Vehicle trajectories in a tandem of intersections

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Let $\mathcal{D}[a,b]$ denote the set of continuously differentiable functions $\gamma : [a,b] \to \mathbb{R}$ satisfying the constraints $0 \leq \dot{\gamma}(t) \leq 1$ and $-\omega \leq \ddot{\gamma}(t) \leq \omega$ for all $t \in [a,b]$. For $\gamma_1 \in \mathcal{D}[a_1,b_1], \gamma_2 \in \mathcal{D}[a_2,b_2]$, when we write $\gamma_1 \leq \gamma_2$ without explicitly mentioning where it applies, we mean $t \in [a_1,b_1] \cap [a_2,b_2]$. We also write $\gamma \leq \min\{\gamma_1,\gamma_2\}$ as a shorthand for $\gamma \leq \gamma_1$ and $\gamma \leq \gamma_2$.

Given some $\gamma \in \mathcal{D}[a,b]$ and some time $\xi \in [a,b]$, consider the *stopping trajectory* $\gamma[\xi]$ that is identical to the original trajectory until ξ, from which it starts decelerating to a full stop, so that at time $t \geq \xi$, the position is given by

$$\gamma[\xi](t) = \gamma(\xi) + \int_\xi^t \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\} d\tau \tag{1a}$$

$$= \gamma(\xi) + \begin{cases} \dot{\gamma}(\xi)(t-\xi) - \omega(t-\xi)^2/2 & \text{for } t \leq \xi + \dot{\gamma}(\xi)/\omega, \\ (\dot{\gamma}(\xi))^2/(2\omega) & \text{for } t \geq \xi + \dot{\gamma}(\xi)/\omega. \end{cases} \tag{1b}$$

This definition guarantees $\gamma[\xi] \in \mathcal{D}[a,\infty)$. Note that a stopping trajectory serves as a lower bound in the sense that, for any $\mu \in \mathcal{D}[c,d]$ such that $\gamma = \mu$ on $[a,\xi] \cap [c,d]$, we have $\gamma \leq \mu$ and $\dot{\gamma} \leq \dot{\mu}$. Furthermore, $\gamma[\xi](t)$ is a non-decreasing function in terms of either of its arguments, while fixing the other. To see this for ξ, fix any t and consider $\xi_1 \leq \xi_2$, then note that $\gamma[\xi_1](t)$ is a lower bound for $\gamma[\xi_2](t)$.

Property 1. *Both $\gamma[\xi](t)$ and $\dot{\gamma}[\xi](t)$ are continuous when considered as functions of (ξ,t).*

Proof. Write $f(\xi,t) := \gamma[\xi](t)$ to emphasize that we are dealing with two variables. Recall that $\dot{\gamma}$ is continuous by assumption, so the equation $\tau = \xi + \dot{\gamma}(\xi)/\omega$ defines a separation boundary of the domain of f. Both cases of (1b) are continuous and they agree at this boundary, so f is continuous on all of its domain. Since $x \mapsto \max\{0,x\}$ is continuous, it is easy to see that also $(\xi,t) \mapsto \dot{\gamma}[\xi](t) = \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\}$ is continuous. □

Because $\gamma[\xi](t)$ is continuous and non-decreasing in ξ, the set $X(t_0,x_0) := \{\xi : \gamma[\xi](t_0) = x_0\}$ is a closed interval (Lemma 3). Define the closed region $\bar{U} := \{(t,x) : \gamma[a](t) \leq x \leq \gamma[b](t)\}$. For each $(t_0,x_0) \in \bar{U}$, there must be some ξ_0 such that $\gamma[\xi_0](t_0) = x_0$, as a consequence of the intermediate value theorem and the above property. We use U to denote $\bar{U}$ without the points on γ, by defining $U := \bar{U} \setminus \{(t,x) : \gamma(t) = x\}$. Next, we prove that $\gamma[\xi_0]$ is actually unique if $(t_0,x_0) \in U$, so we may choose $\xi(t_0,x_0) := \max X(t_0,x_0)$ as the canonical representation of this unique trajectory $\gamma[\xi(t_0,x_0)]$.

