The Distribution Free Newsboy Problem: Review and Extensions

Author(s): Guillermo Gallego and Ilkyeong Moon

Source: *The Journal of the Operational Research Society*, Aug., 1993, Vol. 44, No. 8 (Aug., 1993), pp. 825-834

Published by: Palgrave Macmillan Journals on behalf of the Operational Research Society

Stable URL: https://www.jstor.org/stable/2583894

J. Opl Res. Soc. Vol. 44, No. 8, pp. 825–834
Printed in Great Britain.

0160-5682/93 \$9.00+0.00

The Distribution Free Newsboy Problem: Review and Extensions

GUILLERMO GALLEGO[1] and ILKYEONG MOON[2]
[1]Columbia University, USA, and [2]Pusan National University, Korea

We present here a new, very compact, proof of the optimality of Scarf's ordering rule for the newsboy problem where only the mean and the variance of the demand are known. We then extend the analysis to the recourse case, where there is a second purchasing opportunity; to the fixed ordering cost case, where a fixed cost is charged for placing an order; to the case of random yields; and to the multi-item case, where multiple items compete for a scarce resource.

Key words: inventory, lot sizing, lost sales, newsboy problem

INTRODUCTION

The newsboy problem is to decide the stock quantity of an item when there is a single purchasing opportunity before the start of the selling period and the demand for the item is random. The trade-off is between the risk of overstocking (forcing disposal below the unit purchasing cost) and the risk of understocking (losing the opportunity of making a profit). The newsboy model is often used to aid decision making in the fashion and sporting goods industries, both at the manufacturing and at the retail level. In most cases, the distributional information of the demand is very limited. Sometimes all that is available is an educated guess of the mean and of the variance. There is a tendency to use the normal distribution under these conditions. However, the normal distribution does not provide the best protection against the occurrence of other distributions with the same mean and the same variance.

In 1958 Scarf[1] addressed the newsboy problem where only the mean μ and the variance σ^2 of the demand are known without any further assumptions about the form of the distribution of the demand. Taking a conservative approach, he modelled the problem as that of finding the order quantity that maximizes the expected profit against the worst possible distribution of the demand with the mean μ and the variance σ^2. He showed through a beautiful, but lengthy, mathematical argument that the worst distribution of the demand has positive mass at two points and used this result to obtain a closed form expression for the optimal order quantity. Unfortunately, Scarf's ordering rule is not well known as is evidenced by its absence in modern Operations Management and Operational Research textbooks[2–7].

The purpose of this paper is twofold. The first is to disseminate Scarf's ordering rule; the second is to extend Scarf's ideas in several directions.

We feel that Scarf's ordering rule is of practical value because it is optimal under the conservative approach outlined above when the distributional information is limited to the mean and the variance. Scarf's ordering rule is also easy to use since it only requires the computation of square roots, rather than the inverse of a cumulative distribution function. As we shall see, Scarf's rule is also very easy to remember and, perhaps more importantly, Scarf's rule provides us with an intuitive explanation of when it is profitable (in expectation) to order more (or less) than the expected demand.

Our presentation differs from Scarf's in the choice of the parameters and in the method of proof. We think that our choice of parameters makes Scarf's ordering rule easier to understand and easier to remember. Based on a simple observation relating the positive part of a number to its absolute value, we considerably simplify the proof of Scarf's rule. We also present, in closed form, an extremely simple lower bound on the expected profit with respect to all possible distributions

Correspondence: I. Moon, Department of Industrial Engineering, Pusan National University, Pusan 609–735, Korea

of demand. This lower bound is of interest because it indicates how the parameters influence expected profits, in the worst case.

The rest of the paper is organized as follows. We extend Scarf's ideas to the recourse case, where there is a second purchasing opportunity after observing the demand; to the fixed ordering cost case, where a fixed cost is charged for placing an order; to the case of random yields; and to the multi-item case, where multiple items compete for a scarce resource.

Other works related to the min–max approach include Gallego[8] and Gallego and Moon[9].

