Vehicle trajectory planning in a single lane with minimum following distance and boundary conditions

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Abstract

This section considers a model of a single-lane road on which overtaking is not allowed. Vehicles are modeled as double integrators with bounds on speed and acceleration. Consecutive vehicles must keep some fixed *following distance* to avoid collisions. It is assumed that vehicles enter and exit the lane at predetermined *schedule times*. Whenever a vehicle enters or exits, it must drive at full speed. For an optimization objective that, roughly speaking, minimizes the distance to the end of the lane at all times, we present an algorithm to compute optimal trajectories. Assuming a fixed minimum lane length, we characterize feasibility of this trajectory planning problem in terms of a system of linear inequalities involving only the schedule times.

1 Trajectory planning problem

Vehicles are modeled as double integrators with bounded speed and acceleration, which means that we only consider their longitudinal position on the road. Let $\mathcal{D}[a,b]$ denote the set of valid *trajectories*, which we define to be all continuously differentiable functions $x : [a,b] \to \mathbb{R}$ satisfying the constraints

$$0 \leq \dot{x}(t) \leq 1 \quad \text{and} \quad -\omega \leq \ddot{x}(t) \leq \bar{\omega}, \quad \text{for all } t \in [a,b], \tag{1}$$

for some fixed acceleration bounds $\omega, \bar{\omega} > 0$. Note that we use $\dot{x}$ and $\ddot{x}$ to denote the first and second derivative with respect to time t. When we have a general positive speed upper bound, we can always apply an appropriate scaling of the time axis and the acceleration bounds to obtain the form. Consider positions $A, B \in \mathbb{R}$, such that $B \geq A$, which denote the start[1] and end position of the lane. Let $\bar{D}[a,b] \subset \mathcal{D}[a,b]$ denote the set of trajectories x that satisfy the boundary conditions

$$x(a) = A \;\text{ and }\; x(b) = B. \tag{2}$$

Even further, let $D[a,b] \subset \bar{D}[a,b]$ induce the boundary conditions

$$\dot{x}(a) = \dot{x}(b) = 1. \tag{3}$$

In words, these boundary conditions require that a vehicle arrives to and departs from the lane at predetermined times a and b and do so at full speed.

Let $L > 0$ denote the required *following distance* between consecutive vehicles. Suppose we have N vehicles that are scheduled to traverse the lane. For each vehicle i, let a_i and b_i denote the *schedule time* for entry and exit, respectively. A feasible solution consists of a sequence of trajectories $x_1, \dots, x_N$ such that

$$x_i \in D[a_i, b_i] \qquad \text{for each } i, \tag{4a}$$
$$x_i \leq x_{i-1} - L \qquad \text{for each } i \geq 2, \tag{4b}$$

[1] We could have assumed $A = 0$, but we will later piece together multiple lanes to model intersections.

where we use the shorthand notation $\gamma_1 \leq \gamma_2$ to mean $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in [a_1, b_1] \cap [a_2, b_2]$, given some $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$.

As optimization objective, we will consider

$$\max \sum_{i=1}^{N} \int_{a_i}^{b_i} x_i(t)\, \mathrm{d}t, \tag{5}$$

which, roughly speaking, seeks to keep all vehicles as close to the end of the lane at all times. In particular, we will show that optimal trajectories can be understood as the concatenation of at most four different types of trajectory parts. Based on this observation, we present an algorithm to compute optimal trajectories. Assuming the system parameters $(\omega, \bar{\omega}, A, B, L)$ to be fixed, with lane length $B - A$ sufficiently large, we will show that feasibility of the trajectory optimization problem is completely characterized by a system of linear inequalities in terms of all schedule times a_i and b_i.

Note that objective (5) does not capture energy efficiency in any way. Although this is not desirable in practice, it is precisely this assumption that enables the analysis in this section. Deriving a similar characterization of feasibility and optimal trajectories under an objective that does model energy concumption is an interesting topic for further research.

2 Single vehicle problem

We will first consider a somewhat generalized version of the constraints (4) for a single vehicle i. Therefore, we lighten the notation slightly by dropping the vehicle index i and instead of $x_{i-1} - L$, we assume we are given some arbitrary *lead vehicle boundary* $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$, then we consider the optimization problem

$$\max_{x \in D[a,b]} \int_a^b x(t)\, \mathrm{d}t \quad \text{ such that } \quad x \leq \bar{x}. \tag{6}$$

to which we will refer as the *single vehicle problem*.

2.1 Necessary conditions

For every trajectory $x \in D[a, b]$, we derive two upper bounding trajectories x^1 and $\hat{x}$ and one lower bounding trajectory $\check{x}$, see Figure 1. Using these bounding trajectories, we will then formulate four necessary conditions for feasibility of the single vehicle problem.

Let the *full speed boundary* x^1 be defined as

$$x^1(t) = A + t - a, \tag{7}$$

for all $t \in [a, b]$, then we clearly have $x \leq x^1$. Next, since deceleration is at most ω, we have $\dot{x}(t) \geq \dot{x}(a) - \omega(t - a) = 1 - \omega(t - a)$, which we combine with the speed constraint $\dot{x} \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \omega(t - a)\}$. Hence, we obtain the lower bound

$$x(t) = x(a) + \int_a^t \dot{x}(\tau)\, \mathrm{d}\tau \geq A + \int_a^t \max\{0, 1 - \omega(\tau - a)\}\, \mathrm{d}\tau =: \check{x}(t). \tag{8}$$

Analogously, we derive an upper bound from the fact that acceleration is at most $\bar{\omega}$. Observe that we have $\dot{x}(t) + \bar{\omega}(b - t) \geq \dot{x}(b) = 1$, which we combine with the speed constraint $\dot{x}(t) \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \bar{\omega}(b - t)\}$. Hence, we obtain the upper bound

$$x(t) = x(b) - \int_t^b \dot{x}(\tau)\, \mathrm{d}\tau \leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b - \tau)\}\, \mathrm{d}\tau =: \hat{x}(t). \tag{9}$$

We refer to $\check{x}$ and $\hat{x}$ as the *entry boundary* and *exit boundary*, respectively.

