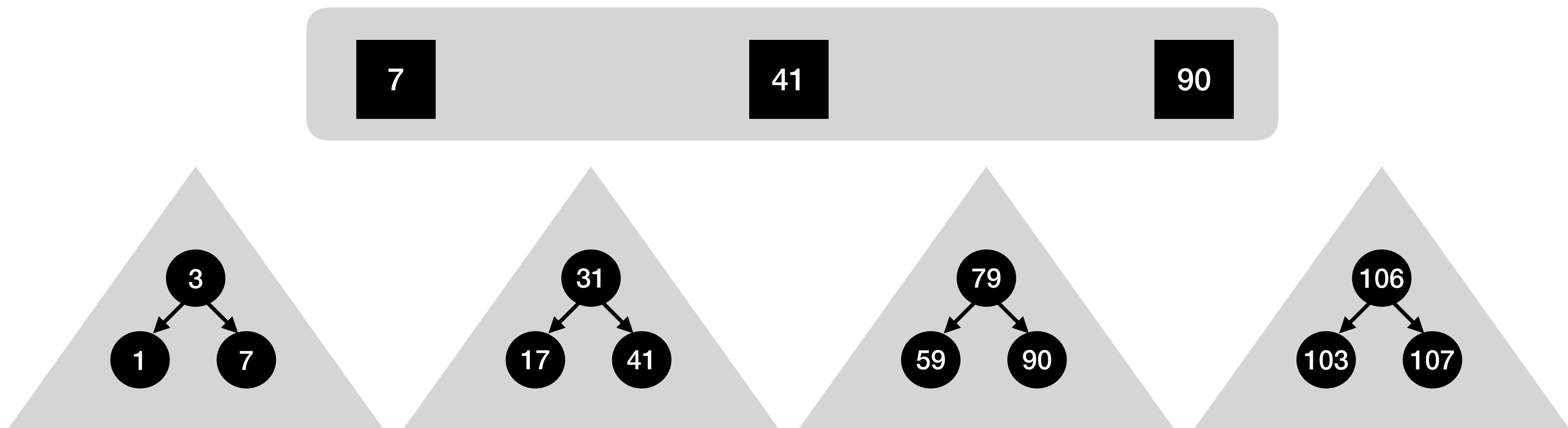


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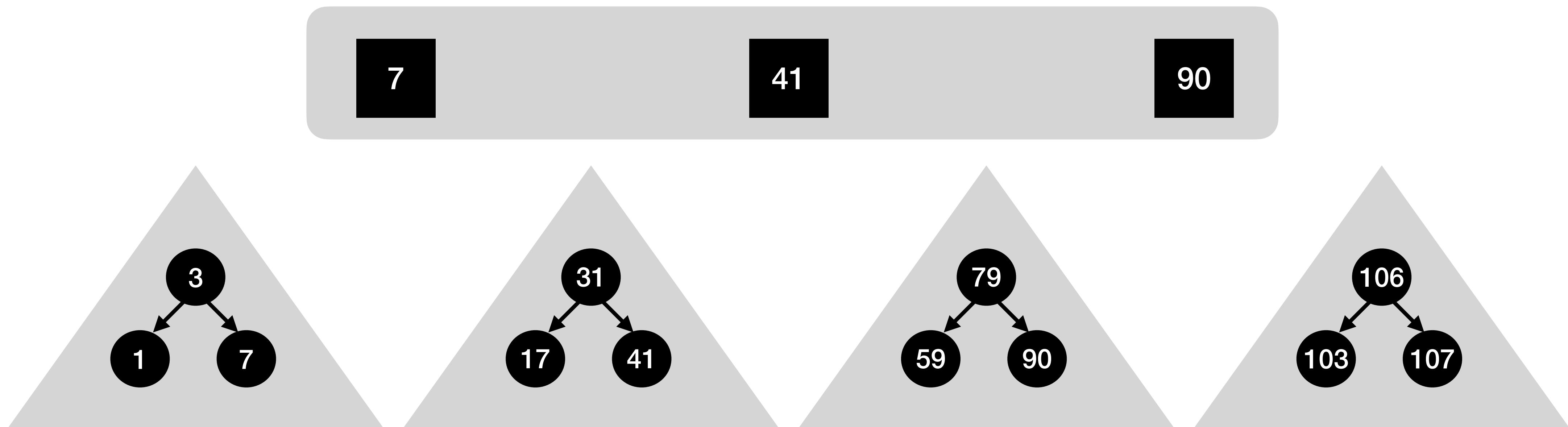
The **y**-fast trie – Successor and Predecessor

Successor(50)

Successor(101)

Predecessor(24)