NEW PROOF OF SCARF'S ORDERING RULE

The data for the newsboy problem are as follows.

$c > 0$	the unit cost,
$p = c(1 + m) > c$	the unit selling price,
$s = (1 - d)c < c$	the unit salvage value,
μ	the expected demand,
σ	the standard deviation of the demand,
Q	the order quantity.

Note that the mark-up m and the discount d are positive parameters that indicate the return (loss) per dollar on units sold (unsold). Let D denote the random demand. We make no assumption on the distribution G of D other than saying that it belongs to the class $\mathcal{G}$ of cumulative distribution functions with mean μ and variance σ^2. In what follows we let $x^+ = \max\{x, 0\}$.

The expected profit can be written as

$$\pi^G(Q) = pE\min(Q, D) + sE(Q - D)^+ - cQ,$$

since $\min(Q, D)$ units are sold, $(Q - D)^+$ are salvaged, and Q units are purchased. Observing that

$$\min(Q, D) = D - (D - Q)^+,$$

and that

$$(Q - D)^+ = (Q - D) + (D - Q)^+,$$

we can write the expected profits as

$$\pi^G(Q) = (p - s)\mu - (c - s)Q - (p - s)E(D - Q)^+,$$

or, using the definition of m and d, as

$$\pi^G(Q) = c\{(m + d)\mu - dQ - (m + d)E(D - Q)^+\}. \tag{1}$$

Evidently, maximizing $\pi^G(Q)$ is equivalent to minimizing

$$dQ + (m + d)E(D - Q)^+, \tag{2}$$

so we concentrate on the latter problem.

Since the distribution G of D is unknown we want to minimize (2) against the worst possible distribution $\mathcal{G}$. To this end, we need the following two lemmas.

Lemma 1

$$E(D - Q)^+ \leqslant \frac{[\sigma^2 + (Q - \mu)^2]^{1/2} - (Q - \mu)}{2}. \tag{3}$$

Proof

Notice that

$$(D - Q)^+ = \frac{|D - Q| + (D - Q)}{2}.$$

The result follows by taking expectations and by using the Cauchy–Schwarz inequality

$$E|D - Q| \leqslant [E(D - Q)^2]^{1/2} = [\sigma^2 + (Q - \mu)^2]^{1/2}.$$

Lemma 2

For every Q, there exists a distribution $G^* \in \mathcal{G}$ where the bound (3) is tight.

Proof

For every Q, consider the two-point cumulative distribution G^* assigning weight

$$\beta = \frac{[\sigma^2 + (Q - \mu)^2]^{1/2} + (Q - \mu)}{2[\sigma^2 + (Q - \mu)^2]^{1/2}}$$

to

$$\mu - \sigma \left[\frac{1 - \beta}{\beta}\right]^{1/2} = Q - [\sigma^2 + (Q - \mu)^2]^{1/2},$$

and weight

$$1 - \beta = \frac{[\sigma^2 + (Q - \mu)^2]^{1/2} - (Q - \mu)}{2[\sigma^2 + (Q - \mu)^2]^{1/2}}$$

to

$$\mu + \sigma \left[\frac{\beta}{1 - \beta}\right]^{1/2} = Q + [\sigma^2 + (Q - \mu)^2]^{1/2}.$$

Clearly (3) holds with equality and it is easy to verify that $G^* \in \mathcal{G}$.

Lemma 1 and Lemma 2 originally appeared in Gallego[8] where it was also shown that the distribution achieving bound (3) is unique. The compact proof of Lemma 2 is due to Gallego[10].

