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Fourier analysis on the quantum E(2) group within an Algebraic framework for quantum group duality

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Oh my God... my middle name is right behind that shrub! I'll finally know what "J." stands for. From this moment forth, I will be known as Homer ... Jay Simpson! It's so beautiful...

Homer J. Simpson

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Oh, thank you, thank you. . .
thank you so much!

Basil Fawlty

Vooraleer van start te gaan, wil ik eerst enkele mensen bedanken die een belangrijke rol gespeeld hebben bij het tot stand komen van dit werkje. In de eerste plaats natuurlijk mijn promotor Fons van Daele, die me zes jaar geleden de kans gaf het wiskundig onderzoek in te stappen—het begin van zes fijne jaren in de afdeling analyse—en wiens suggesties de kiem vormden waaruit later deze thesis zou groeien. En dan is er uiteraard die andere *boy-scout*, Johan Quaegebeur, altijd bereid onze alma-uurtjes op te luisteren met oeverloze discussies over verheven en soms minder verheven onderwerpen; onvermoeibaar kruisvaarder voor het huwelijk ook—ach dat zal de mid-life crisis wel zijn. Het zou mij verder geenszins verwonderen mocht hij op dit eigenste ogenblik het meest-sluwe-aller-plannen bekokstoven om mijn verdediging een ludieke toets te geven.
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Contents

Chapter 1

Introduction

1.1 About quantum groups

Whenever G is a finite group, the space $K(G)$ of complex valued functions on G is a commutative algebra under pointwise operations. The group structure on G yields a so-called *comultiplication* on $K(G)$, which is an algebra homomorphism $\Delta : K(G) \to K(G) \otimes K(G)$ defined by $(\Delta f)(s,t) = f(st)$ for any $f \in K(G)$ and $s, t \in G$. This makes sense because $K(G) \otimes K(G)$ identifies with the algebra of complex functions on $G \times G$. Now the properties of G as a group can be reformulated in terms of this comultiplication. Thus $K(G)$ becomes a finite dimensional commutative Hopf algebra. The axioms of Hopf algebra theory, however, do not depend on the commutativity of the algebra; dropping the commutativity requirement yields the notion of a finite *quantum* group. Such a quantum group can of course only exist on the algebra level—we shall have to abandon the classical idea of a point-set with some operation upon it.

Let us return to the group case: there is another way to obtain a Hopf algebra from a group G, considering the group algebra $\mathbb{C}G$ rather than the function algebra $K(G)$. Now the Hopf algebras $K(G)$ and $\mathbb{C}G$ are in a sense *dual* to each other. In fact, when G is abelian and $\hat{G}$ denotes its dual group, then $\mathbb{C}G$ is actually isomorphic to $K(\hat{G})$ as a Hopf algebra. Even when G is not abelian, we still have this dual pair of Hopf algebras, $K(G)$ and $\mathbb{C}G$. Thus classical Pontryagin duality for finite abelian groups has been extended to the non-abelian case, provided of course that we allow quantum groups to enter the picture.

Unfortunately the above procedure breaks down in case the group G is infinite. If now $K(G)$ denotes the algebra of functions on G with *finite support*, then we still have the identification of $K(G) \otimes K(G)$ with $K(G \times G)$. However Δf will have infinite support unless f is zero. In other words, the comultiplication on $K(G)$ goes outside the algebraic tensor product. We can get around this problem by allowing Δ to take values in the *multiplier* algebra of $K(G) \otimes K(G)$, the latter being nothing but the algebra of *all* functions on $G \times G$. Moreover, the multipliers in the range of Δ turn out to be 'covered' in the sense that $\Delta(f)(1 \otimes g)$ and $(f \otimes 1)\Delta(g)$ are actually in $K(G) \otimes K(G)$ for any $f, g \in K(G)$.

These observations were the starting point for the theory of multiplier Hopf algebras, as initiated by A. Van Daele [39, 40, 41, 42] and developed further in collaboration with B. Drabant, J. Kustermans and Y. Zhang [7, 8, 19, 44, 45]. This theory generalizes the notion of a Hopf algebra to the non-unital case and succesfully incorporates its motivating example of an infinite discrete group. Moreover it provides a setting for the *self-dual* category of regular multiplier Hopf algebras with integrals [41, 43]. The latter category contains all compact & discrete quantum groups [26, 50] and admits the construction of a quantum double [7] within the same category. In [19, 20, 25] it was shown that the so-called *algebraic* quantum groups, i.e. multiplier Hopf *-algebras with a positive integral, can be lifted to the C^*-algebra level; in other words, they have a C^*-envelope fitting into the framework for locally compact quantum groups recently developed by J. Kustermans and S. Vaes [22, 23, 24].

Although the theory of multiplier Hopf algebras constituted a very substantial extension of ordinary Hopf algebra theory, there were some indications that even the concept of a multiplier Hopf algebra was still susceptible to further generalization. Let us consider e.g. the quantum $E(2)$ group, which is most easily described by a particular dual pair $\langle \mathcal{U}_q, \mathcal{A}_q \rangle$ of Hopf *-algebras [13]. This description however, is not quite satisfying in the sense that it does not admit Haar integrals. On the C^*-algebra level, the quantum $E(2)$ group and its dual were first obtained by S.L. Woronowicz [51, 52]. Then S. Baaj studied the regular representation and constructed the Haar weights [3]. Pontryagin duality was investigated by A. Van Daele and S.L. Woronowicz [46]. Although the operator-algebraic setting is undoubtedly the most natural, it is unfortunately also far more complicated than the Hopf algebra picture. Therefore it would be nice if we could have something in between, having the technical simplicity of an algebraic approach, yet still incorporating important features like Haar integrals and Fourier transforms.

A first step in this direction was taken by H.T. Koelink, who focused on the interplay between quantum groups and q-special functions. In particular q-Bessel functions have emerged naturally in his study of the representation theory of quantum $E(2)$. Also work of L. Vainerman [36] and Vaksman-Korogodskiĭ [48] indicated a strong relationship between q-Bessel functions and the quantum $E(2)$ group. In [13] H.T. Koelink constructed a particular dense subalgebra $\mathcal{H}$ within the algebraic dual of the Hopf algebra $\mathcal{U}_q$. What makes this subalgebra so interesting is the fact that, unlike the Hopf algebra $\mathcal{A}_q$, the algebra $\mathcal{H}$ does admit a Haar integral. Now $\mathcal{H}$ turns out to be neither a Hopf algebra or a multiplier Hopf algebra, the reason being that the comultiplication on $\mathcal{H}$ takes values outside the multiplier algebra of $\mathcal{H} \otimes \mathcal{H}$. It seems that we need a more general framework here.

1.2 About this thesis

Every algebra E acts canonically on its dual E' as follows:

$$\langle x, y \triangleright \omega \rangle = \langle xy, \omega \rangle = \langle y, \omega \triangleleft x \rangle$$

for $x, y \in E$ and $\omega \in E'$. These actions will play a *key-role* throughout the whole thesis. In chapter 2 canonical actions will be used to construct *enveloping algebras* by a procedure quite similar to the construction of a multiplier algebra, the main difference being that we now start from the above actions rather than from multiplication. This yields the notion of an *actor*, or synonymously, a *comultiplier*.

The framework developed in chapter 2 will turn out to be indispensable for chapter 3, in which we shall introduce the notion of a *Hopf System*. The latter is intended to provide a unifying algebraic framework for quantum group *duality*, putting into bigger picture many established notions like e.g. locally compact groups, Hopf algebras [11, 27, 38], multiplier Hopf algebras [40] with reduced dual, the duality theory for multiplier Hopf algebras with integrals [41, 42, 43], algebraic quantum groups [19, 20], etc[1]. Also the pair $\langle \mathcal{U}_q, \mathcal{H} \rangle$ mentioned above will fit in this setting.

Again canonical actions are playing a crucial role in developing the theory, because they offer a convenient alternative for the comultiplications used in Hopf algebra theory. In fact comultiplications become almost *obsolete* within our setting, although we will of course consider them from time to time in order to improve the link with Hopf algebra theory.

Maybe the most interesting result in chapter 3 is theorem 3.8.4, which yields a complete though particularly pleasant *characterization* of regular multiplier Hopf algebras with integrals among arbitrary Hopf systems.

One of the most celebrated results in Hopf algebra theory towards applications is probably the so-called *quantum double* construction of Drinfel'd. At the end of chapter 3 we shall construct a quantum double within our category of Hopf Systems; actually the latter category could vaguely be described as the *largest* category admitting a quantum double construction on a purely algebraic level.

Chapter 4 introduces an axiomatic algebraic framework for harmonic analysis, i.e. the study of group duality, Fourier transformation, Plancherel formulas, etc. Also here canonical actions are at the very heart of the theory; in fact the notion of a *Fourier transform* will be defined in terms of these actions.

Eventually chapters 5 and 6 are dealing with the quantum $E(2)$ group, being a concrete example that fits into the framework of chapter 4. The quantum $E(2)$ group is a quantum deformation of the group of Euclidean motions of the plane. In particular we shall explicitly compute the canonical actions for this example; once again they seem to be the main issue here. We believe that it may be interesting to go into a little more detail on this subject:

[1] In this respect we want to draw the reader's attention to appendix D, which contains a schematic overview.

1.3 Harmonic analysis on quantum $E(2)$

On the Hopf *-algebra level the quantum $E(2)$ group can be described by a dual pair $\langle \mathcal{U}_q, \mathcal{A}_q \rangle$ where $\mathcal{A}_q$ should be thought of as a quantized function algebra, whereas $\mathcal{U}_q$ is a quantized universal enveloping algebra of a Lie algebra. Here $\mathcal{U}_q$ is generated by a self-adjoint invertible element a and a normal element b, satisfying the commutation rule $ab = q\, ba$ for some number q with $0 < q < 1$. Similarly $\mathcal{A}_q$ is generated by a unitary α and a normal element β with $\alpha\beta = q\,\beta\alpha$. The comultiplications on $\mathcal{U}_q$ and $\mathcal{A}_q$ are given by

$$\begin{aligned} &\Delta(a) = a \otimes a && \Delta(\alpha) = \alpha \otimes \alpha \\ &\Delta(b) = a \otimes b + b \otimes a^{-1} && \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^{-1} \end{aligned}$$

and eventually the duality is given by

$$\left\langle a^p b^r c^s,\, \alpha^l \beta^m \gamma^n \right\rangle \;=\; \delta_{r,m}\, \delta_{s,n}\, q^{\frac{1}{2}p(l+m-n)}\, q^{\frac{1}{2}l(m+n)}\, [m]_q!\, [n]_q!$$

where $c = b^*$ and $\gamma = -q^{-1}\beta^*$ and $[\,\cdot\,]_q!$ denotes a q^2-factorial (cf. appendix C). The problem with these two Hopf algebras is that they do not admit Haar functionals: indeed on this level we are dealing with *polynomial* functions in the generators, whereas to have Haar integrals we would rather need something like functions *tending to zero* at infinity. The latter however involve a more sophisticated functional calculus, which can be obtained in several ways. One way is to construct a representation on Hilbert space and use operator theory [37, 46, 51, 52, 53]. Another possibility is to use a holomorphic calculus based on *power series* [12, 13]. We shall follow the second approach to give a precise meaning to expressions like

$$\begin{array}{ccc} f(\ln a)\, g(b^*b)\, b^m & \text{Fourier} & \alpha^k \gamma^n\, h(\gamma^*\gamma) \\ & \leftrightarrows & \\ f(\ln a)\, g(b^*b)\, c^m & \text{transforms} & \alpha^k (\gamma^*)^n\, h(\gamma^*\gamma) \end{array}$$

Diagram Functions in the generators and Fourier transforms between them. (1.1)

where f, g and h run through suitable function spaces. On such elements the left Haar integrals are given by

$$\begin{aligned} \varphi\big(f(\ln a)\, g(b^*b)\, b^m\big) &= \delta_{m,0} \sum_{\substack{k,l\,\in\,\mathbb{Z} \\ k-l \text{ even}}} f(k\theta)\, g(\tau q^l)\, q^{k+l} \\ \omega\big(\alpha^k \gamma^n\, h(\gamma^*\gamma)\big) &= \delta_{k,0}\, \delta_{n,0} \sum_{j\in\mathbb{Z}} h(\nu q^{2j})\, q^{2j} \end{aligned}$$

provided of course that f, g and h satisfy appropriate summability conditions. Here τ and ν are arbitrary positive numbers and $\theta = -\frac{1}{2}\ln q$. These Haar integrals then turn out to be positive, faithful, KMS, etc.

Our main objective now, is to construct the *Fourier transforms* which transform elements in the left column of diagram (1.1) into linear combinations of elements in the right column and vice versa. Although the following formula is not very precise[2], it may give a good idea of how such a Fourier transform looks like:

$$f(\ln a)\, g(b^*b)\, b^m$$
$$\downarrow$$
$$\sum_{k\in\mathbb{Z}} (-1)^m q^{-m} q^{\frac{1}{2}m(k-1)} (q^{-1}-q)^m\, f(k\theta)\, \alpha^{k+m}\, \gamma^m\, h_{m,k}(\gamma^*\gamma).$$

Here $h_{m,k}$ is essentially an m-th order q-Hankel transform of g. Explicitly we have

$$h_{m,k}(\nu q^{2r}) \;=\; \sum_{n\in\mathbb{Z}} q^{2n}\, q^{m(n-r)}\, J_m(q^{n+r};q^2)\, g(\tau q^{2n+k})$$

with $J_m(\,\cdot\,;q^2)$ being the little q^2-Bessel function of order m, given by

$$J_m(z;q^2) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k(k-1)}\, q^{2k}\, (q^{2(k+m+1)};q^2)_\infty}{(q^2;q^2)_\infty\, (q^2;q^2)_k}\, z^{2k+m}$$

where $(x;q^2)_k = \prod_{j=0}^{k-1}(1-q^{2j}x)$ is some q-shifted factorial. It is instructive to observe the analogy with classical Hankel transforms, the latter being defined as integral transforms with Bessel function kernels.

So basically this quantum $E(2)$ Fourier transform amounts to a q-Hankel transformation between the functions g and h, which is not completely surprising because Hankel transformation typically appears when Fourier transformation on the plane is expressed in *polar* coordinates, whereas our functional calculus in (b,c) indeed involves some kind of polar decomposition.
Furthermore it turns out that we can only obtain a Fourier transform when the 'dilation' parameters τ and ν are related in a specific way, e.g.

$$\tau = q^{-1} \qquad\qquad \text{and} \qquad\qquad \nu = (q^{-1}-q)^{-2}$$

which is quite remarkable. Once these Fourier transforms are established, we shall prove the *Plancherel* formula. At this point it becomes essential to use that peculiar summation range appearing in the formula for the Haar functional φ, otherwise the Plancherel formula would simply fail. Roughly speaking this means that $f\otimes g$ actually lives on the set

$$\left\{\, (k\theta, \tau q^l) \;\middle|\; k,l\in\mathbb{Z} \text{ with } k-l \text{ even} \,\right\}$$

which reminds us of the *spectral conditions* in the C*-algebraic approach [51]. It also means that it will be convenient to unify f and g into a single object, i.e. to consider $f\otimes g$ as one function in two variables rather than two functions in one variable.

[2] The formula is not completely exact only in the sense that it does not acknowledge the *spectral conditions* and their implications for the summability conditions on f and g.

Eventually we shall use our formulas for the Fourier transforms to calculate the *duality* between functions in the generators of $\mathcal{U}_q$ and $\mathcal{A}_q$ respectively, e.g.

$$\begin{aligned}
&\Big\langle f(\ln a)\, g(b^* b)\, b^m,\, \alpha^l (\gamma^*)^n\, h(\gamma^*\gamma)\Big\rangle \\
&\quad = \delta_{m,n}\, (-1)^m\, q^{-m}\, q^{-\frac{1}{2}m(m+l+1)}\, (q^{-1}-q)^m\, \nu^m\, q^{ml} \ldots \\
&\qquad \ldots \sum_{r,k\in\mathbb{Z}} q^{(m+2)(k+r)}\, J_m(q^{k+r};q^2)\, f(-m\theta - l\theta)\, g(\tau\, q^{2k-m-l})\, h(\nu\, q^{2r+2l}).
\end{aligned}$$

1.4 Conventions

The reader should notice that assumptions are often fixed and kept in force throughout a whole chapter, section or paragraph.

Linear spaces, duality Linear always means $\mathbb{C}$-linear. We use $\otimes$ to denote the algebraic tensor product of linear spaces over $\mathbb{C}$. Whenever V and W are linear spaces, the flip map from $V\otimes W$ onto $W\otimes V$ shall be denoted by χ, hence $\chi(v\otimes w) = w\otimes v$. We always identify $V\otimes\mathbb{C}$ and $\mathbb{C}\otimes V$ with V. The space of all linear functionals on V is denoted by V', the canonical pairing by $\langle\cdot,\cdot\rangle$. In fact most pairings will be denoted this way. We sometimes speak about a 'vector space duality' instead of a 'pairing'. We will almost exclusively deal with *non-degenerate* pairings. Furthermore we shall not be rigorous concerning the *order* in which to write a pairing $\langle\cdot,\cdot\rangle$ between two spaces V and W, in the sense that the pairing of an element $v\in V$ with an element $w\in W$ may be denoted by $\langle v,w\rangle$ or $\langle w,v\rangle$ interchangeably. A triplet $(V,W,\langle\cdot,\cdot\rangle)$ will often be written in shorthand form as $\langle V,W\rangle$. We use the superscript $^\tau$ to denote the (algebraic) transpose of a linear map.

Algebras By an algebra we mean an associative algebra over $\mathbb{C}$, not necessarily having an identity. Multiplication in an algebra E will often be considered as a linear map $m\equiv m_E : E\otimes E\to E : x\otimes y\mapsto xy$. The product in E is said to be *non-degenerate* whenever

$$xE=\{0\} \text{ implies } x=0 \qquad \text{and} \qquad Ex=\{0\} \text{ implies } x=0 \qquad (x\in E).$$

The *opposite* product on E is defined as $m^{\rm op} = m\chi$. The resulting algebra will be denoted by $E^{\rm op}$.

Modules Let E be any algebra. A linear space Ω is a left E-module when it is endowed with a linear map $\mu : E\otimes\Omega\to\Omega$ satisfying $\mu(m\otimes \mathrm{id}) = \mu(\mathrm{id}\otimes\mu)$. We denote $\mu(x\otimes\omega)$ by $x\triangleright\omega$ or sometimes $x\omega$. Right E-modules are defined similarly. If Ω is both a left and right E-module such that $x\triangleright(\omega\triangleleft y) = (x\triangleright\omega)\triangleleft y$ for all $x,y\in E$ and $\omega\in\Omega$, then Ω is said to be an E-bimodule.

Unital modules, reduction A left E-module Ω is said to be *unital* if its structure map μ is surjective. This replaces the usual condition $1 \triangleright \omega = \omega$ for left modules over algebras which do have a unit. We shall write $E \triangleright \Omega$ to denote the range of μ. An E-bimodule Ω is called unital if $E \triangleright \Omega = \Omega = \Omega \triangleleft E$.
Given any E-bimodule Ω, one can pass to the subspace $\Omega_0 = E \triangleright \Omega \triangleleft E$, which is obviously a sub-E-bimodule of Ω. We say that Ω_0 is the *reduction* of Ω as an E-bimodule. Notice that if m_E is surjective, i.e. $E^2 = E$, then Ω_0 is unital. On the other hand, Ω itself is unital if and only if $\Omega = \Omega_0$.

Locally convex spaces We are mainly dealing with weak and inductive topologies [10, 18, 31, 35]. We use the symbol σ when referring to some weak topology. The prefix 'locally convex' will be abbreviated by LC.

Entire functions Let $H(\mathbb{C})$ denote the *-algebra of entire functions, with pointwise multiplication and *-operation $\tilde{}$ defined by $\tilde{g}(z) = \overline{g(\overline{z})}$. We shall use $\mu_n(g)$ to denote the n-th coefficient of the Taylor series expansion of an entire function g at the origin. So whenever $g \in H(\mathbb{C})$ we have $g(z) = \sum_{n=0}^{\infty} \mu_n(g)\, z^n$ for all $z \in \mathbb{C}$. This yields, for every $n \in \mathbb{N}$, a linear functional μ_n on $H(\mathbb{C})$.

Chapter 2

Enveloping algebras

2.1 Actor implementations of an algebra

Abstract 2.1 We introduce the concept of an *actor context* together with some basic examples, and define the notion of an *actor*, which is intended to be an abstract generalization of the actions of an algebra on its dual. Then we give an alternative description of actor contexts in terms of weak comultiplications. Finally we associate to any actor context an *enveloping algebra* of actors. ⋆

2.1.1 Basic definitions

Canonical actions Every algebra E acts on its dual E' as follows:

$$\langle x, y \triangleright \omega\rangle \;=\; \langle xy, \omega\rangle \;=\; \langle y, \omega \triangleleft x\rangle \tag{2.1}$$

for $x, y \in E$ and $\omega \in E'$. With these actions, E' is clearly an E-bimodule.

Definition 2.1.1.1 Consider an algebra E and an E-bimodule Ω together with a non-degenerate pairing $\langle\cdot,\cdot\rangle : E \times \Omega \to \mathbb{C}$. Endow E' with its canonical E-bimodule structure. The pair $(\Omega, \langle\cdot,\cdot\rangle)$ is said to be an *actor implementation* of E if the canonical embedding $\Omega \to E'$ is an E-bimodule morphism. The triplet $(E; \Omega, \langle\cdot,\cdot\rangle)$ will be called an *actor context.* •

Remarks 2.1.1.2

i. We shall always identify Ω with its image in E', so Ω is in fact a sub-E-bimodule of E'. In other words, the above definition requires Ω to be invariant under the actions defined by (2.1).

ii. We are about to introduce several spaces which 'extend' E in some sense. Nevertheless the pairing will *not* be extended accordingly until proposition 2.4.2.1. So we should be careful in dealing with the pairing: for the moment we can only pair Ω with E and—of course—with Ω'.

iii. Both E and Ω are locally convex spaces, if we endow them with the weak topologies induced by the duality. We will sometimes write E_σ or Ω_σ to emphasize the presence of these weak topologies. The above definition could be reformulated in a topological language, requiring E_σ to be a topological[1] algebra. It follows that Ω_σ is a topological[2] E_σ-bimodule. ⋆

Definition 2.1.1.3 Let $\mathbb{E} \equiv (E;\Omega,\langle\cdot,\cdot\rangle)$ be an actor context, and consider linear maps λ and ρ from Ω into Ω. We say that the pair (λ,ρ) is a *pre-actor* for $\mathbb{E}$ when λ is a right E-module morphism and ρ is a left E-module morphism. If moreover

$$\langle x, \rho(\omega \triangleleft y)\rangle \;=\; \langle y, \lambda(x \triangleright \omega)\rangle \tag{2.2}$$

for all $x, y \in E$ and $\omega \in \Omega$, then (λ,ρ) is said to be an *actor* for $\mathbb{E}$. The set of all actors for $\mathbb{E}$ will be denoted by $\mathrm{Act}(\mathbb{E})$, the set of pre-actors by $\mathrm{Pre}(\mathbb{E})$. •

Remarks 2.1.1.4 i. In the above definition, λ can be thought of as a 'left actor' and ρ as a 'right actor', whereas (2.2) expresses their interrelation. Equation (2.2) will be referred to as the *bi-actor* property.

ii. Obviously $\mathrm{Pre}(\mathbb{E})$ is a linear space under pointwise operations, containing $\mathrm{Act}(\mathbb{E})$ as a subspace. Also notice that $(\mathrm{id}_\Omega, \mathrm{id}_\Omega) \equiv 1_{\mathbb{E}}$ belongs to $\mathrm{Act}(\mathbb{E})$.

iii. If Ω is moreover unital as an E-bimodule, i.e. $E \triangleright \Omega = \Omega = \Omega \triangleleft E$, then for (λ,ρ) to be an actor it is sufficient to require just the bi-actor property; the module properties of λ and ρ will hold automatically. Applying the bi-actor property twice, we get for instance

$$\begin{aligned}
\langle x, \rho(z \triangleright (\omega \triangleleft y))\rangle &= \langle x, \rho((z \triangleright \omega) \triangleleft y)\rangle = \langle y, \lambda(x \triangleright (z \triangleright \omega))\rangle \\
&= \langle y, \lambda(xz \triangleright \omega)\rangle = \langle xz, \rho(\omega \triangleleft y)\rangle \\
&= \langle x, z \triangleright \rho(\omega \triangleleft y)\rangle
\end{aligned}$$

for all $x, y, z \in E$ and $\omega \in \Omega$. Since we assume $\Omega \triangleleft E = \Omega$, it follows that ρ is a left E-module morphism; similarly we can treat λ. ⋆

2.1.2 Introducing some examples

We shall follow some examples throughout the development of the theory; for the moment we focus on the actor *contexts* rather than the actors themselves. Our first example has the benefit of its simplicity, but the disadvantage that some features of the general framework will trivialize here:

Example A 2.1.2.1 (function algebras) Let S be a non-empty set and denote by $\mathbb{C}^S$ the algebra of all functions $f: S \to \mathbb{C}$ with pointwise operations. Now let F be any separating algebra of complex functions on S, i.e. a non-zero subalgebra of $\mathbb{C}^S$ which separates the points in S, in the sense that for every $s, s_0 \in S$ with $s \neq s_0$ there exists a function $f \in F$ such that $f(s_0) = 0$ but $f(s) \neq 0$. Since F is an algebra, it follows that for any finite non-empty subset

[1] multiplication in E is separately continuous w.r.t. the $\sigma(E,\Omega)$ topology.
[2] the module structure maps are separately continuous.

S_0 of S and any $s \in S \setminus S_0$ there is an $f \in F$ such that $f(S_0) = \{0\}$ but $f(s) \neq 0$. Next we consider the free linear space $\mathbb{C}S$ generated by the set S. Its canonical basis will be denoted by $\{\delta_s\}_{s \in S}$. Then there is a pairing between $\mathbb{C}^S$ and $\mathbb{C}S$, given by $\langle f, \delta_s \rangle = f(s)$, and moreover we have $(\mathbb{C}S)' \simeq \mathbb{C}^S$. By restriction we get a non-degenerate pairing between F and $\mathbb{C}S$. Now $\mathbb{C}S$ is a bimodule over $\mathbb{C}^S$, hence also over F, the actions being determined by $f \triangleright \delta_s = f(s)\, \delta_s = \delta_s \triangleleft f$. With these ingredients, the triple $(F; \mathbb{C}S, \langle \cdot, \cdot \rangle)$ is obviously an actor context. Notice that $\mathbb{C}S$ is unital as an F-bimodule.
Let's look at a more concrete example of this nature: take a locally compact Hausdorff space X and consider the algebra $K(X)$ of all continuous functions $f : X \to \mathbb{C}$ with compact support. By Urysohn's lemma, $K(X)$ separates the points in X, hence $(K(X); \mathbb{C}X, \langle \cdot, \cdot \rangle)$ is an actor context. $\star$

Example B 2.1.2.2 (C* and W* algebras) Let A be a C^*-algebra [34] and denote its norm-dual by A^*. Then $(A; A^*, \langle \cdot, \cdot \rangle)$ is an actor context. When M is a W^*-algebra with predual M_*, then $(M; M_*, \langle \cdot, \cdot \rangle)$ is an actor context, since multiplication in a W^*-algebra M is separately $\sigma(M, M_*)$ continuous. In particular also $(A^{**}; A^*, \langle \cdot, \cdot \rangle)$ is an actor context. $\star$

Whenever A is a normed algebra, A is also a normed A-bimodule having a dual Banach A-bimodule A^* [5, §I.9.12 and §I.9.13.i-iv]. Explicitly:

Example B 2.1.2.3 (normed algebras) Whenever A is a normed algebra, $(A; A^*, \langle \cdot, \cdot \rangle)$ is an actor context; moreover we have for $x \in A$ and $\omega \in A^*$ that

$$\|x \triangleright \omega\| \;\leq\; \|x\|\, \|\omega\| \qquad \text{and} \qquad \|\omega \triangleleft x\| \;\leq\; \|x\|\, \|\omega\|. \tag{2.3}$$

Proof. Observe that $|\langle y, x \triangleright \omega \rangle| \;=\; |\langle yx, \omega \rangle| \;\leq\; \|y\|\, \|x\|\, \|\omega\|$ for all $y \in A$. ∎

Of course example A is rather trivial, whereas examples B may seem a little artificial. On the other hand, they do provide a familiar setting to illustrate the notions we shall introduce later on, and this indeed will be their main purpose. In §3 though, the theory of actor contexts as developed here shall be used extensively to study our next example, which is intended to offer a framework for generalizing Hopf algebra duality:

Example C 2.1.2.4 (dual pair of algebras) Let $\langle \cdot, \cdot \rangle$ be a non-degenerate vector space duality between two algebras A and B. Now if both

$$\mathbb{A} \equiv (A; B, \langle \cdot, \cdot \rangle) \qquad \text{and} \qquad \mathbb{B} \equiv (B; A, \langle \cdot, \cdot \rangle)$$

are actor contexts, then $\langle A, B \rangle$ is said to be a *dual pair of algebras.* $\star$

The above merely states that A is invariant under the canonical actions of B, and vice versa. Equivalently one could require A and B to be topological algebras in the weak topologies induced by duality (cf. remark 2.1.1.2.iii). In any case, these are very natural conditions to impose on such a dual pair.

Example D 2.1.2.5 Let E be any algebra with non-degenerate product and let $E^\diamond = E \triangleright E' \triangleleft E$ be the reduction of E' as an E-bimodule (§1.4). We say $E^\diamond$ is the *reduced dual* of E. The pairing $\langle E, E^\diamond \rangle$ is non-degenerate since E is assumed to have non-degenerate product. Now $(E; E^\diamond, \langle \cdot, \cdot \rangle)$ is an actor context.

2.1.3 Weak comultiplications

For a while we will take a 'dual' approach towards actor contexts, introducing the notion of a weak comultiplication. Although less convenient from a technical point of view, we believe this to be helpful in understanding the link with Hopf algebra theory [1, 33, 38, 40]. In this paragraph we will use the weak Fubini tensor product, as defined in appendix A. Usually it will be clear from the context to which particular vector space dualities a $\overline{\otimes}$-tensor product is related. Recall $\overline{\otimes}$ is associative (cf. remark A.2.i), so the following makes sense:

Definition 2.1.3.1 Let $\langle E, \Omega\rangle$ be a non-degenerate vector space duality. Then by a *weak comultiplication* on Ω we mean a weakly continuous linear mapping $\Delta : \Omega \to \Omega \mathbin{\overline{\otimes}} \Omega$ which is *coassociative* in the sense that $(\Delta \mathbin{\overline{\otimes}} \mathrm{id})\Delta = (\mathrm{id} \mathbin{\overline{\otimes}} \Delta)\Delta$ (cf. remarks A.2.i and A.3.iv). Observe that this definition involves the duality between $E \otimes E$ and $\Omega \mathbin{\overline{\otimes}} \Omega$. •

Proposition 2.1.3.2 *Let $\langle E, \Omega\rangle$ be a vector space duality. If $\Delta : \Omega \to \Omega \mathbin{\overline{\otimes}} \Omega$ is a weak comultiplication on Ω, then E can be endowed with an algebra structure satisfying*

$$\langle xy, \omega\rangle = \big\langle x \otimes y,\, \Delta(\omega)\big\rangle \tag{2.4}$$

for all $x, y \in E$ and $\omega \in \Omega$. Moreover Ω becomes an E-bimodule, with actions

$$x \triangleright \omega = (\mathrm{id} \mathbin{\overline{\otimes}} f_x)\Delta(\omega) \qquad \text{and} \qquad \omega \triangleleft x = (f_x \mathbin{\overline{\otimes}} \mathrm{id})\Delta(\omega), \tag{2.5}$$

making $\mathbb{E}_\Delta \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ into an actor context. Here f_x stands for $\langle x, \cdot\,\rangle$.

Proof. Since Δ is by assumption weakly continuous, it has a weak transpose $\Delta^* \equiv m : E \otimes E \to E$, defining the product on E. Associativity of m follows easily from the coassociativity of Δ, hence (E, m) is an algebra. The remaining assertions are immediate from equation (A.1) in appendix A. ∎

Example C 2.1.3.3 A non-degenerate dual pair $\langle A, B\rangle$ of Hopf algebras in the sense of [38] is also a dual pair of algebras in the sense of example 2.1.2.4. ⋆

Proposition 2.1.3.4 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be any actor context. Then (2.4) defines a weak comultiplication Δ on Ω. Clearly $\mathbb{E}_\Delta = \mathbb{E}$, so the relations (2.5) between actions and comultiplication will hold.*

Proof. Since multiplication in E_σ is separately continuous, (2.4) defines a map $\Delta : \Omega \to \Omega \mathbin{\overline{\otimes}} \Omega$ which is easily seen to be a weak comultiplication on Ω. ∎

Comultiplications are used extensively in the theory of Hopf algebras; in that theory however, a comultiplication on a linear space Ω is a map from Ω into $\Omega \otimes \Omega$, whereas our *weak* comultiplications are of a more topological nature, in the sense that they are allowed to 'go outside' the *algebraic* tensor product.

Proposition 2.1.3.5 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be an actor context and let Δ be the induced weak comultiplication on Ω. Then the following are equivalent:*

i. $\Delta(\Omega) \subseteq \Omega \otimes \Omega$,

ii. *multiplication in* E_σ *is jointly continuous.*

Proof. $\Omega \otimes \Omega$ identifies with the space of *jointly* continuous bilinear forms on $E \times E$. (cf. the observations preceding definition A.1 in appendix A). ∎

Definition 2.1.3.6 If the conditions of the above proposition are fulfilled, then $\mathbb{E}$ is said to be an *algebraic* actor context. •

Example A 2.1.3.7 The actor context $(F; \mathbb{C}S, \langle\cdot,\cdot\rangle)$ of example A 2.1.2.1 is algebraic, since Δ on $\mathbb{C}S$ is given by $\Delta(\delta_s) = \delta_s \otimes \delta_s$ for all $s \in S$. ⋆

Proposition 2.1.3.8 *Let* $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ *be an actor context,* $\Delta : \Omega \to \Omega \overline{\otimes} \Omega$ *the induced weak comultiplication and* $(\lambda, \rho) \in \mathrm{Pre}(\mathbb{E})$*. Then are equivalent:*

i. $(\lambda, \rho) \in \mathrm{Act}(\mathbb{E})$*, i.e. the pair* (λ, ρ) *satisfies the bi-actor property (2.2)*

ii. $(\overline{\lambda \otimes} \, \mathrm{id})\Delta = (\mathrm{id} \, \overline{\otimes \rho})\Delta$ *(cf. remark A.3.iii).*

Moreover, if these conditions are fulfilled, then

$$(\overline{\lambda \otimes} \, \mathrm{id})\Delta(\Omega) \; = \; (\mathrm{id} \, \overline{\otimes \rho})\Delta(\Omega) \; \subseteq \; \Omega \overline{\otimes} \Omega.$$

Proof. According to the definition of a slice map like $\overline{\lambda \otimes} \, \mathrm{id} : \Omega \overline{\otimes} \Omega \to \overline{\Omega \otimes \Omega}$, we get for all $x, y \in E$ and $\omega \in \Omega$ that

$$\langle y \otimes x, (\overline{\lambda \otimes} \, \mathrm{id})\Delta(\omega)\rangle \overset{(A.3)}{=} \langle y, \lambda(\mathrm{id} \overline{\otimes} f_x)\Delta(\omega)\rangle \overset{(2.5)}{=} \langle y, \lambda(x \triangleright \omega)\rangle.$$

Similarly we get $\langle y \otimes x, (\mathrm{id} \, \overline{\otimes \rho})\Delta(\omega)\rangle = \langle x, \rho(\omega \triangleleft y)\rangle$. The result follows. ∎

Henceforth the emphasis will be on *module* structures rather than comultiplications; as explicit objects the latter are quite redundant, and we will avoid them in developing our theory of actor contexts. Nevertheless it may be instructive to interpret things in terms of the comultiplication from time to time, especially when one wants to appreciate the relationship with Hopf algebra theory [1, 33] or its generalizations [40].

2.1.4 Enveloping algebras

In the following, $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ will be an actor context.

Lemma 2.1.4.1 *Given* $x \in E$ *we define maps* $\lambda_x, \rho_x : \Omega \to \Omega$ *by* $\lambda_x(\omega) = x \triangleright \omega$ *and* $\rho_x(\omega) = \omega \triangleleft x$*. Then* $(\lambda_x, \rho_x) \in \mathrm{Act}(\mathbb{E})$*.*

Proof. The pair (λ_x, ρ_x) is clearly a pre-actor for $\mathbb{E}$, because Ω is an E-bimodule. The bi-actor property (2.2) follows from associativity in E. ∎

Definition 2.1.4.2 Consider two elements $x_1 \equiv (\lambda_1, \rho_1)$ and $x_2 \equiv (\lambda_2, \rho_2)$ in $\mathrm{Pre}(\mathbb{E})$. Clearly $(\lambda_1\lambda_2, \rho_2\rho_1)$ is again a pre-actor for $\mathbb{E}$, which will be denoted by x_1x_2. This defines a product that makes $\mathrm{Pre}(\mathbb{E})$ into a unital algebra. •

Unfortunately the bi-actor property (2.2) is not stable under this operation. In other words, $\mathrm{Act}(\mathbb{E})$ is in general *not* a subalgebra of $\mathrm{Pre}(\mathbb{E})$.

Example B 2.1.4.3 Let G be any infinite discrete abelian group, and consider its group algebra $A = L^1(G)$. Since $L^1(G)$ is a Banach algebra, we have an actor context $\mathbb{A} \equiv (L^1(G); L^1(G)^*, \langle \cdot, \cdot \rangle)$ as in example 2.1.2.3. We claim that in this case $\mathrm{Act}(\mathbb{A})$ is not closed under multiplication. To show this assertion, however, we first need to develop our theory; therefore the actual proof will be deferred until §2.4.4. For the moment we only mention that this is related to the fact that $L^1(G)$ is not *Arens regular* when G is infinite, discrete and abelian [6]. ⋆

Lemma 2.1.4.1 allows us to define a mapping $j : E \to \mathrm{Pre}(\mathbb{E}) : x \mapsto (\lambda_x, \rho_x)$ which is obviously an algebra morphism; moreover $j(E) \subseteq \mathrm{Act}(\mathbb{E})$. Let's investigate whether j is injective:

Lemma 2.1.4.4 *Let $\mathbb{E}$ and j be as above. The following are equivalent:*

i. The spaces $E \triangleright \Omega$ and $\Omega \triangleleft E$ are weakly dense in Ω.

ii. The product in E is non-degenerate.

iii. $(E; \Omega_0, \langle \cdot, \cdot \rangle)$ is again an actor context, Ω_0 being the reduction of Ω (§1.4). In other words, the pairing $\langle E, \Omega_0 \rangle$ is still non-degenerate.

If these assertions hold, then $\mathbb{E}$ is said to be non-degenerate, *and j is injective.*

The proof is easy. Notice that if Ω is unital as an E-bimodule, then a fortiori $\mathbb{E}$ is non-degenerate. When $\mathbb{E}$ is non-degenerate, we can (and will) identify E with its image in $\mathrm{Pre}(\mathbb{E})$, so that E is in fact a subalgebra of $\mathrm{Pre}(\mathbb{E})$.
Now $\mathrm{Pre}(\mathbb{E})$ is an algebra, hence it is a $\mathrm{Pre}(\mathbb{E})$-bimodule under multiplication; since $j : E \to \mathrm{Pre}(\mathbb{E})$ is an algebra morphism, $\mathrm{Pre}(\mathbb{E})$ is as well an E-bimodule, the actions of $z \in E$ on a pre-actor $a \equiv (\lambda, \rho) \in \mathrm{Pre}(\mathbb{E})$ being given by

$$za = j(z)\, a = (\lambda_z \lambda, \rho\rho_z) \qquad \text{and} \qquad az = a\, j(z) = (\lambda\lambda_z, \rho_z \rho).$$

Lemma 2.1.4.5 $\mathrm{Act}(\mathbb{E})$ *is a sub-E-bimodule of* $\mathrm{Pre}(\mathbb{E})$.

Proof. Take $z \in E$ and $a \equiv (\lambda, \rho) \in \mathrm{Act}(\mathbb{E})$. We have to show that za and az, as defined above, are again in $\mathrm{Act}(\mathbb{E})$. Now the pair $(\lambda_z \lambda, \rho\rho_z) = za$ indeed satisfies the bi-actor property: for every $x, y \in E$ and $\omega \in \Omega$ we have

$$\begin{aligned}
\langle x, (\rho\rho_z)(\omega \triangleleft y) \rangle &= \langle x, \rho((\omega \triangleleft y) \triangleleft z) \rangle = \langle x, \rho(\omega \triangleleft yz) \rangle \\
&= \langle yz, \lambda(x \triangleright \omega) \rangle = \langle y, z \triangleright \lambda(x \triangleright \omega) \rangle \\
&= \langle y, (\lambda_z \lambda)(x \triangleright \omega) \rangle
\end{aligned}$$

hence $za \in \mathrm{Act}(\mathbb{E})$. Analogously we prove that $az \in \mathrm{Act}(\mathbb{E})$. ∎

Definition 2.1.4.6 Let $\mathbb{E}$ be any actor context and A a subspace of $\mathrm{Pre}(\mathbb{E})$. Denote

$$\mathrm{Env}(\mathbb{E}\,;A) \;=\; \big\{x \in \mathrm{Pre}(\mathbb{E}) \;\big|\; xA \subseteq A \supseteq Ax\big\}\,.$$

Given $\mathbb{E}$, a natural choice for A would be $A = \mathrm{Act}(\mathbb{E})$, and since this is by far the most important case, we will abbreviate

$$\mathrm{Env}(\mathbb{E}) \;\equiv\; \mathrm{Env}\,\big(\mathbb{E}\,;\mathrm{Act}(\mathbb{E})\big)\,.$$

Remark 2.1.4.7 Notice that if $1_{\mathbb{E}} \in A$, then $\mathrm{Env}(\mathbb{E}\,;A) \subseteq A$. In particular we get $\mathrm{Env}(\mathbb{E}) \subseteq \mathrm{Act}(\mathbb{E})$. ⋆

Lemma 2.1.4.8 *Let $\mathbb{E}$ be any actor context and A a linear subspace of* $\mathrm{Pre}(\mathbb{E})$*. Then* $\mathrm{Env}(\mathbb{E}\,;A)$ *is a unital subalgebra of* $\mathrm{Pre}(\mathbb{E})$*.*

Proof. Take arbitrary $x, y \in \mathrm{Env}(\mathbb{E}\,;A)$. Then by assumption xA, Ax, yA and Ay are all contained in A. For all $a \in A$ we have $(xy)a = x(ya) \in xA \subseteq A$, hence $(xy)A \subseteq A$. Analogously $A(xy) \subseteq Ay \subseteq A$, and we conclude that also xy belongs to $\mathrm{Env}(\mathbb{E}\,;A)$. Obviously $1_{\mathbb{E}} \in \mathrm{Env}(\mathbb{E}\,;A)$. ∎

Proposition 2.1.4.9 *Let $\mathbb{E} \equiv (E;\Omega,\langle\cdot,\cdot\rangle)$ be a non-degenerate actor context and A a sub-E-bimodule of* $\mathrm{Pre}(\mathbb{E})$ *w.r.t. multiplication. Then* $\mathrm{Env}(\mathbb{E}\,;A)$ *is a unital algebra containing E as a subalgebra.*

Proof. We already know that both E and $\mathrm{Env}(\mathbb{E}\,;A)$ are subalgebras of $\mathrm{Pre}(\mathbb{E})$. By assumption we have $EA \subseteq A \supseteq AE$, and hence $E \subseteq \mathrm{Env}(\mathbb{E}\,;A)$. ∎

Corollary 2.1.4.10 *When $\mathbb{E} \equiv (E;\Omega,\langle\cdot,\cdot\rangle)$ is a non-degenerate actor context, then* $\mathrm{Env}(\mathbb{E})$ *is a unital algebra of actors, containing E as a subalgebra, and*

$$E \;\simeq\; j(E) \;\subseteq\; \mathrm{Env}(\mathbb{E}) \;\subseteq\; \mathrm{Act}(\mathbb{E}) \;\subseteq\; \mathrm{Pre}(\mathbb{E}).$$

Proof. Because of lemma 2.1.4.5 we can apply the above with $A = \mathrm{Act}(\mathbb{E})$. ∎

Example A 2.1.4.11 *Recall the actor context $\mathbb{F} \equiv (F;\mathbb{C}S,\langle\cdot,\cdot\rangle)$ of example* A 2.1.2.1*. We now have* $\mathrm{Act}(\mathbb{F}) = \mathrm{Env}(\mathbb{F}) \simeq \mathbb{C}^S$*.*

For the proof we refer to §2.9.1, where we shall study the *commutative* case in general. Notice that, starting from any separating algebra of functions on S, we are able to obtain the algebra of *all* functions on S, by a process which only refers to the underlying set S through the actor context $\mathbb{F}$.

Example B 2.1.4.12 Take $\mathbb{A} \equiv (A;A^*,\langle\cdot,\cdot\rangle)$ as in example 2.1.2.2. A theorem in operator algebra [34, §III.2] states that A^{**} is a W^*-algebra containing A as a subalgebra. It is called the universal enveloping von Neumann algebra of A. Now also $(A^{**};A^*,\langle\cdot,\cdot\rangle)$ is a non-degenerate actor context, hence every $a \in A^{**}$ identifies with an actor (λ_a,ρ_a) for the latter context, which is *a fortiori* an actor for $\mathbb{A}$. In this way we have embedded A^{**} into $\mathrm{Act}(\mathbb{A})$. ⋆

2.2 The duality between actors and multipliers

Abstract 2.2 We investigate the relation between actors and multipliers. It turns out that every multiplier yields an actor, and thus the multiplier algebra can and will be embedded in the enveloping algebra that was introduced in the previous section. However the latter will usually be much larger than the multiplier algebra. ⋆

Definition 2.2.1 If $\mathbb{E} \equiv (E;\Omega,\langle\cdot,\cdot\rangle)$ is a non-degenerate actor context, we write

$$M(\mathbb{E}) \equiv \mathrm{Env}(\mathbb{E}\,;E) = \bigl\{x \in \mathrm{Pre}(\mathbb{E}) \bigm| xE \subseteq E \supseteq Ex\bigr\}.$$

Recall that E is identified with its image $j(E)$ in $\mathrm{Pre}(\mathbb{E})$. •

Our goal is to prove that $M(\mathbb{E})$ is actually—as one expects—isomorphic to the multiplier[3] algebra $M(E)$ of E. To do so, we need an extra assumption: we will require Ω to be unital as an E-bimodule, i.e. $E \triangleright \Omega = \Omega = \Omega \triangleleft E$. Notice that by lemma 2.1.4.4 this condition implies non-degeneracy of the product in E, so we can indeed consider the multiplier algebra of E. Although our actual approach will be different, it's worth noticing that half of the assertion is already known:

Remark 2.2.2 *Let $\mathbb{E} \equiv (E;\Omega,\langle\cdot,\cdot\rangle)$ be an actor context and let Ω be unital. Then $M(\mathbb{E})$ is a unital algebra containing E as an essential two-sided ideal.*

Proof. Apply proposition 2.1.4.9 with $A = j(E)$. Clearly E is a two-sided ideal in $M(\mathbb{E})$, so let's complete this proof showing that it is an *essential* ideal: take e.g. a pre-actor $a \equiv (\lambda,\rho) \in M(\mathbb{E})$ such that $ax = 0$, i.e. $(\lambda\lambda_x, \rho_x\rho) = 0$, for all $x \in E$. Then for all $x, y \in E$ and $\omega \in \Omega$ we have $\lambda(x \triangleright \omega) = \lambda(\lambda_x(\omega)) = 0$ and $\langle x, \rho(y \triangleright \omega)\rangle = \langle x, y \triangleright \rho(\omega)\rangle = \langle xy, \rho(\omega)\rangle = \langle y, \rho_x(\rho(\omega))\rangle = 0$. Using $E \triangleright \Omega = \Omega$ we conclude that both λ and ρ are zero, hence $a = 0$. ■

The other half of our assertion would be the statement that $M(\mathbb{E})$ is in fact the *largest* algebra having this property. Our actual approach however, will be more explicit: we simply construct a natural isomorphism from $M(E)$ onto $M(\mathbb{E})$.

Lemma 2.2.3 *Let $\mathbb{E} \equiv (E;\Omega,\langle\cdot,\cdot\rangle)$ be an actor context such that Ω is unital as an E-bimodule. Then the embedding $j : E \to \mathrm{Pre}(\mathbb{E})$ extends uniquely to a map*

$$\hat{\jmath} : M(E) \to \mathrm{Pre}(\mathbb{E}) : z \mapsto \hat{\jmath}(z) \equiv (\lambda_z, \rho_z)$$

such that the following holds for all $y \in E$, $z \in M(E)$ and $\omega \in \Omega$:

$$\bigl\langle y,\, \lambda_z(\omega)\bigr\rangle \;=\; \langle yz, \omega\rangle \qquad \text{and} \qquad \bigl\langle y,\, \rho_z(\omega)\bigr\rangle \;=\; \langle zy, \omega\rangle. \tag{2.6}$$

Moreover $\hat{\jmath}$ is still an injective algebra morphism.

Proof. Let $z \equiv (L, R)$ be a multiplier of E. Here L and R are linear maps from E into E, having (algebraic) transpositions $L^\tau, R^\tau : E' \to E'$. Now take any $y \in E$ and $\omega \in \Omega$. Identifying Ω with its image in E', we get for all $x \in E$ that

$$\bigl\langle x,\, L^\tau(\omega \triangleleft y)\bigr\rangle = \bigl\langle L(x),\, \omega \triangleleft y\bigr\rangle = \bigl\langle y(zx),\, \omega\bigr\rangle = \bigl\langle (yz)x,\, \omega\bigr\rangle = \bigl\langle x,\, \omega \triangleleft yz\bigr\rangle \tag{2.7}$$

[3]For the theory of multipliers we refer to [40].

hence $L^\tau(\omega \triangleleft y) = \omega \triangleleft yz \in \Omega$. It follows that $L^\tau(\Omega) = L^\tau(\Omega \triangleleft E) \subseteq \Omega$, and analogously $R^\tau(\Omega) \subseteq \Omega$. So we can define linear maps λ_z and ρ_z from Ω into Ω by $\lambda_z(\omega) = R^\tau(\omega)$ and $\rho_z(\omega) = L^\tau(\omega)$, clearly satisfying (2.6). When z happens to be in E itself, this notation is of course compatible with the one we introduced in lemma 2.1.4.1. Now we claim that λ_z is a right E-module morphism; indeed, using (2.6) we get for all $\omega \in \Omega$ and $x, y \in E$ that

$$\big\langle x,\, \lambda_z(\omega \triangleleft y)\big\rangle = \big\langle xz,\, \omega \triangleleft y\big\rangle = \big\langle yxz,\, \omega\big\rangle = \big\langle yx,\, \lambda_z(\omega)\big\rangle = \big\langle x,\, \lambda_z(\omega) \triangleleft y\big\rangle\,.$$

Similarly ρ_z is a left E-module morphism. Hence for every $z \in M(E)$ we have a pre-actor $\hat{\jmath}(z) \equiv (\lambda_z, \rho_z) \in \mathrm{Pre}(\mathbb{E})$ obeying (2.6). The assertion that $\hat{\jmath}$ is an algebra morphism is an easy consequence of (2.6) and associativity in $M(E)$. Injectivity and uniqueness of $\hat{\jmath}$ are obvious. ∎

Proposition 2.2.4 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be an actor context, Ω unital as an E-bimodule, and $\hat{\jmath}$ as above. Then $M(\mathbb{E}) = \hat{\jmath}(M(E)) \subseteq \mathrm{Env}(\mathbb{E}) \subseteq \mathrm{Act}(\mathbb{E})$.*

Proof. We first show that $\mathrm{Act}(\mathbb{E})$ is a sub-$M(E)$-bimodule of $\mathrm{Pre}(\mathbb{E})$, which is quite similar to lemma 2.1.4.5, but now considering $M(E)$ instead of E. So let us take any $z \in M(E)$ and $a \equiv (\lambda, \rho) \in \mathrm{Act}(\mathbb{E})$ and prove that $\hat{\jmath}(z)a$ and $a\hat{\jmath}(z)$ are in $\mathrm{Act}(\mathbb{E})$ again. The pair $(\lambda_z\lambda, \rho\rho_z) = \hat{\jmath}(z)a$ indeed satisfies (2.2), since

$$\big\langle x,\, \rho(\rho_z(\omega \triangleleft y))\big\rangle \overset{(2.7)}{=} \big\langle x,\, \rho(\omega \triangleleft yz)\big\rangle \overset{(2.2)}{=} \big\langle yz,\, \lambda(x \triangleright \omega)\big\rangle \overset{(2.6)}{=} \big\langle y,\, \lambda_z(\lambda(x \triangleright \omega))\big\rangle$$

for all $x, y \in E$ and $\omega \in \Omega$. Here we used $\rho_z(\omega \triangleleft y) = \omega \triangleleft yz$ obtained in (2.7). Similarly for $a\hat{\jmath}(z)$. We conclude that $\hat{\jmath}(M(E)) \subseteq \mathrm{Env}(\mathbb{E})$.
Now we show $M(\mathbb{E}) = \hat{\jmath}(M(E))$. First take any $a \in M(\mathbb{E})$, i.e. let $a \equiv (\lambda, \rho)$ be a pre-actor with $aj(E) \subseteq j(E) \supseteq j(E)a$. Then one can construct linear maps $L, R : E \to E$ such that $j(L(x)) = aj(x)$ and $j(R(x)) = j(x)a$ for any $x \in E$, hence $\lambda_{L(x)} = \lambda\lambda_x$ etc. Now $z \equiv (L, R)$ is a multiplier of E, and for all $x \in E$ we get $\lambda_z\lambda_x = \lambda_{zx} = \lambda_{L(x)} = \lambda\lambda_x$, hence λ_z equals λ on $E \triangleright \Omega = \Omega$. Similarly also $\rho_z = \rho$. We conclude that $a = \hat{\jmath}(z)$. The other inclusion is obvious. ∎

Remarks 2.2.5 i. The above allows us to identify $M(E)$ with a subalgebra $M(\mathbb{E})$ of $\mathrm{Env}(\mathbb{E})$ provided that Ω is unital as an E-bimodule. Moreover (2.6) exhibits a duality between actors and multipliers in the sense that every multiplier can be 'dualized' into an actor; however not every actor for $\mathbb{E}$ need to arise in this way, so $\mathrm{Env}(\mathbb{E})$ is usually larger than $M(E)$. Also notice that $\mathrm{Env}(\mathbb{E})$ depends on the actor implementation, whereas $M(E)$ is completely determined by the algebra E itself. Furthermore we loose a substantial amount of 'affiliation' with E, as shown by example A.

ii. A one-line summary of our results thus far:

$$E \subseteq M(E) \simeq M(\mathbb{E}) \equiv \mathrm{Env}(\mathbb{E}; E) \subseteq \mathrm{Env}(\mathbb{E}) \equiv \mathrm{Env}(\mathbb{E}; \mathrm{Act}(\mathbb{E})) \subseteq \mathrm{Act}(\mathbb{E}).$$

iii. A priori $M(\mathbb{E})$ was merely a set of *pre*-actors satisfying some invariance condition; then it *turned out* to be really a space of *actors*. ⋆

2.3 Actors & functionals

Throughout this paragraph $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ is an actor context.

Abstract 2.3 For a while we adopt a rather topological point of view, but as we proceed, we shall gradually link this to the original algebraic approach. We show how certain linear functionals on Ω naturally induce actors. Eventually we investigate a notion of *invertibility* for actors and—indirectly—for functionals. ⋆

2.3.1 Three natural compatible topologies

Definition 2.3.1.1 A locally convex topology on Ω is said to be $\mathbb{E}$-*compatible* if it is finer than the $\sigma(\Omega, E)$ topology (hence Hausdorff) and making Ω into a *topological* E_σ-bimodule, i.e. the structure maps are separately continuous. •

Definition 2.3.1.2 We introduce the following topologies on Ω:

strict$^\flat$ is synonymous for the *weak* topology on Ω, i.e. the $\sigma(\Omega, E)$ topology.

strict$^\natural$ Consider $x \triangleright (\cdot)$ and $(\cdot) \triangleleft x$, with x running through E, as a family of linear maps from Ω_σ into Ω. These maps induce an inductive LC topology [10, §6.6] on Ω, which will be referred to as the *strict*$^\natural$ topology on Ω. In other words, the strict$^\natural$ topology is the *finest* locally convex topology $\mathcal{I}$ on Ω such that the above mappings are continuous from Ω_σ into $(\Omega, \mathcal{I})$.

strict$^\sharp$ Consider $\omega \triangleleft (\cdot)$ and $(\cdot) \triangleright \omega$, with ω running through Ω, as a family of linear maps from E_σ into Ω. The inductive LC topology on Ω defined by these mappings will be called the *strict*$^\sharp$ topology on Ω.

Whenever Ω is to be considered with one of these topologies, we shall write $\Omega_\flat \equiv \Omega_\sigma$, $\Omega_\natural$ and $\Omega_\sharp$ respectively; their topological duals will be denoted $\Omega^\flat$, $\Omega^\natural$ and $\Omega^\sharp$. So a *sub*script following Ω merely indicates which topology it is endowed with; on the other hand, if Ω is accompanied by a *super*script, we are always referring to some subspace of the algebraic dual Ω'. •

Remarks 2.3.1.3 i. If (λ, ρ) is an actor for $\mathbb{E}$, it follows that λ and ρ are *continuous* mappings from $\Omega_\sharp$ into $\Omega_\flat$.

ii. Let Δ be the weak comultiplication on Ω (proposition 2.1.3.4). Then

$$\begin{aligned}\Omega^\sharp &= \left\{ f \in \Omega' \,\middle|\, \begin{array}{c}\text{for all } \omega \in \Omega, \text{ both } \langle f, (\cdot) \triangleright \omega\rangle \text{ and } \langle f, \omega \triangleleft (\cdot)\rangle \\ \text{are weakly continuous linear functionals on } E\end{array} \right\} \\ &= \left\{ f \in \Omega' \,\middle|\, (\overline{f \otimes} \operatorname{id})\Delta(\Omega) \subseteq \Omega \supseteq (\operatorname{id} \overline{\otimes f})\Delta(\Omega) \right\}.\end{aligned}$$

Slice maps like $\overline{f \otimes} \operatorname{id} : \Omega \overline{\otimes} \Omega \to \overline{\Omega}$ are defined in remark A.3.iii and (A.4). In particular, if $\mathbb{E}$ is algebraic (definition 2.1.3.6) then $\Omega^\sharp = \Omega'$. ⋆

Proposition 2.3.1.4 *The strict*$^\flat$ *and strict*$^\sharp$ *topologies on* Ω *are* $\mathbb{E}$*-compatible. If* Ω *is unital as an* E*-bimodule, then also the strict*$^\natural$ *topology is* $\mathbb{E}$*-compatible.*

Proof. We already knew that Ω_σ is a *topological* E_σ-bimodule (remark 2.1.1.2.iii) hence the strict$^\flat$ topology on Ω is $\mathbb{E}$-compatible. This also implies the strict$^\natural$ and strict$^\sharp$ topologies to be finer than the weak topology. Next we show that $\Omega_\sharp$ is a topological E_σ-bimodule: appealing to the universal property of strict$^\sharp$ as an inductive LC topology, it only remains to show that for all $x \in E$ and $\omega \in \Omega$

$$x \triangleright \big((\cdot) \triangleright \omega\big) \qquad x \triangleright \big(\omega \triangleleft (\cdot)\big) \qquad \big((\cdot) \triangleright \omega\big) \triangleleft x \qquad \big(\omega \triangleleft (\cdot)\big) \triangleleft x$$

are continuous mappings from E_σ into $\Omega_\sharp$, which is easy. Finally, assume Ω to be unital and consider the strict$^\natural$ topology. For all $x \in E$, the mappings $x \triangleright (\cdot)$ and $(\cdot) \triangleleft x$ are weakly-strict$^\natural$ continuous and a fortiori strict$^\natural$ continuous. Now it only remains to show that $(\cdot) \triangleright \omega$ and $\omega \triangleleft (\cdot)$ are continuous mappings from E_σ into $\Omega_\natural$. Using $\Omega = \Omega \triangleleft E$, any $\omega \in \Omega$ can be written as $\omega = \sum_k \omega_k \triangleleft x_k$, hence $(\cdot) \triangleright \omega = \sum_k \big((\cdot) \triangleright \omega_k\big) \triangleleft x_k$ enjoys the desired continuity because the $(\cdot) \triangleleft x_k$ are weakly-strict$^\natural$ continuous. For $\omega \triangleleft (\cdot)$ we need $\Omega = E \triangleright \Omega$. ■

Corollary 2.3.1.5 *The strict$^\sharp$ topology is the finest $\mathbb{E}$-compatible topology on Ω, the strict$^\flat$ topology is the coarsest one. When Ω is unital, the above topologies compare as follows: strict$^\flat$ $\preceq$ strict$^\natural$ $\preceq$ strict$^\sharp$. Consequently $\Omega^\flat \subseteq \Omega^\natural \subseteq \Omega^\sharp$.*

Example A 2.3.1.6 Recall the actor context $(F; \mathbb{C}S, \langle\cdot,\cdot\rangle)$ of example 2.1.2.1. Clearly *every* LC topology on $\mathbb{C}S$ makes $(\cdot) \triangleright \delta_s = \langle\,\cdot, \delta_s\rangle\, \delta_s = \delta_s \triangleleft (\cdot)$ into continuous mappings from F_σ into $\mathbb{C}S$, hence the strict$^\sharp$ topology must be the finest LC topology on $\mathbb{C}S$. In particular $\mathbb{C}S$ is strict$^\sharp$ *complete* (a phenomenon which is likely to become characteristic for *algebraic* actor contexts). ⋆

Example B 2.3.1.7 Let $\mathbb{A} \equiv (A; A^*, \langle\cdot,\cdot\rangle)$ be as in example 2.1.2.3. Then the $\sigma(A^*, A^{**})$ topology is $\mathbb{A}$-compatible because of (2.3), hence $A^{**} \subseteq A^{*\sharp}$. The norm topology will usually not be $\mathbb{A}$-compatible.
In the special case that A is the algebra of compact operators on a Hilbert space, one can show that $A^{**} = A^{*\sharp}$, hence in this case the norm topology on A^* is finer than the strict$^\sharp$ topology (the former being a Mackey topology).

2.3.2 Actors induced by functionals

We already encountered several E-bimodule structures associated with an actor context $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$. Also Ω' can be made into an E-bimodule, the actions of E on Ω' (denoted by juxtaposition) being given by

$$\langle xf, \omega\rangle = \langle f, \omega \triangleleft x\rangle \qquad \text{and} \qquad \langle fx, \omega\rangle = \langle f, x \triangleright \omega\rangle \tag{2.8}$$

for $f \in \Omega'$, $x \in E$ and $\omega \in \Omega$. One could say this E-bimodule structure on Ω' is 'parallel' to the one on $\mathrm{Act}(\mathbb{E})$, but in some sense dual to those on Ω or E'. It is easy to prove that $\Omega^\flat$, $\Omega^\natural$ and $\Omega^\sharp$ are sub-E-bimodules of Ω'.

Proposition 2.3.2.1 *There are unique linear mappings*

i. $\iota : \Omega^\flat \to E$

ii. $\mu : \Omega^\natural \to M(E)$ —*provided that* $\mathbb{E}$ *is non-degenerate*—

iii. $\theta : \Omega^\sharp \to \mathrm{Act}(\mathbb{E}) : f \mapsto \theta(f) \equiv (\lambda_f, \rho_f)$

obeying respectively—for every $x \in E$ *and* $\omega \in \Omega$*,*

i. $\langle \iota(f), \omega\rangle = \langle f, \omega\rangle$ *for all* $f \in \Omega^\flat$

ii. $\langle \mu(f)x, \omega\rangle = \langle f, x \triangleright \omega\rangle$ *and* $\langle x\mu(f), \omega\rangle = \langle f, \omega \triangleleft x\rangle$ *for all* $f \in \Omega^\natural$

iii. $\langle x, \rho_f(\omega)\rangle = \langle f, x \triangleright \omega\rangle$ *and* $\langle x, \lambda_f(\omega)\rangle = \langle f, \omega \triangleleft x\rangle$ *for all* $f \in \Omega^\sharp$*.*

Proof. (i) ι is nothing but the inverse of the canonical embedding $E \hookrightarrow \Omega'$.
(ii) Take any $f \in \Omega^\natural$. From definition 2.3.1.2 it follows that the functionals $fx = \langle f, x \triangleright (\cdot)\rangle$ and $xf = \langle f, (\cdot) \triangleleft x\rangle$ are actually in $\Omega^\flat$ for all $x \in E$, which allows us to define a multiplier $\mu(f)$ of E by $\mu(f)x = \iota(fx)$ and $x\mu(f) = \iota(xf)$.
(iii) Next consider any $f \in \Omega^\sharp$ and $\omega \in \Omega$. This time it follows that $\langle f, (\cdot) \triangleright \omega\rangle$ and $\langle f, \omega \triangleleft (\cdot)\rangle$ are both continuous functionals on E_σ. Hence they identify with elements in Ω, respectively denoted by $\rho_f(\omega)$ and $\lambda_f(\omega)$. So given $f \in \Omega^\sharp$ we have defined linear maps λ_f and ρ_f from Ω into Ω, satisfying (iii). It remains to show that (λ_f, ρ_f) is an actor for $\mathbb{E}$. Now for all $x, y \in E$ and $\omega \in \Omega$ we have

$$\langle x, \lambda_f(\omega \triangleleft y)\rangle = \langle f, (\omega \triangleleft y) \triangleleft x\rangle = \langle f, \omega \triangleleft yx\rangle = \langle yx, \lambda_f(\omega)\rangle = \langle x, \lambda_f(\omega) \triangleleft y\rangle,$$

so λ_f is a right E-module morphism. Similarly ρ_f is a left E-module map and

$$\langle x, \rho_f(\omega \triangleleft y)\rangle = \langle f, x \triangleright (\omega \triangleleft y)\rangle = \langle f, (x \triangleright \omega) \triangleleft y\rangle = \langle y, \lambda_f(x \triangleright \omega)\rangle$$

proves the bi-actor property (2.2). This completes the proof. ■

Remark 2.3.2.2 For any $f \in \Omega^\sharp$, the actor $\theta(f) = (\lambda_f, \rho_f)$ is also given by

$$\lambda_f = (\mathrm{id}\,\overline{\otimes f})\Delta \qquad \text{and} \qquad \rho_f = (\overline{f \otimes}\,\mathrm{id})\Delta$$

(cf. remark 2.3.1.3.ii). This should be compared with (2.5). ⋆

Below we collect some properties of the maps introduced in proposition 2.3.2.1. First observe that $\Omega^\flat$, $\Omega^\natural$, $\Omega^\sharp$, E, $M(E)$ and $\mathrm{Act}(\mathbb{E})$ are all E-bimodules under 'multiplication' in one way or the other. Also recall lemma 2.2.3.

Corollary 2.3.2.3 *The maps* ι*,* μ *and* θ *are* E*-bimodule morphisms;* μ *extends* ι*, whereas* θ *extends* $j \circ \iota$*. If* Ω *is unital, then* μ *and* θ *are injective; moreover* $\Omega^\natural \subseteq \Omega^\sharp$*, and* θ *extends* $\hat{\jmath} \circ \mu$ *accordingly. Up to identification we have* $\iota \subseteq \mu \subseteq \theta$*.*

Proof. Only the last assertion is not completely trivial. So let Ω be unital as an E-bimodule, take any $f \in \Omega^\natural$ and denote $\hat{\jmath}(\mu(f)) \equiv (\lambda, \rho) \in \mathrm{Act}(\mathbb{E})$. By (2.6) we get for all $x \in E$ and $\omega \in \Omega$ that $\langle x, \lambda(\omega)\rangle = \langle x\mu(f), \omega\rangle = \langle f, \omega \triangleleft x\rangle$, and similarly $\langle x, \rho(\omega)\rangle = \langle f, x \triangleright \omega\rangle$. It follows that $\langle f, \omega \triangleleft (\cdot)\rangle$ and $\langle f, (\cdot) \triangleright \omega\rangle$ are continuous functionals on E_σ, for any $\omega \in \Omega$, hence f itself must be strict$^\sharp$ continuous. So $f \in \Omega^\sharp$ and clearly $\theta(f) = (\lambda_f, \rho_f) = (\lambda, \rho) = \hat{\jmath}(\mu(f))$. ■

So the maps μ and θ are injective, provided that Ω is unital, but it would of course be nice if they where actually bijections. For this we need an extra assumption: the existence of a counit (see §2.4).

2.3.3 Invertibility of actors and functionals

Given a non-degenerate actor context $\mathbb{E}$, we have several notions of invertibility. One can for instance consider invertible elements in the unital algebras $M(\mathbb{E})$, $\mathrm{Env}(\mathbb{E})$ or $\mathrm{Pre}(\mathbb{E})$. Below we shall see that it is also possible to define a concept of invertibility within $\mathrm{Act}(\mathbb{E})$, although in general $\mathrm{Act}(\mathbb{E})$ is not really an algebra; in fact the latter notion is the one we will use in chapter 3 to express 'group-like' properties. We start however with a trivial observation:

Lemma 2.3.3.1 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be any actor context, and let $a \equiv (\lambda, \rho)$ be an actor for $\mathbb{E}$. Then the following are equivalent:*

i. a is invertible within $\mathrm{Pre}(\mathbb{E})$ *and a^{-1} belongs to* $\mathrm{Act}(\mathbb{E})$ *again.*

ii. λ and ρ are bijections on Ω and $(\lambda^{-1}, \rho^{-1})$ satisfies the bi-actor property.

Definition 2.3.3.2 Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be any actor context. An *actor* for $\mathbb{E}$ is said to be $\mathbb{E}$-*invertible* when it satisfies the conditions of the previous lemma. On the other hand, a linear *functional* $f : \Omega \to \mathbb{C}$ will be called $\mathbb{E}$-*invertible* if

i. f is strict$^\sharp$ continuous, i.e. f actually belongs to $\Omega^\sharp$, and

ii. $\theta(f)$ is $\mathbb{E}$-invertible as an actor,

with $\theta : \Omega^\sharp \to \mathrm{Act}(\mathbb{E})$ as described in §2.3.2. •

Whenever a pre-actor $a \equiv (\lambda, \rho)$ is invertible in the algebra $\mathrm{Pre}(\mathbb{E})$ or, in other words, when λ and ρ are bijective, we can define an inner automorphism

$$\pi_a : \mathrm{Pre}(\mathbb{E}) \to \mathrm{Pre}(\mathbb{E}) : x \mapsto axa^{-1}.$$

Lemma 2.3.3.3 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be any actor context with Ω unital. Let $a \equiv (\lambda, \rho)$ be an actor for $\mathbb{E}$ such that λ and ρ are bijections, and consider*

i. $\pi_a(E) \subseteq \mathrm{Act}(\mathbb{E})$

ii. a is $\mathbb{E}$-invertible

iii. λ and ρ commute.

Whenever two of these statements are satisfied, all three of them will hold.

Proof. Under the circumstances, (ii) merely states that $(\lambda^{-1}, \rho^{-1})$ satisfies the bi-actor property (2.2). Similarly (i) means that $(\lambda\lambda_x\lambda^{-1},\ \rho^{-1}\rho_x\rho)$ enjoys this property for all $x \in E$. Furthermore λ is a right E-module morphism, hence so is λ^{-1}. We thus obtain, for all $x, y, z \in E$ and $\omega \in \Omega$, the following 'circle':

$$\begin{aligned}
\big\langle x,\, (\rho\lambda^{-1})(y \triangleright \omega \triangleleft z)\big\rangle &= \big\langle x,\, \rho(\lambda^{-1}(y \triangleright \omega) \triangleleft z)\big\rangle &&= \big\langle z,\, \lambda(x \triangleright \lambda^{-1}(y \triangleright \omega))\big\rangle \\
&= \big\langle z,\, (\lambda\lambda_x\lambda^{-1})(y \triangleright \omega)\big\rangle &&\overset{\text{(i)}}{=} \big\langle y,\, (\rho^{-1}\rho_x\rho)(\omega \triangleleft z)\big\rangle \\
&= \big\langle y,\, \rho^{-1}(\rho(\omega \triangleleft z) \triangleleft x)\big\rangle &&\overset{\text{(ii)}}{=} \big\langle x,\, \lambda^{-1}(y \triangleright \rho(\omega \triangleleft z))\big\rangle \\
&= \big\langle x,\, (\lambda^{-1}\rho)(y \triangleright \omega \triangleleft z)\big\rangle &&\overset{\text{(iii)}}{=} \big\langle x,\, (\rho\lambda^{-1})(y \triangleright \omega \triangleleft z)\big\rangle.
\end{aligned}$$

Since by assumption $E \triangleright \Omega \triangleleft E = \Omega$ and $\rho(\Omega \triangleleft E) = \Omega$, the result follows. ∎

2.4 Weakly unital actor contexts

Abstract 2.4 We introduce the notion of a *weakly unital* actor context. The main observation here is that, in the weakly unital case, the correspondence θ between strict$^\sharp$ continuous functionals and actors (cf. §2.3.2) becomes *bijective*. As a consequence, the pairing extends to actors in a natural way. In chapter 3 we shall almost exclusively be dealing with the weakly unital case. ⋆

2.4.1 The counit

Definition 2.4.1.1 Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be any actor context. A functional $\varepsilon \in \Omega'$ is said to be a (left, right, two-sided) *counit* for $\mathbb{E}$ if

$$\langle \varepsilon, x \triangleright \omega \rangle \overset{\text{(left)}}{=} \langle x, \omega \rangle \overset{\text{(right)}}{=} \langle \varepsilon, \omega \triangleleft x \rangle \tag{2.9}$$

for all $\omega \in \Omega$ and $x \in E$. •

Remarks 2.4.1.2 i. A counit ε is both strict$^\natural$ and strict$^\sharp$ continuous as a functional on Ω, so $\varepsilon \in \Omega^\natural$ and $\varepsilon \in \Omega^\sharp$. Furthermore $\theta(\varepsilon) = 1_{\mathbb{E}}$.

ii. From remark 2.3.2.2 we obtain that a counit ε satisfies

$$(\overline{\varepsilon \otimes} \,\mathrm{id})\Delta \;=\; \mathrm{id} \;=\; (\mathrm{id}\, \overline{\otimes \varepsilon})\Delta. \tag{2.10}$$

Conversely, any $\varepsilon \in \Omega'$ obeying (2.10) is a counit for $\mathbb{E}$. Indeed remark 2.3.1.3.ii yields $\varepsilon \in \Omega^\sharp$, hence remark 2.3.2.2 applies and (2.9) follows.

iii. If Ω is unital as an E-bimodule, then a counit for $\mathbb{E}$ is obviously *unique*.