x-fast trie



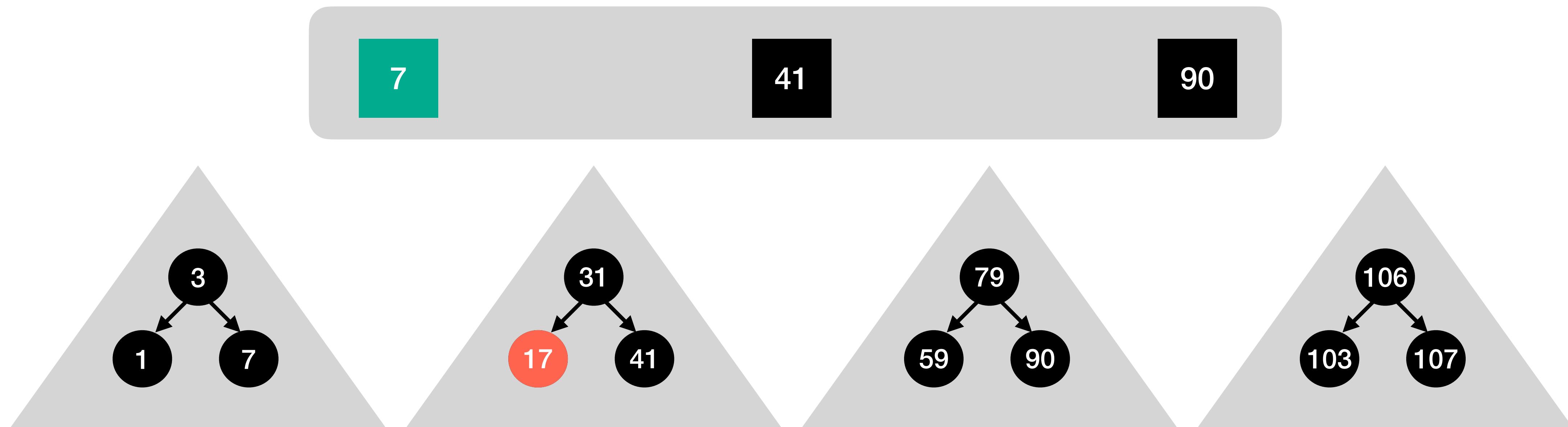
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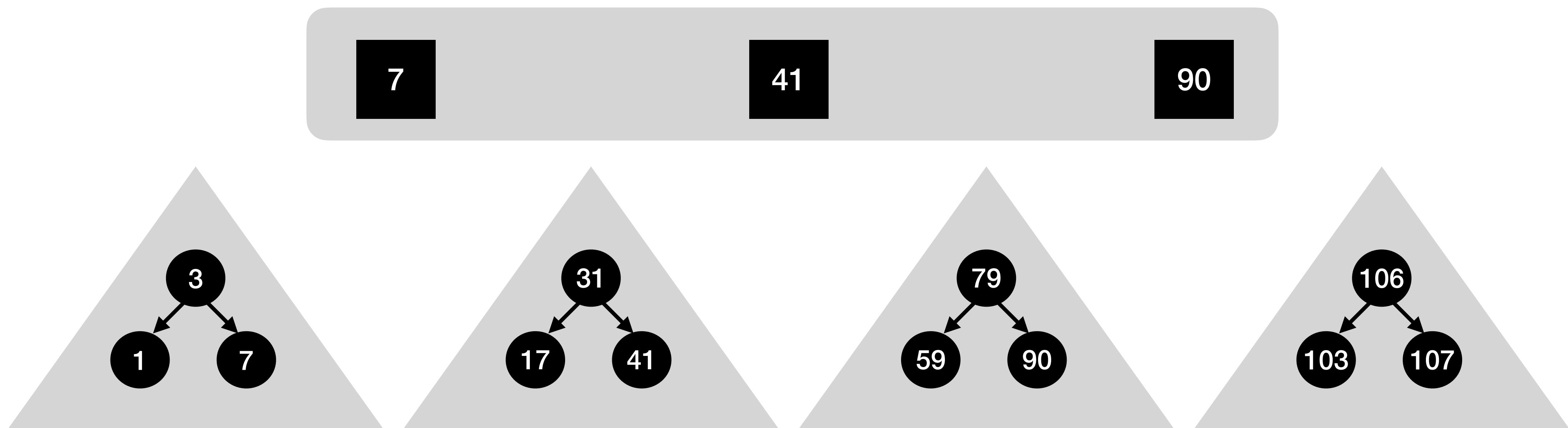
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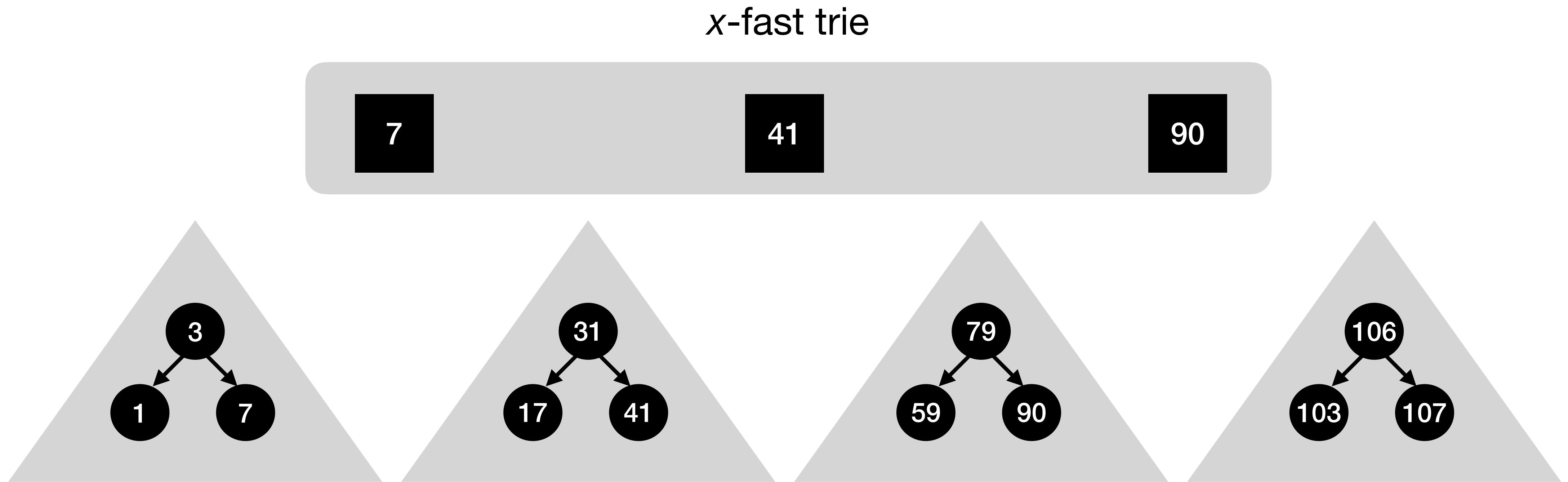
Successor(101)