Combining (2) and (3), our problem is now to minimize the upper bound

$$dQ + (m + d)\frac{[\sigma^2 + (Q - \mu)^2]^{1/2} - (Q - \mu)}{2}. \tag{4}$$

It is easy to verify that (4) is strictly convex in Q. Upon setting the derivative to zero and solving for Q we obtain Scarf's ordering rule:

$$Q^S = \mu + \frac{\sigma}{2}\left(\left[\frac{m}{d}\right]^{1/2} - \left[\frac{d}{m}\right]^{1/2}\right). \tag{5}$$

Thus (5) minimizes (4), and consequently maximizes (1) against the worst possible distribution of the demand. It is worth observing that the order quantity is independent of the unit cost c. This is because the expected profit (1) is homogeneous of degree 1 on the unit cost. Also, notice that (5) calls for an order larger (smaller) than the expected demand if and only if the ratio $m/d > 1$ $(m/d < 1)$. Consequently, in the typical formulation where the salvage value is zero $(d = 1)$, the optimal order size is larger (smaller) than the expected demand if and only if the mark-up is larger (smaller) than one.

Substituting (3) and then (5) into (1) we obtain, for all $G \in \mathcal{G}$, the following lower bound on the optimal expected profit

$$cm\mu\left[1 - \frac{\sigma}{\mu}\left(\frac{d}{m}\right)^{1/2}\right] \leqslant \pi^G(Q^S) \leqslant cm\mu. \tag{6}$$

Remark 1

The lower bound is linearly increasing (decreasing) in μ (σ) and increasing (decreasing) convex in the mark-up m (the discount d).

Remark 2

The lower bound is achieved by the distribution G^* exhibited in Lemma 2 with Q replaced by Q^S. This distribution has weight $m/(m+d)$ at $\mu - \sigma\sqrt{d/m}$ and weight $d/(m+d)$ at $\mu + \sigma(m/d)^{1/2}$.

Noting that $cm\mu$ is the maximum profit when demand is deterministic, we can regard $\frac{\sigma}{\mu}\left(\frac{d}{m}\right)^{1/2}$ as the maximal fractional cost of randomness. Note also that if we were to order μ units, then from (1) and (3), the expected profit would be at least

$$cm\mu\left[1 - \frac{m+d}{2m}\frac{\sigma}{\mu}\right].$$

So if the fraction $\frac{m+d}{2m}\frac{\sigma}{\mu}$ is small, then no great loss is incurred by simply ordering μ units, i.e. by ordering as if the problem were deterministic.

We normally expect demand to be a non-negative random variable. In this case, an order of size zero leads to an expected profit equal to zero. This is a consequence of the fact that for non-negative random variables $ED^+ = \mu$. Thus, we prefer to order zero units whenever ordering may lead to an expected loss in the worst case, i.e. to a negative value in the left side of (6). This happens if $m/d < (\sigma/\mu)^2$. In this case, the ordering rule is modified to order

$$\hat{Q}^S = Q^S \quad \text{if} \quad \frac{m}{d} \geqslant \left(\frac{\sigma}{\mu}\right)^2 \tag{7}$$

holds, and to order $\hat{Q}^S = 0$ otherwise.

To see that this rule is, in fact, optimal note that the two-point distribution exhibited in Lemma 2 is non-negative for

$$Q \geqslant \frac{\mu^2 + \sigma^2}{2\mu}.$$

Over the interval

$$0 \leqslant Q \leqslant \frac{\mu^2 + \sigma^2}{2\mu},$$

the worst distribution of the demand is the one exhibited in Lemma 2 for

$$Q = \frac{\mu^2 + \sigma^2}{2\mu}.$$

Over this interval, and under this two-point distribution, (2) is linear in Q. If condition (7) holds, the slope is negative, so by convexity $\hat{Q}^S = Q^S$. Conversely, if condition (7) fails to hold, the slope in (2) is positive, so the mininium of (2) is attained at $\hat{Q}^S = 0$. In the presence of the modification, the lower bound on the expected profit is given by the positive part of the lower bound in (6).

The above subsumes Scarf's results. The reader should contrast (5) with Scarf's original formula[1]:

$$Q^S = \mu + \frac{\sigma}{2}\frac{1-2a}{[a(1-a)]^{1/2}}$$

where $a = (c-s)/(p-s)$.