Lemma 2.4.1.3 *Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be an actor context such that Ω is unital as an E-bimodule. Then a left counit for $\mathbb{E}$ is also a right counit.*

Proof. Let ε be a left counit for $\mathbb{E}$. Then for all $x, y \in E$ and $\omega \in \Omega$ we have

$$\big\langle \varepsilon, \, (x \triangleright \omega) \triangleleft y \big\rangle = \big\langle \varepsilon, \, x \triangleright (\omega \triangleleft y) \big\rangle = \langle x, \omega \triangleleft y \rangle = \langle yx, \omega \rangle = \langle y, x \triangleright \omega \rangle.$$

Since $E \triangleright \Omega = \Omega$, the above means that ε is also a right counit. ∎

Lemma 2.4.1.4 *Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be an actor context such that Ω is unital as an E-bimodule. Assume there exists an $f \in \Omega^\sharp$ such that ρ_f is bijective (with ρ_f as defined in proposition 2.3.2.1.iii). Then there exists a counit for $\mathbb{E}$.*

Proof. Since ρ_f^{-1} is a left E-module morphism, we get for all $\omega \in \Omega$ and $x \in E$

$$\langle f, \rho_f^{-1}(x \triangleright \omega) \rangle \;=\; \langle f, x \triangleright \rho_f^{-1}(\omega) \rangle \;=\; \langle x, \rho_f(\rho_f^{-1}(\omega)) \rangle \;=\; \langle x, \omega \rangle,$$

hence $\langle f, \rho_f^{-1}(\,\cdot\,) \rangle$ is a left counit for $\mathbb{E}$. Lemma 2.4.1.3 yields the result. ∎

Proposition 2.4.1.5 *Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be an actor context such that Ω is unital as an E-bimodule. Then the following are equivalent:*

i. there exists a counit for $\mathbb{E}$

ii. $\theta : \Omega^\sharp \to \mathrm{Act}(\mathbb{E})$ *is a bijection*

iii. $\mu : \Omega^\natural \to M(E)$ *is a bijection*

iv. E *has a weak approximate identity*

v. E *has a left weak approximate identity*

vi. within $E \otimes \Omega$*, we have* $\ker(\cdot \triangleright \cdot) \subseteq \ker\langle \cdot, \cdot \rangle$

vii. there exists a left counit for $\mathbb{E}$

viii. there exists an $\mathbb{E}$*-invertible functional on* Ω*.*

Proof. We show $(i \Rightarrow ii \Rightarrow iii \Rightarrow i)$ and $(i \Rightarrow iv \Rightarrow v \Rightarrow vi \Rightarrow vii \Rightarrow i)$. Finally we will prove $(i \Rightarrow viii \Rightarrow vii)$.

$(i) \Rightarrow (ii)$. Let ε be a counit for $\mathbb{E}$. Only surjectivity of θ is left to prove, so take any actor $(\lambda, \rho) \in \mathrm{Act}(\mathbb{E})$. Then for all $\omega \in \Omega$ and $x, y \in E$ we have

$$\big\langle \varepsilon,\, \lambda(x \triangleright \omega \triangleleft y)\big\rangle \overset{(2.9)}{=} \big\langle y,\, \lambda(x \triangleright \omega)\big\rangle \overset{(2.2)}{=} \big\langle x,\, \rho(\omega \triangleleft y)\big\rangle \overset{(2.9)}{=} \big\langle \varepsilon,\, \rho(x \triangleright \omega \triangleleft y)\big\rangle\,.$$

By assumption we have $E \triangleright \Omega \triangleleft E = \Omega$, hence $\lambda^\tau(\varepsilon) = \rho^\tau(\varepsilon) \equiv f$ defines one single functional $f \in \Omega'$. Using $f = \lambda^\tau(\varepsilon)$ we get for all $\omega \in \Omega$ and $x \in E$ that

$$\langle f, \omega \triangleleft x\rangle = \big\langle \varepsilon,\, \lambda(\omega \triangleleft x)\big\rangle = \big\langle \varepsilon,\, \lambda(\omega) \triangleleft x\big\rangle = \langle x, \lambda(\omega)\rangle,$$

and analogously we obtain from $f = \rho^\tau(\varepsilon)$ that $\langle f, x \triangleright \omega\rangle = \langle x, \rho(\omega)\rangle$. It follows that $f \in \Omega^\sharp$, and proposition 2.3.2.1.iii yields $\theta(f) = (\lambda_f, \rho_f) = (\lambda, \rho)$.

$(ii) \Rightarrow (iii)$. Again only surjectivity is left to prove; take any $z \in M(E)$. Now Ω is unital, so by proposition 2.2.4 we have $\hat{\jmath}(z) \equiv (\lambda_z, \rho_z) \in \mathrm{Act}(\mathbb{E})$. Since θ is assumed to be bijective, there exists an $f \in \Omega^\sharp$ such that $\theta(f) = \hat{\jmath}(z)$, or in other words, $(\lambda_f, \rho_f) = (\lambda_z, \rho_z)$. Hence we get for all $x \in E$ and $\omega \in \Omega$

$$\langle f, x \triangleright \omega\rangle = \langle x, \rho_f(\omega)\rangle = \langle x, \rho_z(\omega)\rangle \overset{(2.6)}{=} \langle zx, \omega\rangle,$$

and similarly $\langle f, \omega \triangleleft x\rangle = \langle xz, \omega\rangle$. Since zx and xz are in E, it follows that f is strict$^\natural$ continuous, i.e. $f \in \Omega^\natural$, and by proposition 2.3.2.1.ii we have $\mu(f) = z$.

$(iii) \Rightarrow (i)$. Since μ is bijective, there exists an $\varepsilon \in \Omega^\natural$ such that $\mu(\varepsilon) = 1$.

$(i) \Rightarrow (iv)$. Let ε be a counit for $\mathbb{E}$. Since E is weakly dense in Ω', we can take a net $(e_\alpha)_\alpha$ in E such that $\langle e_\alpha, \omega\rangle \to \langle \varepsilon, \omega\rangle$ for all $\omega \in \Omega$. Then we have for any $x \in E$ and $\omega \in \Omega$ that $\langle e_\alpha x, \omega\rangle = \langle e_\alpha, x \triangleright \omega\rangle \to \langle \varepsilon, x \triangleright \omega\rangle = \langle x, \omega\rangle$. It follows that $e_\alpha x \to x$ weakly, and similarly $x e_\alpha \to x$.

$(iv) \Rightarrow (v)$. A fortiori.

$(v) \Rightarrow (vi)$. Let $(e_\alpha)_\alpha$ be a left w.a.i. in E and take any element $\sum_i x_i \otimes \omega_i$ in $E \otimes \Omega$ such that $\sum_i x_i \triangleright \omega_i = 0$. Then $\sum_i \langle e_\alpha x_i, \omega_i\rangle = \sum_i \langle e_\alpha, x_i \triangleright \omega_i\rangle = 0$ for all α, and since $e_\alpha x_i \to x_i$ weakly for every i, we conclude that $\sum_i \langle x_i, \omega_i\rangle = 0$.

$(vi) \Rightarrow (vii)$. Property (vi) allows us to construct a *well-defined* functional ε on $E \triangleright \Omega = \Omega$ satisfying the left part of (2.9).

$(vii) \Rightarrow (i)$. Lemma 2.4.1.3.

$(i) \Rightarrow (viii)$. A counit for $\mathbb{E}$ is clearly $\mathbb{E}$-invertible (cf. remark 2.4.1.2.i).

$(viii) \Rightarrow (vii)$. Lemma 2.4.1.4. ∎

Definition 2.4.1.6 An actor context $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ is said to be *faithful* if it enjoys the following (obviously equivalent) conditions:

i. E^2 is weakly dense in E

ii. the induced comultiplication $\Delta : \Omega \to \Omega \overline{\otimes} \Omega$ is injective

iii. Ω is non-degenerate as a left E-module, in the sense that $\omega \in \Omega$ and $x \triangleright \omega = 0$ for all $x \in E$ implies $\omega = 0$.

Definition 2.4.1.7 An actor context $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ is called *weakly unital* if Ω is unital as an E-bimodule and $\mathbb{E}$ satisfies the eight (equivalent) conditions in proposition 2.4.1.5. We will always use ε to denote the counit. Observe that a weakly unital actor context is a fortiori faithful. •

Example A 2.4.1.8 The actor contexts in example 2.1.2.1 are weakly unital.

Example B 2.4.1.9 Recall example 2.1.2.2. A standard result in C^*-algebra theory states that, given $\omega \in A^*$, there exist $x, y \in A$ and $\varphi, \psi \in A^*$ such that $\varphi(\cdot\, x) = \omega = \psi(y\, \cdot)$, so A^* is unital as an A-bimodule. Now every C^*-algebra has an approximate identity, hence $\mathbb{A}$ is weakly unital. Another way to see this is to consider A^{**}. As a W^*-algebra, A^{**} has an identity, being a counit for $\mathbb{A}$.

Weak unitality is usually a property of an actor context $(E; \Omega, \langle \cdot, \cdot \rangle)$ as a whole. In the next example, however, it amounts to an assumption on the algebra E itself:

Example 2.4.1.10 Let $(E; \Omega, \langle \cdot, \cdot \rangle)$ be a non-degenerate actor context. Assume E to have one-sided (say left) *local units* [8, 21], i.e. for every *finite* subset F of E there exists an $e \in E$ such that $ex = x$ for all $x \in F$. Let Ω_0 denote the reduction of Ω (§1.4). Then $(E; \Omega_0, \langle \cdot, \cdot \rangle)$ is a weakly unital actor context.

Proof. By assumption $(E; \Omega_0, \langle \cdot, \cdot \rangle)$ is again an actor context (lemma 2.1.4.4). Obviously $E^2 = E$, hence Ω_0 is unital. Let us show (vi) in proposition 2.4.1.5 by taking any $\sum_{i=1}^n x_i \otimes \omega_i$ in $E \otimes \Omega_0$ with $\sum_i x_i \triangleright \omega_i = 0$. Considering a local unit w.r.t. the subset $\{x_1, \ldots, x_n\} \subseteq E$, we easily obtain $\sum_i \langle x_i, \omega_i \rangle = 0$. ∎

2.4.2 Extending the pairing

Proposition 2.4.2.1 *Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be a weakly unital actor context, and recall E identifies with a subspace of* $\mathrm{Act}(\mathbb{E})$. *Now there exists a non-degenerate pairing $\langle \mathrm{Act}(\mathbb{E}), \Omega \rangle$ naturally extending the original pairing $\langle E, \Omega \rangle$. Explicitly, for any $a \equiv (\lambda, \rho) \in \mathrm{Act}(\mathbb{E})$, $\omega \in \Omega$ and $f \in \Omega^\sharp$ we have*

$$\langle \varepsilon, \lambda(\omega) \rangle = \langle a, \omega \rangle = \langle \varepsilon, \rho(\omega) \rangle \qquad \text{and} \qquad \langle \theta(f), \omega \rangle = \langle f, \omega \rangle. \tag{2.11}$$

Proof. By assumption, $\theta : \Omega^\sharp \to \mathrm{Act}(\mathbb{E})$ is a bijection, transforming the obvious pairing $\langle \Omega^\sharp, \Omega\rangle$ into the one we are looking for. The first part of (2.11) has already been observed in the proof of $(i \Rightarrow ii)$ in proposition 2.4.1.5. ∎

Survey 2.4.2.2 Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be a *weakly unital* actor context. Then

- $\mathrm{Env}(\mathbb{E}) \equiv \mathrm{Env}\big(\mathbb{E}\,; \mathrm{Act}(\mathbb{E})\big)$ is a unital algebra in which $M(\mathbb{E}) \equiv \mathrm{Env}(\mathbb{E}\,; E)$ is contained as a subalgebra.
- There are *bijections* $\hat{\jmath}$, ι, μ, and θ such that

$$\begin{array}{ccccccccccc}
\Omega^\flat & \subseteq & \Omega^\natural & & & \subseteq & & & \Omega^\sharp & \subseteq & \Omega' \\
\downarrow \iota & & \downarrow \mu & & & & & & \downarrow \theta & & \\
E & \subseteq & M(E) & \stackrel{\hat{\jmath}}{\to} & M(\mathbb{E}) & \subseteq & \mathrm{Env}(\mathbb{E}) & \subseteq & \mathrm{Act}(\mathbb{E}) & \subseteq & \mathrm{Pre}(\mathbb{E}).
\end{array}$$

- $\hat{\jmath}$ is an algebra isomorphism, θ is an E-bimodule isomorphism, $\iota \subseteq \mu \subseteq \theta$.
- We have a unique counit ε. It satisfies $\theta(\varepsilon) = 1_{\mathbb{E}}$ and $\mu(\varepsilon) = 1$.
- The pairing $\langle E, \Omega\rangle$ extends naturally to a pairing $\langle \mathrm{Act}(\mathbb{E}), \Omega\rangle$. A fortiori we have (non-degenerate) pairings $\langle \mathrm{Env}(\mathbb{E}), \Omega\rangle$ and $\langle M(E), \Omega\rangle$.

2.4.3 Some extra properties in the weakly unital case

Throughout this paragraph, $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ is a weakly unital actor context.

Lemma 2.4.3.1 *Let* $a_1 \equiv (\lambda_1, \rho_1)$ *and* $a_2 \equiv (\lambda_2, \rho_2)$ *be any two actors for* $\mathbb{E}$. *If* $a_1 a_2 = (\lambda_1\lambda_2, \rho_2\rho_1)$ *is again an actor for* $\mathbb{E}$, *then*

$$\langle a_1, \lambda_2(\omega)\rangle \;=\; \langle a_1 a_2, \omega\rangle \;=\; \langle a_2, \rho_1(\omega)\rangle \tag{2.12}$$

for all $\omega \in \Omega$. This follows immediately from (2.11).

Lemma 2.4.3.2 *Let* (λ, ρ) *be a pre-actor for* $\mathbb{E}$. *Then* (λ, ρ) *is an actor for* $\mathbb{E}$ *if and only if* $\langle \varepsilon, \lambda(\omega)\rangle = \langle \varepsilon, \rho(\omega)\rangle$ *for all* $\omega \in \Omega$.

Proof. (2.11) yields the 'only if' part. To prove the 'if' part, recall that λ and ρ are respectively a right and a left E-module morphism, and observe that

$$\langle x, \rho(\omega \lhd y)\rangle \stackrel{(2.9)}{=} \langle \varepsilon, \rho(x \rhd \omega \lhd y)\rangle \;=\; \langle \varepsilon, \lambda(x \rhd \omega \lhd y)\rangle \stackrel{(2.9)}{=} \langle y, \lambda(x \rhd \omega)\rangle$$

for all $x, y \in E$ and $\omega \in \Omega$. Hence (λ, ρ) enjoys the bi-actor property. ∎

Lemma 2.4.3.3 *Let* $a \equiv (\lambda, \rho)$ *be an actor for* $\mathbb{E}$ *which is invertible in* $\mathrm{Pre}(\mathbb{E})$, *i.e.* λ *and* ρ *are bijections. If* λ *and* ρ *commute, then* a *is* $\mathbb{E}$*-invertible.*

Proof. We have to show that $a^{-1} = (\lambda^{-1}, \rho^{-1})$ belongs to $\mathrm{Act}(\mathbb{E})$ again. Since ρ commutes with λ, it also commutes with λ^{-1} and hence

$$\langle \varepsilon, \lambda^{-1}\rho(\omega)\rangle = \langle \varepsilon, \rho\lambda^{-1}(\omega)\rangle = \langle \varepsilon, \lambda\lambda^{-1}(\omega)\rangle = \langle \varepsilon, \omega\rangle = \langle \varepsilon, \rho^{-1}\rho(\omega)\rangle$$

for all $\omega \in \Omega$. Since $\rho(\Omega) = \Omega$, lemma 2.4.3.2 yields the result. ∎

Lemma 2.4.3.4 *Let $a_1 \equiv (\lambda_1, \rho_1)$ and $a_2 \equiv (\lambda_2, \rho_2)$ be any two actors for $\mathbb{E}$. If λ_2 commutes with ρ_1, then $a_1 a_2$ is again an actor for $\mathbb{E}$.*

Proof. Applying lemma 2.4.3.2 twice, we observe that for any $\omega \in \Omega$

$$\langle \varepsilon, \lambda_1\lambda_2(\omega)\rangle = \langle \varepsilon, \rho_1\lambda_2(\omega)\rangle = \langle \varepsilon, \lambda_2\rho_1(\omega)\rangle = \langle \varepsilon, \rho_2\rho_1(\omega)\rangle$$

Again by lemma 2.4.3.2, we conclude $a_1 a_2 = (\lambda_1\lambda_2, \rho_2\rho_1)$ is an actor for $\mathbb{E}$. ∎

It is natural to ask for a *converse* to lemma 2.4.3.4. Such a converse does indeed exist, at least in some sense, though we have to be careful:

Lemma 2.4.3.5 *Let $a_1 \equiv (\lambda_1, \rho_1)$ and $a_2 \equiv (\lambda_2, \rho_2)$ be any two actors for $\mathbb{E}$. If there exists a subset $\mathcal{D}$ of $\mathrm{Act}(\mathbb{E})$ such that*

i. $\mathcal{D}$ separates Ω w.r.t. the pairing $\langle \mathrm{Act}(\mathbb{E}), \Omega\rangle$

ii. $a_1\mathcal{D}$ and $\mathcal{D}a_2$ are contained in $\mathrm{Act}(\mathbb{E})$

iii. $a_1\mathcal{D}a_2$ is contained in $\mathrm{Act}(\mathbb{E})$

then λ_2 commutes with ρ_1 (and consequently a_1a_2 is again an actor for $\mathbb{E}$).

Proof. Let $\mathcal{D}$ be such a set and take any $d \equiv (\lambda, \rho) \in \mathcal{D}$. Then $a_1 d = (\lambda_1\lambda, \rho\rho_1)$, $a_1 d a_2 = (\lambda_1\lambda\lambda_2, \rho_2\rho\rho_1)$ and $da_2 = (\lambda\lambda_2, \rho_2\rho)$ are all actors for $\mathbb{E}$. Hence

$$\langle d, \lambda_2\rho_1(\omega)\rangle = \langle \varepsilon, \lambda\lambda_2\rho_1(\omega)\rangle = \langle da_2, \rho_1(\omega)\rangle = \langle \varepsilon, \rho_2\rho\rho_1(\omega)\rangle = \langle a_1 d a_2, \omega\rangle$$

$$= \langle \varepsilon, \lambda_1\lambda\lambda_2(\omega)\rangle = \langle a_1 d, \lambda_2(\omega)\rangle = \langle \varepsilon, \rho\rho_1\lambda_2(\omega)\rangle = \langle d, \rho_1\lambda_2(\omega)\rangle.$$

for any $\omega \in \Omega$. Since this holds for all $d \in \mathcal{D}$, the result follows from (i). ∎

Remark 2.4.3.6 Notice $\mathcal{D} = E$ always satisfies (i) and (ii) in the above lemma. The third condition however, might fail even in case $\mathcal{D} = E$. ⋆

Corollary 2.4.3.7 $\mathrm{Env}(\mathbb{E})$ *equals the following set of actors:*

$$\Big\{(\lambda, \rho) \in \mathrm{Act}(\mathbb{E}) \;\Big|\; \lambda\beta = \beta\lambda \;\text{ and }\; \rho\alpha = \alpha\rho \;\text{ for all }\; (\alpha, \beta) \in \mathrm{Act}(\mathbb{E})\Big\}.$$

Proof. Take any $a \equiv (\lambda, \rho) \in \mathrm{Env}(\mathbb{E})$ and $b \equiv (\alpha, \beta) \in \mathrm{Act}(\mathbb{E})$. Observe that $aEb \subseteq \mathrm{Act}(\mathbb{E})$ and apply lemma 2.4.3.5 with $\mathcal{D} = E$. It follows that α commutes with ρ, and similarly, λ with β. Lemma 2.4.3.4 yields the other inclusion. ∎

Lemma 2.4.3.8 *Let $a \equiv (\lambda, \rho) \in \mathrm{Env}(\mathbb{E})$. Then by the above corollary, λ and ρ commute. Moreover the following are equivalent:*

i. λ and ρ are bijections; in other words, a is invertible within $\mathrm{Pre}(\mathbb{E})$.

ii. a is $\mathbb{E}$-invertible.

iii. a is invertible within the algebra $\mathrm{Env}(\mathbb{E})$.

Proof. (i) $\Rightarrow$ (ii). Lemma 2.4.3.3. (ii) $\Rightarrow$ (iii). Using corollary 2.4.3.7 we obtain: $a \in \mathrm{Env}(\mathbb{E})$ and $a^{-1} \in \mathrm{Act}(\mathbb{E})$ implies $a^{-1} \in \mathrm{Env}(\mathbb{E})$. (iii) $\Rightarrow$ (i). A fortiori. ∎

Lemma 2.4.3.9 *Let* $a \equiv (\lambda, \rho)$ *be an actor for* $\mathbb{E}$ *which is invertible in* $\mathrm{Pre}(\mathbb{E})$. *Let* A *denote the subalgebra of* $\mathrm{Pre}(\mathbb{E})$ *generated by* $\mathrm{Env}(\mathbb{E}) \cup \{a, a^{-1}\}$. *Then the following are equivalent:*

i. $aEa \subseteq \mathrm{Act}(\mathbb{E})$

ii. λ *and* ρ *commute*

iii. $A \subseteq \mathrm{Act}(\mathbb{E})$

iv. a *is* $\mathbb{E}$*-invertible and* $\pi_a(E) = aEa^{-1} \subseteq \mathrm{Act}(\mathbb{E})$.

Proof. (i) $\Rightarrow$ (ii). Lemma 2.4.3.5. (iii) $\Rightarrow$ (i). A fortiori. (ii) $\Rightarrow$ (iii). Assume $\lambda\rho = \rho\lambda$. Lemma 2.4.3.3 yields that a^{-1} is again an actor for $\mathbb{E}$. We show that $a^{n_1} b_1 a^{n_2} b_2 \ldots a^{n_k} b_k$ belongs to $\mathrm{Act}(\mathbb{E})$ for all $k \geq 1$, $b_i \equiv (\alpha_i, \beta_i) \in \mathrm{Env}(\mathbb{E})$ and $n_i \in \mathbb{Z}$, $i = 1, 2, \ldots k$. By corollary 2.4.3.7, the α_i commute with the β_j. Furthermore the α_i also commute with ρ and with ρ^{-1}. Iterating lemma 2.4.3.4 yields the result. This establishes (i $\Leftrightarrow$ ii $\Leftrightarrow$ iii).
(iii) $\Rightarrow$ (iv). A fortiori. (iv) $\Rightarrow$ (ii). Lemma 2.3.3.3. ∎

2.4.4 Arens regularity

Example B 2.4.4.1 Let A be any Banach algebra with an identity, and consider $\mathbb{A} \equiv (A; A^*, \langle\cdot,\cdot\rangle)$ as in example 2.1.2.3. Since A is assumed to have an identity, $\mathbb{A}$ is obviously weakly unital as an actor context, and hence $A^{*\sharp}$ identifies naturally with $\mathrm{Act}(\mathbb{A})$. Now recall that the bidual A^{**} of A as a Banach space is contained in $A^{*\sharp}$ (cf. example 2.3.1.7). On the other hand, R. Arens [2] showed that A^{**} can be made into a Banach algebra in *two* ways, which by definition only coincide when A is *Arens regular*. Now it is easy to see that the two Arens products [2, 5, 6] of functionals $f, g \in A^{**}$ are respectively given by $f\lambda_g = \varepsilon\lambda_f\lambda_g$ and $g\rho_f = \varepsilon\rho_g\rho_f$. It follows that A is Arens regular as a Banach algebra if and only if for all $f, g \in A^{**}$ the product $\theta(f)\theta(g) = (\lambda_f\lambda_g, \rho_g\rho_f)$ belongs to $\mathrm{Act}(\mathbb{A})$ again (cf. lemma 2.4.3.2). In particular, if $\mathrm{Act}(\mathbb{A})$ is closed under multiplication, then A must be Arens regular. Now there exist examples of unital Banach algebras which are not Arens regular[4], and thus we also have examples of weakly unital actor contexts $\mathbb{A}$ with $\mathrm{Act}(\mathbb{A})$ not being closed under multiplication. This proves the assertion in example 2.1.4.3. ⋆

2.5 The enveloping actor context

Throughout this paragraph, $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ is a *weakly unital* actor context.

Abstract 2.5 In paragraph §2.1.4 we introduced the enveloping algebra $\mathrm{Env}(\mathbb{E})$, which is contained in $\mathrm{Act}(\mathbb{E})$. In paragraph §2.4 we obtained an extended pairing between $\mathrm{Act}(\mathbb{E})$ and Ω. We now prove that $(\mathrm{Env}(\mathbb{E}); \Omega, \langle\cdot,\cdot\rangle)$ is again a weakly unital actor context; it turns out that the enveloping algebra associated to the *latter* actor context is simply $\mathrm{Env}(\mathbb{E})$ again. ⋆

[4] e.g. $L^1(G)$ where G is any infinite discrete abelian group [6].

Lemma 2.5.1 *Let F be any separating algebra of actors for $\mathbb{E}$, i.e.*

i. F is a subalgebra of $\mathrm{Pre}(\mathbb{E})$

ii. F is contained in $\mathrm{Act}(\mathbb{E})$

iii. F separates Ω w.r.t. the pairing $\langle \mathrm{Act}(\mathbb{E}), \Omega \rangle$.

Then $\mathbb{F} \equiv (F; \Omega, \langle\cdot,\cdot\rangle)$ is again an actor context, and the counit for $\mathbb{E}$ is still a counit for $\mathbb{F}$. Hence if Ω is moreover unital as an F-bimodule (in particular, if $E \subseteq F$) then $\mathbb{F}$ is weakly unital again.

Proof. From (ii-iii) and proposition 2.4.2.1 we obtain a pairing $\langle F, \Omega\rangle$ which allows us to view Ω as a subspace of F'. Since F is an algebra, its dual F' is canonically endowed with an F-bimodule structure (§2.1.1). In particular, the *canonical* actions (denoted $\triangleright$ and $\triangleleft$) of an $a \in F$ on an $\omega \in \Omega \subseteq F'$ turn out to be implemented by the maps $\lambda, \rho : \Omega \to \Omega$ constituting the actor $a \equiv (\lambda, \rho)$, i.e.

$$a \triangleright \omega = \lambda(\omega) \qquad \text{and} \qquad \omega \triangleleft a = \rho(\omega). \tag{2.13}$$

Let's prove this; take any $b \equiv (\alpha, \beta) \in F \subseteq \mathrm{Act}(\mathbb{E})$ and consider a and ω as above. Now $ba \in F$, for F is an algebra, hence $ba = (\alpha\lambda, \rho\beta) \in \mathrm{Act}(\mathbb{E})$ again. Now (2.11) yields $\langle b, a \triangleright \omega\rangle = \langle ba, \omega\rangle = \langle \varepsilon, \alpha\lambda(\omega)\rangle = \langle b, \lambda(\omega)\rangle$. Similarly we show that $\langle b, \omega \triangleleft a\rangle = \langle b, \rho(\omega)\rangle$ for all $b \in F$ and $\omega \in \Omega$, which proves (2.13). We conclude that Ω is actually a sub-F-bimodule of F', so $\mathbb{F}$ is indeed an actor context. The assertion about the counit follows from (2.11) and (2.13). ∎

Proposition 2.5.2 *Recall lemma 2.1.4.5. Let A be a sub-E-bimodule of* $\mathrm{Act}(\mathbb{E})$ *containing $1_{\mathbb{E}}$. Then $\tilde{\mathbb{E}}_A \equiv \big(\mathrm{Env}(\mathbb{E}; A); \Omega, \langle\cdot,\cdot\rangle\big)$ is again a weakly unital actor context.* In particular, when $A = \mathrm{Act}(\mathbb{E})$ we abbreviate $\tilde{\mathbb{E}} \equiv \big(\mathrm{Env}(\mathbb{E}); \Omega, \langle\cdot,\cdot\rangle\big)$.

Proof. Recall $\mathrm{Env}(\mathbb{E}; A)$ is an algebra (proposition 2.1.4.9, remark 2.1.4.7) with $E \subseteq \mathrm{Env}(\mathbb{E}\,; A) \subseteq A \subseteq \mathrm{Act}(\mathbb{E})$. Apply lemma 2.5.1 with $F = \mathrm{Env}(\mathbb{E}\,; A)$. ∎

Remark 2.5.3 In particular Ω is an $\mathrm{Env}(\mathbb{E})$-bimodule, and (2.13) allows us to write for instance $\lambda(\omega)$ as $a \triangleright \omega$, whenever $a \equiv (\lambda, \rho)$ belongs to $\mathrm{Env}(\mathbb{E})$. ⋆

Proposition 2.5.4 *With A as above, we have $A \subseteq \mathrm{Act}(\tilde{\mathbb{E}}_A) \subseteq \mathrm{Act}(\mathbb{E})$.*

Remark 2.5.5 Although $\tilde{\mathbb{E}}_A$ and $\mathbb{E}$ are two different actor contexts, both actor implementations involved live on the same space Ω. Therefore we can indeed compare $\mathrm{Act}(\tilde{\mathbb{E}}_A)$ and $\mathrm{Act}(\mathbb{E})$ as sets of pairs (λ, ρ) of linear maps on Ω.

Proof. An actor for $\tilde{\mathbb{E}}_A$ is a fortiori an actor for $\mathbb{E}$, because $E \subseteq \mathrm{Env}(\mathbb{E}\,; A)$. Take any $a \equiv (\alpha, \beta) \in A$. By remark 2.1.1.4.iii, it suffices to prove that (α, β) enjoys the bi-actor property w.r.t. $\tilde{\mathbb{E}}_A$. Taking $x_i \equiv (\lambda_i, \rho_i) \in \mathrm{Env}(\mathbb{E}\,; A)$ for $i = 1, 2$, it follows that $x_2 a x_1 = (\lambda_2\alpha\lambda_1, \rho_1\beta\rho_2)$ still belongs to $A \subseteq \mathrm{Act}(\mathbb{E})$ and

$$\big\langle x_1,\, \beta(\omega \triangleleft x_2)\big\rangle \overset{(2.13)}{=} \big\langle x_1,\, \beta\rho_2(\omega)\big\rangle \overset{(2.11)}{=} \big\langle \varepsilon,\, \rho_1\beta\rho_2(\omega)\big\rangle \overset{(2.11)}{=} \big\langle x_2 a x_1,\, \omega\big\rangle$$
$$\overset{(2.11)}{=} \big\langle \varepsilon,\, \lambda_2\alpha\lambda_1(\omega)\big\rangle \overset{(2.11)}{=} \big\langle x_2,\, \alpha\lambda_1(\omega)\big\rangle \overset{(2.13)}{=} \big\langle x_2,\, \alpha(x_1 \triangleright \omega)\big\rangle$$

for all $\omega \in \Omega$. This proves that (α, β) is an actor for $\tilde{\mathbb{E}}_A$. ∎

Corollary 2.5.6 $\mathrm{Act}(\tilde{\mathbb{E}}) = \mathrm{Act}(\mathbb{E})$*, and hence* $\mathrm{Env}(\tilde{\mathbb{E}}) = \mathrm{Env}(\mathbb{E})$*.*

Corollary 2.5.7 *With A as in proposition 2.5.2, $x \equiv (\lambda, \rho) \in \mathrm{Env}(\mathbb{E}\,; A)$ and $(\alpha, \beta) \in A$, we have the following commutation rules:*

$$\lambda\beta = \beta\lambda \qquad \text{and} \qquad \rho\alpha = \alpha\rho. \tag{2.14}$$

Remark. Notice the overlap between (2.14) and corollary 2.4.3.7.

Proof. Proposition 2.5.4 yields that (α, β) is an actor for $\tilde{\mathbb{E}}_A$, hence α is a right $\mathrm{Env}(\mathbb{E}\,; A)$-module morphism. Since $x \in \mathrm{Env}(\mathbb{E}\,; A)$ it follows for all $\omega \in \Omega$ that $(\rho\alpha)(\omega) = \alpha(\omega) \triangleleft x = \alpha(\omega \triangleleft x) = (\alpha\rho)(\omega)$. The other relation is analogous. ∎

Proposition 2.5.8 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ and $\mathbb{F} \equiv (F; \Omega, \langle\cdot,\cdot\rangle)$ be two weakly unital actor contexts, sharing the same counit. If $F \subseteq \mathrm{Env}(\mathbb{E})$ and $E \subseteq \mathrm{Env}(\mathbb{F})$, then $\mathrm{Act}(\mathbb{E}) = \mathrm{Act}(\mathbb{F})$ and consequently $\mathrm{Env}(\mathbb{E}) = \mathrm{Env}(\mathbb{F})$.*

Remark. Both actor implementations live on Ω, hence a comment similar to remark 2.5.5 applies. However we must be very careful here—for instance there are now *two* ways to view the pairing between F and Ω: first we have of course the pairing $\langle F, \Omega\rangle$ given by $\mathbb{F}$. But at the same time we also have $\langle \mathrm{Act}(\mathbb{E}), \Omega\rangle$, and since F is assumed to be a subset of $\mathrm{Act}(\mathbb{E})$ this may cause ambiguity. Fortunately the assumption that $\mathbb{E}$ and $\mathbb{F}$ share the counit, together with (2.11) and (2.13) completely resolves this ambiguity. Similar comments apply w.r.t. the module and algebra structures involved.

Proof. Corollary 2.5.6 yields that every actor for $\mathbb{E}$ is also an actor for $\tilde{\mathbb{E}}$, and then it is *a fortiori* an actor for $\mathbb{F}$ because $F \subseteq \mathrm{Env}(\mathbb{E})$. ∎

Corollary 2.5.9 *Let $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ be a weakly unital actor context and F a subalgebra of $\mathrm{Env}(\mathbb{E})$ containing E. We already know from lemma 2.5.1 that $\mathbb{F} \equiv (F; \Omega, \langle\cdot,\cdot\rangle)$ is a weakly unital actor context with the same counit as $\mathbb{E}$. Now the above proposition yields $\mathrm{Env}(\mathbb{F}) = \mathrm{Env}(\mathbb{E})$.*

2.6 Tensor products

Throughout this paragraph, $\mathbb{E}_1 \equiv (E_1; \Omega_1, \langle\cdot,\cdot\rangle)$ and $\mathbb{E}_2 \equiv (E_2; \Omega_2, \langle\cdot,\cdot\rangle)$ are actor contexts. Observe we have a non-degenerate pairing between $E_1 \otimes E_2$ and $\Omega_1 \otimes \Omega_2$. The following obvious lemmas are recorded to fix notations:

Lemma 2.6.1 $\mathbb{E}_1 \otimes \mathbb{E}_2 \equiv \big(E_1 \otimes E_2;\ \Omega_1 \otimes \Omega_2, \langle\cdot,\cdot\rangle\big)$ *is again an actor context, the left actions being given by $(x_1 \otimes x_2) \triangleright (\omega_1 \otimes \omega_2) = (x_1 \triangleright \omega_1) \otimes (x_2 \triangleright \omega_2)$ for $x_i \in E_i$ and $\omega_i \in \Omega_i$ with $i = 1, 2$. Similarly for the right actions.*

This tensor product construction can be iterated and is obviously associative.

Lemma 2.6.2 *We have well-defined linear maps Φ and Ψ, given by*

$$\begin{array}{ccccccc} \mathrm{Act}(\mathbb{E}_1) \otimes \mathrm{Act}(\mathbb{E}_2) & \xrightarrow{\Phi} & \mathrm{Act}(\mathbb{E}_1 \otimes \mathbb{E}_2) & \qquad & \Omega_1^\sharp \otimes \Omega_2^\sharp & \xrightarrow{\Psi} & (\Omega_1 \otimes \Omega_2)^\sharp \\ (\lambda_1, \rho_1) \otimes (\lambda_2, \rho_2) & \longmapsto & (\lambda_1 \otimes \lambda_2, \rho_1 \otimes \rho_2) & & f_1 \otimes f_2 & \longmapsto & f_1 \otimes f_2. \end{array}$$

Furthermore, let θ_1*,* θ_2 *and* θ *be the mappings associated to respectively* $\mathbb{E}_1$*,* $\mathbb{E}_2$ *and* $\mathbb{E}_1 \otimes \mathbb{E}_2$ *in the sense of proposition 2.3.2.1.iii. Then* $\theta\Psi = \Phi(\theta_1 \otimes \theta_2)$*. Also observe that* Ψ *is injective.*

Proposition 2.6.3 $\mathbb{E}_1 \otimes \mathbb{E}_2$ *is weakly unital if and only if both* $\mathbb{E}_1$ *and* $\mathbb{E}_2$ *are weakly unital.* In this case θ_1, θ_2 and θ are all bijective, hence Φ is injective. If ε_i is the counit for $\mathbb{E}_i$ $(i = 1, 2)$ then $\varepsilon_1 \otimes \varepsilon_2$ is the counit for $\mathbb{E}_1 \otimes \mathbb{E}_2$.

Proof. The 'if' part is trivial; let's prove the 'only if' part. It is not so hard to see that if $\Omega_1 \otimes \Omega_2$ is unital as an $(E_1 \otimes E_2)$-bimodule, then Ω_i is unital as an E_i-bimodule $(i = 1, 2)$. Using (vi) in proposition 2.4.1.5 yields the result. ∎

Lemma 2.6.4 *Assume* $\mathbb{E}_1 \otimes \mathbb{E}_2$ *to be faithful in the sense of definition 2.4.1.6. Take any pre-actor* (λ, ρ) *for* $\mathbb{E}_1 \otimes \mathbb{E}_2$ *and any* $x \in E_1$*. Then* ρ *commutes with* $(x \triangleright \cdot\,) \otimes \mathrm{id}$*. Similarly* λ *commutes with* $(\,\cdot \triangleleft x) \otimes \mathrm{id}$*.*

Proposition 2.6.5 *Again assume* $\mathbb{E}_1 \otimes \mathbb{E}_2$ *to be faithful (definition 2.4.1.6). Then the map* Φ *defined in lemma 2.6.2 restricts to an algebra homomorphism from* $\mathrm{Env}(\mathbb{E}_1) \otimes \mathrm{Env}(\mathbb{E}_2)$ *into* $\mathrm{Env}(\mathbb{E}_1 \otimes \mathbb{E}_2)$*.*

Proof. We will first show that

$$\Phi\left(\mathrm{Env}(\mathbb{E}_1) \otimes 1_{\mathbb{E}_2}\right) \;\subseteq\; \mathrm{Env}(\mathbb{E}_1 \otimes \mathbb{E}_2). \tag{2.15}$$

So taking arbitrary $(\alpha, \beta) \in \mathrm{Env}(\mathbb{E}_1)$ and $(\lambda, \rho) \in \mathrm{Act}(\mathbb{E}_1 \otimes \mathbb{E}_2)$, we have to show that the pairs $\big((\alpha \otimes \mathrm{id})\lambda,\, \rho(\beta \otimes \mathrm{id})\big)$ and $\big(\lambda(\alpha \otimes \mathrm{id}),\, (\beta \otimes \mathrm{id})\rho\big)$ obey the bi-actor property. Let's do this—for instance—for the last one: take any $x_i, y_i \in E_i$ and $\omega_i \in \Omega_i$ $(i = 1, 2)$ and define mappings $\lambda_1, \rho_1 : \Omega_1 \to \Omega_1$ by

$$\begin{aligned}
\lambda_1(\nu) &= \big(\mathrm{id} \otimes \langle y_2, \cdot\,\rangle\big)\; \lambda\left(\nu \otimes (x_2 \triangleright \omega_2)\right) \\
\rho_1(\nu) &= \big(\mathrm{id} \otimes \langle x_2, \cdot\,\rangle\big)\; \rho\left(\nu \otimes (\omega_2 \triangleleft y_2)\right).
\end{aligned}$$

Since (λ, ρ) is an actor for $\mathbb{E}_1 \otimes \mathbb{E}_2$, it follows that (λ_1, ρ_1) is an actor for $\mathbb{E}_1$ (here we need lemma 2.6.4). Consequently also $(\lambda_1\alpha, \beta\rho_1) \in \mathrm{Act}(\mathbb{E}_1)$, hence

$$\begin{aligned}
&\big\langle x_1 \otimes x_2,\, (\beta \otimes \mathrm{id})\rho\left((\omega_1 \otimes \omega_2) \triangleleft (y_1 \otimes y_2)\right)\big\rangle \\
&\quad= \big\langle x_1,\, \beta\left(\mathrm{id} \otimes \langle x_2, \cdot\,\rangle\right) \rho\left((\omega_1 \triangleleft y_1) \otimes (\omega_2 \triangleleft y_2)\right)\big\rangle \\
&\quad= \big\langle x_1,\, \beta\rho_1(\omega_1 \triangleleft y_1)\big\rangle \\
&\quad= \big\langle y_1,\, \lambda_1\alpha(x_1 \triangleright \omega_1)\big\rangle \\
&\quad= \big\langle y_1,\, \left(\mathrm{id} \otimes \langle y_2, \cdot\,\rangle\right) \lambda\left(\alpha(x_1 \triangleright \omega_1) \otimes (x_2 \triangleright \omega_2)\right)\big\rangle \\
&\quad= \big\langle y_1 \otimes y_2,\, \lambda(\alpha \otimes \mathrm{id})\left((x_1 \otimes x_2) \triangleright (\omega_1 \otimes \omega_2)\right)\big\rangle.
\end{aligned}$$

This proves (2.15). Of course a similar result holds when the roles of $\mathbb{E}_1$ and $\mathbb{E}_2$ are reversed. Now it's easy to complete the proof. ∎

Proposition 2.6.6 *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be weakly unital. Given any $\omega \in \Omega_2$, there exists a unique linear map*

$$\Gamma : \mathrm{Act}(\mathbb{E}_1 \otimes \mathbb{E}_2) \to \mathrm{Act}(\mathbb{E}_1) \qquad \textit{such that} \qquad \langle \Gamma(a), \nu \rangle = \langle a, \nu \otimes \omega \rangle$$

for all $a \in \mathrm{Act}(\mathbb{E}_1 \otimes \mathbb{E}_2)$ and all $\nu \in \Omega_1$.

Proof. Fix $\omega \in \Omega_2$ and take any actor $a \equiv (\lambda, \rho)$ for $\mathbb{E}_1 \otimes \mathbb{E}_2$. Now define two linear mappings λ_1 and ρ_1 from Ω_1 into Ω_1 by

$$\begin{aligned} \lambda_1 &= (\mathrm{id} \otimes \varepsilon_2)\, \lambda\, (\,\cdot \otimes \omega) \\ \rho_1 &= (\mathrm{id} \otimes \varepsilon_2)\, \rho\, (\,\cdot \otimes \omega) \end{aligned} \tag{2.16}$$

With lemma 2.6.4 we obtain $(\lambda_1, \rho_1) \in \mathrm{Pre}(\mathbb{E}_1)$. Then lemma 2.4.3.2 yields that $(\lambda_1, \rho_1) \in \mathrm{Act}(\mathbb{E}_1)$. Define $\Gamma(a) = (\lambda_1, \rho_1)$. Now (2.11) yields the result. ∎

Notation 2.6.7 Adopt the above situation, but now consider ω as a weakly continuous functional on E_2. To emphasize this interpretation we shall denote this functional by f_ω. Since Γ naturally extends the *slice* map

$$\mathrm{id} \otimes f_\omega \;:\; E_1 \otimes E_2 \to E_1 \;:\; x_1 \otimes x_2 \;\mapsto\; \langle x_2, \omega \rangle\, x_1,$$

we shall henceforth denote Γ as $\mathrm{id} \,\hat{\otimes}\, f_\omega$. •

Proposition 2.6.8 *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be weakly unital, fix an $\omega \in \Omega_2$ and take an actor $a \equiv (\lambda, \rho)$ for $\mathbb{E}_1 \otimes \mathbb{E}_2$. If λ commutes with $\beta \otimes \mathrm{id}$ and ρ with $\alpha \otimes \mathrm{id}$ for all $(\alpha, \beta) \in \mathrm{Act}(\mathbb{E}_1)$, then $(\mathrm{id} \,\hat{\otimes}\, f_\omega)(a)$ belongs to $\mathrm{Env}(\mathbb{E}_1)$.*

Proof. Corollary 2.4.3.7 and (2.16). ∎

Proposition 2.6.9 Recall definition 2.2.1 and proposition 2.2.4. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be weakly unital, fix an $\omega \in \Omega_2$ and take any $a \in M(\mathbb{E}_1 \otimes \mathbb{E}_2) \simeq M(E_1 \otimes E_2)$. Then $(\mathrm{id} \,\hat{\otimes}\, f_\omega)(a)$ belongs to $M(\mathbb{E}_1) \simeq M(E_1)$.*

Proof. Since $\mathbb{E}_2$ is weakly unital, we may write $\omega = \sum_i y_i \triangleright \xi_i$ with $y_i \in E_2$ and $\xi_i \in \Omega_2$. Using (2.12) we easily obtain that for any $x \in E_1$ and $\nu \in \Omega_1$

$$\big\langle (\mathrm{id} \,\hat{\otimes}\, f_\omega)(a)\, x,\, \nu \big\rangle \;=\; \big\langle a,\, (x \triangleright \nu) \otimes \omega \big\rangle \;=\; \textstyle\sum_i \big\langle a(x \otimes y_i),\, \nu \otimes \xi_i \big\rangle\,.$$

It follows that $(\mathrm{id} \,\hat{\otimes}\, f_\omega)(a)\, x = \sum_i (\mathrm{id} \otimes f_{\xi_i}) \big(a(x \otimes y_i)\big)$ belongs to E_1. Similarly we can deal with $x(\mathrm{id} \,\hat{\otimes}\, f_\omega)(a)$. ∎

2.7 Opposite actor contexts

The statements in the present paragraph are completely trivial. However we believe that dealing with opposite products may be quite tricky notationally, and therefore we shall be rather explicit in this matter. Opposite actor contexts will be an important tool to investigate *regularity* of a Hopf system (cf. §3.4).

Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be any actor context. Then obviously $\mathbb{E}^{\mathrm{op}} \equiv (E^{\mathrm{op}}; \Omega, \langle \cdot, \cdot \rangle)$ is again an actor context, referred to as the *opposite* actor context. When $\mathbb{E}$ is non-degenerate, the trivial anti-isomorphism $\mathrm{id}^{\mathrm{op}} : E \to E^{\mathrm{op}} : x \mapsto x$ extends naturally to an anti-isomorphism

$$\mathrm{id}^{\mathrm{op}} : \mathrm{Pre}(\mathbb{E}) \to \mathrm{Pre}(\mathbb{E}^{\mathrm{op}}) : (\lambda, \rho) \mapsto (\rho, \lambda)$$

mapping $\mathrm{Act}(\mathbb{E})$ onto $\mathrm{Act}(\mathbb{E}^{\mathrm{op}})$ and $\mathrm{Env}(\mathbb{E})$ onto $\mathrm{Env}(\mathbb{E}^{\mathrm{op}})$. It follows that $\mathrm{Env}(\mathbb{E}^{\mathrm{op}}) \simeq \mathrm{Env}(\mathbb{E})^{\mathrm{op}}$. Since confusion is likely, we shall use $\triangleright^{\mathrm{op}}$ and $\triangleleft^{\mathrm{op}}$ to denote the actions associated with $\mathbb{E}^{\mathrm{op}}$. So for $x \in E$ and $\omega \in \Omega$ we have for instance $x \triangleright^{\mathrm{op}} \omega = \omega \triangleleft x$ (observe we have suppressed $\mathrm{id}^{\mathrm{op}}$ in the notation—as we shall usually do henceforth). Of course most properties discussed in the preceding paragraphs are preserved when we pass to the opposite actor context:

- $\mathbb{E}$ is weakly unital if and only if $\mathbb{E}^{\mathrm{op}}$ is weakly unital.
- $\mathbb{E}$ and $\mathbb{E}^{\mathrm{op}}$ induce the same strict$^\flat$, strict$^\natural$ and strict$^\sharp$ topologies on Ω.
- A functional on Ω is $\mathbb{E}$-invertible if and only if it is $\mathbb{E}^{\mathrm{op}}$-invertible.
- If $\mathbb{E}_1$ and $\mathbb{E}_2$ are actor contexts, then ${\mathbb{E}_1}^{\mathrm{op}} \otimes {\mathbb{E}_2}^{\mathrm{op}} = (\mathbb{E}_1 \otimes \mathbb{E}_2)^{\mathrm{op}}$.

In the weakly unital case, $\mathbb{E}$ and $\mathbb{E}^{\mathrm{op}}$ share the counit $\varepsilon : \Omega \to \mathbb{C}$. Furthermore the pairings $\langle \mathrm{Act}(\mathbb{E}), \Omega \rangle$ and $\langle \mathrm{Act}(\mathbb{E}^{\mathrm{op}}), \Omega \rangle$ agree through $\mathrm{id}^{\mathrm{op}}$.

2.8 Involutive actor contexts

Whenever E is a *-algebra, its dual E' is endowed with an involutive operation $\varsigma : E' \to E' : \omega \mapsto \omega^\circ$ defined by $\langle x, \omega^\circ \rangle = \overline{\langle x^*, \omega \rangle}$.

Definition 2.8.1 Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be an actor context. If E is endowed with a *-operation such that E is a *-algebra and such that Ω is $^\circ$-invariant, then $\mathbb{E}$ is said to be an *involutive* actor context. •

Lemma 2.8.2 *Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be an involutive actor context. Then for any $x \in E$ and $\omega \in \Omega$ we have $(x \triangleright \omega)^\circ = \omega^\circ \triangleleft x^*$ and $(\omega \triangleleft x)^\circ = x^* \triangleright \omega^\circ$. Hence*

$$\mathrm{Pre}(\mathbb{E}) \to \mathrm{Pre}(\mathbb{E}) : \ a \equiv (\lambda, \rho) \ \mapsto \ a^* \equiv (\varsigma\rho\varsigma, \varsigma\lambda\varsigma) \tag{2.17}$$

is a well-defined *-operation, making $\mathrm{Pre}(\mathbb{E})$ *into a *-algebra. Thus the map $j : E \to \mathrm{Pre}(\mathbb{E})$ defined beneath lemma 2.1.4.1 becomes a *-algebra morphism. Furthermore* $\mathrm{Act}(\mathbb{E})$ *and* $\mathrm{Env}(\mathbb{E})$ *are *-invariant.*

Proof. Straightforward. ∎

Definition 2.8.3 Let $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$ be an involutive actor context. An actor $a \in \mathrm{Act}(\mathbb{E})$ is said to be *unitary* if $a^* a = 1_{\mathbb{E}} = a a^*$. •

A unitary actor for $\mathbb{E}$ is obviously $\mathbb{E}$-invertible in the sense of definition 2.3.3.2.

Lemma 2.8.4 *Let* $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ *be a weakly unital involutive actor context. Then* $\langle \varepsilon, \omega^\circ \rangle = \overline{\langle \varepsilon, \omega \rangle}$ *and* $\langle a, \omega^\circ \rangle = \overline{\langle a^*, \omega \rangle}$ *for any* $\omega \in \Omega$ *and* $a \in \mathrm{Act}(\mathbb{E})$.

Proof. Observe that for any $x \in E$ and $\omega \in \Omega$

$$\langle \varepsilon, (x \triangleright \omega)^\circ \rangle \;=\; \langle \varepsilon, \omega^\circ \triangleleft x^* \rangle \;=\; \langle x^*, \omega^\circ \rangle \;=\; \overline{\langle x, \omega \rangle} \;=\; \overline{\langle \varepsilon, x \triangleright \omega \rangle}.$$

This proves the first formula since $E \triangleright \Omega = \Omega$. Now take any actor $a \equiv (\lambda, \rho)$ for $\mathbb{E}$ and recall (2.17). Using (2.11) we obtain:

$$\overline{\langle a^*, \omega \rangle} \;=\; \overline{\langle \varepsilon, \rho(\omega^\circ)^\circ \rangle} \;=\; \langle \varepsilon, \rho(\omega^\circ) \rangle \;=\; \langle a, \omega^\circ \rangle. \quad \blacksquare$$

2.9 Special cases

2.9.1 The commutative case

When Ω is a bimodule over some algebra E, we shall use $\mathrm{End}_E(\Omega)$ to denote the algebra of all E-bimodule morphisms from Ω into Ω. If E is *commutative*, then $\mathrm{End}_E(\Omega)$ is again an E-bimodule, the actions (denoted by juxtaposition) of an element $x \in E$ on a morphism $\gamma \in \mathrm{End}_E(\Omega)$ being given by

$$(x\gamma)(\omega) \;=\; x \triangleright \gamma(\omega) \qquad \text{and} \qquad (\gamma x)(\omega) \;=\; \gamma(\omega) \triangleleft x \qquad (\omega \in \Omega).$$

Proposition 2.9.1.1 *Let* $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ *be an actor context,* E *commutative and* Ω *unital as an* E*-bimodule* (§1.4)*. Then we have natural isomorphisms*

$$\mathrm{Act}(\mathbb{E}) \;\simeq\; \mathrm{End}_E(\Omega) \qquad \textit{and} \qquad \mathrm{Env}(\mathbb{E}) \;\simeq\; \mathrm{Center}\big(\mathrm{End}_E(\Omega)\big)$$

of E*-bimodules and algebras respectively (recall lemma 2.1.4.5 and corollary 2.1.4.10). In particular we observe that* $\mathrm{Env}(\mathbb{E})$ *is again commutative.*

Proof. Since $(\Omega, \langle\cdot,\cdot\rangle)$ is an actor implementation of a commutative algebra E, the left and right E-module structures on Ω must coincide, i.e. $x \triangleright \omega = \omega \triangleleft x$ for all $x \in E$ and $\omega \in \Omega$. Let $a \equiv (\lambda, \rho)$ be any actor for $\mathbb{E}$. Since λ is now both a left and right E-module morphism, we get for all $x, y \in E$ and $\omega \in \Omega$ that

$$\langle x, \rho(\omega \triangleleft y) \rangle \overset{(2.2)}{=} \langle y, \lambda(x \triangleright \omega) \rangle \;=\; \langle y, x \triangleright \lambda(\omega) \rangle \;=\; \langle yx, \lambda(\omega) \rangle \;=\; \langle x, \lambda(\omega \triangleleft y) \rangle.$$

Since by assumption $\Omega \triangleleft E = \Omega$, it follows that $\lambda = \rho$. Hence a is of the form (γ, γ) with $\gamma \equiv \lambda = \rho \in \mathrm{End}_E(\Omega)$. Thus we get a well-defined linear bijection

$$\Phi : \mathrm{End}_E(\Omega) \to \mathrm{Act}(\mathbb{E}) : \gamma \mapsto (\gamma, \gamma)$$

which is easily seen to be an isomorphism of E-bimodules, and moreover restricts to an algebra isomorphism of $\mathrm{Center}(\mathrm{End}_E(\Omega))$ onto $\mathrm{Env}(\mathbb{E})$. Indeed, for any $\gamma_1, \gamma_2 \in \mathrm{End}_E(\Omega)$ we observe that $\Phi(\gamma_1)\Phi(\gamma_2) = (\gamma_1\gamma_2, \gamma_2\gamma_1)$ belongs to $\mathrm{Act}(\mathbb{E})$ again if and only if γ_1 and γ_2 commute. The result follows. $\blacksquare$

Example A 2.9.1.2 *Recall the actor context* $\mathbb{F} \equiv (F; \mathbb{C}S, \langle \cdot, \cdot \rangle)$ *of example* A 2.1.2.1. *We now have* $\mathrm{Act}(\mathbb{F}) = \mathrm{Env}(\mathbb{F}) \simeq \mathbb{C}^S$.

Proof. First observe that $\mathbb{F}$ is nothing but a 'restricted' version of the actor context $(\mathbb{C}^S; \mathbb{C}S, \langle \cdot, \cdot \rangle)$. For every function $g \in \mathbb{C}^S$ we consider a linear map $\gamma_g : \mathbb{C}S \to \mathbb{C}S$ defined by $\gamma_g(\delta_s) = g(s)\,\delta_s$. Thus we get an injective algebra morphism $\Gamma : \mathbb{C}^S \to \mathrm{End}_F(\mathbb{C}S) : g \mapsto \gamma_g$. To prove that Γ is surjective, take any $\gamma \in \mathrm{End}_F(\mathbb{C}S)$ and define $g \in \mathbb{C}^S$ by $g(s) = \langle 1, \gamma(\delta_s) \rangle$. Here 1 denotes the identity in $\mathbb{C}^S$. Using the F-bimodule property of γ and the non-degeneracy of $\langle F, \mathbb{C}S \rangle$ it follows easily that $\gamma_g = \gamma$. Now Γ is an algebra isomorphism, hence $\mathrm{End}_F(\mathbb{C}S) \simeq \mathbb{C}^S$ is commutative; proposition 2.9.1.1 yields the result. Also observe that $\Phi\Gamma$ (with Φ as in the above proof) is an algebra isomorphism from $\mathbb{C}^S$ onto $\mathrm{Env}(\mathbb{F})$ extending the natural embedding of F in $\mathrm{Env}(\mathbb{F})$. ∎

2.9.2 Pseudo-discrete actor contexts

Definition 2.9.2.1 A non-degenerate actor context $\mathbb{E}$ is called *pseudo-discrete* if $M(\mathbb{E}) = \mathrm{Act}(\mathbb{E})$. •

In chapter 3, theorem 3.8.4, we shall encounter a large class of pseudo-discrete actor contexts, arising from multiplier Hopf algebras with invariant functionals. The terminology is inspired by the following

Example A 2.9.2.2 *Let* X *be a locally compact Hausdorff space and consider the actor context* $\mathbb{K}_X \equiv (K(X); \mathbb{C}X, \langle \cdot, \cdot \rangle)$ *as introduced in example 2.1.2.1. Then* $\mathbb{K}_X$ *is pseudo-discrete if and only if* X *is discrete.*

Proof. According to example 2.9.1.2 we have $\mathrm{Act}(\mathbb{K}_X) \simeq \mathbb{C}^X$. On the other hand $M(\mathbb{K}_X) \simeq M(K(X)) \simeq C(X)$. Hence $M(\mathbb{K}_X) = \mathrm{Act}(\mathbb{K}_X)$ if and only if every complex function on X is continuous. ∎

Chapter 3

Hopf Systems

3.1 Invertible dual pairs

Abstract 3.1 We introduce the notion of an invertible dual pair of algebras and study its basic properties. All the actor contexts involved turn out to be weakly unital; this will be an important technical advantage throughout the development of the theory. We also introduce antipodes and comultiplications, and show that they take values in the enveloping algebras associated to the appropriate actor contexts (cf. §2.1.4). ⋆

Recall example 2.1.2.4, where we defined the notion of a dual pair of algebras. Whenever $\langle A, B\rangle$ is such a dual pair, we have actor contexts $\mathbb{A} \equiv (A; B, \langle\cdot,\cdot\rangle)$ and $\mathbb{B} \equiv (B; A, \langle\cdot,\cdot\rangle)$. According to lemma 2.6.1 we also have an actor context

$$\mathbb{B} \otimes \mathbb{A} \;\equiv\; \big(B \otimes A;\; A \otimes B, \langle\cdot,\cdot\rangle\big)\,.$$

Now observe that the pairing $\langle\cdot,\cdot\rangle$ between A and B can be viewed as a linear functional $P : A \otimes B \to \mathbb{C} : a \otimes b \mapsto \langle a, b\rangle$. Thus the following makes sense:

Definition 3.1.1 A dual pair of algebras $\langle A, B\rangle$ is called *invertible* whenever its pairing P is $(\mathbb{B} \otimes \mathbb{A})$-invertible in the sense of definition 2.3.3.2. •

So if $\langle A, B\rangle$ is an invertible dual pair of algebras, then $P \in (A \otimes B)^\sharp$ induces a $(\mathbb{B} \otimes \mathbb{A})$-invertible actor $\theta(P) \equiv (\lambda_{P}, \rho_P)$. Here λ_P and ρ_P are bijective linear mappings from $A \otimes B$ onto $A \otimes B$, given by (cf. proposition 2.3.2.1.iii)

$$\begin{aligned} \big\langle d \otimes c,\, \lambda_P(a \otimes b)\big\rangle &= \big\langle a \triangleleft d,\, b \triangleleft c\big\rangle \\ \big\langle d \otimes c,\, \rho_P(a \otimes b)\big\rangle &= \big\langle d \triangleright a,\, c \triangleright b\big\rangle \end{aligned} \tag{3.1}$$

for any $a, c \in A$ and $b, d \in B$. Moreover $(\lambda_P^{-1}, \rho_P^{-1})$ is again an actor for $\mathbb{B} \otimes \mathbb{A}$.

Remark 3.1.2 It may be instructive to express this in *Sweedler* [33] notation: if Δ_A denotes for instance the induced comultiplication[1] on A, then we formally

[1] See §2.1.3 for details.

write $\Delta_A(a) = a_{(1)} \otimes a_{(2)}$ etc. According to remark 2.3.2.2, the mappings λ_P and ρ_P may then be expressed as follows:

$$\begin{aligned} \lambda_P(a \otimes b) &= \langle a_{(2)}, b_{(2)} \rangle \; a_{(1)} \otimes b_{(1)} \\ \rho_P(a \otimes b) &= \langle a_{(1)}, b_{(1)} \rangle \; a_{(2)} \otimes b_{(2)}. \end{aligned}$$

Similar formulas exist for the inverses of λ_P and ρ_P (cf. remark 3.1.7). $\star$

Remark 3.1.3 One should be well aware of the fact that strict$^\sharp$ continuity of $P : A \otimes B \to \mathbb{C}$ is really part of the *assumption* here (cf. definition 2.3.3.2.i). Strict$^\sharp$ continuity of P amounts to the fact that the mappings λ_P and ρ_P given by (3.1) really go into $A \otimes B$ rather than just $(B \otimes A)'$. Invertibility in an actor context was investigated in lemmas 2.3.3.3, 2.4.3.3, 2.4.3.8 and 2.4.3.9, though one should be careful with the results from §2.4.3, since they all *assume* the actor context to be *weakly unital* (see also: proposition 3.1.4 below). $\star$

Proposition 3.1.4 *If $\langle A, B \rangle$ is an invertible dual pair of algebras, then $\mathbb{A}$ and $\mathbb{B}$ are weakly unital in the sense of definition 2.4.1.7.* The counit for $\mathbb{A}$, being a functional on B, will be denoted by ε_B. Similarly ε_A denotes the counit for $\mathbb{B}$.

Proof. First we show that B is unital as an A-bimodule (§1.4) and vice versa; let us for instance prove that $B = B \triangleleft A$. So choose any $b \in B$ and also take some $c \in A$ and $d \in B$ such that $\langle c, d \rangle = 1$. Since ρ_P is surjective, we can write $c \otimes b = \sum_i \rho_P(p_i \otimes q_i)$ with $p_i \in A$ and $q_i \in B$. Then for all $a \in A$

$$\begin{aligned} \langle a, b \rangle &= \big\langle d \otimes a, \; c \otimes b \big\rangle \\ &= \textstyle\sum_i \big\langle d \otimes a, \; \rho_P(p_i \otimes q_i) \big\rangle \\ &= \textstyle\sum_i \big\langle d \triangleright p_i, \; a \triangleright q_i \big\rangle \\ &= \textstyle\sum_i \big\langle (d \triangleright p_i)\, a, \; q_i \big\rangle \\ &= \textstyle\sum_i \big\langle a, \; q_i \triangleleft (d \triangleright p_i) \big\rangle \end{aligned}$$

hence $b = \sum_i q_i \triangleleft (d \triangleright p_i)$ belongs to $B \triangleleft A$. It follows that $A \otimes B$ is unital as a $(B \otimes A)$-bimodule, hence according to proposition 2.4.1.5.viii, $\mathbb{B} \otimes \mathbb{A}$ is weakly unital. Now proposition 2.6.3 yields the result. ∎

Corollary 3.1.5 *Everything in survey 2.4.2.2 and §2.4.3 applies not only to the actor contexts $\mathbb{A}$ and $\mathbb{B}$, but also to all the tensor products constructed from them, e.g. $\mathbb{B} \otimes \mathbb{A}$, $\mathbb{A} \otimes \mathbb{A}$, etc.* We will tacitly adopt all the identifications shown in the diagram in survey 2.4.2.2, e.g. we will no longer explicitly distinguish the functional P from the actor (λ_P, ρ_P) unless confusion is likely. Similarly we identify e.g. the counit ε_A with the actor $1_{\mathbb{B}} = (\mathrm{id}_A, \mathrm{id}_A)$, etc. Furthermore we will extensively rely on (2.11) and (2.12), and whenever it is allowed, we shall exploit the notational convenience offered by (2.13) and remark 2.5.3.

Definition 3.1.6 Let $\langle A, B \rangle$ be any invertible dual pair of algebras. We define linear mappings $S_A : A \to B'$ and $S_B : B \to A'$ requiring that

$$\langle S_A(a), b \rangle \;=\; \langle P^{-1}, a \otimes b \rangle \;=\; \langle a, S_B(b) \rangle \tag{3.2}$$

for all $a \in A$ and $b \in B$. Observe (3.2) makes sense because also P^{-1} is an actor for $\mathbb{B} \otimes \mathbb{A}$ (cf. definition 3.1.1 and corollary 3.1.5). We say that S_A and S_B are the *antipodes* on A and B. •

Remark 3.1.7 Now also the inverses of λ_P and ρ_P may be expressed in Sweedler notation:

$$\begin{aligned} \lambda_P^{-1}(a \otimes b) &= \langle S_A(a_{(2)}), b_{(2)} \rangle \, a_{(1)} \otimes b_{(1)} \\ \rho_P^{-1}(a \otimes b) &= \langle S_A(a_{(1)}), b_{(1)} \rangle \, a_{(2)} \otimes b_{(2)}. \end{aligned}$$

Lemma 3.1.8 *Let $\langle A, B \rangle$ be an invertible dual pair of algebras, and consider also any weakly unital actor context $\mathbb{E} \equiv (E; \Omega, \langle \cdot, \cdot \rangle)$. Now take any pre-actor (α, β) for $\mathbb{E} \otimes \mathbb{B}$ and consider $\mathrm{id} \otimes \lambda_P$ and $\beta \otimes \mathrm{id}$ as mappings on $\Omega \otimes A \otimes B$. Then $\mathrm{id} \otimes \lambda_P$ commutes with $\beta \otimes \mathrm{id}$. Similarly $\mathrm{id} \otimes \rho_P$ commutes with $\alpha \otimes \mathrm{id}$.*

Proof. Take any $a, c \in A$ and $b, d \in B$ and $x \in E$ and $\omega \in \Omega$. Then we can write $\lambda_P(a \otimes b) = \sum_i p_i \otimes q_i$ with $p_i \in A$ and $q_i \in B$. An easy computation (similar to the proof of proposition 3.1.4) shows that $\sum_i \langle c, q_i \rangle \, p_i = (b \triangleleft c) \triangleright a$, hence

$$\begin{aligned} \left\langle x \otimes d \otimes c,\, (\beta \otimes \mathrm{id})(\mathrm{id} \otimes \lambda_P)(\omega \otimes a \otimes b) \right\rangle &= \textstyle\sum_i \left\langle x \otimes d,\, \beta(\omega \otimes p_i) \right\rangle \left\langle c,\, q_i \right\rangle \\ &= \left\langle x \otimes d,\, \beta\left(\omega \otimes (b \triangleleft c) \triangleright a\right) \right\rangle \\ &= \left\langle x \otimes d(b \triangleleft c),\, \beta(\omega \otimes a) \right\rangle. \end{aligned}$$

To obtain the last equality we used an analogue of lemma 2.6.4.
Now write $\beta(\omega \otimes a) = \sum_j \omega_j \otimes a_j$ with $\omega_j \in \Omega$ and $a_j \in A$. Then we have

$$\begin{aligned} \left\langle x \otimes d \otimes c,\, (\mathrm{id} \otimes \lambda_P)(\beta \otimes \mathrm{id})(\omega \otimes a \otimes b) \right\rangle &= \textstyle\sum_j \left\langle x,\, \omega_j \right\rangle \left\langle d \otimes c,\, \lambda_P(a_j \otimes b) \right\rangle \\ &= \textstyle\sum_j \left\langle x,\, \omega_j \right\rangle \left\langle a_j \triangleleft d,\, b \triangleleft c \right\rangle \\ &= \textstyle\sum_j \left\langle x \otimes d(b \triangleleft c),\, \omega_j \otimes a_j \right\rangle \end{aligned}$$

and the result follows. ■

Corollary 3.1.9 *Let $\langle A, B \rangle$ be any invertible dual pair of algebras. If (α, β) is a pre-actor for $\mathbb{B}$, then λ_P commutes with $\beta \otimes \mathrm{id}$, hence also λ_P^{-1} commutes with $\beta \otimes \mathrm{id}$. Similarly ρ_P and ρ_P^{-1} commute with $\alpha \otimes \mathrm{id}$.*

Proof. In the above lemma, let $\mathbb{E}$ be the trivial actor context $(\mathbb{C}; \mathbb{C}, \langle \cdot, \cdot \rangle)$. ■

Proposition 3.1.10 *Let $\langle A, B \rangle$ be an invertible dual pair of algebras. Then the antipodes on A and B can be viewed as linear mappings*

$$S_A : A \to \mathrm{Env}(\mathbb{A}) \qquad\qquad S_B : B \to \mathrm{Env}(\mathbb{B}).$$

In fact, in the sense of notation 2.6.7, we have for any $a \in A$ and $b \in B$ that

$$S_A(a) = (f_a \hat{\otimes} \mathrm{id})(P^{-1}) \qquad\qquad S_B(b) = (\mathrm{id} \hat{\otimes} f_b)(P^{-1}). \tag{3.3}$$

Proof. Combine the above corollary with propositions 2.6.6 and 2.6.8. ■

Notation 3.1.11 We assume the reader to be more or less familiar with the *leg-numbering* notation [4] for tensor products, e.g. if we consider $\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{A}$, then by P_{23} we mean the actor $\Phi(1_{\mathbb{B}}\otimes P)$ with Φ as in lemma 2.6.2, etc. Furthermore we occasionally appeal to proposition 2.6.5, although usually we will suppress the map Φ in our notation. •

Lemma 3.1.12 *Let $\langle A, B\rangle$ be any invertible dual pair of algebras. Then $P_{12}P_{13}$ is an actor for $\mathbb{B}\otimes\mathbb{A}\otimes\mathbb{A}$. Similarly $P_{13}P_{23}$ is an actor for $\mathbb{B}\otimes\mathbb{B}\otimes\mathbb{A}$.*

Proof. Use lemma 3.1.8 with $\mathbb{E} = \mathbb{A}$ and $(\alpha, \beta) = \chi(P)$ where χ is the obvious flip map from $\mathrm{Act}(\mathbb{B}\otimes\mathbb{A})$ into $\mathrm{Act}(\mathbb{A}\otimes\mathbb{B})$. Rearranging the 'legs' a little bit, we conclude that $(\lambda_P)_{13}$ and $(\rho_P)_{12}$ commute. Now invoke lemma 2.4.3.4. ■

Comultiplications Before we enter this subject, we like to emphasize that comultiplications are obsolete in our approach. However we shall pay some attention to them, thus improving the link with Hopf algebra theory.
Whenever $\langle A, B\rangle$ is an invertible dual pair of algebras, we have weakly unital actor contexts $\mathbb{A}$ and $\mathbb{B}$, and therefore also weak comultiplications (§2.1.3, §A)

$$\Delta_A : A \to A \,\overline{\otimes}\, A \qquad \text{and} \qquad \Delta_B : B \to B \,\overline{\otimes}\, B.$$

Recall that $A \,\overline{\otimes}\, A$ and $\mathrm{Act}(\mathbb{A}\otimes\mathbb{A})$ are both contained in $(B\otimes B)'$. Therefore the following makes sense:

Proposition 3.1.13 *Let $\langle A, B\rangle$ be any invertible dual pair of algebras. Then the comultiplications on A and B can be viewed as linear mappings*

$$\Delta_A : A \to \mathrm{Env}(\mathbb{A}\otimes\mathbb{A}) \qquad\qquad \Delta_B : B \to \mathrm{Env}(\mathbb{B}\otimes\mathbb{B}).$$

In fact, in the sense of notation 2.6.7, we have for any $a \in A$ and $b \in B$ that

$$\Delta_A(a) = (f_a \,\hat{\otimes}\, \mathrm{id})(P_{12}P_{13}) \qquad\qquad \Delta_B(b) = (\mathrm{id} \,\hat{\otimes}\, f_b)(P_{13}P_{23}). \tag{3.4}$$

Observe that these formulas make sense because of lemma 3.1.12.

Proof. Let's prove the second formula. Take any $b \in B$ and $a, c \in A$, and write $\lambda_P(c\otimes b) = \sum_i p_i \otimes q_i$ with $p_i \in A$ and $q_i \in B$. An easy computation using (3.1) shows that $\sum_i \langle a, q_i\rangle\, p_i = (b \triangleleft a) \triangleright c$.
Appealing to proposition 2.6.6 (with $\mathbb{E}_1 = \mathbb{B}\otimes\mathbb{B}$ and $\mathbb{E}_2 = \mathbb{A}$) we obtain

$$\begin{aligned}
\big\langle (\mathrm{id} \,\hat{\otimes}\, f_b)(P_{13}P_{23}),\, a\otimes c\big\rangle &= \big\langle P_{13}P_{23},\, a\otimes c\otimes b\big\rangle \\
&= \big\langle P_{13},\, a\otimes \lambda_P(c\otimes b)\big\rangle \\
&= \textstyle\sum_i \big\langle P_{13},\, a\otimes p_i\otimes q_i\big\rangle \\
&= \textstyle\sum_i \big\langle a,\, q_i\big\rangle \big\langle 1_{\mathbb{B}},\, p_i\big\rangle \\
&= \big\langle 1_{\mathbb{B}},\, (b\triangleleft a)\triangleright c\big\rangle \\
&= \big\langle b,\, ac\big\rangle \\
&= \big\langle \Delta_B(b),\, a\otimes c\big\rangle .
\end{aligned}$$

We still have to show that $\Delta_B(b)$ belongs to $\mathrm{Env}(\mathbb{B} \otimes \mathbb{B})$. Therefore, take any actor (α, β) for $\mathbb{B} \otimes \mathbb{B}$. From lemma 3.1.8 we obtain that $\beta \otimes \mathrm{id}$ commutes with $(\lambda_P)_{13}$ and $(\lambda_P)_{23}$, and hence also with $(\lambda_P)_{13}(\lambda_P)_{23}$. Similarly $\alpha \otimes \mathrm{id}$ commutes with $(\rho_P)_{23}(\rho_P)_{13}$. Proposition 2.6.8 yields the result. ∎

Recalling remarks A.2.i and A.3.iv in appendix A, we can construct mappings

$$\mathrm{id} \mathbin{\overline{\otimes}} \Delta_A : B \mathbin{\overline{\otimes}} A \to B \mathbin{\overline{\otimes}} A \mathbin{\overline{\otimes}} A \qquad \text{and} \qquad \Delta_B \mathbin{\overline{\otimes}} \mathrm{id} : B \mathbin{\overline{\otimes}} A \to B \mathbin{\overline{\otimes}} B \mathbin{\overline{\otimes}} A.$$

Obviously the pairing P belongs to $B \mathbin{\overline{\otimes}} A$, hence the following makes sense:

Proposition 3.1.14 *If $\langle A, B\rangle$ is an invertible dual pair of algebras, then*

$$(\mathrm{id} \mathbin{\overline{\otimes}} \Delta_A)(P) = P_{12}P_{13} \qquad \text{and} \qquad (\Delta_B \mathbin{\overline{\otimes}} \mathrm{id})(P) = P_{13}P_{23}.$$

Proof. Recalling the proof of the previous proposition, we obtain that

$$\big\langle P_{13}P_{23},\, a \otimes c \otimes b\big\rangle \;=\; \langle b, ac\rangle \;=\; \big\langle P,\, ac \otimes b\big\rangle \;=\; \big\langle (\Delta_B \mathbin{\overline{\otimes}} \mathrm{id})(P),\, a \otimes c \otimes b\big\rangle$$

for all $b \in B$ and $a, c \in A$. ∎

The following lemma will be useful in constructing a quantum double (§3.9.2).