Predecessor(24)

x-fast trie



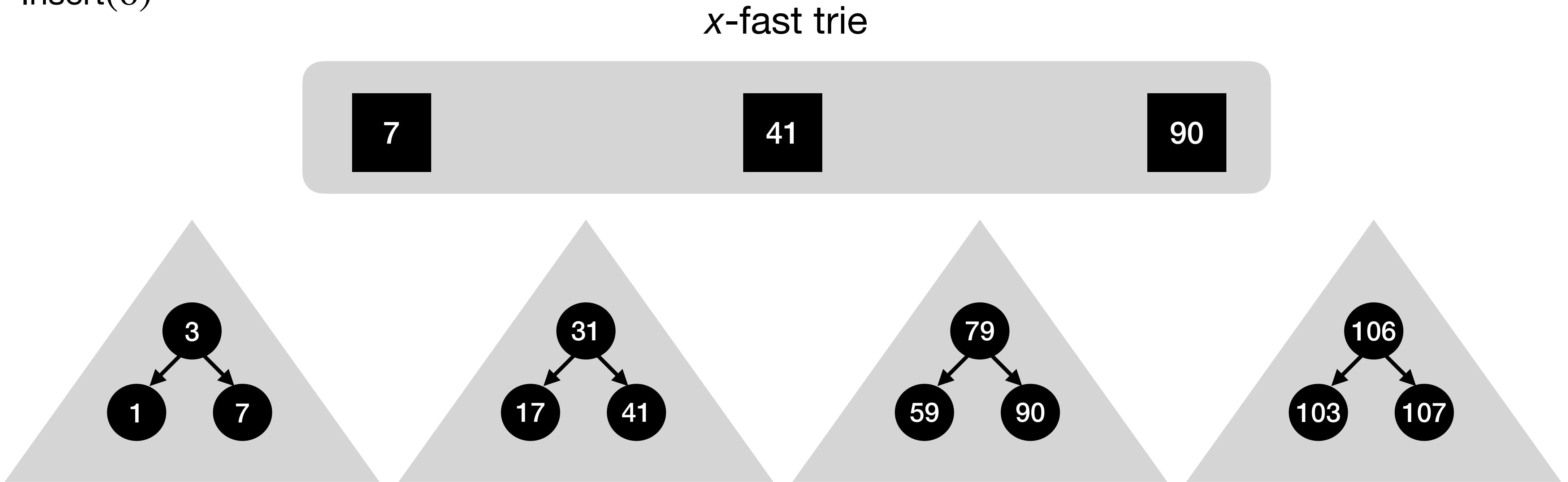
The y -fast trie – Insert and Delete



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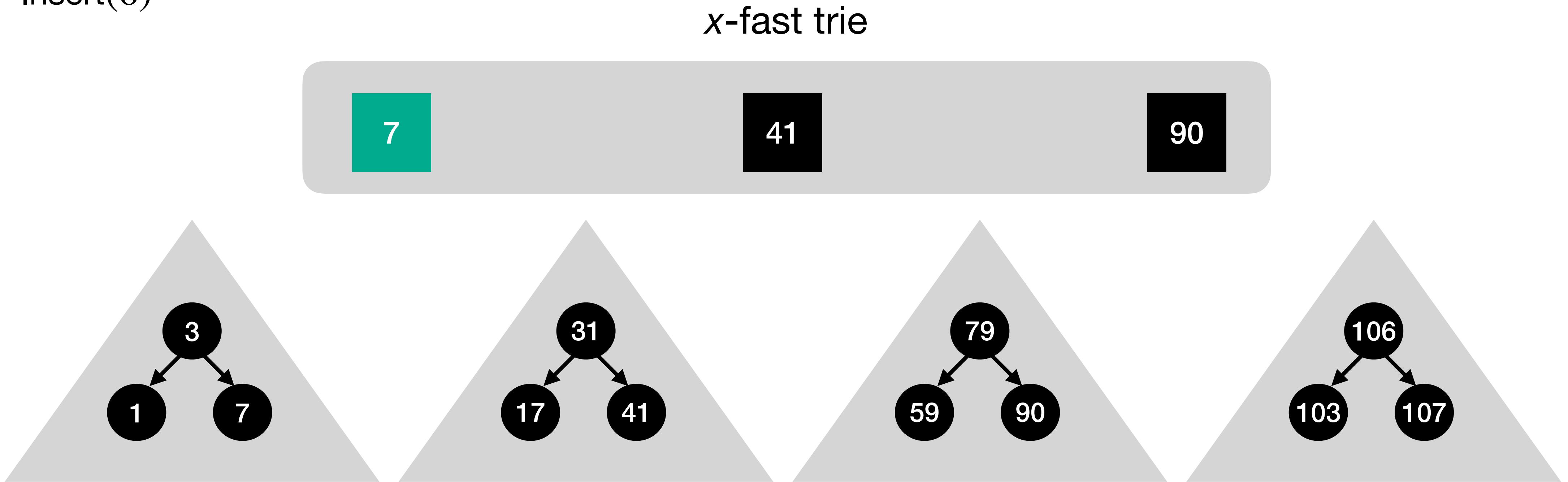
Insert(6)



