Bounded lane capacity

Up to this point, we have not taken into account the fact that lanes between intersection have finite capacity. We need to incorporate this aspect in order to develop a model that could be used for practical applications. Under high traffic loads, lanes with finite buffer capacity can give rise to *blocking* of upstream intersections. Therefore, the traffic controller needs to take into account these additional dynamics.

Recall the single intersection scheduling model

$$
\begin{aligned}
\min_{y} \quad & \sum_{i \in \mathcal{N}} y_i \\
\text{s.t.} \quad & r_i \leq y_i && \text{for } i \in \mathcal{N}, \\
& y_i + \rho_i \leq y_j && \text{for } (i,j) \in \mathcal{C}, \\
& y_i + \sigma_i \leq y_j \text{ or } y_j + \sigma_j \leq y_i && \text{for } \{i,j\} \in \mathcal{D},
\end{aligned}
$$

where we had

$$
\begin{aligned}
\mathcal{D} &= \{(i,j) \in \mathcal{N} : l(i) \neq l(j)\}, \\
\mathcal{C} &= \{(i,j) \in \mathcal{N} : l(i) = l(j),\ k(i) + 1 = k(j)\}.
\end{aligned}
$$

We start the simplest extension of the single intersection model by considering two intersections in tandem. We define a graph (V, E) with *labeled edges* as follows. Let V denote the indices of the intersections. Let E denote the set of ordered triples (l, v, w) for each lane l whose vehicles travel from intersection v to w. Let $d(v, w)$ denote the minimum time necessary to travel between intersections v and w. Let $b(v, w)$ denote the maximum number of vehicles that can be on the lane between intersections v and w. Let $\mathcal{N}(l)$ denote all the vehicles starting on lane l. Let $v_0(i)$ denote the first intersection that vehicle i encounters on its route. Let $\mathcal{R}(l)$ denote the set of intersections visited by vehicles from lane l. Let $y(i, v)$ denote the crossing time of vehicle i at intersection $v \in V$. Let $\mathcal{C}^v$ and $\mathcal{D}^v$ denote the disjunctive and respectively disjunctive pairs for intersection $v \in V$.

Writing `conj(...)` and `disj(...)` for the usual conjunctive and disjunctive constraints, we obtain the following formulation

$$
\begin{aligned}
\min_{y} \quad & \sum_{i \in \mathcal{N}} \sum_{v \in \mathcal{R}(l(i))} y(i, v) && && \text{(1a)} \\
\text{s.t.} \quad & r_i \leq y(i, v_0(i)) && \text{for } i \in \mathcal{N}, && \text{(1b)} \\
& \texttt{conj}(y(i,v), y(j,v)) && \text{for } (i,j) \in \mathcal{C}^v, v \in V, && \text{(1c)} \\
& \texttt{disj}(y(i,v), y(j,v)) && \text{for } \{i,j\} \in \mathcal{D}^v, v \in V, && \text{(1d)} \\
& y(i,v) + d(v,w) \leq y(i,w) && \text{for } i \in \mathcal{N}(l), (l,v,w) \in E, && \text{(1e)} \\
& y(i,w) + \hat{\rho}_i \leq y(j,v) && \text{for } (i,j,v,w) \in \mathcal{F}, && \text{(1f)}
\end{aligned}
$$

where $\mathcal{F}$ is defined as follows

$$
\mathcal{F} = \{(i,j,v,w) : i,j \in \mathcal{N}(l), k(i) + b(v,w) + 1 = k(j), (l,v,w) \in E\}.
$$

Each $(i, j, v, w) \in \mathcal{F}$ represents a pair of vehicles driving on the same lane, for which the first vehicle must have left lane segment (v, w) before vehicle j can enter. The constraints involving $\mathcal{F}$ yield the following property of schedules.

Proposition 1. *Let y be a solution to the network scheduling problem* (1)*. Each lane segment $(l, v, w) \in E$ contains never more than $b(v, w)$ vehicles.*

Proof. Let i be a vehicle that has (v, w) on its route. Define the occupancy interval $D_i = [y(i, v),\, y(i, w)]$, then we say that i occupies (v, w) at some time t whenever $t \in D_i$. Therefore, the number of vehicles in (v, w) at time t equals the number of such intervals containing t.

Suppose at some time t, there are more than $b(v, w)$ vehicles i such that $t \in D_i$. Let i_0 be such that $y(i_0, v) \leq y(i, v)$ for all i such that $t \in D_i$. By the conjunctive constraints at v, we have

$$y(i_0, v) + \rho_{i_0} \leq y(i_1, v) + \rho_{i_1} \leq \cdots \leq y(i_k, v) + \rho_{i_k},$$

where $i_k = (l(i_0), k(i_0) + k)$. To be continued... $\square$