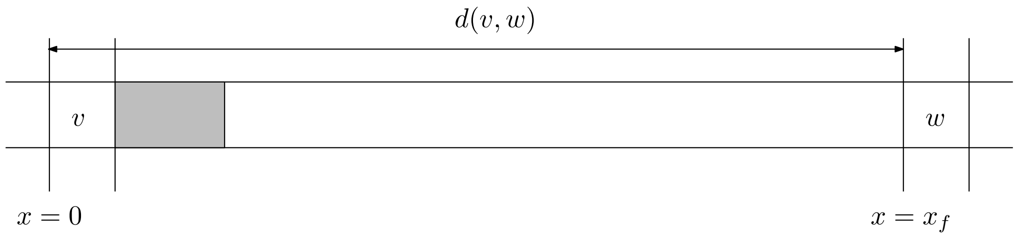


Figure 1: Tandem of two intersections v and w with lane of length $d(v, w)$. The grey rectangle represents some vehicle that just left intersection v.

Trajectory of single vehicle

We need to take into account the fact that a lane between two consecutive intersections has finite capacity for vehicles, in order to model possible blocking effects, also known as *spillback*.

We start with the simplest extension of the single intersection model by considering two intersections in tandem, as illustrated in Figure 1. Let v denote the left intersection and w the right intersection and assume that vehicles drive from left to right. Let the length and width of a vehicle i be denoted by L_i and W_i, respectively. We measure the position of a vehicle at the front bumper and we let position $x = 0$ be at the stopline of intersection v. We denote the position of the stopline at w by $x = x_f = d(v, w)$. To simplify the following discussion, we assume that all vehicles have the same dimensions.

Assumption 1. *All vehicles have the same length $L_i = L$ and width $W_i = W$.*

Now assume that some vehicle is scheduled to cross v at time $t = 0$ and to cross w at some time t_f. Let $y(t)$ denote the trajectory of the predecessor, assuming there is one. In order to keep the vehicle as close to w as possible at every time, we can generate a trajectory by solving the optimal control problem

$$\begin{aligned} \max_{x} \quad & \int_{t=0}^{t_f} x(t)dt \\ \text{s.t.} \quad & 0 \leq \dot{x}(t) \leq v_{\max}, \\ & -a_{\max} \leq \ddot{x}(t) \leq a_{\max}, \\ & y(t) \leq x(t), \\ & x(0) = 0, \qquad x(t_f) = d(v, w), \\ & \dot{x}(0) = v_{\max}, \ \dot{x}(t_f) = v_{\max}. \end{aligned} \tag{1}$$

We believe that the optimal control is always "bang-bang", meaning that there is sequence of disjoint intervals

$$(D_1, A_1, \dots, D_n, A_n)$$

so that the optimal controller is given by

$$a(t) = \begin{cases} -a_{\max} & \text{if } t \in D_k \text{ for some } k, \\ a_{\max} & \text{if } t \in A_k \text{ for some } k. \end{cases}$$

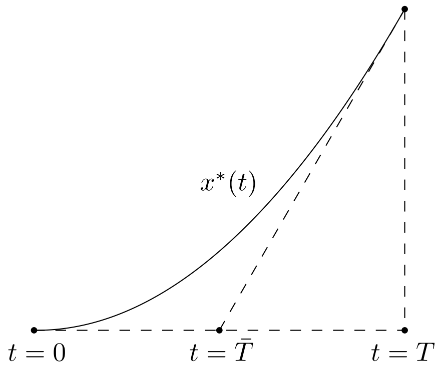


Figure 2: Full acceleration trajectory.

We will start by assuming that the system contains a single vehicle, so that we do not have to worry about keeping a safe distance to vehicle in front of it. We will show how the bang-bang intervals, or *bangs* for short, can be obtained in this situation. Without considering the boundary conditions, it follows from the vehicle dynamics that the time it takes to fully accelerate from zero to maximum velocity is given by

$$T = v_{\max}/a_{\max},$$

with corresponding trajectory x^*, given by

$$x^*(t) = a_{\max}t^2/2 \quad \text{for } 0 \leq t \leq T,$$

as illustrated in Figure 2. First, we introduce a some sort of state transformation that will prove to be helpful. For position x at time t, we define the corresponding *schedule time* by

$$\bar{t}(t, x) := t - x/v_{\max}.$$

In the following, we will use a bar above a symbol when dealing with schedule time. For example, time duration T translated to schedule time, is given by