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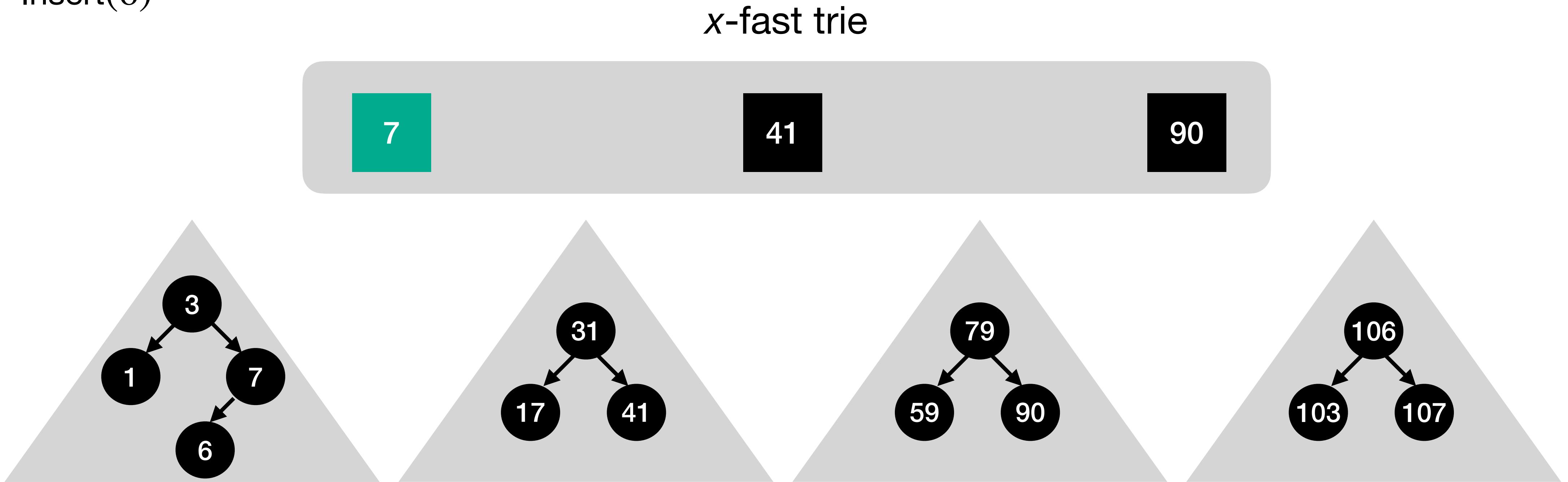
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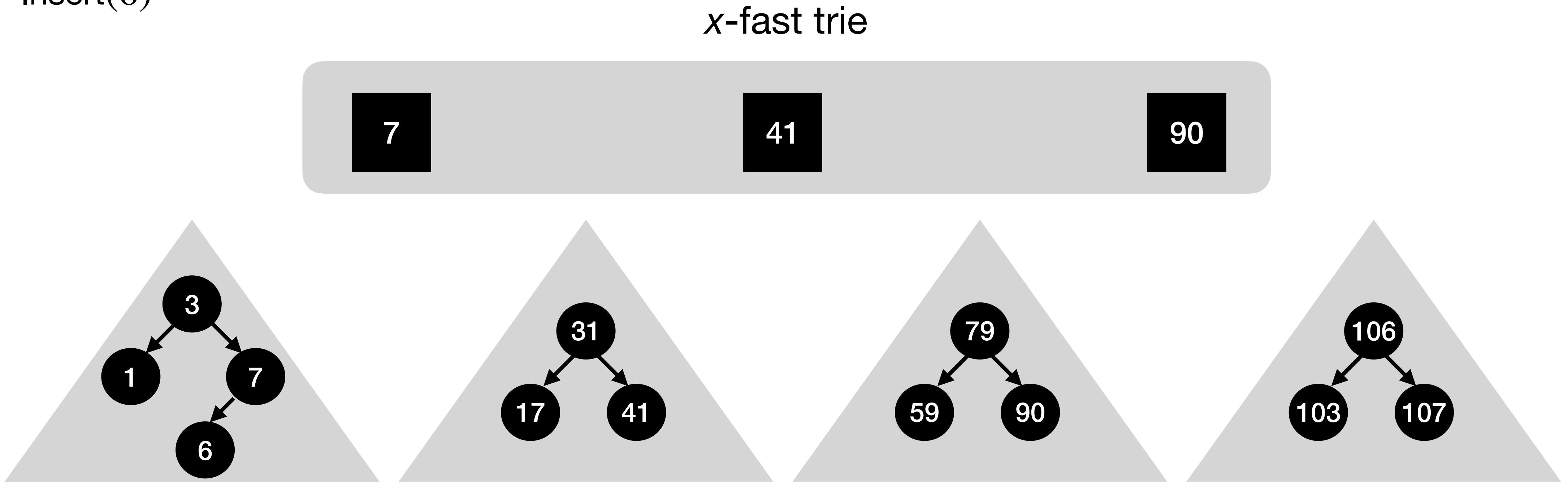
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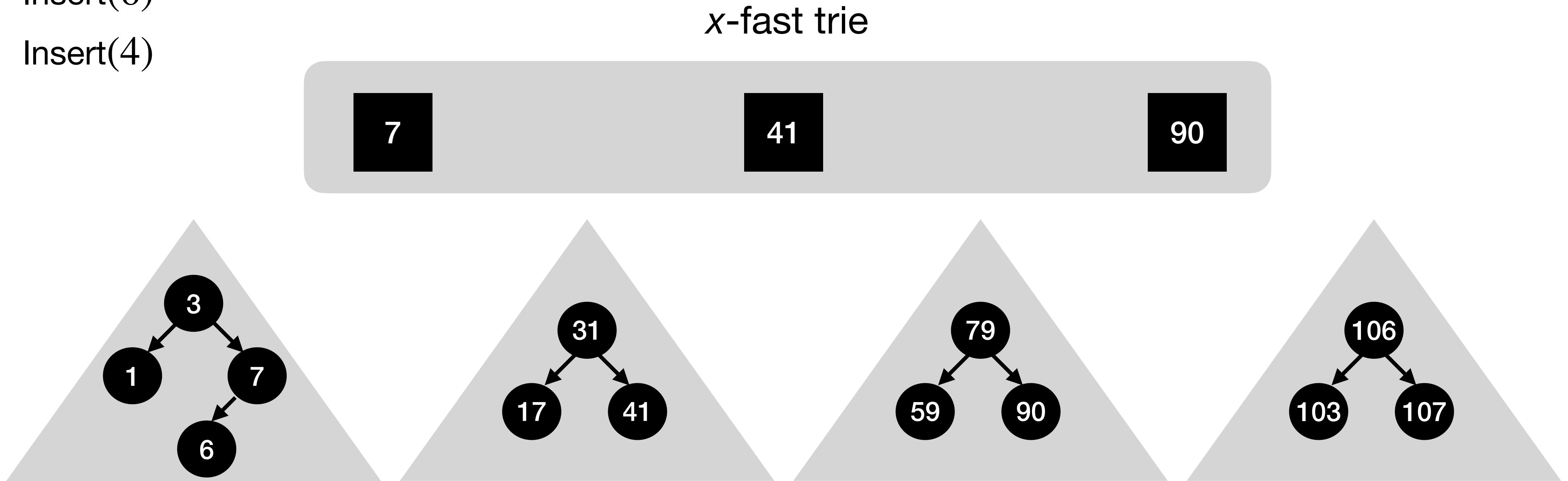


