Stochastic Networks (2MMS40)

Homework Assignment 1 (Loss networks)
Deadline: Tuesday March 7, 2023, 11:59 hrs (upload via Canvas)

This assignment is concerned with loss networks as considered in Section 3.2 of the lecture notes and can be made **in groups** of two students (or three after prior approval).
In the first part of the assignment you will model a real-world situation in terms of a loss network. In the second part of the assignment you will conduct an in-depth analysis of a loss network of your choice.

Modeling component [10 pt]

The Royal Netherlands Navy owns several frigates that are imperative to the organization, in order to contribute to the world-wide safety at sea. A frigate is typically out at sea on mission for several months at a time, at which its (sub)systems are continuously receiving maintenance in order to prevent failures. Fortunately, the ship crew is not complete without a team of maintenance workers that take care of this.

The team consists of smaller groups of maintenance workers where every group has its own specialization. A maintenance request can require one or more workers from one or more groups in order for it to be completed. Whenever a request cannot be handled immediately due to unavailable maintenance workers, it is simply not executed at all, knowing that it may be done the next time such a request comes in. However, skipping maintenance requests leads to higher risks of failures that may not only come with (possibly) high costs, but even jeopardize the mission whenever the failure is so severe that the frigate needs to be docked at the closest (safe) harbor for repair activities. Naturally, the execution of certain maintenance requests is more crucial to prevent high-cost failures than others.

The Royal Netherlands Navy is therefore strongly interested in getting some quantitative insights in the frequencies at which maintenance requests cannot be met. They recently learned that this can be modeled in terms of a 'loss network' for that purpose.

a. Explain how the above situation can be modeled in terms of a loss network, i.e., how to define the customer classes, routes, loads per customer class, the different links and their capacities, etc. Please elaborate on all the assumptions you are making.

b. Can you provide the Royal Netherlands Navy with an approach that minimizes the failure risk/costs? (It is not necessary to give concrete results, a method/line of argument for how to obtain relevant results will be sufficient.)

Erlang fixed-point approximation [30 pt]

In order to answer the questions below, you may consider your personal favorite loss network, just keeping in mind the following two conditions. First of all, the network should involve $L \geq 3$ resources (links) and $K \geq 3$ user classes (routes) i.e., the matrix of capacity requirements $(b_{k,l})_{k=1,\dots,K,l=1,\dots,L}$ should be at least of dimension 3×3. In addition, for at least two pairs of classes the associated routes should intersect but not completely overlap, i.e., have at least one link but not all links in common.

c. Describe the loss network that you consider in terms of the set of links and the set of classes and associated routes, and in particular specify the matrix of capacity requirements.

Assume that class-k users arrive as a Poisson process of rate λ_k and have generally distributed service times with mean β_k. Denote by $\rho_k = \lambda_k \beta_k$ the offered traffic volume of class k, $k = 1, \dots, K$.

d. Provide an expression for the stationary joint distribution of the numbers of active users of the various classes for arbitrary offered traffic volumes ρ_k, $k = 1, \dots, K$, and link capacities C_l, $l = 1, \dots, L$.

e. Show how the blocking probability for class-k users can be expressed in terms of the normalization constant $G = G(C_1, \dots, C_L)$ and the normalization constant $G^k = G(C_1^k, \dots, C_L^k)$ for a network with the same structural properties and traffic volumes where the capacity of link l is reduced to $C_l^k = C_l - b_{kl}$, $l = 1, \dots, L$.

f. Provide a rough indication of the largest link capacities C_l, $l = 1, \dots, L$, for which you are able to numerically calculate the normalization constant $G = G(C_1, \dots, C_L)$.

g. Construct the equations for the Erlang fixed-point approximation, specifically tailored to the particular loss network that you consider.

Consider a series of networks indexed by N where the offered traffic volumes and link capacities grow large in proportion,

$$\frac{1}{N}\rho_k(N) \to \rho_k \text{ as } N \to \infty \text{ for } k = 1, \dots, K,$$

$$\frac{1}{N}C_l(N) \to C_l \text{ as } N \to \infty \text{ for } l = 1, \dots, L.$$

If the matrix of capacity requirements has full rank, it is known for N tending to infinity that the blocking probabilities for the various classes calculated using the Erlang fixed-point approximation will converge to the exact values.

h. Fix a sequence of $(\rho_k(N))_{k,N}$ and $(C_l(N))_{l,N}$ for some values of N. Solve the equations for the Erlang fixed-point approximation (e.g. through iteration) and compute the exact blocking probabilities for the various classes. Can you observe the theoretical convergence? How good are the approximations?