Vehicle dynamics

In control theory, it is common to model motion dynamics of a system in terms of a state vector $x(t) \in \mathbb{R}^n$ and a control input vector $u(t) \in \mathbb{R}^m$, which result in a scalar position $y(t)$ via the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t). \tag{1b}$$

Furthermore, it is common to restrict the state and control trajectories by imposing linear constraints

$$Gx(t) \leq b, \tag{2a}$$
$$Fu(t) \leq d. \tag{2b}$$

In the discussion that follows, each vehicle is modeled as a *double integrator*, with $x(t) = (p(t), v(t))$, where $p(t)$ and $v(t)$ are the scalar position along a predefined path and corresponding velocity, respectively. The three matrices are chosen such that

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \tag{3a}$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t), \tag{3b}$$

which may simply be rewritten as

$$\dot{p}(t) = v(t), \quad \dot{v}(t) = u(t), \quad y(t) = p(t), \tag{4}$$

where we recognize that the control input $u(t)$ corresponds directly to the acceleration of the vehicle. Furthermore, the constraints (2) are chosen such that the acceleration is bound from above and below, so

$$\underline{u} \leq u(t) \leq \overline{u}. \tag{5}$$

For technical reasons, it is assumed the system is *strongly output monotone*, defined as

$$\dot{y}(t) \geq \epsilon, \tag{6}$$

for some $\epsilon > 0$, which means that a vehicle cannot stop or reverse, but can move at an arbitrarily low speed.

Intersection model

Consider an intersection with L lanes. We define the index set

$$\mathcal{I} = \{(l, k) : k \in \{1, \ldots, n_l\},\ l \in \{1, \ldots L\}\}, \tag{7}$$

where n_l denotes the number of vehicles of lane l. To further help with notation, given vehicle index $i = (r, s) \in \mathcal{I}$, we define $l(i) = r$ and $k(i) = s$.

We assume that the position $p_i(t)$ of some vehicle $i \in \mathcal{I}$ corresponds to the physical front of the vehicle. In order to model collision avoidance, we say that a vehicle *occupies the intersection* whenever $p_i(t) \in [L_i, H_i] = \mathcal{E}_i$. The collision avoidance constraints are then given by

$$(p_i(t), p_j(t)) \notin \mathcal{E}_i \times \mathcal{E}_j, \tag{8}$$

for all t and for all pairs of indices $i, j \in \mathcal{I}$ with $l(i) \neq l(j)$, which we collect in the set $\mathcal{D}$. Furthermore, in order to model a safe distance between vehicles on the same lane, we require that

$$p_i(t) - p_j(t) \geq P, \tag{9}$$

for all t and all pairs of indices $i, j \in \mathcal{I}$ such that $l(i) = l(j)$, $k(i) + 1 = k(j)$, which we collect in $\mathcal{C}$. Let $D_i(x_{i,0})$ denote the set of feasible trajectories $x_i(t) = (p_i(t), v_i(t), u_i(t))$ given some initial state $x_{i,0}$ and satisfying the vehicle dynamics given by equations (4), (5) and (6). Given some performance criterion

$$J(x_i) = \int_0^{t_f} \Lambda(x_i(t))dt, \tag{10}$$

where t_f denotes the final time, the coordination problem is formulated as

$$\begin{aligned}
\min_{\mathbf{x}(t)} \quad & \sum_{i \in \mathcal{I}} J(x_i) && && \text{(11a)}\\
\text{s.t.} \quad & x_i \in D_i(x_{i,0}), && \text{for all } i \in \mathcal{I}, && \text{(11b)}\\
& (p_i(t), p_j(t)) \notin \mathcal{E}_i \times \mathcal{E}_j, && \text{for all } (i,j) \in \mathcal{D}, && \text{(11c)}\\
& p_i(t) - p_j(t) \geq P, && \text{for all } (i,j) \in \mathcal{C}, && \text{(11d)}
\end{aligned}$$

where $\mathbf{x}(t) = [\, x_i(t) : i \in \mathcal{I}\,]$.

Exact solution

We discretize problem (11) on a uniform time grid. Let K denote the number of discrete time steps and let Δt denote the time step size. We use the forward Euler integration scheme as follows

$$\begin{aligned}
p_i(t + \Delta t) &= p_i(t) + v_i(t)\Delta t, && \text{(12a)}\\
v_i(t + \Delta t) &= v_i(t) + u_i(t)\Delta t. && \text{(12b)}
\end{aligned}$$

The disjunctive constraints are formulated using the big-M technique by the constraints

$$\begin{aligned}
p_i(t) &\leq L + \delta_i(t)M, && \text{(13a)}\\
H - \gamma_i(t)M &\leq p_i(t), && \text{(13b)}\\
\delta_i(t) + \delta_j(t) + \gamma_i(t) + \gamma_j(t) &\leq 3, && \text{(13c)}
\end{aligned}$$

where $\delta_i(t), \gamma_i(t) \in \{0, 1\}$ for all $i \in \mathcal{I}$ and M is a sufficiently large number. Finally, the follow constraints can simply be enforced at each time step for each pair of consecutive vehicles in $\mathcal{C}$.

Decomposition

The *entry* and *exit* times of vehicle i are given, respectively, by

$$\tau_i = t : p_i(t) = L_i, \quad \xi_i = t : p_i(t) = H_i. \tag{14}$$

Define the optimization problem

$$\begin{aligned} F_i(\tau_i, \xi_i) = \min_{x_i(t)} \; & J(x_i) && \text{(15a)} \\ \text{s.t. } & x_i \in D_i(x_{i,0}), && \text{(15b)} \\ & p_i(\tau_i) = L, && \text{(15c)} \\ & p_i(\xi_i) = H. && \text{(15d)} \end{aligned}$$

Given some vehicle $i = (l, k)$, define

$$\mathcal{N}(i, n) = \{j \in \mathcal{I} : l(j) = l(i),\ k(j) \in \{k(i), \dots, k(i) + n - 1\}\}, \tag{16}$$

to which refer to as the *lane successors* of vehicle i. Now we generalize problem (15a) by defining

$$\begin{aligned} F(i, \tau, \xi, n) = \min_{x_j(t): j \in \mathcal{N}(i,n)} \; & \sum_{j \in \mathcal{N}(i,n)} J(x_j) && \text{(17a)} \\ \text{s.t. } & x_j \in D_j(x_{j,0}), \text{ for all } j \in \mathcal{N}(i, n), && \text{(17b)} \\ & p_i(\tau) = L, && \text{(17c)} \\ & p_j(\xi) = H, \text{ for } j = (l(i), k(i) + n - 1), && \text{(17d)} \\ & p_a(t) - p_b(t) \geq P, \text{ for all } (a, b) \in \mathcal{N}(i, n)^2 \cap \mathcal{C}, && \text{(17e)} \end{aligned}$$

such that $F_i(\tau_i, \xi_i) = F(i, \tau_i, \xi_i, 1)$.

The idea is to generalize scheduling of single vehicle time slots to scheduling of time slots for platoons of consecutive vehicles.