So far we have ignored the constraint $Q \geqslant 0$. We would like to show that if condition (7) is satisfied then $Q^S \geqslant 0$. We will show slightly more. To this end let Q^G be the optimal order quantity when the distribution of the demand is $G \in \mathcal{G}$.

Proposition 1

If (7) holds, then $\pi^G(Q^G) \geqslant 0$ and $Q^G \geqslant 0$ for all $G \in \mathcal{G}$.

Proof

If condition (7) holds then

$$\pi^G(Q^G) \geqslant \pi^G(Q^S) \geqslant cm\mu - c\sigma(md)^{1/2} \geqslant 0$$

So

$$\pi^G(Q^G) = c\{(m+d)\mu - dQ^G - (m+d)E(D-Q^G)^+\} \geqslant 0.$$

Assume for a contradiction that $Q^G < 0$; then, using Jensen's inequality

$$E(D-Q^G)^+ \geqslant E(D-Q^G) = \mu - Q^G,$$

we obtain the following contradiction

$$0 \leqslant \pi(Q^G) \leqslant c\{(m+d)\mu - dQ^G - (m+d)(\mu - Q^G)\} = cmQ^G < 0.$$

If we use the order quantity Q^S instead of Q^G, the expected loss is equal to

$$\pi^G(Q^G) - \pi^G(Q^S).$$

This is the largest amount that we would be willing to pay for the knowledge of G. This quantity can be regarded as the *Expected Value of Additional Information* (EVAI).

Example 1

This problem is taken from Silver and Peterson[7]. The unit cost is \$35.10, the unit selling price is \$50.30, and the unit salvage value is \$25.00. The mean and the standard deviation of the demand are 900 and 122, respectively. We compare the performance of Q^S with Q^N where $N \in \mathcal{G}$ represents the normal distribution. The results are (normal in parenthesis) $Q^S \simeq 925$ (931) and a worst case expected profit of \$12 168 (\$12 488). The EVAI (calculated with the exact values of Q^S and Q^N) is

$$\pi^N(Q^N) - \pi^N(Q^S) = \$12\,488.13 - \$12\,486.66 = \$1.47$$

Example 2

The unit cost is \$40, the unit selling price is \$60, and there is no salvage value. The mean and the standard deviation of the demand are 300 and 200, respectively. Again, we compare the performance of Q^S with Q^N. The results are (normal in parenthesis) $Q^S \simeq 229$ (214) and a worst case expected profit of \$343 (\$1636). The EVAI is

$$\pi^N(Q^N) - \pi^N(Q^S) = \$1636.80 - \$1623.67 = \$13.13$$

For the normal distribution, $N \in \mathcal{G}$, we have found through tabulations that $|Q^N - Q^S| \leqslant 0.0975\sigma$ over the set of problems with $1/9 \leqslant m/d \leqslant 9$. So, for most practical problems where the normal distribution is used, the difference between Scarf's ordering quantity and Q^N is no more than $10\%\sigma$. In fact, we have found through tabulations that

$$\pi^N(Q^N) - \pi^N(Q^S) \leqslant 0.0036c\sigma(md)^{1/2},$$

over the set of problems with $1/9 \leqslant m/d \leqslant 9$. Most real life problems will satisfy condition (7), so over those problems the guarantee is

$$\pi^N(Q^N) - \pi^N(Q^S) \leqslant 0.0036cm\mu,$$

hence using Q^{S} when the demand is normal will result in a loss no larger than in 0.36% of the deterministic profit.

THE RECOURSE CASE

In certain problems, we may have the recourse of placing a second order to satisfy the part of the demand not covered by the first order. Thus, if after ordering Q units we observe D and find that $D > Q$, an additional order is placed for $D - Q$, units. Let $c' = c(1 + e)$ denote the unit cost for items ordered after observing the demand. We assume that $0 < e < m$ because the solution to the other cases are trivial. Indeed, if $e \leqslant 0$ then the first order should be of size zero since the items can be purchased after the demand is known at a unit cost not higher than c. On the other hand, if $e \geqslant m$ then the second order should be of size zero since the unit cost c' is at least as large as the unit selling price p.