$$\bar{T} = \bar{t}(T, x^*(T)) - \bar{t}(0, 0) = T/2.$$

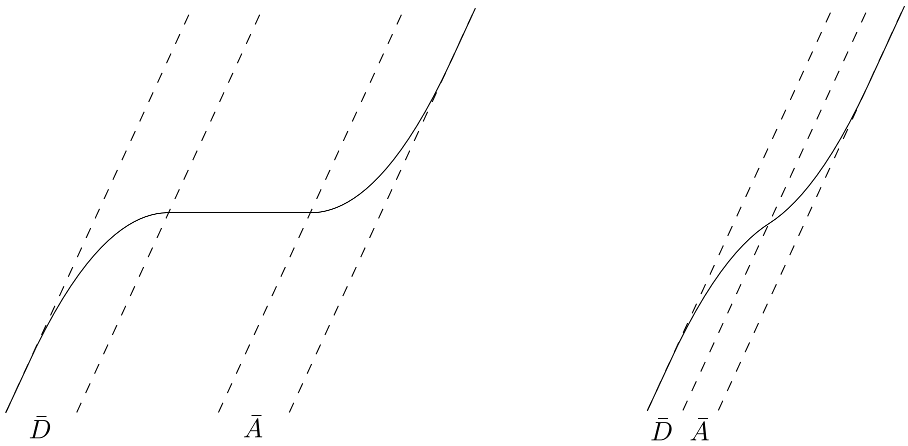
Transformation $\bar{t}$ induces equivalence classes in time-space, corresponding to lines with slope $v_{\max}$.

The crossing of v and w in schedule time are given by $b = \bar{t}(0, 0)$ and $e = \bar{t}(t_f, x_f)$, respectively. Whenever t_f is sufficiently large, it is clear that we need a full deceleration and a full acceleration. Therefore, in schedule time, this would take $2\bar{T}$ time. However, for smaller t_f, time of deceleration and acceleration need to decrease equally. Therefore, writing $(x)^+$ for $\max(x, 0)$, the remaining amount of schedule time in which we have zero acceleration is given by

$$\bar{t}_n = (e - b - 2\bar{T})^+$$

and the length of each bang is

$$\bar{t}_b = (e - b - \bar{t}_n)/2,$$



(a) Positive waiting time $\bar{t}_n > 0$.

(b) Vehicle does not stop.

Figure 3: Optimal trajectory $x(t)$ for a single (isolated) vehicle with corresponding deceleration bang $\bar{D}$ and acceleration bang $\bar{A}$ in schedule time, in case the vehicle waits for some time (a) and in case the vehicle does not come to a full stop (b).

so that the bangs are given by

$$\bar{D} = (b, b + \bar{t}_b),$$
$$\bar{A} = (e - \bar{t}_b, e).$$

The corresponding trajectory are shown in Figure 3.

Back to regular time. We are now left with translating these bangs back to the regular time scale. Let the current velocity be denoted as v_c and suppose $\bar{d}$ denotes the duration of current acceleration bang in schedule time. Consider again the full acceleration trajectory $x^*(t)$ on $0 \leq t \leq T$. Define $t_0 = v_c/a_{\max}$ so that we have $\dot{x}^*(t_0) = v_c$. Next, we find t_1 such that $t_0 \leq t_1 \leq T$ and

$$\bar{t}(t_1, x^*(t_1)) - \bar{t}(t_0, x^*(t_0)) = \bar{d}, \tag{2}$$

such that the duration of the bang in regular time is given by $d = t_1 - t_0$. After some rewriting and substitution of the definitions of x^* and $\bar{t}$ in equation (2), we obtain the quadratic equation

$$-\frac{a_{\max} t_1^2}{2v_{\max}} + t_1 - t_0 + \frac{a_{\max} t_0^2}{2v_{\max}} - \bar{d} = 0,$$

for which we are interested in the solution

$$t_1 = T - \sqrt{T^2 - 2T(t_0 + \bar{d}) + t_0^2}.$$

Similarly, for a deceleration bang of length $\bar{d}$ with current velocity v_c, the duration is given by $d = t_1 - t_0$ where $t_0 = (v_{\max} - v_c)/a_{\max}$ and t_1 is the solution to

$$\bar{t}(t_1, -x^*(-t_1)) - \bar{t}(t_0, -x^*(-t_0)) = \bar{d},$$

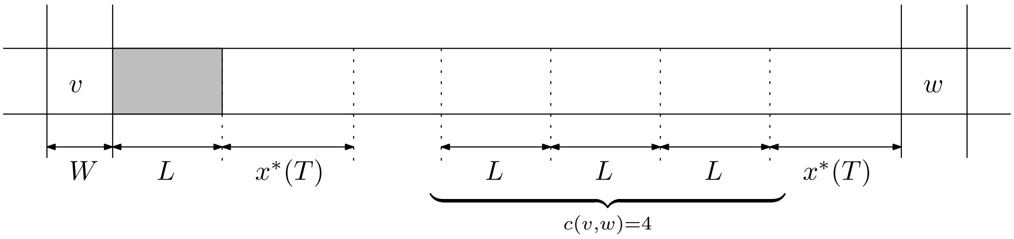


Figure 4: Tandem of intersections with indicated distances used in the capacity calculation.