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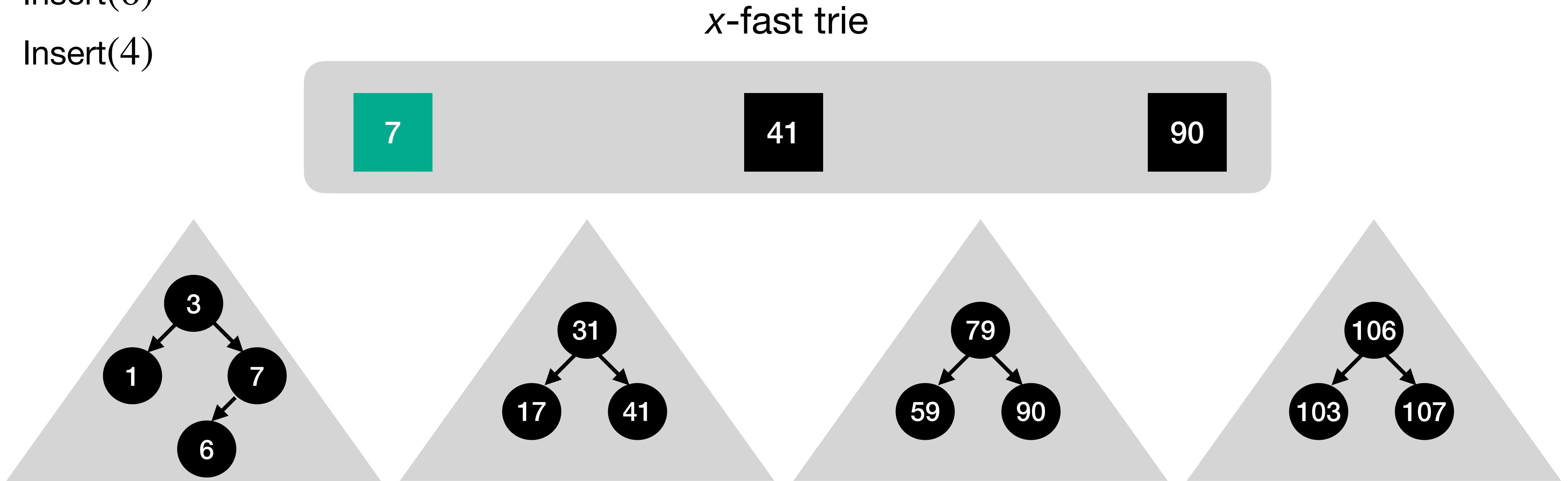


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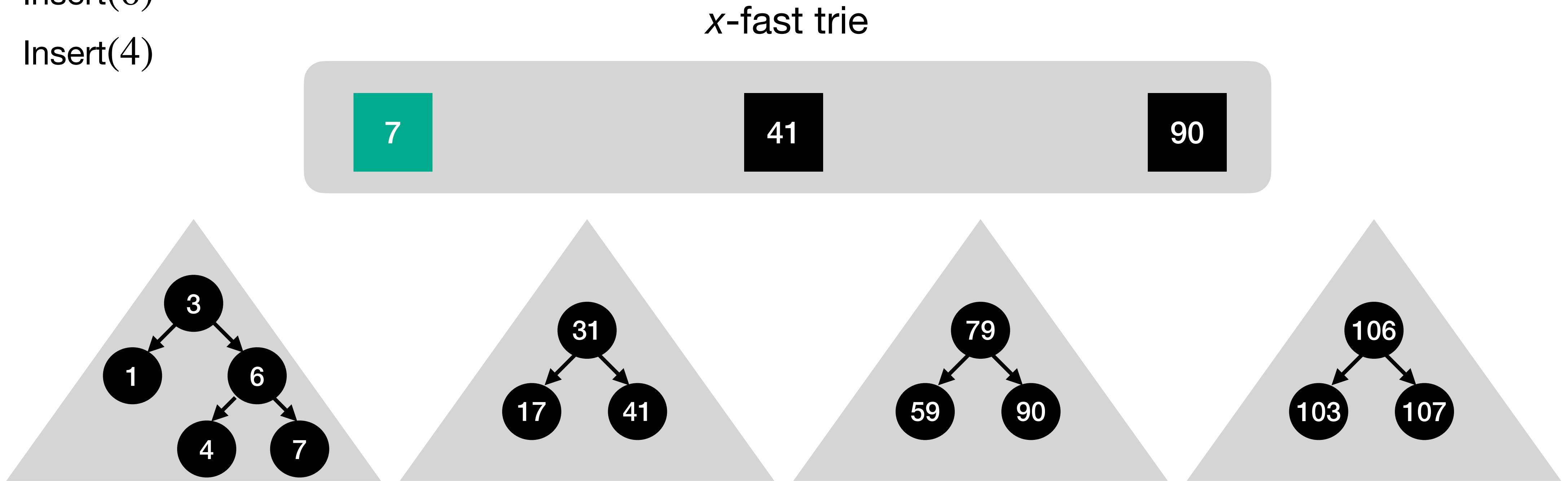