It is clear that under our assumption all the demand will be met. Thus the expected profit, is given by

$$\pi^{G}(Q) = p\mu + sE(Q - D)^{+} - cQ - c'E(D - Q)^{+}.$$

Using again the fact that $(Q - D)^{+} = (Q - D) + (D - Q)^{+}$ and the definition of m, d and e we can write the expected profit as

$$\pi^{G}(Q) = c\{(m + d)\mu - dQ - (e + d)E(D - Q)^{+}\}. \tag{8}$$

Note that the only difference between (1) and (8) is that e appears instead of m in the part involving $E(D - Q)^{+}$. This is intuitively correct since (1) and (8) must agree when $e = m$. This is because if $e = m$ there is no economic incentive to purchase units after observing the demand.

Our goal, as before, is to maximize the expected profit against the worst possible distribution of D. Solving this problem determines what part of the demand should be purchased to stock at unit cost c and what part should be purchased to order at unit cost c'. Using inequality (3) in (8) and following the logic of the previous section, we find that the optimal size of the initial order is given by (5) with e replacing m. Since $e < m$ the size of the first order is smaller when there is a second purchasing opportunity. The lower bound on the expected profit is given by Proposition 2.

Proposition 2

$$\pi^{G}(Q^{S}) \geqslant c(m\mu - \sigma(ed)^{1/2}). \tag{9}$$

Note that the lower bound in (9) is strictly larger than that in (6) so the lower bound on the expected profit is larger when there is a second purchasing opportunity.

If the demand is known to be non-negative, an initial order of size zero, followed by an order equal to the demand, leads to a positive expected profit equal to $c(m - e)\mu$. Thus, the first order should be of size zero unless the expected profit given in (9) is larger than $c(m - e)\mu$. This happens when (7) holds with e replacing m.

To summarize, in the recourse case with non-negative demand, the optimal size of the initial order is

$$\hat{Q}^{S} = \mu + \frac{\sigma}{2}\left(\left[\frac{e}{d}\right]^{1/2} - \left[\frac{d}{e}\right]^{1/2}\right) \quad \text{if} \frac{e}{d} \geqslant \left(\frac{\sigma}{\mu}\right)^{2}$$

holds and $\hat{Q}^{S} = 0$ otherwise.

The size of the second order is, of course, $(D - \hat{Q}^{S})^{+}$. By Proposition 1 the condition $e/d \geqslant (\sigma/\mu)^{2}$ implies that $Q^{G} \geqslant 0$ and $\pi^{G}(Q^{G}) \geqslant c(m - e)\mu$ for all $G \in \mathcal{G}$. Note that m does not enter into the formula of the optimal order size but enters in the lower bound given in (9).

Example 3

The data is as in Example 1. We assume that the items can be purchased after observing the demand at \$40 per unit. The results are (normal in parenthesis) $Q^S \simeq 855$ (845) and a worst case expected profit of \$12 820 (\$13 019). The EVAI is

$$\pi^N(845) - \pi^N(855) \simeq \$13\,019 - \$13\,017 = \$2$$

Example 4

The data is as in Example 2. We assume that the items can be purchased after observing the demand at \$50 per unit. The results are (normal in parenthesis) $Q^S \simeq 150$ (132) and a worst case expected profit of \$2000 (\$3188). The EVAI is

$$\pi^N(132) - \pi^N(150) \simeq \$3200 - \$3188 = \$12.$$

THE FIXED ORDERING COST CASE

Let $I \geqslant 0$ denote the initial inventory and suppose a fixed cost, say A, is charged for placing an order. Let $S = I + Q \geqslant I$, then the expected profit can be written as

$$\pi^G(S) = -A\mathbf{1}_{\{S>I\}} + c\{(m+d)\mu + I - dS - (m+d)E(D-S)^+\}.$$

where $\mathbf{1}$ denotes the indicator function.
Using Lemma 1 and 2, the problem reduces to

$$\min_{S \geqslant I} [A\mathbf{1}_{\{S>I\}} + K(S)]$$

where

$$K(S) = c\left\{dS + \frac{m+d}{2}[[(S-\mu)^2+\sigma^2]^{1/2} - (S-\mu)] - I - (m+d)\mu\right\}.$$