Lemma 3.1.15 *Let $\langle A, B\rangle$ be a dual pair of algebras and assume $\mathbb{A}$ and $\mathbb{B}$ to be weakly unital. If there exist linear bijections λ and ρ from $A \otimes B$ onto $A \otimes B$ such that $(\varepsilon_A \otimes \varepsilon_B)\lambda^{-1} = (\varepsilon_A \otimes \varepsilon_B)\,\rho^{-1}$ and, for all $a \in A$ and $b, d \in B$,*

$$\begin{aligned} (\varepsilon_A \otimes \mathrm{id})\lambda(a \otimes b) &= a \triangleright b & \qquad \lambda\big((a \triangleleft d) \otimes b\big) &= \big((\,\cdot\, \triangleleft d) \otimes \mathrm{id}\big)\lambda(a \otimes b) \\ (\varepsilon_A \otimes \mathrm{id})\,\rho(a \otimes b) &= b \triangleleft a & \qquad \rho\big((d \triangleright a) \otimes b\big) &= \big((d \triangleright \,\cdot\,) \otimes \mathrm{id}\big)\rho(a \otimes b) \end{aligned}$$

then $\langle A, B\rangle$ is an invertible dual pair of algebras, with $\lambda_P = \lambda$ and $\rho_P = \rho$.

Proof. Observe that for any $a, c \in A$ and $b, d \in B$ we have

$$\begin{aligned} \big\langle d \otimes c,\, \lambda(a \otimes b)\big\rangle &= \big\langle c,\, \big(\varepsilon_A(\,\cdot\, \triangleleft d) \otimes \mathrm{id}\big)\lambda(a \otimes b)\big\rangle \\ &= \big\langle c,\, (\varepsilon_A \otimes \mathrm{id})\lambda\big((a \triangleleft d) \otimes b\big)\big\rangle \\ &= \big\langle c,\, (a \triangleleft d) \triangleright b\big\rangle \\ &= \big\langle c(a \triangleleft d),\, b\big\rangle \\ &= \big\langle a \triangleleft d,\, b \triangleleft c\big\rangle\,. \end{aligned}$$

Similarly we show $\big\langle d \otimes c,\, \rho(a \otimes b)\big\rangle = \big\langle d \triangleright a,\, c \triangleright b\big\rangle$. It follows that the pairing $P : A \otimes B \to \mathbb{C}$ is strict$^\sharp$ continuous, and $\theta(P) = (\lambda, \rho)$ is the corresponding actor for $\mathbb{B} \otimes \mathbb{A}$. Eventually lemma 2.4.3.2 yields $(\mathbb{B} \otimes \mathbb{A})$-invertibility. ∎

3.2 Multiplicativity

Abstract 3.2 Given any dual pair of algebras, it would be natural to impose some compatibility condition on the algebra structures involved, i.e. an interaction between the products in both algebras. In the Hopf algebra setting this is usually done by requiring comultiplications to be homomorphisms. However we shall also express this condition of 'multiplicativity' in a way that does not involve comultiplications. This yields the notion of a *Hopf system*, which is the main subject in the present chapter. Eventually we obtain several so-called *twist* maps which can be used later, e.g. to construct a quantum double or to show that the antipodes are anti-multiplicative. $\star$

Let $\langle A, B\rangle$ be any invertible dual pair of algebras. Sometimes it is convenient to write $\langle B \otimes A, A \otimes B\rangle$ as a pairing of $A \otimes B$ with $A \otimes B$ itself. To avoid confusion, the latter will be denoted by $\langle \cdot \underset{\chi}{|} \cdot \rangle$, i.e. for any $x, y \in A \otimes B$ we define $\langle x \underset{\chi}{|} y\rangle = \langle \chi(x), y\rangle$. Here χ denotes the flip $A \otimes B \to B \otimes A$. We shall use the same symbol χ to denote the flip $B \otimes A \to A \otimes B$, confusion being unlikely.

Definition 3.2.1 Let $\langle A, B\rangle$ be any invertible dual pair of algebras. An actor $R \equiv (\lambda, \rho)$ for $\mathbb{B} \otimes \mathbb{A}$ is said to be *multiplicative* if

$$\langle R, xy\rangle \ = \ \big\langle \rho(x) \underset{\chi}{|} \lambda(y)\big\rangle \tag{3.5}$$

for all $x, y \in A \otimes B$. Notice that if R belongs to $\mathrm{Env}(\mathbb{B} \otimes \mathbb{A})$ then (3.5) reads $\langle R, xy\rangle \ = \ \langle x \triangleleft R \underset{\chi}{|} R \triangleright y\rangle$. In particular the pairing P will be multiplicative if

$$\big\langle ac,\, bd\big\rangle \ = \ \big\langle \rho_P(a \otimes b) \underset{\chi}{|} \lambda_P(c \otimes d)\big\rangle \tag{3.6}$$

for all $a, c \in A$ and $b, d \in B$. $\bullet$

Remark 3.2.2 We claim that equation (3.6) indeed amounts to the fact that the comultiplications on A and B are homomorphisms. Although a rigorous proof shall be given below, it may also be instructive to appreciate the meaning of (3.6) in terms of Sweedler notation:

$$\begin{aligned}
\big\langle \rho_P(a \otimes b) \underset{\chi}{|} \lambda_P(c \otimes d)\big\rangle \ &= \ \big\langle \langle a_{(1)}, b_{(1)}\rangle\, a_{(2)} \otimes b_{(2)} \underset{\chi}{|} \langle c_{(2)}, d_{(2)}\rangle\, c_{(1)} \otimes d_{(1)}\big\rangle \\
&= \ \langle a_{(1)}, b_{(1)}\rangle\langle a_{(2)}, d_{(1)}\rangle\langle c_{(1)}, b_{(2)}\rangle\langle c_{(2)}, d_{(2)}\rangle \\
&= \ \langle a, b_{(1)}d_{(1)}\rangle\langle c, b_{(2)}d_{(2)}\rangle \\
&= \ \langle ac, bd\rangle.
\end{aligned}$$

Proposition 3.2.3 *Let $\langle A, B\rangle$ be any invertible dual pair of algebras. Then the following assertions are equivalent:*

i. *P is multiplicative in the sense of the above definition*

ii. *$\Delta_A : A \to \mathrm{Env}(\mathbb{A} \otimes \mathbb{A})$ is an algebra homomorphism*

iii. *$\Delta_B : B \to \mathrm{Env}(\mathbb{B} \otimes \mathbb{B})$ is an algebra homomorphism*

Proof. Take any $a, c \in A$ and $b, d \in B$ and write $\rho_P(a \otimes b) = \sum_i p_i \otimes q_i$ with $p_i \in A$ and $q_i \in B$. Recall that P_{13}, P_{23} and $P_{13}P_{23}$ are actors for $\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A}$, whereas $\Delta_B(d) \otimes 1_{\mathbb{A}}$ belongs to $\mathrm{Env}(\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A})$. Also recall remark 2.5.3.
Now observe the following 'circle' of equalities:

$$\begin{aligned}
\big\langle \Delta_B(bd),\, a \otimes c \big\rangle &\overset{\text{(iii)}}{=} \big\langle \Delta_B(b)\, \Delta_B(d),\, a \otimes c \big\rangle \\
&= \big\langle \Delta_B(b),\, \Delta_B(d) \triangleright (a \otimes c) \big\rangle \\
&= \big\langle (\mathrm{id} \hat{\otimes} f_b)(P_{13}P_{23}),\, \Delta_B(d) \triangleright (a \otimes c) \big\rangle \\
&= \big\langle P_{13}P_{23},\, \big(\Delta_B(d) \triangleright (a \otimes c)\big) \otimes b \big\rangle \\
&= \big\langle P_{13}P_{23}\big(\Delta_B(d) \otimes 1_{\mathbb{A}}\big),\, a \otimes c \otimes b \big\rangle \\
&= \big\langle P_{23}\big(\Delta_B(d) \otimes 1_{\mathbb{A}}\big),\, (\rho_P)_{13}(a \otimes c \otimes b) \big\rangle \\
&= \textstyle\sum_i \big\langle P_{23}\,\big(\Delta_B(d) \otimes 1_{\mathbb{A}}\big),\, p_i \otimes c \otimes q_i \big\rangle \\
&= \textstyle\sum_i \big\langle P_{23},\, \big(\Delta_B(d) \triangleright (p_i \otimes c)\big) \otimes q_i \big\rangle \\
&= \textstyle\sum_i \big\langle 1_{\mathbb{B}} \otimes q_i,\, \Delta_B(d) \triangleright (p_i \otimes c) \big\rangle \\
&= \textstyle\sum_i \big\langle (1_{\mathbb{B}} \otimes q_i)\Delta_B(d),\, p_i \otimes c \big\rangle \\
&= \textstyle\sum_i \big\langle \Delta_B(d),\, p_i \otimes (c \triangleleft q_i) \big\rangle \\
&= \textstyle\sum_i \big\langle p_i(c \triangleleft q_i),\, d \big\rangle \\
&= \textstyle\sum_i \big\langle c \triangleleft q_i,\, d \triangleleft p_i \big\rangle \\
&= \textstyle\sum_i \big\langle q_i \otimes p_i,\, \lambda_P(c \otimes d) \big\rangle \\
&= \big\langle \rho_P(a \otimes b) \underset{\chi}{|} \lambda_P(c \otimes d) \big\rangle \\
&\overset{\text{(i)}}{=} \big\langle ac,\, bd \big\rangle \\
&= \big\langle \Delta_B(bd),\, a \otimes c \big\rangle
\end{aligned}$$

This proves (i) $\Leftrightarrow$ (iii). Similarly we prove (i) $\Leftrightarrow$ (ii). ∎

Definition 3.2.4 An invertible dual pair of algebras is called a *Hopf system* whenever its pairing is multiplicative in the sense of definition 3.2.1. •

Proposition 3.2.5 *If $\langle A, B\rangle$ is a Hopf system, then its counits ε_A and ε_B are algebra homomorphisms into $\mathbb{C}$.*

Proof. Take any $a, c \in A$ and $b, d \in B$. Since ρ_P is surjective, we can write $a \otimes b = \sum_i \rho_P(p_i \otimes q_i)$ with $p_i \in A$ and $q_i \in B$. Then

$$\begin{aligned}
\big\langle a(c \triangleleft b),\, d \big\rangle &= \big\langle c \triangleleft b,\, d \triangleleft a \big\rangle \\
&= \big\langle b \otimes a,\, \lambda_P(c \otimes d) \big\rangle \\
&= \textstyle\sum_i \big\langle \rho_P(p_i \otimes q_i) \underset{\chi}{|} \lambda_P(c \otimes d) \big\rangle \\
&= \textstyle\sum_i \big\langle p_i c,\, q_i d \big\rangle \\
&= \textstyle\sum_i \big\langle p_i c \triangleleft q_i,\, d \big\rangle
\end{aligned}$$

and hence $a(c \triangleleft b) = \sum_i p_i c \triangleleft q_i$. It follows that

$$\begin{aligned}
\big\langle 1_{\mathbb{B}},\, a(c \triangleleft b)\big\rangle &= \textstyle\sum_i \big\langle 1_{\mathbb{B}},\, p_i c \triangleleft q_i\big\rangle \\
&= \textstyle\sum_i \big\langle p_i c,\, q_i\big\rangle \\
&= \textstyle\sum_i \big\langle p_i,\, c \triangleright q_i\big\rangle \\
&= \textstyle\sum_i \big\langle P,\, p_i \otimes (c \triangleright q_i)\big\rangle \\
&= \textstyle\sum_i \big\langle P(1_{\mathbb{B}} \otimes c),\, p_i \otimes q_i\big\rangle \\
&= \big\langle 1_{\mathbb{B}} \otimes c,\, a \otimes b\big\rangle \\
&= \big\langle 1_{\mathbb{B}},\, a\big\rangle \big\langle 1_{\mathbb{B}},\, c \triangleleft b\big\rangle .
\end{aligned}$$

Since $A \triangleleft B = A$, the above proves that $\varepsilon_A \simeq 1_{\mathbb{B}}$ is a homomorphism. ∎

Definition 3.2.6 Consider two algebras (A, m_A) and (B, m_B). A linear map $T : B \otimes A \to A \otimes B$ is said to be an (A, B)-*twisting* if

$$\begin{aligned}
T(m_B \otimes \mathrm{id}) &= (\mathrm{id} \otimes m_B)(T \otimes \mathrm{id})(\mathrm{id} \otimes T) \\
T(\mathrm{id} \otimes m_A) &= (m_A \otimes \mathrm{id})(\mathrm{id} \otimes T)(T \otimes \mathrm{id}).
\end{aligned}$$

The following result can be found in almost every textbook or paper [11, 47] covering topics like the quantum double or the quantum Yang-Baxter equation:

Proposition 3.2.7 *If $T : B \otimes A \to A \otimes B$ is an (A, B)-twisting between two algebras A and B, then the vector space $A \otimes B$ can be made into an algebra with product $(m_A \otimes m_B)(\mathrm{id} \otimes T \otimes \mathrm{id})$.* This algebra will be denoted by $A \otimes_T B$.

Lemma 3.2.8 *Consider two algebras A and B, and let $R : A \otimes B \to A \otimes B$ be a linear map. Then $R\chi$ is an (A, B)-twisting if and only if*

$$R(\mathrm{id} \otimes m_B) = (\mathrm{id} \otimes m_B) R_{12} R_{13} \tag{3.7}$$

$$R(m_A \otimes \mathrm{id}) = (m_A \otimes \mathrm{id}) R_{23} R_{13}. \tag{3.8}$$

Furthermore, if R is a bijection and if $R\chi$ is an (A, B)-twisting, then $R^{-1}\chi$ is an $(A^{\mathrm{op}}, B^{\mathrm{op}})$-twisting.

Lemma 3.2.9 *Consider two algebras A and B, and let $R, Q : A \otimes B \to A \otimes B$ be linear maps such that $R\chi$ and $Q\chi$ are (A, B)-twistings. If $R_{13}Q_{12} = Q_{12}R_{13}$ and $R_{13}Q_{23} = Q_{23}R_{13}$, then $RQ\chi$ is again an (A, B)-twisting.*

Proposition 3.2.10 *Let $\langle A, B\rangle$ be any Hopf system. Then*

i. $\lambda_P \chi$ *is an* (A^{op}, B)*-twisting*

ii. $\rho_P \chi$ *is an* (A, B^{op})*-twisting.*

Proof. Take any $a, x \in A$ and $b, d, y \in B$ and write $\lambda_P(a \otimes d) = \sum_i p_i \otimes q_i$ and $x \otimes y = \sum_k \rho_P(v_k \otimes w_k)$ with $p_i, v_k \in A$ and $q_i, w_k \in B$. Also observe that $(q_i \triangleright x) \otimes y = \sum_k \rho_P\big((q_i \triangleright v_k) \otimes w_k\big)$ because of lemma 2.6.4. Now we have

$$\begin{aligned}
&\big\langle y \otimes x,\, (\mathrm{id} \otimes m_B)(\lambda_P)_{12}(\lambda_P)_{13}(a \otimes b \otimes d)\big\rangle \\
&\quad= \textstyle\sum_i \big\langle y \otimes x,\, (\mathrm{id} \otimes m_B)\big(\lambda_P(p_i \otimes b) \otimes q_i\big)\big\rangle \\
&\quad= \textstyle\sum_i \big\langle y \otimes (q_i \triangleright x),\, \lambda_P(p_i \otimes b)\big\rangle \\
&\quad= \textstyle\sum_{i,k} \big\langle \rho_P\big((q_i \triangleright v_k) \otimes w_k\big) \underset{\chi}{|} \lambda_P(p_i \otimes b)\big\rangle \\
&\quad= \textstyle\sum_{i,k} \big\langle (q_i \triangleright v_k)p_i,\, w_k b\big\rangle \\
&\quad= \textstyle\sum_{i,k} \big\langle q_i \triangleright v_k,\, p_i \triangleright w_k b\big\rangle \\
&\quad= \textstyle\sum_{i,k} \big\langle \rho_P(v_k \otimes w_k b),\, q_i \otimes p_i\big\rangle \\
&\quad= \textstyle\sum_k \big\langle \rho_P(v_k \otimes w_k b) \underset{\chi}{|} \lambda_P(a \otimes d)\big\rangle \\
&\quad= \textstyle\sum_k \big\langle v_k a,\, w_k b d\big\rangle \\
&\quad= \textstyle\sum_k \big\langle \rho_P(v_k \otimes w_k) \underset{\chi}{|} \lambda_P(a \otimes bd)\big\rangle \\
&\quad= \big\langle y \otimes x,\, \lambda_P(\mathrm{id} \otimes m_B)(a \otimes b \otimes d)\big\rangle
\end{aligned}$$

which proves (3.7) for $R = \lambda_P$. The other cases are similar. ∎

Corollary 3.2.11 *$\rho_P \lambda_P^{-1} \chi$ is an (A, B^{op})-twisting.*

Proof. Both $\rho_P \chi$ and $\lambda_P^{-1} \chi$ are (A, B^{op})-twistings. Analogous to the proof of lemma 3.1.12 we obtain that $(\rho_P)_{13}$ commutes with $(\lambda_P)_{12}$ and hence with $(\lambda_P^{-1})_{12}$. Similarly $(\rho_P)_{13}$ commutes with $(\lambda_P^{-1})_{23}$. Now invoke lemma 3.2.9. ∎

Remark 3.2.12 Recall that the pairing P is an actor for the weakly unital actor context $\mathbb{B} \otimes \mathbb{A}$. Now if P enjoys one of the equivalent conditions (i-iv) in lemma 2.4.3.9, e.g. if λ_P and ρ_P commute, then the map $\rho_P \lambda_P^{-1}$ appearing in the previous corollary is actually *dual* to the 'inner' homomorphism

$$\pi_P : B \otimes A \to \mathrm{Act}(\mathbb{B} \otimes \mathbb{A}) : y \mapsto P y P^{-1}$$

in the sense that $\langle \pi_P(y), x\rangle = \langle y, \rho_P \lambda_P^{-1}(x)\rangle$ for all $y \in B \otimes A$ and $x \in A \otimes B$. In fact the twist map $\rho_P \lambda_P^{-1} \chi$ is the one that will be used later (see §3.9) to construct a *quantum double* for the pair $\langle A, B^{\mathrm{op}}\rangle$. ⋆

3.3 Antipodes

Proposition 3.3.1 *If $\langle A, B\rangle$ is a Hopf system, then its antipodes*

$$S_A : A \to \mathrm{Env}(\mathbb{A}) \qquad\qquad S_B : B \to \mathrm{Env}(\mathbb{B}).$$

are anti-homomorphisms.

Proof. Take any $a, c \in A$ and $b \in B$ and write $\rho_P^{-1}(a \otimes b) = \sum_i p_i \otimes q_i$ with $p_i \in A$ and $q_i \in B$. Now observe that

$$\begin{aligned}
\big\langle S_A(a)\, S_A(c),\, b\big\rangle &= \big\langle S_A(a),\, S_A(c) \triangleright b\big\rangle \\
&= \big\langle P^{-1},\, a \otimes \big(S_A(c) \triangleright b\big)\big\rangle \\
&= \big\langle P^{-1}\left(1_{\mathbb{B}} \otimes S_A(c)\right),\, a \otimes b\big\rangle \\
&= \big\langle 1_{\mathbb{B}} \otimes S_A(c),\, \rho_P^{-1}(a \otimes b)\big\rangle \\
&= \textstyle\sum_i \big\langle 1_{\mathbb{B}},\, p_i\big\rangle \big\langle S_A(c),\, q_i\big\rangle \\
&= \textstyle\sum_i \big\langle 1_{\mathbb{B}},\, p_i\big\rangle \big\langle P^{-1},\, c \otimes q_i\big\rangle \\
&= \textstyle\sum_i \big\langle 1_{\mathbb{B}} \otimes 1_{\mathbb{B}} \otimes 1_{\mathbb{A}},\, (\rho_P^{-1})_{13}(c \otimes p_i \otimes q_i)\big\rangle \\
&= \big\langle 1_{\mathbb{B}} \otimes 1_{\mathbb{B}} \otimes 1_{\mathbb{A}},\, (\rho_P^{-1})_{13}(\rho_P^{-1})_{23}(c \otimes a \otimes b)\big\rangle \\
&\overset{(*)}{=} \big\langle 1_{\mathbb{B}} \otimes 1_{\mathbb{A}},\, (m_A \otimes \mathrm{id})(\rho_P^{-1})_{13}(\rho_P^{-1})_{23}(c \otimes a \otimes b)\big\rangle \\
&\overset{(\sharp)}{=} \big\langle 1_{\mathbb{B}} \otimes 1_{\mathbb{A}},\, \rho_P^{-1}(m_A \otimes \mathrm{id})(c \otimes a \otimes b)\big\rangle \\
&= \big\langle P^{-1},\, ca \otimes b\big\rangle \\
&= \big\langle S_A(ca),\, b\big\rangle\,.
\end{aligned}$$

Observe how $(*)$ relies on $\varepsilon_A \simeq 1_{\mathbb{B}}$ being a homomorphism (proposition 3.2.5). Proposition 3.2.10.ii implies (3.8) with $R = \rho_P$. This yields $(\sharp)$. ∎

Lemma 3.3.2 *If $\langle A, B\rangle$ is a Hopf system, then $S_B(B) \triangleright A = A = A \triangleleft S_B(B)$. In other words, A is unital as an $S_B(B)$-bimodule* (recall that $S_B(B)$ is a subalgebra of $\mathrm{Env}(\mathbb{B})$ because of proposition 3.3.1).

Proof. Choose any $a \in A$ and $d \in B$. Also take $c \in A$ and $b \in B$ with $\langle c, b\rangle = 1$, and write $\lambda_P(a \otimes b) = \sum_i p_i \otimes q_i$ with $p_i \in A$ and $q_i \in B$. Then we have

$$\begin{aligned}
\langle a, d\rangle &= \big\langle d \otimes c,\, a \otimes b\big\rangle \\
&= \textstyle\sum_i \big\langle d \otimes c,\, \lambda_P^{-1}(p_i \otimes q_i)\big\rangle \\
&= \textstyle\sum_i \big\langle P^{-1},\, (p_i \otimes q_i) \triangleleft (d \otimes c)\big\rangle \\
&= \textstyle\sum_i \big\langle p_i \triangleleft d,\, S_B(q_i \triangleleft c)\big\rangle \\
&= \textstyle\sum_i \big\langle p_i,\, d\, S_B(q_i \triangleleft c)\big\rangle \\
&= \textstyle\sum_i \big\langle S_B(q_i \triangleleft c) \triangleright p_i,\, d\big\rangle
\end{aligned}$$

and hence $a = \sum_i S_B(q_i \triangleleft c) \triangleright p_i$. The other case is similar. ∎

Recall that $S_A(A)$ is a subspace of B' (cf. definition 3.1.6). On the other hand B' is a B-bimodule under canonical actions (§2.1.1) so the following makes sense:

Lemma 3.3.3 *Let $\langle A, B\rangle$ be a Hopf system. Then $S_A(A)$ is a sub-B-bimodule of B'. In fact we have for any $a \in A$ and $b \in B$ that*

$$S_A(a) \triangleleft b \;=\; S_A\big(S_B(b) \triangleright a\big)\,. \tag{3.9}$$

Proof. Anti-multiplicativity of S_B yields that for any $a \in A$ and $b, d \in B$

$$\begin{aligned}\big\langle S_A(a) \triangleleft b,\, d\big\rangle &= \big\langle S_A(a),\, bd\big\rangle \\ &= \big\langle a,\, S_B(bd)\big\rangle \\ &= \big\langle a,\, S_B(d)\, S_B(b)\big\rangle \\ &= \big\langle S_B(b) \triangleright a,\, S_B(d)\big\rangle \\ &= \big\langle S_A\left(S_B(b) \triangleright a\right),\, d\big\rangle\end{aligned}$$

and (3.9) follows. Similarly for the *left* actions of B on $S_A(A)$. ■

Corollary 3.3.4 *Let $\langle A, B\rangle$ be a Hopf system. If we assume that $S_A(A) \subseteq A$, then $\varepsilon_A S_A = \varepsilon_A$. Similarly, $S_B(B) \subseteq B$ implies $\varepsilon_B S_B = \varepsilon_B$.*

Proof. Assume $S_A(A) \subseteq A$. Then we may apply ε_A to (3.9), yielding

$$\varepsilon_A S_A\left(S_B(b) \triangleright a\right) \;=\; \big\langle S_A(a),\, b\big\rangle \;=\; \big\langle a,\, S_B(b)\big\rangle \;=\; \varepsilon_A\left(S_B(b) \triangleright a\right).$$

According to lemma 3.3.2 we have $S_B(B) \triangleright A = A$, so the result follows. ■

3.4 Regularity

Abstract 3.3 Regularity for a Hopf system is mainly about the behaviour of the antipodes. It is well-known that a Hopf algebra (A, Δ) has an *invertible* antipode if and only if also $(A, \chi\Delta)$ is a Hopf algebra; in this case the antipodes for (A, Δ) and $(A, \chi\Delta)$ are each others inverse. Below we shall obtain a similar result within the Hopf system framework. ⋆

Recall §2.7. If $\langle A, B\rangle$ is a dual pair of algebras in the sense of example 2.1.2.4, then so is $\langle A, B^{\mathrm{op}}\rangle$, obviously. The latter involves the actor contexts[2]

$$\mathbb{A} \equiv (A; B, \langle\cdot,\cdot\rangle) \qquad \text{and} \qquad \mathbb{B}^{\mathrm{op}} \equiv (B^{\mathrm{op}}; A, \langle\cdot,\cdot\rangle). \tag{3.10}$$

It is not so clear however, whether the *invertibility* of $\langle A, B\rangle$ does imply the invertibility of $\langle A, B^{\mathrm{op}}\rangle$. On the other hand we do have the following:

Lemma 3.4.1 *Let $\langle A, B\rangle$ be a dual pair of algebras. Then $\langle A, B\rangle$ is invertible if and only if $\langle A^{\mathrm{op}}, B^{\mathrm{op}}\rangle$ is invertible. Similarly, $\langle A, B^{\mathrm{op}}\rangle$ is invertible if and only if $\langle A^{\mathrm{op}}, B\rangle$ is invertible.*

Let us assume—for a moment—that both $\langle A, B\rangle$ and $\langle A, B^{\mathrm{op}}\rangle$ are invertible. Then the pairing between A and B also identifies with an invertible actor for

$$\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A} \;\equiv\; \left(B^{\mathrm{op}} \otimes A;\; A \otimes B, \langle\cdot,\cdot\rangle\right),$$

[2] The superscript $^{\mathrm{op}}$ accompanying A or B will only be written when it really matters, e.g. in (3.10) we wrote $\mathbb{A} \equiv (A; B, \langle\cdot,\cdot\rangle)$ rather than $(A; B^{\mathrm{op}}, \langle\cdot,\cdot\rangle)$ because the product on B is *irrelevant* at this point.

say $P^{\mathrm{op}} = (\lambda_P^{\mathrm{op}}, \rho_P^{\mathrm{op}})$. So in addition to λ_P and ρ_P, we have linear bijections λ_P^{op} and ρ_P^{op} from $A \otimes B$ onto $A \otimes B$, given by

$$\begin{array}{rclcl} \big\langle d \otimes c,\, \lambda_P^{\mathrm{op}}(a \otimes b)\big\rangle & = & \big\langle P^{\mathrm{op}},\, (a \triangleleft^{\mathrm{op}} d) \otimes (b \triangleleft c)\big\rangle & = & \big\langle d \triangleright a,\, b \triangleleft c\big\rangle \\ \big\langle d \otimes c,\, \rho_P^{\mathrm{op}}(a \otimes b)\big\rangle & = & \big\langle P^{\mathrm{op}},\, (d \triangleright^{\mathrm{op}} a) \otimes (c \triangleright b)\big\rangle & = & \big\langle a \triangleleft d,\, c \triangleright b\big\rangle \end{array} \tag{3.11}$$

for any $a, c \in A$ and $b, d \in B$. Furthermore, according to lemma 3.4.1, we now have actually *four* invertible dual pairs,

$$\langle A, B\rangle \qquad \langle A, B^{\mathrm{op}}\rangle \qquad \langle A^{\mathrm{op}}, B\rangle \qquad \langle A^{\mathrm{op}}, B^{\mathrm{op}}\rangle$$

each of which induces an antipode either on A or A^{op}, say *respectively*

$$S_A,\, S_A^{\mathrm{op}} : A \to \mathrm{Env}(\mathbb{A}) \qquad\qquad {}^{\mathrm{op}}S_A,\, {}^{\mathrm{op}}S_A^{\mathrm{op}} : A^{\mathrm{op}} \to \mathrm{Env}(\mathbb{A}^{\mathrm{op}})$$

and something similar for the antipodes on B and B^{op}. Now it is not so hard to see that for instance ${}^{\mathrm{op}}S_A^{\mathrm{op}} = S_A$ and ${}^{\mathrm{op}}S_A = S_A^{\mathrm{op}}$, provided that we either consider these antipodes simply as linear maps from A into B', or otherwise take into account the appropriate $\mathrm{id}^{\mathrm{op}}$ anti-isomorphisms of §2.7. Therefore only S_A, S_B and $S_A^{\mathrm{op}}, S_B^{\mathrm{op}}$ shall be used henceforth, although the above may be instructive if one wants to appreciate the *symmetry* between A and B. Anyway,

$$S_A^{\mathrm{op}} : A \to \mathrm{Env}(\mathbb{A}) \qquad \text{and} \qquad S_B^{\mathrm{op}} : B^{\mathrm{op}} \to \mathrm{Env}(\mathbb{B}^{\mathrm{op}})$$

are the antipodes associated with $\langle A, B^{\mathrm{op}}\rangle$.

Lemma 3.4.2 *If $\langle A, B\rangle$ and $\langle A, B^{\mathrm{op}}\rangle$ are invertible dual pairs of algebras, then*

$$\begin{array}{ll} (\lambda_P^{\mathrm{op}})_{21} \textit{ commutes with } (\lambda_P)_{23} & (\lambda_P^{\mathrm{op}})_{32} \textit{ commutes with } (\rho_P)_{12} \\ (\rho_P^{\mathrm{op}})_{21} \textit{ commutes with } (\rho_P)_{23} & (\rho_P^{\mathrm{op}})_{32} \textit{ commutes with } (\lambda_P)_{12} \end{array} \tag{3.12}$$

Proof. Recall that $P^{\mathrm{op}} = (\lambda_P^{\mathrm{op}}, \rho_P^{\mathrm{op}})$ is an actor for $\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A}$, hence $(\rho_P^{\mathrm{op}}, \lambda_P^{\mathrm{op}})$ is an actor for $(\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A})^{\mathrm{op}} = \mathbb{B} \otimes \mathbb{A}^{\mathrm{op}}$ (cf. §2.7). So if we define $\alpha = \chi \rho_P^{\mathrm{op}} \chi$ and $\beta = \chi \lambda_P^{\mathrm{op}} \chi$, then (α, β) is an actor for $\mathbb{A}^{\mathrm{op}} \otimes \mathbb{B}$. Now invoke lemma 3.1.8 with $\mathbb{E} = \mathbb{A}^{\mathrm{op}}$. This yields the commutation rules in the left column of (3.12). Those in the right column are obtained similarly from an appropriate analogue of lemma 3.1.8 (involving $\mathbb{B} \otimes \mathbb{A} \otimes \mathbb{E}$ rather than $\mathbb{E} \otimes \mathbb{B} \otimes \mathbb{A}$) with $\mathbb{E} = \mathbb{B}^{\mathrm{op}}$. ∎

Lemma 3.4.3 *Let $\langle A, B\rangle$ be any Hopf system. If $\langle A, B^{\mathrm{op}}\rangle$ is invertible, then $\langle A, B^{\mathrm{op}}\rangle$ is again a Hopf system.*

Proof. This is immediately clear from proposition 3.2.3. However, we did say that comultiplications are in fact obsolete in our theory, so we shall give a proof not relying on comultiplications: using the previous lemma (and rearranging the 'legs' if necessary) we obtain for all $a, c \in A$ and $b, d \in B$ that

$$\big\langle \rho_P(a \otimes b) \underset{\chi}{|} \lambda_P(c \otimes d)\big\rangle \quad = \quad \big\langle P_{14}^{\mathrm{op}} P_{32}^{\mathrm{op}},\, (\rho_P)_{12}\, (\lambda_P)_{34}\, (a \otimes b \otimes c \otimes d)\big\rangle$$

$$
\begin{aligned}
&= \big\langle \varepsilon,\, (\rho_P^{\mathrm{op}})_{14}\, (\lambda_P^{\mathrm{op}})_{32}\, (\rho_P)_{12}\, (\lambda_P)_{34}\, (a\otimes b\otimes c\otimes d) \big\rangle \\
&\overset{(3.12)}{=} \big\langle \varepsilon,\, (\rho_P)_{12}\, (\lambda_P)_{34}\, (\rho_P^{\mathrm{op}})_{14}\, (\lambda_P^{\mathrm{op}})_{32}\, (a\otimes b\otimes c\otimes d) \big\rangle \\
&\overset{(*)}{=} \big\langle \varepsilon,\, (\rho_P)_{14}\, (\lambda_P)_{32}\, (\rho_P^{\mathrm{op}})_{12}\, (\lambda_P^{\mathrm{op}})_{34}\, (a\otimes d\otimes c\otimes b) \big\rangle \\
&= \big\langle P_{14}\, P_{32},\, (\rho_P^{\mathrm{op}})_{12}\, (\lambda_P^{\mathrm{op}})_{34}\, (a\otimes d\otimes c\otimes b) \big\rangle \\
&= \big\langle \rho_P^{\mathrm{op}}(a\otimes d) \underset{\chi}{|} \lambda_P^{\mathrm{op}}(c\otimes b) \big\rangle
\end{aligned}
$$

where $\varepsilon \equiv \varepsilon_A \otimes \varepsilon_B \otimes \varepsilon_A \otimes \varepsilon_B$ (recall that $\mathbb{B}$ and $\mathbb{B}^{\mathrm{op}}$ share the counit ε_A). In $(*)$ we interchanged legs 2 and 4 (notice that this does not affect ε). ∎

Lemma 3.4.4 *If $\langle A, B\rangle$ is a Hopf system, then for any $a, c \in A$ and $b, d \in B$*

$$
\begin{aligned}
\big\langle d\otimes c,\, (\mathrm{id}\otimes S_B)\rho_P^{-1}(a\otimes b) \big\rangle &= \big\langle d \triangleright a,\, S_B(b) \triangleleft c \big\rangle \\
\big\langle d\otimes c,\, (\mathrm{id}\otimes S_B)\lambda_P^{-1}(a\otimes b) \big\rangle &= \big\langle a \triangleleft d,\, c \triangleright S_B(b) \big\rangle .
\end{aligned}
$$

Proof. We show the second formula. Anti-multiplicativity of S_A yields

$$
\begin{aligned}
\big\langle a \triangleleft d,\, c \triangleright S_B(b) \big\rangle &= \big\langle (a \triangleleft d)c,\, S_B(b) \big\rangle \\
&= \big\langle S_A(c)\, S_A(a \triangleleft d),\, b \big\rangle \\
&= \big\langle S_A(a \triangleleft d),\, b \triangleleft S_A(c) \big\rangle \\
&= \big\langle P^{-1},\, (a \triangleleft d) \otimes \big(b \triangleleft S_A(c)\big) \big\rangle \\
&= \big\langle \big(d \otimes S_A(c)\big)\, P^{-1},\, a\otimes b \big\rangle \\
&= \big\langle d\otimes c,\, (\mathrm{id}\otimes S_B)\lambda_P^{-1}(a\otimes b) \big\rangle .
\end{aligned}
$$

The other formula is similar. ∎

Lemma 3.4.5 *Let $\langle A, B\rangle$ be any Hopf system. If S_B is a bijection from B onto B, then $\langle A, B^{\mathrm{op}}\rangle$ is invertible. By symmetry, the same conclusion applies when S_A is a bijection from A onto A (cf. lemma 3.4.1).*

Proof. Assume S_B to be a bijection from B onto B. First we have to show that the pairing between A and B indeed induces an *actor* for $\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A}$, i.e. that the functional $A\otimes B \to \mathbb{C} : a\otimes b \mapsto \langle a, b\rangle$ is strict$^\sharp$ continuous within the actor context $\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A}$ (cf. remark 3.1.3). In particular we have to prove that given any $a \in A$ and $b \in B$ there exist $x, y \in A\otimes B$ such that (cf. equation 3.11)

$$
\big\langle d \triangleright a,\, b \triangleleft c \big\rangle = \big\langle d\otimes c,\, x \big\rangle \qquad\qquad \big\langle a \triangleleft d,\, c \triangleright b \big\rangle = \big\langle d\otimes c,\, y \big\rangle .
$$

for all $c \in A$ and $d \in B$. From lemma 3.4.4 it is clear that

$$
x = (\mathrm{id}\otimes S_B)\rho_P^{-1}\big(a\otimes S_B^{-1}(b)\big) \qquad\qquad y = (\mathrm{id}\otimes S_B)\lambda_P^{-1}\big(a\otimes S_B^{-1}(b)\big)
$$

will do the job. So the pairing indeed corresponds to an actor for $\mathbb{B}^{\mathrm{op}} \otimes \mathbb{A}$, say $(\lambda_P^{\mathrm{op}}, \rho_P^{\mathrm{op}})$, where

$$
\begin{aligned}
\lambda_P^{\mathrm{op}} &= (\mathrm{id}\otimes S_B)\rho_P^{-1}(\mathrm{id}\otimes S_B^{-1}) \\
\rho_P^{\mathrm{op}} &= (\mathrm{id}\otimes S_B)\lambda_P^{-1}(\mathrm{id}\otimes S_B^{-1})
\end{aligned}
\tag{3.13}
$$

are bijections from $A \otimes B$ onto $A \otimes B$. Now it only remains to show that the pair $\big((\lambda_P^{\rm op})^{-1}, (\rho_P^{\rm op})^{-1}\big)$ is still an actor for $\mathbb{B}^{\rm op} \otimes \mathbb{A}$. To prove this, we shall rely on lemma 2.4.3.2. By assumption (λ_P, ρ_P) is an actor for $\mathbb{B} \otimes \mathbb{A}$, and hence $(\varepsilon_A \otimes \varepsilon_B)\lambda_P = (\varepsilon_A \otimes \varepsilon_B)\rho_P$. Recall that $\mathbb{B}$ and $\mathbb{B}^{\rm op}$ share the counit ε_A, and also recall that $\varepsilon_B S_B = \varepsilon_B$ (corollary 3.3.4). Taking inverses in (3.13) yields

$$\begin{array}{rcl} (\lambda_P^{\rm op})^{-1} & = & (\mathrm{id} \otimes S_B)\rho_P(\mathrm{id} \otimes S_B^{-1}) \\ (\rho_P^{\rm op})^{-1} & = & (\mathrm{id} \otimes S_B)\lambda_P(\mathrm{id} \otimes S_B^{-1}) \end{array} \tag{3.14}$$

If we now apply $\varepsilon_A \otimes \varepsilon_B$, we obtain $(\varepsilon_A \otimes \varepsilon_B) \circ (\lambda_P^{\rm op})^{-1} = (\varepsilon_A \otimes \varepsilon_B) \circ (\rho_P^{\rm op})^{-1}$. The result follows from lemma 2.4.3.2. ∎

Lemma 3.4.6 *Let $\langle A, B\rangle$ be any Hopf system, and assume also $\langle A, B^{\rm op}\rangle$ to be a Hopf system. If $S_B(B) \subseteq B$, then $S_B^{\rm op} S_B = \mathrm{id}$.*

Proof. Combining lemma 3.4.4 with (3.11) we obtain

$$\begin{array}{rcl} \lambda_P^{\rm op}(\mathrm{id} \otimes S_B) & = & (\mathrm{id} \otimes S_B)\rho_P^{-1} \\ \rho_P^{\rm op}(\mathrm{id} \otimes S_B) & = & (\mathrm{id} \otimes S_B)\lambda_P^{-1}, \end{array} \tag{3.15}$$

which is completely similar to (3.13), though now obtained in a different setting. By assumption $\lambda_P^{\rm op}$ and $\rho_P^{\rm op}$ are bijective, hence e.g. the second formula in (3.15) can be rewritten as $(\mathrm{id} \otimes S_B)\lambda_P = (\rho_P^{\rm op})^{-1}(\mathrm{id} \otimes S_B)$, and thus we get for any $a, c \in A$ and $b, d \in B$ that

$$\begin{array}{rcl} \big\langle d \otimes c,\, (\mathrm{id} \otimes S_B)\lambda_P(a \otimes b)\big\rangle & = & \big\langle d \otimes c,\, (\rho_P^{\rm op})^{-1}\big(a \otimes S_B(b)\big)\big\rangle \\ & = & \big\langle (P^{\rm op})^{-1},\, (d \triangleright^{\rm op} a) \otimes \big(c \triangleright S_B(b)\big)\big\rangle \\ & = & \big\langle S_A^{\rm op}(a \triangleleft d),\, c \triangleright S_B(b)\big\rangle \\ & = & \big\langle S_A^{\rm op}(a \triangleleft d)\, c,\, S_B(b)\big\rangle \\ & = & \big\langle c,\, S_B(b) \triangleleft S_A^{\rm op}(a \triangleleft d)\big\rangle\,. \end{array}$$

On the other hand we also have

$$\begin{array}{rcl} \big\langle d \otimes c,\, (\mathrm{id} \otimes S_B)\lambda_P(a \otimes b)\big\rangle & = & \big\langle \big(d \otimes S_A(c)\big)\, P,\, a \otimes b\big\rangle \\ & = & \big\langle P,\, (a \otimes b) \triangleleft \big(d \otimes S_A(c)\big)\big\rangle \\ & = & \big\langle a \triangleleft d,\, b \triangleleft S_A(c)\big\rangle \\ & = & \big\langle S_A(c)\,(a \triangleleft d),\, b\big\rangle \\ & = & \big\langle S_A(c),\, (a \triangleleft d) \triangleright b\big\rangle \\ & = & \big\langle c,\, S_B\big((a \triangleleft d) \triangleright b\big)\big\rangle\,. \end{array}$$

Since $A \triangleleft B = A$, it follows that for all $a \in A$ and $b \in B$

$$S_B(b) \triangleleft S_A^{\rm op}(a) \;=\; S_B(a \triangleright b). \tag{3.16}$$

Observe that it was essential to assume $S_B(B) \subseteq B$, in order to be able to make the above computations. For the same reason we may now apply ε_B to (3.16). Together with the fact that $\varepsilon_B S_B = \varepsilon_B$ (corollary 3.3.4) we get

$$\big\langle a,\, S_B^{\mathrm{op}}(S_B(b))\big\rangle \;=\; \big\langle S_A^{\mathrm{op}}(a),\, S_B(b)\big\rangle \;=\; \big\langle a,\, b\big\rangle$$

and we conclude that $S_B^{\mathrm{op}}(S_B(b)) = b$. ■

Of course a similar result applies to the antipodes on A, by symmetry. On the other hand, interchanging the roles of $\langle A, B\rangle$ and $\langle A, B^{\mathrm{op}}\rangle$ yields

Corollary 3.4.7 *If $S_B^{\mathrm{op}}(B) \subseteq B$, then $S_B S_B^{\mathrm{op}} = \mathrm{id}$.*

Corollary 3.4.8 *When both $S_B(B) \subseteq B$ and $S_B^{\mathrm{op}}(B) \subseteq B$, then S_B and S_B^{op} are bijections from B onto B. Furthermore $S_B^{-1} = S_B^{\mathrm{op}}$.*

Proposition 3.4.9 *Let $\langle A, B\rangle$ be a Hopf system. The following are equivalent:*

i. S_A and S_B are bijections, resp. from A onto A and from B onto B.

ii. The pair $\langle A, B^{\mathrm{op}}\rangle$ is a Hopf system and the antipodes S_A and S_A^{op} leave A invariant, whereas S_B and S_B^{op} leave B invariant, e.g. $S_A(A) \subseteq A$, etc.

Proof. (i) ⇒ (ii). According to lemma 3.4.5, the pair $\langle A, B^{\mathrm{op}}\rangle$ is invertible, and therefore a Hopf system (cf. lemma 3.4.3). Now lemma 3.4.6 yields $S_B^{\mathrm{op}} S_B = \mathrm{id}$, and analogously we have $S_A^{\mathrm{op}} S_A = \mathrm{id}$. Since by assumption $S_A(A) = A$ and $S_B(B) = B$, it follows that $S_A^{\mathrm{op}}(A) = A$ and $S_B^{\mathrm{op}}(B) = B$.
(ii) ⇒ (i). Corollary 3.4.8 and its analogue for the antipodes on A. ■

Definition 3.4.10 A Hopf system $\langle A, B\rangle$ is said to be *regular* whenever it enjoys the equivalent conditions (i-ii) in the previous proposition. •

An example of a non-regular Hopf system will be given in example 3.7.3.3.

3.5 Hopf *-systems

Proposition 3.5.1 *Let $\langle A, B\rangle$ be a dual pair of algebras (cf. example 2.1.2.4) and moreover assume A and B to be endowed with *-operations making them into *-algebras. Then the following are equivalent:*

i. $\mathbb{A}$ and $\mathbb{B}$ are involutive (§2.8) actor contexts. The pairing $P : A \otimes B \to \mathbb{C}$ is strict$^\sharp$ continuous w.r.t. $\mathbb{B} \otimes \mathbb{A}$ and the corresponding actor is unitary in the sense of definition 2.8.3, and multiplicative as in definition 3.2.1.

ii. The pair $\langle A, B\rangle$ is a regular Hopf system and $\langle a, b^*\rangle = \overline{\langle S_A(a)^*, b\rangle}$ for any $a \in A$ and $b \in B$.

Proof. First assume (i). Then $\langle A, B\rangle$ is clearly a Hopf system, and $P^{-1} = P^*$. Using lemma 2.8.4 it follows that for all $a \in A$ and $b \in B$

$$\big\langle P^*,\, a \otimes b^\circ \big\rangle \;=\; \overline{\big\langle P,\, (a \otimes b^\circ)^\circ \big\rangle} \;=\; \overline{\langle a^\circ, b\rangle} \;=\; \langle a, b^*\rangle. \tag{3.17}$$

On the other hand, recall that $S_A(A) \subseteq \mathrm{Env}(\mathbb{A})$, so again using lemma 2.8.4, we obtain

$$\big\langle P^{-1},\, a \otimes b^\circ \big\rangle \;=\; \big\langle S_A(a),\, b^\circ \big\rangle \;=\; \overline{\big\langle S_A(a)^*,\, b \big\rangle} \tag{3.18}$$

and hence $\langle a, b^*\rangle = \overline{\langle S_A(a)^*, b\rangle}$. This formula can be rewritten as $S_A(a) = (a^\circ)^*$. It follows that S_A is a bijection from A onto A, and analogously S_B is a bijection from B onto B. Hence $\langle A, B\rangle$ is regular, which proves (ii).
Now assume (ii). First observe that, under the circumstances, the formula $\langle a, b^*\rangle = \overline{\langle S_A(a)^*, b\rangle}$ automatically implies its S_B-analogue: indeed, replacing a with $S_A^{-1}(a^*)$ and b with $S_B(b)^*$, we obtain for all $a \in A$ and $b \in B$

$$\langle a^*, b\rangle \;=\; \big\langle S_A^{-1}(a^*),\, S_B(b) \big\rangle \;=\; \overline{\big\langle a,\, S_B(b)^* \big\rangle}.$$

It follows that $a^\circ = S_A(a)^*$ and $b^\circ = S_B(b)^*$. Hence A and B are $^\circ$-invariant, or in other words, $\mathbb{A}$ and $\mathbb{B}$ are involutive actor contexts. Equations (3.17) and (3.18) still hold and imply that $P^{-1} = P^*$, so P is unitary. This proves (i). ∎

Definition 3.5.2 A pair $\langle A, B\rangle$ satisfying the conditions in proposition 3.5.1 is said to be a *Hopf* **-system*. •

Proposition 3.5.3 *If $\langle A, B\rangle$ is a Hopf *-system, then the comultiplications Δ_A and Δ_B, and the counits ε_A and ε_B, are all *-homomorphisms. Furthermore $S_A(S_A(a)^*)^* = a^{\circ\circ} = a$ and $S_B(S_B(b)^*)^* = b^{\circ\circ} = b$ for any $a \in A$ and $b \in B$.*

Proof. Take any $a \in A$ and $b, d \in B$. Recall that $b^\circ = S_B(b)^*$ and $d^\circ = S_B(d)^*$. Using lemma 2.8.4 and the anti-multiplicativity of S_B, we obtain

$$\begin{aligned}
\big\langle \Delta_A(a)^*,\, b \otimes d \big\rangle &= \overline{\big\langle \Delta_A(a),\, b^\circ \otimes d^\circ \big\rangle} \\
&= \overline{\big\langle \Delta_A(a),\, S_B(b)^* \otimes S_B(d)^* \big\rangle} \\
&= \overline{\big\langle a,\, S_B(b)^* S_B(d)^* \big\rangle} \\
&= \overline{\big\langle a,\, S_B(bd)^* \big\rangle} \\
&= \big\langle a^*,\, bd \big\rangle \\
&= \big\langle \Delta_A(a^*),\, b \otimes d \big\rangle
\end{aligned}$$

hence $\Delta_A(a)^* = \Delta_A(a^*)$. Combining corollary 3.3.4 with lemma 2.8.4, it follows that $\varepsilon_A(a^*) = \overline{\varepsilon_A(a)}$. Similarly for Δ_B and ε_B. ∎

3.6 Invariant functionals

Abstract 3.4 We define the notion of an invariant functional and show that *strong* invariance follows automatically. Furthermore we introduce the KMS property, which helps to overcome some difficulties caused by the non-commutativity of the algebra. We conclude with some trivial observations on normalization. ⋆

Definition 3.6.1 Let $\langle A, B\rangle$ be any Hopf system. A linear functional φ on A is called *left invariant* if $\varphi(a \triangleleft b) = \varphi(a)\,\varepsilon_B(b)$ for all $a \in A$ and $b \in B$. *Right* invariance is defined analogously—involving *left* actions of B on A. Similarly for functionals on B. •

Invariant functionals are often referred to as Haar functionals, because they constitute the non-commutative analogue of Haar integrals on locally compact groups. If $\langle A, B\rangle$ is a Hopf system for which (A, Δ_A) happens to be a regular multiplier Hopf algebra, then the above notion of invariance is easily seen to be equivalent with the one in [41].

Proposition 3.6.2 *Let $\langle A, B\rangle$ be any Hopf system, and let φ be a left invariant functional on A. Then for all $a, c \in A$ and $b \in B$ we have*

$$\varphi\left(a(c \triangleleft b)\right) \;=\; \varphi\big(\big(a \triangleleft S_B(b)\big)\, c\big)\,.$$

This property of φ will be referred to as *strong* left invariance.

Proof. Write $a \otimes b = \sum_i \rho_P(p_i \otimes q_i)$ with $p_i \in A$, $q_i \in B$. For all $d \in B$ we have

$$\begin{aligned}
\textstyle\sum_i \big\langle \varepsilon_B(q_i)\, p_i,\, d\big\rangle &= \textstyle\sum_i \big\langle d \otimes 1_{\mathbb{A}},\, p_i \otimes q_i\big\rangle \\
&= \big\langle d \otimes 1_{\mathbb{A}},\, \rho_P^{-1}(a \otimes b)\big\rangle \\
&= \big\langle P^{-1}(d \otimes 1_{\mathbb{A}}),\, a \otimes b\big\rangle \\
&= \big\langle P^{-1},\, (d \triangleright a) \otimes b\big\rangle \\
&= \big\langle d \triangleright a,\, S_B(b)\big\rangle \\
&= \big\langle a \triangleleft S_B(b),\, d\big\rangle
\end{aligned}$$

and hence

$$\textstyle\sum_i \varepsilon_B(q_i)\, p_i \;=\; a \triangleleft S_B(b). \tag{3.19}$$

In the proof of proposition 3.2.5 we obtained—in the same circumstances—the equation $a(c \triangleleft b) = \sum_i p_i c \triangleleft q_i$. Applying φ and invoking left invariance yields

$$\varphi\big(a(c \triangleleft b)\big) \;=\; \textstyle\sum_i \varphi(p_i c \triangleleft q_i) \;=\; \sum_i \varphi(p_i c)\, \varepsilon_B(q_i) \;\stackrel{(3.19)}{=}\; \varphi\big(\big(a \triangleleft S_B(b)\big)\, c\big)\,. \quad \blacksquare$$

Definition 3.6.3 A linear functional φ on an algebra E is said to be *faithful* if $E \times E \to \mathbb{C} : (x, y) \mapsto \varphi(xy)$ is non-degenerate as a bilinear form.
When E is a *-algebra, φ is called *hermitian* if $\varphi(x^*) = \overline{\varphi(x)}$ for any $x \in E$. Eventually φ is said to be *positive* whenever $\varphi(x^* x) \geq 0$ for any $x \in E$. •

It follows easily from the Cauchy-Schwarz inequality that a positive linear functional φ on a *-algebra is faithful if and only if $\varphi(x^* x) = 0$ implies $x = 0$.

Definition 3.6.4 A faithful linear functional φ on an algebra E is said to be *weakly* KMS whenever there exists a linear bijection $\sigma : E \to E$ such that $\varphi(xy) = \varphi(y\sigma(x))$ for all $x, y \in E$. •

Remark 3.6.5 The faithfulness assumption on φ implies σ to be *unique*. The assumed bijectivity of σ accounts for the 'two-sidedness' of our KMS property, in the sense that also $\varphi(xy) = \varphi(\sigma^{-1}(y)\, x)$ for $x, y \in E$. ⋆

Proposition 3.6.6 *Let E be any algebra and φ a faithful linear functional on E which is weakly* KMS *with respect to $\sigma : E \to E$. Then*

i. σ is an automorphism of the algebra E,

ii. φ is σ-invariant on E^2.

Notice that (ii) makes sense because E^2 itself is σ-invariant by virtue of (i).
If, moreover, E is a *-algebra and φ is hermitian, then we also have

iii. $\sigma(\sigma(x)^*)^* = x$ for any $x \in E$.

The proofs are similar to the special case treated in [41]. Because of (i) and uniqueness, σ is said to be *the* KMS automorphism of φ and denoted by σ_φ.

The following will be useful when we introduce Fourier contexts in chapter 4:

Lemma 3.6.7 *Let φ and ψ be non-trivial hermitian linear functionals on any *-algebra E, and let $S : E \to E$ be a linear map such that $S(S(x^*)^*) = x$ for all $x \in E$. If there exist $\lambda, \mu \in \mathbb{C}$ such that $\varphi S = \lambda\psi$ and $\psi S = \mu\varphi$, then $\lambda\overline{\mu} = 1$.*

Proof. Observe that, for any $x \in E$,

$$\varphi(x) = \varphi\big(S(S(x^*)^*)\big) = \lambda\, \psi\big(S(x^*)^*\big) = \lambda\, \overline{\psi(S(x^*))} = \lambda\, \overline{\mu\varphi(x^*)} = \lambda\overline{\mu}\, \varphi(x). \blacksquare$$

Corollary 3.6.8 *In the previous lemma, ψ can be replaced with a positive multiple of ψ in such a way that there exists a single complex number ζ such that, with this new normalization, $\varphi S = \zeta\psi$ and $\psi S = \zeta\varphi$. It follows that $|\zeta| = 1$.*

Proof. Consider $\psi' = |\lambda|\psi$ and take $\zeta = |\lambda|^{-1}\lambda$. From $\overline{\lambda}\mu = 1$ we get $\psi' S = \zeta\varphi$ and $\varphi S = \zeta\psi'$. Now replace ψ with ψ'. Clearly $|\zeta| = 1$. ■

3.7 Examples

Abstract 3.5 We show how Hopf algebras [11, 38] and multiplier Hopf algebras [40] fit into the Hopf system picture. Also the duality theory for multiplier Hopf algebras with integrals as developed by A. Van Daele [41, 42, 43] fits into this framework. Then we introduce a special subclass of Hopf systems, the so-called *algebraic* ones, having an 'algebraic' comultiplication[3] on at least one side; these are the kind of Hopf systems that enter e.g. the algebraic description of the quantum $E(2)$ group covered in chapter 5. For the moment we stick to a fairly trivial but very familiar (hence instructive) example derived from the group $(\mathbb{R}, +)$. We conclude with an example of a *non-regular* Hopf system. ⋆

[3] i.e. taking values in the algebraic tensor product, rather than some envelope of it. See definition 2.1.3.6.

3.7.1 Multiplier Hopf algebras

Lemma 3.7.1.1 *Let $\langle A, B\rangle$ be a dual pair of algebras. Assume B to be unital as an A-bimodule. Let A be endowed with a comultiplication $\Delta : A \to M(A\otimes A)$ making (A,Δ) into a multiplier Hopf algebra. If for any $a, c \in A$ and $b, d \in B$*

$$\big\langle T_1(a\otimes c),\, b\otimes d\big\rangle \;=\; \big\langle a,\, b(c\triangleright d)\big\rangle \tag{3.20}$$

$$\big\langle T_2(a\otimes c),\, b\otimes d\big\rangle \;=\; \big\langle c,\, (b\triangleleft a)d\big\rangle \tag{3.21}$$

then $\langle A, B\rangle$ is a Hopf system. If (A,Δ) is regular as a multiplier Hopf algebra and if $S_B(B) = B$, then $\langle A, B\rangle$ is regular in the sense of definition 3.4.10.

Remarks 3.7.1.2

i. The comultiplication, counit and antipode associated with A in the sense of [40] are consistent[4] with the objects Δ_A, ε_A and S_A derived from the Hopf system $\langle A, B\rangle$. For the comultiplication and the counit, this assertion is easily shown; for the antipode it follows from the proof below.

ii. Observe that conditions (3.20) and (3.21) are actually equivalent. They express that the comultiplication Δ is dual to the product in B. Here T_1 and T_2 are the bijections from $A\otimes A$ onto $A\otimes A$ as defined in [40]. Explicitly:

$$T_1(a\otimes c) \;=\; \Delta(a)(1\otimes c) \qquad\qquad T_2(a\otimes c) \;=\; (a\otimes 1)\Delta(c).$$

Proof. Before we start, recall definition 3.1.1 and remark 3.1.3. Take any $a, c, e \in A$ and $b, d \in B$, and write $T_1(a\otimes e) = \sum_i p_i \otimes r_i$. Now (3.20) yields

$$\begin{aligned}
\big\langle a\triangleleft d,\, (e\triangleright b)\triangleleft c\big\rangle &= \big\langle a,\, d(e\triangleright b\triangleleft c)\big\rangle\\
&= \big\langle T_1(a\otimes e),\, d\otimes(b\triangleleft c)\big\rangle\\
&= \textstyle\sum_i \langle p_i, d\rangle\langle r_i, b\triangleleft c\rangle\\
&= \textstyle\sum_i \langle p_i, d\rangle\langle c, r_i\triangleright b\rangle\\
&= \textstyle\sum_i \big\langle d\otimes c,\, p_i\otimes(r_i\triangleright b)\big\rangle
\end{aligned}$$

Since $A\triangleright B = B$, it follows that given any $a \in A$ and $b \in B$, there exists an $x \in A\otimes B$ such that for all $c \in A$ and $d \in B$

$$\big\langle d\otimes c,\, x\big\rangle \;=\; \big\langle a\triangleleft d,\, b\triangleleft c\big\rangle\,.$$

Similarly we show the existence of an $y \in A\otimes B$, only depending on a and b, such that

$$\big\langle d\otimes c,\, y\big\rangle \;=\; \big\langle d\triangleright a,\, c\triangleright b\big\rangle\,.$$

This means that the pairing $P : A\otimes B \to \mathbb{C}$ is indeed strict$^\sharp$ continuous within the actor context $\mathbb{B}\otimes\mathbb{A}$, hence identifying with an actor $P \simeq (\lambda_P, \rho_P)$ for $\mathbb{B}\otimes\mathbb{A}$. Now let $\mu_L : A\otimes B \to B$ denote the left action of A on B, and observe we have actually shown that (cf. equations 3.1)

$$\lambda_P(\mathrm{id}\otimes\mu_L) \;=\; (\mathrm{id}\otimes\mu_L)(T_1\otimes\mathrm{id}).$$

[4] i.e. if we take into account that $M(A) \subseteq \mathrm{Env}(\mathbb{A})$ and $M(A\otimes A) \subseteq \mathrm{Env}(\mathbb{A}\otimes\mathbb{A})$.

Because μ_L and T_1 are surjective, it follows that $\lambda_P(A\otimes B) = A\otimes B$. The next step is to show that λ_P is injective. Therefore, consider the linear mapping η from $A\otimes B$ into $(B\otimes A)'$ defined by

$$\big\langle d\otimes c,\, \eta(a\otimes b)\big\rangle \;=\; \big\langle cS(a\triangleleft d),\, b\big\rangle \tag{3.22}$$

where $S : A \to M(A)$ is the antipode associated to the MHA (A,Δ). Again take any $a, c, e \in A$ and $b, d \in B$, and write $T_1(a\otimes e) = \sum_i p_i \otimes r_i$ as before. An easy computation shows that $T_1\big((a\triangleleft d)\otimes e\big) = \sum_i (p_i \triangleleft d)\otimes r_i$, hence

$$\textstyle\sum_i S(p_i\triangleleft d)\, r_i \;\overset{(*)}{=}\; \sum_i (\varepsilon\otimes \mathrm{id})\, T_1^{-1}\big((p_i\triangleleft d)\otimes r_i\big) \;=\; \varepsilon(a\triangleleft d)\, e \;=\; \langle a, d\rangle\, e$$

where ε is the counit on A. In $(*)$ we used [40, definition 4.1]. Thus we obtain

$$\begin{aligned}
\big\langle d\otimes c,\, \eta\lambda_P\big(a\otimes(e\triangleright b)\big)\big\rangle &= \textstyle\sum_i \big\langle d\otimes c,\, \eta\big(p_i\otimes(r_i\triangleright b)\big)\big\rangle \\
&= \textstyle\sum_i \big\langle cS(p_i\triangleleft d)r_i,\, b\big\rangle \\
&= \langle a,d\rangle\langle ce, b\rangle \\
&= \big\langle d\otimes c,\, a\otimes(e\triangleright b)\big\rangle
\end{aligned}$$

and it follows that $\eta\lambda_P$ is the identity map on $A\otimes B$, hence λ_P is a bijection from $A\otimes B$ onto $A\otimes B$. Similarly also ρ_P is bijective. Now we would like to use lemma 2.4.3.2 to prove that $(\lambda_P^{-1}, \rho_P^{-1})$ is again an actor for $\mathbb{B}\otimes\mathbb{A}$. To do this, however, we first need to establish weak unitality: using (3.20) and (3.21) together with the surjectivity of T_1 and T_2, it is not so hard to see that A is unital as a B-bimodule. From lemma 2.4.1.4 it follows that $\mathbb{B}\otimes\mathbb{A}$ is weakly unital, hence so are $\mathbb{A}$ and $\mathbb{B}$ (proposition 2.6.3). Now $S(A)\subseteq M(A)\subseteq \mathrm{Env}(\mathbb{A})$ can be paired with B, and (3.22) can be rewritten as

$$\big\langle d\otimes c,\, \lambda_P^{-1}(a\otimes b)\big\rangle \;=\; \big\langle c,\, S(a\triangleleft d)\triangleright b\big\rangle\,.$$

Fix $b\in B$ for a while; using weak unitality we obtain for any $a\in A$ and $d\in B$

$$\big\langle d,\, (\mathrm{id}\otimes\varepsilon_B)\lambda_P^{-1}(a\otimes b)\big\rangle \;=\; \big\langle f,\, a\triangleleft d\big\rangle$$

where $f : A\to\mathbb{C}$ is defined by $f(x) = \langle S(x), b\rangle$. Analogously we obtain

$$\big\langle d,\, (\mathrm{id}\otimes\varepsilon_B)\,\rho_P^{-1}(a\otimes b)\big\rangle \;=\; \big\langle f,\, d\triangleright a\big\rangle\,.$$

It follows that $f\in A^\sharp$, hence $\theta(f) = (\lambda_f, \rho_f)$ is an actor for $\mathbb{B}$ given by

$$\lambda_f(a) \;=\; (\mathrm{id}\otimes\varepsilon_B)\lambda_P^{-1}(a\otimes b) \qquad\qquad \rho_f(a) \;=\; (\mathrm{id}\otimes\varepsilon_B)\,\rho_P^{-1}(a\otimes b).$$

(cf. proposition 2.3.2.1.iii). Applying ε_A and using (2.11) we obtain

$$(\varepsilon_A\otimes\varepsilon_B)\lambda_P^{-1}(a\otimes b) \;=\; \langle S(a), b\rangle \;=\; (\varepsilon_A\otimes\varepsilon_B)\,\rho_P^{-1}(a\otimes b). \tag{3.23}$$

Lemma 2.4.3.2 yields that $(\lambda_P^{-1}, \rho_P^{-1})$ is an actor for $\mathbb{B}\otimes\mathbb{A}$, which proves that $\langle A, B\rangle$ is an invertible dual pair of algebras. From proposition 3.2.3 it follows

that $\langle A, B\rangle$ is a Hopf system. Comparing (3.23) with (3.2) and (2.11), we conclude that S coincides with the antipode S_A derived from the Hopf system $\langle A, B\rangle$ (as announced in remark 3.7.1.2.i).
Recall from [40] that the antipode on a *regular* multiplier Hopf algebra A is a bijection from A onto A. Assuming $S_B(B) = B$, it follows that S_B is a bijection from B onto B. Hence $\langle A, B\rangle$ is regular. ∎

Proposition 3.7.1.3 *Let (A, Δ) be any multiplier Hopf algebra. Let $A^\diamond$ be the reduced dual of A (defined as $A^\diamond = A \triangleright A' \triangleleft A$). Then $\langle A, A^\diamond\rangle$ is a Hopf system. If (A, Δ) is regular, then $\langle A, A^\diamond\rangle$ is regular. Finally, if (A, Δ) is a multiplier Hopf *-algebra, then $\langle A, A^\diamond\rangle$ is a Hopf *-system.*

Proof. Recall example 2.1.2.5. Since the product in A is non-degenerate, it follows that $\mathbb{A} \equiv (A; A^\diamond, \langle\cdot,\cdot\rangle)$ is an actor context. Furthermore $A^\diamond$ is unital as an A-bimodule because $A^2 = A$. In [40, §6] it was shown that $A^\diamond$ can be made into an algebra in such a way that for all $a, c_i, e_i \in A$ and $\omega_i \in A'$ $(i = 1, 2)$

$$\big\langle (c_1 \otimes c_2)\Delta(a)(e_1 \otimes e_2),\, \omega_1 \otimes \omega_2 \big\rangle \;=\; \big\langle a,\, \nu_1\nu_2 \big\rangle \tag{3.24}$$

where $\nu_i = e_i \triangleright \omega_i \triangleleft c_i$ $(i = 1, 2)$. Now it is easy to see that also $(A^\diamond; A, \langle\cdot,\cdot\rangle)$ is an actor context, and hence $\langle A, A^\diamond\rangle$ is a dual pair of algebras. For this pair, the conditions (3.20) and (3.21) can be derived from (3.24). Lemma 3.7.1.1 then yields that $\langle A, A^\diamond\rangle$ is a Hopf system. If (A, Δ) is regular, then $S : A \to A$ is a bijection, hence so is its transpose $S^\tau : A' \to A'$. Using the anti-multiplicativity of S, it is not so hard to show that for all $c, e \in A$ and $\omega \in A'$

$$S^\tau\big(S(c) \triangleright \omega \triangleleft S(e)\big) \;=\; e \triangleright S^\tau(\omega) \triangleleft c.$$

It follows that $S^\tau(A^\diamond) = A^\diamond$, hence $\langle A, A^\diamond\rangle$ is regular.
Recall from [40] that a multiplier Hopf *-algebra is automatically regular, and furthermore $A^\diamond$ can be made into a *-algebra [40, proposition 6.6] in such a way that $\langle A, A^\diamond\rangle$ becomes a Hopf *-system. ∎

Corollary 3.7.1.4 *If A is a Hopf algebra, then $\langle A, A'\rangle$ is a Hopf system. When A has invertible antipode, then $\langle A, A'\rangle$ is regular. If A is a *-Hopf algebra, then $\langle A, A'\rangle$ is a Hopf *-system.*

Proposition 3.7.1.5 *Let (A, Δ) be a regular multiplier Hopf algebra with non-trivial invariant functionals. Let $(\hat{A}, \hat{\Delta})$ be the dual object defined in [42, 41]. Then $\langle A, \hat{A}\rangle$ is a regular Hopf system.*

Proof. Let φ be a non-trivial left invariant functional on A, and consider the Fourier transform $A \to \hat{A} : a \mapsto \hat{a} = \varphi(\cdot\, a)$. The duality between A and $\hat{A}$, given by $\langle a, \hat{c}\rangle = \varphi(ac)$, is non-degenerate because φ is faithful. Now it is easy to show that e.g.[5] $a \triangleright \hat{c} = \widehat{ac}$. Since $A^2 = A$ it follows that $\hat{A}$ is unital as an A-bimodule. On the other hand we have for instance $\hat{c} \triangleright a = (\mathrm{id} \otimes \varphi)T_1(a \otimes c)$. It follows that A is an $\hat{A}$-bimodule, and hence $\langle A, \hat{A}\rangle$ is a dual pair of algebras. The remaining conditions of lemma 3.7.1.1 are easily verified. ∎

[5] for the right action we need to use another Fourier transform, of course.

3.7.2 Algebraic Hopf systems

Definition 3.7.2.1 Let $(A, \Delta, \varepsilon, S)$ be any Hopf algebra, B any algebra and $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{C}$ a non-degenerate vector space duality. Embed B in A'. If

i. B is invariant under canonical actions of A on A', and

ii. for all $a \in A$ and $b, d \in B$ we have $\langle a, bd \rangle = \langle \Delta(a), b \otimes d \rangle$

then the pair $\langle \underline{A}\,, B \rangle$ is said to be an *algebraic Hopf system.* If moreover

iii. A is a Hopf *-algebra and B is a *-algebra,

iv. B is $^\circ$-invariant (§2.8) within A'

v. for any $a \in A$ and $b \in B$ we have $\langle a, b^* \rangle = \overline{\langle S(a)^*, b \rangle}$

then $\langle \underline{A}\,, B \rangle$ is called an *algebraic Hopf *-system.* •

Here the *underlining* in $\langle \underline{A}\,, B \rangle$ is merely a convenient way to indicate which one of the two algebras involved is a genuine Hopf algebra; indeed we wouldn't like this to depend on the *order* in which we write the pairing. The terminology is justified by the following:

Proposition 3.7.2.2 *An algebraic Hopf system $\langle \underline{A}\,, B \rangle$ in the sense of the above definition is indeed a Hopf system as defined in the beginning of chapter 3. An algebraic Hopf *-system is a Hopf *-system in the sense of definition 3.5.2.*

Proof. Observe that A is invariant under canonical actions of B on B' because $\Delta(A) \subseteq A \otimes A$ (cf. proposition 2.1.3.2). Hence $\langle A, B \rangle$ is a dual pair of algebras. Recall that A has an identity; lemma 3.7.1.1 yields that $\langle A, B \rangle$ is a Hopf system. Assume that $\langle \underline{A}\,, B \rangle$ moreover enjoys conditions (iii-iv-v) of definition 3.7.2.1. We claim that $\langle A, B \rangle$ is regular as a Hopf system: indeed S_A is bijective because A is a Hopf *-algebra, whereas (v) can be rewritten as $S_B(b^\circ) = b^*$, hence S_B is a bijection from B onto B. The result follows. ∎

Remark 3.7.2.3 An algebraic Hopf system $\langle \underline{A}\,, B \rangle$ might fail to be regular, even when A has invertible antipode; see example 3.7.3.3. ⋆

Example 3.7.2.4 Let G be any locally compact group, and let $\mathbb{C}G$ denote its group algebra, with canonical basis $\{\delta_s\}_{s \in G}$. It is well-known that $(\mathbb{C}G, \Delta, \varepsilon, S)$ is a Hopf algebra with

$$\Delta(\delta_s) \;=\; \delta_s \otimes \delta_s \qquad \varepsilon(\delta_s) \;=\; 1 \qquad S(\delta_s) \;=\; \delta_{s^{-1}} \qquad (s \in G).$$

Also consider the space $K(G)$ of all continuous complex functions on G with compact support. $K(G)$ becomes an algebra under pointwise multiplication. Defining the pairing as in example 2.1.2.1, the pair $\big\langle K(G),\, \underline{\mathbb{C}G} \big\rangle$ is easily seen to be a regular (algebraic) Hopf system with non-trivial invariant functionals on both sides: indeed integration w.r.t. the Haar measures on G yields invariant functionals on $K(G)$, whereas on $\mathbb{C}G$ we have a Haar functional φ given by $\varphi(\delta_s) = 0$ if $s \neq e$ and $\varphi(\delta_e) = 1$, where e denotes the identity in G. ⋆

Example 3.7.2.5 *If $\langle A, B \rangle$ is a dual pair of Hopf *-algebras [38] with non-degenerate pairing, then $\langle \underline{A}\,, \underline{B} \rangle$ is an algebraic Hopf *-system.* ⋆

3.7.3 A familiar example

Let's have a look at probably the most familiar example of a (non-compact, non-discrete) locally compact group: $(\mathbb{R}, +)$.
Let R be the unital *-algebra generated by a single self-adjoint element h. Then R can be made into a Hopf *-algebra by defining

$$\Delta(h) = h \otimes 1 + 1 \otimes h \qquad \varepsilon(h) = 0 \qquad S(h) = -h.$$

In fact R is nothing but the algebra of polynomial functions on $\mathbb{R}$, endowed with the usual comultiplication, counit and antipode:

$$(\Delta f)(s,t) = f(s+t) \qquad \varepsilon(f) = f(0) \qquad (Sf)(t) = f(-t) \tag{3.25}$$

for $s, t \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ a polynomial function. Clearly Δf is a polynomial in two variables, hence identifying with an element in $R \otimes R$. Next we define a pairing of R with R itself, as follows:

$$\langle h^n, h^m \rangle \;=\; \delta_{n,m}\, n!\, (-i)^n \qquad (n, m \in \mathbb{N}). \tag{3.26}$$

Then $\langle R, R\rangle$ is a dual pair of Hopf *-algebras [38]. Now we wish to consider all entire[6] functions rather than just polynomial ones, so recall the conventions concerning entire functions (§1.4). The main observation in this respect, is that (3.25) no longer defines an 'algebraic' comultiplication: indeed, if f is an entire function, then Δf will generally not identify with an element in the *algebraic* tensor product $H(\mathbb{C}) \otimes H(\mathbb{C})$. In other words, we cannot make $H(\mathbb{C})$ into a Hopf algebra. Nevertheless we can extend the pairing (3.26) on one side, defining a bilinear form $\langle \cdot, \cdot \rangle : R \times H(\mathbb{C}) \to \mathbb{C}$ by

$$\langle h^n, f \rangle \;=\; \mu_n(f)\, n!\, (-i)^n \qquad (f \in H(\mathbb{C}),\, n \in \mathbb{N}). \tag{3.27}$$

Example 3.7.3.1 *$\big\langle \underline{R}\,,\, H(\mathbb{C}) \big\rangle$ is an algebraic Hopf *-system. The actions of R on $H(\mathbb{C})$ are essentially given by differentiation:*

$$h \triangleright f \;=\; -if' \;=\; f \triangleleft h. \tag{3.28}$$

Proof. Non-degeneracy of (3.27) is clear. Also (3.28) and conditions (i-iii-v) of definition 3.7.2.1 are easily verified. (ii) involves the binomial theorem and the rules for multiplying Taylor series, whereas (iv) follows from $f^\circ(z) = \tilde{f}(-z)$. ∎

Now let $S(\mathbb{R})$ denote classical Schwartz space and define

$$\mathcal{S}_H \;=\; \left\{ f \in H(\mathbb{C}) \,\middle|\, f_{|\mathbb{R}} \in S(\mathbb{R}) \right\}.$$

Clearly $\mathcal{S}_H$ is a *-subalgebra of $H(\mathbb{C})$, containing e.g. the Hermite functions.