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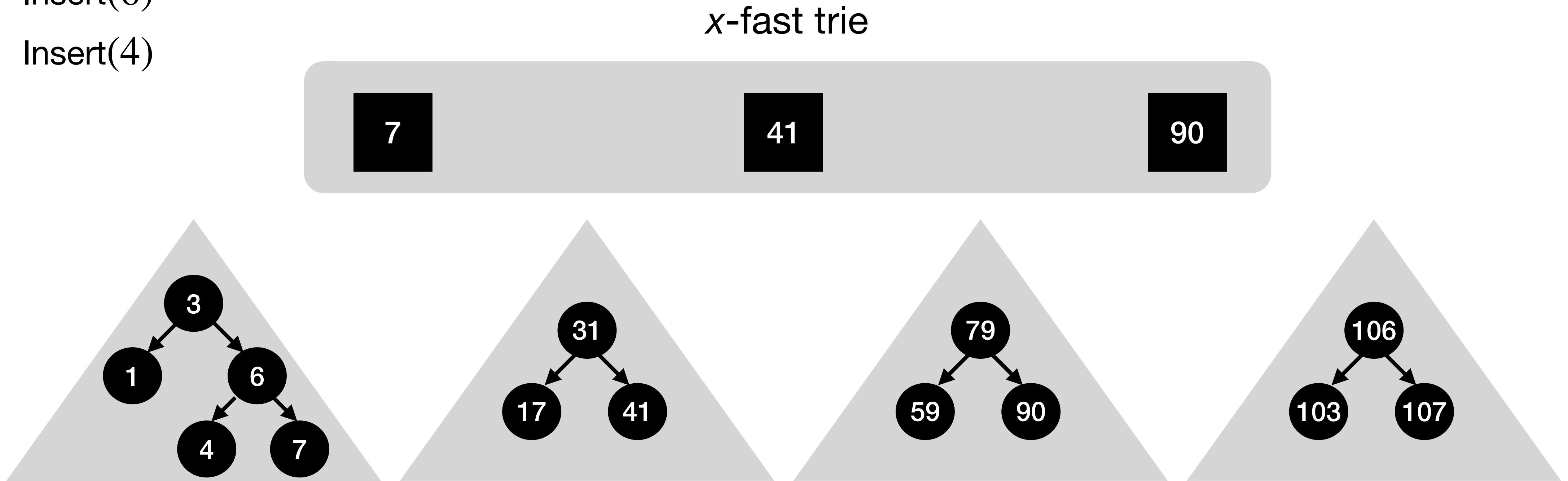


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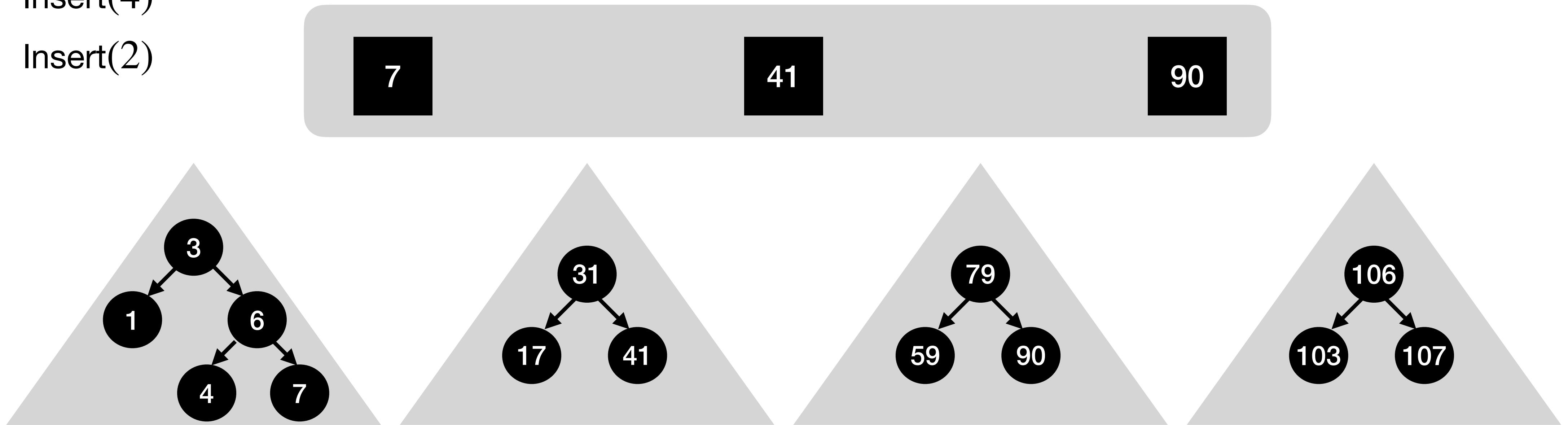
The y -fast trie – Insert and Delete

Insert(6)

Insert(4)

Insert(2)