Let S^* denote the unconstrained minimizer of $K(S)$. From the result of the previous section we know that

$$S^* = \mu + \frac{\sigma}{2}\left(\left[\frac{m}{d}\right]^{1/2} - \left[\frac{d}{m}\right]^{1/2}\right)$$

and that

$$K(S^*) = -c(I + m\mu - \sigma(md)^{1/2}).$$

Clearly, an order should be placed if $I < S^*$ and $K(I) > A + K(S^*)$. Since $K(S)$ is strictly convex and is not bounded from above, there exists a unique $s^* < S^*$ satisfying

$$K(s^*) = A + K(S^*).$$

After some algebra we obtain

$$s^* = \mu + \frac{(m-d)\hat{A} - (m+d)[\hat{A}^2 - md\sigma^2]^{1/2}}{2md} \tag{10}$$

where

$$\hat{A} = \sigma(md)^{1/2} + \frac{A}{c}.$$

The ordering rule is: order up to $S^*(Q^* = S^* - I)$ units if $I < s^*$ and do not order otherwise.

Example 5

We continue Example 1. Suppose there is a fixed ordering cost, say $A = \$500$, then $(s^*, S^*) = (824{,}925)$ using (10). That is, the optimal policy is to order up to 925 units if the initial inventory is less than 824 and not to order otherwise.

THE RANDOM YIELD PROBLEM

Consider a production environment where the decision to release Q units for production results in $G(Q)$ good units, where $G(Q)$ is a random variable. We are particularly interested in the situation where each unit released for production has the same probability, say ρ, of being good. Thus, if Q is an integer, the yield $G(Q)$ is a binomial random variable with mean $Q\rho$ and variance $Q\rho\bar{\rho}$ where $\bar{\rho} \equiv 1 - \rho$. This model can also be used in a non-manufacturing setting, when an order for Q units results in the delivery of exactly Q units, each of which is good with probability ρ.

We are interested in determining the optimal order quantity to maximize expected profit against the worst possible distribution of the demand. As before, we transform the profit maximization problem into one of cost minimization. To do this, we define the mark-up m, and the discount d, relative to the expected unit cost $\hat{c} \equiv c/\rho$. Thus, $p = \hat{c}(1 + m)$ and $s = \hat{c}(1 - d)$. With this notation, the resulting cost minimization problem is equivalent to

$$\min_{Q} \left\{ dQ + (m + d)\frac{1}{\rho}E(D - G(Q))^+ \right\}. \tag{11}$$

We assume that the yield $G(Q)$ is independent of the demand D. Since we know the distribution of $G(Q)$, our knowledge of the distribution of $D - G(Q)$ is more than just its mean $\mu - Q\rho$ and its variance $\sigma^2 + Q\rho\bar{\rho}$. However, the upper bound on (11) obtained by conditioning on $G(Q) = q$, and applying Lemma 1 for each value of q, exceeds the upper bound on (11) resulting from a direct application of Lemma 1. Thus, applying Lemma 1 to $D - G(Q)$, we obtain

$$E(D - G(Q))^+ \leqslant \frac{[\sigma^2 + Q\rho\bar{\rho} + (\rho Q - \mu)^2]^{1/2} - (\rho Q - \mu)}{2}. \tag{12}$$

Minimizing the resulting upper bound on the expected cost with respect to Q, we obtain

$$Q^{\mathrm{S}} = \frac{1}{\rho}\left\{ \mu - \frac{\bar{\rho}}{2} + \frac{1}{2}\left(\left[\frac{m}{d}\right]^{1/2} - \left[\frac{d}{m}\right]^{1/2} \right) \left(\sigma^2 + \mu^2 - \left(\frac{\bar{\rho}}{2} - \mu\right)^2 \right)^{1/2} \right\}. \tag{13}$$