Example 3.7.3.2 *$\big\langle \underline{R}\,,\, \mathcal{S}_H \big\rangle$ is an algebraic Hopf *-system. Lebesgue integration over $\mathbb{R}$ yields a functional $\int_{\mathbb{R}}$ on $\mathcal{S}_H$ which is both left and right invariant.*

[6] Of course entire functions are defined on all of $\mathbb{C}$, but since they are determined by their values on $\mathbb{R}$, we can still think of them as functions on our group $\mathbb{R}$.

Proof. $\mathcal{S}_H$ is invariant under differentiation, hence also under the actions (3.28). The pairing is still non-degenerate. The fundamental theorem of calculus yields

$$\int_{\mathbb{R}} f \triangleleft h \;=\; \int_{\mathbb{R}} h \triangleright f \;=\; -i \int_{\mathbb{R}} f' \;=\; 0 \;=\; \varepsilon(h) \int_{\mathbb{R}} f \qquad (f \in \mathcal{S}_H).$$

Hence $\int_{\mathbb{R}}$ is both left and right invariant in the sense of definition 3.6.1. ∎

A non-regular Hopf system Let E be the subspace of $H(\mathbb{C})$ spanned by the functions $\mathbb{C} \to \mathbb{C} : x \mapsto x^n \exp(\alpha x)$ with $n \in \mathbb{N}$ and $\alpha > 0$. Clearly E is a *-subalgebra of $H(\mathbb{C})$. Since E is not invariant under the antipode, we obtain

Example 3.7.3.3 $\langle \underline{R}\,, E \rangle$ *is an algebraic Hopf system, but it is not regular in the sense of definition 3.4.10.*

3.8 Multiplier Hopf systems

Abstract 3.6 In this section, the Hopf systems arising from regular multiplier Hopf algebras with integrals (cf. proposition 3.7.1.5) shall be characterized among arbitrary Hopf systems. The criterion distinguishing the so-called *multiplier* Hopf systems from the general ones is particularly pleasant: the pairing is required to be an invertible *multiplier* rather than merely an invertible *actor*. The main results in this section are proposition 3.8.2 and theorem 3.8.4. Proving the latter involves the *construction* of invariant functionals. It also turns out that the actor contexts involved are *pseudo-discrete* in the sense of definition 2.9.2.1, which is quite remarkable. ⋆

Recall definition 2.2.1 and proposition 2.2.4. Whenever $\mathbb{E} \equiv (E; \Omega, \langle\cdot,\cdot\rangle)$ is a weakly unital actor context, we shall identify $M(\mathbb{E})$ with $M(E)$.

Definition 3.8.1 A Hopf system $\langle A, B \rangle$ is said to be a *multiplier* Hopf system whenever P and P^{-1} belong to $M(B \otimes A)$. •

Proposition 3.8.2 *Let (A, Δ) be a regular multiplier*[7] *Hopf algebra with non-trivial invariant functionals. Let $(\hat{A}, \hat{\Delta})$ be the dual object defined in [42, 41]. Then $\langle A, \hat{A} \rangle$ is a regular multiplier Hopf system.*

Proof. In proposition 3.7.1.5 we showed that $\langle A, \hat{A} \rangle$ is a regular Hopf system. Let φ be a non-trivial left invariant functional on A, and consider the Fourier transform[8] $\mathcal{F}_L : A \to \hat{A} : a \mapsto \hat{a} = \varphi(a\,\cdot\,)$. Now take any $a, c, e \in A$ and $\omega \in \hat{A}$. Using the 'strong left invariance' formula from [41], we obtain

$$\hat{e} \triangleright a \;=\; (\mathrm{id} \otimes \varphi)\big((1 \otimes e)\Delta(a)\big) \;=\; S_A(\mathrm{id} \otimes \varphi)\big(\Delta(e)(1 \otimes a)\big)\,.$$

[7] At this point the terminology may be slightly misleading: be aware that the notion of a multiplier Hopf algebra *generalizes* ordinary Hopf algebras, whereas multiplier Hopf systems are *special* among ordinary Hopf systems.

[8] Be aware that in [41] the Fourier transform $\hat{a}$ was defined to be $\varphi(\,\cdot\,a)$ rather than $\varphi(a\,\cdot\,)$.

Since (A,Δ) is assumed to be regular, S_A is a bijection from A onto A, so we may write $(S_A^{-1}(c)\otimes 1)\Delta(e)=\sum_i p_i\otimes r_i$ with $p_i,r_i\in A$. Now observe that

$$\begin{aligned}
\big\langle P(\hat{e}\otimes c),\, a\otimes\omega\big\rangle &= \big\langle \hat{e}\triangleright a,\, c\triangleright\omega\big\rangle \\
&= \big\langle S_A(\mathrm{id}\otimes\varphi)\big(\Delta(e)(1\otimes a)\big)\,c,\,\omega\big\rangle \\
&= \big\langle S_A(\mathrm{id}\otimes\varphi)\big((S_A^{-1}(c)\otimes 1)\Delta(e)(1\otimes a)\big)\,,\,\omega\big\rangle \\
&= \textstyle\sum_i \big\langle S_A(\mathrm{id}\otimes\varphi)(p_i\otimes r_i a),\,\omega\big\rangle \\
&= \textstyle\sum_i \varphi(r_i a)\,\big\langle S_A(p_i),\,\omega\big\rangle\,.
\end{aligned}$$

It follows that $P(\hat{e}\otimes c)=\sum_i \widehat{r_i}\otimes S_A(p_i)$ belongs to $\hat{A}\otimes A$ again. Observe that we have actually shown that the map $\hat{A}\otimes A\to\hat{A}\otimes A : x\mapsto Px$ equals

$$(\mathcal{F}_L\otimes S_A)\,\chi\,T_2\,\chi(\mathcal{F}_L\otimes S_A)^{-1}$$

where χ denotes the flip map; recall that $\mathcal{F}_L$, S_A and T_2 are all bijective [41]. It follows that $x\mapsto Px$, and analogously, $x\mapsto xP$, are bijections from $\hat{A}\otimes A$ onto $\hat{A}\otimes A$, hence P and P^{-1} belong to $M(\hat{A}\otimes A)$. ∎

Lemma 3.8.3 *If $\langle A,B\rangle$ is a multiplier Hopf system, then the comultiplications and antipodes can be viewed as linear mappings*

$$\begin{array}{llll}
\Delta_A : A \to M(A\otimes A) & \qquad\qquad & S_A : A \to M(A) \\
\Delta_B : B \to M(B\otimes B) & & S_B : B \to M(B)
\end{array}$$

Here Δ_A and Δ_B are homomorphisms, S_A and S_B are anti-homomorphisms.

Proof. Combine proposition 2.6.9 with (3.4) and (3.3). ∎

So if $\langle A,B\rangle$ is a multiplier Hopf system, it is natural to ask whether (A,Δ_A) and (B,Δ_B) are multiplier Hopf algebras [40]. In this respect we announce:

Theorem 3.8.4 *If $\langle A,B\rangle$ is a regular multiplier Hopf system, then (A,Δ_A) and (B,Δ_B) are regular multiplier Hopf algebras admitting non-trivial invariant functionals. Moreover $B\simeq\hat{A}$ and $A\simeq\hat{B}$ in the sense of [42, 41]. Furthermore, the actor contexts $\mathbb{A}$ and $\mathbb{B}$ are pseudo-discrete in the sense of definition 2.9.2.1.*

Since the proof of this theorem is rather involved, we shall break it up into smaller units. The first step is to prove the *existence* of non-trivial invariant functionals (cf. proposition 3.8.10). This will involve the following:

Definition 3.8.5 Let $\langle A,B\rangle$ be any Hopf system. Let Γ_L and Γ_R be the unique linear mappings from $A\otimes B$ into $(A\otimes B)'$ such that for all $x,y\in A\otimes B$

$$\big\langle\Gamma_L(x),\,y\big\rangle \;=\; \big\langle P,\,xy\big\rangle \;=\; \big\langle x,\,\Gamma_R(y)\big\rangle\,. \tag{3.29}$$

Recall the actions $\triangleright$ and $\triangleleft$ of an algebra E on its dual E' were defined by (2.1). The same formulas can be used to define canonical actions of $M(E)$ on E' (provided the product in E is non-degenerate). Thus the following makes sense:

Lemma 3.8.6 *Whenever $\langle A, B\rangle$ is a Hopf system, we have:*

i. Γ_L *and* Γ_R *are respectively*[9] *right and left* $M(A\otimes B)$*-module morphisms:*

$$\Gamma_L(xm) = \Gamma_L(x) \triangleleft m \qquad\qquad \Gamma_R(mx) = m \triangleright \Gamma_R(x) \tag{3.30}$$

for all $x\in A\otimes B$ *and* $m \in M(A\otimes B)$.

ii. Let $\lambda_P^\tau, \rho_P^\tau : (A\otimes B)' \to (A\otimes B)'$ *denote the algebraic transposes of* λ_P *and* ρ_P*. Considering* $B\otimes A$ *as a subspace of* $(A\otimes B)'$*, we have*

$$\Gamma_L = \lambda_P^\tau \chi \rho_P \qquad\qquad \Gamma_R = \rho_P^\tau \chi \lambda_P. \tag{3.31}$$

iii. Γ_L *and* Γ_R *are injective and have weakly dense range.*

Proof. (3.30) follows immediately from (3.29). The multiplicativity of P (in the sense of definition 3.2.1) yields that the mappings in (3.31) indeed satisfy (3.29). Assertion (iii) follows from (3.31) and the fact that λ_P and ρ_P are bijective. ∎

Lemma 3.8.7 *Let $\langle A, B\rangle$ be any Hopf system. The following are equivalent:*

i. The pair $\langle A, B\rangle$ is a multiplier Hopf system.

ii. The mappings Γ_L and Γ_R are bijections from $A\otimes B$ onto $B\otimes A$.

iii. The mappings λ_P^τ and ρ_P^τ restrict to bijections from $B\otimes A$ onto $B\otimes A$.

Proof. (ii ⇔ iii) follows easily from (3.31). On the other hand, it is not hard to see that for any $x\in B\otimes A$ we have

$$\begin{aligned} Px &= \rho_P^\tau(x) & \qquad P^{-1}x &= (\rho_P^{-1})^\tau(x) \\ xP &= \lambda_P^\tau(x) & \qquad xP^{-1} &= (\lambda_P^{-1})^\tau(x). \end{aligned}$$

These formulas make sense because e.g. $Px \in \mathrm{Act}(\mathbb{B}\otimes\mathbb{A}) \subseteq (A\otimes B)'$ etc. Now (i ⇔ iii) becomes obvious. ∎

Lemma 3.8.8 *If $\langle A, B\rangle$ is a regular multiplier Hopf system, then so is $\langle A, B^{\mathrm{op}}\rangle$.*

Proof. Take transposes of (3.13) and (3.14). Then use the previous lemma. ∎

Remark 3.8.9 If Γ_L and Γ_R end up in $B\otimes A$ (as they do in lemma 3.8.7.ii) then the actions $\triangleright$ and $\triangleleft$ appearing in (3.30) may also be interpreted within the 'enveloping' actor context $\big(\mathrm{Env}(\mathbb{A}\otimes\mathbb{B}); B\otimes A, \langle\cdot,\cdot\rangle\big)$ of $\mathbb{A}\otimes\mathbb{B}$, as explained by remark 2.5.3 and proposition 2.2.4. ⋆

Proposition 3.8.10 *Let $\langle A, B\rangle$ be a multiplier Hopf system. Then there exist non-trivial invariant functionals on A and B (in the sense of definition 3.6.1).*

[9] Be aware that the subscripting of the Γ's *anti*-corresponds to their module properties.

Proof. Denote $\varepsilon_A \otimes \varepsilon_B$ by ε. Since Γ_L and Γ_R are bijections from $A \otimes B$ onto $B \otimes A$, we may define linear functionals φ and ψ on $B \otimes A$ by $\varphi = \varepsilon\,\Gamma_L^{-1}$ and $\psi = \varepsilon\,\Gamma_R^{-1}$. From proposition 3.2.5 we obtain that $\varepsilon : A \otimes B \to \mathbb{C}$ is an algebra homomorphism, and hence it extends to a homomorphism $\tilde{\varepsilon} : M(A \otimes B) \to \mathbb{C}$. From (3.30) it follows that for any $x \in A \otimes B$ and $m \in M(A \otimes B)$

$$\varphi\left(\Gamma_L(x) \triangleleft m\right) \;=\; \varphi\left(\Gamma_L(xm)\right) \;=\; \varepsilon(xm) \;=\; \varepsilon(x)\,\tilde{\varepsilon}(m) \;=\; \varphi\big(\Gamma_L(x)\big)\,\tilde{\varepsilon}(m).$$

Since $\Gamma_L(A \otimes B) = B \otimes A$, we have $\varphi(z \triangleleft m) = \varphi(z)\,\tilde{\varepsilon}(m)$ for all $z \in B \otimes A$. In particular we observe that φ is a left invariant functional on $B \otimes A$ with respect to the Hopf system $\langle B \otimes A, A \otimes B\rangle$. Similarly ψ turns out to be right invariant. Notice that φ is non-trivial, because ε is non-trivial. Now take $a_0 \in A$ and $b_0 \in B$ with $\varphi(b_0 \otimes a_0) = 1$, and define functionals φ_A on A and φ_B on B by

$$\varphi_A \;=\; \varphi(b_0 \otimes \,\cdot\,) \qquad\text{and}\qquad \varphi_B \;=\; \varphi(\,\cdot\, \otimes a_0).$$

Then φ_A and φ_B are non-trivial, and for any $a \in A$ and $b \in B$ we have e.g.

$$\varphi_A(a \triangleleft b) \;=\; \varphi\left((b_0 \otimes a) \triangleleft (1 \otimes b)\right) \;=\; \varphi(b_0 \otimes a)\,\tilde{\varepsilon}(1 \otimes b) \;=\; \varphi_A(a)\,\varepsilon_B(b),$$

so φ_A is left invariant. Right invariant functionals are obtained similarly. ■

Definition 3.8.11 Let $\langle A, B\rangle$ be any Hopf system. Given any left invariant functionals φ_A and φ_B on respectively A and B, we define linear mappings

$$\mathcal{F}_L : A \to A' : a \mapsto \varphi_A(a\,\cdot\,) \qquad\qquad \mathcal{G}_L : B \to B' : b \mapsto \varphi_B(b\,\cdot\,).$$

Similarly, given any right invariant functionals ψ_A and ψ_B we define

$$\mathcal{F}_R : A \to A' : a \mapsto \psi_A(\,\cdot\, a) \qquad\qquad \mathcal{G}_R : B \to B' : b \mapsto \psi_B(\,\cdot\, b).$$

These mappings are often called *Fourier transforms.* Be aware they depend on the choice of the invariant functionals[10]. •

Lemma 3.8.12 *Let $\langle A, B\rangle$ be a multiplier Hopf system. Let φ_A, ψ_A and φ_B, ψ_B be any*[11] *non-trivial invariant functionals, respectively left and right, on A and on B. Consider the associated Fourier transforms as in definition 3.8.11. Then*

$$(\mathrm{id} \otimes \varphi_A)\,\Gamma_L \;=\; \mathcal{F}_L \otimes \varepsilon_B \qquad\qquad (\varphi_B \otimes \mathrm{id})\,\Gamma_L \;=\; \varepsilon_A \otimes \mathcal{G}_L$$
$$(\mathrm{id} \otimes \psi_A)\,\Gamma_R \;=\; \mathcal{F}_R \otimes \varepsilon_B \qquad\qquad (\psi_B \otimes \mathrm{id})\,\Gamma_R \;=\; \varepsilon_A \otimes \mathcal{G}_R.$$

Proof. Take any $a \in A$ and $b \in B$ and write $\Gamma_L(a \otimes b) = \sum_i q_i \otimes p_i$ with $p_i \in A$ and $q_i \in B$. From (3.29) we get for all $c \in A$ and $d \in B$ that

$$\big\langle \Gamma_L(a \otimes b),\, c \otimes d \big\rangle \;=\; \big\langle ac,\, bd \big\rangle \;=\; \big\langle ac \triangleleft b,\, d \big\rangle$$

[10] uniqueness of invariant functionals not yet being established at this moment.

[11] So a priori not necessarily the ones constructed in the proof of proposition 3.8.10.

and hence $\sum_i \langle c, q_i\rangle\, p_i = ac \triangleleft b$. Now we proceed as follows:

$$\begin{aligned}\big\langle c,\ (\mathrm{id}\otimes\varphi_A)\,\Gamma_L(a\otimes b)\big\rangle &= \textstyle\sum_i \langle c, q_i\rangle\,\varphi_A(p_i)\\ &= \varphi_A(ac\triangleleft b)\\ &= \varphi_A(ac)\,\varepsilon_B(b)\\ &= \big\langle c,\ \mathcal{F}_L(a)\big\rangle\,\varepsilon_B(b)\end{aligned}$$

and the result follows. The other cases are similar. ∎

Corollary 3.8.13 *$\mathcal{F}_L(A) = \mathcal{F}_R(A) = B$ and $\mathcal{G}_L(B) = \mathcal{G}_R(B) = A$.*

Corollary 3.8.14 *There exists a complex scalar μ such that $\varphi_B\mathcal{F}_L = \mu\,\varepsilon_A$ and $\varphi_A\mathcal{G}_L = \mu\,\varepsilon_B$. Moreover $(\varphi_B\otimes\varphi_A)\,\Gamma_L = \mu(\varepsilon_A\otimes\varepsilon_B)$.*

Proof. Observe that $\varphi_B\mathcal{F}_L\otimes\varepsilon_B = (\varphi_B\otimes\varphi_A)\,\Gamma_L = \varepsilon_A\otimes\varphi_A\mathcal{G}_L$. ∎

Theorem 3.8.15 *Let $\langle A, B\rangle$ be any multiplier Hopf system. Then there exist non-trivial invariant functionals on A and B, unique up to a scalar.*
We will always denote the invariant functionals by $\varphi_A, \psi_A, \varphi_B, \psi_B$ as above.

Proof. Existence was shown in proposition 3.8.10. So let's take e.g. two left invariant functionals on A, say φ_A and φ'_A. We know there exists a non-trivial left invariant functional on B, say φ_B. According to corollary 3.8.14, both $(\varphi_B\otimes\varphi_A)\,\Gamma_L$ and $(\varphi_B\otimes\varphi'_A)\,\Gamma_L$ are scalar multiples of $\varepsilon_A\otimes\varepsilon_B$. Since Γ_L maps $A\otimes B$ onto $B\otimes A$, it follows that φ_A and φ'_A are scalar multiples. ∎

Regularity Recall lemma 3.8.8. In addition to the Fourier transforms in definition 3.8.11, we also have those associated to $\langle A, B^{\mathrm{op}}\rangle$. Explicitly:

$$\begin{aligned}\mathcal{F}_L^{\mathrm{op}}(a) &= \psi_A(a\,\cdot\,) \qquad & \mathcal{G}_L^{\mathrm{op}}(b) &= \varphi_B(\,\cdot\,b)\\ \mathcal{F}_R^{\mathrm{op}}(a) &= \varphi_A(\,\cdot\,a) \qquad & \mathcal{G}_R^{\mathrm{op}}(b) &= \psi_B(b\,\cdot\,)\end{aligned}$$

for $a\in A$ and $b\in B$. Now corollary 3.8.13 can be improved as follows:

Lemma 3.8.16 *If $\langle A, B\rangle$ is a regular multiplier Hopf system, then all the Fourier transforms defined above are bijections between A and B.*

Proof. Applying corollary 3.8.13 both to $\langle A, B\rangle$ and $\langle A, B^{\mathrm{op}}\rangle$ yields

$$\begin{aligned}\mathcal{F}_L(A) &= \mathcal{F}_R(A) = \mathcal{F}_L^{\mathrm{op}}(A) = \mathcal{F}_R^{\mathrm{op}}(A) = B\\ \mathcal{G}_L(B) &= \mathcal{G}_R(B) = \mathcal{G}_L^{\mathrm{op}}(B) = \mathcal{G}_R^{\mathrm{op}}(B) = A.\end{aligned}$$

Now observe that $\mathcal{F}_L$ and $\mathcal{F}_R^{\mathrm{op}}$ are each others transpose in the sense that

$$\big\langle \mathcal{F}_L(a),\ c\big\rangle = \varphi_A(ac) = \big\langle a,\ \mathcal{F}_R^{\mathrm{op}}(c)\big\rangle$$

for all $a, c\in A$, hence $\mathcal{F}_L$ and $\mathcal{F}_R^{\mathrm{op}}$ are injective. The other cases are similar. ∎

Recall lemma 3.8.3. Given any multiplier Hopf system $\langle A, B\rangle$ we can define linear mappings T_1^A and T_2^A from $A\otimes A$ into $M(A\otimes A)$ by

$$T_1^A(a\otimes c) \;=\; \Delta_A(a)(1\otimes c) \qquad\qquad T_2^A(a\otimes c) \;=\; (a\otimes 1)\Delta_A(c).$$

Similarly we may define mappings T_1^B and T_2^B from $B\otimes B$ into $M(B\otimes B)$.

Lemma 3.8.17 *Let $\langle A, B\rangle$ be any regular multiplier Hopf system, and recall that $\mathcal{F}_R$ is a bijection from A onto B. We now have*

$$T_1^A \;=\; (\mathrm{id}\otimes\mathcal{F}_R^{-1})\,\lambda_P\,(\mathrm{id}\otimes\mathcal{F}_R).$$

It follows that T_1^A is actually a bijection from $A\otimes A$ onto $A\otimes A$.

Proof. Take any $a, c\in A$ and $b, d\in B$ and write $\lambda_P\big(a\otimes\mathcal{F}_R(c)\big) = \sum_i p_i\otimes q_i$ with $p_i\in A$ and $q_i\in B$. Using (3.1) we obtain that for any $e\in A$

$$\begin{aligned}
\textstyle\sum_i \big\langle b\otimes e,\, p_i\otimes q_i\big\rangle \;&=\; \big\langle a\triangleleft b,\, \mathcal{F}_R(c)\triangleleft e\big\rangle\\
&=\; \big\langle e(a\triangleleft b),\, \mathcal{F}_R(c)\big\rangle\\
&=\; \psi_A\big(e(a\triangleleft b)c\big)\\
&=\; \big\langle e,\, \mathcal{F}_R\big((a\triangleleft b)c\big)\big\rangle
\end{aligned}$$

and hence $\sum_i \langle p_i, b\rangle\, q_i \;=\; \mathcal{F}_R\big((a\triangleleft b)c\big)$. It follows that

$$\begin{aligned}
\big\langle(\mathrm{id}\otimes\mathcal{F}_R^{-1})\,\lambda_P\,(\mathrm{id}\otimes\mathcal{F}_R)(a\otimes c),\, b\otimes d\big\rangle \;&=\; \textstyle\sum_i \big\langle p_i\otimes\mathcal{F}_R^{-1}(q_i),\, b\otimes d\big\rangle\\
&=\; \textstyle\sum_i \big\langle\mathcal{F}_R^{-1}\big(\langle p_i, b\rangle\, q_i\big)\,,\, d\big\rangle\\
&=\; \big\langle(a\triangleleft b)c,\, d\big\rangle\\
&=\; \big\langle a,\, b(c\triangleright d)\big\rangle\\
&=\; \big\langle\Delta_A(a),\, b\otimes(c\triangleright d)\big\rangle\\
&=\; \big\langle T_1^A(a\otimes c),\, b\otimes d\big\rangle
\end{aligned}$$

which completes the proof. ∎

Proof of theorem 3.8.4 From lemma 3.8.17 and its analogues for T_2^A, T_1^B and T_2^B, it follows easily that (A, Δ_A) and (B, Δ_B) are multiplier Hopf algebras. Since $\langle A, B\rangle$ is assumed to be regular, we may replace[12] $\langle A, B\rangle$ by $\langle A, B^{\mathrm{op}}\rangle$. It follows that (A, Δ_A) and (B, Δ_B) are regular as a multiplier Hopf algebra. Recalling theorem 3.8.15 and 3.8.16, we conclude that $B\simeq\hat{A}$ and $A\simeq\hat{B}$. Finally, pseudo-discreteness of e.g. $\mathbb{B} = (\hat{A}; A, \langle\cdot,\cdot\rangle)$ follows from a result in [21], stating that $M(\mathbb{B})\simeq M(\hat{A})$ identifies naturally with the space

$$\Big\{f\in A'\;\Big|\;(\mathrm{id}\otimes f)\Delta_A(a)\in A \text{ and } (f\otimes\mathrm{id})\Delta_A(a)\in A, \text{ for any } a\in A\Big\}.$$

According to remark 2.3.1.3.ii, the latter is nothing but $A^\sharp\simeq\mathrm{Act}(\mathbb{B})$. ∎

[12]cf. lemma 3.8.8.

3.9 The quantum double of a Hopf system

Abstract 3.7 Given any regular Hopf system we shall construct a quantum double, provided an extra condition is fulfilled; this extra assumption is called balancedness and will be investigated in the first paragraph. Then we show that our quantum double is again a regular Hopf system. In particular the construction applies to multiplier Hopf systems, and in that case the quantum double turns out to be again a multiplier Hopf system. Thus we obtain an alternative for the approach in [7]. $\star$

3.9.1 Balanced Hopf systems

Let $\langle A, B\rangle$ be any Hopf system, and recall the mappings Γ_L and Γ_R introduced in definition 3.8.5. Now it is not hard to see that Γ_L and Γ_R actually take values in the weak[13] Fubini tensorproduct $B \overline{\otimes} A$. On the other hand[14] $B \overline{\otimes} A$ identifies naturally with the spaces $L(A)$ and $L(B)$ of weakly continuous linear operators on A and B respectively. Explicitly, for any $a, p \in A$ and $b, q \in B$ we have

$$\big\langle \Gamma_L(a \otimes b),\, p \otimes q \big\rangle \;=\; \langle ap, bq\rangle \;=\; \langle p, bq \triangleleft a\rangle,$$

hence $\Gamma_L(a \otimes b)$ identifies with the operator $(b\,\cdot) \triangleleft a$ on B. Analogously:

	$L(A)$	$L(B)$
$\Gamma_L(a \otimes b)$	$(a\,\cdot) \triangleleft b$	$(b\,\cdot) \triangleleft a$
$\Gamma_R(a \otimes b)$	$b \triangleright (\cdot\, a)$	$a \triangleright (\cdot\, b)$

So the range of Γ_L identifies with the subspace of $L(B)$ spanned by the operators $(b\,\cdot) \triangleleft a$ with $a \in A$ and $b \in B$. We shall denote this subspace of $L(B)$ by $\mathcal{C}_L(B)$. Similarly we define $\mathcal{C}_R(B)$, $\mathcal{C}_L(A)$ and $\mathcal{C}_R(A)$.[15]
Combining lemma 3.2.8 and proposition 3.2.10, we obtain that $T_L = \rho_P^{-1}\chi$ is an (A^{op}, B)-twisting, whereas $T_R = \lambda_P^{-1}\chi$ is an (A, B^{op})-twisting. Hence according to proposition 3.2.7 we have twisted tensorproducts $A^{\mathrm{op}} \otimes_{T_L} B$ and $A \otimes_{T_R} B^{\mathrm{op}}$. Furthermore $L(A)$ and $L(B)$ are algebras under composition of operators, so the following makes sense:

Proposition 3.9.1.1 *Let $\langle A, B\rangle$ be any Hopf system. Then the mappings*

$$\Gamma_L : A^{\mathrm{op}} \otimes_{T_L} B \to L(B) \simeq L(A)^{\mathrm{op}} \quad \text{and} \quad \Gamma_R : A \otimes_{T_R} B^{\mathrm{op}} \to L(B) \simeq L(A)^{\mathrm{op}}$$

are algebra homomorphisms. It follows that $\mathcal{C}_L(B)$ and $\mathcal{C}_R(B)$ are subalgebras of $L(B)$. Similarly $\mathcal{C}_L(A)$ and $\mathcal{C}_R(A)$ are subalgebras of $L(A)$.

Proof. Take any $a, c, p \in A$ and $b, d, q \in B$ and observe that

$$\big\langle p,\, \big(\Gamma_L(a \otimes b) \circ \Gamma_L(c \otimes d)\big)\,(q)\big\rangle \quad = \quad \big\langle p,\, \big(\Gamma_L(a \otimes b)\big)\,(dq \triangleleft c)\big\rangle$$

[13] w.r.t. the duality $\langle A, B\rangle$, of course. See appendix A for details.
[14] cf. remark A.2.ii in appendix A.
[15] These spaces are comparable to the algebra $\mathcal{C}(V)$ associated to a multiplicative unitary V in the sense of [4].

$$
\begin{aligned}
&= \big\langle p,\, b(dq \triangleleft c) \triangleleft a\big\rangle \\
&= \big\langle ap \triangleleft b,\, dq \triangleleft c\big\rangle \\
&= \big\langle b \otimes c,\, \lambda_P(ap \otimes dq)\big\rangle \\
&= \big\langle \rho_P\big(\rho_P^{-1}(c \otimes b)\big) \underset{\chi}{|} \lambda_P(ap \otimes dq)\big\rangle \\
&\overset{(3.5)}{=} \big\langle P,\, \rho_P^{-1}(c \otimes b)\,(ap \otimes dq)\big\rangle \\
&= \big\langle P,\, \big(T_L(b \otimes c)\,(a \otimes d)\big)\,(p \otimes q)\big\rangle \\
&\overset{(3.29)}{=} \big\langle \Gamma_L\big((a \otimes b) \rtimes (c \otimes d)\big),\, p \otimes q\big\rangle
\end{aligned}
$$

where $\rtimes$ denotes the product in $A^{\mathrm{op}} \otimes_{T_L} B$. The other cases are similar. ■

Let us consider a second approach to these $\mathcal{C}$-algebras: from (3.31) it is clear that the range of Γ_L is also equal to $\lambda_P^\tau(B \otimes A) = (B \otimes A)P$. Now observe that

$$\big\langle (b \otimes a)P,\, p \otimes q\big\rangle \;=\; \big\langle b \otimes a,\, \lambda_P(p \otimes q)\big\rangle \;=\; \big\langle p \triangleleft b,\, q \triangleleft a\big\rangle \;=\; \big\langle p,\, b(q \triangleleft a)\big\rangle$$

for all $a, p \in A$ and $b, q \in B$. It follows that $\mathcal{C}_L(B)$ is also spanned by the operators $b(\,\cdot \triangleleft a)$ with $a \in A$ and $b \in B$. Defining operators $\eta(a)$ and $\pi(b)$ on B by $\eta(a)q = q \triangleleft a$ and $\pi(b)q = bq$, our results can be summarized as follows:

$$\mathrm{Range}(\Gamma_L) \;\simeq\; \mathcal{C}_L(B) \;=\; \eta(A)\,\pi(B) \;=\; \pi(B)\,\eta(A) \;\subseteq\; L(B).$$

Similar results hold for the other $\mathcal{C}$-algebras. Another interesting observation is that when $F(B)$ denotes the ideal of all *finite rank* operators in $L(B)$, then lemma 3.8.7 can easily be reformulated as follows:

Lemma 3.9.1.2 *A Hopf system $\langle A, B\rangle$ is a multiplier Hopf system if and only if $\mathcal{C}_L(B) = F(B) = \mathcal{C}_R(B)$.*

Whenever $\langle A, B\rangle$ is a Hopf system, we have an algebra homomorphism

$$\pi_P : B \otimes A \to \mathrm{Pre}(\mathbb{B} \otimes \mathbb{A}) : y \mapsto PyP^{-1}$$

(cf. remark 3.2.12 and lemma 2.4.3.9). In order to construct a quantum double, however, we shall need π_P to be actually an *auto*morphism of $B \otimes A$:

Lemma 3.9.1.3 *Let $\langle A, B\rangle$ be any Hopf system. The following are equivalent:*

i. π_P *is an automorphism of* $B \otimes A$

ii. $(B \otimes A)P = P(B \otimes A)$

iii. $\mathrm{Range}(\Gamma_L) = \mathrm{Range}(\Gamma_R)$

iv. $\mathcal{C}_L(B) = \mathcal{C}_R(B)$.

If $\langle A, B\rangle$ is regular, then the above assertions are moreover equivalent with:

v. The range of Γ_L is invariant under $S_B \,\overline{\otimes}\, S_A^{-1}$ and $S_B^{-1} \,\overline{\otimes}\, S_A$.

vi. $S_B \mathcal{C}_L(B) S_B^{-1} \; = \; \mathcal{C}_L(B)$.

Observe that (v) and (vi) make sense because S_A and S_B are invertible elements in $L(A)$ and $L(B)$ respectively.

Proof. (i $\Leftrightarrow$ ii) is obvious. (ii $\Leftrightarrow$ iii) follows from (3.31). (iii $\Leftrightarrow$ iv) holds by definition. Now assume $\langle A, B\rangle$ to be regular. We claim that

$$(S_B \mathbin{\overline{\otimes}} S_A^{-1})\, \Gamma_L (S_A \otimes S_B^{-1}) \; = \; \Gamma_R .$$

Indeed, for all $a, p \in A$ and $b, q \in B$ we have

$$\begin{aligned}
&\big\langle (S_B \mathbin{\overline{\otimes}} S_A^{-1})\, \Gamma_L (S_A \otimes S_B^{-1})(a \otimes b),\, p \otimes q \big\rangle \\
&\quad = \; \big\langle \Gamma_L \big(S_A(a) \otimes S_B^{-1}(b) \big) \,,\, S_A(p) \otimes S_B^{-1}(q) \big\rangle \\
&\quad = \; \big\langle S_A(a)\, S_A(p),\, S_B^{-1}(b)\, S_B^{-1}(q) \big\rangle \\
&\quad = \; \big\langle pa,\, qb \big\rangle \\
&\quad = \; \big\langle \Gamma_R(a \otimes b),\, p \otimes q \big\rangle
\end{aligned}$$

It follows that the range of Γ_R equals the range of $(S_B \mathbin{\overline{\otimes}} S_A^{-1})\Gamma_L$. This proves (iii $\Leftrightarrow$ v). Remark A.3.v in appendix A yields (v $\Leftrightarrow$ vi). ∎

Definition 3.9.1.4 A Hopf system $\langle A, B\rangle$ is said to be *balanced* whenever it enjoys the equivalent conditions (i-iv) in lemma 3.9.1.3. •

Observe that e.g. every multiplier Hopf system is balanced. Also notice that if $\langle A, B\rangle$ is a balanced Hopf system, then $P \in \mathrm{Act}(\mathbb{B} \otimes \mathbb{A})$ enjoys condition (iv) in lemma 2.4.3.9. In particular it follows that λ_P and ρ_P commute.

Remark 3.9.1.5 The notion of balancedness is similar to S. Montgomery's RL-condition emerging in the duality theory for smash products [28, §9.4]. When $\langle A, B\rangle$ is a dual pair of Hopf algebras, then A is said to satisfy the RL-condition with respect to B if the right action $(\cdot) \triangleleft a$ of any $a \in A$ on B can be expressed as a linear combination of operators $b(c \triangleright \cdot\,)$ with $c \in A$ and $b \in B$. So the RL-condition roughly means that left actions can be expressed in terms of right actions. Our condition of balancedness seems to be more symmetric, but unfortunately also more restrictive. On the other hand, balancedness will appear to be a very obvious requirement in our quantum double construction below. ⋆

3.9.2 Construction of a quantum double

Throughout this section we shall adopt the following

Setting 3.9.2.1 Let $\langle A, B\rangle$ be a balanced regular Hopf system. Let R be the bijective linear map from $A \otimes B$ onto $A \otimes B$ given by $R = \rho_P \lambda_P^{-1}$. Recall that $T = R\chi$ is an (A, B^{op})-twisting (corollary 3.2.11) and let $X = A \otimes_T B^{\mathrm{op}}$ be the corresponding twisted tensor product algebra as defined in proposition 3.2.7. Let $\star$ denote the product in X. Explicitly, for $a, c \in A$ and $b, d \in B$ we have

$$(a \otimes b) \star (c \otimes d) \;=\; (m_A \otimes m_B^{\mathrm{op}})(\mathrm{id} \otimes T \otimes \mathrm{id})(a \otimes b \otimes c \otimes d) \;=\; (a \otimes d)R(c \otimes b).$$

On the other hand, let $Y = B \otimes A$ be the ordinary tensor product algebra, and consider the pairing $\langle X, Y\rangle$ given by the obvious vector space duality between $A \otimes B$ and $B \otimes A$. Since $\langle A, B\rangle$ is assumed to be *balanced*, we have an automorphism $\pi_P : Y \to Y : y \mapsto PyP^{-1}$ which is dual to R in the sense that

$$\big\langle x,\, \pi_P(y)\big\rangle \;=\; \big\langle x,\, PyP^{-1}\big\rangle \;=\; \big\langle \rho_P \lambda_P^{-1}(x),\, y\big\rangle \;=\; \big\langle R(x),\, y\big\rangle$$

for all $x \in X$ and $y \in Y$. It follows that

$$\alpha \;=\; (\mathrm{id}_X \otimes \pi_P^{-1})\,(\lambda_P)_{13}\,(\mathrm{id}_X \otimes \pi_P)\,(\lambda_P^{\mathrm{op}})_{42} \tag{3.32}$$

$$\beta \;=\; (\mathrm{id}_X \otimes \pi_P^{-1})\,(\rho_P^{\mathrm{op}})_{42}\,(\mathrm{id}_X \otimes \pi_P)\,(\rho_P)_{13} \tag{3.33}$$

are linear bijections from $X \otimes Y$ onto $X \otimes Y$. Here the leg-numbering notation is to be considered w.r.t. $A \otimes B \otimes B \otimes A$. •

Our strategy will be the following: we want to use lemma 3.1.15 to show that $\langle X, Y\rangle$ is indeed an invertible dual pair of algebras. So first we shall verify the conditions of this lemma. Then we shall prove multiplicativity and regularity.

Since $X = A \otimes_T B^{\mathrm{op}}$ is an algebra, it acts canonically on its dual X'. To avoid confusion with the actions $\triangleright$ and $\triangleleft$ of the ordinary tensor product algebra $A \otimes B$, we shall denote the former ones by $\looparrowright$ and $\looparrowleft$. The question arises whether

$$\mathbb{X} \;\equiv\; \big(X; Y, \langle\cdot,\cdot\rangle\big) \qquad\text{and}\qquad \mathbb{Y} \;\equiv\; \big(Y; X, \langle\cdot,\cdot\rangle\big) \tag{3.34}$$

are actor contexts. For the second one, the answer is obvious since $\mathbb{Y} = \mathbb{B} \otimes \mathbb{A}$. Moreover $\mathbb{A}$ and $\mathbb{B}$ are known to be weakly unital, hence so is $\mathbb{Y}$. The counit for $\mathbb{Y}$ is given by $\varepsilon_X = \varepsilon_A \otimes \varepsilon_B$. Now let us investigate the actions of X on Y:

Lemma 3.9.2.2 *Adopt setting 3.9.2.1. For all $x \in X$ and $y \in Y$ we have*

$$x \looparrowright y \;=\; (\varepsilon_X \otimes \mathrm{id}_Y)\,\alpha(x \otimes y) \tag{3.35}$$

$$y \looparrowleft x \;=\; (\varepsilon_X \otimes \mathrm{id}_Y)\,\beta(x \otimes y). \tag{3.36}$$

Proof. Take any $a, c, p \in A$ and $b, d, q \in B$. Straightforward computations show that $(\mathrm{id} \otimes \varepsilon_B)\lambda_P^{\mathrm{op}}(c \otimes b) = c \triangleleft b$ and $(\varepsilon_A \otimes \mathrm{id})\lambda_P = \mu_L$, where $\mu_L : A \otimes B \to B$ denotes the left action $\triangleright$ of A on B. Now write $R^{-1}(p \otimes q) = \sum_i v_i \otimes w_i$ with $v_i \in A$ and $w_i \in B$, and observe that

$$\begin{aligned}
&\big\langle p \otimes q,\, (\varepsilon_X \otimes \mathrm{id}_Y)\,\alpha(a \otimes b \otimes d \otimes c)\big\rangle \\
&\overset{(3.32)}{=} \;\big\langle R^{-1}(p \otimes q),\, (\varepsilon_A \otimes \varepsilon_B \otimes \mathrm{id}_Y)\,(\lambda_P)_{13}\,(\mathrm{id}_X \otimes \pi_P)\,(\lambda_P^{\mathrm{op}})_{42}\,(a \otimes b \otimes d \otimes c)\big\rangle \\
&= \;\textstyle\sum_i \big\langle v_i \otimes w_i,\, (\mu_L \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \pi_P)\,\big(a \otimes d \otimes (c \triangleleft b)\big)\big\rangle
\end{aligned}$$

$$\begin{aligned}
&= \textstyle\sum_i \left\langle v_i a \otimes w_i,\ \pi_P\big(d \otimes (c \triangleleft b)\big)\right\rangle \\
&= \textstyle\sum_i \left\langle R(m_A \otimes \mathrm{id})(v_i \otimes a \otimes w_i),\ d \otimes (c \triangleleft b)\right\rangle \\
&\overset{(3.8)}{=} \textstyle\sum_i \left\langle (m_A \otimes \mathrm{id})R_{23}R_{13}(v_i \otimes a \otimes w_i),\ d \otimes (c \triangleleft b)\right\rangle \\
&= \left\langle (m_A \otimes \mathrm{id})R_{23}(p \otimes a \otimes q),\ d \otimes (c \triangleleft b)\right\rangle \\
&= \left\langle (p \otimes b)R(a \otimes q),\ d \otimes c\right\rangle \\
&= \left\langle (p \otimes q) \star (a \otimes b),\ d \otimes c\right\rangle \\
&= \left\langle p \otimes q,\ (a \otimes b) \looparrowright (d \otimes c)\right\rangle .
\end{aligned}$$

This proves (3.35). The second formula is shown analogously. ■

Corollary 3.9.2.3 *$X \looparrowright Y = Y = Y \looparrowleft X$.*

Lemma 3.9.2.4 *Let $\langle A, B\rangle$ be a regular Hopf system. For any $y \in B \otimes A$, the elements $P^{-1}y$, $yP^{-1} \in \mathrm{Act}(\mathbb{B} \otimes \mathbb{A})$ can also be viewed as elements in $B \mathbin{\overline{\otimes}} A$. Applying the slice map $\mathrm{id}\,\overline{\otimes\,\varepsilon_A} : B \mathbin{\overline{\otimes}} A \to \overline{B} \equiv A'$ (cf. remark A.3.iii) we get*

$$(\mathrm{id}\,\overline{\otimes\,\varepsilon_A})(yP^{-1}) \;=\; (\mathrm{id} \otimes \varepsilon_A)(y) \;=\; (\mathrm{id}\,\overline{\otimes\,\varepsilon_A})(P^{-1}y), \tag{3.37}$$

which means that

$$\varepsilon_A(f_a \mathbin{\overline{\otimes}} \mathrm{id})(yP^{-1}) \;=\; \varepsilon_A(f_a \otimes \mathrm{id})(y) \;=\; \varepsilon_A(f_a \mathbin{\overline{\otimes}} \mathrm{id})(P^{-1}y)$$

for all $a \in A$. Here f_a denotes the functional $\langle a, \cdot\,\rangle$ on B. See also (A.4).

Proof. Take any $a, c \in A$ and $b, d \in B$ and observe that

$$\left\langle (d \otimes c)P^{-1},\ a \otimes b\right\rangle \;=\; \left\langle P^{-1},\ (a \triangleleft d) \otimes (b \triangleleft c)\right\rangle \;=\; \begin{cases} \langle c\, S_A(a \triangleleft d), b\rangle \\ \langle a, d\, S_B(b \triangleleft c)\rangle. \end{cases}$$

It follows that $(d \otimes c)P^{-1}$ belongs to $B \mathbin{\overline{\otimes}} A$ and

$$(f_a \mathbin{\overline{\otimes}} \mathrm{id})\,\big((d \otimes c)P^{-1}\big) \;=\; c\, S_A(a \triangleleft d).$$

Now apply ε_A and invoke proposition 3.2.5 and corollary 3.3.4. This yields

$$\varepsilon_A(f_a \mathbin{\overline{\otimes}} \mathrm{id})\,\big((d \otimes c)P^{-1}\big) \;=\; \varepsilon_A(c)\, \varepsilon_A\big(S_A(a \triangleleft d)\big) \;=\; \varepsilon_A(c)\, \langle a, d\rangle.$$

The other case is similar. ■

Lemma 3.9.2.5 *Adopt setting 3.9.2.1 and recall $\pi_P(B \otimes A) = B \otimes A$. We have*

$$(\mathrm{id} \otimes \varepsilon_A)\, \pi_P \;=\; \mathrm{id} \otimes \varepsilon_A \qquad \textit{and} \qquad (\varepsilon_B \otimes \mathrm{id})\, \pi_P \;=\; \varepsilon_B \otimes \mathrm{id}$$

and hence $(\varepsilon_B \otimes \varepsilon_A)\, \pi_P \;=\; \varepsilon_B \otimes \varepsilon_A$.

Proof. Take $y \in B \otimes A$. Using (3.37) twice, we obtain

$$(\mathrm{id} \otimes \varepsilon_A)\, \pi_P(y) \;=\; (\mathrm{id}\,\overline{\otimes\,\varepsilon_A})\,\big(P^{-1}(PyP^{-1})\big) \;=\; (\mathrm{id}\,\overline{\otimes\,\varepsilon_A})(yP^{-1}) \;=\; (\mathrm{id} \otimes \varepsilon_A)(y).$$

The other case is similar. ■

Corollary 3.9.2.6 *Defining $\varepsilon_Y = \varepsilon_B \otimes \varepsilon_A$, we have for all $x \in X$ and $y \in Y$:*

$$(\varepsilon_X \otimes \varepsilon_Y)\,\alpha(x \otimes y) \;=\; \langle x, y\rangle \;=\; (\varepsilon_X \otimes \varepsilon_Y)\,\beta(x \otimes y). \tag{3.38}$$

Proof. Consider (3.32) and (3.33), apply $\varepsilon_A \otimes \varepsilon_B \otimes \varepsilon_B \otimes \varepsilon_A$ and recall that e.g. $(\varepsilon_A \otimes \varepsilon_B)\lambda_P(a \otimes b) = \langle a, b\rangle$ etc. (cf. equation (2.11) in §2.4). Lemma 3.9.2.5 helps us to get rid of the π_P maps, and the result follows. ∎

Proposition 3.9.2.7 *Adopt setting 3.9.2.1. Then the triplets $\mathbb{X}$ and $\mathbb{Y}$ defined in (3.34) are both weakly unital actor contexts, with counits $\varepsilon_Y = \varepsilon_B \otimes \varepsilon_A$ and $\varepsilon_X = \varepsilon_A \otimes \varepsilon_B$. In particular $\langle X, Y\rangle$ is a dual pair of algebras (example 2.1.2.4).*

Proof. For $\mathbb{Y}$ the result has already been obtained (cf. before lemma 3.9.2.2). Corollary 3.9.2.3 yields that also $\mathbb{X}$ is an actor context, and moreover that Y is unital as an X-bimodule. Combining (3.38) with (3.35) and (3.36), we may conclude that $\varepsilon_Y = \varepsilon_B \otimes \varepsilon_A$ is indeed a counit for $\mathbb{X}$. ∎

Lemma 3.9.2.8 *For any $y \in Y$, the mapping α defined in (3.32) commutes with $(\,\cdot \triangleleft y) \otimes \mathrm{id}_Y$. Similarly β will commute with $(y \triangleright \cdot\,) \otimes \mathrm{id}_Y$.*

Proof. Straightforward; use the module properties of λ_P, ρ_P, λ_P^{op} and ρ_P^{op} in the sense of lemma 2.6.4. ∎

Lemma 3.9.2.9 *Adopt setting 3.9.2.1. For any $x \in X$ and $y \in Y$ we have*

$$(\varepsilon_X \otimes \varepsilon_Y)\,\alpha^{-1}(x \otimes y) \;=\; \big\langle R(S_A \otimes S_B^{\mathrm{op}})(x),\, y\big\rangle \;=\; (\varepsilon_X \otimes \varepsilon_Y)\,\beta^{-1}(x \otimes y). \tag{3.39}$$

Proof. This is very much like the proof of corollary 3.9.2.6. First take inverses of (3.32) and (3.33). Then apply $\varepsilon_A \otimes \varepsilon_B \otimes \varepsilon_B \otimes \varepsilon_A$ and recall that e.g.

$$(\varepsilon_A \otimes \varepsilon_B)(\lambda_P^{\mathrm{op}})^{-1}(a \otimes b) \;=\; \big\langle (P^{\mathrm{op}})^{-1},\, a \otimes b\big\rangle \;=\; \big\langle a,\, S_B^{\mathrm{op}}(b)\big\rangle$$

etc. Using lemma 3.9.2.5, the result follows easily. ∎

Proposition 3.9.2.10 *Adopt setting 3.9.2.1. Then $\langle X, Y\rangle$ is an invertible dual pair of algebras. If $Q \equiv (\lambda_Q, \rho_Q)$ denotes the actor for $\mathbb{Y} \otimes \mathbb{X}$ corresponding to the functional $X \otimes Y \to \mathbb{C} : x \otimes y \mapsto \langle x, y\rangle$, then $\lambda_Q = \alpha$ and $\rho_Q = \beta$.*

Proof. Recall proposition 3.9.2.7, equations (3.35) and (3.36), lemma 3.9.2.8 and equation (3.39). Now invoke lemma 3.1.15. ∎

Notation 3.9.2.11 There are now *two* invertible dual pairs involved: $\langle A, B\rangle$ and $\langle X, Y\rangle$. Therefore we must be careful with the notation. As before, P will denote the actor for $\mathbb{B} \otimes \mathbb{A}$ induced by the pairing in $\langle A, B\rangle$. To avoid conflicts, the actor for $\mathbb{Y} \otimes \mathbb{X}$ derived from the pairing in $\langle X, Y\rangle$ will be denoted by Q as in the previous proposition. Furthermore the mappings α and β defined in (3.32) and (3.33) shall henceforth be denoted by λ_Q and ρ_Q respectively.

3.9.3 Multiplicativity

Again adopt setting 3.9.2.1 and recall §3.2. By now we know that $\langle X, Y\rangle$ is an invertible dual pair of algebras, but of course we want it to be a Hopf system. The shortest way towards multiplicativity is to show that the comultiplication

$$\Delta_Y : Y = B \otimes A \;\to\; \mathrm{Env}(\mathbb{Y} \otimes \mathbb{Y}) = \mathrm{Env}(\mathbb{B} \otimes \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{A})$$

is an algebra homomorphism and then invoke proposition 3.2.3. Since Δ_Y is dual to $m_X = (m_A \otimes m_B^{\mathrm{op}})(\mathrm{id} \otimes T \otimes \mathrm{id})$, we have $\Delta_Y = \Lambda\Phi(\Delta_B \otimes \chi\Delta_A)$ where

$$\Delta_A : A \to \mathrm{Env}(\mathbb{A} \otimes \mathbb{A}) \qquad \text{and} \qquad \Delta_B : B \to \mathrm{Env}(\mathbb{B} \otimes \mathbb{B})$$

are the comultiplications (cf. proposition 3.1.13) on A and B,

$$\Phi : \mathrm{Env}(\mathbb{B} \otimes \mathbb{B}) \otimes \mathrm{Env}(\mathbb{A} \otimes \mathbb{A}) \;\to\; \mathrm{Env}(\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{A})$$

is the embedding established in proposition 2.6.5, and

$$\Lambda : \mathrm{Pre}(\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{A}) \;\to\; \mathrm{Pre}(\mathbb{B} \otimes \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{A})$$

is given by $\Lambda = \chi_{23}(P_{23} \cdot P_{23}^{-1})$. Of course the flip map involved here should be extended properly to the pre-actor algebra, but this does not present a problem. On the other hand we do have to be careful with Λ, since we need

$$\big\langle (\mathrm{id} \otimes T \otimes \mathrm{id})(u),\, v \big\rangle \;=\; \big\langle u,\, \Lambda(v) \big\rangle \tag{3.40}$$

to hold for any $u \in A \otimes B \otimes A \otimes B$ and $v \in \mathrm{Env}(\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{A})$. For the right hand side of (3.40) to make sense, however, Λ should map $\mathrm{Env}(\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{A})$ into $\mathrm{Act}(\mathbb{B} \otimes \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{A})$. Fortunately $P_{23} \in \mathrm{Act}(\mathbb{B} \otimes \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{A})$ enjoys (iv) of lemma 2.4.3.9, and statement (iii) of this lemma will ensure (3.40).
Now Δ_A, Δ_B, Φ and Λ are all algebra homomorphisms, hence so is Δ_Y.

The above argument, however, is not very loyal to our duality approach; indeed it would be nice if we could show multiplicativity (in the sense of definition 3.2.1) *directly* from the formulas (3.32) and (3.33) for λ_Q and ρ_Q (cf. notation 3.9.2.11) i.e. without the intervention of comultiplications. In this respect we have:

Proposition 3.9.3.1 *Adopt setting 3.9.2.1 and notation 3.9.2.11. Then*

$$\big\langle \rho_Q(x_1 \otimes y_1) \underset{\chi}{|} \lambda_Q(x_2 \otimes y_2) \big\rangle \;=\; \big\langle x_1 \star x_2,\, y_1 y_2 \big\rangle$$

for all $x_1, x_2 \in X$ *and* $y_1, y_2 \in Y$.

Proof. Take any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, and denote

$$\xi \;=\; \big\langle \rho_Q(x_1 \otimes y_1) \underset{\chi}{|} \lambda_Q(x_2 \otimes y_2) \big\rangle\,.$$

Before we start, let us agree that the leg-numbering notation always refers to the lowest level, i.e. the legs are always in A or in B (rather than in X or Y).

From time to time we shall use the fact that $\pi_P : Y \to Y$ is multiplicative, which will be indicated with an asterisk $(*)$. Observe that for any $x \in X$ and $y \in Y$

$$
\begin{aligned}
&\big\langle \rho_Q(x_1 \otimes y_1) \underset{\chi}{\mid} (\mathrm{id}_X \otimes \pi_P^{-1})(x \otimes y) \big\rangle \\
&\quad = \big\langle \rho_Q(x_1 \otimes y_1),\, \pi_P^{-1}(y) \otimes x \big\rangle \\
&\quad = \big\langle \pi_P^{-1}(y) \triangleright x_1,\, x \looparrowright y_1 \big\rangle \\
&\quad = \big\langle x_1,\, m_Y\big((x \looparrowright y_1) \otimes \pi_P^{-1}(y)\big) \big\rangle \\
&\quad \overset{(3.35)}{=} \big\langle x_1,\, m_Y(\varepsilon_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_Y)(\lambda_Q \otimes \mathrm{id}_Y)\big(x \otimes y_1 \otimes \pi_P^{-1}(y)\big) \big\rangle \\
&\quad = \big\langle x_1,\, (\varepsilon_X \otimes m_Y)(\lambda_Q \otimes \pi_P^{-1})(x \otimes y_1 \otimes y) \big\rangle \\
&\quad \overset{(3.32)}{=} \big\langle x_1,\, (\varepsilon_X \otimes m_Y)(\mathrm{id}_X \otimes \pi_P^{-1} \otimes \pi_P^{-1}) \\
&\qquad\qquad (\lambda_P)_{13}(\mathrm{id}_X \otimes \pi_P \otimes \mathrm{id}_Y)(\lambda_P^{\mathrm{op}})_{42}(x \otimes y_1 \otimes y) \big\rangle \\
&\quad \overset{(*)}{=} \big\langle x_1,\, \pi_P^{-1}(\varepsilon_X \otimes m_Y)(\lambda_P)_{13}(\mathrm{id}_X \otimes \pi_P \otimes \mathrm{id}_Y)(\lambda_P^{\mathrm{op}})_{42}(x \otimes y_1 \otimes y) \big\rangle \\
&\quad = \big\langle R^{-1}(x_1),\, (\varepsilon_X \otimes m_Y)(\lambda_P)_{13}(\pi_P)_{34}(\lambda_P^{\mathrm{op}})_{42}\chi_{35}\chi_{46}(x \otimes y \otimes y_1) \big\rangle .
\end{aligned}
$$

Now we replace $x \otimes y$ with $(\lambda_P)_{13}(\mathrm{id}_X \otimes \pi_P)(\lambda_P^{\mathrm{op}})_{42}(x_2 \otimes y_2)$ and obtain:

$$
\begin{aligned}
\xi &= \big\langle R^{-1}(x_1), (\varepsilon_X \otimes m_Y)(\lambda_P)_{13}(\pi_P)_{34}(\lambda_P^{\mathrm{op}})_{42}(\lambda_P)_{15}(\pi_P)_{56}(\lambda_P^{\mathrm{op}})_{62}(x_2 \otimes y_1 \otimes y_2) \big\rangle \\
&= \big\langle R^{-1}(x_1), (\varepsilon_X \otimes m_Y)(\lambda_P)_{13}(\lambda_P)_{15}(\pi_P)_{34}(\pi_P)_{56}(\lambda_P^{\mathrm{op}})_{42}(\lambda_P^{\mathrm{op}})_{62}(x_2 \otimes y_1 \otimes y_2) \big\rangle
\end{aligned}
$$

Recall that $\lambda_P\chi$ is an (A^{op}, B)-twisting (proposition 3.2.10). Using (3.7) we get[16]

$$
(\mathrm{id}_X \otimes m_{B\otimes A})(\lambda_P)_{13}(\lambda_P)_{15} \;=\; (\lambda_P)_{13}(\mathrm{id}_X \otimes m_{B\otimes A}) \tag{3.41}
$$

and our computation proceeds as follows:

$$
\begin{aligned}
\xi \;&=\; \big\langle R^{-1}(x_1),\, (\varepsilon_X \otimes \mathrm{id}_Y)(\lambda_P)_{13}(\mathrm{id}_X \otimes m_Y) \\
&\qquad\qquad (\pi_P)_{34}(\pi_P)_{56}(\lambda_P^{\mathrm{op}})_{42}(\lambda_P^{\mathrm{op}})_{62}(x_2 \otimes y_1 \otimes y_2) \big\rangle \\
&\overset{(*)}{=} \big\langle R^{-1}(x_1),\, (\mu_L \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \varepsilon_B \otimes \pi_P m_Y)(\lambda_P^{\mathrm{op}})_{42}(\lambda_P^{\mathrm{op}})_{62}(x_2 \otimes y_1 \otimes y_2) \big\rangle .
\end{aligned}
$$

We used the fact that $(\varepsilon_A \otimes \mathrm{id})\lambda_P = \mu_L : A \otimes B \to B$ is the left action $\triangleright$ of A on B. For mere notational convenience we shall assume x_1, x_2, y_1 and y_2 to be simple tensors, say $x_i = a_i \otimes b_i$ and $y_i = d_i \otimes c_i$ $(i = 1, 2)$. Furthermore we write $\lambda_P^{\mathrm{op}}(c_2 \otimes b_2) = \sum_k p_k \otimes q_k$ and proceed:

$$
\begin{aligned}
\xi \;=\; \textstyle\sum_k \big\langle R^{-1}(x_1),\, (\mu_L \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \varepsilon_B \otimes \pi_P m_Y) \\
\qquad (\lambda_P^{\mathrm{op}})_{42}(a_2 \otimes q_k \otimes d_1 \otimes c_1 \otimes d_2 \otimes p_k) \big\rangle .
\end{aligned}
$$

An easy computation shows that $(\mathrm{id} \otimes \varepsilon_B)\lambda_P^{\mathrm{op}}(c_1 \otimes q_k) = c_1 \triangleleft q_k$, hence

$$
\xi \;=\; \textstyle\sum_k \big\langle R^{-1}(a_1 \otimes b_1),\, (\mu_L \otimes \mathrm{id}_A)\big[a_2 \otimes \pi_P\big(d_1 d_2 \otimes (c_1 \triangleleft q_k) p_k\big)\big] \big\rangle
$$

[16] The presence of m_A in (3.41) is irrelevant and does not interfere with the use of (3.7).

$$\begin{aligned}
&= \sum_k \left\langle (m_A^{\mathrm{op}} \otimes \mathrm{id}_B) R_{23}^{-1}(a_2 \otimes a_1 \otimes b_1),\, \pi_P\big(d_1 d_2 \otimes (c_1 \triangleleft q_k) p_k\big)\right\rangle \\
&\overset{(3.8)}{=} \sum_k \left\langle R^{-1}(m_A^{\mathrm{op}} \otimes \mathrm{id}_B) R_{13}(a_2 \otimes a_1 \otimes b_1),\, \pi_P\big(d_1 d_2 \otimes (c_1 \triangleleft q_k) p_k\big)\right\rangle \\
&= \sum_k \left\langle (m_A^{\mathrm{op}} \otimes \mathrm{id}_B) R_{13}(a_2 \otimes a_1 \otimes b_1),\, d_1 d_2 \otimes (c_1 \triangleleft q_k) p_k\right\rangle .
\end{aligned}$$

Now observe that for any $b \in B$ we have

$$\begin{aligned}
\sum_k \left\langle (c_1 \triangleleft q_k) p_k,\, b\right\rangle &= \sum_k \left\langle c_1 \triangleleft q_k,\, p_k \triangleright b\right\rangle \\
&= \sum_k \left\langle \rho_P^{\mathrm{op}}(c_1 \otimes b),\, q_k \otimes p_k\right\rangle \\
&= \left\langle \rho_P^{\mathrm{op}}(c_1 \otimes b) \underset{\chi}{|} \lambda_P^{\mathrm{op}}(c_2 \otimes b_2)\right\rangle \\
&= \left\langle c_1 c_2,\, b_2 b\right\rangle
\end{aligned}$$

and hence $\sum_k (c_1 \triangleleft q_k) p_k = c_1 c_2 \triangleleft b_2$. We proceed:

$$\begin{aligned}
\xi &= \left\langle (m_A^{\mathrm{op}} \otimes \mathrm{id}_B) R_{13}(a_2 \otimes a_1 \otimes b_1),\, d_1 d_2 \otimes c_1 c_2 \triangleleft b_2\right\rangle \\
&= \left\langle (a_1 \otimes b_2) R(a_2 \otimes b_1),\, d_1 d_2 \otimes c_1 c_2\right\rangle \\
&= \left\langle x_1 \star x_2,\, y_1 y_2\right\rangle .
\end{aligned}$$

This completes the proof. ∎

3.9.4 The quantum double as a Hopf system

Theorem 3.9.4.1 *Let $\langle A, B\rangle$ be a balanced*[17] *regular Hopf system and define $R = \rho_P \lambda_P^{-1}$ and $T = R\chi$. Then $\langle A \otimes_T B^{\mathrm{op}}, B \otimes A\rangle \equiv \langle X, Y\rangle$ is a regular Hopf system. Counits and antipodes are given by*

$$\begin{aligned}
\varepsilon_X &= \varepsilon_A \otimes \varepsilon_B & \qquad S_X &= R(S_A \otimes S_B^{\mathrm{op}}) \\
\varepsilon_Y &= \varepsilon_B \otimes \varepsilon_A & \qquad S_Y &= (S_B \otimes S_A^{\mathrm{op}})\, \pi_P
\end{aligned} \tag{3.42}$$

Furthermore, if φ_A and φ_B are left invariant functionals on A and on B, then $\varphi_A \otimes \varphi_B$ is a left invariant functional on $A \otimes_T B^{\mathrm{op}}$.

Proof. Propositions 3.9.2.10 and 3.9.3.1 yield that $\langle X, Y\rangle$ is a Hopf system, whereas (3.42) follows from proposition 3.9.2.7 and lemma 3.9.2.9. Obviously the antipodes on X and Y are bijective, hence $\langle X, Y\rangle$ is regular. The result on invariant functionals follows since $\mathbb{Y} = \mathbb{B} \otimes \mathbb{A}$. ∎

Definition 3.9.4.2 The Hopf system $\langle A \otimes_T B^{\mathrm{op}}, B \otimes A\rangle$ in the above theorem is called the *quantum double* of $\langle A, B^{\mathrm{op}}\rangle$. •

Notice that we have chosen to associate our quantum double to $\langle A, B^{\mathrm{op}}\rangle$ rather than to $\langle A, B\rangle$. Thus our definition is compatible with the ones in literature [11, 27, 47].

Proposition 3.9.4.3 *If $\langle A, B\rangle$ is a regular multiplier Hopf system, then so is the quantum double $\langle A \otimes_T B^{\mathrm{op}}, B \otimes A\rangle$.*

[17] see definition 3.9.1.4

Proof. First recall that multiplier Hopf systems are balanced, so we do have a quantum double. Now take transposes of (3.32) and (3.33) and recall that π_P is dual to R. Lemmas 3.8.7 and 3.8.8 yield the result. ∎

Corollary 3.9.4.4 *If we adopt the setting of proposition 3.8.2, then*

$$\Big(A \otimes_T \hat{A}^{\mathrm{op}},\ \chi_{23}(\Delta \otimes \hat{\Delta})\Big) \qquad \textit{and} \qquad \Big(\hat{A} \otimes A,\ \chi_{23}(\pi_P)_{23}(\hat{\Delta} \otimes \chi\Delta)\Big)$$

are again regular multiplier Hopf algebras with invariant functionals.

Remarks 3.9.4.5 i. By assumption P and P^{-1} belong to $M(\hat{A} \otimes A)$. Thus $(\pi_P)_{23}$ can be considered as an automorphism of $M(\hat{A} \otimes \hat{A} \otimes A \otimes A)$.

ii. We have to be a little bit careful with the comultiplication on $A \otimes_T \hat{A}^{\mathrm{op}}$. A priori $\chi_{23}(\Delta \otimes \hat{\Delta})$ is to be considered merely as a map from $A \otimes \hat{A}$ into $(\hat{A} \otimes A \otimes \hat{A} \otimes A)'$. It is then part of the assertion that this comultiplication really ends up in the multiplier algebra of $(A \otimes_T \hat{A}^{\mathrm{op}}) \otimes (A \otimes_T \hat{A}^{\mathrm{op}})$.

iii. The two objects obtained in this corollary are dual to each other and can be considered as quantum doubles of $(A, \chi\Delta)$ within the category of regular multiplier Hopf algebras with invariant functionals [7]. ⋆

Proof. According to proposition 3.9.4.3, the pair $\langle A \otimes_T \hat{A}^{\mathrm{op}}, \hat{A} \otimes A \rangle$ is again a regular multiplier Hopf system. Theorem 3.8.4 yields the result. ∎

Chapter 4

Algebraic harmonic analysis

4.1 Fourier contexts

Abstract 4.1 We introduce an axiomatic context in which to do harmonic analysis on an algebraic level, the axioms being inspired to a large extend by the quantum $E(2)$ example that is studied in the next chapter. Minimalizing the set of axioms is not our first concern: we mainly want to outline a setting providing all the ingredients needed to do Fourier analysis. Within this context, we define the notion of a *Fourier transform* and establish its uniqueness. The underlying structure of such a 'Fourier context' basically consists of three algebraic Hopf *-systems—one of which is actually an ordinary dual pair of Hopf *-algebras. $\star$

Definition 4.1.1 Let $\langle \mathfrak{A}, \mathfrak{B} \rangle$ be a non-degenerate dual pair of Hopf *-algebras and let $\langle \underline{\mathfrak{A}}\,, B \rangle$ and $\langle A, \underline{\mathfrak{B}} \rangle$ be algebraic Hopf *-systems (definition 3.7.2.1). Then both A and $\mathfrak{A}$ can be considered as subalgebras of $\mathfrak{B}'$, hence elements of A can be multiplied with elements of $\mathfrak{A}$ within $\mathfrak{B}'$ (similarly for B and $\mathfrak{B}$ within $\mathfrak{A}'$). Assume there exist invariant functionals on A and B, say respectively φ_A, φ_B (left invariant) and ψ_A, ψ_B (right invariant). Furthermore, let A_0 and B_0 be non-trivial subspaces of A and B respectively. Now assume the following:

i. Under multiplication, A is an $\mathfrak{A}$-bimodule and B is a $\mathfrak{B}$-bimodule.

ii. A_0 is a sub-$\mathfrak{B}$-bimodule of A (under the actions induced by duality) as well as a sub-$\mathfrak{A}$-bimodule (under multiplication). Similarly for B_0.

iii. A_0 and B_0 are invariant under * and $^\circ$, or equivalently, under * and $S^{\pm 1}$.

iv. The invariant functionals are all hermitian, positive and faithful.

v. The invariant functionals also enjoy the following faithfulness properties: for any $a \in A_0$ and $b \in B_0$ we have

$$\begin{array}{lll} \varphi_A(\mathfrak{A}a) = \{0\} \quad \Rightarrow \quad a = 0 & \qquad & \varphi_B(\mathfrak{B}b) = \{0\} \quad \Rightarrow \quad b = 0 \\ \psi_A(\mathfrak{A}a) = \{0\} \quad \Rightarrow \quad a = 0 & \qquad & \psi_B(\mathfrak{B}b) = \{0\} \quad \Rightarrow \quad b = 0 \end{array}$$

vi. The invariant functionals are all weakly KMS. The corresponding KMS automorphisms shall be denoted by σ_{φ_A} etc.

vii. Moreover assume φ_A to be $\mathfrak{A}$-KMS on A_0 in the sense that

$$\sigma_{\varphi_A}(A_0) = A_0 \qquad \text{and} \qquad \varphi_A(a\alpha) = \varphi_A\big(\alpha\, \sigma_{\varphi_A}(a)\big)$$

for any $a \in A_0$ and $\alpha \in \mathfrak{A}$.
Similarly ψ_A is $\mathfrak{A}$-KMS on A_0, whereas φ_B and ψ_B are $\mathfrak{B}$-KMS on B_0.

viii. There exist complex numbers ζ_A and ζ_B of modulus 1 such that

$$\varphi_A S = \zeta_A \psi_A \qquad \psi_A S = \zeta_A \varphi_A \qquad \varphi_B S = \zeta_B \psi_B \qquad \psi_B S = \zeta_B \varphi_B .$$

ix. There exists a $\delta_A \in \mathfrak{A}$ such that $\varphi_A S = \varphi_A(\,\cdot\,\delta_A)$. Similarly there exists a $\delta_B \in \mathfrak{B}$ such that $\varphi_B S = \varphi_B(\,\cdot\,\delta_B)$. δ_A and δ_B are called *modular* elements.

Under these circumstances, $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ is said to be a *Fourier context*. •

Let's agree to keep the notation as short as possible, e.g. if $A = A_0$, $B = B_0$, $\varphi_A = \psi_A$ and $\varphi_B = \psi_B$, then we abbreviate the above by $\big(A, \varphi_A;\ \mathfrak{A}, \mathfrak{B};\ B, \varphi_B\big)$.

It is also worth noticing that the above definition is *self-dual*, in the sense that it is completely symmetric between $\mathfrak{A}$ and $\mathfrak{B}$, between A and B, etc.

Definition 4.1.2 Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be any Fourier context. A linear map $F_{RL} : A_0 \to B_0$ is said to be an *RL Fourier transform* if

i. F_{RL} is a left $\mathfrak{B}$-module morphism; explicitly, for any $\beta \in \mathfrak{B}$ and $a \in A_0$ we have $F_{RL}(\beta \triangleright a) = \beta\, F_{RL}(a)$.

ii. $\varphi_B\big(F_{RL}(a)\big) = \langle a, 1_{\mathfrak{B}} \rangle$ for any $a \in A_0$.

Similarly we can define LR, LL and RR[1] Fourier transforms from A_0 to B_0 and vice versa, i.e. from B_0 to A_0. •

Remarks 4.1.3 i. Requiring F_{RL} to be a $\mathfrak{B}$-module morphism expresses a well-known fact from classical Fourier analysis: Fourier transformation interchanges differentiation and multiplication.

ii. The main reason to allow these Fourier transforms to be only defined on subspaces A_0 or B_0 rather than all of A or B, is that in general A_0 and B_0 need not to be *algebras*, e.g. in the case of the quantum $E(2)$ group it certainly won't be easy to find candidates for A_0 and B_0 which are moreover algebras (cf. chapter 6).

[1] The first L/R subscript *anti*-corresponds to the fact that the Fourier transform under consideration is a left/right $\mathfrak{B}$-module morphism, the second L/R corresponds to the Left/Right invariant functional we are dealing with in (ii).

iii. Observe that the conditions (v) in definition 4.1.1 are only 'one-sided'. However, we still have the *-operations[2] to change sides. These conditions will ensure that, given a particular Fourier context, its Fourier transforms are *unique* (cf. lemma 4.1.5 below).

iv. Obviously the above set of axioms for a Fourier context is far from minimal. For instance condition (viii) should normally be a proposition rather than an assumption; also the different types of KMS conditions are likely to be related, etc. However, our main interest here is just to provide all the ingredients we need for doing Fourier analysis, without getting involved in difficult (though important) matters like uniqueness and faithfulness of Haar functionals. Anyway, in our quantum $E(2)$ example all the features required by definition 4.1.1 shall be available.

v. Observe that requiring ζ_A and ζ_B to have modulus 1 does not really harm generality, because of corollary 3.6.8.

vi. Recall that A_0 and B_0 were merely assumed to be non-trivial, whereas maybe it would have been more natural to impose some *density* requirement on them. Certainly the latter would not harm, but it turns out that A_0 and B_0 *automatically* contain sufficiently many elements—check out lemma 4.1.4.ii for details.

vii. Fourier transforms also emerged in the duality theory for multiplier Hopf algebras with an integral [41, 43] and multiplier Hopf systems (definition 3.8.11). In this respect we mention that e.g. the RL Fourier transform in definition 4.1.2 is comparable to the *inverse* of $a \mapsto \hat{a} = \varphi(\,\cdot\, a)$. $\star$

Let's collect some properties of Fourier contexts in the following

Lemma 4.1.4 *Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be a Fourier context.*

i. The Haar functionals enjoy the following notions of strong left invariance: (e.g.) for any $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$ and $b, d \in B$ we have

$$\begin{aligned} \varphi_B\big(b\,(d \triangleleft \alpha)\big) &= \varphi_B\big((b \triangleleft S(\alpha))\,d\big) \\ \varphi_B\big(\beta\,(d \triangleleft \alpha)\big) &= \varphi_B\big((\beta \triangleleft S(\alpha))\,d\big) \\ \varphi_B\big(b\,(\beta \triangleleft \alpha)\big) &= \varphi_B\big((b \triangleleft S(\alpha))\,\beta\big) \end{aligned}$$

ii. Besides (iv) and (v) in definition 4.1.1, the invariant functionals also enjoy the following faithfulness property: (e.g.)

$$\beta \in \mathfrak{B} \quad \text{and} \quad \varphi_B(B_0\beta) = \{0\} \qquad \Longrightarrow \qquad \beta = 0.$$

iii. The modular elements δ_A and δ_B in definition 4.1.1.ix are unique.

iv. Also ψ_A and ψ_B have a modular property: (e.g.) $\psi_A S = \psi_A(S(\delta_A)\,\cdot\,)$

[2] or the antipodes, cf. condition (viii) of definition 4.1.1.