Property 2. *For $(t_0,x_0) \in U$, if $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0) = x_0$, then $\gamma[\xi_1] = \gamma[\xi_2]$.*

Proof. Suppose $t_0 < \xi_i$, then $x_0 = \gamma[\xi_i](t_0) = \gamma(t_0)$ contradicts the assumption $(t_0,x_0) \in U$. Therefore, assume $\xi_1 \leq \xi_2 < t_0$, without loss of generality. Since $\gamma[\xi_1] = \gamma[\xi_2]$ on $[a,\xi_1]$, note that we have the lower bounds $\gamma[\xi_1] \leq \gamma[\xi_2]$ and $\dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]$. We must have $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, because otherwise $\gamma[\xi_1] > \gamma[\xi_2]$ somewhere in a sufficiently small neighborhood of t_0, which contradicts the first lower bound.

Since $\dot{\gamma}[\xi_1](\xi_1) \leq \dot{\gamma}[\xi_2](\xi_2)$, it is clear that $\ddot{\gamma}[\xi_1](t) \geq \ddot{\gamma}[\xi_2](t)$, for $t \geq \xi_2$. This implies that $\dot{\gamma}[\xi_1](t) \geq \dot{\gamma}_2(t)$ for $t \geq t_0$. This in turn implies that $\dot{\gamma}[\xi_1](t) = \dot{\gamma}[\xi_2](t)$ and thus $\gamma[\xi_1](t) = \gamma[\xi_2](t)$ for $t \geq t_0$.

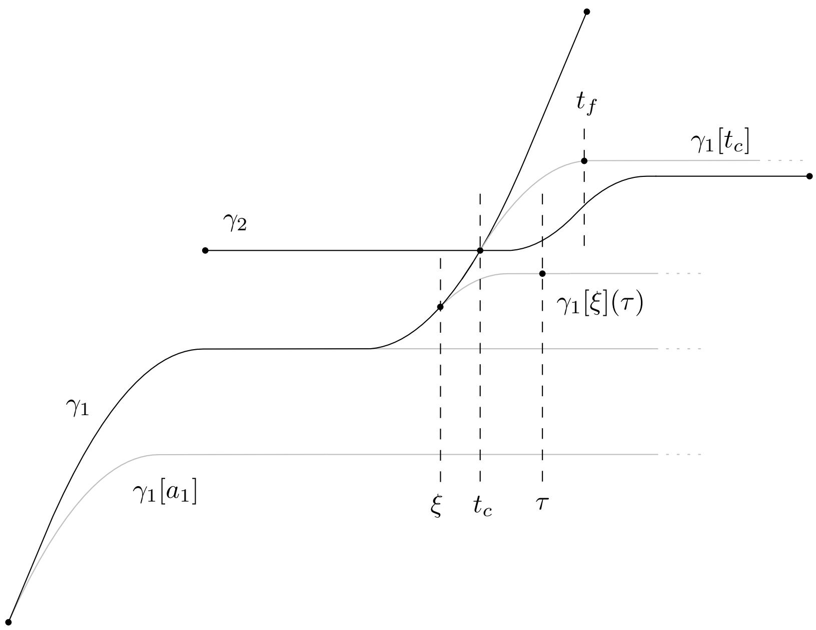

Figure 1: Sketch of the proof of Lemma 1.

It remains to show that $\gamma[\xi_1] = \gamma[\xi_2]$ on $[\xi_1, t_0]$, so consider the smallest $t^* \in (\xi_1, t_0)$ such that $\gamma[\xi_2](t^*) > \gamma[\xi_1](t^*)$. Since $\dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]$, this implies that $\gamma[\xi_2](t) > \gamma[\xi_1](t)$ for all $t \geq t^*$, but this contradicts our assumption that $\gamma[\xi_2](t_0) = \gamma[\xi_1](t_0)$. □

Lemma 1. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be intersecting at exactly one time t_c and assume $\dot{\gamma}_1(t_c) > \dot{\gamma}_2(t_c)$, then under the conditions*

(i) $\gamma_2 \geq \gamma_1[a_1]$,

(ii) $b_2 \geq t_f := t_c + \dot{\gamma}_1(t_c)/\omega$,

there is a unique trajectory ψ such that

- $\psi = \gamma_1[\xi]$, *for some* $\xi < t_c$,
- $\psi(\tau) = \gamma_2(\tau)$ *and* $\dot{\psi}(\tau) = \dot{\gamma}_2(\tau)$, *for some* $\tau > t_c$,
- $\psi \leq \gamma_2$.