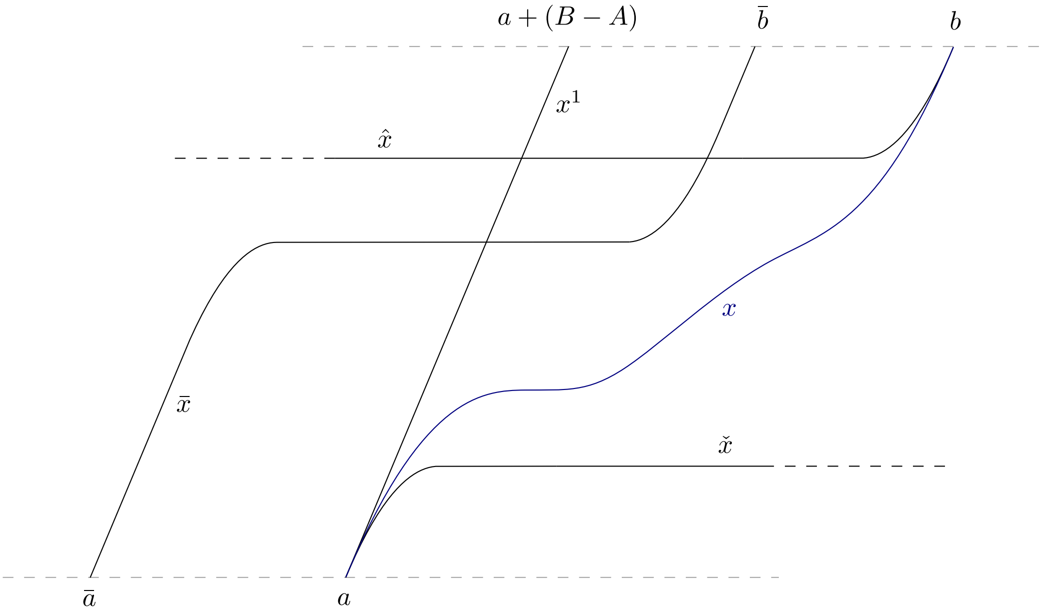

Figure 1: Illustration of the four bounding trajectories $\bar{x}, x^1, \hat{x}, \check{x}$ that bound feasible trajectories from above and below. We also drew an example of a feasible trajectory x in blue. The horizontal axis represents time and the vertical axis corresponds to the position on the lane, so the vertical dashed grey lines correspond to the start and end of the lane.

Lemma 1. *Let $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$ and assume there exists a trajectory $x \in D[a, b]$ such that $x \leq \bar{x}$, then the following conditions must hold*

(i)	$b - a \geq B - A,$	*(full speed constraint)*
(ii)	$\bar{b} \leq b,$	*(downstream order constraint)*
(iii)	$\bar{a} \leq a,$	*(upstream order constraint)*
(iv)	$\bar{x} \geq \check{x}.$	*(entry space constraint)*

Proof. Each of the conditions corresponds somehow to one of the four bounding trajectories defined above. Observe that $x^1(t) = B$ for $t = a + (B - A)$, which can be interpreted as the earliest time of departure from the lane. This shows that $b \geq a + (B - A)$, which is equivalent with (i). When either (ii) or (iii) is violated, the constraint $x \leq \bar{x}$ conflicts with one of the boundary conditions $x(a) = A$ or $x(b) = B$. To see that (iv) must hold, suppose that $\bar{x}(\tau) < \check{x}(\tau)$ for some time τ. Since $\bar{a} \leq a$, this means that $\bar{a}$ must intersect $\check{a}$ from above. Therefore, any trajectory that satisfies $x \leq \bar{x}$ must also intersect $\check{a}$ from above, which contradicts the assumption that x was a feasible solution. □

We show that the boundaries $\hat{x}$ and $\check{x}$ together could yield yet another necessary condition. It is straightforward to verify from equations (8) and (9) that $\hat{x}(t) \geq B - 1/(2\bar{\omega})$ and $\check{x}(t) \leq A + 1/(2\omega)$. Therefore, whenever $B - A < 1/(2\bar{\omega}) + 1/(2\omega)$, these boundaries intersect for certain values of a and b. Because the exact condition is somewhat cumbersome to characterize, we avoid this case by simply assuming that the lane length is sufficiently large.

Assumption 1. *The length of the lane satisfies $B - A \geq 1/(2\omega) + 1/(2\bar{\omega})$.*

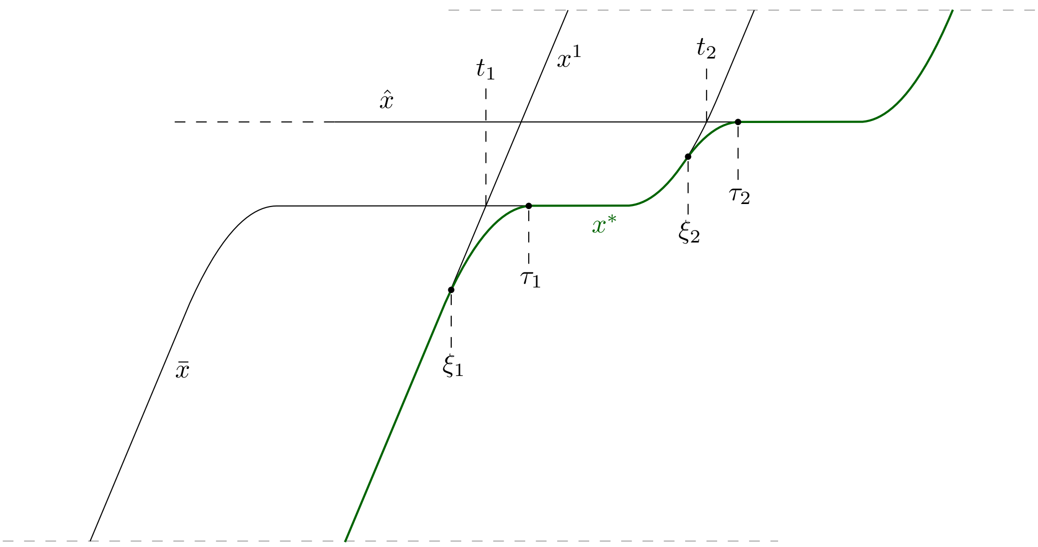

Figure 2: The minimum boundary γ, induced by three upper boundaries $\bar{x}$, $\hat{x}$ and x^1, is smoothened around time t_1 and t_2, where the derivative is discontinuous, to obtain the smooth optimal trajectory x^*, drawn in green. The times ξ_i and τ_i correspond to the start and end of the connecting deceleration as defined in Section 2.2.2.