Proof. Let $\mathcal{B}$ be the *-subalgebra of $\mathfrak{A}'$ generated by B and $\mathfrak{B}$. In fact $\mathcal{B}$ is then also the linear span of $B \cup \mathfrak{B}$ within $\mathfrak{A}'$, and $\langle \underline{\mathfrak{A}}, \mathcal{B} \rangle$ is obviously still an algebraic Hopf *-system. Now proposition 3.6.2 yields (i).
To show (ii), take any $\beta \in \mathfrak{B}$ and assume that $\varphi_B(B_0\beta) = \{0\}$. Using strong left invariance we observe that, for any $\alpha \in \mathfrak{A}$, $\eta \in \mathfrak{B}$ and $b_0 \in B_0$,

$$\varphi_B\big(\eta b_0\,(\beta \triangleleft \alpha)\big) \;=\; \varphi_B\big((\eta b_0 \triangleleft S(\alpha))\,\beta\big) = 0.$$

Here we used that $\eta b_0 \triangleleft S(\alpha)$ still belongs to B_0. So $\varphi_B\big(\mathfrak{B}\, b_0(\beta \triangleleft \alpha)\big) = \{0\}$ and hence $b_0(\beta \triangleleft \alpha) = 0$. It follows that $\langle 1_{\mathfrak{A}}, b_0\rangle\langle \alpha, \beta\rangle = \langle 1_{\mathfrak{A}}, b_0(\beta \triangleleft \alpha)\rangle = 0$ for all $\alpha \in \mathfrak{A}$ and $b_0 \in B_0$. Now the non-degeneracy of the pairings $\langle \mathfrak{A}, \mathfrak{B}\rangle$ and $\langle \mathfrak{A}, B\rangle$ yields $\beta = 0$. Indeed, if β were non-zero, then we would have $\langle 1_{\mathfrak{A}}, B_0\rangle = \{0\}$ and hence $\langle \alpha, b_0\rangle = \langle 1_{\mathfrak{A}}, \alpha \triangleright b_0\rangle = 0$ for all $\alpha \in \mathfrak{A}$ and $b_0 \in B_0$, which would mean that B_0 is trivial. This completes the proof of (ii). Now (iii) follows easily from (ii). Eventually, using 4.1.1.vi-vii-viii-ix, we get

$$(\psi_A S)(a) \;=\; \zeta_A\, \varphi_A(a) \;=\; \zeta_A\, \varphi_A(S^{-1}(a)\,\delta_A) \;=\; (\psi_A S)(S^{-1}(a)\,\delta_A) \;=\; \psi_A(S(\delta_A)a)$$

for all $a \in A$, which proves (iv). ∎

Lemma 4.1.5 *Let* $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ *be a Fourier context. For any linear mapping* $F_{RL} : A_0 \to B_0$ *the following are equivalent:*

i. F_{RL} *is an* RL *Fourier transform,*

ii. $\varphi_B\big(\beta F_{RL}(a)\big) = \langle a, \beta\rangle$ *for all* $\beta \in \mathfrak{B}$ *and* $a \in A_0$.

Similar properties hold for LR, LL and RR Fourier transforms. As a corollary we observe that *Fourier transforms are unique.*

Proof. (i $\Rightarrow$ ii). For all $\beta \in \mathfrak{B}$ and $a \in A_0$, we have

$$\varphi_B\big(\beta F_{RL}(a)\big) \;=\; \varphi_B\big(F_{RL}(\beta \triangleright a)\big) \;=\; \langle \beta \triangleright a, 1_{\mathfrak{B}}\rangle \;=\; \langle a, \beta\rangle.$$

(ii $\Rightarrow$ i). For all $\beta, \eta \in \mathfrak{B}$ and $a \in A_0$, we have

$$\varphi_B\big(\eta\beta F_{RL}(a)\big) \;=\; \langle a, \eta\beta\rangle \;=\; \langle \beta \triangleright a, \eta\rangle \;=\; \varphi_B\big(\eta F_{RL}(\beta \triangleright a)\big)\,.$$

Observe that $\beta F_{RL}(a)$ and $F_{RL}(\beta \triangleright a)$ belong to B_0, hence from definition 4.1.1.v it follows that $F_{RL}(\beta \triangleright a) = \beta F_{RL}(a)$. Furthermore, $\varphi_B(F_{RL}(a)) = \langle a, 1_{\mathfrak{B}}\rangle$ is just the $\beta = 1_{\mathfrak{B}}$ case of (ii). Uniqueness follows from (ii) and definition 4.1.1.v. ∎

Recall that Fourier transforms from A_0 to B_0 are, by definition, $\mathfrak{B}$-module morphisms. Now *strong* invariance (cf. lemma 4.1.4.i) of the Haar functionals implies these Fourier transforms to have $\mathfrak{A}$-module properties as well:

Lemma 4.1.6 *Take a Fourier context as above and let* $F_{LL} : A_0 \to B_0$ *be an* LL *Fourier transform. Then* $F_{LL}\, S^{-1} : A_0 \to B_0$ *is a right* $\mathfrak{A}$*-module morphism, i.e. for all* $a \in A_0$ *and* $\alpha \in \mathfrak{A}$ *we have:*

$$(F_{LL}\, S^{-1})(a\alpha) \;=\; (F_{LL}\, S^{-1})(a) \triangleleft \alpha.$$

Proof. Using an LL-analogue of lemma 4.1.5, we obtain for any $a \in A_0$, $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$ that

$$\begin{array}{rcl} \varphi_B\big((F_{LL}S^{-1})(a\alpha)\,\beta\big) & = & \big\langle S^{-1}(\alpha)\,S^{-1}(a),\,\beta\big\rangle \\ & = & \big\langle S^{-1}(a),\,\beta \triangleleft S^{-1}(\alpha)\big\rangle \\ & = & \varphi_B\Big((F_{LL}S^{-1})(a)\;\big(\beta \triangleleft S^{-1}(\alpha)\big)\Big) \\ & \overset{(*)}{=} & \varphi_B\Big(\big((F_{LL}S^{-1})(a) \triangleleft \alpha\big)\,\beta\Big)\,. \end{array}$$

Here $(*)$ relies on *strong left invariance* of φ_B (cf. lemma 4.1.4.i). The result follows from the faithfulness assumptions in definition 4.1.1.v. ■

Example 4.1.7 Recall examples 3.7.3.1 and 3.7.3.2. One may ask whether

$$\Big(\mathcal{S}_H,\,\tfrac{1}{\sqrt{2\pi}}\textstyle\int_{\mathbb{R}};\,R,R;\,\mathcal{S}_H,\,\tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big)$$

is a Fourier context; the answer is no, for the following reason: condition (v) in definition 4.1.1 specializes to the *moment problem*

$$f \in \mathcal{S}_H \;\text{ and} \int_{\mathbb{R}} t^m f(t)\,dt = 0 \;\text{ for all }\; m \in \mathbb{N} \qquad \overset{?}{\Longrightarrow} \qquad f = 0 \tag{4.1}$$

which has *negative* answer for our Schwartz-like space $\mathcal{S}_H$. Indeed, it is easy to construct a non-zero function $g \in C^\infty(\mathbb{R})$ with compact support and such that g and all its derivatives vanish at the origin. Now if $f = \hat{g}$ denotes the (usual) Fourier transform of g, then $f \in S(\mathbb{R})$ because $g \in C_c^\infty(\mathbb{R}) \subseteq S(\mathbb{R})$. Furthermore f is entire, for g has compact support. It follows that f is a non-zero function in $\mathcal{S}_H$ with vanishing moments. This argument immediately reveals a second reason to reject $\mathcal{S}_H$ as the domain for our Fourier transforms: indeed Fourier transformation simply doesn't map $\mathcal{S}_H$ into itself. ⋆

Example 4.1.8 Let $\mathcal{E}$ be the subspace of $H(\mathbb{C})$ spanned by the functions

$$\mathbb{C} \to \mathbb{C} \,:\, x \mapsto x^m \exp(-\lambda x^2 + \zeta x) \qquad (m \in \mathbb{N},\, \lambda \in \mathbb{R}_0^+,\, \zeta \in \mathbb{C}).$$

Notice that $\mathcal{E}$ is actually a *-subalgebra of $\mathcal{S}_H$. We claim that

$$\Big(\mathcal{E} \subseteq \mathcal{S}_H,\,\tfrac{1}{\sqrt{2\pi}}\textstyle\int_{\mathbb{R}};R,\,R;\,\mathcal{E} \subseteq \mathcal{S}_H,\,\tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big) \tag{4.2}$$

is a Fourier context.

Proof. The observation that $\mathcal{S}_H$ is invariant under multiplication with polynomials yields (i) of definition 4.1.1. Furthermore $\mathcal{E}$ is invariant under differentiation and therefore under the actions (3.28), which means that $\mathcal{E}$ is a R-bimodule w.r.t. these actions. Moreover $\mathcal{E}$ is obviously invariant under multiplication with polynomials, which proves (ii). To prove (v) we need positive answer to the moment problem for $\mathcal{E}$, analogous to (4.1) in the previous example, which is easily established using the fact that Hermite functions form an ONB for the Hilbert space $L^2(\mathbb{R})$. The remaining items are obvious. ■

Proposition 4.1.9 *Let* $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ *be any Fourier context and let there exist an* RL *Fourier transform* $F_{RL} : A_0 \to B_0$*. Then we automatically have the other three Fourier transforms from* A_0 *to* B_0*, e.g.*[3]

$$F_{LR} \;=\; \zeta_B\, S^{-1} F_{RL} S^{-1} \qquad\qquad F_{LL} \;=\; \sigma_{\varphi_B}^{-1} F_{RL} \qquad\qquad F_{RR} \;=\; \sigma_{\psi_B} F_{LR}.$$

Another way to construct e.g. F_{RR} *from* F_{RL} *is to exploit the* **-structure:*

$$F_{RR} \;=\; \zeta_B * S F_{RL} * \tag{4.3}$$

Eventually the presence of modular elements yields formulas like

$$\zeta_B\, F_{RL}(a) \;=\; \delta_B\, \sigma_{\varphi_B}\big(F_{LR}(a)\big) \qquad\qquad \textit{for any } a \in A_0. \tag{4.4}$$

Proof. According to lemma 4.1.5, we have for any $\beta \in \mathfrak{B}$ and $a \in A_0$ that

$$\varphi_B\big(S(\beta)\, F_{RL}(S^{-1}(a))\big) \;=\; \big\langle S^{-1}(a),\, S(\beta)\big\rangle \;=\; \langle a, \beta\rangle.$$

With $\varphi_B S = \zeta_B \psi_B$ and the fact that the antipode (on $\mathfrak{A}' \supseteq \mathfrak{B}, B_0$) is an anti-homomorphism, we obtain

$$\zeta_B\, \psi_B\big((S^{-1} F_{RL} S^{-1})(a)\, \beta\big) \;=\; \langle a, \beta\rangle$$

which (by an analogue of lemma 4.1.5) means precisely that $\zeta_B\, S^{-1} F_{RL} S^{-1}$ is an LR Fourier transform from A_0 to B_0. This shows the first formula. The second one is an immediate consequence of the $\mathfrak{B}$-KMS property of φ_B on B_0 (cf. definition 4.1.1.vii). Indeed for any $\beta \in \mathfrak{B}$ and $a \in A_0$ we have

$$\varphi_B\big(\sigma_{\varphi_B}^{-1}(F_{RL}(a))\, \beta\big) \;=\; \varphi_B\big(\beta\, F_{RL}(a)\big) \;=\; \langle a, \beta\rangle.$$

The third formula is analogous to the second one. To prove (4.3) we start again from lemma 4.1.5. Using the fact that φ_B is hermitian, we obtain

$$\varphi_B\big(F_{RL}(a^*)^*\, S(\beta)\big) \;=\; \overline{\varphi_B\big(S(\beta)^*\, F_{RL}(a^*)\big)} \;=\; \overline{\big\langle a^*,\, S(\beta)^*\big\rangle} \;=\; \langle a, \beta\rangle$$

for all $\beta \in \mathfrak{B}$ and $a \in A_0$. Using $\varphi_B S = \zeta_B \psi_B$ we obtain

$$\zeta_B \psi_B\big(\beta\, S^{-1}(F_{RL}(a^*)^*)\big) \;=\; \langle a, \beta\rangle$$

and (4.3) follows. Once again using the $\mathfrak{B}$-KMS property of φ_B on B_0, we get

$$\varphi_B\big(\beta\, \delta_B\, \sigma_{\varphi_B}(F_{LR}(a))\big) \;=\; \varphi_B\big(F_{LR}(a)\, \beta \delta_B\big) \;=\; \zeta_B\, \psi_B\big(F_{LR}(a)\, \beta\big) \;=\; \zeta_B \langle a, \beta\rangle$$

for any $\beta \in \mathfrak{B}$ and $a \in A_0$. This proves (4.4). ∎

Remark 4.1.10 More generally we have $S F_{PQ} S = \zeta_B F_{\bar{P}\bar{Q}}$ for any pair of labels P, Q in the label set $\{L, R\}$. Here $^-$ is defined by $\bar{L} = R$ and $\bar{R} = L$. ⋆

[3] recall that antipodes and KMS automorphisms are bijective on A_0 and B_0 because of (iii) and (vii) in definition 4.1.1.

4.2 Duality

Abstract 4.2 Whenever $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ is a Fourier context, we have several 'Hopf type' dualities, as represented in the diagram below:

$$\begin{array}{ccccc} A_0 & \subseteq & A & \overset{\text{dual}}{\longleftrightarrow} & \mathfrak{B} \\ ?\updownarrow & & ?\updownarrow & & \updownarrow \text{dual} \\ B_0 & \subseteq & B & \overset{\text{dual}}{\longleftrightarrow} & \mathfrak{A} \end{array}$$

Thus the question arises whether this situation induces a natural duality between A and B, or more likely, between A_0 and B_0. This will be investigated below. ⋆

Lemma 4.2.1 *Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be a Fourier context and assume there exist Fourier transforms F_{LL}, F_{RL} ... from A_0 to B_0. Then*

$$\varphi_B\big(F_{LL}(a)\,b\big) \overset{(i)}{=} \varphi_B\big(b\,F_{RL}(a)\big) \overset{(ii)}{=} \psi_B\big(F_{LR}(a)\,b\big) \overset{(iii)}{=} \psi_B\big(b\,F_{RR}(a)\big) \overset{(\mathrm{def})}{=} \langle a, b\rangle$$

for all $a \in A_0$ and $b \in B_0$. This defines a vector space duality between A_0 and B_0, which we shall once again denote by $\langle \cdot, \cdot \rangle$. If the above Fourier transforms are bijections, then this pairing $\langle A_0, B_0 \rangle$ is non-degenerate.

Proof. (i) follows immediately from the KMS property of φ_B and the formula $\sigma_{\varphi_B} F_{LL} = F_{RL}$ shown in proposition 4.1.9. (iii) is analogous, whereas (ii) relies on the modular elements: indeed, using the KMS property of φ_B we obtain

$$\zeta_B\,\varphi_B\big(b\,F_{RL}(a)\big) \overset{(4.4)}{=} \varphi_B\big(b\,\delta_B\,\sigma_{\varphi_B}(F_{LR}(a))\big) = \varphi_B\big(F_{LR}(a)\,b\,\delta_B\big) = \zeta_B\,\psi_B\big(F_{LR}(a)\,b\big)$$

for all $a \in A_0$ and $b \in B_0$. Now let's assume F_{RL} to be bijective and prove the non-degeneracy statement: first, let $a \in A_0$ such that $\langle a, b\rangle = 0$ for all $b \in B_0$. Then in particular $\langle a, F_{RL}(a)^* \rangle = 0$ and hence $\varphi_B\big(F_{RL}(a)^*\,F_{RL}(a)\big) = 0$. Since φ_B is faithful, it follows that $F_{RL}(a) = 0$ and hence $a = 0$. Next, let $b \in B_0$ such that $\langle a, b\rangle = 0$ for all $a \in A_0$. This means that $\varphi_B(bB_0) = \{0\}$ and in particular $\varphi_B(bb^*) = 0$, hence $b = 0$. ■

Recall that A_0 and B_0 are bimodules over both the algebras $\mathfrak{A}$ and $\mathfrak{B}$, either under multiplication or under the actions induced by the dualities $\langle \mathfrak{A}, B\rangle$ and $\langle A, \mathfrak{B}\rangle$. The following lemma tells us that the 'induced' pairing $\langle A_0, B_0\rangle$ is compatible with these bimodule structures. Also antipodes and *-structures satisfy the usual relations:

Lemma 4.2.2 *The pairing $\langle A_0, B_0\rangle$ defined in lemma 4.2.1 enjoys*

$$\langle \alpha a, b\rangle = \langle a, b \triangleleft \alpha\rangle \qquad \langle a\alpha, b\rangle = \langle a, \alpha \triangleright b\rangle \tag{4.5}$$

$$\langle a, \beta b\rangle = \langle a \triangleleft \beta, b\rangle \qquad \langle a, b\beta\rangle = \langle \beta \triangleright a, b\rangle \tag{4.6}$$

for any $a \in A_0$, $b \in B_0$, $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$. Furthermore we have

$$\langle S(a), b\rangle = \langle a, S(b)\rangle \qquad \textit{and} \qquad \langle a, b^*\rangle = \overline{\langle S(a)^*, b\rangle}.$$

Proof. Combining lemma 4.1.6 with lemma 4.1.4.i we obtain

$$\begin{array}{rcl}\langle \alpha a, b\rangle &=& \varphi_B\big(F_{LL}(\alpha a)\, b\big)\\ &=& \varphi_B\big((F_{LL}S^{-1})\,\big(S(a)S(\alpha)\big)\, b\big)\\ &=& \varphi_B\big(\big[(F_{LL}S^{-1})(S(a)) \triangleleft S(\alpha)\big]\, b\big)\\ &=& \varphi_B\big(\big[F_{LL}(a) \triangleleft S(\alpha)\big]\, b\big)\\ &=& \varphi_B\big(F_{LL}(a)\,(b \triangleleft \alpha)\big)\\ &=& \langle a, b \triangleleft \alpha\rangle\end{array}$$

The second formula in (4.5) is similar, whereas the ones in (4.6) are trivial, e.g.

$$\langle a, \beta b\rangle \;=\; \varphi_B\big(F_{LL}(a)\,\beta b\big) \;=\; \varphi_B\big(F_{LL}(a \triangleleft \beta)\, b\big) \;=\; \langle a \triangleleft \beta, b\rangle.$$

Using $SF_{RL}S = \zeta_B F_{LR}$ we obtain

$$\langle S(a), b\rangle = \varphi_B\big(b\, F_{RL}S(a)\big) = \zeta_B\, \varphi_B\big(b\, S^{-1}F_{LR}(a)\big) = \psi_B\big(F_{LR}(a)\, S(b)\big) = \langle a, S(b)\rangle$$

The last formula follows easily from (4.3). ∎

4.3 Inversion formulas

Until now we only considered Fourier transforms from A_0 to B_0. However our theory is clearly fully symmetric, hence one could as well consider Fourier transformation from B_0 to A_0. To avoid any confusion about the left-right nomenclature, we give an example below, although it is completely analogous to definition 4.1.2.

Definition 4.3.1 Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be any Fourier context. A linear map $G_{LR} : B_0 \to A_0$ is called an LR *Fourier transform* if

i. G_{LR} is a *right* $\mathfrak{A}$-module morphism; explicitly, for any $\alpha \in \mathfrak{A}$ and $b \in B_0$ we have $G_{LR}(b \triangleleft \alpha) \;=\; G_{LR}(b)\, \alpha$

ii. $\psi_A\big(G_{LR}(b)\big) \;=\; \langle 1_{\mathfrak{A}}, b\rangle$ for any $b \in B_0$.

So the subscript L in G_{LR} *anti*-corresponds to the fact we are dealing with a *right* module morphism in (i), whereas the R in G_{LR} refers to the *right* invariant functional ψ_A in (ii). •

Proposition 4.3.2 *Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be any Fourier context and let $F_{LL} : A_0 \to B_0$ be a bijective* LL *Fourier transform. Assume there exists a scalar $\lambda \in \mathbb{C}$ such that $\langle 1_{\mathfrak{A}}, F_{LL}(a)\rangle = \lambda\, \varphi_A(a)$ for any $a \in A_0$. Then*

$$G_{LR} \;=\; \lambda \zeta_A^{-1}\, S F_{LL}^{-1}$$

is an LR *Fourier transform from B_0 to A_0.*

Proof. Condition (i) of definition 4.3.1 is fulfilled because of lemma 4.1.6, whereas (ii) follows from our assumption. ∎

Remark 4.3.3 The assumption that $\langle 1_{\mathfrak{A}}, F_{LL}(\,\cdot\,)\rangle$ is a scalar multiple of φ_A (restricted to A_0) is a very natural one—in fact it would hold automatically if we would have shown an appropriate result on uniqueness of Haar functionals (an important though difficult matter we didn't elaborate on). Indeed the functional $\langle 1_{\mathfrak{A}}, F_{LL}(\,\cdot\,)\rangle$ on A_0 behaves just like a left invariant functional:

$$\big\langle 1_{\mathfrak{A}},\, F_{LL}(a \triangleleft \beta)\big\rangle \;=\; \big\langle 1_{\mathfrak{A}},\, F_{LL}(a)\,\beta\big\rangle \;=\; \varepsilon(\beta)\,\big\langle 1_{\mathfrak{A}},\, F_{LL}(a)\big\rangle$$

for all $a \in A_0$ and $\beta \in \mathfrak{B}$. Here we used that F_{LL} is a right $\mathfrak{B}$-module morphism.

4.4 Plancherel contexts

Abstract 4.3 We shall make no attempt to prove Plancherel formulas in general; in the quantum $E(2)$ example, Plancherel formulas will be obtained directly from the orthogonality relations for Hahn-Exton q-Bessel functions (cf. chapter 6). Nevertheless it is worth the effort to make some general observations. ⋆

Lemma 4.4.1 *Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be a Fourier context. Assume we have* LL, LR, RL *and* RR *Fourier transforms from A_0 to B_0, say F_{LL}, F_{LR}, etc. Consider the following Plancherel identities:*

LL. $\varphi_A(xy^*) \;=\; \varphi_B\big(F_{LL}(x)\,F_{LL}(y)^*\big)$ *for all* $x, y \in A_0$

LR. $\varphi_A(xy^*) \;=\; \psi_B\big(F_{LR}(x)\,F_{LR}(y)^*\big)$ *for all* $x, y \in A_0$

RL. $\psi_A(y^*x) \;=\; \varphi_B\big(F_{RL}(y)^*\,F_{RL}(x)\big)$ *for all* $x, y \in A_0$

RR. $\psi_A(y^*x) \;=\; \psi_B\big(F_{RR}(y)^*\,F_{RR}(x)\big)$ *for all* $x, y \in A_0$

To avoid confusion: we do not *assume* these identities to hold—we merely intend to study their *mutual* relationship:

If $\zeta_A = \zeta_B$, then (LR) is equivalent with (RL). Conversely, if both (LR) and (RL) are satisfied simultaneously, then $\zeta_A = \zeta_B$. A similar statement holds for the pair (LL-RR).

If both (RR) and (RL) are satisfied simultaneously, then we are dealing with the rather special situation where $\zeta_B = 1$ and $\sigma_{\psi_A}(x) = S^{-2}(x)$ for all $x \in A_0$.

Proof. Take any $x, y \in A_0$ and observe the following 'circle' of identities:

$$\begin{aligned}
\psi_A(y^*x) \quad &= \quad \zeta_A\,\varphi_A\big(S^{-1}(x)\,S(y)^*\big) \\
&\overset{\text{(LR)}}{=} \quad \zeta_A\,\psi_B\big(F_{LR}(S^{-1}(x))\,F_{LR}(S(y))^*\big) \\
&= \quad \zeta_A\zeta_B\,\varphi_B\big((\underbrace{SF_{LR}S}_{\zeta_B\,F_{RL}})(y)^*\,(\underbrace{S^{-1}F_{LR}S^{-1}}_{\zeta_B^{-1}F_{RL}})(x)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \zeta_A\zeta_B\,\overline{\zeta_B}\,\zeta_B^{-1}\,\varphi_B\big(F_{RL}(y)^*\,F_{RL}(x)\big)\\
&\overset{(\mathrm{RL})}{=} \zeta_A\zeta_B^{-1}\,\psi_A(y^*x)\\
&\overset{(\sharp)}{=} \psi_A(y^*x)
\end{aligned}
$$

here $(\sharp)$ stands for $\zeta_A = \zeta_B$. To show the last statement, consider

$$
\begin{aligned}
\psi_A(y^*x) &\overset{(\mathrm{RR})}{=} \psi_B\big(F_{RR}(y)^*\,F_{RR}(x)\big)\\
&\overset{(4.3)}{=} \psi_B\big((\zeta_B * SF_{RL}*)(y)^*\,(\zeta_B * SF_{RL}*)(x)\big)\\
&= \psi_B\big(\overline{\zeta_B}\,(SF_{RL})(y^*)\,\zeta_B(SF_{RL})(x^*)^*\big)\\
&= \psi_B S\left(S^{-1}((SF_{RL})(x^*)^*)\,F_{RL}(y^*)\right)\\
&= \zeta_B\,\varphi_B\big((S^2F_{RL})(x^*)^*\,F_{RL}(y^*)\big)\\
&= \zeta_B\,\varphi_B\big((\zeta_B^2\,F_{RL}S^{-2})(x^*)^*\,F_{RL}(y^*)\big)\\
&= \zeta_B\,\overline{\zeta_B^2}\,\varphi_B\big(F_{RL}(S^{-2}(x^*))^*\,F_{RL}(y^*)\big)\\
&\overset{(\mathrm{RL})}{=} \zeta_B^{-1}\,\psi_A\big(S^{-2}(x^*)^*\,y^*\big)\\
&= \zeta_B^{-1}\,\psi_A\big(S^2(x)\,y^*\big)\\
&= \zeta_B^{-1}\,\psi_A\big(y^*\,\sigma_{\psi_A}(S^2(x))\big)\,.
\end{aligned}
$$

So if both (RR) and (RL) hold simultaneously, then $\sigma_{\psi_A}(S^2(x)) = \zeta_B\,x$ for any $x \in A_0$. Since σ_{ψ_A} and S^2 are algebra automorphisms of A, we may expect that $\zeta_B = 1$. However at the moment we do not know whether $\sigma_{\psi_A}S^2 = \zeta_B$ holds on *all* of A. Therefore, we need a little trick: using proposition 3.6.6 (or invoking KMS twice) we obtain for any $x, y \in A_0$ that

$$\psi_A\big(\sigma_{\psi_A}(S^2(x))\,\sigma_{\psi_A}(S^2(y))\big) \;=\; \psi_A\big(S^2(x)\,S^2(y)\big) \;=\; \psi_A\big(S^2(xy)\big) \;=\; \zeta_A^2\,\psi_A(xy).$$

On the other hand we have

$$\psi_A\big(\sigma_{\psi_A}(S^2(x))\,\sigma_{\psi_A}(S^2(y))\big) \;=\; \psi_A\left((\zeta_B\,x)\,(\zeta_B\,y)\right) \;=\; \zeta_B^2\,\psi_A(xy)$$

and since ψ_A is faithful and A_0 is non-trivial, it follows that $\zeta_A^2 = \zeta_B^2$. Similarly we can show that $\zeta_A^2\,\psi_A(xyz) = \zeta_B^3\,\psi_A(xyz)$ for any triple $x, y, z \in A_0$, hence $\zeta_A^2 = \zeta_B^3$, and eventually $\zeta_B = 1$. ∎

Remark 4.4.2 The above lemma shows the various Plancherel formulas are not independent, which isn't very surprising. The lemma however also indicates that in general we may not expect *all* Fourier transforms to obey a Plancherel identity. It turns out[4] that we should focus mainly on the (LR-RL) pair. ⋆

Lemma 4.4.3 *Adopt the setting of proposition 4.3.2. Then proposition 4.1.9 (and analogue for inverse transforms) will provide all the Fourier transforms from A_0 to B_0 and vice versa. Assume that $|\lambda| = 1$. Now if F_{RL} satisfies its Plancherel formula, then so does G_{LR}.*

[4] e.g. in our quantum $E(2)$ example, or in the theory of algebraic quantum groups [41].

Proof. Using $SF_{RR}S = \zeta_B F_{LL}$ and (4.3) we get

$$SF_{LL}^{-1} \;=\; S\,\zeta_B\,S^{-1}F_{RR}^{-1}S^{-1} \;=\; \zeta_B\,F_{RR}^{-1}S^{-1} \;=\; *\,F_{RL}^{-1}S^{-1}*S^{-1} \;=\; *\,F_{RL}^{-1}*$$

and hence, from proposition 4.3.2,

$$G_{LR} \;=\; \lambda\zeta_A^{-1}\,SF_{LL}^{-1} \;=\; \lambda\zeta_A^{-1}\,*\,F_{RL}^{-1}* \tag{4.7}$$

Using the Plancherel formula for F_{RL} (cf. lemma 4.4.1) we get for all $x, y \in B_0$

$$\psi_A\big(G_{LR}(x)\,G_{LR}(y)^*\big) \;=\; \big|\lambda\zeta_A^{-1}\big|^2\;\psi_A\left(F_{RL}^{-1}(x^*)^*\,F_{RL}^{-1}(y^*)\right) \;=\; \varphi_B(xy^*)$$

which is exactly the Plancherel formula for G_{LR}. ∎

Definition 4.4.4 Let ω be a positive linear functional on a *-algebra A. Then a GNS pair $(\mathcal{H}, \Lambda)$ for (A, ω) is a Hilbert space $\mathcal{H}$ together with a linear map $\Lambda : A \to \mathcal{H}$ such that $\Lambda(A)$ is dense in $\mathcal{H}$ and $\omega(y^*x) = \langle \Lambda(x) \,|\, \Lambda(y)\rangle$ for all $x, y \in A$. Here $\langle\cdot\,|\,\cdot\rangle$ denotes the scalar product on $\mathcal{H}$. It is clear that such a GNS pair for (A, ω) exists and that it is unique up to unitary equivalence. Also observe that Λ is injective if and only if ω is faithful. •

Definition 4.4.5 Let $\mathfrak{F} \equiv (A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be a Fourier context. Assume there *exist* LL, LR, RL and RR Fourier transforms from A_0 to B_0 and vice versa. Recall they are uniquely determined by $\mathfrak{F}$ (lemma 4.1.5). Assume moreover:

i. All these Fourier transforms are *bijections* between A_0 and B_0.

ii. The (LR) and (RL) Fourier transforms (from A_0 to B_0 *and* vice versa) satisfy their respective Plancherel formulas; see lemmas 4.4.1 and 4.4.3 for details. In particular the scaling constants ζ_A and ζ_B are equal. Therefore, let's abbreviate $\zeta_A = \zeta_B \equiv \zeta$.

iii. For any $a \in A_0$ the following (equivalent—see proposition 4.1.9) conditions hold:

$$\begin{array}{ll} \langle 1_{\mathfrak{A}}, F_{LR}(a)\rangle = \varphi_A(a) & \langle 1_{\mathfrak{A}}, F_{LL}(a)\rangle = \zeta\,\varphi_A(a) \\ \langle 1_{\mathfrak{A}}, F_{RL}(a)\rangle = \psi_A(a) & \langle 1_{\mathfrak{A}}, F_{RR}(a)\rangle = \zeta\,\psi_A(a) \end{array} \tag{4.8}$$

and similar formulas for the inverse transforms. This amounts to the condition in proposition 4.3.2 with $\lambda = \zeta$.

iv. If $(\mathcal{H}, \Lambda)$ is a GNS pair for (A, φ_A) or (A, ψ_A) then $\Lambda(A_0)$ is dense in $\mathcal{H}$. Similarly, if $(\mathcal{H}, \Lambda)$ is a GNS pair for (B, φ_B) or (B, ψ_B) then $\Lambda(B_0)$ is dense.

Under these circumstances, $\mathfrak{F}$ is said to be a *Plancherel context*. •

Of course the above set of axioms is again very far from minimal; however the dependencies between these assumptions should be clear from propositions 4.1.9 and 4.3.2 and lemmas 4.4.1 and 4.4.3. For instance the analogue of (4.8) for the inverse transforms (i.e. from B_0 to A_0) will hold automatically: from (4.8) and proposition 4.3.2 we get $G_{LR} = SF_{LL}^{-1}$ and together with $\varphi_B F_{LL} = \langle\,\cdot\,, 1_{\mathfrak{B}}\rangle$ we obtain that $\langle G_{LR}(b), 1_{\mathfrak{B}}\rangle = \varphi_B(b)$ for any $b \in B_0$. The latter formula is indeed the 'inverse transform' analogue of (4.8).

Remark 4.4.6 In an operator-theoretic approach to quantum groups [22], one usually prefers to work on a single Hilbert space; this would mean we need to identify the GNS objects for A and B with one another. Here Plancherel contexts could enter the picture, yielding a gateway from the purely algebraic level into the C^*-algebra world.
Another motivation to study this notion is the subject of harmonic analysis itself: it provides a framework for Fourier transformation, algebraic in nature, but nonetheless general enough to contain interesting examples like the quantum $E(2)$ group (cf. chapters 5 and 6).
Of course one should also consider the possibility of representing A and B as *-algebras of bounded operators on the respective GNS Hilbert spaces, which is our motivation to include axiom (iv) in the above definition; this is however beyond the subject of the present text. $\star$

Example 4.4.7 Recall example 4.1.8. We claim that (4.2) is a Plancherel context. The usual Fourier transform on $\mathbb{R}$, $f \mapsto \hat{f}$ with

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-ikx}\, dx, \tag{4.9}$$

maps $\mathcal{E}$ bijectively onto itself, and it is easy to prove that (4.9) indeed defines Fourier transforms in the sense of definition 4.1.2 (cf. remarks 4.1.3.i). Observe that $(L^2(\mathbb{R}), \mathrm{id})$ is a GNS pair for $\left(\mathcal{S}_H, \int_{\mathbb{R}}\right)$. Clearly $\mathcal{E}$ is dense in $L^2(\mathbb{R})$, which proves (iv) of definition 4.4.5. $\star$

4.5 Duality revisited

Let $(A_0 \subseteq A, \varphi_A, \psi_A; \mathfrak{A}, \mathfrak{B}; B_0 \subseteq B, \varphi_B, \psi_B)$ be a Plancherel context. Above we introduced a natural duality between A_0 and B_0, which was induced by the Fourier transforms from A_0 to B_0 (see lemma 4.2.1). But of course, because of the symmetry we could as well use Fourier transforms from B_0 to A_0 to obtain such a pairing. So the question arises: do we get the same pairing?

Lemma 4.5.1 *For any $a \in A_0$ and $b \in B_0$ we have*

$$\psi_A\big(G_{LR}(b)\, a\big) \;=\; \langle a, b\rangle \;=\; \varphi_B\big(b\, F_{RL}(a)\big)\,.$$

Proof. The conditions of proposition 4.3.2 are fulfilled (cf. definition 4.4.5.iii), hence (4.7) should hold with $\lambda = \zeta$, i.e. $G_{LR} = *\, F_{RL}^{-1} *$. It follows that

$$\psi_A\big(G_{LR}(b)\, a\big) \;=\; \psi_A\big(F_{RL}^{-1}(b^*)^*\, F_{RL}^{-1}(F_{RL}(a))\big) \;=\; \varphi_B\big(b\, F_{RL}(a)\big)$$

for any $a \in A_0$ and $b \in B_0$. We used the Plancherel formula for F_{RL}. ∎

Chapter 5

A Fourier context for quantum $E(2)$

Conventions Throughout this chapter and the next one, q denotes a fixed real number with $0 < q < 1$. Furthermore, let us agree that comultiplications, counits and antipodes will always be denoted by Δ, ε and S, respectively.

5.1 Introduction

Abstract 5.1 We recall the description of the quantum $E(2)$ group and its Pontryagin dual on the Hopf *-algebra level. In particular we introduce a dual pair $\langle \mathcal{U}_q, \mathcal{A}_q \rangle$ of Hopf *-algebras [12, 13, 46, 52]. Furthermore we introduce the operators $\Gamma, \Phi, \Omega, \Psi$ which will be used throughout chapters 5 and 6. ⋆

5.1.1 Introducing two Hopf *-algebras

Let $\mathcal{A}_q$ be the algebra with unit 1, generators $\alpha, \beta, \gamma, \delta$ and relations

$$\alpha\beta = q\,\beta\alpha \qquad \alpha\gamma = q\,\gamma\alpha \qquad \beta\delta = q\,\delta\beta \qquad \gamma\delta = q\,\delta\gamma \qquad \beta\gamma = \gamma\beta$$

$$\alpha\delta = 1 = \delta\alpha.$$

We make $\mathcal{A}_q$ into a *-algebra as follows:

$$\begin{array}{ll} \alpha^* = \delta & \beta^* = -q\,\gamma \\ \gamma^* = -q^{-1}\beta & \delta^* = \alpha \end{array}$$

In other words, α is unitary whereas β is normal. Eventually $\mathcal{A}_q$ is made into a Hopf *-algebra by defining Δ, ε and S on the generators:

$$\begin{array}{ll} \Delta(\alpha) = \alpha \otimes \alpha & \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta \\ \Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma & \Delta(\delta) = \delta \otimes \delta \end{array}$$

$$
\begin{aligned}
&\varepsilon(\alpha) = 1 \qquad \varepsilon(\beta) = 0 && S(\alpha) = \delta \qquad S(\beta) = -q^{-1}\beta \\
&\varepsilon(\gamma) = 0 \qquad \varepsilon(\delta) = 1 && S(\gamma) = -q\,\gamma \qquad S(\delta) = \alpha
\end{aligned}
$$

Next, let $\mathcal{U}_q$ be the algebra with unit 1, generators a, b, c, d and relations

$$ab = q\,ba \qquad ac = q^{-1}ca \qquad bc = cb \qquad ad = 1 = da \tag{5.1}$$

Notice bc is central in $\mathcal{U}_q$. The *-structure on $\mathcal{U}_q$ is given by

$$
\begin{aligned}
a^* &= a \qquad & b^* &= c \\
c^* &= b \qquad & d^* &= d
\end{aligned}
$$

Now also $\mathcal{U}_q$ becomes a Hopf *-algebra by defining

$$
\begin{aligned}
\Delta(a) &= a \otimes a \qquad & \Delta(b) &= a \otimes b + b \otimes d \\
\Delta(c) &= a \otimes c + c \otimes d \qquad & \Delta(d) &= d \otimes d
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(a) = 1 \qquad \varepsilon(b) = 0 && S(a) = d \qquad S(b) = -q^{-1}b \\
&\varepsilon(c) = 0 \qquad \varepsilon(d) = 1 && S(c) = -q\,c \qquad S(d) = a
\end{aligned}
$$

It is not difficult to show that $\mathcal{A}_q$ and $\mathcal{U}_q$ are indeed Hopf *-algebras. In [12, 13] $\mathcal{A}_q$ was denoted by $\mathcal{A}_q(\widetilde{M}(2))$ and $\mathcal{U}_q$ by $\mathcal{U}_q(\mathfrak{m}(2))$. Since we are dealing with these two Hopf *-algebras exclusively, we prefer the shorthand form.

5.1.2 The pairing

We want the pairing $\langle \mathcal{U}_q, \mathcal{A}_q \rangle$ to take the following values on the generators:

$$
\begin{aligned}
&\langle a, \alpha \rangle = q^{\frac{1}{2}} \qquad \langle a, \delta \rangle = q^{-\frac{1}{2}} \qquad \langle b, \beta \rangle = 1 \\
&\langle c, \gamma \rangle = 1 \qquad \langle d, \alpha \rangle = q^{-\frac{1}{2}} \qquad \langle d, \delta \rangle = q^{\frac{1}{2}} \\
&\langle x, \xi \rangle = 0 \quad \text{for other choices } x, \xi \text{ of generators.}
\end{aligned}
\tag{5.2}
$$

Observe that $\{\alpha^l \beta^m \gamma^n \mid l \in \mathbb{Z};\, m, n \in \mathbb{N}\}$ is a basis for the linear space $\mathcal{A}_q$ whereas $\{a^p b^r c^s \mid p \in \mathbb{Z};\, r, s \in \mathbb{N}\}$ is a basis for $\mathcal{U}_q$. Of course we assume $\alpha^{-n} = \delta^n$ and $a^{-n} = d^n$ for $n \in \mathbb{N}$. In [12, 13] H. T. Koelink calculated the full pairing on these basis elements; we turn his result into a definition:

Definition 5.1.2.1 Let $\langle \cdot, \cdot \rangle : \mathcal{U}_q \times \mathcal{A}_q \to \mathbb{C}$ be the bilinear form such that

$$\langle a^p b^r c^s, \alpha^l \beta^m \gamma^n \rangle \;=\; \delta_{r,m}\, \delta_{s,n}\, q^{\frac{1}{2}p(l+m-n)}\, q^{\frac{1}{2}l(m+n)}\, [m]_q!\, [n]_q! \tag{5.3}$$

for all $p, l \in \mathbb{Z}$ and $m, n, r, s \in \mathbb{N}$. For the q-factorials, see appendix C. •

Proposition 5.1.2.2 *The pairing defined in (5.3) is non-degenerate, it makes $\langle \mathcal{U}_q, \mathcal{A}_q \rangle$ into a duality of Hopf *-algebras and specializes to (5.2) on generators.*

Proof. See [12, 13, 38]. ■

5.1.3 Introducing shift & multiplication operators

Throughout the remainder of this text we set $\theta = -\frac{1}{2}\ln q$. Since $0 < q < 1$ we have $\theta > 0$, and $e^{-\theta} = q^{\frac{1}{2}}$. Let $F(\mathbb{Z}\theta)$ be the *-algebra of all complex functions on $\mathbb{Z}\theta = \{k\theta \mid k \in \mathbb{Z}\} \subseteq \mathbb{R}$, with pointwise operations.

Definition 5.1.3.1 We define two linear mappings $\Gamma, \Phi : F(\mathbb{Z}\theta) \to F(\mathbb{Z}\theta)$ by

$$(\Gamma f)(x) = f(x - \theta) \qquad \text{and} \qquad (\Phi f)(x) = e^x f(x)$$

for $f \in F(\mathbb{Z}\theta)$ and $x \in \mathbb{Z}\theta$. •

Obviously Γ and Φ are bijective, which will allow us occasionally to write powers Γ^k or Φ^k for any $k \in \mathbb{Z}$. Actually Γ is a *-isomorphism, and the commutation rule $\Gamma\Phi = q^{\frac{1}{2}}\,\Phi\Gamma$ is easily verified.

Now recall the *-algebra $H(\mathbb{C})$ of entire functions (§1.4). Again we introduce two linear operators, Ω and Ψ, which play a role similar to Γ and Φ above:

Definition 5.1.3.2 Let $\Omega, \Psi : H(\mathbb{C}) \to H(\mathbb{C})$ be defined by

$$(\Omega g)(z) = g(qz) \qquad \text{and} \qquad (\Psi g)(z) = z\,g(z)$$

For $g \in H(\mathbb{C})$ and $z \in \mathbb{C}$. •

Here Ω is a bijection, but Ψ is not, so one should be aware not to take *negative powers* of Ψ. Since q is real, Ω is a *-isomorphism. Notice that $\Omega\Psi = q\,\Psi\Omega$ involves a factor q, whereas the Γ, Φ commutation rule involved $q^{\frac{1}{2}}$.

5.2 The full extended algebra $\mathcal{U}_q^{\text{ext}}$

Abstract 5.2 In order to host 'functions' of the generators of $\mathcal{U}_q$, we shall replace $\mathcal{U}_q$ by a larger *-algebra $\mathcal{U}_q^{\text{ext}}$ and then accordingly extend the pairing with $\mathcal{A}_q$. On the other hand, we recall that $\mathcal{A}_q'$ is a *-algebra as well, being the algebraic dual of a Hopf *-algebra. The *-algebra $\mathcal{U}_q^{\text{ext}}$ then turns out to be isomorphic to $\mathcal{A}_q'$. We also compute the antipode and the canonical actions of $\mathcal{A}_q$ on $\mathcal{U}_q^{\text{ext}}$ induced by the duality.

5.2.1 Construction

Definition 5.2.1.1 Let $\mathcal{U}_q^{\text{ext}}$ be the linear space of all functions from $\mathbb{N}\times\mathbb{N}$ into $F(\mathbb{Z}\theta)$. Hence an element

$$f : \mathbb{N}\times\mathbb{N} \to F(\mathbb{Z}\theta) : (r,s) \mapsto f(r,s) \equiv f_{r,s}$$

of $\mathcal{U}_q^{\text{ext}}$ can be considered as an indexed family $(f_{r,s})_{r,s\in\mathbb{N}}$ of functions on $\mathbb{Z}\theta$. •

Proposition 5.2.1.2 *$\mathcal{U}_q^{\text{ext}}$ becomes a *-algebra under the following operations:*

$$\begin{aligned}(fg)_{k,l} &= \sum_{r=0}^{k}\sum_{s=0}^{l} f_{r,s}\,\Gamma^{2(s-r)}g_{k-r,l-s} \\ (f^*)_{k,l} &= \Gamma^{2(l-k)}\overline{f_{l,k}}\end{aligned} \tag{5.4}$$

where $f,g \in \mathcal{U}_q^{\text{ext}}$ and $k,l \in \mathbb{N}$.

Proof. Straightforward. ∎

Remark 5.2.1.3 In a *strictly formal* sense, an element $f \in \mathcal{U}_q^{\text{ext}}$ could be thought of as a formal power series

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f_{r,s}(\ln a)\, b^r c^s$$

where a,b,c satisfy the relations (5.1). Formal multiplication and conjugation of such series and applying the commutation rules then yields (5.4). ⋆

Define for every $p \in \mathbb{Z}$ a function $e_p \in F(\mathbb{Z}\theta)$ by $e_p(x) = e^{px}$ for $x \in \mathbb{Z}\theta$. Next, define a linear map j from $\mathcal{U}_q$ into $\mathcal{U}_q^{\text{ext}}$ by $j(a^p b^m c^n)_{r,s} = \delta_{r,m}\,\delta_{s,n}\,e_p$ for $p \in \mathbb{Z}$ and $m,n,r,s \in \mathbb{N}$. Then it is straightforward to show

Lemma 5.2.1.4 *The map j is a *-algebra embedding of $\mathcal{U}_q$ in $\mathcal{U}_q^{\text{ext}}$.*

5.2.2 Extending the pairing

Above we have embedded $\mathcal{U}_q$ into a larger *-algebra. Now we shall accordingly extend the pairing with $\mathcal{A}_q$.

Definition 5.2.2.1 Let $\langle\cdot,\cdot\rangle : \mathcal{U}_q^{\text{ext}} \times \mathcal{A}_q \to \mathbb{C}$ be the unique bilinear form such that

$$\left\langle f,\, \alpha^l\beta^m\gamma^n\right\rangle \;=\; (\Gamma^{l+m-n} f_{m,n})(0)\, q^{\frac{1}{2}l(m+n)}\,[m]_q!\,[n]_q! \tag{5.5}$$

for all $f \in \mathcal{U}_q^{\text{ext}}$, $l \in \mathbb{Z}$ and $m,n \in \mathbb{N}$. •

Proposition 5.2.2.2 *The pairing $\langle\mathcal{U}_q^{\text{ext}},\mathcal{A}_q\rangle$ defined above is non-degenerate, and in view of the embedding $\mathcal{U}_q \overset{j}{\hookrightarrow} \mathcal{U}_q^{\text{ext}}$, it is compatible with the pairing $\langle\mathcal{U}_q,\mathcal{A}_q\rangle$ defined in (5.3). Furthermore*

$$\langle fg,\xi\rangle = \langle f\otimes g, \Delta(\xi)\rangle \qquad \text{and} \qquad \langle f^*,\xi\rangle = \overline{\langle f, S(\xi)^*\rangle} \tag{5.6}$$

for all $f,g \in \mathcal{U}_q^{\text{ext}}$ and $\xi \in \mathcal{A}_q$.

Proof. The first two statements are easily verified. The proof of (5.6) involves the q-binomial theorem and techniques similar to those used in [12, 13] to compute the pairing. ∎

Since we now have a non-degenerate pairing $\langle \mathcal{U}_q^{\text{ext}}, \mathcal{A}_q \rangle$, we can consider $\mathcal{U}_q^{\text{ext}}$ as a linear subspace of the algebraic dual $\mathcal{A}_q'$ of $\mathcal{A}_q$. Since $\mathcal{A}_q$ is a Hopf *-algebra, also $\mathcal{A}_q'$ carries a *-algebra structure. In fact we have the following:

Proposition 5.2.2.3 *$\mathcal{U}_q^{\text{ext}}$ and $\mathcal{A}_q'$ are isomorphic *-algebras.*

Proof. In view of (5.6) it suffices to show the canonical embedding of $\mathcal{U}_q^{\text{ext}}$ in $\mathcal{A}_q'$ is surjective. So let's take any $\omega \in \mathcal{A}_q'$ and define $f \in \mathcal{U}_q^{\text{ext}}$ by

$$f_{m,n}(k\theta) \;=\; \frac{q^{\frac{1}{2}(k+m-n)(m+n)}}{[m]_q!\,[n]_q!} \,\left\langle \omega,\, \alpha^{-k-m+n}\beta^m\gamma^n \right\rangle$$

for $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}$. Then f corresponds canonically to ω. ∎

5.2.3 Actions & antipode

In (2.1) we defined the actions of an algebra on its dual. Together with proposition 5.2.2.3 this yields actions of $\mathcal{A}_q$ on $\mathcal{U}_q^{\text{ext}}$, making $\mathcal{U}_q^{\text{ext}}$ into an $\mathcal{A}_q$-bimodule. It suffices to compute the actions of the generators α, β and γ.

Proposition 5.2.3.1 *For all $f \in \mathcal{U}_q^{\text{ext}}$ and $r, s \in \mathbb{N}$ we have*

$$\begin{aligned}
(\alpha \triangleright f)_{r,s} &= q^{-\frac{1}{2}(r+s)}\,\Gamma f_{r,s} \\
(f \triangleleft \alpha)_{r,s} &= q^{\frac{1}{2}(r+s)}\,\Gamma f_{r,s} \\
(\beta \triangleright f)_{r,s} &= q^{-\frac{1}{2}(r-s)}\,[r+1]_q\,\Phi\Gamma f_{r+1,s} \\
(f \triangleleft \beta)_{r,s} &= q^{\frac{1}{2}(r-s)}\,[r+1]_q\,\Phi^{-1}\Gamma f_{r+1,s} \\
(\gamma \triangleright f)_{r,s} &= q^{-\frac{1}{2}(r-s)}\,[s+1]_q\,\Phi\Gamma^{-1} f_{r,s+1} \\
(f \triangleleft \gamma)_{r,s} &= q^{\frac{1}{2}(r-s)}\,[s+1]_q\,\Phi^{-1}\Gamma^{-1} f_{r,s+1}
\end{aligned}$$

Proof. Straightforward calculation based on formula (5.5) for the pairing. ∎

For any $h \in F(\mathbb{Z}\theta)$ we define $h^\bullet \in F(\mathbb{Z}\theta)$ by $h^\bullet(x) = h(-x)$ for $x \in \mathbb{Z}\theta$. Once again, it is straightforward to prove the following:

Proposition 5.2.3.2 *Define a linear map $S : \mathcal{U}_q^{\text{ext}} \to \mathcal{U}_q^{\text{ext}}$ by*

$$S(f)_{r,s} \;=\; (-q)^{s-r}\,\Gamma^{2(s-r)}(f^\bullet_{r,s}) \qquad\qquad (f \in \mathcal{U}_q^{\text{ext}};\ r, s \in \mathbb{N}).$$

Then $\langle S(f), \xi \rangle = \langle f, S(\xi) \rangle$ for all $f \in \mathcal{U}_q^{\text{ext}}$ and $\xi \in \mathcal{A}_q$.

5.3 A functional calculus in 2 variables

Abstract 5.3 We construct subspaces of $\mathcal{U}_q^{\text{ext}}$ spanned by elements which are formally of type $f(\ln a)\, g(bc)\, b^m$ and $f(\ln a)\, g(bc)\, c^m$, where f and g run trough suitable spaces of functions and $m \in \mathbb{N}$. We compute the actions of $\mathcal{A}_q$ on these particular elements; idem for the antipode. Eventually we obtain an algebraic Hopf *-system in the sense of definition 3.7.2.1. ⋆

5.3.1 Construction

Recall the conventions concerning entire functions (cf. §1.4).

Definition 5.3.1.1 Define a linear mapping $\Upsilon : F(\mathbb{Z}\theta) \otimes H(\mathbb{C}) \to \mathcal{U}_q^{\text{ext}}$ by

$$\Upsilon(f \otimes g)_{r,s} \;=\; \delta_{r,s}\, \mu_r(g) f \qquad\qquad (r, s \in \mathbb{N}).$$

Remark 5.3.1.2 In the spirit of remark 5.2.1.3, an element in $\mathcal{U}_q^{\text{ext}}$ of the form $\Upsilon(f \otimes g)$ could be interpreted *formally* as

$$f(\ln a)\, g(c^* c) \;=\; \sum_{n=0}^{\infty} \mu_n(g)\, f(\ln a)\, b^n c^n.$$

Υ could be thought of as a 'functional calculus' in 2 variables:

Proposition 5.3.1.3 *The map Υ is an injective *-algebra morphism.*

Proof. Immediate from definition 5.3.1.1, formulas (5.4) and the rule for multiplying Taylor series. ∎

If we identify $\mathcal{U}_q$ with its image in $\mathcal{U}_q^{\text{ext}}$ as in lemma 5.2.1.4, then the following makes sense:

Lemma 5.3.1.4 *For all $f \in F(\mathbb{Z}\theta)$, $g \in H(\mathbb{C})$ and $r, s, m \in \mathbb{N}$ we have*

$$\begin{aligned}
\big(\Upsilon(f \otimes g)\, b^m\big)_{r,s} &= \delta_{r,s+m}\, \mu_s(g) f \\
\big(\Upsilon(f \otimes g)\, c^m\big)_{r,s} &= \delta_{r+m,s}\, \mu_r(g) f
\end{aligned}$$

Proof. Combine (5.4) with lemma 5.2.1.4 and definition 5.3.1.1. ∎

With the same techniques one can show the following commutation rules:

Proposition 5.3.1.5 *For all $f \in F(\mathbb{Z}\theta)$, $g \in H(\mathbb{C})$, $m \in \mathbb{N}$ and $p \in \mathbb{Z}$ we have*

$$\begin{aligned}
b^m\, \Upsilon(f \otimes g) &= \Upsilon(\Gamma^{-2m} f \otimes g)\, b^m \\
c^m\, \Upsilon(f \otimes g) &= \Upsilon(\Gamma^{2m} f \otimes g)\, c^m
\end{aligned}$$

$$\begin{aligned}
\Upsilon(f \otimes g)\, (bc)^m &= \Upsilon(f \otimes \Psi^m g) &&= (bc)^m\, \Upsilon(f \otimes g) \\
\Upsilon(f \otimes g)\, a^p &= \Upsilon(\Phi^p f \otimes g) &&= a^p\, \Upsilon(f \otimes g)
\end{aligned} \tag{5.7}$$

5.3.2 Actions & antipode

Besides the well-known q-derivative (see appendix C) we shall also need the following q-difference operator:

Definition 5.3.2.1 For any $m \in \mathbb{N}$ we define $\nabla_q^{(m)} : H(\mathbb{C}) \to H(\mathbb{C})$ by

$$\nabla_q^{(m)} \;=\; \frac{1}{q-q^{-1}} \left(q^m \, \Omega - q^{-m} \, \Omega^{-1} \right). \tag{5.8}$$

Proposition 5.3.2.2 *For all $f \in F(\mathbb{Z}\theta)$, $g \in H(\mathbb{C})$ and $m \in \mathbb{N}$ we have*

$$\begin{array}{rcll}
\alpha \triangleright \Upsilon(f \otimes g)\, b^m &=& q^{-\frac{1}{2}m}\, \Upsilon\big(\Gamma f \otimes \Omega^{-1} g\big)\, b^m & \\
\alpha \triangleright \Upsilon(f \otimes g)\, c^m &=& q^{-\frac{1}{2}m}\, \Upsilon\big(\Gamma f \otimes \Omega^{-1} g\big)\, c^m & \\
\beta \triangleright \Upsilon(f \otimes g)\, b^m &=& q^{-\frac{1}{2}(m-1)}\, \Upsilon\big(\Phi\Gamma f \otimes \nabla_q^{(m)} g\big)\, b^{m-1} & (m \geq 1) \\
\beta \triangleright \Upsilon(f \otimes g)\, c^m &=& q^{\frac{1}{2}(m+1)}\, \Upsilon\big(\Phi\Gamma f \otimes \Omega^{-1} D_{q^2} g\big)\, c^{m+1} & \\
\gamma \triangleright \Upsilon(f \otimes g)\, b^m &=& q^{-\frac{1}{2}(m+1)}\, \Upsilon\big(\Phi\Gamma^{-1} f \otimes \Omega^{-1} D_{q^2} g\big)\, b^{m+1} & \\
\gamma \triangleright \Upsilon(f \otimes g)\, c^m &=& q^{\frac{1}{2}(m-1)}\, \Upsilon\big(\Phi\Gamma^{-1} f \otimes \nabla_q^{(m)} g\big)\, c^{m-1} & (m \geq 1) \\
\Upsilon(f \otimes g)\, b^m \triangleleft \alpha &=& q^{\frac{1}{2}m}\, \Upsilon(\Gamma f \otimes \Omega g)\, b^m & \\
\Upsilon(f \otimes g)\, c^m \triangleleft \alpha &=& q^{\frac{1}{2}m}\, \Upsilon\big(\Gamma f \otimes \Omega g\big)\, c^m & \\
\Upsilon(f \otimes g)\, b^m \triangleleft \beta &=& q^{\frac{1}{2}(m-1)}\, \Upsilon\big(\Phi^{-1}\Gamma f \otimes \nabla_q^{(m)} g\big)\, b^{m-1} & (m \geq 1) \\
\Upsilon(f \otimes g)\, c^m \triangleleft \beta &=& q^{-\frac{1}{2}(m+1)}\, \Upsilon\big(\Phi^{-1}\Gamma f \otimes \Omega^{-1} D_{q^2} g\big)\, c^{m+1} & \\
\Upsilon(f \otimes g)\, b^m \triangleleft \gamma &=& q^{\frac{1}{2}(m+1)}\, \Upsilon\big(\Phi^{-1}\Gamma^{-1} f \otimes \Omega^{-1} D_{q^2} g\big)\, b^{m+1} & \\
\Upsilon(f \otimes g)\, c^m \triangleleft \gamma &=& q^{-\frac{1}{2}(m-1)}\, \Upsilon\big(\Phi^{-1}\Gamma^{-1} f \otimes \nabla_q^{(m)} g\big)\, c^{m-1} & (m \geq 1)
\end{array}$$

Proof. Combine proposition 5.2.3.1 with lemma 5.3.1.4. The calculations are not too short but straightforward. ∎

Proposition 5.3.2.3 *Recall proposition 5.2.3.2. For all $f \in F(\mathbb{Z}\theta)$, $g \in H(\mathbb{C})$ and $m \in \mathbb{N}$ we have*

$$\begin{array}{rcl}
S\big(\Upsilon(f \otimes g)\big) &=& \Upsilon(f^\bullet \otimes g) \\
S\big(\Upsilon(f \otimes g)\, b^m\big) &=& (-q)^{-m}\, \Upsilon\big(\Gamma^{-2m}(f^\bullet) \otimes g\big)\, b^m \\
S\big(\Upsilon(f \otimes g)\, c^m\big) &=& (-q)^{m}\, \Upsilon\big(\Gamma^{2m}(f^\bullet) \otimes g\big)\, c^m \\
S^2\big(\Upsilon(f \otimes g)\, b^m\big) &=& q^{-2m}\, \Upsilon(f \otimes g)\, b^m \\
S^2\big(\Upsilon(f \otimes g)\, c^m\big) &=& q^{2m}\, \Upsilon(f \otimes g)\, c^m
\end{array}$$

5.3.3 The space $\mathcal{U}_q(\mathfrak{L})$

Whenever $\mathfrak{L}$ is a subspace of $F(\mathbb{Z}\theta) \otimes H(\mathbb{C})$, we define $\mathcal{U}_q(\mathfrak{L})$ to be the following subspace of $\mathcal{U}_q^{\mathrm{ext}}$

$$\mathcal{U}_q(\mathfrak{L}) \;=\; \mathrm{span}\left(\Upsilon(\mathfrak{L})\, b^{\mathbb{N}} \,\cup\, \Upsilon(\mathfrak{L})\, c^{\mathbb{N}} \right). \tag{5.9}$$

From definition 5.3.1.1 and lemma 5.3.1.4 it is clear that $\mathcal{U}_q(\mathfrak{L})$ is actually a *direct sum* of linear spaces:

$$\Big(\textstyle\bigoplus_{m=1}^{\infty} \Upsilon(\mathfrak{L})\, b^m\Big) \;\oplus\; \Upsilon(\mathfrak{L}) \;\oplus\; \Big(\textstyle\bigoplus_{n=1}^{\infty} \Upsilon(\mathfrak{L})\, c^n\Big) \tag{5.10}$$

It follows immediately that any $x \in \mathcal{U}_q(\mathfrak{L})$ can be *uniquely* written as

$$x \;=\; \sum_{m=1}^{\infty} \Upsilon(X_m)\, b^m \;+\; \Upsilon(X_0) \;+\; \sum_{n=1}^{\infty} \Upsilon(X_{-n})\, c^n$$

with only finitely many non-zero $X_m \in \mathfrak{L}$ $(m \in \mathbb{Z})$. Concerning *uniqueness*, observe that e.g. $\Upsilon(X)\, b^m = 0$ for some $X \in \mathfrak{L}$ and $m \in \mathbb{N}$ implies that

$$\Upsilon\big((\mathrm{id} \otimes \Psi^m)X\big) \;\stackrel{(5.7)}{=}\; \Upsilon(X)\, b^m c^m \;=\; 0,$$

and recall that Υ is injective. Since also Ψ is injective, we conclude that $X = 0$.

Proposition 5.3.3.1 *If $\mathfrak{L}$ is a self-adjoint subspace of $F(\mathbb{Z}\theta) \otimes H(\mathbb{C})$ which is invariant under $\Gamma \otimes \Omega^{\pm 1}$, $\Gamma^{-1} \otimes \Omega^{\pm 1}$, $\Gamma^{\pm 2} \otimes \mathrm{id}$, $\Phi^{\pm 1} \otimes \mathrm{id}$, $\mathrm{id} \otimes \Psi$, $\mathrm{id} \otimes D_{q^2}$ and ${}^{\bullet} \otimes \mathrm{id}$, then $\mathcal{U}_q(\mathfrak{L})$ is*

i. a sub-$\mathcal{A}_q$-bimodule of $\mathcal{U}_q^{\mathrm{ext}}$

ii. a $\mathcal{U}_q$-bimodule under multiplication within $\mathcal{U}_q^{\mathrm{ext}}$

iii. invariant under $S^{\pm 1}$ and *

If moreover $\mathfrak{L}$ is an algebra, then so is $\mathcal{U}_q(\mathfrak{L})$.

Proof. Notice $\Omega^{\pm 1}$-invariance in the second factor implies $\nabla_q^{(m)}$-invariance. The results follow easily from propositions 5.3.1.3, 5.3.1.5, 5.3.2.2 and 5.3.2.3. ∎

Corollary 5.3.3.2 *Under the assumptions of the previous proposition, including the 'moreover' part, $\langle \mathcal{U}_q(\mathfrak{L})\, ,\, \underline{\mathcal{A}_q} \rangle$ is an algebraic Hopf *-system*[1]*, provided that $\mathcal{U}_q(\mathfrak{L})$ separates $\mathcal{A}_q$ within the duality.*

Remark 5.3.3.3 It is worth mentioning that the above construction is susceptible to some further finetuning: instead of starting from a single space $\mathfrak{L}$, one could as well consider an indexed family $(\mathfrak{L}_m)_{m\in\mathbb{Z}}$ of subspaces of $F(\mathbb{Z}\theta) \otimes H(\mathbb{C})$. Instead of (5.10) we would then use

$$\mathcal{U}_q\big((\mathfrak{L}_m)_{m\in\mathbb{Z}}\big) \;=\; \Big(\textstyle\bigoplus_{m=1}^{\infty} \Upsilon(\mathfrak{L}_m)\, b^m\Big) \;\oplus\; \Upsilon(\mathfrak{L}_0) \;\oplus\; \Big(\textstyle\bigoplus_{n=1}^{\infty} \Upsilon(\mathfrak{L}_{-n})\, c^n\Big)\,.$$

To obtain an analogue of proposition 5.3.3.1, one should require the spaces $\mathfrak{L}_m$ to enjoy the same invariance conditions that were imposed on $\mathfrak{L}$ in proposition 5.3.3.1, except for invariance under $\mathrm{id} \otimes D_{q^2}$, which is to be replaced by

$$(\mathrm{id} \otimes D_{q^2})\mathfrak{L}_m \;\subseteq\; \mathfrak{L}_{m+1} \qquad \text{and} \qquad (\mathrm{id} \otimes D_{q^2})\mathfrak{L}_{-m} \;\subseteq\; \mathfrak{L}_{-m-1}$$

[1]See definition 3.7.2.1.

for all $m \geq 0$. Moreover one should add the condition that

$$(\mathrm{id} \otimes \nabla_q^{(m)})\mathfrak{L}_m \subseteq \mathfrak{L}_{m-1} \qquad \text{and} \qquad (\mathrm{id} \otimes \nabla_q^{(m)})\mathfrak{L}_{-m} \subseteq \mathfrak{L}_{-m+1}$$

for all $m \geq 1$. Self-adjointness of $\mathfrak{L}$ could be replaced by a condition like $\mathfrak{L}_{-m}^* = \mathfrak{L}_m$ for any $m \in \mathbb{Z}$. Furthermore, to reproduce item (ii) of proposition 5.3.3.1, we would need some nesting properties like

$$(\mathrm{id} \otimes \Psi)\mathfrak{L}_{m+1} \subseteq \mathfrak{L}_m \subseteq \mathfrak{L}_{m+1}$$

for $m \geq 0$, and similarly for negative indices. Eventually, if we want $\mathcal{U}_q\big((\mathfrak{L}_m)_{m\in\mathbb{Z}}\big)$ to be an *algebra*, then we would need something like $\mathfrak{L}_m\mathfrak{L}_n \subseteq \mathfrak{L}_{m+n}$ for any $m, n \in \mathbb{Z}$, rather than requiring all the $\mathfrak{L}_m$ to be algebras in themselves; so one could say we are dealing with some $\mathbb{Z}$-graded structure here. It is very likely that all the results in the following sections can be adapted to this $\mathbb{Z}$-graded approach, yielding a theory which would be richer, but unfortunately also more complicated. Therefore we shall not elaborate on this point an stick to our original approach. Nevertheless it might be worth doing the exercise, especially in view of remark 6.2.6.5 in chapter 6. ⋆

5.4 Haar functionals on $\mathcal{U}_q^{\mathrm{ext}}$

Abstract 5.4 We construct positive and faithful Haar functionals on $\mathcal{U}_q(\mathfrak{L})$ for a suitable choice of $\mathfrak{L}$. In their definition we recognize the *spectral conditions* of [37, 51]. Furthermore we compute KMS automorphisms and modular elements. ⋆

Throughout this section we fix a real number τ with $\tau > 0$.

5.4.1 Construction & invariance

We start by making a specific choice for the space $\mathfrak{L}$ appearing in §5.3.3.

Assumptions 5.4.1.1 Whenever r is a real number with $r > 0$, we assume $\mathcal{G}_r$ to be a non-trivial self-adjoint subspace of $H(\mathbb{C})$ such that

i. $\mathcal{G}_r$ is invariant under $\Omega^{\pm 2}$, Ψ and D_{q^2}

ii. $\sum_{k\in\mathbb{Z}} g(rq^{2k})\, q^{2k}$ is absolutely summable for any $g \in \mathcal{G}_r$.

$\mathcal{G}_r$ is *not* assumed to be an algebra unless explicitly stated. •

Example 5.4.1.2 A possible choice for $\mathcal{G}_r$ could be the following: first define

$$\mathcal{S}_r(\mathbb{R}^+; q) \;=\; \Big\{ g \in H(\mathbb{C}) \;\Big|\; \text{for all } n \in \mathbb{N}, \text{ the set } \big\{\, g(rq^k)\, q^{nk} \big\}_{k\in\mathbb{Z}} \text{ is bounded} \Big\}.$$

Now it is not hard to prove[2] that $\mathcal{S}_r(\mathbb{R}^+; q^2)$ satisfies our requirements on $\mathcal{G}_r$. Moreover $\mathcal{S}_r(\mathbb{R}^+; q^2)$ is in fact an *algebra*, and furthermore it is easy to prove that it is actually equal to the 'Schwartz-like' space that was used in [12, 13] for similar purposes. ⋆

[2] To show (ii) and D_{q^2}-invariance, one should take into account the following observation: if g is entire, then so is $D_{q^2}g$. Hence g and $D_{q^2}g$ are bounded on the interval $[0, r]$.

Given a space $\mathcal{G}_\tau$ satisfying assumptions 5.4.1.1, we define subspaces $\mathcal{G}_\tau^{\text{even}}$ and $\mathcal{G}_\tau^{\text{odd}}$ of $H(\mathbb{C})$ by $\mathcal{G}_\tau^{\text{even}} = \mathcal{G}_\tau$ and $\mathcal{G}_\tau^{\text{odd}} = \Omega\mathcal{G}_\tau$. Now it is easy to see that both $\mathcal{G}_\tau^{\text{even}}$ and $\mathcal{G}_\tau^{\text{odd}}$ are self-adjoint and invariant under $\Omega^{\pm 2}$, Ψ and D_{q^2}. In this respect, observe that $D_{q^2}\Omega = q\,\Omega D_{q^2}$.

Let us agree that ${}^{\text{even}}_{\text{odd}}$ means: even and odd *respectively*. Henceforth $K^{\text{even}}_{\text{odd}}(\mathbb{Z}\theta)$ will denote the space of all functions on $\mathbb{Z}\theta$ with finite support contained in $\mathbb{Z}_{\text{even}\atop\text{odd}}\theta$. Here $\mathbb{Z}_{\text{even}\atop\text{odd}}$ denotes the ${}^{\text{even}}_{\text{odd}}$ integers. Clearly $K^{\text{even}}_{\text{odd}}(\mathbb{Z}\theta)$ are self-adjoint subalgebras of $F(\mathbb{Z}\theta)$ which are moreover invariant under $\Gamma^{\pm 2}$, $\Phi^{\pm 1}$ and ${}^\bullet$.

Notice that $K^{\text{even}}(\mathbb{Z}\theta)$ and $K^{\text{odd}}(\mathbb{Z}\theta)$ are in direct sum within $F(\mathbb{Z}\theta)$, and consequently, $K^{\text{even}}(\mathbb{Z}\theta) \otimes \mathcal{G}_\tau^{\text{even}}$ and $K^{\text{odd}}(\mathbb{Z}\theta) \otimes \mathcal{G}_\tau^{\text{odd}}$ are in direct sum within $F(\mathbb{Z}\theta) \otimes H(\mathbb{C})$. Therefore we can define

$$\mathfrak{L}(\mathcal{G}_\tau) \;=\; \big(K^{\text{even}}(\mathbb{Z}\theta) \otimes \mathcal{G}_\tau^{\text{even}}\big) \oplus \big(K^{\text{odd}}(\mathbb{Z}\theta) \otimes \mathcal{G}_\tau^{\text{odd}}\big) \;\subseteq\; K(\mathbb{Z}\theta) \otimes H(\mathbb{C}). \tag{5.11}$$

Now it is easy to verify that $\mathfrak{L} = \mathfrak{L}(\mathcal{G}_\tau)$ satisfies the conditions of proposition 5.3.3.1, except for the 'moreover' part, the latter being satisfied as well *provided* we assume $\mathcal{G}_\tau$ to be an algebra; indeed if $\mathcal{G}_\tau$ is an algebra, then so is $\mathfrak{L}(\mathcal{G}_\tau)$, because $K^{\text{even}}(\mathbb{Z}\theta)$ and $K^{\text{odd}}(\mathbb{Z}\theta)$ annihilate one another under multiplication.

Remarks 5.4.1.3 i. The fact we actually have a *direct* sum in (5.11) comes merely from the 'first leg' of the tensor product. Indeed $\mathcal{G}_\tau^{\text{even}}$ and $\mathcal{G}_\tau^{\text{odd}}$ need not to be in a direct sum position within $H(\mathbb{C})$. To illustrate this, consider the example $\mathcal{G}_\tau = \mathcal{S}_\tau(\mathbb{R}^+; q^2)$ and observe that

$$\mathcal{S}_\tau^{\text{even}}(\mathbb{R}^+; q^2) \,\cap\, \mathcal{S}_\tau^{\text{odd}}(\mathbb{R}^+; q^2) \;=\; \mathcal{S}_\tau(\mathbb{R}^+; q). \tag{5.12}$$

ii. Notice that in (5.11) *both* the even and odd components are necessary: otherwise $\mathfrak{L}(\mathcal{G}_\tau)$ would not be invariant under $\Gamma \otimes \Omega^{\pm 1}$ or $\Gamma^{-1} \otimes \Omega^{\pm 1}$ (which was an essential condition in proposition 5.3.3.1).

iii. Why so complicated? Can't we simply take $\mathfrak{L} = K(\mathbb{Z}\theta) \otimes \mathcal{S}_\tau(\mathbb{R}^+; q)$? First observe that the latter space is actually *smaller* than $\mathfrak{L}\big(\mathcal{S}_\tau(\mathbb{R}^+; q^2)\big)$ because of (5.12). Furthermore it indeed satisfies all the requirements of proposition 5.3.3.1, and as a matter of fact one would get quite far with it. However, this is *not at all* the proper choice if we want to do harmonic analysis, as we will explain later (cf. remarks 5.4.1.9 and 6.4.2.2). $\star$

We are about to invoke corollary 5.3.3.2. In this respect only one condition remains to be shown: $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$ should separate $\mathcal{A}_q$ within the duality. In order not to slow down our construction, we defer this technical matter to appendix B. The result of all this is the following:

Proposition 5.4.1.4 *Whenever $\mathcal{G}_\tau$ is a subalgebra of $H(\mathbb{C})$ satisfying assumptions 5.4.1.1, the pair $\big\langle \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)\,,\, \underline{\mathcal{A}_q} \big\rangle$ is an algebraic Hopf *-system*[3].

[3]See definition 3.7.2.1.

The next step is to construct Haar functionals on $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$.