Proof.

- Identify for which parameters $\xi < t_c < \tau$ we have $\gamma_1[\xi](\tau) = \gamma_2(\tau)$ and $\dot{\gamma}_1[\xi](\tau) = \dot{\gamma}_2(\tau)$.
  - Define set U and functions $X(t, x)$ and $\xi(t, x)$ as we did for γ above, now for γ_1.
  - For each $\tau > t_c$, observe that $(\tau, \gamma_2(\tau)) \in U$. This means that there is some $\xi = \xi(\tau, \gamma_2(\tau))$ such that $\psi_\tau := \gamma_1[\xi]$ is the unique trajectory such that $\psi_\tau(\tau) = \gamma_2(\tau)$. Therefore, we proceed to characterize the set T of values of $\tau > t_c$ for which also $\dot{\psi}_\tau(\tau) = \dot{\gamma}_2(\tau)$. More explicitly, we have $T := \{\tau > t_c : \dot{\gamma}_1[\xi(\tau, \gamma_2(\tau))](\tau) = \dot{\gamma}_2(\tau)\}$.
  - To this end, we define the auxiliary function $g(t, x) := \dot{\gamma}_1[\xi(t, x)](t)$, which gives the slope of the unique stopping trajectory through each point $(t, x) \in U$.
- Function g is continuous in (t, x). We use the notation $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$.

- We will write $f_x(\xi, t) = \gamma_1[\xi](t)$, $f_v(\xi, t) = \dot{\gamma}_1[\xi](t)$ and $h_t(\xi) = \gamma_1[\xi](t)$ to emphasize the quantities that we treat as variables. Observe that $h_t^{-1}(x) = X(t, x)$.
- Let $x_0 = f_x(\xi_0, \tau_0)$ and $v_0 = f_v(\xi_0, \tau_0)$ for some ξ_0 and τ_0 and pick some arbitrary $\varepsilon > 0$. Note that $\xi_0 \in [\xi_1, \xi_2] := h_{\tau_0}^{-1}(x_0)$. We apply the ε-δ definition of continuity to each of the endpoints of this interval. Let $i \in \{1, 2\}$, then there exist $\delta_i > 0$ such that $\xi \in N_{\delta_i}(\xi_i), \tau \in N_{\delta_i}(\tau_0)$ implies $f_v(\xi, \tau) \in N_\varepsilon(v_0)$. Let $\delta = \min\{\delta_1, \delta_2\}$ and define $N_1 := (\xi_1 - \delta, \xi_2 + \delta)$ and $N_2 := N_\delta(\tau_0)$. Let $\xi \in N_1$ and $\tau \in N_2$, then $f_v(\xi, \tau) \in N_\varepsilon(v_0)$. This is obvious when ξ is chosen to be in one of $N_{\delta_i}(\xi_i)$. Otherwise, we must have $\xi \in [\xi_1, \xi_2]$, in which case $f_v(\xi, \tau) = f_v(\xi_1, \tau) \in N_\varepsilon(v_0)$.
- Because $h_{\tau_0}(\xi)$ is continuous, the image $I := h_{\tau_0}(N_1)$ must be an interval containing x_0, with $\inf I = h_{\tau_0}(\xi_1 - \delta)$ and $\sup I = h_{\tau_0}(\xi_2 + \delta)$.
- We argue that I contains x_0 in its interior. For sake of contradiction, suppose $x_0 = \max I$, then $h_{\tau_0}(\xi_2 + \delta') = x_0$, for each $\delta' \in (0, \delta)$, because h_{τ_0} is non-decreasing, but this contradicts the definition of ξ_2. Similarly, when $x_0 = \min I$, then $h_{\tau_0}(\xi_1 - \delta') = x_0$, for each $\delta' \in (0, \delta)$, which contradicts the definition of ξ_1.
- Define $\nu := \min\{x_0 - \inf I, \sup I - x_0\}$ and $N_3 := (x_0 - \nu/2, x_0 + \nu/2)$. Because $h_\tau(\xi)$ is also continuous in τ, there exists a neighborhood $N_2^* \subset N_2$ of τ_0 such that for every $\tau \in N_2^*$, we have