2.2 Constructing the optimal trajectory

Assuming the four conditions of Lemma 1 hold, we will construct an optimal solution x^* for the single vehicle problem, thereby showing that these conditions are thus also sufficient for feasibility. First, we construct x^* by combining the upper boundaries $\bar{x}$, $\hat{x}$ and x^1 in a certain way to obtain a smooth trajectory satisfying $x^* \in D[a, b]$. We show that x^* is still an upper boundary for any other feasible solution, which shows that it is optimal.

The starting point of the construction is the *minimum boundary* $\gamma : [a, b] \to \mathbb{R}$ induced by the upper boundaries, defined as

$$\gamma(t) := \min\{\bar{x}(t), \hat{x}(t), x^1(t)\}. \tag{10}$$

Obviously, γ is still a valid upper boundary for any feasible solution, but in general, γ may have a discontinuous derivative at some[2] isolated points in time.

Definition 1. *Let $\mathcal{P}[a, b]$ be the set of functions $\mu : [a, b] \to \mathbb{R}$ for which there is a finite subdivision $a = t_0 < \cdots < t_{n+1} = b$ such that the truncation $\mu|_{[t_i,t_{i+1}]} \in \mathcal{D}[t_i, t_{i+1}]$ is a smooth trajectory, for each $i \in \{0, \ldots, n\}$, and for which the one-sided limits of $\dot{\mu}$ satisfy*

$$\dot{\mu}(t_i^-) := \lim_{t \uparrow t_i} \dot{\mu}(t) > \lim_{t \downarrow t_i} \dot{\mu}(t) =: \dot{\mu}(t_i^+), \tag{11}$$

for each $i \in \{1, \ldots, n\}$. We refer to such μ as a piecewise trajectory (with downward bends).

It is not difficult to see from Figure 1 that, under the necessary conditions, γ satisfies the above definition, so $\gamma \in \mathcal{P}[a, b]$. In other words, γ consists of a number of pieces that are smooth and satisfy the vehicle dynamics, with possibly some sharp bend downwards where these pieces come together. Next, we present a simple procedure to smoothen out this kind of discontinuity by decelerating from the original trajectory somewhat before some t_i, as illustrated in Figure 2. We show that this procedure can be repeated at every point of discontinuity.

[2] In fact, it can be shown that, under the necessary conditions, there are at most two of such discontinuities.

2.2.1 Deceleration boundary

In order to formalize the smoothing procedure, we will first define some parameterized family of functions to model the deceleration part. Recall the derivation of $\check{x}$ in equation (8) and the discussion preceding it, which we will now generalize a bit. Let $x \in \mathcal{D}[a,b]$ be some smooth trajectory, then observe that $\dot{x}(t) \geq \dot{x}(\xi) - \omega(t-\xi)$ for all $t \in [a,b]$. Combining this with the constraint $\dot{x}(t) \in [0,1]$, this yields

$$\dot{x}(t) \geq \max\{0,\, \min\{1,\, \dot{x}(\xi) - \omega(t-\xi)\}\} =: \{\dot{x}(\xi) - \omega(t-\xi)\}_{[0,1]}, \tag{12}$$

where use $\{\cdot\}_{[0,1]}$ as a shorthand for this clipping operation. Hence, for any $t \in [a,b]$, we obtain the following lower bound

$$x(t) = x(\xi) + \int_{\xi}^{t} \dot{x}(\tau)\,\mathrm{d}\tau \geq x(\xi) + \int_{\xi}^{t} \{\dot{x}(\xi) - \omega(\tau-\xi)\}_{[0,1]}\,\mathrm{d}\tau =: x[\xi](t), \tag{13}$$

where we will refer to the right-hand side as the *deceleration boundary* of x at ξ. Observe that this definition indeed generalizes the definition of $\check{x}$, because we have $\check{x} = (x[a])|_{[a,b]}$, so $x[a]$ restricted to the interval $[a,b]$.

Note that $x[\xi]$ depends on x only through the two real numbers $x(\xi)$ and $\dot{x}(\xi)$. It will be convenient later to rewrite the right-hand side of (13) as

$$x^{-}[p,v,\xi](t) = p + \int_{\xi}^{t} \{v - \omega(\tau-\xi)\}_{[0,1]}\,\mathrm{d}\tau, \tag{14}$$

such that $x[\xi](t) = x^{-}[x(\xi), \dot{x}(\xi), \xi](t)$. We can expand the integral in this expression further by carefully handling the clipping operation. Observe that the expression within the clipping operation reaches the bounds 1 and 0 for $\delta_1 := \xi - (1-v)/\omega$ and $\delta_0 := \xi + v/\omega$, respectively. Using this notation, a straightforward calculation shows that

$$x^{-}[p,v,\xi](t) = p + \begin{cases} (1-v)^2/(2\omega) + (t-\xi) & \text{for } t \leq \delta_1, \\ v(t-\xi) - \omega(t-\xi)^2/2 & \text{for } t \in [\delta_1, \delta_0], \\ v^2/(2\omega) & \text{for } t \geq \delta_0. \end{cases} \tag{15}$$

It is easily verified that these three cases coincide at $t \in \{\delta_1, \delta_0\}$, which justifies the overlap there. Furthermore, since x and $\dot{x}$ are continuous by assumption, this shows that $x[\xi](t) = x^{-}[x(\xi), \dot{x}(\xi), \xi](t)$ is continuous as a function of either of its arguments[3]. Assuming $0 \leq v \leq 1$, it can be verified that for every $t \in \mathbb{R}$, we have $\ddot{x}^{-}[p,v,\xi](t) \in \{-\omega, 0\}$ and $\dot{x}^{-}[p,v,\xi](t) \in [0,1]$ due to the clipping operation, so that $x^{-}[p,v,\xi] \in \mathcal{D}(-\infty,\infty)$.