given by

$$t_1 = -T + \sqrt{T^2 + 2T(t_0 + \bar{d}) + t_0^2}.$$

Using the above formulas, we can translate a sequence of bangs in schedule time to regular time. However, we need to be careful whenever the velocity becomes $v_{\max}$, because the regular time is not unique in this case. Therefore, we specify a *target position* x_t, which fixes the regular time in these cases as follows. Let $D = (b, e)$ be some deceleration bang, then the start of the first deceleration bang in regular time is given by

$$b + (x_t - x^*(T))/v_{max}.$$

The rest of the sequence of bangs in schedule time can now be translated as follows, assuming that the velocity will not reach $v_{\max}$ until the last acceleration bang. We need to argue or prove why this assumption holds. Intuitively, if the assumption does not hold, the whole trajectory could be “shifted up” in the graph, which would decrease the objective. We keep track of the current time t_c and velocity v_c. Each time we process a bang $\bar{A}$ or $\bar{D}$, we update $v_c \leftarrow v_c \pm a_{\max} d$ accordingly, and $t \leftarrow t + d$, where d is the regular bang duration, computed using the formulas from above. This way, we obtain the sequence of bangs in regular time, which completely define the corresponding controller.

Multiple vehicles

We will now consider the case when the system contains multiple vehicles. Now, we need to take into account the safe following constraints. This also causes additional constraints to the crossing time scheduling problem, since the lane only provides space for a limited number of vehicles, which we investigate first.

Capacity

Consider again the tandem of two intersection. Suppose that we want to design the tandem network such that at least $c(v, w)$ vehicles can enter and decelerate to some waiting position, from which it is also possible to accelerate again to full speed before crossing w. Vehicles are required to drive at full speed $v = v_{\max}$ as long as they occupy any intersection. Therefore, a vehicle crossing v can only start decelerating after $x(t) \geq L + W$, so the earliest position where a vehicle can come to a stop is $x = L + W + x^*(T)$. Because vehicles need to gain maximum speed $v = v_{\max}$ before reaching w, the latest position where a vehicle can wait is $x_f - x^*(T)$. Hence, in order to accomodate for $c(v, w)$ waiting vehicles, the length of the lane must satisfy

$$d(v, w) \geq L + W + 2x^*(T) + (c(v, w) - 1)L, \tag{3}$$

as illustrated in Figure 4. Conversely, given the lane length $d(v, w)$, the corresponding capacity is given by

$$c(v, w) = \texttt{floor}\left(\frac{d(v, w) - W - 2x^*(T)}{L}\right), \tag{4}$$

where $\texttt{floor}(x)$ denotes the largest integer smaller than or equal to x.

Remark 1. *Without Assumption 1, we cannot derive such a simple expression fo the capacity, because it would depend on the specific combination of lengths of the vehicles that arrived to the system.*

Bang merging

Now assume we have a feasible crossing time schedule, that satisfies the buffer constraints.

We first describe the "start-stop" trajectory of a single vehicle, without considering safe following constraints. Observe that L is the minimal distance between two consecutive vehicles. From a waiting position, we move to the next waiting position that is exactly L units further on the lane. It is clear that we need equal acceleration and deceleration $\bar{D} = \bar{A}$. By symmetry, the vehicle moves $L/2$ during both acceleration and deceleration. Assume $L/2 < x^*(T)$, then we find $\bar{A}$. Still need to generalize this. Let t_0 be such that $x^*(t_0) = L/2$, then

$$t_0 = \sqrt{L/a_{\max}},$$

with corresponding schedule time

$$\hat{t} = \bar{t}(t_0, L/2) = t_0 - \frac{L}{2v_{\max}}.$$

Therefore, we have

$$\begin{aligned} \bar{D} &= (b, b + \hat{t}), \\ \bar{A} &= (b + \hat{t}, b + 2\hat{t}). \end{aligned}$$

Safe following constraints. We now see how the trajectory of the predecessor influences the bangs of the current vehicle. Define

$$\delta = L/v_{\max}.$$

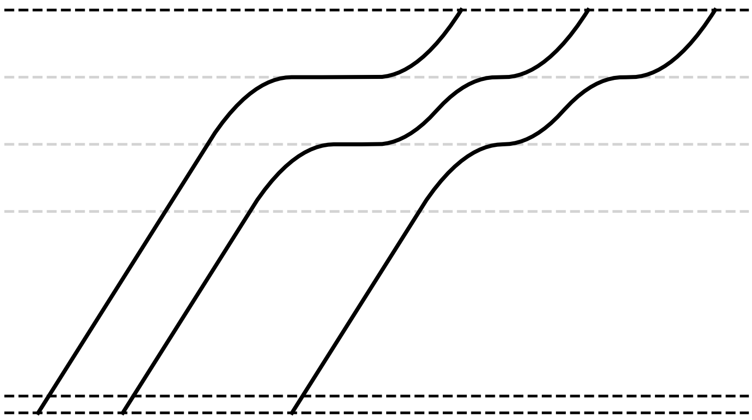


Figure 5: Buffering vehicles.