Optimal Waiting Time

Consider two lanes 1 and 2, where at the current time $t = 0$, some vehicle $j = 0$ has just been completed at lane 1. We know that another vehicle $j = 2$ will arrive to lane somewhere in the future at $r = r_1$, which is a random variable. Furthermore, some other vehicle $j = 2$ is available (so $r_2 \leq 0$) for service on lane 2. Assume that all processing times are $p_j = p = 1$. We need to choose how long we wait for vehicle $j = 1$, before switching to lane 2. Let y_j denote the scheduled starting (crossing) time of the vehicles. The goal is to minimize

$$\sum_j y_j. \tag{1}$$

Let the waiting time be denoted as $x \geq 0$, then our problem may be stated as

$$\min_{x \geq 0} \mathbb{E}\left[\Pi(x, r)\right], \tag{2}$$

where the value function Π is given by

$$\Pi(x, r) = \begin{cases} r + (r + 1 + s) & \text{if } x \geq r, \\ \max(x, s) + \max(r, \max(x, s) + 1 + s) & \text{if } x < r. \end{cases} \tag{3}$$

We calculate

$$\begin{aligned}
\mathbb{E}[\Pi(x, R)] &= \int_0^\infty \Pi(x, r) dF(r) \\
&= \int_0^x 2r + s + 1 \, dF(r) + \int_x^\infty \max(r, \max(x, s) + 1 + s) \, dF(r) \\
&= (s + 1)F(x) + 2\int_0^x r \, dF(r) \\
&\quad + \max(x, s)(1 - F(x)) \\
&\quad + (\max(x, s) + s + 1) \left(F(\max(x, s) + s + 1) - F(x)\right) \\
&\quad + \int_{\max(x,s)+s+1}^\infty r \, dF(r),
\end{aligned}$$

where the part involving $\max(x, s)$ can be verified by considering the cases $x < s$ and $x > s$ separately.

Exponential interarrival times

We can now optimize this expression as function of x. For example, assume that $R \sim \text{Exp}(\lambda)$, with $F(r) = 1 - e^{\lambda r}$ for $r \geq 0$. Using the fact that

$$\int_A^B r dF(r) = \left(-B - \frac{1}{\lambda}\right) e^{-\lambda B} - \left(-A - \frac{1}{\lambda}\right) e^{-\lambda A}, \tag{4}$$

we obtain the explicit expression

$$
\begin{aligned}
\mathbb{E}[\Pi(x,R)] = &(s+1)(1-e^{-\lambda x}) \\
&+ 2\left(\frac{1}{\lambda} - \left(x + \frac{1}{\lambda}\right)e^{-\lambda x}\right) \\
&+ \max(x,s)e^{-\lambda x} \\
&+ (\max(x,s)+s+1)\left(-e^{-\lambda x} + e^{-\lambda(\max(x,s)+s+1)}\right) \\
&+ \left(\max(x,s)+s+1+\frac{1}{\lambda}\right)e^{-\lambda(\max(x,s)+s+1)} \\
= &s+1+\frac{2}{\lambda} \\
&+ \left(\max(x,s)-s-1-\frac{2}{\lambda}\right)e^{-\lambda x} \\
&- 2xe^{-\lambda x} \\
&+ \left(2(\max(x,s)+s+1)+\frac{1}{\lambda}\right)e^{-\lambda(\max(x,s)+s+1)}.
\end{aligned}
$$

For $x < s$, this simplifies to

$$\mathbb{E}[\Pi(x,R)] = s+1+\frac{2}{\lambda} - \left(1+\frac{2}{\lambda}\right)e^{-\lambda x} - 2xe^{-\lambda x} + \left(4s+2+\frac{1}{\lambda}\right)e^{-\lambda(2s+1)},$$

with derivative

$$(\lambda + 2\lambda x)e^{-\lambda x}.$$

For $x > s$, this simplifies to

$$\mathbb{E}[\Pi(x,R)] = s+1+\frac{2}{\lambda} + \left(x-s-1-\frac{2}{\lambda}\right)e^{-\lambda x} - 2xe^{-\lambda x} + \left(2x+2s+2+\frac{1}{\lambda}\right)e^{-\lambda(x+s+1)},$$

with derivative

$$(\lambda x + \lambda s + \lambda - 3)e^{-\lambda x} + (1 - 2\lambda x - 2\lambda s - 2\lambda)e^{-\lambda(x+s+1)}.$$

We can probably show that these two derivatives are always negative and positive, respectively, so that the minimum is always achieved at $x = s$.

Simulation

Simulation verifies that $x = s$ is indeed optimal, see `wait.py`.