Definition 5.4.1.5 Let us define functionals $\lambda_{\substack{\mathrm{even}\\\mathrm{odd}}}$ and $\rho_{\substack{\mathrm{even}\\\mathrm{odd}}}$ on $K^{\substack{\mathrm{even}\\\mathrm{odd}}}(\mathbb{Z}\theta)$ by

$$\lambda_{\substack{\mathrm{even}\\\mathrm{odd}}}(f) \;=\; \sum_{k\,\in\,\mathbb{Z}_{\substack{\mathrm{even}\\\mathrm{odd}}}} f(k\theta)\,q^k \qquad \text{and} \qquad \rho_{\substack{\mathrm{even}\\\mathrm{odd}}}(f) \;=\; \sum_{k\,\in\,\mathbb{Z}_{\substack{\mathrm{even}\\\mathrm{odd}}}} f(k\theta)\,q^{-k}$$

whereas on $\mathcal{G}_\tau^{\substack{\mathrm{even}\\\mathrm{odd}}}$ we define linear functionals $\chi_{\substack{\mathrm{even}\\\mathrm{odd}}}$ by

$$\chi_{\substack{\mathrm{even}\\\mathrm{odd}}}(g) \;=\; \sum_{l\,\in\,\mathbb{Z}_{\substack{\mathrm{even}\\\mathrm{odd}}}} g(\tau q^l)\,q^l.$$

Lemma 5.4.1.6 *We have*

$$\begin{aligned} \lambda_{\mathrm{even}}\Gamma &= q\lambda_{\mathrm{odd}} & \qquad \rho_{\mathrm{even}}\Gamma &= q^{-1}\rho_{\mathrm{odd}} \\ \lambda_{\mathrm{odd}}\Gamma &= q\lambda_{\mathrm{even}} & \qquad \rho_{\mathrm{odd}}\Gamma &= q^{-1}\rho_{\mathrm{even}} \end{aligned}$$

$$\begin{aligned} \chi_{\mathrm{even}}\Omega &= q^{-1}\chi_{\mathrm{odd}} & \qquad & \\ \chi_{\mathrm{odd}}\Omega &= q^{-1}\chi_{\mathrm{even}} & \qquad & \chi_{\substack{\mathrm{even}\\\mathrm{odd}}}\nabla_q^{(1)} = 0. \end{aligned}$$

This makes sense because $K^{\mathrm{even}}(\mathbb{Z}\theta) = \Gamma K^{\mathrm{odd}}(\mathbb{Z}\theta)$ and $\mathcal{G}_\tau^{\mathrm{even}} = \Omega\mathcal{G}_\tau^{\mathrm{odd}}$.

Definition 5.4.1.7 In view of (5.10) we can define linear functionals φ and ψ on $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$ as follows: putting φ and ψ equal to zero on

$$\Upsilon\big(\mathfrak{L}(\mathcal{G}_\tau)\big)\,b^{\mathbb{N}_0} \qquad \text{and} \qquad \Upsilon\big(\mathfrak{L}(\mathcal{G}_\tau)\big)\,c^{\mathbb{N}_0}, \tag{5.13}$$

it only remains to define φ and ψ on $\Upsilon\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$. In this respect we require

$$\varphi\Upsilon \;=\; (\lambda_{\mathrm{even}}\otimes\chi_{\mathrm{even}})\oplus(\lambda_{\mathrm{odd}}\otimes\chi_{\mathrm{odd}}) \tag{5.14}$$

$$\psi\Upsilon \;=\; (\rho_{\mathrm{even}}\otimes\chi_{\mathrm{even}})\oplus(\rho_{\mathrm{odd}}\otimes\chi_{\mathrm{odd}}) \tag{5.15}$$

where we considered Υ restricted to $\mathfrak{L}(\mathcal{G}_\tau)$ as defined in (5.11). •

Proposition 5.4.1.8 *The functionals φ and ψ as defined above are respectively left and right invariant (cf. definition 3.6.1) for $\big\langle\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)\,,\,\underline{\mathcal{A}_q}\big\rangle$.*

Proof. Let's consider the case of φ. Since $\varepsilon(\alpha) = 1$ and $\varepsilon(\beta) = \varepsilon(\gamma) = 0$, it suffices to show

$$\begin{aligned} \varphi\left(\Upsilon(X)\,\{b \text{ or } c\}^m \triangleleft \alpha\right) &= \varphi\left(\Upsilon(X)\,\{b \text{ or } c\}^m\right) \\ \varphi\left(\Upsilon(X)\,\{b \text{ or } c\}^m \triangleleft \beta\right) &= 0 \\ \varphi\left(\Upsilon(X)\,\{b \text{ or } c\}^m \triangleleft \gamma\right) &= 0 \end{aligned}$$

for $X \in \mathfrak{L}(\mathcal{G}_\tau)$ and $m \in \mathbb{N}$. Observe the action of α does not alter the exponent of b or c, whereas it is lowered or raised by 1 under the actions of β and γ

(cf. proposition 5.3.2.2). Since φ vanishes on (5.13) only a few cases remain to be investigated:

$$\begin{aligned}
\varphi\big(\Upsilon(X) \triangleleft \alpha\big) &= \varphi\big(\Upsilon\big(\Gamma\otimes\Omega\big)\, X\big) &&= \varphi\big(\Upsilon(X)\big)\\
\varphi\big(\Upsilon(X)\, b \triangleleft \beta\big) &= \varphi\big(\Upsilon\big(\Phi^{-1}\Gamma\otimes\nabla_q^{(1)}\big)\, X\big) &&= 0\\
\varphi\big(\Upsilon(X)\, c \triangleleft \gamma\big) &= \varphi\big(\Upsilon\big(\Phi^{-1}\Gamma^{-1}\otimes\nabla_q^{(1)}\big)\, X\big) &&= 0.
\end{aligned}$$

Here we used proposition 5.3.2.2, lemma 5.4.1.6 and of course (5.14). ∎

Remark 5.4.1.9 Elements in $F(\mathbb{Z}\theta)\otimes H(\mathbb{C})$ can be considered as functions in 2 variables, i.e. from $\mathbb{Z}\theta\times\mathbb{C}$ into $\mathbb{C}$. When definition 5.4.1.5 is plugged into (5.14) we thus obtain the following expression for φ on $\Upsilon\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$:

$$\varphi\big(\Upsilon(X)\big) = \sum_{(k,l)\,\in\,\mathfrak{S}} X(k\theta,\tau q^l)\, q^{k+l}. \tag{5.16}$$

Here $\mathfrak{S} = (\mathbb{Z}_{\text{even}}\times\mathbb{Z}_{\text{even}})\cup(\mathbb{Z}_{\text{odd}}\times\mathbb{Z}_{\text{odd}})$ and $X\in\mathfrak{L}(\mathcal{G}_\tau)$ was implicitly identified with the corresponding function on $\mathbb{Z}\theta\times\mathbb{C}$. This formula however, isn't always convenient because of the peculiar nature of the summation range; one might ask why we don't simply take a sum over $\mathbb{Z}^2$ anyway, and indeed, as far as the construction of invariant functionals is concerned, there's absolutely no need to work with $\mathfrak{S}$ instead of $\mathbb{Z}^2$ (cf. remark 5.4.1.3.iii). Nevertheless there are good reasons to use $\mathfrak{S}$: in chapter 6 we shall construct Fourier transforms and prove a Plancherel formula which relates the Haar functionals φ and ψ with the Haar functional on the dual as constructed in [12, 13, see also §5.5 below]. Precisely at this point it becomes essential to use this particular 'parity' structure, otherwise our proof for this Plancherel formula would fail. Moreover only (5.11) will supply sufficiently many elements which behave 'nicely' under Fourier transformation. Another motivation comes from the operator-algebraic versions of the quantum $E(2)$ group [46, 51, 52]. The so-called 'spectral conditions' [37, 51] emerging at this operator-theoretical level are indeed closely related to our choice $\mathfrak{S}$ for the summation range in (5.16). ⋆

5.4.2 Positivity & faithfulness

Let $\mathcal{G}_\tau$ be a subalgebra of $H(\mathbb{C})$ satisfying assumptions 5.4.1.1.

Lemma 5.4.2.1 *Take any $X, Y\in\mathfrak{L}(\mathcal{G}_\tau)$ and $r,s\in\mathbb{N}$. Put $m=\min\{r,s\}$. Then we have*

$$\big(\Upsilon(Y)\, b^r\big)^*\big(\Upsilon(X)\, b^s\big) = \Upsilon\big((\Gamma^{2r}\otimes\Psi^m)(Y^*X)\big)\, b^{s-m}c^{r-m} \tag{5.17}$$

$$\big(\Upsilon(Y)\, b^r\big)^*\big(\Upsilon(X)\, c^s\big) = \Upsilon\big((\Gamma^{2r}\otimes\mathrm{id})(Y^*X)\big)\, c^{r+s} \tag{5.18}$$

$$\big(\Upsilon(Y)\, c^r\big)^*\big(\Upsilon(X)\, b^s\big) = \Upsilon\big((\Gamma^{-2r}\otimes\mathrm{id})(Y^*X)\big)\, b^{r+s} \tag{5.19}$$

$$\big(\Upsilon(Y)\, c^r\big)^*\big(\Upsilon(X)\, c^s\big) = \Upsilon\big((\Gamma^{-2r}\otimes\Psi^m)(Y^*X)\big)\, b^{r-m}\, c^{s-m} \tag{5.20}$$

Proof. Combine the commutation rules of proposition 5.3.1.5 with the fact that Υ is a *-algebra morphism (proposition 5.3.1.3). ∎

Corollary 5.4.2.2 *By definition the functionals φ and ψ vanish on the elements (5.18) and (5.19) unless both r and s are zero. Moreover φ and ψ also vanish on the elements (5.17) and (5.20) except when $r = s = m$. In the latter case we actually have for instance*

$$\begin{aligned}\varphi\Big(\big(\Upsilon(Y)\,b^m\big)^*\big(\Upsilon(X)\,b^m\big)\Big) &= q^{2m}\,\varphi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(Y^*X)\Big)\\ \varphi\Big(\big(\Upsilon(Y)\,c^m\big)^*\big(\Upsilon(X)\,c^m\big)\Big) &= q^{-2m}\,\varphi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(Y^*X)\Big)\end{aligned}$$

Proof. Observe (5.17) and (5.20) only involve *even* (!) powers of Γ. On the other hand, from lemma 5.4.1.6 it follows that $\lambda_{\substack{\text{even}\\ \text{odd}}}\Gamma^2 = q^2\,\lambda_{\substack{\text{even}\\ \text{odd}}}$. ∎

Definition 5.4.2.3 For any $m \in \mathbb{N}$ we define a sesquilinear form $\langle\cdot\,|\,\cdot\rangle_\varphi^{(m)}$ on $\mathfrak{L}(\mathcal{G}_\tau)\times\mathfrak{L}(\mathcal{G}_\tau)$ by

$$\langle X\,|\,Y\rangle_\varphi^{(m)} \;=\; \varphi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(Y^*X)\Big)$$

Lemma 5.4.2.4 *The sesquilinear forms defined above are scalar products.*

Proof. First recall (5.11) and observe that the spaces $K^{\text{even}}(\mathbb{Z}\theta)\otimes\mathcal{G}_\tau^{\text{even}}$ and $K^{\text{odd}}(\mathbb{Z}\theta)\otimes\mathcal{G}_\tau^{\text{odd}}$ are mutually orthogonal w.r.t. $\langle\cdot\,|\,\cdot\rangle_\varphi^{(m)}$. Therefore they can be treated separately. Let's for instance take any $X = \sum_{i=1}^r f_i\otimes g_i$ with $f_1,\dots,f_r$ in $K^{\text{even}}(\mathbb{Z}\theta)$ and $g_1,\dots,g_r$ in $\mathcal{G}_\tau^{\text{even}}$. Then

$$\begin{aligned}\langle X\,|\,X\rangle_\varphi^{(m)} &= \sum_{i,j=1}^r \varphi\Big(\Upsilon\big(\overline{f_i}f_j\otimes\Psi^m(\widetilde{g}_i g_j)\big)\Big)\\
&= \sum_{i,j=1}^r \lambda_{\text{even}}\big(\overline{f_i}f_j\big)\,\chi_{\text{even}}\big(\Psi^m(\widetilde{g}_i g_j)\big)\\
&= \sum_{i,j=1}^r\ \sum_{k\,\in\,\mathbb{Z}_{\text{even}}}\big(\overline{f_i}f_j\big)(k\theta)\,q^k \sum_{l\,\in\,\mathbb{Z}_{\text{even}}}\big(\Psi^m(\widetilde{g}_i g_j)\big)\,(\tau q^l)\,q^l\\
&= \sum_{k,l\,\in\,\mathbb{Z}_{\text{even}}}\ \sum_{i,j=1}^r \overline{f_i(k\theta)\,g_i(\tau q^l)}\ f_j(k\theta)\,g_j(\tau q^l)\ \tau^m\,q^{l(m+1)+k}\\
&= \sum_{k,l\,\in\,\mathbb{Z}_{\text{even}}}\left|\sum_{i=1}^r f_i(k\theta)\,g_i(\tau q^l)\right|^2 \tau^m\,q^{l(m+1)+k}\end{aligned}$$

Notice that the summation over k in the above expressions de facto runs over a *finite* number of terms, so the only infinite sum involved here, is the summation over l. Clearly the sums converge absolutely, because of the assumptions on $\mathcal{G}_\tau$. The last expression above is obviously non-negative; it is zero if and *only* if

$$\Big(\textstyle\sum_{i=1}^r f_i(k\theta)\,g_i\Big)\,(\tau q^l) = 0 \qquad \text{for all } k,l\in\mathbb{Z}_{\text{even}}. \tag{5.21}$$

Since $\sum_{i=1}^{r} f_i(k\theta)\, g_i$ is entire and vanishes on a set having a point of accumulation, it follows that $\sum_{i=1}^{r} f_i(k\theta)\, g_i = 0$ for any $k \in \mathbb{Z}_{\text{even}}$, and hence $X = 0$. The 'odd' case is similar. ∎

Proposition 5.4.2.5 *φ and ψ are hermitian, positive and faithful.*

Proof. Choose any element $x \in \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$. Recalling (5.10) we can write

$$x \;=\; \sum_{m=0}^{\infty} \Upsilon(X_m)\, b^m \;+\; \sum_{n=1}^{\infty} \Upsilon(Y_n)\, c^n \tag{5.22}$$

with only finitely many non-zero $X_m, Y_n \in \mathfrak{L}(\mathcal{G}_\tau)$. Corollary 5.4.2.2 and definition 5.4.2.3 yield

$$\varphi(x^* x) \;=\; \sum_{m=0}^{\infty} q^{2m}\, \langle X_m \,|\, X_m\rangle_\varphi^{(m)} \;+\; \sum_{n=1}^{\infty} q^{-2n}\, \langle Y_n \,|\, Y_n\rangle_\varphi^{(m)}$$

Now positivity and faithfulness of φ follows since the $\langle \cdot \,|\, \cdot\rangle_\varphi^{(m)}$ are scalar products (cf. lemma 5.4.2.4). Furthermore φ is hermitian because Υ is a *-morphism. ∎

5.4.3 KMS properties & modular element

Recall definitions 3.6.4 and 4.1.1. Again (besides assumptions 5.4.1.1) we assume $\mathcal{G}_\tau$ to be an *algebra*.

Proposition 5.4.3.1 *The Haar functionals φ and ψ are weakly KMS. The KMS-automorphisms are given by $\sigma_\varphi = S^2$ and $\sigma_\psi = S^{-2}$. Moreover φ and ψ are $\mathcal{U}_q$-KMS in the sense that for instance $\varphi(x\xi) = \varphi(\xi\sigma_\varphi(x))$ for any $\xi \in \mathcal{U}_q$ and $x \in \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$. Furthermore, we have $\varphi S = \psi$ and $\psi S = \varphi$. Here S is the antipode on $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$. Eventually we observe that the modular element for φ is given by $a^4 \in \mathcal{U}_q$.*

Proof. We have to show that $\varphi(xy) = \varphi(yS^2(x))$ for any $x, y \in \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$. Now in view of (5.10) it suffices to consider

$$x = \Upsilon(X)\,\{b \text{ or } c\}^r \qquad \text{and} \qquad y = \Upsilon(Y)\,\{b \text{ or } c\}^s \tag{5.23}$$

for $r, s \in \mathbb{N}$ and $X, Y \in \mathfrak{L}(\mathcal{G}_\tau)$. From propositions 5.3.1.3, 5.3.1.5 and 5.3.2.3, together with the invariance properties of $\mathfrak{L}(\mathcal{G}_\tau)$, it follows easily that both $\varphi(xy)$ and $\varphi(yS^2(x))$ vanish for most combinations of x and y of type (5.23). Only the following cases remain to be shown:

$$\varphi\Big(\Upsilon(X)\, b^m\; \Upsilon(Y)\, c^m\Big) \;=\; \varphi\Big(\Upsilon(Y)\, c^m\; S^2\big(\Upsilon(X)\, b^m\big)\Big) \tag{5.24}$$

$$\varphi\Big(\Upsilon(X)\, c^m\; \Upsilon(Y)\, b^m\Big) \;=\; \varphi\Big(\Upsilon(Y)\, b^m\; S^2\big(\Upsilon(X)\, c^m\big)\Big) \tag{5.25}$$

for all $m \in \mathbb{N}$. Now, again with propositions 5.3.1.3, 5.3.1.5 and 5.3.2.3, and together with the fact that $\mathfrak{L}(\mathcal{G}_\tau)$ is a (commutative!) algebra which is invariant under $\Gamma^{\pm 2} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Psi$, we obtain

$$\begin{aligned}
\varphi\Big(\Upsilon(X)\, b^m\, \Upsilon(Y)\, c^m\Big) &= \varphi\Big(\Upsilon(X)\,\Upsilon\big((\Gamma^{-2m} \otimes \mathrm{id})Y\big)\, b^m c^m\Big) \\
&= \varphi\Big(\Upsilon\Big\{\big(\Gamma^{-2m} \otimes \mathrm{id}\big)\Big(\big[(\Gamma^{2m} \otimes \mathrm{id})X\big]\, Y\Big)\Big\}\,(bc)^m\Big) \\
&= \varphi\Big(\Upsilon\Big\{\big(\Gamma^{-2m} \otimes \Psi^m\big)\Big(\big[(\Gamma^{2m} \otimes \mathrm{id})X\big]\, Y\Big)\Big\}\Big) \\
&\overset{(*)}{=} q^{-2m}\,\varphi\Big(\Upsilon\Big\{\big(\mathrm{id} \otimes \Psi^m\big)\Big(\big[(\Gamma^{2m} \otimes \mathrm{id})X\big]\, Y\Big)\Big\}\Big) \\
&= q^{-2m}\,\varphi\Big(\Upsilon\big(Y(\Gamma^{2m} \otimes \mathrm{id})X\big)\,(bc)^m\Big) \\
&= q^{-2m}\,\varphi\Big(\Upsilon(Y)\,\Upsilon\,\big((\Gamma^{2m} \otimes \mathrm{id})X\big)\, c^m b^m\Big) \\
&= q^{-2m}\,\varphi\Big(\Upsilon(Y)\, c^m\, \Upsilon(X)\, b^m\Big) \\
&= \varphi\Big(\Upsilon(Y)\, c^m\, S^2\big(\Upsilon(X)\, b^m\big)\Big)
\end{aligned}$$

In $(*)$ we used that

$$\varphi\left(\Upsilon(\Gamma^k \otimes \mathrm{id})Z\right) \;=\; q^k\,\varphi\left(\Upsilon(Z)\right)$$

for any $Z \in \mathfrak{L}(\mathcal{G}_\tau)$ and any *even* integer k. This proves (5.24). Similarly (5.25). Also the $\mathcal{U}_q$-KMS statement is quite straightforward to prove; then again, the computation differs significantly from the one above, so let's have a look anyway: for any $X \in \mathfrak{L}(\mathcal{G}_\tau)$, $p \in \mathbb{Z}$ and $m, r, s \in \mathbb{N}$ we have

$$\begin{aligned}
\varphi\Big(\Upsilon(X)\, b^m\, a^p b^r c^s\Big) &= q^{-mp}\,\varphi\Big(\Upsilon(X)\, a^p b^m b^r c^s\Big) \\
&= q^{-mp}\,\varphi\Big(\Upsilon\big((\Phi^p \otimes \mathrm{id})X\big)\, b^{m+r} c^s\Big) \\
&= \delta_{m+r,s}\, q^{-mp}\,\varphi\Big(\Upsilon\big((\Phi^p \otimes \Psi^{m+r})X\big)\Big) \\
&= \delta_{m+r,s}\, q^{-mp}\, q^{-2m}\,\varphi\Big(\Upsilon\big((\Gamma^{2m} \otimes \mathrm{id})(\Phi^p \otimes \Psi^{m+r})X\big)\Big) \\
&= \delta_{m+r,s}\, q^{-(s-r)p}\, q^{-2m}\,\varphi\Big((bc)^{m+r}\,\Upsilon\big((\Gamma^{2m}\Phi^p \otimes \mathrm{id})X\big)\Big) \\
&= q^{(r-s)p}\, q^{-2m}\,\varphi\Big(b^{m+r} c^s\,\Upsilon\big((\Gamma^{2m}\Phi^p \otimes \mathrm{id})X\big)\Big) \\
&= q^{(r-s)p}\, q^{-2m}\,\varphi\Big(b^r c^s\,\Upsilon\big((\Phi^p \otimes \mathrm{id})X\big)\, b^m\Big) \\
&= q^{(r-s)p}\, q^{-2m}\,\varphi\Big(b^r c^s a^p\,\Upsilon(X)\, b^m\Big) \\
&= q^{-2m}\,\varphi\Big(a^p b^r c^s\,\Upsilon(X)\, b^m\Big) \\
&= \varphi\Big(a^p b^r c^s\, S^2\big(\Upsilon(X)\, b^m\big)\Big).
\end{aligned}$$

Notice the above computation again involved several invariance requirements on $\mathfrak{L}(\mathcal{G}_\tau)$, which are all satisfied here, of course. The remaining cases are similar. The behaviour w.r.t. the antipode ($\varphi S = \psi$ etc.) is an easy consequence of proposition 5.3.2.3 and the fact that $\lambda_{\substack{\text{even}\\\text{odd}}}\bullet = \rho_{\substack{\text{even}\\\text{odd}}}$. Eventually the formula for the modular element follows easily from one of the commutation rules in proposition 5.3.1.5. ■

5.5 Haar functionals on $\mathcal{A}_q^{\text{ext}}$

Abstract 5.5 The subject of the present section has already been studied by H.T. Koelink [12, 13]. Some of his results have been included below to fix the notations, which have been slightly modified to match the language of the previous sections; furthermore the particular choice of function space[4] will be postponed until chapter 6. In fact the construction of Haar functionals in the previous sections is *dual* to the story of [12, 13] sketched below, and actually both constructions proceed in a similar way, except on one point: [12, 13] involves a functional calculus in one variable, whereas in §5.3 we were forced into taking 'functions' in two variables. Our main goal, however, is to study the *interplay* between both pictures; at this point, Fourier analysis on the quantum $E(2)$ group will emerge (see §5.6 and chapter 6). ⋆

5.5.1 Functional calculus

In order to host 'functions' of the generators of $\mathcal{A}_q$, we shall first replace $\mathcal{A}_q$ by a larger *-algebra $\mathcal{A}_q^{\text{ext}}$. From (5.3) it is clear that any formal series

$$\sum_{\substack{l\in\mathbb{Z}\\ \text{finitely many}}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w_{\,l,m,n}\,\alpha^l\beta^m\gamma^n \qquad (w_{\,l,m,n}\in\mathbb{C}) \tag{5.26}$$

can be interpreted in a natural way as a well-defined element in $\mathcal{U}_q'$. Moreover, the commutation rules between the generators of $\mathcal{A}_q$ imply that the formal product and * of such formal series is again of this form; the resulting *-algebra will be denoted by $\mathcal{A}_q^{\text{ext}}$. Recall that $\mathcal{U}_q'$, being the algebraic dual of a Hopf *-algebra, is a *-algebra as well. Now it is not so hard to prove that we indeed have *-algebra embeddings $\mathcal{A}_q \hookrightarrow \mathcal{A}_q^{\text{ext}} \hookrightarrow \mathcal{U}_q'$. Furthermore, also the antipode on $\mathcal{A}_q$ extends to a map $S:\mathcal{A}_q^{\text{ext}}\to\mathcal{A}_q^{\text{ext}}$ in a straightforward manner.

The space of elements of type (5.26) with $w_{\,l,m,n}=0$ whenever $l+m+n$ is odd obviously constitutes a subalgebra of $\mathcal{A}_q^{\text{ext}}$. This subalgebra of 'even' elements in $\mathcal{A}_q^{\text{ext}}$ shall be denoted by $\mathcal{A}_q^{\text{even}}$. In [12, 13] it was explained that $\mathcal{A}_q^{\text{ext}}$ should in fact be considered as a quantized algebra of functions on a *double cover* of the $E(2)$ group, whereas $\mathcal{A}_q^{\text{even}}$ corresponds to the actual quantum $E(2)$ group.

Definition 5.5.1.1 Whenever $\mathcal{G}$ is a subspace of $H(\mathbb{C})$ we define the following

[4] in [12, 13] a Schwartz-like space was chosen.

subspaces of $\mathcal{A}_q^{\rm ext}$:

$$\begin{aligned} \mathcal{A}_q(\mathcal{G}) &= \operatorname{span}\left\{ \begin{array}{l} \alpha^l \gamma^m \, g(\gamma^*\gamma) \\ \alpha^l \, (\gamma^*)^m \, g(\gamma^*\gamma) \end{array} \;\middle|\; \begin{array}{l} l \in \mathbb{Z},\ m \in \mathbb{N}, \\ g \in \mathcal{G} \end{array} \right\} \\ \mathcal{A}_q^{\rm even}(\mathcal{G}) &= \mathcal{A}_q(\mathcal{G}) \cap \mathcal{A}_q^{\rm even}. \end{aligned}$$

Here, of course, $g(\gamma^*\gamma)$ is merely a transparent way to write the formal series

$$g(\gamma^*\gamma) \;=\; \sum_{n=0}^{\infty} \mu_n(g)\,(\gamma^*\gamma)^n \;=\; \sum_{n=0}^{\infty} (-q^{-1})^n\, \mu_n(g)\, \beta^n \gamma^n \tag{5.27}$$

which is indeed of type (5.26) and obviously the 'functional calculus'

$$H(\mathbb{C}) \to \mathcal{A}_q^{\rm ext} \,:\, g \mapsto g(\gamma^*\gamma)$$

is an injective *-algebra morphism. Furthermore it is straightforward to show

Lemma 5.5.1.2 *For all $g \in \mathcal{G}$, $l \in \mathbb{Z}$ and $m \in \mathbb{N}$,*

$$\alpha^l \, g(\gamma^*\gamma) \;=\; (\Omega^{2l} g)(\gamma^*\gamma)\, \alpha^l$$

$$(\gamma^*\gamma)^m \, g(\gamma^*\gamma) \;=\; (\Psi^m g)(\gamma^*\gamma) \;=\; g(\gamma^*\gamma)\,(\gamma^*\gamma)^m.$$

$g(\gamma^*\gamma)$ commutes with γ and γ^*.

Lemma 5.5.1.3 *$\mathcal{A}_q(\mathcal{G})$ can be considered as a direct sum:*

$$\bigoplus_{l\in\mathbb{Z}} \left\{ \left(\bigoplus_{m=1}^{\infty} \alpha^l \gamma^m \, \mathcal{G}(\gamma^*\gamma) \right) \oplus \alpha^l \, \mathcal{G}(\gamma^*\gamma) \oplus \left(\bigoplus_{m=1}^{\infty} \alpha^l (\gamma^*)^m \, \mathcal{G}(\gamma^*\gamma) \right) \right\}.$$

Proof. This is clear from (5.3) and (5.27). ∎

Corollary 5.5.1.4 Using this direct sum structure and (5.27) it follows that

$$\mathcal{A}_q^{\rm even}(\mathcal{G}) \;=\; \operatorname{span}\left\{ \begin{array}{l} \alpha^l \gamma^m \, g(\gamma^*\gamma) \\ \alpha^l \, (\gamma^*)^m \, g(\gamma^*\gamma) \end{array} \;\middle|\; \begin{array}{l} l \in \mathbb{Z},\ m \in \mathbb{N} \ \text{ with } l+m \text{ even}, \\ g \in \mathcal{G} \end{array} \right\}.$$

The following summarizes some results which were already present in [12, 13]:

Proposition 5.5.1.5 *If $\mathcal{G}$ is a non-trivial self-adjoint subspace of $H(\mathbb{C})$ which is invariant under $\Omega^{\pm 2}$, Ψ and D_{q^2}, then $\mathcal{A}_q(\mathcal{G})$ is*

i. a sub-$\mathcal{U}_q$-bimodule of $\mathcal{U}_q'$ under canonical actions (2.1)

ii. an $\mathcal{A}_q$-bimodule under multiplication within $\mathcal{A}_q^{\rm ext}$

iii. invariant under $S^{\pm 1}$ and *

iv. weakly dense in $\mathcal{U}_q'$ (i.e. separates $\mathcal{U}_q$ within the duality).

If moreover $\mathcal{G}$ is an algebra, then so is $\mathcal{A}_q(\mathcal{G})$.

Proof. (i) is shown through explicit calculation [12, 13] of the actions of $\mathcal{U}_q$ on $\mathcal{A}_q(\mathcal{G})$, involving $\{\Omega^{\pm 2}, D_{q^2}\}$-invariance of $\mathcal{G}$. The proof of (ii-iii) relies on lemma 5.5.1.2. (ii) requires $\{\Omega^{\pm 2}, \Psi\}$-invariance, whereas (iii) only involves self-adjointness and $\{\Omega^{\pm 2}\}$-invariance. (iv) is similar to lemma B.1. The 'moreover' part follows from the fact that our functional calculus is a *-algebra morphism and involves $\{\Omega^{\pm 2}, \Psi\}$-invariance of $\mathcal{G}$. ∎

Corollary 5.5.1.6 *Under the assumptions of the previous proposition, including the 'moreover' part, $\langle \underline{\mathcal{U}_q}\, , \mathcal{A}_q(\mathcal{G}) \rangle$ is an algebraic Hopf *-system*[5].

The antipode Under the assumptions of proposition 5.5.1.5, the antipode S on $\mathcal{A}_q(\mathcal{G})$ is actually a bijection. Furthermore it is straightforward to prove that

$$\begin{aligned}
S\big(\alpha^l \gamma^m\, g(\gamma^*\gamma)\big) &= (-q)^m\, q^{ml}\; \alpha^{-l}\gamma^m\, (\Omega^{2l} g)(\gamma^*\gamma) \\
S\big(\alpha^l (\gamma^*)^m\, g(\gamma^*\gamma)\big) &= (-q)^{-m}\, q^{ml}\; \alpha^{-l}(\gamma^*)^m\, (\Omega^{2l} g)(\gamma^*\gamma) \\
S^2\big(\alpha^l \gamma^m\, g(\gamma^*\gamma)\big) &= q^{2m}\, \alpha^l \gamma^m\, g(\gamma^*\gamma) \\
S^2\big(\alpha^l (\gamma^*)^m\, g(\gamma^*\gamma)\big) &= q^{-2m}\, \alpha^l (\gamma^*)^m\, g(\gamma^*\gamma)
\end{aligned}$$

5.5.2 The Haar functional

Throughout this paragraph we fix a real number ν with $\nu > 0$. Furthermore, let $\mathcal{G}_\nu$ be any sub*algebra* of $H(\mathbb{C})$ satisfying assumptions 5.4.1.1.

Definition 5.5.2.1 In view of lemma 5.5.1.3 it is possible to define a functional ω on $\mathcal{A}_q(\mathcal{G}_\nu)$ as follows: for any $l \in \mathbb{Z}$, $m \in \mathbb{N}$ and $g \in \mathcal{G}_\nu$, we set

$$\begin{aligned}
&\left.\begin{aligned} &\omega\big(\alpha^l \gamma^m\, g(\gamma^*\gamma)\big) = 0 \\ &\omega\big(\alpha^l (\gamma^*)^m\, g(\gamma^*\gamma)\big) = 0 \end{aligned}\right\} \qquad \text{if either } l \neq 0 \text{ or } m \neq 0 \\
&\omega\big(g(\gamma^*\gamma)\big) \;=\; \sum_{k\in\mathbb{Z}} g(\nu q^{2k})\, q^{2k} \qquad \text{otherwise.}
\end{aligned} \tag{5.28}$$

Proposition 5.5.2.2 *The functional ω as defined above is both left and right invariant (see definition 3.6.1 and corollary 5.5.1.6). Moreover, ω is hermitian, positive and faithful.*

Proposition 5.5.2.3 *ω is weakly KMS. The KMS-automorphism σ_ω is given by $\sigma_\omega = a^{-2} \triangleright (\,\cdot\,) \triangleleft a^{-2}$. Explicitly, for $g \in \mathcal{G}_\nu$, $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have:*

$$\sigma_\omega\Big(\alpha^l\, (\gamma \textit{ resp. } \gamma^*)^m\, g(\gamma^*\gamma)\Big) \;=\; q^{-2l}\, \alpha^l\, (\gamma \textit{ resp. } \gamma^*)^m\, g(\gamma^*\gamma).$$

Moreover ω is $\mathcal{A}_q$-KMS in the sense that $\omega(y\eta) = \omega(\eta\, \sigma_\omega(y))$ for any $\eta \in \mathcal{A}_q$ and $y \in \mathcal{A}_q(\mathcal{G}_\nu)$. Furthermore we have $\omega S = \omega$, and hence the modular element associated with ω is trivial.

[5] See definition 3.7.2.1.

Proof. The KMS-automorphism was already computed in [12, 13]. Let's have a closer look at the $\mathcal{A}_q$-KMS property. We have to consider $\eta = \alpha^p \beta^r \gamma^s$ and $y = \alpha^l\,(\gamma \text{ or } \gamma^*)^m\, g(\gamma^*\gamma)$ for any $g \in \mathcal{G}_\nu$, $l, p \in \mathbb{Z}$ and $m, r, s \in \mathbb{N}$. We have e.g.

$$
\begin{aligned}
&\omega\Big(\alpha^l \gamma^m g(\gamma^*\gamma)\ \alpha^p \beta^r \gamma^s\Big) \\
&\quad= \omega\Big(\alpha^l \gamma^m \alpha^p\, (\Omega^{-2p} g)(\gamma^*\gamma)\, \beta^r \gamma^s\Big) \\
&\quad= q^{-pm}\, \omega\Big(\alpha^{l+p} \beta^r \gamma^{m+s}\, (\Omega^{-2p} g)(\gamma^*\gamma)\Big) \\
&\quad= \delta_{l+p,0}\, \delta_{r,m+s}\, q^{-pm}\, \omega\Big((-q\gamma^*\gamma)^r\, (\Omega^{-2p} g)(\gamma^*\gamma)\Big) \\
&\quad= \delta_{l+p,0}\, \delta_{r,m+s}\, q^{-pm}\, (-q)^r\, \omega\big((\Psi^r \Omega^{-2p} g)(\gamma^*\gamma)\big) \\
&\quad= \delta_{l+p,0}\, \delta_{r,m+s}\, q^{-pm}\, (-q)^r\, q^{2pr}\, \omega\big((\Omega^{-2p} \Psi^r g)(\gamma^*\gamma)\big) \\
&\quad\overset{(*)}{=} \delta_{l+p,0}\, \delta_{r,m+s}\, q^{-pm}\, (-q)^r\, q^{2pr}\, q^{2p}\, \omega\big((\Psi^r g)(\gamma^*\gamma)\big) \\
&\quad= \delta_{l+p,0}\, \delta_{r,m+s}\, q^{-pm}\, q^{2pr}\, q^{2p}\, \omega\big((-q\gamma^*\gamma)^r\, g(\gamma^*\gamma)\big) \\
&\quad= \delta_{l+p,0}\, \delta_{r,m+s}\, q^{-(-l)(r-s)}\, q^{2(-l)r}\, q^{2(-l)}\, \omega\big(\beta^r \gamma^{m+s}\, g(\gamma^*\gamma)\big) \\
&\quad= \delta_{l+p,0}\, q^{-l(r+s)}\, q^{-2l}\, \omega\big(\beta^r \gamma^{m+s}\, g(\gamma^*\gamma)\big) \\
&\quad= q^{-l(r+s)}\, q^{-2l}\, \omega\Big(\alpha^{l+p} \beta^r \gamma^s\, \gamma^m\, g(\gamma^*\gamma)\Big) \\
&\quad= q^{-2l}\, \omega\Big(\alpha^p \beta^r \gamma^s \alpha^l \gamma^m\, g(\gamma^*\gamma)\Big) \\
&\quad= \omega\Big(\alpha^p \beta^r \gamma^s\, \sigma_\omega\big(\alpha^l \gamma^m\, g(\gamma^*\gamma)\big)\Big).
\end{aligned}
$$

In $(*)$ we used the following property of the Haar functional:

$$
\omega\big((\Omega^{2n} g)(\gamma^*\gamma)\big) \;=\; q^{-2n}\, \omega\big(g(\gamma^*\gamma)\big)
$$

for $g \in \mathcal{G}_\nu$ and $n \in \mathbb{Z}$. The last statement (i.e. $\omega S = \omega$) follows immediately from the formulas for the antipode on $\mathcal{A}_q(\mathcal{G}_\nu)$. ∎

5.6 Towards a Fourier context for quantum $E(2)$

Abstract 5.6 We give a short preview on chapter 6, in which shall we study harmonic analysis on the quantum $E(2)$ group, in terms the objects introduced in the present chapter. But first we combine the results of the previous sections in the following

Theorem 5.6.1 *Fix real numbers q, τ, ν with $0 < q < 1$ and $\tau, \nu > 0$, and let $\mathcal{G}_\tau$, $\mathcal{G}'_\tau$, $\mathcal{G}_\nu$ and $\mathcal{G}'_\nu$ be subspaces of $H(\mathbb{C})$ satisfying assumptions 5.4.1.1. Moreover assume that*

i. $\mathcal{G}'_\tau \subseteq \mathcal{G}_\tau$ *and* $\mathcal{G}'_\nu \subseteq \mathcal{G}_\nu$

ii. $\mathcal{G}_\tau$ *and* $\mathcal{G}_\nu$ *are subalgebras of* $H(\mathbb{C})$

iii. $\mathcal{G}'_\tau$ *and* $\mathcal{G}'_\nu$ *have faithful moments w.r.t. Jackson* q^2*-integration [16], in the following sense: (e.g.) for any* $g \in \mathcal{G}'_\nu$ *we have*

$$\sum_{k\in\mathbb{Z}} (\Psi^m g)(\nu q^{2k})\, q^{2k} = 0 \qquad \textit{for all } \ m \in \mathbb{N} \qquad \Longrightarrow \qquad g = 0.$$

Furthermore, let φ, ψ *and* ω *be the invariant functionals on* $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$ *and* $\mathcal{A}_q(\mathcal{G}_\nu)$ *as constructed in the previous sections. Then*

$$\Big(\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}'_\tau)\big) \subseteq \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)\,, \varphi, \psi; \mathcal{U}_q, \mathcal{A}_q;\, \mathcal{A}_q(\mathcal{G}'_\nu) \subseteq \mathcal{A}_q(\mathcal{G}_\nu), \omega\Big)$$

is a Fourier context (cf. definition 4.1.1).

Proof. Most of definition 4.1.1 follows from propositions 5.1.2.2, 5.3.3.1, 5.4.1.4, 5.4.1.8, 5.4.2.5, 5.4.3.1, 5.5.1.5, 5.5.2.2, 5.5.2.3, corollary 5.5.1.6 and lemma B.1. Only item (v) of definition 4.1.1 still requires some explanation. Since this is rather technical and not very interesting, we defer its proof to appendix B. ∎

In chapter 6 we shall use the Schwartz-like spaces we defined in example 5.4.1.2, in particular $\mathcal{G}_\tau = \mathcal{S}_\tau(\mathbb{R}^+; q^2)$ and $\mathcal{G}_\nu = \mathcal{S}_\nu(\mathbb{R}^+; q^2)$. We also construct suitable subspaces $\mathcal{E}_\tau$ and $\mathcal{E}_\nu$ of 'nice' functions (e.g. functions of q^2-Exponential type). Then we construct Fourier transforms (cf. definition 4.1.2) having *little* q^2*-Bessel* functions as kernel, prove Plancherel formulas, inversion formulas etc. Eventually we obtain a Plancherel context (cf. definition 4.4.5)

$$\Big(\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big) \subseteq \mathcal{U}_q\big(\mathfrak{L}(\mathcal{S}_\tau(\mathbb{R}^+; q^2))\big)\,, \varphi, \psi; \mathcal{U}_q, \mathcal{A}_q;\, \mathcal{A}_q(\mathcal{E}_\nu) \subseteq \mathcal{A}_q\big(\mathcal{S}_\nu(\mathbb{R}^+; q^2)\big)\,, \omega\Big)$$

provided the numbers q, τ and ν satisfy particular relations.

Chapter 6

Harmonic analysis on quantum $E(2)$

6.1 Little q-Bessel functions

Abstract 6.1 Little q-Bessel functions were investigated in [17, 32] and first related to the quantum $E(2)$ group in [13, 48]. In [17] they were used as q-integral kernels in order to construct q-Hankel transformation. Little q-Bessel functions are usually defined in terms of q-hypergeometric series. For our purposes, we can rely entirely on a few elementary properties summarized below, so we shall never have to deal with their definition directly; for the sake of completeness however, we give an *ad hoc* definition below. For the definition of the q-shifted factorials and q-Exponential function we refer to appendix C. ⋆

Lemma 6.1.1 *Whenever $n \in \mathbb{Z}$ and $q \in \mathbb{R}$ with $0 < q < 1$, the power series*

$$J_n(z;q) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{\frac{1}{2}k(k-1)}\, q^k\, (q^{k+n+1};q)_\infty}{(q;q)_\infty\,(q;q)_k}\, z^{2k+n} \tag{6.1}$$

defines an entire function in z, having a zero of order $|n|$ at the origin.
This function is said to be the *little q-Bessel* function of order n.

Proof. First notice that (6.1) is indeed a *power* series—not a genuine Laurent series—when n is negative, because $(q^{k+n+1};q)_\infty = 0$ whenever $n < -k \leq 0$. So if $n < 0$, the $(k = |n|)$-term will be the first non-zero term in the series. To prove that (6.1) converges for all $z \in \mathbb{C}$, we appeal to the ratio test: fix any $z \in \mathbb{C}$ and let a_k denote the k-term of the above series, then a_k is non-zero (provided $k \geq |n|$ when n is negative) and

$$\left|\frac{a_{k+1}}{a_k}\right| \;=\; q^{k+1}\,\frac{(q^{k+n+2};q)_\infty}{(q^{k+n+1};q)_\infty}\,\frac{(q;q)_k}{(q;q)_{k+1}}\,|z|^2 \;=\; \frac{q^{k+1}\,|z|^2}{(1-q^{k+n+1})\,(1-q^{k+1})},$$

which tends to zero as $k \to \infty$, since $0 < q < 1$. ∎

Remark 6.1.2 Notice little q-Bessel functions take real values on the real line, since all the coefficients in (6.1) are real. It should be noted that it is also possible to consider little q-Bessel functions of non-integer order. However, those of integer order have some interesting properties not shared by those of non-integer order. Furthermore, given q we shall usually deal with q^2-Bessel functions henceforth—rather than with q-Bessel functions.

Proposition 6.1.3 *Little q^2-Bessel functions satisfy the following recurrence and symmetry relations, for all $n, m \in \mathbb{Z}$ and $z \in \mathbb{C}$,*

$$z\, J_{n-1}(z;q^2) = J_n(z;q^2) - q^n\, J_n(qz;q^2) \tag{6.2}$$

$$-q^{-n} z\, J_{n+1}(z;q^2) = J_n(q^{-1}z;q^2) - q^{-n} J_n(z;q^2) \tag{6.3}$$

$$J_n(q^m;q^2) = J_m(q^n;q^2) \tag{6.4}$$

$$J_{-n}(z;q^2) = (-q)^n\, J_n(q^n z;q^2) \tag{6.5}$$

and respectively Hansen-Lommel and q-Hankel orthogonality relations

$$\sum_{k\in\mathbb{Z}} q^{2k}\, J_{n+k}(z;q^2)\, J_{m+k}(z;q^2) = \delta_{n,m}\, q^{-2n} \qquad (|qz| < 1) \tag{6.6}$$

$$\sum_{k\in\mathbb{Z}} q^{2k}\, J_r(q^{n+k};q^2)\, J_r(q^{m+k};q^2) = \delta_{n,m}\, q^{-2n} \qquad (r \in \mathbb{Z}) \tag{6.7}$$

where the sums in (6.6) and (6.7) are absolutely convergent. We also mention the following estimate: for every $m \in \mathbb{Z}$ there exists a positive number C_m such that

$$\left|J_m(q^k;q^2)\right| \le C_m \begin{cases} q^{km} & (k \ge 0) \\ q^{-km}\, q^{k(k-1)} & (k \le 0) \end{cases} \tag{6.8}$$

$$\le C_m\, q^{|k|m} \tag{6.9}$$

for all $k \in \mathbb{Z}$.

Proof. The recurrence relations (6.2) and (6.3) are equivalent with formulas (4.8) and (4.6) in [15], the latter being in turn related to q-derivatives of little q-Bessel functions. It is however not too hard to derive (6.2) directly from the power series expression given in [17, proof of prop. 2.1]. The other relations can be found in [12, 13, 17, 32]. Be aware that (6.6) is only valid when $|qz| < 1$, and observe how (6.7) follows from (6.6) via (6.4) for $r \ge 0$, whereas proving the $r < 0$ case of (6.7) moreover involves (6.5). Also notice (6.5) interchanges (6.2) and (6.3). ∎

6.2 Holomorphic q-Hankel transformation

Abstract 6.2 q-analogues of Hankel transformation based on little q^2-Bessel functions first appeared explicitly in [17], whereas in [36] they were mentioned in relation to the quantum $E(2)$ group. The starting point for our theory will be the definition proposed in [17], which we will recall below—reformulated in an L^2-language. Then we study the possibility of transforming *entire* functions into entire functions, a feature which is crucial[1] for our applications to the quantum $E(2)$ group. Furthermore we investigate the behaviour of q-Hankel transformation w.r.t. q-differentiation and multiplication. Eventually we search for *eigenfunctions* of q-Hankel transformation; these turn out to be functions of q^2-Exponential type. ⋆

6.2.1 The L^2-theory

Let $\mathbb{R}_q^+ \equiv (\mathbb{R}_q^+, m_q)$ be the discrete set $\{q^k \mid k \in \mathbb{Z}\}$ endowed with the measure m_q which assigns weight q^{2k} to the point q^k. Thus integration w.r.t. m_q yields

$$\int_{\mathbb{R}_q^+} f\, dm_q \;=\; \sum_{k\in\mathbb{Z}} f(q^k)\, q^{2k} \qquad \text{for} \quad f \in L^1(\mathbb{R}_q^+).$$

Now the orthogonality relations (6.7) can be reformulated in an L^2-language:

Proposition 6.2.1.1 *Fix any $m \in \mathbb{Z}$ and define functions $e_k^{(m)}$ on $\mathbb{R}_q^+$ by*

$$e_k^{(m)}(x) \;=\; q^k\, J_m(q^k x; q^2) \qquad (k \in \mathbb{Z},\ x \in \mathbb{R}_q^+).$$

Then $(e_k^{(m)})_{k\in\mathbb{Z}}$ is an orthonormal basis (ONB) in the Hilbert space $L^2(\mathbb{R}_q^+)$.

On the other hand, we have a canonical ONB for $L^2(\mathbb{R}_q^+)$, say $(d_k)_{k\in\mathbb{Z}}$ with $d_k = q^{-k}\delta_{q^k}$. Here δ_{q^k} denotes the characteristic function of the singleton $\{q^k\}$.

Definition 6.2.1.2 Given $m \in \mathbb{Z}$, let H_m be the unitary transformation of $L^2(\mathbb{R}_q^+)$ which maps the ONB $(e_k^{(m)})_{k\in\mathbb{Z}}$ into the ONB $(d_k)_{k\in\mathbb{Z}}$. Explicitly:

$$H_m f \;=\; \sum_{k\in\mathbb{Z}} \langle f \mid e_k^{(m)} \rangle\, d_k,$$

for $f \in L^2(\mathbb{R}_q^+)$ and $k \in \mathbb{Z}$, or equivalently,

$$(H_m f)(q^k) \;=\; q^{-k}\langle f \mid e_k^{(m)} \rangle \;=\; \sum_{n\in\mathbb{Z}} q^{2n} J_m(q^{n+k}; q^2) f(q^n). \tag{6.10}$$

H_m is said to be the *q-Hankel transform* of order m. •

Proposition 6.2.1.3 $H_m^2 = \mathrm{id}$ *for all $m \in \mathbb{Z}$.*

Proof. It suffices to show that $H_m d_j = e_j^{(m)}$ for all $j \in \mathbb{Z}$. Now observe that

$$\begin{aligned}(H_m d_j)(q^k) &= \textstyle\sum_{n\in\mathbb{Z}} q^{2n}\, J_m(q^{n+k}; q^2)\, d_j(q^n) \\ &= q^{2j}\, J_m(q^{j+k}; q^2)\, q^{-j} \\ &= q^j\, J_m(q^j q^k; q^2) \\ &= e_j^{(m)}(q^k)\end{aligned}$$

for all $k \in \mathbb{Z}$. ∎

[1] in the sense that we want to stick to our algebraic description of the quantum $E(2)$ group, as outlined in chapter 5, the algebraic nature of which depends heavily on the use of entire functions in constructing the functional calculus.

6.2.2 Holomorphic q-Hankel transformation

In the previous section we considered functions living on a discrete space $\mathbb{R}_q^+$. Now we shall try to 'interpolate' these functions on $\mathbb{R}_q^+$ by entire functions, whenever possible.
But first let us recall the Schwartz-like spaces $\mathcal{S}_r(\mathbb{R}^+; q)$ of entire functions, as defined in example 5.4.1.2. In case $r = 1$, the subscript r will be suppressed in our notation. Also recall the operators Γ, Φ, Ω and Ψ introduced in §5.1.3. Let's add one more operator $K : H(\mathbb{C}) \to H(\mathbb{C})$, given by $(Kf)(z) = f(z^2)$. Then the commutation rules $\Omega K = K\Omega^2$ and $\Psi^2 K = K\Psi$ are easily verified. Also notice that K maps $\mathcal{S}(\mathbb{R}^+; q^2)$ into $\mathcal{S}(\mathbb{R}^+; q)$.
Furthermore, let R_q denote the restriction map from $H(\mathbb{C})$ into the space of all functions on $\mathbb{R}_q^+$. Since $\mathbb{R}_q^+$ admits a point of accumulation, the identity theorem for holomorphic functions implies R_q to be injective. Observe that R_q maps $\mathcal{S}(\mathbb{R}^+; q)$ into $L^2(\mathbb{R}_q^+)$. The proof of this involves the fact that entire functions are bounded on e.g. the unit interval; also recall that $\mathbb{R}_q^+ \equiv (\mathbb{R}_q^+, m_q)$ was endowed with a particular measure—*not* the counting measure.

Abstract 6.2.2.1 Our next aim is to find, for any $m \in \mathbb{Z}$, a sufficiently large subspace $\mathcal{R}_m$ of $\mathcal{S}(\mathbb{R}^+; q^2)$ together with a linear map $\mathfrak{H}_m$ from $\mathcal{R}_m$ into $\mathcal{R}_m$, such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
\mathcal{R}_m & \xrightarrow{\mathrm{id}} & \mathcal{S}(\mathbb{R}^+; q^2) & \xrightarrow{K} & \mathcal{S}(\mathbb{R}^+; q) & \xrightarrow{\Psi^{|m|}} & \mathcal{S}(\mathbb{R}^+; q) & \xrightarrow{R_q} & L^2(\mathbb{R}_q^+) \\
{\scriptstyle \mathfrak{H}_m}\big\downarrow & & & & & & & & \big\downarrow{\scriptstyle H_m} \\
\mathcal{R}_m & \xrightarrow[\mathrm{id}]{} & \mathcal{S}(\mathbb{R}^+; q^2) & \xrightarrow[K]{} & \mathcal{S}(\mathbb{R}^+; q) & \xrightarrow[\Psi^{|m|}]{} & \mathcal{S}(\mathbb{R}^+; q) & \xrightarrow[R_q]{} & L^2(\mathbb{R}_q^+)
\end{array}
$$

Diagram Holomorphic q-Hankel transform (6.11)

The map $\mathfrak{H}_m$ will then be called the *holomorphic* q-Hankel transform of order m, because it transforms entire functions into entire functions. However the above scheme involves more than merely the passage to entire functions: the operators K and $\Psi^{|m|}$ appearing in the diagram will ensure the system $(\mathfrak{H}_m)_{m\in\mathbb{Z}}$ to have the proper behaviour w.r.t. q^2-differentiation (cf. proposition 6.2.4.6). Iterating the above diagram immediately reveals that $\mathfrak{H}_m^2 = \mathrm{id}$ (recall that $H_m^2 = \mathrm{id}$, cf. proposition 6.2.1.3). ⋆

Definition 6.2.2.2 For all $m \in \mathbb{Z}$ we define $\mathcal{H}_m : \mathcal{S}(\mathbb{R}^+; q^2) \to L^2(\mathbb{R}_q^+)$ by

$$\mathcal{H}_m = H_m R_q \Psi^{|m|} K.$$

A pair (f, g) of entire functions is said to be an $(m; q)$-Hankel pair whenever

i. $f \in \mathcal{S}(\mathbb{R}^+; q^2)$ and $\mathcal{H}_m f = R_q g$

ii. g is an even (resp. odd) function whenever m is even (resp. odd)

iii. g has a zero at the origin of order at least $|m|$.

Finally we define $\mathcal{R}_m$ to be the following subspace of $\mathcal{S}(\mathbb{R}^+; q^2)$:

$$\mathcal{R}_m \;=\; \left\{ f \in \mathcal{S}(\mathbb{R}^+; q^2) \;\middle|\; \begin{array}{l} \text{there exists an entire function } g \text{ such} \\ \text{that } (f,g) \text{ is an } (m;q)\text{-Hankel pair} \end{array} \right\}$$

Remark 6.2.2.3 The maps $\mathcal{H}_m$ and the notion of an $(m;q)$-Hankel pair shall have no role in the eventual picture; however they will turn out to be very convenient in formulating some intermediate results. Roughly speaking we want to consider entire functions whose q-Hankel transforms on $\mathbb{R}_q^+$ admit interpolation by entire functions. The above approach may seem to be rather descriptive and certainly not very constructive at this moment, since it does not provide a criterion to determine whether a function f belongs to $\mathcal{R}_m$ or not—at least not without computing $\mathcal{H}_m f$ first. In this way however, we want to avoid (or at least postpone) some technical questions which do not have too high priority from our perspective. Just to give a hint of the kind of trouble we run into when we actually try to *construct* such an $(m;q)$-Hankel pair (f,g), let's write out in detail the equation $\mathcal{H}_m f = R_q g$ appearing in the above definition:

$$g(q^k) \;=\; (H_m R_q \Psi^{|m|} K f)(q^k) \;=\; \sum_{n\in\mathbb{Z}} q^{2n} J_m(q^{n+k};q^2)\, q^{n|m|} f(q^{2n}) \qquad (6.12)$$

for any $k \in \mathbb{Z}$. Formally replacing q^k with $z \in \mathbb{C}$ we obtain

$$g(z) \;=\; \sum_{n\in\mathbb{Z}} q^{2n} J_m(q^n z;q^2)\, q^{n|m|} f(q^{2n}) \qquad \text{for any } z \in \mathbb{C}. \qquad (6.13)$$

Now we want (6.13) to define an entire function—so we need the sum to converge uniformly on compact sets—and that's precisely the point where things become very tricky. Indeed there turns out to be a huge difference between (6.12) and (6.13) on the technical level. The fact is that our little q^2-Bessel functions are behaving very well as far as we evaluate them in in q-powers only, whereas they are not quite so innocent in the rest of the complex plane. For instance, the orthogonality relation (6.7) reveals that $J_m(q^{n+k};q^2)$ tends to zero as $n \to -\infty$, but in general, when $z \in \mathbb{C}$ is not a q-power, $J_m(q^n z;q^2)$ will grow very rapidly when $n \to -\infty$.
However, the 'interpolation' strategy we have chosen for will allow us to proceed quite far with only *partial* answers (like for instance lemma 6.2.5.1) to the question of convergence mentioned above, and therefore avoiding a lot of trouble. Indeed precisely those results that we are looking for will come almost for free in our approach, but of course we also pay a price: some properties one might expect to hold could be hard to prove within our setting (cf. remark 6.2.6.2). Let's conclude the present remark by stating that we *won't* give a full explicit description of the spaces $\mathcal{R}_m$, whereas we *will* show—constructively—that they contain plenty of elements. ⋆

6.2.3 q-Hankel transforms and q-differentiation

Recall the difference operators D_{q^2} and $\nabla_q^{(m)}$ on $H(\mathbb{C})$ emerging in the formulas of proposition 5.3.2.2. Notice that $\mathcal{S}(\mathbb{R}^+;q^2)$ is not $\nabla_q^{(m)}$-invariant, though it certainly is $\nabla_q^{(m)}\Omega$-invariant. Furthermore $\mathcal{S}(\mathbb{R}^+;q^2)$ is D_{q^2}-invariant, as was already observed in example 5.4.1.2. Therefore the following makes sense:

Lemma 6.2.3.1 *For all $f \in \mathcal{S}(\mathbb{R}^+;q^2)$ and $m,k \in \mathbb{Z}$ with $m \geq 1$ we have*

$$(\mathcal{H}_m D_{q^2} f)(q^k) = -\frac{q^{-1}}{q^{-1}-q}\, q^k\, (\mathcal{H}_{m-1}\Omega^2 f)(q^k) \tag{6.14}$$

$$(\mathcal{H}_{m-1}\nabla_q^{(m)}\Omega f)(q^k) = \frac{1}{q^{-1}-q}\, q^{-m}\, q^k\, (\mathcal{H}_m f)(q^k) \tag{6.15}$$

$$(\mathcal{H}_{-m} D_{q^2} f)(q^k) = \frac{q^{-1}}{q^{-1}-q}\, q^k\, (\mathcal{H}_{-m+1} f)(q^k) \tag{6.16}$$

$$(\mathcal{H}_{-m+1}\nabla_q^{(m)}\Omega f)(q^k) = -\frac{1}{q^{-1}-q}\, q^m\, q^k\, (\mathcal{H}_{-m}\Omega^2 f)(q^k) \tag{6.17}$$

Proof. Take any $f \in \mathcal{S}(\mathbb{R}^+;q^2)$. Assuming $m \geq 1$, we have for all $k \in \mathbb{Z}$

$$\begin{aligned}
&(\mathcal{H}_m D_{q^2} f)(q^k)\\
&= (H_m R_q \Psi^{|m|} K D_{q^2} f)(q^k)\\
&= \sum_{n\in\mathbb{Z}} q^{2n} J_m(q^{n+k};q^2)\,(R_q\Psi^{|m|}KD_{q^2}f)(q^n)\\
&= \sum_{n\in\mathbb{Z}} q^{2n} J_m(q^{n+k};q^2)\, q^{nm}\,(D_{q^2}f)(q^{2n})\\
&= \sum_{n\in\mathbb{Z}} q^{2n} J_m(q^{n+k};q^2)\, q^{nm}\,\frac{f(q^{2n})-f(q^2q^{2n})}{(1-q^2)q^{2n}}\\
&\overset{(\sharp)}{=} \frac{1}{1-q^2}\Bigg(\sum_{n\in\mathbb{Z}} J_m(q^{n+k};q^2)\, q^{nm}\, f(q^{2n})\\
&\qquad\qquad - \sum_{n\in\mathbb{Z}} J_m(q^{n+k};q^2)\, q^{nm}\, f\big(q^{2n+2}\big)\Bigg)\\
&\overset{(*)}{=} \frac{1}{1-q^2}\sum_{n\in\mathbb{Z}}\Big(q^m J_m(q^{n+1+k};q^2) - J_m(q^{n+k};q^2)\Big)\, q^{nm} f\big(q^{2n+2}\big)\\
&\overset{(6.2)}{=} -\frac{1}{1-q^2}\sum_{n\in\mathbb{Z}} q^{n+k} J_{m-1}(q^{n+k};q^2)\, q^{nm}\,(\Omega^2 f)(q^{2n})\\
&= -\frac{1}{1-q^2}\, q^k \sum_{n\in\mathbb{Z}} q^{2n} J_{m-1}(q^{n+k};q^2)\, q^{n(m-1)}\,(\Omega^2 f)(q^{2n})\\
&= -\frac{q^{-1}}{q^{-1}-q}\, q^k\,(\mathcal{H}_{m-1}\Omega^2 f)(q^k).
\end{aligned}$$

$(*)$ relies on replacing n by $n+1$ in only *one* of the two summations in the RHS of $(\sharp)$. In this respect it is crucial (!) that both sums in themselves converge absolutely[2]. To see this, consider the following: as far as $n \to +\infty$ is concerned, absolute summability follows from the fact that f and $J_m(\,\cdot\,; q^2)$ are bounded on a neighborhood of the origin, together with the assumption that $m \geq 1$. To deal with $n \to -\infty$, we have to use that Kf is in the Schwartz-like space $\mathcal{S}(\mathbb{R}^+; q)$, which roughly means that $f(q^{2n}) = (Kf)(q^n)$ tends to zero very rapidly when $n \to -\infty$. Furthermore we need some bound for $J_m(q^{n+k}; q^2)$ when $n \to -\infty$, which can easily be obtained from the orthogonality relations (6.7). We conclude that $(\sharp)$ indeed constitutes a valid operation. In the above computation we also used recurrence relation (6.2) of proposition 6.1.3. This proves (6.14).
The proof of (6.17) proceeds similarly:

$$
\begin{aligned}
&\left(\mathcal{H}_{-m+1}\nabla_q^{(m)}\Omega f\right)(q^k) \\
&\quad= \left(H_{-m+1}R_q\Psi^{|-m+1|}K\nabla_q^{(m)}\Omega f\right)(q^k) \\
&\quad= \sum_{n\in\mathbb{Z}} q^{2n} J_{-m+1}(q^{n+k};q^2)\, q^{n(m-1)} \left(\nabla_q^{(m)}\Omega f\right)(q^{2n}) \\
&\quad= \frac{1}{q-q^{-1}} \sum_{n\in\mathbb{Z}} q^{2n} J_{-m+1}(q^{n+k};q^2)\, q^{n(m-1)} \Big(q^m f\left(q^2 q^{2n}\right) - q^{-m} f(q^{2n})\Big) \\
&\quad= \frac{1}{q-q^{-1}} \sum_{n\in\mathbb{Z}} q^{2n}\, q^{n(m-1)} \Big(q^m J_{-m+1}(q^{n+k};q^2) \\
&\qquad\qquad - qJ_{-m+1}(q^{n+1+k};q^2)\Big)\, f\!\left(q^{2n+2}\right) \\
&\quad= \frac{1}{q-q^{-1}} \sum_{n\in\mathbb{Z}} q^{2n}\, q^{n(m-1)}\, q^m \Big(J_{-m+1}(q^{n+k};q^2) \\
&\qquad\qquad - q^{-m+1} J_{-m+1}(q^{n+1+k};q^2)\Big)\, (\Omega^2 f)(q^{2n}) \\
&\quad\overset{(6.2)}{=} \frac{1}{q-q^{-1}} \sum_{n\in\mathbb{Z}} q^{2n}\, q^{n(m-1)}\, q^m\, q^{n+k} J_{-m}(q^{n+k};q^2)\, (\Omega^2 f)(q^{2n}) \\
&\quad= -\frac{1}{q^{-1}-q}\, q^m\, q^k\, (\mathcal{H}_{-m}\Omega^2 f)(q^k).
\end{aligned}
$$

Formulas (6.15) and (6.16) are shown similarly, this time relying on recurrence relation (6.3). ∎

6.2.4 Construction and basic properties

We are about to construct the holomorphic q-Hankel transform announced in abstract 6.2.2.1. We also translate the intermediate results of lemma 6.2.3.1 into there final form (proposition 6.2.4.6) and prove some basic properties.

[2]To appreciate this, do consider for a moment the $m = 0$ case which we are excluding in the text above. Indeed, if m were zero, the sums in the RHS of $(\sharp)$ would actually *diverge* as $n \to +\infty$, unless $f(0) = 0$, since $J_0(0; q^2) \neq 0$. Nevertheless the LHS of $(\sharp)$ would still exist.

Lemma 6.2.4.1 *Take any $m \in \mathbb{Z}$. If (f,g) is an $(m;q)$-Hankel pair, then g belongs to $\mathcal{S}(\mathbb{R}^+;q)$.*

Proof. First consider the case $m \geq 0$. Let (f,g) be an $(m;q)$-Hankel pair and take any $n \in \mathbb{N}$. Iterating equation (6.14) n times yields, for all $k \in \mathbb{Z}$,

$$\left(\mathcal{H}_{m+n}\left(D_{q^2}\Omega^{-2}\right)^n f\right)(q^k) \;=\; \left(-\frac{q^{-1}}{q^{-1}-q}\right)^n q^{nk}\left(\mathcal{H}_m f\right)(q^k).$$

It follows that for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$

$$q^{nk} g(q^k) \;=\; (q^2-1)^n \left(\mathcal{H}_{m+n} f_n\right)(q^k)$$

where $f_n = (D_{q^2}\Omega^{-2})^n f$. These f_n are still in $\mathcal{S}(\mathbb{R}^+;q^2)$ because $\mathcal{S}(\mathbb{R}^+;q^2)$ is invariant under $\Omega^{\pm 2}$ and D_{q^2}. Since $\mathcal{H}_{m+n}$ maps $\mathcal{S}(\mathbb{R}^+;q^2)$ into $L^2(\mathbb{R}_q^+)$, we have

$$\sum_{k\in\mathbb{Z}} |q^{nk} g(q^k)|^2\, q^{2k} \;<\; \infty$$

for any $n \in \mathbb{N}$. From this one can easily derive that g belongs to $\mathcal{S}(\mathbb{R}^+;q)$.
In case m is negative we proceed analogously, but now using (6.16). ∎

Proposition 6.2.4.2 *For any $m \in \mathbb{Z}$ there exists a unique linear map $\mathfrak{H}_m$ from $\mathcal{R}_m$ into $\mathcal{R}_m$, such that diagram (6.11) in abstract 6.2.2.1 commutes. In particular, if (f,g) is an $(m;q)$-Hankel pair, then $f \in \mathcal{R}_m$ and $\Psi^{|m|}K\mathfrak{H}_m f = g$. Furthermore we have $\mathfrak{H}_m^2 = \mathrm{id}$.*

Proof. Uniqueness is obvious (recall that R_q is injective). To prove existence, take any $m \in \mathbb{Z}$ and $f \in \mathcal{R}_m$. By definition there exists an entire function g such that (f,g) is an $(m;q)$-Hankel pair. Now the function

$$\mathbb{C}_0 \to \mathbb{C} : z \mapsto \frac{1}{z^{|m|}}\, g(z)$$

is holomorphic, and has a removable singularity at the origin because of (iii) in definition 6.2.2.2. Furthermore this function will always be *even* because of (ii) in definition 6.2.2.2. Hence there exists an entire function, denoted $\mathfrak{H}_m f$, such that

$$(\mathfrak{H}_m f)(z^2) \;=\; \frac{1}{z^{|m|}}\, g(z) \qquad\qquad (z \in \mathbb{C}_0)$$

or equivalently, $\Psi^{|m|}K\mathfrak{H}_m f = g$. Lemma 6.2.4.1 yields that $g \in \mathcal{S}(\mathbb{R}^+;q)$ and then it follows easily that $\mathfrak{H}_m f$ belongs to $\mathcal{S}(\mathbb{R}^+;q^2)$. But we actually want $\mathfrak{H}_m f \in \mathcal{R}_m$, and in this respect we claim that $(\mathfrak{H}_m f, \Psi^{|m|}Kf)$ is again an $(m;q)$-Hankel pair. Indeed, items (ii-iii) of definition 6.2.2.2 are obviously fulfilled; to complete (i), observe that

$$\begin{aligned}
\mathcal{H}_m \mathfrak{H}_m f &= H_m R_q \Psi^{|m|} K \mathfrak{H}_m f \\
&= H_m R_q g \\
&= H_m \mathcal{H}_m f \\
&= H_m^2 R_q \Psi^{|m|} K f \\
&= R_q \Psi^{|m|} K f.
\end{aligned}$$

We used that $H_m^2 = \mathrm{id}$ (proposition 6.2.1.3). Thus we have constructed a linear map $\mathfrak{H}_m$ from $\mathcal{R}_m$ into $\mathcal{R}_m$ satisfying all our requirements. ■

Proposition 6.2.4.3 *For any $m \in \mathbb{Z}$, the space $\mathcal{R}_m$ is $\Omega^{\pm 2}$-invariant and*

$$\mathfrak{H}_m \Omega^2 \;=\; q^{-2(|m|+1)}\, \Omega^{-2} \mathfrak{H}_m. \tag{6.18}$$

Proof. Let (f,g) be any $(m;q)$-Hankel pair; we claim that $(\Omega^2 f,\, q^{-|m|-2}\Omega^{-1} g)$ is an $(m;q)$-Hankel pair as well. First observe $\mathcal{S}(\mathbb{R}^+; q^2)$ is $\Omega^{\pm 2}$-invariant. Now from $\Psi^{|m|} K \Omega^2 = q^{-|m|} \Omega \Psi^{|m|} K$ it follows that, for all $k \in \mathbb{Z}$,

$$\begin{aligned}
(\mathcal{H}_m \Omega^2 f)(q^k) &= (H_m R_q \Psi^{|m|} K \Omega^2 f)(q^k) \\
&= q^{-|m|}\, (H_m R_q \Omega \Psi^{|m|} K f)(q^k) \\
&= q^{-|m|} \sum_{n \in \mathbb{Z}} q^{2n} J_m(q^{n+k}; q^2)\, (R_q \Omega \Psi^{|m|} K f)(q^n) \\
&\overset{(*)}{=} q^{-|m|} \sum_{n \in \mathbb{Z}} q^{2(n-1)} J_m(q^{n-1+k}; q^2)\, (\Psi^{|m|} K f)(q^{n-1} q) \\
&= q^{-|m|-2}\, (\mathcal{H}_m f)(q^{k-1}) \\
&= q^{-|m|-2}\, (\Omega^{-1} g)(q^k).
\end{aligned}$$

In (*) we replaced the summation index n by $n-1$. This proves item (i) of definition 6.2.2.2, whereas items (ii-iii) are obvious here. Proposition 6.2.4.2 yields that $\Omega^2 f$ belongs to $\mathcal{R}_m$ and

$$\begin{aligned}
\Psi^{|m|} K \mathfrak{H}_m \Omega^2 f &= q^{-|m|-2}\, \Omega^{-1} g \\
&= q^{-|m|-2}\, \Omega^{-1} \Psi^{|m|} K \mathfrak{H}_m f \\
&= q^{-|m|-2}\, q^{-|m|}\, \Psi^{|m|} K \Omega^{-2} \mathfrak{H}_m f.
\end{aligned}$$

We used the commutation rules for Ψ, K, and Ω. Canceling $\Psi^{|m|} K$ yields the result. ■

Recall that $H(\mathbb{C})$ is endowed with a *-operation $\tilde{}$ (cf. §1.4).

Proposition 6.2.4.4 *Take any $m \in \mathbb{Z}$. The space $\mathcal{R}_m$ is $\tilde{}$-invariant. In fact, $\tilde{}$ commutes with $\mathfrak{H}_m$.*

Proof. Observe that $\tilde{}$ commutes with Ψ and K. Furthermore H_m obviously commutes with complex conjugation, since little q-Bessel functions take real values on the real line. We also have $R_q \tilde{f} = \overline{R_q f}$ for any $f \in H(\mathbb{C})$. It is clear that if (f,g) is an $(m;q)$-Hankel pair, then so is $(\tilde{f}, \tilde{g})$. The result now follows easily from proposition 6.2.4.2. ■

Proposition 6.2.4.5 *Take any $m \in \mathbb{Z}$. Then $\mathcal{R}_{-m} = \mathcal{R}_m$ and*

$$\mathfrak{H}_{-m} \;=\; (-q)^m q^{m|m|}\, \Omega^{2m} \mathfrak{H}_m.$$

In this respect we also have two auxiliary results worth mentioning:

$$(H_{-m}f)(q^k) = (-q)^m (H_m f)(q^{k+m}) \tag{6.19}$$
$$(\mathcal{H}_{-m}g)(q^k) = (-q)^m (\mathcal{H}_m g)(q^{k+m}) \tag{6.20}$$

for all $f \in L^2(\mathbb{R}_q^+)$, *all* $g \in \mathcal{S}(\mathbb{R}^+; q^2)$ *and any* $k \in \mathbb{Z}$.