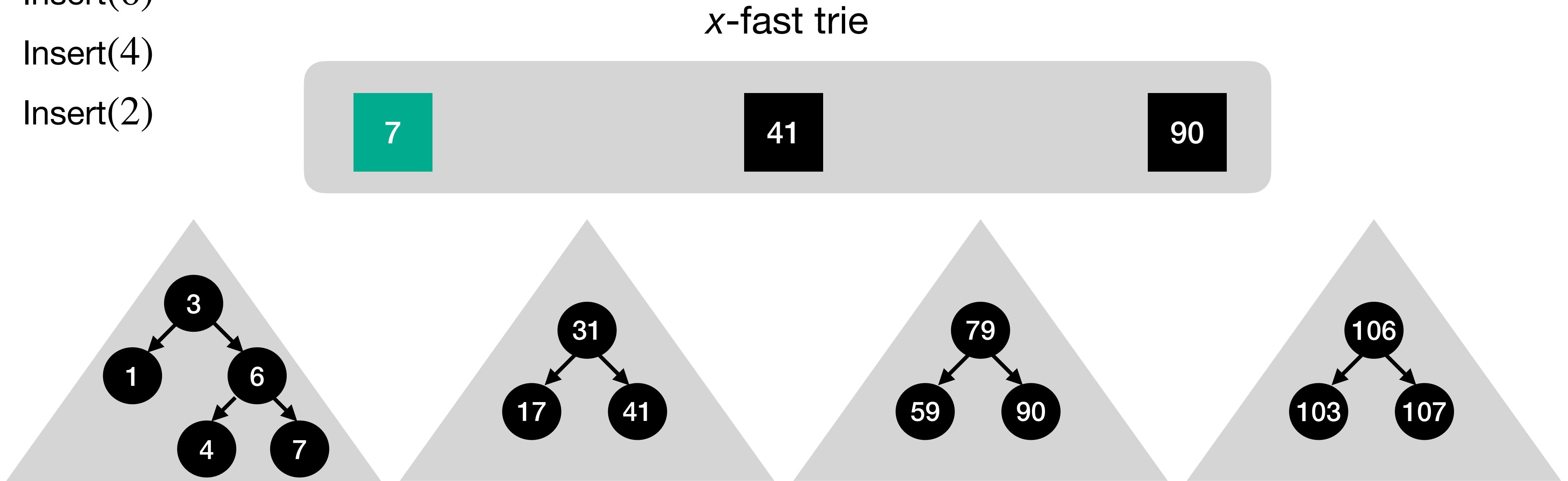
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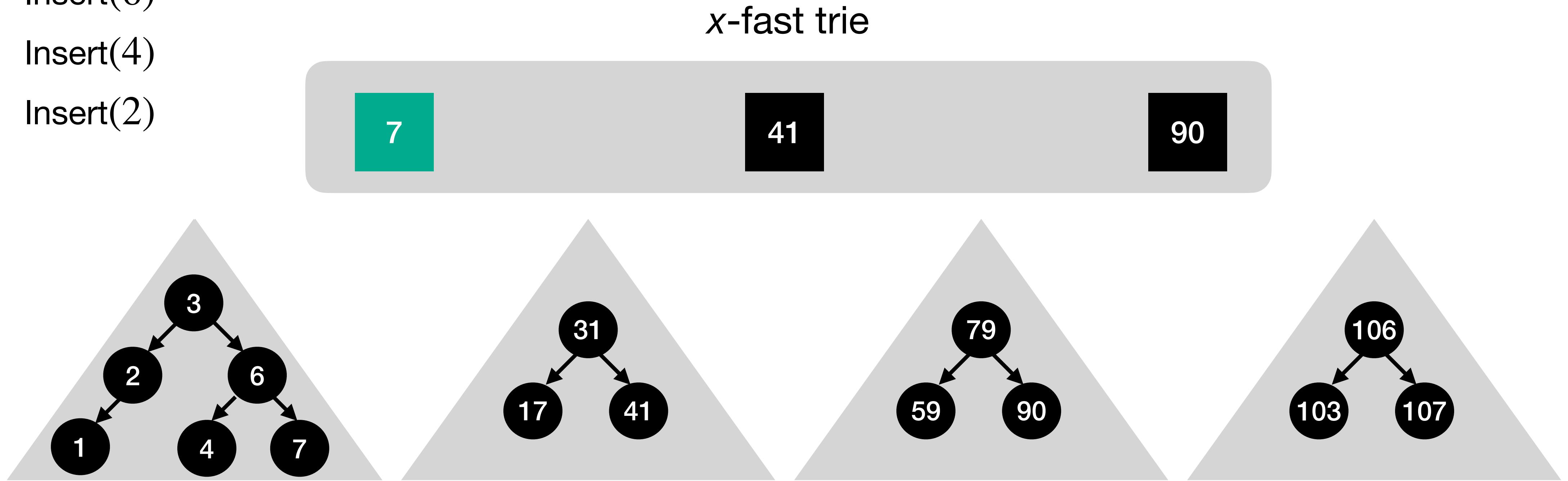
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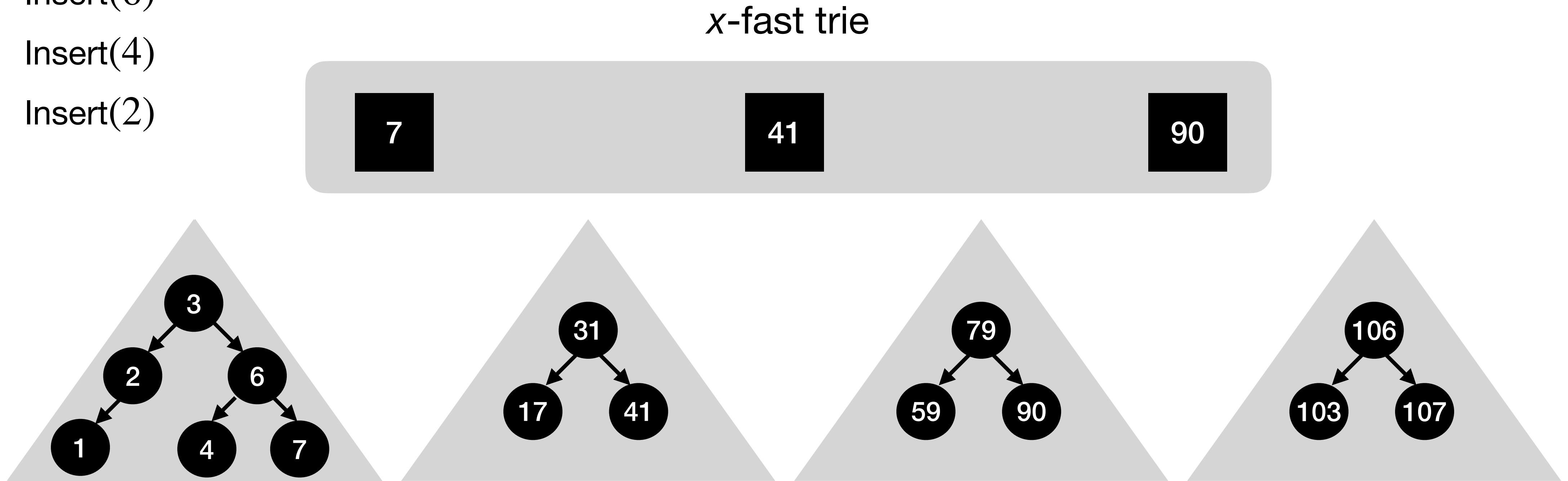
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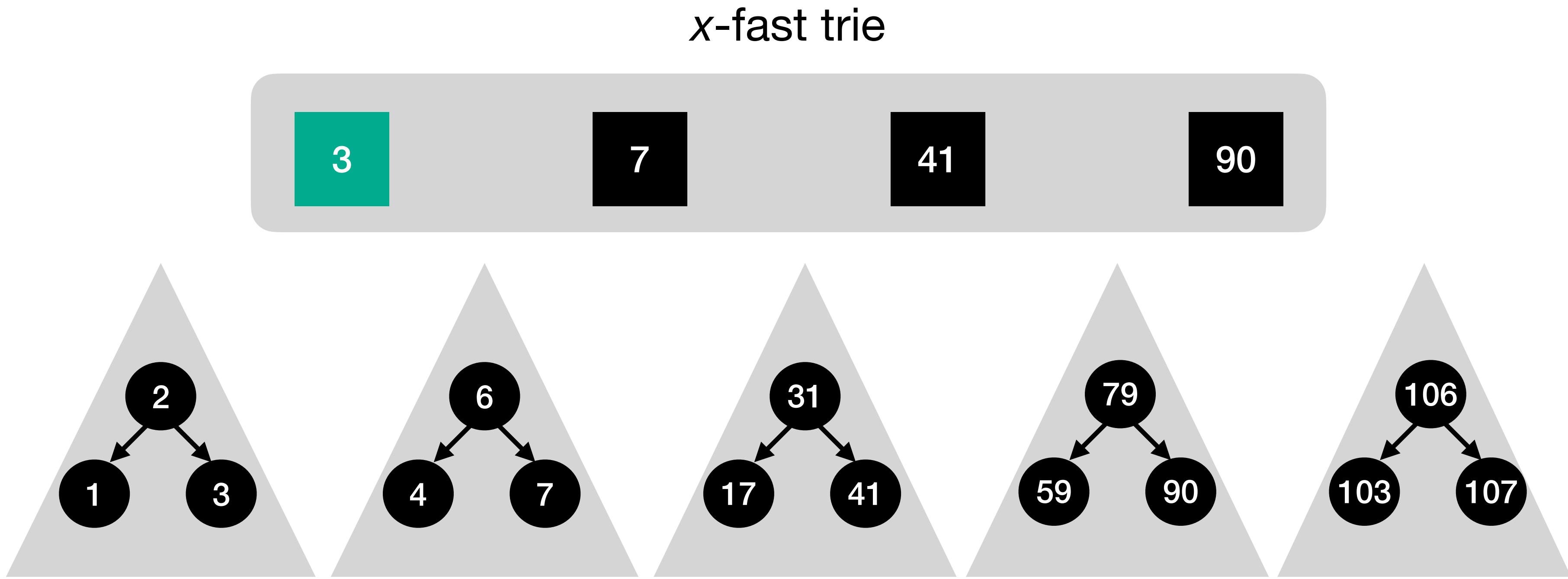
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The y-fast trie – Amortisation