$$h_\tau(\xi_1 - \delta) \leq h_{\tau_0}(\xi_1 - \delta) + \nu/2 = \inf I + \nu/2 < x_0 - \nu/2,$$
$$h_\tau(\xi_2 + \delta) \geq h_{\tau_0}(\xi_2 + \delta) - \nu/2 = \sup I - \nu/2 > x_0 + \nu/2,$$

  which shows that $h_\tau(N_1) \supset N_3$. It follows that $h_\tau^{-1}(N_3) \subset N_1$.
- Finally, take any $\tau \in N_2^*$ and $x \in N_3$, then there exists some $\xi \in N_1$ such that $h_\tau(\xi) = x$ and $g(\tau, x) = f_v(\max h_\tau^{-1}(x), \tau) = f_v(\xi, \tau) \in N_\varepsilon(v_0)$.

- Function g is non-decreasing and Lipschitz continuous in x.

  - Let $x_1 \leq x_2$ and τ such that $g(\tau, x_1)$ and $g(\tau, x_2)$ are defined. There must be $\xi_1 \leq \xi_2$ such that $h_\tau(\xi_1) = x_1$ and $h_\tau(\xi_2) = x_2$ and we have

$$\begin{aligned} g(\tau, x_1) = \dot{\gamma}_1[\xi_1](\tau) &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} = \dot{\gamma}_1[\xi_2](\tau) = g(\tau, x_2). \end{aligned}$$

  - Furthermore, we have $\dot{\gamma}_1(\xi_2) \leq \dot{\gamma}_1(\xi_1) + \omega(\xi_2 - \xi_1)$, so that

$$\begin{aligned} g(\tau, x_2) &= \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) + \omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1) + 2\omega(\xi_2 - \xi_1)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} + 2\omega(\xi_2 - \xi_1) = g(\tau, x_1) + 2\omega(\xi_2 - \xi_1). \end{aligned}$$

    Together with the above non-decreasing property, this shows that g is Lipschitz continuous in x, with Lipschitz constant 2ω.

- Observe that T can also be written as $T = \{\tau > t_c : \dot{\gamma}_2(\tau) = g(\tau, \gamma_2(\tau))\}$, so continuity of g shows that it is a closed set (Lemma 3). It is not necessarily connected (see Figure ...), so it is the union of a sequence of disjoint closed intervals $T_1, T_2, \ldots, T_n$.

  - Write $\tau_i := \min T_i$ and let $\psi_i := \gamma_1[\xi(\tau_i, \gamma_2(\tau_i))]$ be the unique stopping trajectory through $(\tau_i, \gamma_2(\tau_i))$. By definition, we have $\dot{\gamma}_2(\tau) = g(\tau, \gamma_2(\tau)) = \dot{\psi}_i(\tau)$, for every $\tau \in T_i$. Since $g(t, x)$ is continuous in t and Lipschitz continuous in x, it is a consequence of the existence and uniqueness theorem (Picard-Lindelöf) that $\gamma_2 = \psi_i$ on T_i. Hence, we have $\gamma_1[\xi(\tau, \gamma_2(\tau))] = \psi_i$ for any $\tau \in T_i$, so ψ_i can be seen as the canonical candidate trajectory corresponding to T_i.

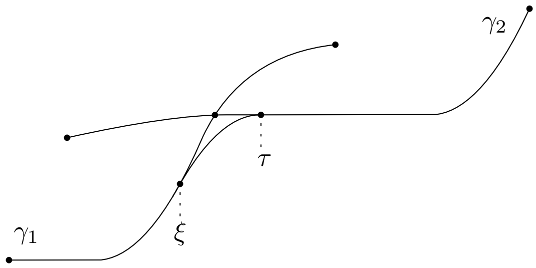

Figure 2: Two non-decreasing functions joined by a part with maximal negative curvature.