Following the development as before, we obtain a lower bound on the expected profit over all possible distributions of demand.

$$\hat{c}m\mu - \hat{c}\left[\sigma^2 + \mu^2 - \left(\frac{\bar{\rho}}{2} - \mu\right)^2 \right]^{1/2} (md)^{1/2} - \hat{c}(m - d)\bar{\rho}. \tag{14}$$

Notice that (13) and (14) reduce to (5) and (6) when $\rho = 1$.

THE MULTI-PRODUCT CASE (STOCHASTIC PRODUCT MIX PROBLEM)

Consider now a multi-item problem in the presence of a budget constraint. This is typically the problem faced at the time when purchasing or production decisions are made in the fashion and sporting goods industries where the purchasing manager must allocate his budget among competing items. Note that in the case of production, there may be a capacity constraint rather than a budget constraint. See Silver and Peterson[7] for the description of this problem. This problem is sometimes called a stochastic product mix problem (Johnson and Montgomery[5]).

Let $c_i, p_i = c_i(1 + m_i)$, and $s_i = c_i(1 - d_i)$ be the unit cost, the unit selling price, and the unit salvage value where m_i and d_i denote the mark-up and the discount on item $i = 1, \ldots, N$. Let μ_i and σ_i^2 denote the mean and the variance of the demand for item $i = 1, \ldots, N$. Suppose that the cost of purchasing all the items cannot exceed a predetermined budget of B dollars. We want to find the order quantities that maximize the expected profit against the worst possible distribution of the demand without exceeding the budget constraint. The problem can be formulated as follows:

$$\min_{Q_1,\ldots,Q_N} \sum_{i=1}^{N} c_i \left\{ d_i Q_i + (m_i + d_i) \frac{[(\sigma_i^2 + (Q_i - \mu_i)^2)^{1/2} - (Q_i - \mu_i)]}{2} \right\}$$

$$\text{subject to} \sum_{i=1}^{N} c_i Q_i \leqslant B. \tag{15}$$

We form the Lagrangian function

$$L(Q_1, \ldots, Q_N, \lambda) = \sum_{i=1}^{N} c_i \left\{ d_i Q_i + (m_i + d_i) \frac{[(\sigma_i^2 + (Q_i - \mu_i)^2)^{1/2} - (Q_i - \mu_i)]}{2} \right\} - \lambda \left[\sum_{i=1}^{N} c_i Q_i - B \right],$$

where λ is a Lagrange multiplier associated with the budget constraint. By computing $\frac{\partial L}{\partial Q_i} = 0$ for all i, we see that the solution is of the form:

$$Q_i(\lambda) = \mu_i + \frac{\sigma_i}{2} \left(\left[\frac{m_i - \lambda}{d_i + \lambda} \right]^{1/2} - \left[\frac{d_i + \lambda}{m_i - \lambda} \right]^{1/2} \right) \quad \text{if} \quad \frac{m_i - \lambda}{d_i + \lambda} \geqslant \frac{\sigma_i^2}{\mu_i^2} \tag{16}$$

and 0 otherwise.

The problem is to find the smallest non-negative λ such that $Q_i(\lambda)$ satisfies (15). The following algorithm is essentially a line search to find the optimal value of λ.

Algorithm

Step 1. Check if $Q_i(0)$ $(\lambda = 0)$ satisfies the budget constraint (15). If it satisfies the constraint, the solution is optimal, stop. Else go to Step 2.

Step 2. Start from an arbitrary $\lambda > 0$, set $\varepsilon > 0$.

Step 3. If $\frac{m_i - \lambda}{d_i + \lambda} \geqslant \frac{\sigma_i^2}{\mu_i^2}$, *set* $Q_i(\lambda)$ as in (16). Else set $Q_i(\lambda) = 0$.