Piecewise trajectories. Let $\mu \in \mathcal{P}[a,b]$ be some piecewise trajectory and let $a = t_0 < \cdots < t_{n+1} = b$ denote the corresponding subdivision as in Definition 1, then we generalize the definition of a deceleration boundary to μ. Whenever $\xi \in [a,b] \setminus \{t_1, \dots, t_n\}$, we just define $\mu[\xi] := x^{-}[\mu(\xi), \dot{\mu}(\xi), \xi]$. However, when $\xi \in \{t_1, \dots, t_n\}$, the derivative $\dot{\mu}(\xi)$ is not defined, so we to use the left-sided limit instead, by defining $\mu[\xi] := x^{-}[\mu(\xi), \dot{\mu}(\xi^{-}), \xi]$.

Remark 1. Please note that we cannot just replace x with μ in inequality (13) to obtain a similar bound for μ on the its full interval $[a,b]$. Instead, we get the following *piecewise lower bounding* property. Consider some interval $I \in \{[a,t_1], (t_1,t_2], \dots, (t_n,b]\}$, then what remains true is that $\xi \in I$ implies $\mu(t) \geq \mu[\xi](t)$ for every $t \in I$.

[3]Even more, it can be shown that $x[\xi](t)$ is continuous as a function of (ξ, t).

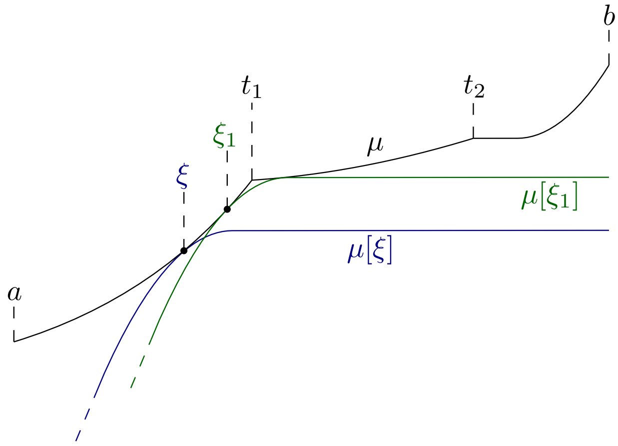

Figure 3: Illustration of some piecewise trajectory $\mu \in \mathcal{P}[a, b]$ with some deceleration boundary $\mu[\xi]$ at time ξ in blue and the unique connecting deceleration $\mu[\xi_1]$ in green. We truncated both deceleration boundaries for a more compact figure. This particular μ has a discontinuous derivative at times t_1 and t_2. The careful observer may notice that μ cannot occur as the minimum boundary defined in (10), but please note that the class of piecewise trajectories $\mathcal{P}[a, b]$ is just slightly more general than necessary for our current purposes.

2.2.2 Smoothing procedure

Let $\mu \in \mathcal{P}[a, b]$ be some piecewise trajectory and let $a = t_0 < \cdots < t_{n+1} = b$ denote the subdivision as in Definition 1. We first show how to smoothen the discontinuity at t_1 and then argue how to repeat this process for the remaining times t_i.

Assumption 2. *Throughout the following discussion, we assume $\mu \geq \mu[a]$ and $\mu \geq \mu[b]$.*

Our aim is to choose some time $\xi \in [a, t_1]$ from which the vehicle starts fully decelerating, such that $\mu[\xi] \leq \mu$ and such that $\mu[\xi]$ touches μ at some time $\tau \in [t_1, b]$ tangentially. We will show there is a unique trajectory $\mu[\xi]$ that satisfies these requirements and refer to it as the *connecting deceleration*, see Figure 3 for an example.

Touching. Recall Remark 1, which asserts that we have $\mu[\xi] \leq \mu$ on $[a, t_1]$ for any $\xi \in [a, t_1]$. After the discontinuity, so on the interval $[t_1, b]$, we want $\mu[\xi] \leq \mu$ and equality at least somewhere, so we measure the relative position of $\mu[\xi]$ with respect to μ here, by considering

$$d(\xi) := \min_{t \in [t_1, b]} \mu(t) - \mu[\xi](t). \tag{16}$$

Since $\mu(t)$ and $\mu[\xi](t)$ are both continuous as a function of t on the interval $[t_1, b]$, this minimum actually exists (extreme value theorem). Furthermore, since d is the minimum of a continuous function over a closed interval, it is continuous as well (see Lemma A.1). Observe that $d(a) \geq 0$, because $\mu \geq \mu[a]$ by assumption. By definition of t_1, we have $\dot{\mu}(t_1^-) > \dot{\mu}(t_1^+)$, from which it follows that $\mu(t) < \mu[t_1](t)$ for $t \in (t_1, t_1 + \epsilon)$ for some small $\epsilon > 0$, which shows that $d(t_1) < 0$. By the intermediate value theorem, there is $\xi_1 \in [a, t_1)$ such that $d(\xi_1) = 0$.

Uniqueness. It turns out that ξ_1 itself is not necessarily unique, which we explain below. Instead, we are going to show that the connecting deceleration $\mu[\xi_1]$ is unique. More precisely, given any other $\xi \in [a, t_1)$ such that $d(\xi) = 0$, we will show that $\mu[\xi] = \mu[\xi_1]$.