- Show that ψ_1 exists. We write $s(\tau) := g(\tau, \gamma_2(\tau))$.
  - Case 1: Suppose $\gamma_2(t_f) \le \gamma_1[t_c](t_f)$, then because g is non-decreasing in x, we have $g(t_f, \gamma_2(t_f)) \le g(t_f, \gamma_1[t_c](t_f)) = \dot\gamma_1(t_c) - \omega(t_f - t_c) = 0$ by definition of t_f, so $s(t_f) = 0$.
  - Case 2: Otherwise, (it follows from Lemma ...) that γ_2 crosses $\gamma_1[t_c]$ at some time $t_d \in (t_c, t_f)$, such that $\dot\gamma_2(t_d) > \gamma_1[t_c](t_d) = s(t_d)$.
  - We have $\gamma_1[a_1](t) \le \gamma_2(t) \le \gamma_1[t_c](t)$ for $t \in \{t_f, t_d\}$, so the intermediate value theorem guarantees that $s(t)$ actually exists for the above two cases, because there is some ξ such that $\gamma_2(t) = f_x(\xi, t) = \gamma_1[\xi](t)$ and thus $s(t) = g(t, \gamma_2(t)) = f_v(\max h_t^{-1}(\gamma_2(t)), t) = f_v(\xi, t) = \dot\gamma_1[\xi](t)$ exists.
  - In both of the above two cases, we have $\dot\gamma_2(t_c) < \dot\gamma_1(t_c) = s(t_c)$ and $\dot\gamma_2(t_d) \ge s(t_d)$ for some $t_d \in (t_c, t_f]$. Hence, as a consequence of the intermediate value theorem, there must be some smallest $\tau_1 \in (t_c, t_d]$ such that $\dot\gamma_2(\tau_1) = s(\tau_1)$.
- Show that $\psi_i \le \gamma_2$ if and only if $i = 1$.
  - Show that $\psi_i(t) > \gamma_2(t)$ for some t implies $i \ge 2$.
    - Suppose ψ_i crosses γ_2 at some $t_x \in (t_c, \tau_i)$, so $\dot\gamma_2(t_x) < \dot\gamma_1[\xi_i](t_x) = s(t_x)$.
    - By definition of τ_i, we have $\dot\gamma_2(\tau_i) = s(\tau_i)$.
    - Argue that we must have $\dot\gamma_2(t) > s(t)$ for some $t \in (t_x, \tau_i)$, otherwise $\gamma_2(\tau_i) < \gamma_1[\xi_i](\tau_i)$.
    - Therefore, this shows there exists $t \in (t_x, \tau_i)$ such that $\dot\gamma_2(t) = s(t)$, which means that $i \ge 2$.
  - Show $i \ge 2$ implies $\psi_i(t) > \gamma_2(t)$ for some t.
  - In conclusion, $\psi := \psi_1$ is the unique trajectory satisfying the stated requirements.

□

Remark 1. Note that condition (i) in Lemma 1 is necessary, because Figure ... shows a configuration in which the trajectories cannot be joined. Condition (ii) is not necessary, because Figure ... shows a valid configuration that can be joined, but which does not satisfy this condition.

Suppose we have two trajectories that cross each other exactly once. In the following, we will investigate conditions under which, roughly speaking, these trajectories can be glued together to form a smooth trajectory by introducing a stopping trajectory in between, as illustrated in Figure 2. Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ and let $[\xi, \tau] \subset [a_1, b_1] \cap [a_2, b_2]$

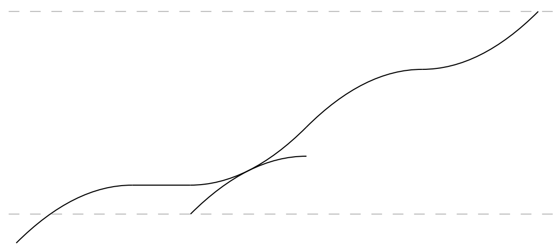

Figure 3: Illustration of “buffer constraint”.

parameterize the interval over which we introduce deceleration, then we define the candidate trajectory

$$(\gamma_1 * \gamma_2)[\xi,\tau](t) := \begin{cases} \gamma_1(t) & \text{for } t \in [a_1, \xi], \\ \gamma_1[\xi](t) & \text{for } t \in [\xi,\, \tau], \\ \gamma_2(t) & \text{for } t \in [\tau, b_2]. \end{cases}$$

Suppose γ_1 and γ_2 intersect at exactly one time t_c, but do so tangentially, i.e., such that $\dot{\gamma}_1(t_c) = \dot{\gamma}_2(t_c)$, then clearly $\gamma_1 * \gamma_2 := (\gamma_1 * \gamma_2)[t_c, t_c]$ satisfies $\gamma_1 * \gamma_2 \in \mathcal{D}[a_1, b_2]$ and $\gamma_1 * \gamma_2 \leq \min\{\gamma_1, \gamma_2\}$. Next, we present conditions under which we can choose such ξ and τ in the case when γ_1 and γ_2 do not intersect tangentially.