Step 4. If $\sum_{i=1}^{N} c_i Q_i(\lambda) < B - \varepsilon$, decrease λ and go to Step 3.

If $\sum_{i=1}^{N} c_i Q_i^{\lambda} > B + \varepsilon$, increase λ and go to Step 3.

If $-\varepsilon \leqslant \sum_{i=1}^{N} c_i Q_i^{\lambda} - B \leqslant \varepsilon$, stop.

Example 6

Consider the problem of a merchandise manager for a department store who must purchase items for a special sale. He is considering four different items for stock, but is not certain of the sales potential for any item. In establishing his inventory levels prior to the start of the sale, he cannot exceed his budget of \$80 000. The sale is of short duration, so there is no opportunity to reorder. The relevant data are as follows: $\boldsymbol{c} = (35.1, 25.0, 28.0, 4.8)$, $\boldsymbol{p} = (50.3, 40.0, 32.0, 6.1)$, $\boldsymbol{s} = (25.0, 12.5, 15.1, 2.0)$, $\boldsymbol{\mu} = (900, 800, 1200, 2300)$, $\boldsymbol{\sigma} = (122, 200, 170, 200)$.

Using the algorithm, the optimal order quantities are (normal in parenthesis) 881 (871), 772 (758), 698 (729), and 2123 (2094). The optimal Lagrangian value is 0.127 (0.141). The worst case expected profit is \$26 391 (\$27 622). The value of the distributional information when demand is normally distributed is

$$\pi^{\mathrm{N}}(871, 758, 729, 2094) - \pi^{\mathrm{N}}(881, 772, 698, 2123) \simeq \$27\,622 - \$27\,333 = \$289.$$

CONCLUDING REMARKS

We have presented a new, compact proof of the optimality of Scarf's ordering rule. We have also extended Scarf's approach in several directions. We hope that this paper will help disseminate Scarf's ordering rule and that it will stimulate new research on robust inventory policies. It can be conjectured from numerical examples that Scarf's ordering rule is robust. Both theoretical and experimental investigations on the robustness of Scarf's ordering rule might be an interesting research problem. Another interesting extension of this paper would be to study (s, S) policies over an infinite horizion where only the mean and the variance of the demand are known.

Acknowledgements—The authors acknowledge the helpful comments made by two anonymous referees. The work of Guillermo Gallego was partially supported by the National Science Foundation under grant DDM-91-09636. The work of Ilkyeong Moon was partially supported by the Korean Science Foundation under grant 931-1000-033-1.

REFERENCES

1. H. Scarf (1958) A min-max solution of an inventory problem. In *Studies in The Mathematical Theory of Inventory and Production*. (K. Arrow, S. Karlin and H. Scarf, Eds) pp 201–209. Stanford University Press, California.
2. G. Hadley and T. Whitin (1963) *Analysis of Inventory Systems*. Prentice-Hall, New Jersey.
3. A. Hax and D. Candea (1984) *Production and Inventory Management*. Prentice-Hall, New Jersey.
4. F. Hillier and G. Lieberman (1990) *Introduction to Operations Research*. 5th edn, Holden-Day, California.
5. L. Johnson and D. Montgomery (1974) *Operations Research in Production Planning, Scheduling, and Inventory Control*. John Wiley, New York.
6. S. Nahmias (1989) *Production and Operations Analysis*. Irwin, Illinois.
7. E. Silver and R. Peterson (1985) *Decision Systems for Inventory Management and Production Planning*. 2nd ed, John Wiley, New York.
8. G. Gallego (1990) A minmax distribution free procedure for the (Q, r) inventory model. Working Paper. Department of IE & OR, Columbia University. (To appear in *Opns Res. Lett.*).
9. G. Gallego and I. Moon (1991) Minmax distribution free procedures for the inventory models with a mixture of backorders and lost sales. Working Paper. Department of IE & OR, Columbia University.
10. G. Gallego (1991) Tight bounds on tail probabilities and other expectations. Working Paper. Department of IE & OR, Columbia University.