- If we **do not split** upon Insert, we pay $O(\log \log U)$.
- If we do, we pay $O(\log U)$. But note that to split, we must have done $O(\log U)$ insertions to a tree, hence on average we pay:

$$\frac{O(\log U) \cdot O(\log \log U) + O(\log U)}{O(\log U)} = O(\log \log U).$$

- Logic for Delete: if a tree becomes too small, we merge it with the next tree. Split again the resulting tree if necessary and update the x-fast trie.

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- Logic for Delete: if a tree becomes too small, we merge it with the next tree. Split again the resulting tree if necessary and update the x-fast trie.
- We have a **sorting algorithm** that runs in $O(n \log \log U)$ amortized-time by simply inserting each integer in a y-fast trie! Radix Sort takes $O(n \cdot (\log U)/r)$ time. Sorting with an y-fast trie is better for $r = o(\log U/\log \log U)$.

Summary

	x-fast trie	y-fast trie
Min/Max	$O(1)$	$O(1)$
Member	$O(1)$	$O(\log \log U)$
Predecessor/Successor	$O(\log \log U)$	$O(\log \log U)$
Subset	$O(\log \log U + \text{Subset})$	$O(\log \log U + \text{Subset})$
Insert/Delete	$O(\log U)$	$O(\log \log U)$
Space	$O(n \log U)$	$\Theta(n)$
Main tools	Binary search on key's prefixes stored into a hash table.	Sparsification and amortisation. A suboptimal structure might be ok if used on few elements.

The last question is...

- Why the names “x” and “y” ?!
- Quote:

*“...the more you think about what the B in B-Tree means,
the better you understand B-Trees!”*

References

- Dan E. Willard. *Log-logarithmic worst-case range queries are possible in space $\Theta(N)$* , Information Processing Letters, 17 (2): 81-84, 1983.
- P. van Emde Boas, *Preserving order in a forest in less than logarithmic time*, FOCS, 75-84, 1975.
- P. van Emde Boas; R. Kaas; E. Zijlstra, *Design and implementation of an efficient priority queue*, Math. Syst. Theory, 99–127, 1977.
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