The first step is to establish that the level set

$$X := \{\xi \in [a, t_1) : d(\xi) = 0\} \tag{17}$$

is a closed interval. To this end, we show that d is non-increasing on $[a, t_1)$, which together with continuity implies the desired result (see Lemma A.2). To show that d is non-increasing, it suffices to show that $\mu[\xi](t)$ is non-decreasing as a function of ξ, for every $t \in [t_1, b]$. We can do this by computing the partial derivative of $\mu[\xi]$ with respect to ξ and verifying it is non-negativity. Recall the definition of $\mu[\xi]$, based on x^- in equation (15). Using similar notation, we write $\delta_1(\xi) = \xi - (1 - \dot{\mu}(\xi))/\omega$ and $\delta_0(\xi) = \xi + \dot{\mu}(\xi)/\omega$ and compute

$$\frac{\partial}{\partial \xi}\mu[\xi](t) = \dot{\mu}(\xi) + \begin{cases} \ddot{\mu}(\xi)(\dot{\mu}(\xi) - 1)/\omega - 1 & \text{for } t \leq \delta_1(\xi), \\ \ddot{\mu}(\xi)(t - \xi) - \dot{\mu}(\xi) + \omega(t - \xi) & \text{for } t \in [\delta_1(\xi), \delta_0(\xi)], \\ \ddot{\mu}(\xi)\dot{\mu}(\xi)/\omega & \text{for } t \geq \delta_0(\xi). \end{cases} \tag{18}$$

It is easily verified that the cases match at $t \in \{\delta_1(\xi), \delta_0(\xi)\}$, which justifies the overlaps there. Consider any $\xi \in [a, t_1)$ and $t \in [t_1, b]$, then we always have $\delta_1(\xi) \leq \xi \leq t$, so we only have to verify the second and third case:

$$\frac{\partial}{\partial \xi}\mu[\xi](t) = (\ddot{\mu}(\xi) + \omega)(t - \xi) \geq 0 \qquad \text{for } t \in [\delta_1(\xi), \delta_0(\xi)], \tag{19a}$$

$$\frac{\partial}{\partial \xi}\mu[\xi](t) \geq \dot{\mu}(\xi) + (-\omega)\dot{\mu}(\xi)/\omega = 0 \qquad \text{for } t \geq \delta_0(\xi). \tag{19b}$$

This concludes the argument for X being a closed interval.

Assuming ξ to be fixed, observe that there is equality in (19a) for some $t \in [\delta_1(\xi), \delta_0(\xi)]$ if and only if there is equality in (19b) for some other $t' \geq \delta_0(\xi)$. Note that this happens precisely when $\ddot{\mu}(\xi) = -\omega$. Therefore, whenever μ is fully deceleration, so $\dot{\mu}(t) = -\omega$ on some open interval $U \subset (a, t_1)$, we have $(\partial/\partial\xi)\mu[\xi](t) = 0$ for all $t \geq \delta_1(\xi)$. This essentially means that any choice of $\xi \in U$ produces the same trajectory $\mu[\xi]$. Please see Figure 4 for an example of this case. This observation is key to the remaining uniqueness argument.

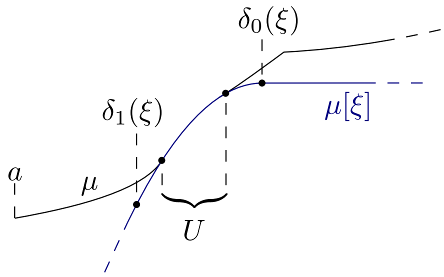

Figure 4: Example of a piecewise trajectory μ with a part of full deceleration over some interval U such that any choice of $\xi \in U$ produces the same deceleration boundary $\mu[\xi]$, which naturally coincides with μ on U.

Since X is a closed interval, we may define $\xi_0 = \min X$. Consider any $\xi' \in X$ with $\xi' > \xi_0$, then we show $\mu[\xi'](t) = \mu[\xi_0](t)$ for all $t \in [\xi_0, b]$. For sake of contradiction, suppose there is some $t' \in [\xi_0, b]$ such that $\mu[\xi'](t') > \mu[\xi_0](t')$, then there must be some open interval $U \subset (\xi_0, \xi')$ such that

$$\frac{\partial}{\partial \xi}\mu[\xi](t') > 0 \text{ for all } \xi \in U. \tag{20}$$

However, we argued in the previous paragraph that this actually holds for any $t' \geq \delta_1(\xi)$. In particular, let $t^* \in [t_1, b]$ be such that $\mu(t^*) = \mu[\xi_0](t^*)$, then $t^* \geq t_1 \geq \xi \geq \delta_1(\xi)$, so (20) yields $\mu[\xi'](t^*) > \mu[\xi_0](t^*)$, but then $d(\xi') > d(\xi_0) = 0$, so $\xi' \notin X$, a contradiction.

Touching tangentially. It remains to show that μ and $\mu[\xi_0]$ touch tangentially somewhere on $[t_1, b]$. Let $\tau_1 \in [t_1, b]$ be the smallest time such that $\mu(\tau_1) - \mu[\xi_0](\tau_1) = d(\xi_0) = 0$ and consider the following three cases.

First of all, note that $\tau_1 = t_1$ is not possible, because this would require

$$\dot{\mu}(t_1^+) > \dot{\mu}[\xi_0](t_1^+) = \dot{\mu}[\xi_0](t_1), \tag{21}$$

but since μ is a piecewise trajectory, we must have $\dot{\mu}(t_1^-) > \dot{\mu}(t_1^+) > \dot{\mu}[\xi_0](t_1)$. This shows that $\mu(t_1 - \epsilon) < \mu[\xi_0](t_1 - \epsilon)$, for some small $\epsilon > 0$, which contradicts $\mu[\xi_0] \leq \mu$.

Suppose $\tau_1 \in (t_1, b)$, then recall the definition of $d(\xi_0)$ and observe that the usual first-order necessary condition (derivative zero) for local minima requires $\dot{\mu}(\tau_1) = \dot{\mu}[\xi_0](\tau_1)$.