Proof. Equation (6.19) follows easily from (6.5), whereas (6.20) is an immediate consequence of the former. Now let (f, g) be any $(m; q)$-Hankel pair; then it is easy to see that $(f, (-q)^m \Omega^m g)$ is a $(-m; q)$-Hankel pair. Hence according to proposition 6.2.4.2 we have $f \in \mathcal{R}_{-m}$ and

$$\begin{aligned} \Psi^{|-m|} K \mathfrak{H}_{-m} f &= (-q)^m \, \Omega^m g \\ &= (-q)^m \, \Omega^m \Psi^{|m|} K \mathfrak{H}_m f \\ &= (-q)^m \, q^{m|m|} \, \Psi^{|m|} K \Omega^{2m} \mathfrak{H}_m f. \end{aligned}$$

Canceling $\Psi^{|m|} K$ yields the result. ∎

The next proposition will play a key-role henceforth:

Proposition 6.2.4.6 *Take any* $m \in \mathbb{N}$ *with* $m \geq 1$. *Then*

$$\begin{aligned} D_{q^2} \mathcal{R}_{m-1} &\subseteq \mathcal{R}_m \\ \nabla_q^{(m)} \Omega \, \mathcal{R}_m &\subseteq \mathcal{R}_{m-1} \\ D_{q^2} \mathcal{R}_{-m+1} &\subseteq \mathcal{R}_{-m} \\ \nabla_q^{(m)} \Omega^{-1} \mathcal{R}_{-m} &\subseteq \mathcal{R}_{-m+1} \end{aligned}$$

and accordingly:

$$\begin{aligned} \mathfrak{H}_m D_{q^2} f &= -\frac{q^{-1}}{q^{-1} - q} \, \mathfrak{H}_{m-1} \Omega^2 f & (f \in \mathcal{R}_{m-1}) \\ \mathfrak{H}_{m-1} \nabla_q^{(m)} \Omega f &= \frac{1}{q^{-1} - q} \, q^{-m} \, \Psi \mathfrak{H}_m f & (f \in \mathcal{R}_m) \\ \mathfrak{H}_{-m} D_{q^2} f &= \frac{q^{-1}}{q^{-1} - q} \, \mathfrak{H}_{-m+1} f & (f \in \mathcal{R}_{-m+1}) \\ \mathfrak{H}_{-m+1} \nabla_q^{(m)} \Omega^{-1} f &= -\frac{1}{q^{-1} - q} \, q^m \, \Psi \mathfrak{H}_{-m} f & (f \in \mathcal{R}_{-m}) \end{aligned}$$

Proof. Take any $f \in \mathcal{R}_{m-1}$. Then according to proposition 6.2.4.3, also $\Omega^2 f$ belongs to $\mathcal{R}_{m-1}$. Now let $(\Omega^2 f, g)$ be the corresponding $(m-1; q)$-Hankel pair. With formula (6.14) of lemma 6.2.3.1 it follows easily that

$$\left(D_{q^2} f, \, -\frac{q^{-1}}{q^{-1} - q} \, \Psi g \right)$$

is an $(m;q)$-Hankel pair, and hence $D_{q^2}f$ belongs to $\mathcal{R}_m$. Furthermore, since m and $m-1$ are positive, we have (cf. proposition 6.2.4.2)

$$\Psi^m K \mathfrak{H}_m D_{q^2} f \;=\; -\frac{q^{-1}}{q^{-1}-q}\,\Psi g \;=\; -\frac{q^{-1}}{q^{-1}-q}\,\Psi\Psi^{m-1}K\mathfrak{H}_{m-1}\Omega^2 f$$

Canceling $\Psi^m K$ yields the first formula. The other formulas are more or less analogous—let's also have a look at the last one: take any $f \in \mathcal{R}_{-m}$ and let (f,g) be a $(-m;q)$-Hankel pair. From (6.17) we obtain that

$$\left(\nabla_q^{(m)}\Omega^{-1}f,\; -\frac{1}{q^{-1}-q}\,q^m\,\Psi g\right)$$

is a $(-m+1;q)$-Hankel pair, hence $\nabla_q^{(m)}\Omega^{-1}f \in \mathcal{R}_{-m+1}$ and

$$\begin{aligned}
\Psi^{|-m+1|}K\mathfrak{H}_{-m+1}\nabla_q^{(m)}\Omega^{-1}f &= -\frac{1}{q^{-1}-q}\,q^m\,\Psi g \\
&= -\frac{1}{q^{-1}-q}\,q^m\,\underbrace{\Psi\Psi^{|-m|}K}_{\Psi^{m-1}K\Psi}\mathfrak{H}_{-m}f
\end{aligned}$$

Canceling $\Psi^{m-1}K$ yields the last formula. ■

Remark 6.2.4.7 It is instructive to observe how the combination of propositions 6.2.4.5 and 6.2.4.3 transforms the first two formulas of proposition 6.2.4.6 into the remaining two and vice versa. In a similar way (6.20) interchanges (6.14-6.16) as well as (6.15-6.17). ⋆

6.2.5 The spaces $\mathcal{R}_m$ do contain many functions

The above theory would of course collapse in case the spaces $\mathcal{R}_m$ would turn out to be trivial; fortunately we have the following:

Lemma 6.2.5.1 *Let f be an entire function having the property that there exists an integer n_0 such that $f(q^{2n}) = 0$ for all integers $n < n_0$. Then $f \in \mathcal{R}_m$ for any $m \in \mathbb{Z}$.*

Proof. Observe that our assumptions imply f to be in $\mathcal{S}(\mathbb{R}^+;q^2)$ in the first place. Then fix any $m \in \mathbb{Z}$. We claim that the series

$$g(z) \;=\; \frac{1}{z^{|m|}}\sum_{n\in\mathbb{Z}} q^{2n}\,J_m(q^n z;q^2)\,q^{|m|n}\,f(q^{2n}) \qquad (6.21)$$

converges[3] absolutely for all $z \in \mathbb{C}_0$ and defines an *entire* function g (the singularity at the origin being removable). Once this fact established, it will be

[3]Notice that convergence in (6.21) is quite a different matter than for instance in (6.10). Indeed in (6.21) our q-Bessel functions are to be evaluated in any complex number, whereas (6.10) only deals with powers of q. See also remark 6.2.2.3

easy to see that $(f, \Psi^{|m|} g)$ is an $(m;q)$-Hankel pair, and hence $f \in \mathcal{R}_m$. So let's investigate the above series; according to lemma 6.1.1 it is possible to define, for any integer n, an entire function g_n such that

$$g_n(z) \;=\; (q^n z)^{-|m|}\, J_m(q^n z; q^2)\, f(q^{2n})$$

for $z \in \mathbb{C}_0$. The series (6.21) can now be rewritten as follows:

$$g \;=\; \sum_{n=n_0}^{\infty} q^{2n(|m|+1)}\, g_n \tag{6.22}$$

Since every g_n is entire, it suffices to show the latter series converges uniformly on compact sets. Therefore, pick any number $r > 0$ and let $D(r)$ denote the disk $\{z \in \mathbb{C} \mid |z| \leq r\}$. Once again appealing to lemma 6.1.1, it is clear that there exists a bound, say $M_r > 0$, such that $|x^{-|m|} J_m(x; q^2)| \leq M_r$ whenever $0 < |x| \leq q^{n_0} r$. On the other hand also f is entire, hence bounded on compact sets; so we can find a bound $N > 0$ such that $|f(x)| \leq N$ for all x in the interval $[0, q^{2n_0}]$. Since $0 < q < 1$, it follows that $|g_n(z)| \leq M_r N$ for any $n \geq n_0$ and any $z \in D(r)$. In other words, the family $\{g_n\}_{n=n_0}^{\infty}$ is uniformly bounded on $D(r)$. Since $\sum_{n=n_0}^{\infty} q^{2(|m|+1)n}$ is a convergent geometric series, (6.22) yields an entire function g which satisfies (6.21). Now we still have to show that $(f, \Psi^{|m|} g)$ is indeed an $(m;q)$-Hankel pair. Only item (ii) of definition 6.2.2.2 requires some explanation; from the power series (6.1) of the little q-Bessel functions, it is clear that $J_m(\,\cdot\,; q^2)$ is even (resp. odd) whenever m is even (resp. odd). It follows that g_n is always even (for any n) and consequently also g is even. Eventually, $\Psi^{|m|} g$ satisfies item (ii) of definition 6.2.2.2, whereas (i) is straightforward and (iii) is obvious. ∎

Remark 6.2.5.2 With a more thorough study using the proper estimates for the little q-Bessel functions involved here, it may be possible to relax the conditions on the function f a little bit, yet still ensuring the proper convergence in (6.21). From the theory of entire functions and canonical products, it is nevertheless clear that there exist plenty of functions satisfying the conditions of the previous lemma. The q-analogue of the exponential function described in appendix C provides an important example: ⋆

Corollary 6.2.5.3 *Let $E^{\bullet}_{q^2}$ denote the entire function given by*

$$E^{\bullet}_{q^2}(z) \;=\; E_{q^2}(-z) \;=\; (z;q^2)_\infty \;=\; \prod_{k=0}^{\infty} (1 - q^{2k} z).$$

Then $E^{\bullet}_{q^2} \in \mathcal{R}_m$ for any $m \in \mathbb{Z}$.

6.2.6 One single core susceptible to q-Hankel transforms of any order

The problem with the system $(\mathcal{R}_m, \mathfrak{H}_m)_{m \in \mathbb{Z}}$ is that we have to keep track of the order m at any time. It would certainly be convenient to have some kind of order-independent domain on which all the $\mathfrak{H}_m$ can act nicely; so let's try to find such a 'core'.

Proposition 6.2.6.1 *If $n, m \in \mathbb{Z}$ and $|n| \leq |m|$, then $\mathcal{R}_n \subseteq \mathcal{R}_m$.*

Proof. Recalling proposition 6.2.4.5, it suffices to prove that $\mathcal{R}_{m-1} \subseteq \mathcal{R}_m$ for all $m \geq 1$. So let's take any $m \in \mathbb{N}$ with $m \geq 1$ and any $g \in \mathcal{R}_{m-1}$. Then $\Omega^{-2}\mathfrak{H}_{m-1}\, g$ still belongs to $\mathcal{R}_{m-1}$, hence we can apply the first formula of proposition 6.2.4.6 with $f = \Omega^{-2}\mathfrak{H}_{m-1}\, g$, yielding

$$\mathfrak{H}_m D_{q^2}\Omega^{-2}\mathfrak{H}_{m-1}\, g \;=\; -\frac{q^{-1}}{q^{-1}-q}\,\mathfrak{H}_{m-1}\Omega^2\Omega^{-2}\mathfrak{H}_{m-1}\, g \;=\; -\frac{q^{-1}}{q^{-1}-q}\, g. \tag{6.23}$$

We conclude that g belongs to $\mathcal{R}_m$. ∎

Remark 6.2.6.2 At present we lack an example which shows the inclusions in proposition 6.2.6.1 to be *strict* inclusions, and as a matter of fact the spaces $\mathcal{R}_m$ for various $m \in \mathbb{Z}$ are not unlikely to coincide. For instance from the second formula in proposition 6.2.4.6 one can derive that $\Psi\mathcal{R}_m \subseteq \mathcal{R}_{m-1}$ for any $m \geq 1$, which yields an indication for the reversed inclusions. In our approach to holomorphic q-Hankel transformation however, it won't be easy to settle this question; but do we really care?[4] ⋆

Definition 6.2.6.3 We introduce two more subspaces of $H(\mathbb{C})$ as follows:

$$\begin{aligned} \mathcal{R} &= \bigcap_{m\in\mathbb{Z}} \mathcal{R}_m \\ \mathcal{E} &= \left\{ f \in \mathcal{R} \;\middle|\; \mathfrak{H}_m f,\, D_{q^2}^m f \in \mathcal{R} \text{ for all } m \in \mathbb{N} \right\} \end{aligned}$$

Observe that $\mathcal{E} \subseteq \mathcal{R} \subseteq \mathcal{S}(\mathbb{R}^+; q^2)$. Also notice the definition of $\mathcal{E}$ only refers to *positive* order q-Hankel transforms, which is justifiable in view of propositions 6.2.4.5 and 6.2.4.3. Of course $\mathcal{R}$ is nothing but $\mathcal{R}_0$ (cf. proposition 6.2.6.1). However we prefer to use this new symbol $\mathcal{R}$ (rather than $\mathcal{R}_0$) to emphasize that it will be considered in relation to q-Hankel transforms of *any* order—not just $\mathfrak{H}_0$. Now the problem with $\mathcal{R}$ is that it might (cf. remark 6.2.6.2) not be invariant under $\mathfrak{H}_m$ when $m \neq 0$, and moreover it is not clear either whether $\mathcal{R}$ is invariant under D_{q^2}. This is of course exactly the reason why we have introduced the space $\mathcal{E}$. Indeed we can prove the following:

Proposition 6.2.6.4 *$\mathcal{E}$ is invariant under $\mathfrak{H}_m$ for any $m \in \mathbb{Z}$. Furthermore, $\mathcal{E}$ is also invariant under $\tilde{}$, $\Omega^{\pm 2}$, Ψ and D_{q^2}.*

[4] no, not really; it will become clear very soon that we can perfectly well proceed without knowing whether the $\mathcal{R}_m$ coincide or not.

Proof. Obviously $\mathcal{R}$ is $\Omega^{\pm 2}$-invariant (cf. proposition 6.2.4.3). From (6.18) and the commutation rule $D_{q^2}\Omega^2 = q^2\Omega^2 D_{q^2}$, it follows that also $\mathcal{E}$ is invariant under $\Omega^{\pm 2}$.
Next we will show D_{q^2}-invariance; take any $f \in \mathcal{E}$ and put $g = D_{q^2} f$. Choose any $m \in \mathbb{N}$ and first assume that $m \geq 1$. Since $f \in \mathcal{R} \subseteq \mathcal{R}_{m-1}$, it follows from the first formula in proposition 6.2.4.6 that $\mathfrak{H}_m g = (\text{scalar})\, \mathfrak{H}_{m-1}\Omega^2 f$. Now the latter belongs to $\mathcal{R}$ because $f \in \mathcal{E}$. Now consider the $m = 0$ case. Since $f \in \mathcal{E}$, we have $g = D_{q^2} f \in \mathcal{R} = \mathcal{R}_0$, and hence $\mathfrak{H}_0 g \in \mathcal{R}$. So we have shown that $\mathfrak{H}_m g \in \mathcal{R}$ for any $m \in \mathbb{N}$, whereas clearly $D_{q^2}^m g = D_{q^2}^{m+1} f \in \mathcal{R}$ for any $m \in \mathbb{N}$. We conclude that $g \in \mathcal{E}$.
Now we will show by induction on $m \in \mathbb{N}$ that $\mathcal{E}$ is invariant under $\mathfrak{H}_m$. Notice it is sufficient to consider positive m only, because of proposition 6.2.4.5. Let's first consider the $m = 0$ case; take any $f \in \mathcal{E}$, put $g = \mathfrak{H}_0 f$ and observe that $g \in \mathcal{R}_0 = \mathcal{R}$. We shall prove that g belongs to $\mathcal{E}$ again; therefore, choose any $n \in \mathbb{N}$ and consider $\mathfrak{H}_n g$ and $D_{q^2}^n g$. When $n = 0$, we obtain $\mathfrak{H}_0 g = \mathfrak{H}_0^2 f = f \in \mathcal{R}$. To deal with $n \geq 1$, rewrite the first formula in proposition 6.2.4.6 as

$$\mathfrak{H}_n \;=\; (\text{scalar})\, D_{q^2}\Omega^{-2}\mathfrak{H}_{n-1} \tag{6.24}$$

(see also: equation (6.23) above). Iterating this n times yields

$$\mathfrak{H}_n \;=\; (\text{scalar})\, (D_{q^2}\Omega^{-2})^n \mathfrak{H}_0 \;=\; (\text{scalar})\, \Omega^{-2n} D_{q^2}^n \mathfrak{H}_0 \tag{6.25}$$

We used the commutation rule for D_{q^2} and Ω^{-2}. The exact value of the scalars involved here could be computed easily, though they are not relevant for our purposes (except for the fact they are all non-zero). Since g belongs to $\mathcal{R}_0$ we may apply (6.25) to it, yielding

$$\mathfrak{H}_n g \;=\; (\text{scalar})\, \Omega^{-2n} D_{q^2}^n f.$$

Now the latter is in $\mathcal{R}$ because $f \in \mathcal{E}$. We have shown that $\mathfrak{H}_n g \in \mathcal{R}$ for all $n \in \mathbb{N}$. On the other hand, (6.25) is also useful in the other direction:

$$D_{q^2}^n g \;=\; D_{q^2}^n \mathfrak{H}_0 f \;=\; (\text{scalar})\, \Omega^{2n}\mathfrak{H}_n f.$$

It follows that $D_{q^2}^n g \in \mathcal{R}$ for all $n \in \mathbb{N}$, which completes the $m = 0$ case. Now we proceed by induction on m. Let's assume $\mathcal{E}$ to be invariant under $\mathfrak{H}_{m-1}$ for some $m \geq 1$, and prove that this implies invariance under $\mathfrak{H}_m$. Applying (6.24) to an arbitrarily chosen $f \in \mathcal{E}$ yields

$$\mathfrak{H}_m f \;=\; (\text{scalar})\, D_{q^2}\Omega^{-2}\mathfrak{H}_{m-1} f.$$

By hypothesis $\mathfrak{H}_{m-1} f$ belongs to $\mathcal{E}$ again, and since we have already shown that $\mathcal{E}$ is invariant under Ω^{-2} and D_{q^2}, the result follows.
Invariance under $\tilde{}$ follows easily from proposition 6.2.4.4 together with the fact that $\tilde{}$ commutes with D_{q^2}. It only remains to prove that $\mathcal{E}$ is Ψ-invariant. Therefore consider the $m = 1$ case of the second formula in proposition 6.2.4.6. Plugging in (5.8) and canceling some scalars, the formula can be rewritten as

$$\mathfrak{H}_0(\text{id} - q^2\Omega^2)\mathfrak{H}_1 g \;=\; \Psi g$$

for $g \in \mathcal{R}_1$ and a fortiori for $g \in \mathcal{E}$. Since we have already established that $\mathcal{E}$ is invariant under $\mathfrak{H}_0$, $\mathfrak{H}_1$ and Ω^2, the result follows. ∎

Remark 6.2.6.5 Again (cf. remark 6.2.2.3) our approach is far from being explicit: still we lack any criterion which describes the spaces $\mathcal{R}$ and $\mathcal{E}$ in a direct way; we merely *define* $\mathcal{E}$ to be the largest space on which one can happily take iterated q-Hankel transforms of any order. It is likely that one could do better than this, unraveling all the details of q-Hankel transformation: for instance we can perhaps obtain a richer theory if we stick to the original system $(\mathcal{R}_m, \mathfrak{H}_m)_{m\in\mathbb{Z}}$, keeping track of the orders at any time. . . (cf. remark 5.3.3.3). In this respect, observe the striking similarity between the properties of $(\mathcal{R}_m)_{m\in\mathbb{Z}}$ given in proposition 6.2.4.6, and the conditions on the 'second leg' of $(\mathfrak{L}_m)_{m\in\mathbb{Z}}$ in remark 5.3.3.3). Our main concern however, is to proceed into harmonic analysis on the quantum $E(2)$ group. Nevertheless we will show—constructively—that $\mathcal{E}$ contains an important class of functions. ⋆

6.2.7 Eigenfunctions of q-Hankel transformation

Below we shall prove that the function $z \mapsto E_{q^2}(-q^2 z)$ belongs to the space $\mathcal{E}$ constructed in the previous paragraph, and moreover, that it constitutes an eigenfunction of all positive order holomorphic q-Hankel transforms; it follows that $\mathcal{E}$ is non-trivial. This function of q^2-exponential type plays a role similar to the Gaussian[5] in ordinary Fourier analysis on the real line.

Lemma 6.2.7.1 *Let $E_q^\bullet$ denote the entire function given by $E_q^\bullet(z) = (z;q)_\infty$. Then*

$$D_q E_q^\bullet \;=\; -\frac{1}{1-q}\,\Omega E_q^\bullet.$$

Proof. We could easily derive this from the power series (C.2) but it is even more convenient to use the product representation: indeed (C.1) yields

$$E_q^\bullet(z) \;=\; (1-z)E_q^\bullet(qz)$$

for all $z \in \mathbb{C}$, and hence

$$(D_q E_q^\bullet)(z) \;=\; \frac{E_q^\bullet(z) - E_q^\bullet(qz)}{(1-q)z} \;=\; \frac{-z\,E_q^\bullet(qz)}{(1-q)z} \;=\; -\frac{1}{1-q}\,(\Omega E_q^\bullet)(z).$$

for $z \neq 0$. Extending the result to $z = 0$ by continuity completes the proof. ∎

Corollary 6.2.7.2 *Replacing q by q^2 and (consequently!) Ω by Ω^2, we get*

$$D_{q^2} E_{q^2}^\bullet \;=\; -\frac{1}{1-q^2}\,\Omega^2 E_{q^2}^\bullet.$$

[5] and as a matter of fact, this q^2-exponential does amount to a q-Gaussian if one takes into account the '*squaring* operator' K appearing in diagram (6.11).

Proposition 6.2.7.3 $\Omega^2 E^{\bullet}_{q^2} = (\,\cdot\, q^2; q^2)_\infty$ *satisfies the q^2-differential equation*

$$D_{q^2} f \;=\; -\frac{q^2}{1-q^2}\,\Omega^2 f \qquad\qquad (f \text{ entire}). \tag{6.26}$$

Moreover the solution of this q^2-differential equation is unique in the sense that any entire function f satisfying (6.26) must be a scalar multiple of $\Omega^2 E^{\bullet}_{q^2}$.

Proof. Combining corollary 6.2.7.2 with the obvious commutation rule for D_{q^2} and Ω^2 yields

$$D_{q^2}\Omega^2 E^{\bullet}_{q^2} \;=\; q^2\Omega^2 D_{q^2} E^{\bullet}_{q^2} \;=\; -\frac{q^2}{1-q^2}\,\Omega^2\Omega^2 E^{\bullet}_{q^2}.$$

To prove the uniqueness statement, let's evaluate (6.26) in q^{2n} for any $n \in \mathbb{N}$. We get

$$\frac{f(q^{2n}) - f(q^2 q^{2n})}{(1-q^2)\, q^{2n}} \;=\; -\frac{q^2}{1-q^2}\, f(q^2 q^{2n})$$

Putting $x_n = f(q^{2n})$ for $n \in \mathbb{N}$, we obtain a sequence $(x_n)_{n=0}^\infty$ of complex numbers satisfying the following recurrence relation:

$$x_n \;=\; (1 - q^{2n} q^2)\, x_{n+1}.$$

Iterating this relation yields

$$x_0 \;=\; (1-q^2)(1-q^2q^2)\ldots(1-q^{2(n-1)}q^2)\, x_n \;=\; (q^2;q^2)_n\, x_n. \tag{6.27}$$

It follows that the sequence $(x_n)_n$ is completely determined by the value of x_0. In other words, (6.26) and $f(1)$ determine the value of f at the points q^{2n} ($n \in \mathbb{N}$). Since $q^{2n} \to 0$ as $n \to \infty$, we may invoke the identity theorem for holomorphic functions and draw the conclusion (notice that $f(1) = 0$ implies $f = 0$, and observe that (6.26) is *linear* in f). ∎

It is also instructive to observe (C.1) implies that

$$x_n \;=\; (q^{2n}q^2; q^2)_\infty \;=\; (\Omega^2 E^{\bullet}_{q^2})(q^{2n})$$

indeed satisfies the above recurrence relation (6.27).

Lemma 6.2.7.4 *Let m be any non-negative integer. If f is a solution of (6.26) then so is $\mathfrak{H}_m f$* (notice that f belongs to $\mathcal{R}$ because of proposition 6.2.7.3 and corollary 6.2.5.3, so the statement makes sense).

Proof. With the commutation rule $D_{q^2}\Omega^2 = q^2\,\Omega^2 D_{q^2}$ the first formula in proposition 6.2.4.6 can be rewritten in the form

$$D_{q^2}\mathfrak{H}_{m-1} = -\frac{q^2}{1-q^2}\,\Omega^2\mathfrak{H}_m \qquad\qquad (m \geq 1) \tag{6.28}$$

Replacing m with $m+1$ and applying this to a solution f of (6.26) yields

$$D_{q^2}\mathfrak{H}_m f \;=\; -\frac{q^2}{1-q^2}\,\Omega^2\mathfrak{H}_{m+1}f \;\overset{(6.26)}{=}\; \Omega^2\mathfrak{H}_{m+1}\Omega^{-2}D_{q^2}f \;=\; q^2\Omega^2\mathfrak{H}_{m+1}D_{q^2}\Omega^{-2}f$$

for any $m \geq 0$. Once again applying the first formula in proposition 6.2.4.6 yields

$$D_{q^2}\mathfrak{H}_m f \;=\; -\frac{q^2}{1-q^2}\,\Omega^2\mathfrak{H}_m f \tag{6.29}$$

which means that $\mathfrak{H}_m f$ is a solution to (6.26). ∎

Remark 6.2.7.5 The above lemma does not extend to negative m. To see this, apply proposition 6.2.4.5 to (6.29). The reason for this remarkable lack of symmetry is not so clear at the moment. ⋆

Proposition 6.2.7.6 *$\Omega^2 E^{\bullet}_{q^2}$ is an eigenfunction of all $\mathfrak{H}_m$ with $m \geq 0$.*

Proof. Combine the above lemma with proposition 6.2.7.3. ∎

Below we shall compute the corresponding eigenvalues and search for other eigenfunctions as well. Before we proceed, however, we wish to emphasize that our core $\mathcal{E}$ is finally known to be non-trivial—indeed it does already contain an important class of functions:

Corollary 6.2.7.7 *For any polynomial P and integer k, the entire function*

$$\mathbb{C} \to \mathbb{C} : \; z \;\mapsto\; P(z)\, E_{q^2}(-q^{2k}z) \tag{6.30}$$

belongs to the space $\mathcal{E}$ of definition 6.2.6.3.

Proof. We already established that $\Omega^2 E^{\bullet}_{q^2}$ belongs to $\mathcal{R}$ (cf. corollary 6.2.5.3) and from proposition 6.2.7.6 it is clear that $\mathfrak{H}_m\Omega^2 E^{\bullet}_{q^2}$ is in $\mathcal{R}$ as well, for any $m \in \mathbb{N}$. Furthermore, the condition of lemma 6.2.5.1 is obviously invariant under q^2-differentiation, hence $D^m_{q^2}\Omega^2 E^{\bullet}_{q^2}$ still belongs to $\mathcal{R}$, for all $m \in \mathbb{N}$.
We conclude that $\Omega^2 E^{\bullet}_{q^2}$ is contained in $\mathcal{E}$. Eventually, the invariance properties of $\mathcal{E}$ (cf. proposition 6.2.6.4) yield the result. ∎

Notation 6.2.7.8 Henceforth ξ_0 will denote the entire function $\Omega^2 E^{\bullet}_{q^2}$ that was investigated above, so $\xi_0(z) = (q^2 z; q^2)_\infty$ for any $z \in \mathbb{C}$. Furthermore we denote

$$\xi_k \;=\; q^k\,\Omega^{2k}\xi_0 \tag{6.31}$$

for any $k \in \mathbb{Z}$. Observe that the ξ_k belong to the space $\mathcal{E}$ (cf. previous corollary).

Lemma 6.2.7.9 *$\Psi\xi_0 = (\mathrm{id} - \Omega^{-2})\xi_0$ and consequently, for any $k \in \mathbb{Z}$,*

$$\Psi\xi_k \;=\; q^{-2k}\,(\xi_k - q\,\xi_{k-1}). \tag{6.32}$$

It follows that the functions in (6.30) belong to the linear span of $\{\xi_k \mid k \in \mathbb{Z}\}$.

Proof. The first formula follows from a straightforward computation using the product representation (C.1). Equation (6.32) is an immediate consequence. ■

Lemma 6.2.7.10 *$\mathfrak{H}_m \xi_0 = \xi_0$ for all $m \in \mathbb{N}$.*

Proof. We already know from proposition 6.2.7.6 that ξ_0 is an eigenfunction of $\mathfrak{H}_m$ for all $m \in \mathbb{N}$, so it only remains to show that the corresponding eigenvalues λ_m are all equal to 1. First we compute the value of λ_0. To do so, we shall rely on proposition 6.3.2 from the *next* section—this may be slightly unpleasant, but it does not really raise a problem since the proof of the latter result is obviously independent of the present lemma and its consequences. Thus we obtain

$$(1-q^2)\,(\mathfrak{H}_0\xi_0)(0) \;=\; \int_0^\infty \xi_0(x)\,d_{q^2}x.$$

For details we refer to §6.3. Now recall corollary 6.2.7.2, where we observed that $\xi_0 = -(1-q^2)\,D_{q^2}E^{\bullet}_{q^2}$. It follows that

$$(\mathfrak{H}_0\xi_0)(0) \;=\; -\int_0^\infty (D_{q^2}E^{\bullet}_{q^2})(x)\,d_{q^2}x \;=\; E^{\bullet}_{q^2}(0) \;=\; (0;q^2)_\infty \;=\; 1.$$

Observe how we used the relation between q^2-integration and q^2-differentiation, combined with the following facts:

(i) $E^{\bullet}_{q^2}(q^{2n})$ vanishes for all negative integers n.

(ii) $\lim_{n\to+\infty} E^{\bullet}_{q^2}(q^{2n}) \;=\; E^{\bullet}_{q^2}(0)$ since $E^{\bullet}_{q^2}$ is entire and a fortiori continuous.

Using $\xi_0(0) = 1$ and $\mathfrak{H}_0\xi_0 = \lambda_0\xi_0$ we get $\lambda_0 = 1$. To show $\lambda_m = 1$ for all $m \in \mathbb{N}$, we proceed by induction: using the 2nd formula in proposition 6.2.4.6, we obtain

$$\begin{aligned}
\Psi\xi_0 &= \Psi\mathfrak{H}_{m+1}\mathfrak{H}_{m+1}\xi_0 \\
&= \lambda_{m+1}\,\Psi\mathfrak{H}_{m+1}\xi_0 \\
&= \lambda_{m+1}\,(q^{-1}-q)\,q^{m+1}\,\mathfrak{H}_m\nabla_q^{(m+1)}\Omega\xi_0 \\
&\overset{(5.8)}{=} -\lambda_{m+1}\,q^{m+1}\,\mathfrak{H}_m(q^{m+1}\,\Omega - q^{-m-1}\,\Omega^{-1})\Omega\xi_0 \\
&= -\lambda_{m+1}\,\mathfrak{H}_m(q^{2(m+1)}\,\Omega^2 - \mathrm{id})\xi_0 \\
&\overset{(6.18)}{=} -\lambda_{m+1}\,(\Omega^{-2} - \mathrm{id})\mathfrak{H}_m\xi_0 \\
&= \lambda_{m+1}\lambda_m\,(\mathrm{id} - \Omega^{-2})\xi_0 \\
&= \lambda_{m+1}\lambda_m\,\Psi\xi_0.
\end{aligned}$$

The last equality relies on lemma 6.2.7.9. It follows that $\lambda_{m+1}\lambda_m = 1$ for all $m \in \mathbb{N}$. This completes the proof. ■

Corollary 6.2.7.11 *An eigenfunction for the q-Hankel transformation H_m on $L^2(\mathbb{R}_q^+)$ is given by $f_m = R_q\Psi^m K\xi_0$ and the eigenvalue is 1. Explicitly we have:*

$$f_m(q^n) \;=\; q^{nm}\,(q^{2n+2};q^2)_\infty$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. In terms of little q^2-Bessel functions, this means that

$$\sum_{n \in \mathbb{Z}} q^{(2+m)n} \, (q^{2n+2}; q^2)_\infty \, J_m(q^{n+k}; q^2) \;=\; q^{km} \, (q^{2k+2}; q^2)_\infty.$$

Proof. The first assertion follows immediately from the above lemma, together with diagram (6.11) in abstract 6.2.2.1. The last formula follows from (6.10). ■

Proposition 6.2.7.12 $\mathfrak{H}_m \, \xi_k = q^{-2mk} \, \xi_{-k}$ *for all $m \in \mathbb{N}$ and $k \in \mathbb{Z}$.*

Proof. Simply combine lemma 6.2.7.10 with proposition 6.2.4.3. ■

Corollary 6.2.7.13 *For any $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, the functions*

$$q^{mk} \, \xi_k \;\pm\; q^{-mk} \, \xi_{-k}$$

are eigenfunctions of $\mathfrak{H}_m$ with respectively eigenvalues ± 1.

Notice that ± 1 are the only possible eigenvalues because $\mathfrak{H}_m^2 = \mathrm{id}$.

6.2.8 More interesting functions in the space $\mathcal{E}$

In the previous section we constructed a system $(\xi_k)_{k \in \mathbb{Z}}$ of q^2-exponential type functions within the space $\mathcal{E}$, having a particularly pleasant behaviour with respect to our holomorphic q-Hankel transforms:

$$\mathfrak{H}_m \, \xi_k \;=\; q^{-2mk} \, \xi_{-k} \tag{6.33}$$

for all $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. Basically these ξ_k are dilations of a q^2-exponential over the q^2-grid. We also observed (cf. lemma 6.2.7.9) that the functions in (6.30) can be written as finite linear combinations of the ξ_k.
One might object however, that the examples of functions in $\mathcal{E}$ given thus far, all vanish in the points q^{-2n} for sufficiently large $n \in \mathbb{N}$. Therefore we are about to consider *infinite* combinations like

$$\hat{a} \;=\; \sum_{k \in \mathbb{Z}} a_k \, \xi_k \tag{6.34}$$

where $a \equiv (a_k)_{k \in \mathbb{Z}}$ is an appropriate sequence of complex numbers. Since we want the function $\hat{a}$ to be entire, the question arises which conditions on the sequence $(a_k)_k$ can ensure the series (6.34) to converge uniformly on compact sets. Thus one might worry to run into the same kind of trouble as sketched in remark 6.2.2.3. This however shall not be the case: gaining control over the convergence of (6.34) will turn out to be much more realistic than trying to cope with (6.13). We start with some general definitions concerning sequences of complex numbers:

Definition 6.2.8.1 Let $\mathbb{C}^{\mathbb{Z}}$ denote the linear space of all $\mathbb{Z}$-indexed sequences of complex numbers, and define the following linear operations on this space: for $a \equiv (a_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ we consider

$$\begin{array}{lrcl} \text{Shift} & (Sa)_k & = & a_{k+1} \\ \text{Multiplication} & (Ma)_k & = & q^k\, a_k \\ \text{Reflection} & (Ra)_k & = & a_{-k} \end{array}$$

Clearly S, M and R are bijections from $\mathbb{C}^{\mathbb{Z}}$ onto $\mathbb{C}^{\mathbb{Z}}$. Furthermore we introduce a linear q-difference operator D on $\mathbb{C}^{\mathbb{Z}}$ by

$$D \;=\; M^{-2}(\mathrm{id} - q^{-1}S).$$

Notation 6.2.8.2 For any $n \in \mathbb{N}$, let P_n denote the following polynomial:

$$P_n(x) \;=\; (q^2x;q^2)_n \;=\; (1-q^2x)(1-q^4x)\ldots(1-q^{2n}x).$$

Lemma 6.2.8.3 *For all $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have:*

$$|P_n(z)| \;\le\; P_n(-|z|) \;\le\; \big(1+q^2|z|\big)^n \;\le\; \big(1+|z|\big)^n$$

Proof. Straightforward. ∎

Definition 6.2.8.4 For any $r > 0$, $n \in \mathbb{N}$ and $a \equiv (a_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, we denote

$$B(a;r,n;q) \;=\; \sup\Big\{\, |a_{-k}|\, q^{-nk}\, P_k(-q^{-2k}r) \;\Big|\; k \in \mathbb{N}\Big\}$$

where the supremum of an *unbounded* subset of $\mathbb{R}^+$ is understood to be ∞. •

Definition 6.2.8.5 Define subspaces $\mathfrak{X}_q$ and $\mathfrak{Y}_q$ of the sequence space $\mathbb{C}^{\mathbb{Z}}$ by

$$\begin{aligned} \mathfrak{X}_q &= \Big\{ a \in \mathbb{C}^{\mathbb{Z}} \;\Big|\; B(a;r,n;q) < \infty \text{ for all } r > 0 \text{ and } n \in \mathbb{N}\Big\} \\ \mathfrak{Y}_q &= \mathfrak{X}_q \cap R\mathfrak{X}_q \end{aligned}$$

Remarks 6.2.8.6

i. Take any sequence $a \equiv (a_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$. Since $1 \le P_k(x)$ whenever $x \le 0$, it follows that

$$|a_{-k}|\, q^{-nk} \;\le\; B(a;1,n;q)$$

for all $k, n \in \mathbb{N}$. So if $a \in \mathfrak{X}_q$ then $a_{-k}\, q^{-nk} \to 0$ as $k \to +\infty$. This means that a_{-k} tends to zero *very rapidly* when $k \to +\infty$.

ii. Definition 6.2.8.4 only involves the entries a_{-k} for $k \in \mathbb{N}$. In other words, it only deals with *negative* indices, and therefore $\mathfrak{X}_q$ will not be invariant under reflection; that is why we introduced $\mathfrak{Y}_q$. So if $(a_k)_{k\in\mathbb{Z}}$ belongs to $\mathfrak{Y}_q$ then a_k will tend to zero 'very rapidly' when $k \to \pm\infty$.

iii. Any sequence with only *finitely* many non-zero entries is contained in $\mathfrak{Y}_q$.

Lemma 6.2.8.7 *The spaces $\mathfrak{X}_q$ and $\mathfrak{Y}_q$ are invariant under the operations S and M, as well as under their inverses. $\mathfrak{Y}_q$ is moreover invariant under R. As a corollary, observe that $\mathfrak{Y}_q$ is also invariant under the difference operator D.*

Proof. Take any $r > 0$, $n \in \mathbb{N}$ and $a \in \mathbb{C}^{\mathbb{Z}}$. Observe that

$$B(a; r, n; q) \quad \leq \quad B(Ma; r, n; q) \quad = \quad B(a; r, n+1; q).$$

This yields M-invariance. To show S-invariance, observe that for all $k \in \mathbb{N}$

$$q^{-n(k+1)}\, P_{k+1}(-q^{-2(k+1)} r) \quad = \quad q^{-nk}\, q^{-n}\, (1+r)\, P_k\big(-q^{-2k}\, q^{-2} r\big)$$

and hence

$$B(Sa; r, n; q) \quad = \quad q^{-n}\, (1+r)\, B\big(a; q^{-2} r, n; q\big)\,.$$

Thus we have shown that $\mathfrak{X}_q$ is invariant under $S^{\pm 1}$ and $M^{\pm 1}$. The invariance properties of $\mathfrak{Y}_q$ then follow immediately from obvious commutation rules like $SR = RS^{-1}$ and $MR = RM^{-1}$. ∎

Example 6.2.8.8 Let us give an example showing that $\mathfrak{X}_q$ and $\mathfrak{Y}_q$ also contain sequences with an *infinite* number of non-zero entries. Take any sequence $(t_n)_{n \in \mathbb{N}}$ of complex numbers such that $t_n \to 0$ as $n \to \infty$. Then define

$$a_k \;=\; \big(t_{|k|}\big)^{|k|}\, q^{2k^2}$$

for all $k \in \mathbb{Z}$. Fix any $n \in \mathbb{N}$ and $r > 0$. Now assume $k \geq 0$ and observe that

$$\begin{aligned} |a_{-k}|\, q^{-nk}\, P_k(-q^{-2k} r) \quad &\leq \quad |t_k|^k\, q^{2k^2}\, q^{-nk}\, \big(1 + q^{-2k} r\big)^k \\ &\leq \quad \left(|t_k|\, q^{-n}\, \frac{1 + q^{-2k} r}{q^{-2k}}\right)^k . \end{aligned}$$

Since

$$\lim_{k \to +\infty} \frac{1 + q^{-2k} r}{q^{-2k}} \;=\; r \qquad \text{and} \qquad \lim_{k \to +\infty} t_k \;=\; 0,$$

it follows easily that $B(a; r, n; q) < \infty$ for any $r > 0$ and all $n \in \mathbb{N}$. In other words, the sequence $(a_k)_{k \in \mathbb{Z}}$ belongs to $\mathfrak{X}_q$. This sequence moreover belongs to $\mathfrak{Y}_q$ because $a_{-k} = a_k$ for all k. ⋆

Lemma 6.2.8.9 *For any $n \in \mathbb{N}$ and $z \in \mathbb{C}$ we have $\xi_0(z) = P_n(z)\, \xi_0(q^{2n} z)$.*

Proof. Straightforward, using the product representation (C.1). ∎

Lemma 6.2.8.10 *If a sequence $a \equiv (a_k)_{k \in \mathbb{Z}}$ belongs to $\mathfrak{Y}_q$ then*

$$\hat{a} \;=\; \sum_{k \in \mathbb{Z}} a_k\, \xi_k \tag{6.35}$$

converges uniformly on compact sets to an entire function.

Proof. Plugging the definition of the ξ_k (notation 6.2.7.8) into (6.35) we obtain, for any $z \in \mathbb{C}$,

$$\sum_{k\in\mathbb{Z}} a_k\, q^k\, \xi_0(q^{2k} z) \tag{6.36}$$

Obviously there's no challenge in establishing uniform convergence on compact sets as far as the $k \to +\infty$ part is concerned—so let's investigate the behaviour of the above series for $k \to -\infty$. Considering only the terms for negative $k \in \mathbb{Z}$, we obtain

$$\sum_{n=0}^{\infty} a_{-n}\, q^{-n}\, \xi_0(q^{-2n} z)$$

and using lemma 6.2.8.9, we get

$$\sum_{n=0}^{\infty} q^n\, a_{-n}\, q^{-2n}\, P_n(q^{-2n} z)\, \xi_0(z)$$

or shortly

$$\xi_0(z) \sum_{n=0}^{\infty} q^n f_n(z) \tag{6.37}$$

with

$$f_n(z) \;=\; a_{-n}\, q^{-2n}\, P_n(q^{-2n} z).$$

Now fix any $r > 0$ and let $D(r) \subseteq \mathbb{C}$ denote the closed disk with radius r around the origin. Using lemma 6.2.8.3 we obtain for all $n \in \mathbb{N}$ and $z \in D(r)$ that

$$|f_n(z)| \;\le\; |a_{-n}|\, q^{-2n}\, P_n(-q^{-2n} r) \;\le\; B(a; r, 2; q).$$

So if $a \in \mathfrak{Y}_q$ then (6.37) converges uniformly on $D(r)$ for any $r > 0$. ■

Definition 6.2.8.11 The above lemma yields a well-defined linear mapping

$$\Theta : \mathfrak{Y}_q \to H(\mathbb{C}) : a \mapsto \hat{a},$$

which shall be referred to as the Θ-*transform*. •

Our main goal for the remainder of this section is to prove that the range $\widehat{\mathfrak{Y}}_q$ of the Θ-transform is contained in the common domain $\mathcal{E}$ of all holomorphic q-Hankel transforms.

Lemma 6.2.8.12 *For any $a \in \mathfrak{Y}_q$ we have:*

$$\begin{aligned} \Psi\, \hat{a} &= \Theta D a \\ D_{q^2}\, \hat{a} &= \frac{1}{q - q^{-1}}\, \Theta S^{-1} M^2 a \\ \Omega^2\, \hat{a} &= q^{-1}\, \Theta S^{-1} a \end{aligned}$$

Notice that the above makes sense because of lemma 6.2.8.7.

Proof. First observe that it is allowed to push the operators Ψ, D_{q^2} and Ω^2 through the summation in (6.35). This merely involves *pointwise* convergence because Ψ, D_{q^2} and Ω^2 are essentially pointwise operations[6].
Now the actual computations are straightforward: the proof of the first formula involves (6.32), the second one follows easily from proposition 6.2.7.3, and the last one is an immediate consequence of (6.31). ∎

Corollary 6.2.8.13 *The range $\widehat{\mathfrak{Y}}_q$ of the Θ-transform is invariant under the operators Ψ, D_{q^2} and $\Omega^{\pm 2}$.*

Lemma 6.2.8.14 *If $a \equiv (a_k)_{k\in\mathbb{Z}}$ belongs to $\mathfrak{Y}_q$ then $\hat{a}$ belongs to $\mathcal{S}(\mathbb{R}^+;q^2)$.*

Proof. Since $\widehat{\mathfrak{Y}}_q$ is invariant under the multiplication operator Ψ, it suffices to show that $\hat{a}$ is *bounded* on the q^2-grid whenever $a \in \mathfrak{Y}_q$. Using (6.36) together with the fact that $\xi_0(q^{2j})$ vanishes for negative j, we obtain for any $n \in \mathbb{Z}$ that

$$\hat{a}(q^{2n}) \;=\; \sum_{k=-n}^{+\infty} a_k\, q^k\, \xi_0\big(q^{2k+2n}\big)\,.$$

Since ξ_0 is entire, it follows that

$$N \;\equiv\; \sup_{0\le x\le 1} |\xi_0(x)| \;<\; \infty, \tag{6.38}$$

and hence

$$\big|\hat{a}(q^{2n})\big| \;\;\le\;\; N \sum_{k=-n}^{+\infty} |a_k|\, q^k \;\;\le\;\; N \sum_{k\in\mathbb{Z}} |a_k|\, q^k \;\;<\;\; \infty.$$

The latter series is independent of n and converges because a is assumed to be in $\mathfrak{Y}_q$ (cf. remarks 6.2.8.6.i-ii). ∎

Lemma 6.2.8.15 *If $a \equiv (a_k)_{k\in\mathbb{Z}}$ belongs to $\mathfrak{Y}_q$ then for any $m \in \mathbb{N}$, the pair*

$$\big(\hat{a},\, \Psi^m K\Theta M^{2m} Ra\big)$$

is an $(m;q)$-Hankel pair (cf. definition 6.2.2.2).

Observe that for the above to make sense, we really need $\mathfrak{Y}_q$ to be invariant under the reflection operator R.

Proof. Take any $a \in \mathfrak{Y}_q$ and $m \in \mathbb{N}$. Observe that both $\hat{a}$ and $\Psi^m K\Theta M^{2m} Ra$ are entire because of lemma 6.2.8.7 and lemma 6.2.8.10. We also proved that $\hat{a}$ belongs to $\mathcal{S}(\mathbb{R}^+;q^2)$. Now fix any $j \in \mathbb{Z}$ and define for any $k, n \in \mathbb{Z}$ a number

$$\zeta_{n,k} \;=\; q^{2n}\, J_m(q^{n+j};q^2)\, q^{mn}\, a_k\, q^k\, \xi_0\big(q^{2n+2k}\big)\,.$$

[6] The situation would be quite different (and more complicated) if one would apply e.g. the holomorphic q-Hankel transform $\mathfrak{H}_m$ to $\hat{a}$. See also lemma 6.2.8.15.

Using (6.8) and (6.38) we obtain

$$\begin{aligned} |\zeta_{n,k}| &\leq q^{2n}\, C_m\, q^{-m(n+j)}\, q^{(n+j)(n+j-1)}\, q^{mn}\, |a_k|\, q^k\, N \\ &= \underbrace{C_m\, N\, q^{j(j-m-1)}}_{\text{independent of } n,k}\, q^{n(n+2j+1)}\, |a_k|\, q^k \end{aligned}$$

for all $k, n \in \mathbb{Z}$ with $n + j \leq 0$, and

$$|\zeta_{n,k}| \leq \underbrace{C_m\, N\, q^{mj}}_{\text{independent of } n,k}\, q^{2n(m+1)}\, |a_k|\, q^k$$

whenever $n + j \geq 0$. It follows that

$$\sum_{n,k\in\mathbb{Z}} \zeta_{n,k}$$

is absolutely summable, which justifies the following computation:

$$\begin{aligned} (\mathcal{H}_m \hat{a})(q^j) &= (H_m R_q \Psi^m K \hat{a})(q^j) \\ &\overset{(6.10)}{=} \sum_{n\in\mathbb{Z}} q^{2n}\, J_m(q^{n+j}; q^2)\, (\Psi^m K \hat{a})(q^n) \\ &\overset{(6.35)}{=} \sum_{n\in\mathbb{Z}} q^{2n}\, J_m(q^{n+j}; q^2)\, q^{mn} \sum_{k\in\mathbb{Z}} a_k\, \xi_k(q^{2n}) \\ &= \sum_{n,k\in\mathbb{Z}} \zeta_{n,k} \\ &= \sum_{k\in\mathbb{Z}} a_k \sum_{n\in\mathbb{Z}} q^{2n}\, J_m(q^{n+j}; q^2)\, (\Psi^m K \xi_k)(q^n) \\ &\overset{(6.10)}{=} \sum_{k\in\mathbb{Z}} a_k\, (H_m R_q \Psi^m K \xi_k)(q^j) \\ &\overset{(6.11)}{=} \sum_{k\in\mathbb{Z}} a_k\, (\Psi^m K \mathfrak{H}_m \xi_k)(q^j) \\ &\overset{(6.33)}{=} \sum_{k\in\mathbb{Z}} a_k\, q^{-2mk}\, (\Psi^m K \xi_{-k})(q^j) \\ &= \sum_{k\in\mathbb{Z}} a_{-k}\, q^{2mk}\, q^{mj}\, \xi_k(q^{2j}) \\ &= q^{mj} \sum_{k\in\mathbb{Z}} (M^{2m} R a)_k\, \xi_k(q^{2j}) \\ &\overset{(6.35)}{=} q^{mj}\, (\Theta M^{2m} R a)(q^{2j}) \\ &= (\Psi^m K \Theta M^{2m} R a)(q^j). \end{aligned}$$

This shows item (i) of definition 6.2.2.2, whereas (ii) and (iii) are obvious. ∎

Corollary 6.2.8.16 *The range $\widehat{\mathfrak{Y}}_q$ of the Θ-transform is contained in $\mathcal{E}$ and*

$$\mathfrak{H}_m \Theta \;=\; \Theta M^{2m} R \tag{6.39}$$

for all $m \in \mathbb{N}$.

Proof. Take any $a \in \mathfrak{Y}_q$. Combining the above lemma with proposition 6.2.4.2, it follows that $\hat{a} \in \mathcal{R}_m$ for all $m \in \mathbb{N}$. Furthermore we obtain

$$\Psi^m K \mathfrak{H}_m \hat{a} \;=\; \Psi^m K \Theta M^{2m} R a.$$

Canceling $\Psi^m K$ yields (6.39). By now we know already that $\widehat{\mathfrak{Y}}_q \subseteq \mathcal{R}$. Putting together lemma 6.2.8.7, corollary 6.2.8.13 and (6.39) we get $\widehat{\mathfrak{Y}}_q \subseteq \mathcal{E}$. ∎

Remark 6.2.8.17 In a purely formal way, (6.39) could have been obtained immediately from (6.33) and (6.34). In this respect, we emphasize that the proof of lemma 6.2.8.15 is not at all about the computation itself—it is about its *justification*: indeed the essence of this lemma lies within interchanging the two summations. The reason why this approach works where (6.13) had failed, is because the proof of lemma 6.2.8.15 requires the q^2-Bessel functions to be evaluated only in the points q^k with k integer. ⋆

Questions 6.2.8.18 Several problems are still open:

i. We know that $\widehat{\mathfrak{Y}}_q \subseteq \mathcal{E}$. The question arises whether the inclusion is *strict*.

ii. Is $\mathcal{E}$ an algebra? Clearly the functions described in (6.30) do not constitute an algebra. Nevertheless $\mathcal{E}$ or $\widehat{\mathfrak{Y}}_q$ may be algebras.

iii. Can one obtain—within the appropriate L^2-setting—an *orthogonal* family of eigenfunctions for the holomorphic q-Hankel transforms? ⋆

6.3 The q-moment problem for the space $\mathcal{R}$

Abstract 6.3 The q-moment problem will play a key-role in establishing *uniqueness* of Fourier transforms for quantum E(2). Moreover, one of the intermediate results in the present paragraph shall also be crucial for the *construction* of these Fourier transforms. In classical moment problems the question typically reads as follows: suppose a function of some particular class has vanishing moments; may we then conclude that the function itself vanishes everywhere? In this paragraph however we shall consider q-moments, i.e. moments computed w.r.t. a well-known q-analogue of the integral. Restricting to functions in the space $\mathcal{R}$ of definition 6.2.6.3, our q-moment problem has positive answer. ⋆

Definition 6.3.1 Let f be any complex valued function defined on some subset of $\mathbb{C}$ containing the points q^n with $n \in \mathbb{Z}$. The q-integral (or *Jackson* integral) of f is then defined by

$$\int_0^\infty f(x)\, d_q x \;=\; (1-q) \sum_{n\in\mathbb{Z}} f(q^n)\, q^n$$

provided the summation in the RHS converges absolutely. •

Notice this is different from integration on $\mathbb{R}_q^+ \equiv (\mathbb{R}_q^+, m_q)$ as introduced at the beginning of section §6.2. Furthermore we shall actually be dealing with q^2-integration rather than q-integration.

First we establish a link between q^2-moments and q-Hankel transformation:

Proposition 6.3.2 *Take any $m \in \mathbb{N}$ and $f \in \mathcal{R}_m$. Then*

$$(1-q^2)\,(q^2;q^2)_m\,(\mathfrak{H}_m f)(0) \;=\; \int_0^\infty x^m f(x)\,d_{q^2}x.$$

Observe that the q^2-integral is well-defined since f belongs to $\mathcal{S}(\mathbb{R}^+;q^2)$.

Proof. Let (f,g) be an $(m;q)$-Hankel pair. In the proof of proposition 6.2.4.2 we observed that $(\mathfrak{H}_m f)(z^2) = z^{-m} g(z)$ for all $z \in \mathbb{C}_0$. In particular we obtain for any $k \in \mathbb{Z}$ that

$$\begin{aligned}
(\mathfrak{H}_m f)(q^{2k}) &= q^{-mk}\, g(q^k) \\
&= q^{-mk}\,(H_m R_q \Psi^m K f)(q^k) \\
&\overset{(6.10)}{=} \sum_{n\in\mathbb{Z}} q^{-m(n+k)} J_m(q^{n+k};q^2)\, q^{2(m+1)n} f(q^{2n}) \\
&= \sum_{n\in\mathbb{Z}} \mathfrak{I}_m(q^{n+k})\, q^{2(m+1)n} f(q^{2n})
\end{aligned}$$

where $\mathfrak{I}_m$ is the entire function satisfying

$$\mathfrak{I}_m(z) \;=\; z^{-m} J_m(z;q^2)$$

for all $z \in \mathbb{C}_0$ (the singularity at the origin being removable, cf. lemma 6.1.1). For any $k,n \in \mathbb{Z}$, let $t_{n,k}$ denote the number

$$t_{n,k} \;=\; \mathfrak{I}_m(q^{n+k})\, q^{2(m+1)n} f(q^{2n}).$$

Since $\mathfrak{H}_m f$ is entire, it is a fortiori continuous at the origin, and therefore

$$(\mathfrak{H}_m f)(0) \;=\; \lim_{k\to+\infty} (\mathfrak{H}_m f)(q^{2k}) \;=\; \lim_{k\to+\infty} \Big(\textstyle\sum_{n\in\mathbb{Z}} t_{n,k}\Big) \tag{6.40}$$

Now comes the tricky part: we have to compute the above limit, which amounts to interchanging the limit and the summation. To do so, we rely on dominated convergence and the estimate (6.9) for the little q-Bessel functions: indeed, whenever $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $k+n \geq 0$, we have

$$|t_{n,k}| \;\leq\; C_m\, q^{2(m+1)n}\, |f(q^{2n})|.$$

On the other hand, if $k+n \leq 0$, then $n \leq -k \leq 0$, hence $q^{-k} \leq q^n$ and

$$\begin{aligned}
|t_{n,k}| &\leq C_m\, q^{-2m(n+k)}\, q^{2(m+1)n}\, |f(q^{2n})| \\
&= C_m\, q^{-2mk}\, q^{2n}\, |f(q^{2n})| \\
&\leq C_m\, q^{2mn}\, q^{2n}\, |f(q^{2n})|.
\end{aligned}$$

In both cases it follows that

$$|t_{n,k}| \;\leq\; C_m\, B_m$$

where

$$B_m \;=\; \sup\Big\{\, |f(q^{2n})|\, q^{2n(m+1)} \;\Big|\; n \in \mathbb{Z} \Big\} \;<\; \infty$$

because f belongs to $\mathcal{S}(\mathbb{R}^+; q^2)$. Thus (6.40) becomes

$$\begin{aligned} (\mathfrak{H}_m f)(0) &= \sum_{n \in \mathbb{Z}} \lim_{k \to +\infty} t_{n,k} \\ &= \mathfrak{J}_m(0) \sum_{n \in \mathbb{Z}} q^{2mn}\, f(q^{2n})\, q^{2n} \\ &= \frac{\mathfrak{J}_m(0)}{1-q^2} \int_0^\infty x^m\, f(x)\, d_{q^2}x. \end{aligned}$$

Now it only remains to compute $\mathfrak{J}_m(0)$. It is clear however that $\mathfrak{J}_m(0)$ is nothing but the $(k=0)$-coefficient in the power series (6.1) defining the little q^2-Bessel functions (q replaced with q^2). Thus we obtain

$$\mathfrak{J}_m(0) \;=\; \frac{(q^{2(m+1)}; q^2)_\infty}{(q^2; q^2)_\infty} \;=\; \frac{1}{(q^2; q^2)_m}.$$

This completes the proof. ∎

Lemma 6.3.3 *If f is an entire function such that $(D_q^m f)(0) = 0$ for all $m \in \mathbb{N}$, then $f = 0$.*

Proof. Let's first introduce the following notion of q-factorials: for $n \in \mathbb{N}$, put

$$[\![n]\!]_q \;=\; \frac{1-q^n}{1-q} \qquad\qquad [\![0]\!]_q! \;=\; 1 \qquad\qquad [\![n]\!]_q! \;=\; [\![1]\!]_q [\![2]\!]_q \cdots [\![n]\!]_q.$$

Now let $f(z) = \sum_{k=0}^\infty a_k\, z^k$ be the power series of f around the origin. Then

$$(D_q f)(z) \;=\; \frac{1}{(1-q)z} \left(\sum_{k=0}^\infty a_k\, z^k \;-\; \sum_{k=0}^\infty a_k\, (qz)^k \right) \;=\; \sum_{k=1}^\infty [\![k]\!]_q\, a_k\, z^{k-1}$$

for all $z \in \mathbb{C}_0$ (and hence also for $z = 0$). Iterating this $m \in \mathbb{N}$ times and then evaluating at the origin, we obtain $(D_q^m f)(0) = [\![m]\!]_q!\, a_m$. The result follows. ∎

Lemma 6.3.4 *If $f \in \mathcal{R}$ such that $(\mathfrak{H}_m f)(0) = 0$ for all $m \in \mathbb{N}$, then $f = 0$.*

Proof. Recall the formula (6.25) which was derived from proposition 6.2.4.6, and apply it to f:

$$\mathfrak{H}_m f \;=\; (\text{scalar})\, \Omega^{-2m} D_{q^2}^m \mathfrak{H}_0 f$$

With our assumptions it follows that $(D_{q^2}^m \mathfrak{H}_0 f)(0) = 0$ for all $m \in \mathbb{N}$. Because $\mathfrak{H}_0 f$ is entire, lemma 6.3.3 (q replaced with q^2) yields the result. ∎

Combining the above lemma with proposition 6.3.2, we get

Theorem 6.3.5 *If $f \in \mathcal{R}$ has vanishing q^2-moments, i.e.*

$$\int_0^\infty x^m f(x)\, d_{q^2}x \;=\; 0$$

for all $m \in \mathbb{N}$, then $f = 0$.

6.4 Fourier transforms for quantum $E(2)$

Abstract 6.4 We introduce a Fourier context for the quantum $E(2)$ group and then explicitly construct its Fourier transforms, involving holomorphic q-Hankel transformation. Of course we then have to *prove* that the formulas we propose really are the Fourier transforms; in this respect propositions 6.2.4.6 and 6.3.2 will play a key-role. It also turns out that the dilation and quantization parameters τ, ν and q cannot be chosen independently. ⋆

6.4.1 A Fourier context for the quantum $E(2)$ group

Henceforth the positive numbers τ and ν will be given by

$$\tau \;=\; q^{-1} \qquad \text{and} \qquad \nu \;=\; (q^{-1}-q)^{-2}. \tag{6.41}$$

We shall need some dilation operators on $H(\mathbb{C})$ similar to Ω, though for an entirely different purpose: for any $r > 0$, let Λ_r be the linear mapping on $H(\mathbb{C})$ given by $(\Lambda_r g)(z) = g(r^{-1}z)$. We are mainly interested in the cases $r = \tau$ and $r = \nu$, and in particular we have $\Lambda_\tau = \Omega$. Also observe the commutation rule $\Lambda_r \Psi = r^{-1}\Psi\Lambda_r$. Accordingly, we shall need dilated versions of the space $\mathcal{E}$ in definition 6.2.6.3. Therefore define $\mathcal{E}_r = \Lambda_r\mathcal{E}$, for any $r > 0$. Notice that $\mathcal{E}_r$ is contained in $\mathcal{S}_r(\mathbb{R}^+; q^2)$. Furthermore $\mathcal{E}_r$ is non-trivial since $\mathcal{E}$ is non-trivial (cf. corollary 6.2.7.7).

We are about to invoke theorem 5.6.1, which will provide a setting for doing harmonic analysis on the quantum $E(2)$ group. From the above observations and proposition 6.2.6.4, it follows that $\mathcal{G}'_r \equiv \mathcal{E}_r$ satisfies assumptions 5.4.1.1, for any $r > 0$ and in particular for $r = \tau, \nu$. On the other hand, we already know from example 5.4.1.2 that $\mathcal{G}_r \equiv \mathcal{S}_r(\mathbb{R}^+; q^2)$ is a subalgebra of $H(\mathbb{C})$ satisfying assumptions 5.4.1.1. Eventually theorem 6.3.5 yields that $\mathcal{E}_\tau$ and $\mathcal{E}_\nu$ enjoy condition (iii) of theorem 5.6.1. We conclude:

Theorem 6.4.1.1 *The system*

$$\Big(\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big) \subseteq \mathcal{U}_q\big(\mathfrak{L}(\mathcal{S}_\tau(\mathbb{R}^+; q^2))\big)\,, \varphi, \psi;\, \mathcal{U}_q, \mathcal{A}_q;\; \mathcal{A}_q(\mathcal{E}_\nu) \subseteq \mathcal{A}_q\big(\mathcal{S}_\nu(\mathbb{R}^+; q^2)\big)\,, \omega\Big)$$

is a Fourier context (cf. chapter 4).

6.4.2 Parity structure: an important slice map

Let ϕ_0 be the linear functional on $F(\mathbb{Z}\theta)$ defined by $\phi_0(f) = f(0)$. Now we consider the slice map $\phi_0 \otimes \mathrm{id}$ from $F(\mathbb{Z}\theta) \otimes H(\mathbb{C})$ into $H(\mathbb{C})$. Although trivial, the following lemma is essential for the construction of Fourier transforms and crucial in understanding the 'parity' structure (cf. remark 5.4.1.9) that entered the construction of the Haar functionals φ and ψ. Recall $\mathfrak{L}(\mathcal{E}_\tau)$ is defined as

$$\mathfrak{L}(\mathcal{E}_\tau) \;=\; \Big(K^{\mathrm{even}}(\mathbb{Z}\theta) \otimes \mathcal{E}_\tau^{\mathrm{even}}\Big) \;\oplus\; \Big(K^{\mathrm{odd}}(\mathbb{Z}\theta) \otimes \mathcal{E}_\tau^{\mathrm{odd}}\Big). \tag{6.42}$$

Lemma 6.4.2.1 *The slice map* $\phi_0 \otimes \mathrm{id}$ *maps* $\mathfrak{L}(\mathcal{E}_\tau)$ *into* $\mathcal{E}_\tau$*.*

Proof. Clearly ϕ_0 annihilates $K^{\mathrm{odd}}(\mathbb{Z}\theta)$ and therefore only the 'even' component of (6.42) is able to survive $\phi_0 \otimes \mathrm{id}$. The result follows since $\mathcal{E}_\tau^{\mathrm{even}} = \mathcal{E}_\tau$. ∎

Remark 6.4.2.2 We claim that (6.42) offers the most natural way towards the above result. To appreciate this a little more, let's try to come up with an alternative: let's suppose for instance, that we would replace $\mathfrak{L}\big(\mathcal{S}_\tau(\mathbb{R}^+; q^2)\big)$ by the space $K(\mathbb{Z}\theta) \otimes \mathcal{S}_\tau(\mathbb{R}^+; q)$ proposed in remark 5.4.1.3.iii. In view of (5.12) the corresponding alternative for $\mathfrak{L}(\mathcal{E}_\tau)$ would be $K(\mathbb{Z}\theta) \otimes \big(\mathcal{E}_\tau^{\mathrm{even}} \cap \mathcal{E}_\tau^{\mathrm{odd}}\big)$. Also with these choices it should be possible to produce a Fourier context similar to the one of theorem 6.4.1.1 above, and moreover the latter choice would qualify w.r.t. lemma 6.4.2.1. However we run into trouble by taking the *intersection* of $\mathcal{E}_\tau^{\mathrm{even}}$ and $\mathcal{E}_\tau^{\mathrm{odd}}$. The point is that in this operation we probably loose all the interesting functions supplied by corollary 6.2.7.7, and in fact $\mathcal{E}_\tau^{\mathrm{even}} \cap \mathcal{E}_\tau^{\mathrm{odd}}$ might as well be trivial—the question is still open. Anyway it should be clear by now that (6.42) is much more natural.
One more try: could we take $\mathfrak{L}(\mathcal{E}_\tau)$ to be the even component in (6.42) only? Again the answer is negative, since then $\mathfrak{L}(\mathcal{E}_\tau)$ would no longer be invariant under $\Gamma \otimes \Omega^{\pm 1}$ or $\Gamma^{-1} \otimes \Omega^{\pm 1}$, which was essential in proposition 5.3.3.1. So there seems to be no valuable alternative for (6.42) and moreover, our choice shall be manifestly confirmed by further results (cf. §6.5 and §6.6). ⋆

6.4.3 Yet another 'H'

In this paragraph we introduce and study a family $\mathbb{H}_m^{(k)}$ (with $m, k \in \mathbb{Z}$) of linear transforms from $\mathfrak{L}(\mathcal{E}_\tau)$ into $\mathcal{E}_\nu$, making diagram (6.43) below commute. Basically these $\mathbb{H}_m^{(k)}$ can still be thought of as some kind of q-Hankel transform.

Definition 6.4.3.1 Whenever $m, k \in \mathbb{Z}$, let $\kappa_q(m,k)$ denote the scalar

$$\kappa_q(m,k) \;=\; (-1)^m q^{-m} q^{\frac{1}{2}|m|(k-1)} (q^{-1} - q)^{|m|}.$$

Furthermore let $\mathbb{H}_m^{(k)}$ be the following linear map from $\mathfrak{L}(\mathcal{E}_\tau)$ into $\mathcal{E}_\nu$:

$$\mathbb{H}_m^{(k)} \;=\; \kappa_q(m,k)\, \Lambda_\nu\, \mathfrak{H}_m\, \Lambda_\tau^{-1}\, (\phi_0 \otimes \mathrm{id})\, (\Gamma^{-1} \otimes \Omega)^k_{|\mathfrak{L}(\mathcal{E}_\tau)}$$

That this definition makes sense can be seen from the following diagram:

$$\begin{tikzcd}
\mathfrak{L}(\mathcal{E}_\tau) \arrow[r, "(\Gamma^{-1}\otimes\Omega)^k"] \arrow[d, "\mathbb{H}_m^{(k)}"'] & \mathfrak{L}(\mathcal{E}_\tau) \arrow[r, "\phi_0\otimes\mathrm{id}"] & \mathcal{E}_\tau \arrow[r, "\Lambda_\tau^{-1}"] & \mathcal{E} \arrow[d, "\mathfrak{H}_m"] \\
\mathcal{E}_\nu & \mathcal{E}_\nu \arrow[l] & \mathcal{E}_\nu \arrow[l, "\kappa_q(m,k)"] & \mathcal{E} \arrow[l, "\Lambda_\nu"]
\end{tikzcd} \tag{6.43}$$

Diagram definition of $\mathbb{H}_m^{(k)}$

This diagram in itself does make several statements: clearly lemma 6.4.2.1 is involved, as well as proposition 6.2.6.4, stating that $\mathcal{E}$ is invariant under all $\mathfrak{H}_m$. Furthermore we need $\mathfrak{L}(\mathcal{E}_\tau)$ to be invariant under $\Gamma^{\pm1}\otimes\Omega^{\mp1}$, which is an immediate consequence of (6.42). In this respect, notice how $\Gamma^{\pm1}\otimes\Omega^{\mp1}$ interchanges[7] the even and odd components in (6.42). •

Lemma 6.4.3.2 *For any $m,k\in\mathbb{Z}$ we have*

$$\kappa_q(m,k) = q^{\frac12|m|}\,\kappa_q(m,k-1) \tag{6.44}$$

$$(q^{-1}-q)\,\kappa_q(m-1,k) = -q^{\frac32-\frac12k}\,\kappa_q(m,k) \qquad \text{if } m\geq 1 \tag{6.45}$$

$$(q^{-1}-q)\,\kappa_q(-m+1,k) = -q^{-\frac12-\frac12k}\,\kappa_q(-m,k) \qquad \text{if } m\geq 1. \tag{6.46}$$

Proof. Straightforward. ■

In §5.4.1 we observed that a space like $\mathfrak{L}(\mathcal{E}_\tau)$ satisfies the conditions of proposition 5.3.3.1. In particular, $\mathfrak{L}(\mathcal{E}_\tau)$ is invariant under

$$\Gamma^{\pm1}\otimes\Omega^{\mp1} \qquad \mathrm{id}\otimes\Omega^{\pm2} \qquad \Phi^{\pm1}\otimes\mathrm{id} \qquad \mathrm{id}\otimes\Omega^{\pm1}\nabla_q^{(m)} \qquad \mathrm{id}\otimes D_{q^2}.$$

So the question arises how the $\mathbb{H}_m^{(k)}$ behave under these operations:

Lemma 6.4.3.3 *For any $m,k\in\mathbb{Z}$ we have*

$$\begin{array}{lcll}
\mathbb{H}_m^{(k)}(\Gamma\otimes\Omega^{-1}) & = & q^{\frac12|m|}\,\mathbb{H}_m^{(k-1)} & \\
\mathbb{H}_m^{(k)}(\Phi\otimes\mathrm{id}) & = & q^{-\frac12k}\,\mathbb{H}_m^{(k)} & \\
\mathbb{H}_m^{(k)}(\mathrm{id}\otimes\Omega^2) & = & q^{-2|m|-2}\,\Omega^{-2}\,\mathbb{H}_m^{(k)} & \\
\mathbb{H}_{m-1}^{(k)}(\mathrm{id}\otimes\Omega\nabla_q^{(m)}) & = & -q^{\frac32-\frac12k-m}\,\Psi\,\mathbb{H}_m^{(k)} & \text{if } m\geq 1 \\
\mathbb{H}_{-m+1}^{(k)}(\mathrm{id}\otimes\Omega^{-1}\nabla_q^{(m)}) & = & q^{-\frac12-\frac12k+m}\,\Psi\,\mathbb{H}_{-m}^{(k)} & \text{if } m\geq 1 \\
\mathbb{H}_{m+1}^{(k)}(\mathrm{id}\otimes D_{q^2}) & = & q^{-\frac32-\frac12k}\,\mathbb{H}_m^{(k)}(\mathrm{id}\otimes\Omega^2) & \text{if } m\geq 0 \\
\mathbb{H}_{-m-1}^{(k)}(\mathrm{id}\otimes D_{q^2}) & = & -q^{\frac12-\frac12k}\,\mathbb{H}_{-m}^{(k)} & \text{if } m\geq 0.
\end{array}$$

[7] which is just one of the reasons why *both* components should be present.

Proof. The first formula is an immediate consequence of (6.44) whereas the second follows from $\phi_0\Phi = \phi_0$ and the commutation rule $\Gamma^{-k}\Phi = q^{-\frac{1}{2}k}\,\Phi\Gamma^{-k}$. The third formula corresponds to (6.18). The remaining formulas are all consequences of proposition 6.2.4.6 and lemma 6.4.3.2. Let's for instance check the last but one: using commutation rules like $\Omega^k D_{q^2} = q^{-k}\, D_{q^2}\Omega^k$, we get

$$\begin{aligned}
&\mathbb{H}^{(k)}_{m+1}(\mathrm{id}\otimes D_{q^2})\\
&\quad= \kappa_q(m+1,k)\;\Lambda_\nu\,\mathfrak{H}_{m+1}\,\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k(\mathrm{id}\otimes D_{q^2})\\
&\quad= q^{-k}\,\tau^{-1}\,\kappa_q(m+1,k)\;\Lambda_\nu\,\mathfrak{H}_{m+1}\,D_{q^2}\,\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k\\
&\quad\overset{(*)}{=} -q^{-k}q\,\frac{q^{-1}}{q^{-1}-q}\,\kappa_q(m+1,k)\;\Lambda_\nu\,\mathfrak{H}_m\Omega^2\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k\\
&\quad\overset{(6.45)}{=} q^{-k}\,q^{-\frac{3}{2}+\frac{1}{2}k}\,\kappa_q(m,k)\;\Lambda_\nu\,\mathfrak{H}_m\,\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k(\mathrm{id}\otimes\Omega^2)\\
&\quad= q^{-\frac{3}{2}-\frac{1}{2}k}\,\mathbb{H}^{(k)}_m\,(\mathrm{id}\otimes\Omega^2)
\end{aligned}$$

for $m \geq 0$. In $(*)$ we used the first formula of proposition 6.2.4.6. Also notice that the value of τ was involved in the computation. In order to appreciate the particular choice (6.41) we made for ν, we shall also have a closer look at for instance the fifth formula in the list: for any $m \geq 1$ we have

$$\begin{aligned}
&\mathbb{H}^{(k)}_{-m+1}(\mathrm{id}\otimes\Omega^{-1}\nabla_q^{(m)})\\
&\quad= \kappa_q(-m+1,k)\;\Lambda_\nu\,\mathfrak{H}_{-m+1}\,\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k(\mathrm{id}\otimes\Omega^{-1}\nabla_q^{(m)})\\
&\quad= \kappa_q(-m+1,k)\;\Lambda_\nu\,\mathfrak{H}_{-m+1}\,\nabla_q^{(m)}\Omega^{-1}\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k\\
&\quad\overset{(\sharp)}{=} -\frac{1}{q^{-1}-q}\,q^m\,\kappa_q(-m+1,k)\;\Lambda_\nu\,\Psi\mathfrak{H}_{-m}\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k.
\end{aligned}$$

In $(\sharp)$ we used the last formula of proposition 6.2.4.6. Now the parameter ν gets involved, since we need to use the commutation rule

$$\Lambda_\nu\Psi \;=\; \nu^{-1}\Psi\Lambda_\nu \;=\; (q^{-1}-q)^2\Psi\Lambda_\nu$$

yielding

$$\begin{aligned}
\cdots &= -q^m\,(q^{-1}-q)\kappa_q(-m+1,k)\;\Psi\Lambda_\nu\mathfrak{H}_{-m}\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k\\
&\overset{(6.46)}{=} q^m\,q^{-\frac{1}{2}-\frac{1}{2}k}\,\kappa_q(-m,k)\;\Psi\Lambda_\nu\mathfrak{H}_{-m}\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k\\
&= q^{-\frac{1}{2}-\frac{1}{2}k+m}\;\Psi\,\mathbb{H}^{(k)}_{-m}.
\end{aligned}$$

The other cases are similar. ∎

Remark 6.4.3.4 Observe that it is really essential to take $\nu = (q^{-1}-q)^{-2}$ in order to get rid of all the $(q^{-1}-q)$ factors involved in the last computation. Indeed merely adjusting[8] the definition of the scalars κ_q wouldn't yield the

[8] for instance, $\kappa_q(m,k) = \ldots(q^{-1}-q)^{-|m|}$ instead of $\ldots(q^{-1}-q)^{|m|}$

proper effect, since then these $(q^{-1}-q)$ factors would simply emerge somewhere else (more precisely, in the last two formulas of lemma 6.4.3.3). Soon it will become clear *why* we actually want these $(q^{-1}-q)$ factors to cancel (cf. proof of proposition 6.4.4.2). ⋆

Lemma 6.4.3.5 *Take any $X \in \mathfrak{L}(\mathcal{E}_\tau)$ and $m \in \mathbb{Z}$. Then $\mathbb{H}^{(k)}_m X = 0$ except for finitely many $k \in \mathbb{Z}$.*

Proof. By definition $\mathfrak{L}(\mathcal{E}_\tau) \subseteq K(\mathbb{Z}\theta) \otimes H(\mathbb{C})$. Writing $X = \sum_i f_i \otimes g_i$ with $f_i \in K(\mathbb{Z}\theta)$ and $g_i \in H(\mathbb{C})$, we get

$$(\phi_0 \otimes \mathrm{id})\,(\Gamma^{-1} \otimes \Omega)^k X \;=\; \textstyle\sum_i \; f_i(k\theta)\,\Omega^k g_i.$$

Since all the f_i have finite support, the result follows. ∎

6.4.4 Explicit construction of Fourier transforms

Definition 6.4.4.1 Define linear mappings F_R and F_L from $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$ into $\mathcal{A}_q(\mathcal{E}_\nu)$ by

$$\begin{aligned}
F_R\big(\Upsilon(X)\,b^m\big) &= \sum_{k\in\mathbb{Z}} \alpha^{k+m}\,\gamma^m \left(\mathbb{H}^{(k)}_m X\right)(\gamma^*\gamma) \\
F_R\big(\Upsilon(X)\,c^m\big) &= \sum_{k\in\mathbb{Z}} \alpha^{k-m}\,(\gamma^*)^m \left(\mathbb{H}^{(k)}_{-m} X\right)(\gamma^*\gamma) \\
F_L\big(\Upsilon(X)\,b^m\big) &= \sum_{k\in\mathbb{Z}} (q^2\alpha)^{k+m}\,\gamma^m \left(\mathbb{H}^{(k)}_m X\right)(\gamma^*\gamma) \\
F_L\big(\Upsilon(X)\,c^m\big) &= \sum_{k\in\mathbb{Z}} (q^2\alpha)^{k-m}\,(\gamma^*)^m \left(\mathbb{H}^{(k)}_{-m} X\right)(\gamma^*\gamma)
\end{aligned}$$

for any $X \in \mathfrak{L}(\mathcal{E}_\tau)$ and $m \in \mathbb{N}$. Observe the summations over $k \in \mathbb{Z}$ are in fact *finite* sums because of lemma 6.4.3.5. Also notice the formulae are in agreement when $m = 0$. •

Proposition 6.4.4.2 *The maps F_R and F_L are Fourier transforms w.r.t. the Fourier context established in theorem 6.4.1.1. More precisely F_R is both an* RL *and an* RR *transform, whereas F_L is an* LR *and* LL *transform. In particular this means that (cf. definition 4.1.2)*

i. F_R and F_L are respectively left and right $\mathcal{A}_q$-module morphisms, and

ii. $\omega F_R = \langle\,\cdot\,,1\rangle = \omega F_L$.