When it exists, we will denote the unique ψ from Lemma 1 as $\gamma_1 * \gamma_2$.

Remark 2. Based on Lemma 1, a numerical method could be developed to compute ξ and τ. However, the trajectories that we consider contain more structure that allows a simpler algorithm.

Lemma 2. *Let $\gamma_1 \in \mathcal{D}[a_1, b_2]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ such that $\psi := \gamma_1 * \gamma_2$ exists and let ξ and τ denote the joining times. For all $\gamma \in \mathcal{D}[a, b]$ such that $\gamma \leq \min\{\gamma_1, \gamma_2\}$, we have $\gamma \leq \psi$.*

Proof. We obviously have $\gamma \leq \psi$ on $[a_1, \xi] \cup [\tau, b_2]$, so consider the interval (ξ, τ) of the joining deceleration part. Suppose there exists some $t_d \in (\xi, \tau)$ such that $\gamma(t_d) > \psi(t_d)$. Because $\gamma(\xi) \leq \psi(\xi)$, this means that γ must intersect ψ at least once in $[\xi, t_d)$, so let $t_c := \sup\{t \in [\xi, t_d) : \gamma(t) = \psi(t)\}$ be the latest time of intersection such that $\gamma \geq \psi$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{\gamma}(t_v) > \dot{\psi}(t_v)$, otherwise

$$\gamma(t_d) = \gamma(t_c) + \int_{t_c}^{t_d} \dot{\gamma}(t)dt \leq \psi(t_c) + \int_{t_c}^{t_d} \dot{\psi}(t)dt = \psi(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{\gamma}(t) \geq \dot{\gamma}(t_v) - \omega(t - t_v) > \dot{\psi}(t_v) - \omega(t - t_v) = \dot{\psi}(t).$$

It follows that $\gamma(\tau) > \psi(\tau)$, which contradicts $\gamma \leq \gamma_2$. $\square$

Lemma 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $y \in \mathbb{R}^m$, then the level set $N := f^{-1}(\{y\})$ is a closed subset of $\mathbb{R}^n$.*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. $\square$

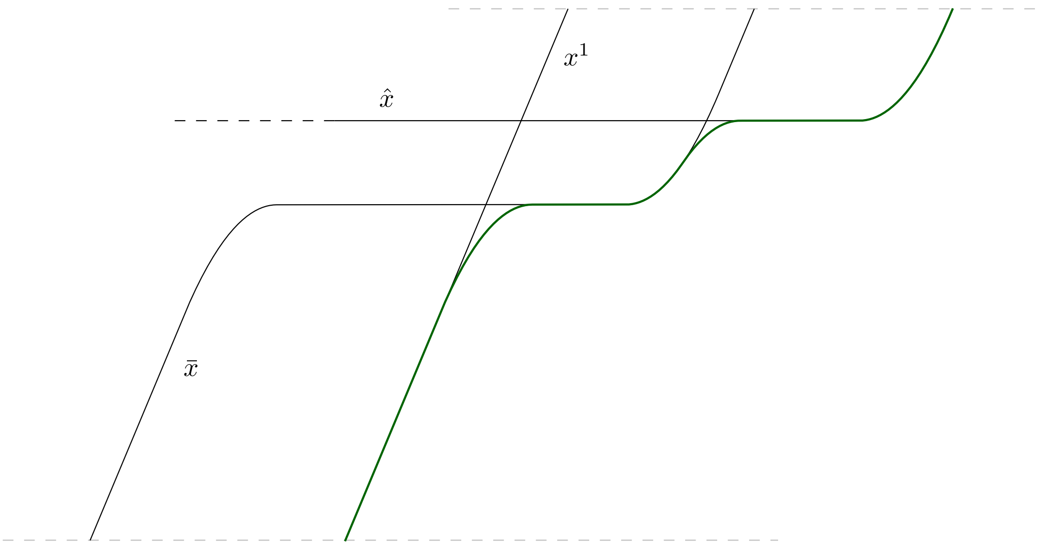

Figure 4: Sketch of how the three boundaries are joined to form the optimal trajectory.