Finally, consider $\tau_1 = b$. Observe that $\dot{\mu}(b) > \dot{\mu}[\xi_0](b)$, would contradict minimality of $\tau_1 = b$. Therefore, suppose $\dot{\mu}(b) < \dot{\mu}[\xi_0](b)$, then $\dot{\mu}[b](b) = \dot{\mu}(b) < \dot{\mu}[\xi_0](b)$, so

$$\dot{\mu}[b](t) \leq \dot{\mu}[\xi_0](t) \text{ for } t \leq b, \tag{22}$$

but then $\mu[b](t) > \mu[\xi_0](t)$ for $t < b$. In particular, for $t = \xi_0$, this shows $\mu[b](\xi_0) > \mu[\xi_0](\xi_0) = \mu(\xi_0)$, which contradicts the assumption $\mu[b] \leq \mu$ of Assumption 2.

Repeat for remaining discontinuities. Before we proceed, let us summarize what we have established so far. The times $\xi_0 \in [a, t_1)$ and $\tau_1 \in (t_1, b]$ have been chosen such that

$$\mu[\xi_0] \leq \mu \quad \text{for} \quad t \in [\xi_0, \tau_1], \tag{23a}$$

$$\dot{\mu}[\xi_0](\xi_0) = \dot{\mu}(\xi_0) \quad \text{and} \quad \dot{\mu}[\xi_0](\tau_1) = \dot{\mu}(\tau_1). \tag{23b}$$

Instead of ξ_0, it will be convenient later to choose $\xi_1 := \max X$ as the representative of the unique connecting deceleration. We can now use $\mu[\xi_1]_{[\xi_1, \tau_1]}$ to replace μ at $[\xi_1, \tau_1]$ to obtain a trajectory without the discontinuity at t_1. More precisely, we define

$$\mu_1(t) = \begin{cases} \mu(t) & \text{for } t \in [a, \xi_1] \cup [\tau_1, b], \\ \mu[\xi_1](t) & \text{for } t \in [\xi_1, \tau_1]. \end{cases} \tag{24}$$

From the way we constructed $\mu[\xi_1]$, it follows from (23) that we have $\mu_1 \in \mathcal{P}[a, b]$, but without the discontinuity at t_1. Observe that a single connecting deceleration may cover more than one discontinuity, as illustrated in Figure 5. Note that we must have $\dot{\mu}_1(a) = \dot{\mu}_1(b) = 1$. Moreover, it is not difficult to see that μ_1 must still satisfy Assumption 2, so that we can keep repeating the exact same process, obtaining connecting decelerations $(\xi_2, \tau_2), (\xi_3, \tau_3), \dots$ and the corresponding piecewise trajectories $\mu_2, \mu_3 \dots$ to remove any remaining discontinuities until we end up with a smooth trajectory $\mu^* \in \mathcal{D}[a, b]$. We emphasize again that $\dot{\mu}^*(a) = \dot{\mu}(a)$ and $\dot{\mu}^*(b) = \dot{\mu}(b)$.

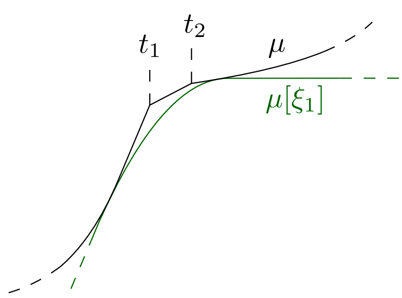

Figure 5: Part of a piecewise trajectory μ on which a single connecting deceleration covers the two discontinuities at t_1 and t_2 at once.

2.2.3 Optimality after smoothing

Let us return to the minimum boundary γ defined in (10). From Figure 1 and the necessary conditions, it is clear that γ must satisfy $\gamma(a) = A$, $\gamma(b) = B$ and $\dot{\gamma}(a) = \dot{\gamma}(b) = 1$, so whenever we have $\gamma \in \mathcal{D}[a, b]$, we automatically have $\gamma \in D[a, b]$ so that γ is already an optimal solution. Otherwise, we perform the smoothing procedure presented above to obtain the smoothed trajectory $\gamma^* \in \mathcal{D}[a, b]$. The next lemma shows that it is still an upper boundary for any feasible trajectory, which proves that $x^* := \gamma^*$ is an optimal solution to the single vehicle problem.

Lemma 2. *Let $\mu \in \mathcal{P}[a, b]$ be a piecewise trajectory and let $\mu^* \in \mathcal{D}[a, b]$ denote the result after smoothing. All trajectories $x \in \mathcal{D}[a, b]$ that are such that $x \leq \mu$, must satisfy $x \leq \mu^*$.*

Proof. Consider some interval (ξ, τ) where we introduced some connecting deceleration boundary. Suppose there exists some $t_d \in (\xi, \tau)$ such that $x(t_d) > \mu(t_d)$. Because $x(\xi) \leq \mu(\xi)$, this means that x must intersect μ at least once in $[\xi, t_d)$, so let $t_c := \sup\{t \in [\xi, t_d) : x(t) = \mu(t)\}$ be the latest time of intersection such that $x \geq \mu$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{x}(t_v) > \dot{\mu}(t_v)$, otherwise

$$x(t_d) = x(t_c) + \int_{t_c}^{t_d} \dot{x}(t)\,\mathrm{d}t \leq \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t)\,\mathrm{d}t = \mu(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{x}(t) \geq \dot{x}(t_v) - \omega(t - t_v) > \dot{\mu}(t_v) - \omega(t - t_v) = \dot{\mu}(t).$$

It follows that $x(\tau) > \mu(\tau)$, which contradicts the assumption. $\square$