These Fourier transforms are unique within the setting of theorem 6.4.1.1.

Proof. *Uniqueness* of Fourier transforms is a property of any Fourier context (cf. lemma 4.1.5). Nevertheless it may be an instructive exercise to observe how

in this particular case uniqueness of Fourier transforms eventually amounts to the q-moment problem for the space $\mathcal{R}$ as treated in §6.2.
Now let us show (ii). Observe that for any $X \in \mathfrak{L}(\mathcal{E}_\tau)$ and $m \in \mathbb{N}$ we have

$$\begin{aligned}
&\omega\Big(F_R\big(\Upsilon(X)\, b^m\big)\Big) \\
&= \omega\left(\sum_{k\in\mathbb{Z}} \alpha^{k+m}\,\gamma^m\,\Big(\mathbb{H}_m^{(k)}\,X\Big)\,(\gamma^*\gamma)\right) \\
&= \delta_{m,0}\ \omega\left[\Big(\mathbb{H}_0^{(0)}\,X\Big)\,(\gamma^*\gamma)\right] \\
&= \delta_{m,0}\sum_{n\in\mathbb{Z}}\Big(\mathbb{H}_0^{(0)}X\Big)\,(\nu q^{2n})\,q^{2n} \\
&= \delta_{m,0}\sum_{n\in\mathbb{Z}}\kappa_q(0,0)\ \big(\Lambda_\nu\,\mathfrak{H}_0\,\Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})X\big)\,(\nu q^{2n})\,q^{2n} \\
&= \delta_{m,0}\,\frac{1}{1-q^2}\int_0^\infty\big(\mathfrak{H}_0\,\Lambda_\tau^{-1}(\phi_0\otimes\mathrm{id})X\big)\,(x)\,d_{q^2}x \\
&= \delta_{m,0}\,(q^2;q^2)_0\,\big(\mathfrak{H}_0\mathfrak{H}_0\,\Lambda_\tau^{-1}(\phi_0\otimes\mathrm{id})X\big)\,(0) \\
&= \delta_{m,0}\,X(0,0) \\
&= \langle\Upsilon(X)\,b^m,1\rangle.
\end{aligned}$$

Here we used the definitions of F_R, ω, $\mathbb{H}_0^{(0)}$, κ_q and q^2-integration. We also used proposition 6.3.2 and $\mathfrak{H}_0^2 = \mathrm{id}$. The last equality is clear from definition 5.2.2.1 and lemma 5.3.1.4. Similarly we deal with $\Upsilon(X)c^m$ and with F_L.

To verify the *module* properties, it suffices to check the action of the generators α, β and γ, both on elements of type $\Upsilon(X)\,b^m$ and of type $\Upsilon(X)\,c^m$, and both for F_R and F_L. All together this yields 12 cases to check; we shall have a closer look on only 4 of them. However the remaining 8 cases are more or less analogous: they are all consequences of proposition 5.3.2.2 and lemma 6.4.3.3. In what follows, X is an arbitrary element in $\mathfrak{L}(\mathcal{E}_\tau)$ and $m \in \mathbb{N}$. Now consider for instance the following case:

$$\begin{aligned}
F_R\big(\alpha \triangleright \Upsilon(X)\,b^m\big) &\overset{(A)}{=} q^{-\frac{1}{2}m}\,F_R\Big(\Upsilon\big((\Gamma\otimes\Omega^{-1})X\big)\ b^m\Big) \\
&\overset{(F)}{=} \sum_{k\in\mathbb{Z}} q^{-\frac{1}{2}m}\,\alpha^{k+m}\,\gamma^m\,\Big(\mathbb{H}_m^{(k)}\,(\Gamma\otimes\Omega^{-1})X\Big)\,(\gamma^*\gamma) \\
&\overset{(H)}{=} \sum_{k\in\mathbb{Z}} q^{-\frac{1}{2}m}q^{\frac{1}{2}|m|}\,\alpha^{k+m}\,\gamma^m\,\Big(\mathbb{H}_m^{(k-1)}X\Big)\,(\gamma^*\gamma) \\
&\overset{(S)}{=} \sum_{k\in\mathbb{Z}} \alpha^{k+1+m}\,\gamma^m\,\Big(\mathbb{H}_m^{(k)}X\Big)\,(\gamma^*\gamma) \\
&\overset{(F)}{=} \alpha\,F_R\big(\Upsilon(X)\,b^m\big)\,.
\end{aligned}$$

Let's explain the labeling: (A) refers to the formulas for the $\mathcal{A}_q$-actions as computed in proposition 5.3.2.2. (F) refers to the definition of our Fourier transforms, i.e. definition 6.4.4.1. (H) links to the properties of the maps $\mathbb{H}_m^{(k)}$ as given in lemma 6.4.3.3, and eventually (S) means that the summation variable k is substituted with $k+1$. In what follows, (C) will indicate that we use some commutation rule involving the generators, for instance $\alpha\beta = q\,\beta\alpha$.

$$
\begin{aligned}
&F_R\big(\beta \triangleright \Upsilon(X)\,c^m\big)\\
&\overset{(A)}{=} q^{\frac{1}{2}(m+1)}\,F_R\Big(\Upsilon\big((\Phi\otimes\mathrm{id})(\Gamma\otimes\Omega^{-1})(\mathrm{id}\otimes D_{q^2})X\big)\,c^{m+1}\Big)\\
&\overset{(F)}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(m+1)}\,\alpha^{k-m-1}\,(\gamma^*)^{m+1}\\
&\qquad\qquad\Big(\mathbb{H}_{-m-1}^{(k)}(\Phi\otimes\mathrm{id})(\Gamma\otimes\Omega^{-1})(\mathrm{id}\otimes D_{q^2})X\Big)(\gamma^*\gamma)\\
&\overset{(H)}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(m+1)}q^{-\frac{1}{2}k}\,\alpha^{k-m-1}(\gamma^*)^{m+1}\Big(\mathbb{H}_{-m-1}^{(k)}(\Gamma\otimes\Omega^{-1})(\mathrm{id}\otimes D_{q^2})X\Big)(\gamma^*\gamma)\\
&\overset{(H)}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(m+1-k)}\,q^{\frac{1}{2}|-m-1|}\,\alpha^{k-m-1}\,(\gamma^*)^{m+1}\Big(\mathbb{H}_{-m-1}^{(k-1)}\big(\mathrm{id}\otimes D_{q^2}\big)X\Big)(\gamma^*\gamma)\\
&\overset{(H)}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(m+1-k)}\,q^{\frac{1}{2}(m+1)}(-1)\,q^{\frac{1}{2}-\frac{1}{2}(k-1)}\,\alpha^{k-m-1}(\gamma^*)^{m+1}\Big(\mathbb{H}_{-m}^{(k-1)}X\Big)(\gamma^*\gamma)\\
&= \sum_{k\in\mathbb{Z}} (-1)\,q^{m-k+2}\alpha^{k-m-1}\,(-q^{-1}\beta)\,(\gamma^*)^m\,\Big(\mathbb{H}_{-m}^{(k-1)}X\Big)(\gamma^*\gamma)\\
&\overset{(C)}{=} \sum_{k\in\mathbb{Z}} \beta\,\alpha^{k-m-1}\,(\gamma^*)^m\,\Big(\mathbb{H}_{-m}^{(k-1)}X\Big)(\gamma^*\gamma)\\
&\overset{(S)}{=} \beta\,F_R\big(\Upsilon(X)\,c^m\big)\,.
\end{aligned}
$$

Our next case deals with right actions and is a little more involved:

$$
\begin{aligned}
&F_L\big(\Upsilon(X)\,c^m \triangleleft \alpha\big)\\
&\overset{(A)}{=} q^{\frac{1}{2}m}\,F_L\Big(\Upsilon\big((\Gamma\otimes\Omega)X\big)\,c^m\Big)\\
&\overset{(F)}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}m}\,q^{2(k-m)}\,\alpha^{k-m}\,(\gamma^*)^m\,\Big(\mathbb{H}_{-m}^{(k)}(\Gamma\otimes\Omega^{-1})(\mathrm{id}\otimes\Omega^2)X\Big)(\gamma^*\gamma)\\
&\overset{(H)}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}m}\,q^{2(k-m)}\,q^{\frac{1}{2}|-m|}\,\alpha^{k-m}\,(\gamma^*)^m\,\Big(\mathbb{H}_{-m}^{(k-1)}(\mathrm{id}\otimes\Omega^2)X\Big)(\gamma^*\gamma)\\
&\overset{(H)}{=} \sum_{k\in\mathbb{Z}} q^{2(k-m)}\,q^m\,q^{-2|-m|-2}\,\alpha^{k-m}\,(\gamma^*)^m\,\Big(\Omega^{-2}\mathbb{H}_{-m}^{(k-1)}X\Big)(\gamma^*\gamma)\\
&\overset{(S)}{=} \sum_{k\in\mathbb{Z}} q^{2(k+1-m)}\,q^{-m-2}\,\alpha^{k+1-m}\,(\gamma^*)^m\,\Big(\Omega^{-2}\mathbb{H}_{-m}^{(k)}X\Big)(\gamma^*\gamma)\\
&= \sum_{k\in\mathbb{Z}} q^{2(k-m)}\,q^{-m}\,\alpha^{k-m}\,\alpha\,(\gamma^*)^m\,\Big(\Omega^{-2}\mathbb{H}_{-m}^{(k)}X\Big)(\gamma^*\gamma)
\end{aligned}
$$

$$\stackrel{\text{(C)}}{=} \sum_{k\in\mathbb{Z}} q^{2(k-m)}\,\alpha^{k-m}\,(\gamma^*)^m\,\Big(\mathbb{H}_{-m}^{(k)}X\Big)(\gamma^*\gamma)\,\alpha$$
$$\stackrel{\text{(F)}}{=} F_L\big(\Upsilon(X)\,c^m\big)\,\alpha.$$

In the above computation, besides $\alpha\beta = q\,\beta\alpha$, the label (C) also refers to the commutation rule in lemma 5.5.1.2.
Let us consider one more example: assuming $m \geq 1$, we have

$$F_L\big(\Upsilon(X)\,b^m \triangleleft \beta\big)$$
$$\stackrel{\text{(A)}}{=} q^{\frac{1}{2}(m-1)}\,F_L\Big(\Upsilon\big((\Phi^{-1}\otimes\mathrm{id})(\Gamma\otimes\Omega^{-1})(\mathrm{id}\otimes\Omega\nabla_q^{(m)})X\big)\,b^{m-1}\Big)$$
$$\stackrel{\text{(F)}}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(m-1)}\,(q^2\alpha)^{k+m-1}\,\gamma^{m-1} \Big(\mathbb{H}_{m-1}^{(k)}(\Phi^{-1}\otimes\mathrm{id})(\Gamma\otimes\Omega^{-1})(\mathrm{id}\otimes\Omega\nabla_q^{(m)})X\Big)(\gamma^*\gamma)$$
$$\stackrel{\text{(H)}}{=} \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(m-1)}\,q^{2(k+m-1)}\,q^{\frac{1}{2}k}\,q^{\frac{1}{2}|m-1|}\,\alpha^{k+m-1}\,\gamma^{m-1} \Big(\mathbb{H}_{m-1}^{(k-1)}\,\big(\mathrm{id}\otimes\Omega\nabla_q^{(m)}\big)\,X\Big)(\gamma^*\gamma)$$
$$\stackrel{\text{(H)}}{=} \sum_{k\in\mathbb{Z}} q^{m-1}\,q^{2(k+m-1)}\,q^{\frac{1}{2}k}\,(-1)\,q^{\frac{3}{2}-\frac{1}{2}(k-1)-m}\,\alpha^{k+m-1}\,\gamma^{m-1} \Big(\Psi\,\mathbb{H}_{m}^{(k-1)}X\Big)(\gamma^*\gamma)$$
$$\stackrel{\text{(M)}}{=} \sum_{k\in\mathbb{Z}} (-1)\,q^{-1}\,q^{2(k+m)}\,\alpha^{k+m-1}\,\gamma^{m-1}\,(\gamma^*\gamma)\,\Big(\mathbb{H}_{m}^{(k-1)}X\Big)(\gamma^*\gamma)$$
$$\stackrel{\text{(C)}}{=} \sum_{k\in\mathbb{Z}} (-1)\,q^{-1}\,q^{2(k+m)}\,\alpha^{k+m-1}\,\gamma^{m}\,(-q^{-1}\beta)\,\Big(\mathbb{H}_{m}^{(k-1)}X\Big)(\gamma^*\gamma)$$
$$\stackrel{\text{(C)}}{=} \sum_{k\in\mathbb{Z}} q^{-2}\,q^{2(k+m)}\,\alpha^{k+m-1}\,\gamma^{m}\,\Big(\mathbb{H}_{m}^{(k-1)}X\Big)(\gamma^*\gamma)\,\beta$$
$$\stackrel{\text{(S)}}{=} F_L\big(\Upsilon(X)\,b^m\big)\,\beta.$$

Here (M) refers to an obvious property of our functional calculus, involving the multiplication operator Ψ (cf. lemma 5.5.1.2). ∎

6.4.5 Even-odd structure and double covering

Eventually we shall establish a link between the even-odd structure in (6.42) and the algebra $\mathcal{A}_q^{\mathrm{even}}$ which is the one to be considered when we really want to distinguish the quantum $E(2)$ group from its double cover—see §5.5 for details.

Lemma 6.4.5.1 *The above Fourier transforms F_R and F_L map*

$$\mathcal{U}_q\big(K^{\mathrm{even}}(\mathbb{Z}\theta)\otimes\mathcal{E}_\tau^{\mathrm{even}}\big) \qquad \textit{into} \qquad \mathcal{A}_q^{\mathrm{even}}(\mathcal{E}_\nu).$$

Proof. Take any $X \in K^{\mathrm{even}}(\mathbb{Z}\theta) \otimes \mathcal{E}_\tau^{\mathrm{even}}$ and recall the observations made in the proof of lemma 6.4.2.1. It is then clear from definition 6.4.3.1 that $\mathbb{H}_m^{(k)} X = 0$ whenever k is odd, since $\Gamma^{-1} \otimes \Omega$ interchanges even and odd parts of (6.42). This means that in the sums appearing in definition 6.4.4.1, only the terms corresponding to even k will survive. Corollary 5.5.1.4 yields the result. ∎

This investigation shall be continued at the end of §6.6.

6.5 Plancherel formulas

Abstract 6.5 We prove Plancherel formulas for the Fourier transforms constructed in the previous section, relating the Haar functional on the quantum $E(2)$ group with the ones on its Pontryagin dual. At the heart of the proof we encounter—not quite surprisingly—the orthogonality relations (6.7) for little q-Bessel functions. ⋆

The following lemma is of the same nature as lemma 5.4.2.1.

Lemma 6.5.1 *Take any $X, Y \in \mathfrak{L}(\mathcal{E}_\tau)$ and $r, s \in \mathbb{N}$. Put $m = \min\{r, s\}$. Then*

$$\begin{aligned} F_R\big(\Upsilon(Y)\, b^r\big)^* \, F_R\big(\Upsilon(X)\, b^s\big) &= \sum_{k,l\in\mathbb{Z}} (q^{-r}\alpha)^{l-k+s-r} \, (\gamma^*)^{r-m} \, \gamma^{s-m} \\ &\qquad \Big\{\Psi^m \Big[\Big(\Omega^{-2(l-k+s-r)} \, \mathbb{H}_r^{(k)} Y\Big)^{\sim}\Big(\mathbb{H}_s^{(l)} X\Big)\Big]\Big\}(\gamma^*\gamma) \\ F_R\big(\Upsilon(Y)\, b^r\big)^* \, F_R\big(\Upsilon(X)\, c^s\big) &= \sum_{k,l\in\mathbb{Z}} (q^{-r}\alpha)^{l-k-s-r} \, (\gamma^*)^{r+s} \\ &\qquad \Big[\Big(\Omega^{-2(l-k-s-r)} \, \mathbb{H}_r^{(k)} Y\Big)^{\sim}\Big(\mathbb{H}_{-s}^{(l)} X\Big)\Big](\gamma^*\gamma) \end{aligned}$$

and similar formulas for

$$F_R\big(\Upsilon(Y)\, c^r\big)^* \, F_R\big(\Upsilon(X)\, b^s\big)$$
$$F_R\big(\Upsilon(Y)\, c^r\big)^* \, F_R\big(\Upsilon(X)\, c^s\big).$$

Notice the summations over $k, l \in \mathbb{Z}$ are in fact *finite* sums (cf. lemma 6.4.3.5).

Proof. Using the commutation rules $\alpha\beta = q\,\beta\alpha$ and lemma 5.5.1.2, we obtain

$$\begin{aligned} &F_R\big(\Upsilon(Y)\, b^r\big)^* \, F_R\big(\Upsilon(X)\, b^s\big) \\ &= \left(\sum_{k\in\mathbb{Z}} \alpha^{k+r}\, \gamma^r \Big(\mathbb{H}_r^{(k)} Y\Big)(\gamma^*\gamma)\right)^* \left(\sum_{l\in\mathbb{Z}} \alpha^{l+s}\, \gamma^s \Big(\mathbb{H}_s^{(l)} X\Big)(\gamma^*\gamma)\right) \\ &= \sum_{k,l\in\mathbb{Z}} \Big(\mathbb{H}_r^{(k)} Y\Big)^{\sim}(\gamma^*\gamma)\, (\gamma^*)^r \, \alpha^{-k-r} \, \alpha^{l+s} \, \gamma^s \Big(\mathbb{H}_s^{(l)} X\Big)(\gamma^*\gamma) \\ &= \sum_{k,l\in\mathbb{Z}} q^{-r(l-k+s-r)} \, \alpha^{l-k+s-r} \, (\gamma^*)^r \, \gamma^s \\ &\qquad \Big[\Big(\Omega^{-2(l-k+s-r)} \, \mathbb{H}_r^{(k)} Y\Big)^{\sim}\Big(\mathbb{H}_s^{(l)} X\Big)\Big](\gamma^*\gamma) \end{aligned}$$

With lemma 5.5.1.2 the first formula follows. The other ones are similar. ∎

For the other Fourier transform, the situation is slightly different—although the proof is based on the same techniques:

Lemma 6.5.2 *Take any $X, Y \in \mathfrak{L}(\mathcal{E}_\tau)$ and $r, s \in \mathbb{N}$. Put $m = \min\{r, s\}$. Then*

$$F_L\big(\Upsilon(X)\, b^r\big)\; F_L\big(\Upsilon(Y)\, b^s\big)^* \;=\; \sum_{k,l\in\mathbb{Z}} q^{2(k+l+r+s)}\, q^{(r+s)(l+s)}$$
$$\alpha^{k-l+r-s}\,\gamma^{r-m}\,(\gamma^*)^{s-m}\Big\{\Psi^m\,\Omega^{2(l+s)}\,\Big[\Big(\mathbb{H}_r^{(k)}X\Big)\Big(\mathbb{H}_s^{(l)}Y\Big)^{\sim}\Big]\Big\}\,(\gamma^*\gamma).$$

and similar formulas for

$$F_L\big(\Upsilon(X)\, b^r\big)\; F_L\big(\Upsilon(Y)\, c^s\big)^*$$
$$F_L\big(\Upsilon(X)\, c^r\big)\; F_L\big(\Upsilon(Y)\, b^s\big)^*$$
$$F_L\big(\Upsilon(X)\, c^r\big)\; F_L\big(\Upsilon(Y)\, c^s\big)^*$$

The essence of our Plancherel formula lies within the proof of the next lemma. Recall that we had fixed a Haar functional ω on $\mathcal{A}_q\big(\mathcal{S}_\nu(\mathbb{R}^+; q^2)\big)$ and observe the following makes sense because $\mathcal{S}_\nu(\mathbb{R}^+; q^2)$ and $\mathcal{A}_q\big(\mathcal{S}_\nu(\mathbb{R}^+; q^2)\big)$ are *-algebras, containing respectively $\mathcal{E}_\nu$ and $\mathcal{A}_q(\mathcal{E}_\nu)$.

Lemma 6.5.3 *Take any $X, Y \in \mathfrak{L}(\mathcal{E}_\tau)$ and $r, s \in \mathbb{N}$. Then ω vanishes on*

$$F_R\big(\Upsilon(Y)\, b^r\big)^*\, F_R\big(\Upsilon(X)\, c^s\big) \qquad\qquad F_R\big(\Upsilon(Y)\, c^r\big)^*\, F_R\big(\Upsilon(X)\, b^s\big)$$
$$F_L\big(\Upsilon(X)\, b^r\big)\; F_L\big(\Upsilon(Y)\, c^s\big)^* \qquad\qquad F_L\big(\Upsilon(X)\, c^r\big)\; F_L\big(\Upsilon(Y)\, b^s\big)^*$$

unless both r and s are zero. Moreover ω also vanishes on

$$F_R\big(\Upsilon(Y)\, b^r\big)^*\, F_R\big(\Upsilon(X)\, b^s\big) \qquad\qquad F_R\big(\Upsilon(Y)\, c^r\big)^*\, F_R\big(\Upsilon(X)\, c^s\big)$$
$$F_L\big(\Upsilon(X)\, b^r\big)\; F_L\big(\Upsilon(Y)\, b^s\big)^* \qquad\qquad F_L\big(\Upsilon(X)\, c^r\big)\; F_L\big(\Upsilon(Y)\, c^s\big)^*$$

except when $r = s = m$. In the latter case we actually have

$$\omega\Big(F_R\big(\Upsilon(Y)\, b^m\big)^*\, F_R\big(\Upsilon(X)\, b^m\big)\Big) \;=\; q^{-2m}\,\psi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(Y^*X)\Big) \tag{6.47}$$
$$\omega\Big(F_R\big(\Upsilon(Y)\, c^m\big)^*\, F_R\big(\Upsilon(X)\, c^m\big)\Big) \;=\; q^{2m}\,\psi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(Y^*X)\Big) \tag{6.48}$$
$$\omega\Big(F_L\big(\Upsilon(X)\, b^m\big)\; F_L\big(\Upsilon(Y)\, b^m\big)^*\Big) \;=\; \varphi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(XY^*)\Big) \tag{6.49}$$
$$\omega\Big(F_L\big(\Upsilon(X)\, c^m\big)\; F_L\big(\Upsilon(Y)\, c^m\big)^*\Big) \;=\; \varphi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(XY^*)\Big)\,. \tag{6.50}$$

Proof. All statements follow from the definition of ω and the previous lemmas. Only the last set of formulas is not immediately clear. Therefore, observe that for all $m \in \mathbb{N}$

$$
\begin{aligned}
&\omega\Big(F_R\big(\Upsilon(Y)\,b^m\big)^* F_R\big(\Upsilon(X)\,b^m\big)\Big)\\
&= \omega\left(\sum_{k\in\mathbb{Z}}\left\{\Psi^m\left[\left(\mathbb{H}_m^{(k)}Y\right)^{\sim}\left(\mathbb{H}_m^{(k)}X\right)\right]\right\}(\gamma^*\gamma)\right)\\
&= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(\nu q^{2n})^m\left[\left(\mathbb{H}_m^{(k)}Y\right)^{\sim}\left(\mathbb{H}_m^{(k)}X\right)\right](\nu q^{2n})\,q^{2n}\\
&= \sum_{k\in\mathbb{Z}}\kappa_q(m,k)^2\sum_{n\in\mathbb{Z}}q^{2n}\,(\nu q^{2n})^m\,(\Lambda_\nu\mathfrak{H}_m\,g_k)^{\sim}(\nu q^{2n})\,\,(\Lambda_\nu\mathfrak{H}_m\,f_k)(\nu q^{2n})
\end{aligned}
$$

Here $f_k, g_k \in \mathcal{E}$ are given by (cf. definition 6.4.3.1)

$$
\begin{aligned}
f_k &= \Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k X\\
g_k &= \Lambda_\tau^{-1}\,(\phi_0\otimes\mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k Y
\end{aligned}
$$

for any $k\in\mathbb{Z}$. Below $\langle\cdot\,|\,\cdot\rangle$ will denote the scalar product on $L^2(\mathbb{R}_q^+, m_q)$ as introduced at the beginning of §6.2. Recall H_m was a unitary[9] operator on this Hilbert space (cf. definition 6.2.1.2). We proceed with our computation:

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\sum_{n\in\mathbb{Z}}q^{2n}\,\overline{(R_q\Psi^m K\mathfrak{H}_m\,g_k)(q^n)}\,(R_q\Psi^m K\mathfrak{H}_m\,f_k)(q^n)\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\sum_{n\in\mathbb{Z}}q^{2n}\,\overline{(\mathcal{H}_m g_k)(q^n)}\,(\mathcal{H}_m f_k)(q^n)\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\,\langle\mathcal{H}_m f_k\,|\,\mathcal{H}_m g_k\rangle\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\,\,\big\langle H_m R_q\Psi^m K f_k\,\big|\,H_m R_q\Psi^m K g_k\big\rangle\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\,\,\big\langle R_q\Psi^m K f_k\,\big|\,R_q\Psi^m K g_k\big\rangle\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\sum_{n\in\mathbb{Z}}q^{2n}\,\overline{(\Psi^m K g_k)(q^n)}\,(\Psi^m K f_k)(q^n)\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\sum_{n\in\mathbb{Z}}q^{2n}\,q^{2nm}\,\overline{g_k(q^{2n})}\,f_k(q^{2n})\\
&= \sum_{k\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\sum_{n\in\mathbb{Z}}q^{2n}\,q^{2nm}\,\overline{Y(k\theta,\tau q^{2n+k})}\,X(k\theta,\tau q^{2n+k})\\
&= \sum_{k,n\in\mathbb{Z}}\nu^m\,\kappa_q(m,k)^2\,q^{2n}\,q^{2nm}\,(Y^*X)(k\theta,\tau q^{2n+k})\\
&= \sum_{(k,l)\in\mathfrak{S}}\nu^m\,\kappa_q(m,k)^2\,q^{l-k}\,q^{(l-k)m}\,(Y^*X)(k\theta,\tau q^l)
\end{aligned}
$$

[9] Notice that unitarity of H_m amounts to the orthogonality relations (6.7).

where

$$\mathfrak{S} \;=\; (\mathbb{Z}_{\text{even}} \times \mathbb{Z}_{\text{even}}) \cup (\mathbb{Z}_{\text{odd}} \times \mathbb{Z}_{\text{odd}}). \tag{6.51}$$

So in the last step we performed a rearrangement of our summation variables, according to $l = 2n + k$. Furthermore X and Y were interpreted as functions in two variables—in the obvious way. Before we proceed with the computation, let's have a closer look at some of the scalars involved here:

$$\begin{aligned}
&\nu^m \,\kappa_q(m,k)^2 \, q^{(l-k)m} \\
&\quad= \; (q^{-1}-q)^{-2m} \Big((-1)^m \, q^{-m} \, q^{\frac{1}{2}|m|(k-1)} \, (q^{-1}-q)^{|m|} \Big)^2 \, q^{(l-k)m} \\
&\quad= \; q^{-3m} \, q^{lm} \\
&\quad= \; q^{-2m} \, (\tau q^l)^m .
\end{aligned}$$

Notice that we used our particular choices for τ and ν. Eventually we obtain

$$\begin{aligned}
\omega\Big(F_R\big(\Upsilon(Y)\, b^m\big)^* \, F_R\big(\Upsilon(X)\, b^m\big)\Big) \;&=\; q^{-2m} \sum_{(k,l)\in\mathfrak{S}} (\tau q^l)^m \, (Y^*X)(k\theta, \tau q^l)\, q^{-k+l} \\
&=\; q^{-2m} \, \psi\Big(\Upsilon(\,\mathrm{id} \otimes \Psi^m)\,(Y^*X)\Big) .
\end{aligned}$$

Here we used the ψ-analogue of (5.16). This proves (6.47). Equation (6.48) is shown similarly, though one should be careful with the signs, since here *negative* order q-Hankel transforms are involved:

$$\begin{aligned}
&\omega\Big(F_R\big(\Upsilon(Y)\, c^m\big)^* \, F_R\big(\Upsilon(X)\, c^m\big)\Big) \\
&\qquad\qquad= \; \omega\left(\sum_{k\in\mathbb{Z}} \Big\{ \Psi^m \Big[\Big(\mathbb{H}^{(k)}_{-m} Y\Big)^{\sim} \Big(\mathbb{H}^{(k)}_{-m} X\Big) \Big] \Big\} (\gamma^*\gamma) \right) \;=\; \ldots
\end{aligned}$$

So now we have to use $\kappa_q(-m,k)$ instead of $\kappa_q(m,k)$.
To prove (6.49) we start from lemma 6.5.2:

$$\begin{aligned}
&\omega\,\Big(F_L\big(\Upsilon(X)\, b^m\big) \; F_L\big(\Upsilon(Y)\, b^m\big)^*\Big) \\
&\quad= \; \omega\left(\sum_{k\in\mathbb{Z}} q^{4(k+m)} \, q^{2m(k+m)} \Big\{ \Psi^m \, \Omega^{2(k+m)} \Big[\Big(\mathbb{H}^{(k)}_m X\Big) \Big(\mathbb{H}^{(k)}_m Y\Big)^{\sim} \Big] \Big\} \, (\gamma^*\gamma) \right) \\
&\quad= \; \sum_{k\in\mathbb{Z}} q^{4(k+m)} q^{2m(k+m)} \sum_{n\in\mathbb{Z}} q^{2n} (\nu q^{2n})^m \Big[\Big(\mathbb{H}^{(k)}_m X\Big) \Big(\mathbb{H}^{(k)}_m Y\Big)^{\sim} \Big] \Big(\nu q^{2(n+k+m)}\Big) \\
&\quad\overset{(*)}{=} \; \sum_{k\in\mathbb{Z}} q^{4(k+m)} \, q^{2m(k+m)} \sum_{n\in\mathbb{Z}} q^{2(n-k-m)} \, \nu^m \, q^{2(n-k-m)m} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Big[\Big(\mathbb{H}^{(k)}_m X\Big) \Big(\mathbb{H}^{(k)}_m Y\Big)^{\sim} \Big] \, (\nu q^{2n}) \\
&\quad= \; \sum_{k\in\mathbb{Z}} q^{2(k+m)} \sum_{n\in\mathbb{Z}} q^{2n} \, (\nu q^{2n})^m \, \Big[\Big(\mathbb{H}^{(k)}_m X\Big) \Big(\mathbb{H}^{(k)}_m Y\Big)^{\sim} \Big] \, (\nu q^{2n})
\end{aligned}$$

$$
\begin{aligned}
&\overset{(\sharp)}{=} \sum_{k\in\mathbb{Z}} q^{2(k+m)}\,\nu^m\,\kappa_q(m,k)^2 \sum_{n\in\mathbb{Z}} q^{2n}\,q^{2nm}\,(XY^*)(k\theta,\tau q^{2n+k}) \\
&= \sum_{(k,l)\in\mathfrak{S}} q^{2(k+m)}\,q^{-2m}\,(\tau q^l)^m\,(XY^*)(k\theta,\tau q^l)\,q^{-k+l} \\
&= \sum_{(k,l)\in\mathfrak{S}} (\tau q^l)^m\,(XY^*)(k\theta,\tau q^l)\,q^{k+l} \\
&= \varphi\Big(\Upsilon(\mathrm{id}\otimes\Psi^m)\,(X\,Y^*)\Big)\,.
\end{aligned}
$$

In $(*)$ we substituted n with $n-k-m$. From $(\sharp)$ on, the computation proceeds completely analogous to the one made above in proving (6.47). Thus we have shown (6.49). Equation (6.50) is analogous. ∎

Remark 6.5.4 Observe how the 'spectral conditions' (cf. remark 5.4.1.9) incorporated by (6.51) emerge naturally in the above computation, although here it has nothing to do with any operator theoretic considerations whatsoever; this is quite remarkable. Also notice that—once again—the particular values for the 'dilation' parameters τ and ν chosen in (6.41) seem to be quite indispensable in our computations. ⋆

Theorem 6.5.5 *The Fourier transforms F_R and F_L given in definition 6.4.4.1 enjoy the following Plancherel formulas:*

$$\psi(y^*x) \;=\; \omega\big(F_R(y)^*F_R(x)\big) \qquad \text{and} \qquad \varphi(xy^*) \;=\; \omega\big(F_L(x)\,F_L(y)^*\big)$$

for any $x,y\in\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$.

Proof. Recall that any $x,y\in\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$ can be written as

$$
\begin{aligned}
x &= \sum_{m=0}^{\infty}\Upsilon(X_m)\,b^m + \sum_{m=1}^{\infty}\Upsilon(X_{-m})\,c^m \\
y &= \sum_{m=0}^{\infty}\Upsilon(Y_m)\,b^m + \sum_{m=1}^{\infty}\Upsilon(Y_{-m})\,c^m
\end{aligned}
$$

with only *finitely many* non-zero $X_n,Y_n\in\mathfrak{L}(\mathcal{E}_\tau)$. Applying consecutively the ψ-analogon of (5.16) and the previous lemma, we obtain

$$\psi(y^*x) \;=\; \sum_{n\in\mathbb{Z}} q^{-2n}\,\psi\Big(\Upsilon(\,\mathrm{id}\otimes\Psi^{|n|})(Y_n^*X_n)\Big) \;=\; \omega\big(F_R(y)^*F_R(x)\big)\,.$$

Notice how the $n\geq 0$ and $n<0$ cases have been conveniently merged into a single summation over all $n\in\mathbb{Z}$.
The second Plancherel formula is analogous; however, instead of the formulas in lemma 5.4.2.1 we need analogous formulas like for instance

$$\big(\Upsilon(X)\,b^m\big)\,\big(\Upsilon(Y)\,b^m\big)^* \;=\; \Upsilon(\mathrm{id}\otimes\Psi^m)\,(XY^*) \;=\; \big(\Upsilon(X)\,c^m\big)\,\big(\Upsilon(Y)\,c^m\big)^*$$

which are equally easy to prove. ∎

Remark 6.5.6 Notice that in dealing with F_R we have used expressions of type y^*x whereas concerning F_L we considered expressions of type xy^*. It is an instructive exercise to see what goes wrong when e.g. one tries to use y^*x in dealing with F_L. $\star$

6.6 Inversion formulas

Definition 6.6.1 Let δ_0 be the function $\delta_0 : \mathbb{Z}\theta \to \mathbb{C}$ given by $\delta_0(k\theta) = \delta_{k,0}$ for $k \in \mathbb{Z}$. Clearly δ_0 belongs to $K^{\text{even}}(\mathbb{Z}\theta)$. Next let T_0 denote the map from $H(\mathbb{C})$ into $F(\mathbb{Z}\theta) \otimes H(\mathbb{C})$ given by tensoring with δ_0 on the left, i.e. $T_0(g) = \delta_0 \otimes g$. Finally, for any $k \in \mathbb{Z}$, let P_k denote the map $P_k : F(\mathbb{Z}\theta) \to F(\mathbb{Z}\theta)$ given by $(P_k f)(l\theta) = \delta_{k,l} f(l\theta)$ for $l \in \mathbb{Z}$ and $f \in F(\mathbb{Z}\theta)$. $\bullet$

The following lemma is trivial but important; it is similar to lemma 6.4.2.1.

Lemma 6.6.2 *The mapping T_0 maps $\mathcal{E}_\tau$ into $K^{\text{even}}(\mathbb{Z}\theta) \otimes \mathcal{E}_\tau^{\text{even}} \subseteq \mathfrak{L}(\mathcal{E}_\tau)$.*

Definition 6.6.3 First recall definition 6.4.3.1. For any $m, k \in \mathbb{Z}$, we define a linear mapping $\tilde{\mathbb{H}}_m^{(k)}$ from $\mathcal{E}_\nu$ into $\mathfrak{L}(\mathcal{E}_\tau)$ as follows:

$$\tilde{\mathbb{H}}_m^{(k)} \;=\; \kappa_q(m,k)^{-1}\,(\Gamma^{-1} \otimes \Omega)^{-k}\, T_0\, \Lambda_\tau\, \mathfrak{H}_m\, \Lambda_\nu^{-1}.$$

Observe that because of the previous lemma, $\tilde{\mathbb{H}}_m^{(k)}$ indeed ends up in $\mathfrak{L}(\mathcal{E}_\tau)$. $\bullet$

Lemma 6.6.4 *Take $m, k, l \in \mathbb{Z}$. Then*

$$\mathbb{H}_m^{(k)}\,\tilde{\mathbb{H}}_m^{(l)} \;=\; \delta_{k,l}\,\mathrm{id} \qquad \textit{and} \qquad \tilde{\mathbb{H}}_m^{(k)}\,\mathbb{H}_m^{(k)} \;=\; P_k \otimes \mathrm{id}$$

and hence

$$\sum_{k\in\mathbb{Z}} \tilde{\mathbb{H}}_m^{(k)}\,\mathbb{H}_m^{(k)} X \;=\; X$$

for any $X \in \mathfrak{L}(\mathcal{E}_\tau)$.

Notice that the last formula makes sense because of lemma 6.4.3.5.

Proof. Recall that $\mathfrak{H}_m^2 = \mathrm{id}$ and observe that

$$\begin{aligned} T_0(\phi_0 \otimes \mathrm{id}) &= P_0 \otimes \mathrm{id} \\ (\phi_0 \otimes \mathrm{id})(\Gamma^{-1} \otimes \Omega)^{k-l} T_0 &= \delta_{k,l}\,\mathrm{id}. \end{aligned}$$ ■

Proposition 6.6.5 *The Fourier transforms F_R and F_L (cf. definition 6.4.4.1) are bijections from $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$ onto $\mathcal{A}_q(\mathcal{E}_\nu)$. Their inverses are given by*

$$\begin{aligned}
F_R^{-1}\Big(\alpha^l \gamma^m\, g(\gamma^*\gamma)\Big) &= \Upsilon\Big(\tilde{\mathbb{H}}_m^{(l-m)}\, g\Big)\, b^m \\
F_R^{-1}\Big(\alpha^l (\gamma^*)^m\, g(\gamma^*\gamma)\Big) &= \Upsilon\Big(\tilde{\mathbb{H}}_{-m}^{(l+m)}\, g\Big)\, c^m \\
F_L^{-1}\Big(\alpha^l \gamma^m\, g(\gamma^*\gamma)\Big) &= q^{-2l}\, \Upsilon\Big(\tilde{\mathbb{H}}_m^{(l-m)}\, g\Big)\, b^m \\
F_L^{-1}\Big(\alpha^l (\gamma^*)^m\, g(\gamma^*\gamma)\Big) &= q^{-2l}\, \Upsilon\Big(\tilde{\mathbb{H}}_{-m}^{(l+m)}\, g\Big)\, c^m
\end{aligned}$$

Lemma 6.6.6 *We have* $\langle 1_{\mathcal{U}_q}, F_L(x)\rangle = \varphi(x)$ *for any* $x \in \mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$.

Proof. Using proposition 6.3.2 we get for all $X \in \mathfrak{L}(\mathcal{E}_\tau)$ and $m \in \mathbb{N}$ that

$$\begin{aligned}
&\big\langle 1_{\mathcal{U}_q},\, F_L\big(\Upsilon(X)\, b^m\big)\big\rangle \\
&\quad= \sum_{k\in\mathbb{Z}} \Big\langle 1_{\mathcal{U}_q},\, (q^2\alpha)^{k+m}\,\gamma^m\,\Big(\mathbb{H}^{(k)}_m\, X\Big)\,(\gamma^*\gamma)\Big\rangle \\
&\quad= \sum_{k\in\mathbb{Z}} \delta_{m,0}\, q^{2k}\,\Big(\mathbb{H}^{(k)}_0\, X\Big)\,(0) \\
&\quad= \sum_{k\in\mathbb{Z}} \delta_{m,0}\, q^{2k}\,\Big(\mathfrak{H}_0\,\Lambda_\tau^{-1}\,(\phi_0\otimes \mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k X\Big)\,(0) \\
&\quad= \sum_{k\in\mathbb{Z}} \delta_{m,0}\, q^{2k}\frac{1}{1-q^2}\int_0^\infty \Big(\Lambda_\tau^{-1}\,(\phi_0\otimes \mathrm{id})\,(\Gamma^{-1}\otimes\Omega)^k X\Big)\,(x)\, d_{q^2}x \\
&\quad= \sum_{k\in\mathbb{Z}} \delta_{m,0}\, q^{2k}\frac{1}{1-q^2}\int_0^\infty X(k\theta,\,\tau q^k x)\, d_{q^2}x \\
&\quad= \sum_{k\in\mathbb{Z}} \delta_{m,0}\, q^{2k}\sum_{n\in\mathbb{Z}} X(k\theta,\,\tau q^k q^{2n})\, q^{2n} \\
&\quad= \delta_{m,0}\sum_{(k,l)\,\in\,\mathfrak{S}} X(k\theta,\,\tau q^l)\, q^{k+l} \\
&\quad= \varphi\big(\Upsilon(X)\, b^m\big)\,.
\end{aligned}$$

At the last but one equality we rearranged the sums according to the rule $k+2n=l$. Similarly we can treat $\Upsilon(X)\, c^m$. ∎

The previous lemma provides the condition in proposition 4.3.2, yielding

Proposition 6.6.7 $G_{LR} = SF_L^{-1}$ *is an* LR *Fourier transform from* $\mathcal{A}_q(\mathcal{E}_\nu)$ *to* $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$.

From the general observations in chapter 4 we also obtain all the other Fourier transforms from $\mathcal{A}_q(\mathcal{E}_\nu)$ to $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$, as well as the appropriate Plancherel identities for G_{LR} and G_{RL} (observe however that G_{LL} and G_{RR} do *not* obey a Plancherel formula).

Even-odd structure and double covering, continued Let's complete the investigation that was taken up in §6.4.5.

Lemma 6.6.8 *Take* $m, k \in \mathbb{Z}$. *If* k *is even, then the range of* $\tilde{\mathbb{H}}^{(k)}_m$ *is contained in* $K^{\mathrm{even}}(\mathbb{Z}\theta)\otimes\mathcal{E}_\tau^{\mathrm{even}}$.

Proof. Combining lemma 6.6.2 and definition 6.6.3, we obtain

$$\tilde{\mathbb{H}}^{(k)}_m(\mathcal{E}_\nu) \;=\; (\Gamma^{-1}\otimes\Omega)^{-k}\, T_0(\mathcal{E}_\tau) \;\subseteq\; (\Gamma^{-1}\otimes\Omega)^{-k}\,\big(K^{\mathrm{even}}(\mathbb{Z}\theta)\otimes\mathcal{E}_\tau^{\mathrm{even}}\big)\,.$$

This yields the result, for $\Gamma^{-1}\otimes\Omega$ interchanges even and odd parts of (6.42). ∎

Lemma 6.6.9 *The inverses F_R^{-1} and F_L^{-1} of the Fourier transforms, as given in proposition 6.6.5, map*

$$\mathcal{A}_q^{\text{even}}(\mathcal{E}_\nu) \qquad \textit{into} \qquad \mathcal{U}_q\big(K^{\text{even}}(\mathbb{Z}\theta) \otimes \mathcal{E}_\tau^{\text{even}}\big)\,.$$

Proof. When F_R^{-1} and F_L^{-1} are invoked on elements of $\mathcal{A}_q^{\text{even}}(\mathcal{E}_\nu)$ only, then in the formulas of proposition 6.6.5, we may assume $l \pm m$ to be even. Now the previous lemma completes the proof. ∎

Combining the above lemma with lemma 6.4.5.1, we obtain

Proposition 6.6.10 *The Fourier transforms F_R and F_L restrict to bijections*

$$\textit{from} \qquad \mathcal{U}_q\big(K^{\text{even}}(\mathbb{Z}\theta) \otimes \mathcal{E}_\tau^{\text{even}}\big) \qquad \textit{onto} \qquad \mathcal{A}_q^{\text{even}}(\mathcal{E}_\nu).$$

6.7 Conclusions

Theorem 6.7.1 *Take $0 < q < 1$ and define $\tau = q^{-1}$ and $\nu = (q^{-1} - q)^{-2}$. Then the system*

$$\Big(\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big) \subseteq \mathcal{U}_q\big(\mathfrak{L}(\mathcal{S}_\tau(\mathbb{R}^+; q^2))\big)\,, \varphi, \psi; \mathcal{U}_q, \mathcal{A}_q;\ \mathcal{A}_q(\mathcal{E}_\nu) \subseteq \mathcal{A}_q\big(\mathcal{S}_\nu(\mathbb{R}^+; q^2)\big)\,, \omega\Big)$$

is a Plancherel context (cf. definition 4.4.5).

Proof. Let's show conditions (i-iv) of definition 4.4.5. Condition (i) follows from propositions 6.4.4.2, 6.6.5 and 6.6.7. (ii) follows from theorem 6.5.5 together with some results in chapter 4. (iii) was shown in lemma 6.6.6. Also (iv) is not so hard to prove. ∎

Further investigation From lemmas 4.2.1 and 4.5.1 we now know there is a natural duality between $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$ and $\mathcal{A}_q(\mathcal{E}_\nu)$, for instance, $\langle x, y\rangle = \omega(F_L(x)\, y)$ for $x \in \mathcal{U}_q\big(\mathfrak{L}(\mathcal{E}_\tau)\big)$ and $y \in \mathcal{A}_q(\mathcal{E}_\nu)$. This formula for the pairing can be made very explicit simply by plugging in definitions 6.4.4.1, 6.4.3.1, 6.2.2.2 and 6.2.1.2, together with proposition 6.2.4.2 and the definition of the Haar functional ω. This should yield some very concrete expressions for the pairing, e.g.

$$\begin{aligned}\Big\langle \Upsilon(X)\, b^m,\ \alpha^l (\gamma^*)^n\, g(\gamma^*\gamma)\Big\rangle \quad &= \quad \delta_{m,n}\, \kappa_q(m, -m-l)\, \nu^m\, q^{ml} \dots \\ \dots \sum_{r,k\in\mathbb{Z}} q^{(m+2)(k+r)}\, J_m(q^{k+r}; q^2)\, g(\nu\, q^{2r+2l})\, X\big(-m\theta - l\theta,\ \tau\, q^{2k-m-l}\big)\end{aligned}$$

for any and $l \in \mathbb{Z}$, $m, n \in \mathbb{N}$, $g \in \mathcal{E}_\nu$ and $X \in \mathfrak{L}(\mathcal{E}_\tau)$. The interesting point about this formula is that it no longer depends on the use of *entire* functions, i.e. X and g may be any functions living on the appropriate (discrete!) sets, provided they satisfy some elementary summability criterion. So it may be possible to take the above formula as the *point to start* the construction of the quantum $E(2)$ group, i.e. to make it into a *definition* for the pairing...

Appendix A

Weak Fubini tensor products

Let $\langle E_i, \Omega_i \rangle$ $(i = 1, \ldots n)$ be a finite number of vector space dualities, i.e. for every index i we consider two linear spaces E_i and Ω_i and a non-degenerate pairing $\langle \cdot, \cdot \rangle : E_i \times \Omega_i \to \mathbb{C}$. Endow E_i and Ω_i with the weak topologies induced by the duality. We have canonical embeddings $\Omega_i \hookrightarrow E_i' \equiv \overline{\Omega_i}$ and

$$\Omega_1 \otimes \ldots \Omega_n \;\hookrightarrow\; E_1' \otimes \ldots E_n' \;\hookrightarrow\; (E_1 \otimes \ldots E_n)' \;\equiv\; \overline{\Omega_1 \otimes \ldots \Omega_n}$$

and analogously $E_1 \otimes \ldots E_n \hookrightarrow (\Omega_1 \otimes \ldots \Omega_n)'$. The obvious pairing between $E_1 \otimes \ldots E_n$ and $\Omega_1 \otimes \ldots \Omega_n$ is still non-degenerate. Now $(E_1 \otimes \ldots E_n)'$ identifies with the space of all multilinear forms on $E_1 \times \ldots E_n$, e.g. a simple tensor $\omega_1 \otimes \ldots \omega_n \in \Omega_1 \otimes \ldots \Omega_n$ corresponds to the form

$$E_1 \times \ldots E_n \to \mathbb{C} \, : \, (x_1, \ldots x_n) \;\mapsto\; \langle x_1, \omega_1 \rangle \ldots \langle x_n, \omega_n \rangle.$$

Observe that all multilinear forms which arise from elements in the algebraic tensor product $\Omega_1 \otimes \ldots \Omega_n$ are *jointly* continuous w.r.t. the weak topologies on $E_1, \ldots E_n$. Moreover, in the $n = 2$ case, a standard result (see [35, §42.4], [18, §II.41.3.10], or [10, §15.3.6]) states that $\Omega_1 \otimes \Omega_2$ is actually linearly isomorphic to the space of all jointly continuous bilinear forms on $E_1 \times E_2$. Weakening the condition of *joint* continuity leads us to the following:

Definition A.1 Consider $\langle E_i, \Omega_i \rangle$ as above $(i = 1, \ldots n)$. The linear space of all *separately* continuous multilinear forms on $E_1 \times \ldots E_n$ will be denoted by $\Omega_1 \mathbin{\overline{\otimes}} \ldots \Omega_n$, and referred to as the *weak Fubini* tensor product of the spaces Ω_i w.r.t. the dualities $\langle E_i, \Omega_i \rangle$. •

Remarks A.2 i. It's easy to see that $\overline{\otimes}$ is associative, i.e.

$$(\Omega_1 \mathbin{\overline{\otimes}} \Omega_2) \mathbin{\overline{\otimes}} \Omega_3 \;\simeq\; \Omega_1 \mathbin{\overline{\otimes}} \Omega_2 \mathbin{\overline{\otimes}} \Omega_3 \;\simeq\; \Omega_1 \mathbin{\overline{\otimes}} (\Omega_2 \mathbin{\overline{\otimes}} \Omega_3)$$

though we have to take some care with the interpretation: besides $\langle E_i, \Omega_i\rangle$ $(i = 1, 2, 3)$ we also need to consider the dualities $\langle E_1 \otimes E_2, \Omega_1 \mathbin{\overline{\otimes}} \Omega_2\rangle$ and $\langle E_2 \otimes E_3, \Omega_2 \mathbin{\overline{\otimes}} \Omega_3\rangle$ in order to deal with the iterated $\overline{\otimes}$-products.

ii. A weak Fubini tensor product can also be thought of as a space of *weakly continuous* linear operators. In fact we have

$$L(E_1, \Omega_2) \simeq \Omega_1 \mathbin{\overline{\otimes}} \Omega_2 \simeq L(E_2, \Omega_1),$$

the identification of any $\omega \in \Omega_1 \mathbin{\overline{\otimes}} \Omega_2$ with operators $S_\omega \in L(E_1, \Omega_2)$ and $T_\omega \in L(E_2, \Omega_1)$ being given by

$$\langle x_2, S_\omega x_1\rangle = \langle x_1 \otimes x_2, \omega\rangle = \omega(x_1, x_2) = \langle x_1, T_\omega x_2\rangle$$

for all $x_1 \in E_1$ and $x_2 \in E_2$. Notice S_ω and T_ω are each others transpose. Within these identifications, the elements of the *algebraic* tensor product $\Omega_1 \otimes \Omega_2$ are in 1-1-correspondence with the *finite rank* operators; simple tensors correspond to rank one operators.

iii. One should always be aware of the fact that weak Fubini tensor products are related to the vector space dualities involved. ⋆

A nice feature of weak Fubini tensor products is the fact they admit extension of *slice maps,* as we will see below; first we have a closer look at a special case:

Slicing with continuous functionals Let $\langle E, \Omega\rangle$ and $\langle F, \Gamma\rangle$ be two vector space dualities and consider an $x \in E$. Since E identifies with the space Ω^* of weakly continuous functionals on Ω, the element x corresponds to a functional $\langle x, \cdot\rangle \equiv f_x \in \Omega^*$. When $\phi \in \Omega \mathbin{\overline{\otimes}} \Gamma$ is a separately continuous bilinear form on $E \times F$, then $\phi(x, \cdot)$ is by assumption a weakly continuous functional on F. Hence $\phi(x, \cdot)$ identifies with a unique element in Γ, which will be denoted by $(f_x \mathbin{\overline{\otimes}} \mathrm{id})(\phi)$. Thus we obtain a linear map $f_x \mathbin{\overline{\otimes}} \mathrm{id} : \Omega \mathbin{\overline{\otimes}} \Gamma \to \Gamma$ determined by

$$\big\langle y,\, (f_x \mathbin{\overline{\otimes}} \mathrm{id})(\phi)\big\rangle = \phi(x, y) = \big\langle x \otimes y,\, \phi\big\rangle \tag{A.1}$$

for $y \in F$ and $\phi \in \Omega \mathbin{\overline{\otimes}} \Gamma$. This map obviously extends the ordinary algebraic slice map $f_x \otimes \mathrm{id}$, and of course one could also consider slicing 'from the right'.

General slice maps Consider vector space dualities $\langle E_i, \Omega_i\rangle$ and $\langle F_i, \Gamma_i\rangle$, and linear maps $\Lambda_i : \Omega_i \to \overline{\Gamma_i}$ $(i = 1, 2)$. Assume Λ_2 to be weakly continuous, i.e. having transpose $\Lambda_2^* : F_2 \to E_2$. Notice that the Λ_i are allowed to take values in the weak completions $\overline{\Gamma_i} \equiv F_i'$ and moreover, that Λ_1 is not assumed to be continuous. Now it is easy to see that

$$\Lambda_1 \otimes \Lambda_2 : \Omega_1 \otimes \Omega_2 \to \overline{\Gamma_1} \otimes \overline{\Gamma_2} \equiv F_1' \otimes F_2'$$

has a canonical extension to the weak Fubini tensor product, say

$$\overline{\Lambda_1 \otimes} \Lambda_2 : \Omega_1 \mathbin{\overline{\otimes}} \Omega_2 \to \overline{\Gamma_1 \otimes \Gamma_2} \equiv (F_1 \otimes F_2)', \tag{A.2}$$

such that

$$\big\langle y_1 \otimes y_2,\, (\overline{\Lambda_1 \otimes}\, \Lambda_2)(\phi)\big\rangle \;=\; \big\langle y_1,\, \Lambda_1(\mathrm{id}\,\overline{\otimes}\, f_x)(\phi)\big\rangle$$

for $\phi \in \Omega_1 \,\overline{\otimes}\, \Omega_2$, $y_1 \in F_1$, $y_2 \in F_2$, and $x \equiv \Lambda_2^*(y_2) \in E_2$.

Remarks A.3 i. Notice we preferred to write $\overline{\Lambda_1 \otimes}\, \Lambda_2$ rather than $\Lambda_1 \,\overline{\otimes}\, \Lambda_2$. The former notation is intended to remind to the fact that Λ_1 is not assumed to be continuous, while the latter is reserved for a 'proper' weak Fubini tensor product of *two* continuous linear maps (cf. remark iv).

ii. Of course the roles of Λ_1 and Λ_2 can be switched—what matters is that at least *one* of the two maps involved is weakly continuous.

iii. In particular we can take $\langle E_2, \Omega_2\rangle \equiv \langle F_2, \Gamma_2\rangle$ and $\Lambda_2 \equiv \mathrm{id}$, thus obtaining a slice map $\overline{\Lambda_1 \otimes}\, \mathrm{id} : \Omega_1 \,\overline{\otimes}\, \Omega_2 \to \overline{\Gamma_1 \otimes \Omega_2}$, obeying

$$\big\langle x \otimes y,\, (\overline{\Lambda_1 \otimes}\, \mathrm{id})(\phi)\big\rangle \;=\; \big\langle x,\, \Lambda_1(\mathrm{id}\,\overline{\otimes}\, f_y)(\phi)\big\rangle \tag{A.3}$$

for $\phi \in \Omega_1 \,\overline{\otimes}\, \Omega_2$, $x \in F_1$ and $y \in E_2$. In the special case that Λ_1 is just a linear *functional* (i.e. if $\Gamma_1 = \mathbb{C} = F_1$), say $\Lambda_1 \equiv g : \Omega_1 \to \mathbb{C}$, then we obtain a slice map $\overline{g \otimes}\, \mathrm{id} : \Omega_1 \,\overline{\otimes}\, \Omega_2 \to \overline{\Omega_2}$, determined by

$$\big\langle y,\, (\overline{g \otimes}\, \mathrm{id})(\phi)\big\rangle \;=\; g(\mathrm{id}\,\overline{\otimes}\, f_y)(\phi). \tag{A.4}$$

iv. When Λ_1 and Λ_2 are actually *both* weakly continuous mappings from Ω_i into $\overline{\Gamma_i}$ ($i = 1, 2$ resp.) then we have a priori two extensions $\overline{\Lambda_1 \otimes}\, \Lambda_2$ and $\Lambda_1 \overline{\otimes\, \Lambda_2}$ which are in fact equal, since they are both restrictions of

$$\overline{\Lambda_1 \otimes \Lambda_2} \equiv (\Lambda_1^* \otimes \Lambda_2^*)^\tau :\; \overline{\Omega_1 \otimes \Omega_2} \;\to\; \overline{\Gamma_1 \otimes \Gamma_2}.$$

If moreover $\Lambda_i(\Omega_i) \subseteq \Gamma_i$ ($i = 1, 2$), then also the transposed mappings $\Lambda_i^* : F_i \to E_i$ are weakly continuous. Only in this case we have a true $\overline{\otimes}$ of continuous linear maps, being a weakly continuous mapping

$$\Lambda_1 \,\overline{\otimes}\, \Lambda_2 :\; \Omega_1 \,\overline{\otimes}\, \Omega_2 \to \Gamma_1 \,\overline{\otimes}\, \Gamma_2 :\; \phi \;\mapsto\; \phi\big(\Lambda_1^*(\cdot), \Lambda_2^*(\cdot)\big) \tag{A.5}$$

between the $\overline{\otimes}$-products of the spaces involved.

v. In view of remark A.2.ii, the action of the map in (A.5) can be interpreted in terms of *composition* of continuous linear operators: let $\Lambda_i : \Omega_i \to \Gamma_i$ ($i = 1, 2$) be weakly continuous, hence having transposes $\Lambda_i^* : F_i \to E_i$, and take any $\omega \in \Omega_1 \,\overline{\otimes}\, \Omega_2$. Identifying $\Omega_1 \,\overline{\otimes}\, \Omega_2$ and $\Gamma_1 \,\overline{\otimes}\, \Gamma_2$ with spaces of continuous operators, we get

$$\Lambda_2 S_\omega \Lambda_1^* \;\simeq\; (\Lambda_1 \,\overline{\otimes}\, \Lambda_2)(\omega) \;\simeq\; \Lambda_1 T_\omega \Lambda_2^*.$$

vi. If Λ_1 and Λ_2 are as in (A.2), i.e. with Λ_2 being continuous, but now moreover $\Lambda_2(\Omega_2) \subseteq \Gamma_2$, then it's easy to show the following Fubini property:

$$(\overline{\Lambda_1 \otimes}\, \mathrm{id})\, (\mathrm{id}\,\overline{\otimes}\, \Lambda_2) \;=\; \overline{\Lambda_1 \otimes}\, \Lambda_2 \;=\; (\overline{\mathrm{id} \otimes \Lambda_2})\, (\overline{\Lambda_1 \otimes}\, \mathrm{id})$$

vii. We conclude with a little warning: consider linear maps $\Lambda_i : \Omega_i \to \Gamma_i$ $(i = 1, 2)$, *none* of them weakly continuous. Then according to the above it is still possible to construct slice maps on weak Fubini tensor products and, though not true in general, it might accidentally happen that

$$(\overline{\Lambda_1 \otimes}\, \mathrm{id})(\Omega_1 \mathbin{\overline{\otimes}} \Omega_2) \subseteq \Gamma_1 \mathbin{\overline{\otimes}} \Omega_2 \qquad \text{and} \qquad (\mathrm{id}\, \overline{\otimes \Lambda_2})(\Omega_1 \mathbin{\overline{\otimes}} \Omega_2) \subseteq \Omega_1 \mathbin{\overline{\otimes}} \Gamma_2,$$

which would allow us to *formulate* a Fubini-type property for these maps. It does however not imply that such a property actually *holds.* $\star$

Appendix B

Some technical results

Lemma B.1 *If $\mathcal{G}_\tau$ satisfies assumptions 5.4.1.1, then $\mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)$ separates $\mathcal{A}_q$ within the duality.*

Proof. Take any $\xi \in \mathcal{A}_q$, say $\xi = \sum_{l,m,n} w_{l,m,n}\, \alpha^l \beta^m \gamma^n$ with

$$w : \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \to \mathbb{C} : (l,m,n) \mapsto w_{l,m,n}$$

having finite support, such that $\big\langle \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}_\tau)\big)\,,\, \xi \big\rangle = \{0\}$. We shall prove that ξ must be zero. We know that $\langle \Upsilon(f \otimes g)\, b^k, \xi\rangle = 0$ for all $f \in K^{\substack{\text{even}\\ \text{odd}}}(\mathbb{Z}\theta)$, all $g \in \mathcal{G}^{\substack{\text{even}\\ \text{odd}}}$ and all $k \in \mathbb{N}$. According to (5.5) and lemma 5.3.1.4, this means

$$\sum_{l,m,n} w_{l,m,n}\, \delta_{m,n+k}\, \mu_n(g)\, f\big(-(l+m-n)\theta\big)\; q^{\frac{1}{2}l(m+n)}\, [m]_q!\, [n]_q! \;=\; 0.$$

Eliminating summation over m, we get

$$\sum_l q^{\frac{1}{2}lk}\, f\big(-(l+k)\theta\big) \sum_n w_{l,n+k,n}\, \mu_n(g)\, q^{ln}\, [n+k]_q!\, [n]_q! \;=\; 0,$$

still for all f, g and k as above. Now fix any $k_0 \in \mathbb{N}$ and $l_0 \in \mathbb{Z}$ for a while. Let's assume for instance that $k_0 + l_0$ is even (the odd case is similar). Then we can take $f \in K^{\text{even}}(\mathbb{Z}\theta)$ to be the function which has value 1 in $-(k_0+l_0)\theta$ and zero otherwise. It follows that for all $g \in \mathcal{G}_\tau^{\text{even}}$

$$\sum_{n \in \mathbb{N}} w_{l_0,n+k_0,n}\, \mu_n(g)\, q^{l_0 n}\, [n+k_0]_q!\, [n]_q! \;=\; 0.$$

Now we set $N(l_0,k_0) = \max\{n \in \mathbb{N} \mid w_{l_0,n+k_0,n} \neq 0\}$. Merely from the fact $\mathcal{G}_\tau^{\text{even}}$ is non-trivial and $\{\Psi, D_{q^2}\}$-invariant it follows easily that, given any $n_0 \in \mathbb{N}$ with $n_0 \le N(l_0,k_0)$, there exists an element in $\mathcal{G}_\tau^{\text{even}}$, say g_{n_0}, such that

$$\mu_n\left(g_{n_0}\right) \;=\; \begin{cases} 0 & \text{if } n \neq n_0 \text{ and } n \le N(l_0,k_0), \\ 1 & \text{if } n = n_0. \end{cases}$$

Hence for all $n_0 \in \mathbb{N}$ with $n_0 \leq N(l_0, k_0)$ we have

$$w_{l_0,n_0+k_0,n_0}\, q^{l_0 n_0}\, [n_0 + k_0]_q!\, [n_0]_q! \;=\; 0.$$

Now this must hold for any $k_0 \in \mathbb{N}$ and $l_0 \in \mathbb{Z}$, and after cancelling all non-zero factors we eventually conclude that $w_{l_0,n_0+k_0,n_0} = 0$ for all $n_0, k_0 \in \mathbb{N}$ and $l_0 \in \mathbb{Z}$. In other words, $w_{l,m,n} = 0$ for $m, n \in \mathbb{N}$ with $m \geq n$ and all $l \in \mathbb{Z}$.
Analogously $\langle \Upsilon(f \otimes g)\, c^k, \xi\rangle = 0$ (for f, g and k as before) implies $w_{l,m,n} = 0$ for $m, n \in \mathbb{N}$ with $m \leq n$ and all $l \in \mathbb{Z}$. Hence $w = 0$. ∎

Completing the proof of theorem 5.6.1 First take any $x \in \mathcal{U}_q\big(\mathfrak{L}(\mathcal{G}'_\tau)\big)$ and assume $\varphi(x\,\mathcal{U}_q) = \{0\}$ (cf. remark 4.1.3.iii). Expand x as in (5.22), but now with $X_m, Y_n \in \mathfrak{L}(\mathcal{G}'_\tau)$. Fix any $m_0 \in \mathbb{N}$ and consider, for any $p \in \mathbb{Z}$ and $r \in \mathbb{N}$,

$$\begin{aligned} x\, a^p b^r c^{m_0+r} \;&=\; \sum_{m=0}^{\infty} q^{-pm}\, \Upsilon\big((\Phi^p \otimes \mathrm{id})X_m\big)\, b^{m+r} c^{m_0+r} \\ &\quad+ \sum_{n=1}^{\infty} q^{pn}\, \Upsilon\big((\Phi^p \otimes \mathrm{id})Y_n\big)\, b^r c^{n+m_0+r} \end{aligned}$$

Here we have used proposition 5.3.1.5. Next recall how (5.7) allows us to handle powers of bc. Since φ vanishes on (5.13), only one term of the above expression will survive applying φ. We obtain

$$0 \;=\; \varphi\big(x\, a^p b^r c^{m_0+r}\big) \;=\; q^{-pm_0}\, \varphi\Big(\Upsilon\big((\Phi^p \otimes \Psi^{m_0+r})X_{m_0}\big)\Big) \tag{B.1}$$

for all $p \in \mathbb{Z}$ and $r \in \mathbb{N}$. Since $X_{m_0} \in \mathfrak{L}(\mathcal{G}'_\tau)$, we can write

$$X_{m_0} \;=\; \Big(\textstyle\sum_i\, f_i^{\mathrm{even}} \otimes g_i^{\mathrm{even}}\Big) \,\oplus\, \Big(\textstyle\sum_j\, f_j^{\mathrm{odd}} \otimes g_j^{\mathrm{odd}}\Big)$$

where i and j run through finite index sets, the $f_{i \,\mathrm{or}\, j}^{\substack{\mathrm{even}\\ \mathrm{odd}}}$ belong to $K^{\substack{\mathrm{even}\\ \mathrm{odd}}}(\mathbb{Z}\theta)$ and the $g_{i \,\mathrm{or}\, j}^{\substack{\mathrm{even}\\ \mathrm{odd}}}$ belong to $\mathcal{G}'^{\substack{\mathrm{even}\\ \mathrm{odd}}}_\tau$ (recall that $\substack{\mathrm{even}\\ \mathrm{odd}}$ means: even and odd *respectively*). The f_i^{even} and f_j^{odd} can and *will* be assumed to be linearly independent.
Now (B.1) becomes:

$$\textstyle\sum_i\, \lambda_{\mathrm{even}}(\Phi^p f_i^{\mathrm{even}})\, \chi_{\mathrm{even}}(\Psi^{m_0+r} g_i^{\mathrm{even}}) \;+\; \sum_j\, \lambda_{\mathrm{odd}}(\Phi^p f_j^{\mathrm{odd}})\, \chi_{\mathrm{odd}}(\Psi^{m_0+r} g_j^{\mathrm{odd}}) \;=\; 0.$$

Defining

$$F_r \;=\; \textstyle\sum_i\, \chi_{\mathrm{even}}\big(\Psi^{m_0+r} g_i^{\mathrm{even}}\big)\, f_i^{\mathrm{even}} \;+\; \sum_j\, \chi_{\mathrm{odd}}\big(\Psi^{m_0+r} g_j^{\mathrm{odd}}\big)\, f_j^{\mathrm{odd}},$$

we obtain functions F_r on $\mathbb{Z}\theta$ with finite support, enjoying $\sum_{k\in\mathbb{Z}} e^{pk\theta} F_r(k\theta) = 0$ for all $p \in \mathbb{Z}$ and $r \in \mathbb{N}$. It follows easily that $F_r = 0$, for any $r \in \mathbb{N}$, and hence $\chi_{\mathrm{even}}(\Psi^{m_0+r} g_i^{\mathrm{even}}) = 0$ and $\chi_{\mathrm{odd}}(\Psi^{m_0+r} g_j^{\mathrm{odd}}) = 0$, or equivalently

$$\sum_{n\in\mathbb{Z}} \big(\Psi^r(\Psi^{m_0} g_i^{\mathrm{even}})\big)\, (\tau q^{2n})\, q^{2n} = 0$$

and

$$\sum_{n\in\mathbb{Z}} \big(\Psi^r(\Psi^{m_0} g_j^{\rm odd})\big)\,(\tau q^{2n+1})\, q^{2n+1} = 0,$$

for all i, j and all $r \in \mathbb{N}$. Now from assumption (iii) it follows already that $\Psi^{m_0} g_i^{\rm even} = 0$ and hence $g_i^{\rm even} = 0$ for all i. Furthermore, since $\Omega\Psi^r = q^r\, \Psi^r\Omega$, we have

$$q^{r+1} \sum_{n\in\mathbb{Z}} \big(\Psi^r(\Omega\Psi^{m_0} g_j^{\rm odd})\big)\,(\tau q^{2n})\, q^{2n} = 0$$

for all $r \in \mathbb{N}$. Hence also $g_j^{\rm odd} = 0$ for all j. This proves $X_{m_0} = 0$. Analogously, considering $\varphi\big(x\, a^p b^{n_0+r} c^r\big) = 0$ for $n_0 \in \mathbb{N}_0$, it follows that $Y_{n_0} = 0$. Since m_0 and n_0 were chosen arbitrarily, we conclude that $x = 0$.
On the other hand, let $\xi \in \mathcal{A}_q(\mathcal{G}'_\nu)$ be such that $\omega(\mathcal{A}_q\xi) = \{0\}$. We can write

$$\xi = \sum_{l\in\mathbb{Z}} \alpha^l \left(\sum_{m=0}^{\infty} \gamma^m\, g_{m,l}(\gamma^*\gamma) \;+\; \sum_{n=1}^{\infty} (\gamma^*)^n\, h_{n,l}(\gamma^*\gamma) \right).$$

with only finitely many non-zero $g_{m,l}$, $h_{n,l} \in \mathcal{G}'_\nu$. Using (iii) we get $\xi = 0$. ∎

Appendix C

q-calculus

Fix any non-zero complex number q. For any $m \in \mathbb{N}$, the *q-number* $[m]_q$ is defined by

$$[m]_q \;=\; \frac{q^m - q^{-m}}{q - q^{-1}}.$$

We also consider the *q-factorial* $[m]_q!$ given by

$$[m]_q! \;=\; [1]_q[2]_q \ldots [m]_q \qquad\qquad [0]_q! = 1,$$

and the *q-shifted factorial* $(a;q)_m$ defined by

$$(a;q)_m \;=\; \prod_{k=0}^{m-1}(1 - aq^k) \qquad\qquad (a;q)_0 = 1$$

for any $a \in \mathbb{C}$. These two are closely related by the formula [11]

$$[m]_q! \;=\; \frac{q^{-\frac{1}{2}m(m-1)}}{(1-q^2)^m}\,(q^2;q^2)_m.$$

The q-Exponential function Let q be any real number with $0 < q < 1$.

Definition C.1 Take any $z \in \mathbb{C}$ and recall the q-shifted factorials $(z;q)_m$ as introduced above. Since $0 < q < 1$, the limit $m \to \infty$ is well defined and yields

$$(z;q)_\infty \;=\; \lim_{m\to\infty}(z;q)_m \;=\; \prod_{k=0}^{\infty}(1 - q^k z). \tag{C.1}$$

Now the q-Exponential $E_q : \mathbb{C} \to \mathbb{C}$ is defined by $E_q(z) = (-z;q)_\infty$. •

Remark C.2 According to Weierstraß theory of entire functions and canonical products, (C.1) defines an entire function in z, having simple zeros at the points q^{-k} for $k \in \mathbb{N}$, and no zeros elsewhere. Notice that it is essential that $0 < q < 1$ to ensure the convergence of (C.1). It should be noted that there exist other q-analogues of the exponential function as well, but the above one is the most suitable for our purposes. ⋆

Proposition C.3 *The q-Exponential E_q is entire and has the following power series at the origin:*

$$E_q(z) \;=\; \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\, z^n \tag{C.2}$$

for any $z \in \mathbb{C}$. Furthermore $E_q(-q^{-k}) = (q^{-k};q)_\infty = 0$ for all $k \in \mathbb{N}$.

Proof. See above remark; also (C.2) is a standard result in q-calculus. ∎

The q-derivative Whenever $f \in H(\mathbb{C})$ we consider the function

$$\mathbb{C}_0 \to \mathbb{C} : z \mapsto \frac{f(z) - f(qz)}{(1-q)z} \tag{C.3}$$

which is holomorphic on $\mathbb{C}_0$ and obviously has a removable singularity at the origin. Hence (C.3) actually defines an *entire* function again, which we denote by $D_q f$. This yields a linear operator $D_q : H(\mathbb{C}) \to H(\mathbb{C})$.

Appendix D

Inheritance schemes & Lookup tables

On the following pages one will find a schematic overview which visualizes the hierarchy of the many notions introduced in chapters 2 and 3. The arrows in these schemes could be read as '... inherits the features and properties of ...'

Each inheritance scheme is followed immediately by a scheme of exactly the same layout, providing a lookup table with references to related definitions.

D.1 Actor contexts

$$
\begin{tikzcd}
& \boxed{\begin{array}{c}\text{weakly unital}\\ \text{actor context}\end{array}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{\begin{array}{c}\text{faithful}\\ \text{actor context}\end{array}} \arrow[dr] & \boxed{\begin{array}{c}\text{actor context}\\ (E,\Omega;\langle\cdot,\cdot\rangle)\\ \text{with } \Omega \text{ unital}\end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c}\text{non-degenerate}\\ \text{actor context}\end{array}} \arrow[dl] \\
& \boxed{\text{actor context}} &
\end{tikzcd}
$$

definition 2.4.1.7
proposition 2.4.1.5
summary 2.4.2.2
paragraph §2.4.3
proposition 3.1.4

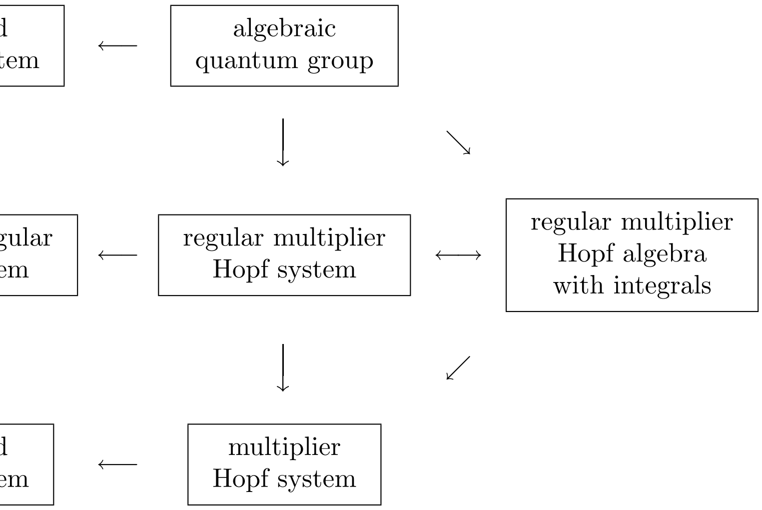
definition 2.4.1.6
lemma 2.6.4

paragraphs
§1.4 and §2.2

lemma 2.1.4.4

definition 2.1.1.1

D.2 Hopf systems


algebraic quantum group
regular multiplier Hopf system
regular multiplier Hopf algebra with integrals
multiplier Hopf system

2
1.4

[19, 20]

10
1.4

definition 3.8.1
lemma 3.8.8
proposition3.8.2

[41, 42, 43]
proposition 3.7.1.5
theorem 3.8.4

1.4

definition 3.8.1

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