3 Computing optimal trajectories

Recall the original trajectory optimization problem

$$\begin{aligned} G(a, b) := \max \quad & \sum_{i=1}^{N} \int_{a_i}^{b_i} x_i(t)\,\mathrm{d}t, && \text{(25a)} \\ \text{s.t.} \quad & x_i \in D[a_i, b_i] \qquad \text{for each } i \in \{1, \dots, N\}, && \text{(25b)} \\ & x_i \leq x_{i-1} - L \qquad \text{for each } i \in \{2, \dots, N\}, && \text{(25c)} \end{aligned}$$

where we use a and b to denote the vectors of arrival and departure times. We now show how this problem decomposes into a sequence of instances of the single vehicle problems. Let the optimal solution of the single vehicle problem be denoted as

$$x^*(\alpha, \beta, \bar{x}) := \underset{x \in D[\alpha, \beta]}{\arg\max} \int_{\alpha}^{\beta} x(t) \qquad \text{such that } x \leq \bar{x} \tag{26}$$

and let $F(\alpha, \beta, \bar{x})$ denote the corresponding objective value. Consider the safe following constraint (25c). We show how to model this as a boundary $\bar{x}_i \in \bar{D}[\bar{a}_i, \bar{b}_i]$ such that we can apply the single vehicle problem. It is clear from Figure 6 that inequality (25c) only applies on some subinterval $I_i \subset [a_{i-1}, b_{i-1}]$. More specifically, by defining

$$x_i^{-1}(p) := \inf\{t : x_i(t) = p\}, \tag{27}$$

it is easily seen that $I_i = [x_{i-1}^{-1}(L), b_i]$. However, since $\dot{x}_{i-1}(b_{i-1}) = 1$, we can, roughly speaking, extend the boundary until $b_{i-1} + L$, by defining

$$\bar{x}_i := \begin{cases} x_{i-1}(t) - L & \text{for } t \in [x_{i-1}^{-1}(L), b_{i-1}], \\ t - b_{i-1} - L & \text{for } t \in [b_{i-1}, b_{i-1} + L]. \end{cases} \tag{28}$$

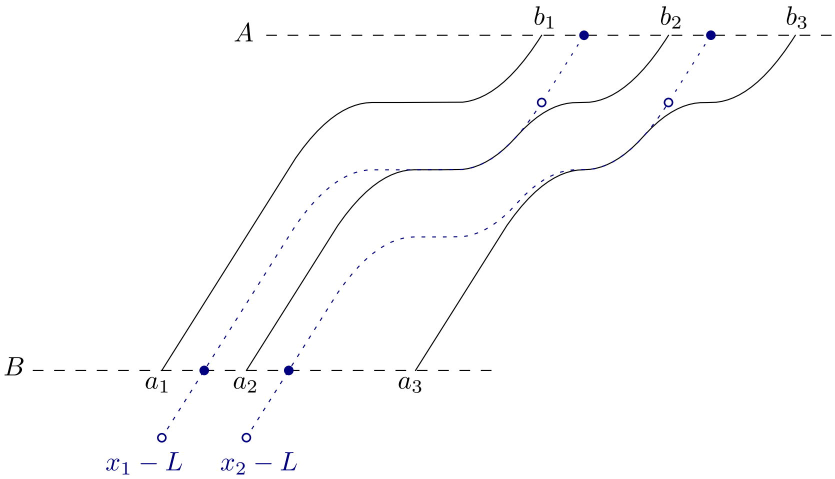

Figure 6: Optimal trajectories x_i for three vehicles. The dotted blue trajectories between the little open circles illustrates the safe following constraints (25c). The dotted blue trajectories between the solid dots are the following boundaries $\bar{x}_2 \in \bar{D}[\bar{a}_2, \bar{b}_2]$ and $\bar{x}_3 \in \bar{D}[\bar{a}_3, \bar{b}_3]$.

Now, it is then clear that the optimal trajectories x_i can be recursively computed as

$$x_1 = x^*(a_1, b_1, \varnothing), \tag{29a}$$

$$x_i = x^*(a_i, b_i, \bar{x}_i), \quad \text{for } i \geq 2, \tag{29b}$$

where we use the notation $\bar{x} = \varnothing$ to denote the single vehicle problem without the boundary constraint. Alternatively, we could think about this as having some $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$ with very small $\bar{a} \ll a$ and $\bar{b} \ll b$. The corresponding objective value is simply given by

$$G(a, b) = F(a_1, b_1, \varnothing) + \sum_{i=2}^{N} F(a_i, b_i, \bar{x}_i). \tag{30}$$

3.1 Alternating trajectories

Due to the recursive nature of the problem, we will see that optimal trajectories possess a particularly simple structure, which enables a very simple computation.

Definition 2. *Let a trajectory $\gamma \in \mathcal{D}[a, b]$ be called* alternating *if for all $t \in [a, b]$, we have*

$$\ddot{\gamma}(t) \in \{-\omega, 0, \bar{\omega}\} \quad \text{and} \quad \ddot{\gamma}(t) = 0 \implies \dot{\gamma}(t) \in \{0, 1\}. \tag{31}$$

We now argue that each vehicle's optimal trajectory x_i is alternating. First, consider $x_1 = x^*(a_1, b_1, \varnothing)$, which is constructed by joining $x^1[x_1]$ and $\hat{x}[x_1]$ together by smoothing. Observe that both boundaries are alternating by definition. Let $\gamma_1(t) = \min\{x^1[x_1](t), \hat{x}[x_1](t)\}$ be the minimum boundary, then it is clear that the smoothened $x_1 = \gamma_1^*$ must also be alternating, because we only added a part of deceleration at some interval $[\xi, \tau]$, which clearly satisfies $\ddot{\gamma}_1^*(t) = -\omega$ for $t \in [\xi, \tau]$. Assume that x_{i-1} is alternating, we can similarly argue that x_i is alternating. Again, let $\gamma_i(t) = \min\{\bar{x}[x_{i-1}], \hat{x}[x_i](t), x^1[x_i](t)\}$ be the minimum boundary. After adding the required decelerations for smoothing, it is clear that $x_i = \gamma_i^*$ must also be alternating.

Observe that an alternating trajectory $\gamma \in \mathcal{D}[a, b]$ can be described as a sequence of four types of consecutive repeating phases, see Figure 7 for an example. In general, there exists a partition of $[a, b]$, denoted by

$$a = t_{f1} \leq t_{d1} \leq t_{s1} \leq t_{a1} \leq t_{f2} \leq t_{d2} \leq t_{s2} \leq t_{a2} \leq \cdots \leq t_{f,n+1} = b,$$

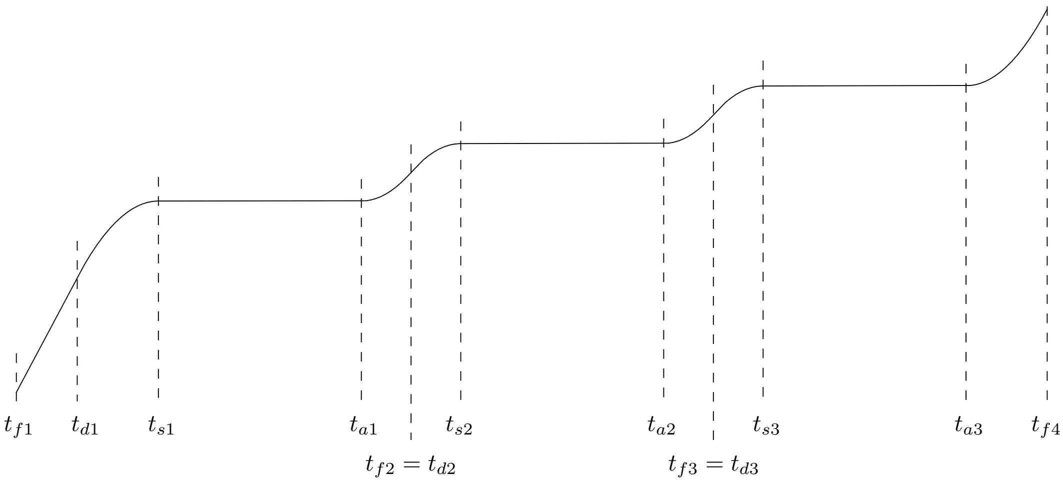

Figure 7: Example of an alternating vehicle trajectory with its defining time intervals. The particular shape of this trajectory is due to two leading vehicles, which causes the two start-stop *bumps* around the times where these leading vehicles depart from the lane.

such that we have the consecutive intervals

$$\begin{aligned} F_i &:= [t_{f,i}, t_{d,i}] \quad \text{(full speed)}, & S_i &:= [t_{s,i}, t_{a,i}] \quad \text{(stopped)}, \\ D_i &:= [t_{d,i}, t_{s,i}] \quad \text{(deceleration)}, & A_i &:= [t_{a,i}, t_{f,i+1}] \quad \text{(acceleration)}, \end{aligned}$$

such that on these intervals, γ satisfies

$$\begin{aligned} \dot{\gamma}(t) &= 1 \quad \text{for } t \in F_i, & \dot{\gamma}(t) &= 0 \quad \text{for } t \in S_i, \\ \ddot{\gamma}(t) &= -\omega \quad \text{for } t \in D_i, & \ddot{\gamma}(t) &= \bar{\omega} \quad \text{for } t \in A_i. \end{aligned}$$

We will define parameterized functions x^1, x^-, x^0, x^+ to describe γ on each of these four types of intervals. In the next section, we will show that this makes the smoothing procedure particularly simple.

3.2 Calculating smoothing times

Derive x^+ similarly to how we derived x^- when we introduced the deceleration boundary.

It can be shown that smoothing introduces a part of deceleration x^- only between the four pairs of partial trajectories

$$x^+ \to x^+, \qquad x^+ \to x^0, \qquad x^1 \to x^+, \qquad x^1 \to x^0.$$

Algorithm 1 Computing connecting deceleration for alternating trajectories.

Let i such that I_i is the latest such that $t_1 < I_i$.
Let j such that I_j is the earliest such that $t_1 > I_j$.

4 Feasibility as system of linear inequalities

Show that the follow constraint $a_i \geq \bar{a}_i$ can be written in terms of a_{i-1}.

We need to express the entry space constraint, condition (iv) in Lemma 1, in terms of the schedule times. Recall that this conditions requires that

$$\bar{x}_i \geq \check{x}_i. \tag{32}$$

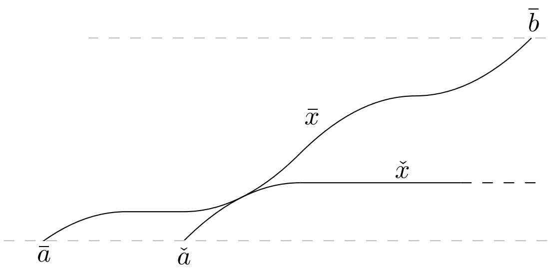

Figure 8: Illustration of entry space constraint and the induced minimum entry time $\check{a}$.

We will show that this condition can be rewritten in the form

$$a_i \geq \check{a}_i(a), \tag{33}$$

where $\check{a}_i(a, b)$ denotes some expression of the schedule times $a_1, \ldots, a_n$.

In conclusion, feasibility is expressed through the system of linear inequalities

$$b_i - a_i \geq B - A \qquad \text{for all } i \in \{1, \ldots, N\}, \tag{34a}$$

$$a_i \geq a_{i-1} - 1/\omega \qquad \text{for all } i \in \{2, \ldots, N\}, \tag{34b}$$

$$b_i \geq b_{i-1} - 1/\omega \qquad \text{for all } i \in \{2, \ldots, N\}, \tag{34c}$$

$$a_i \leq \check{a}_i(a) \qquad \text{for all } i \in \{2, \ldots, N\}. \tag{34d}$$

A Miscellaneous

Lemma A.1. *Let $f : X \times Y \to \mathbb{R}$ be some continuous function. If Y is compact, then the function $g : X \to \mathbb{R}$, defined as $g(x) = \inf\{f(x, y) : y \in Y\}$, is also continuous.*

Lemma A.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $y \in \mathbb{R}^m$, then the level set $N := f^{-1}(\{y\})$ is a closed subset of $\mathbb{R}^n$.*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. $\square$