Trajectory optimization for vehicles in a lane model with minimum following distance and boundary conditions

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Abstract

This section considers a model of a single-lane road on which overtaking is not allowed. Vehicles are modeled as double integrators with bounds on speed and acceleration. Consecutive vehicles must keep some fixed *following distance* to avoid collisions. It is assumed that vehicles enter and exit the lane at predetermined *schedule times*. Whenever a vehicle enters or exits, it must drive at full speed. For an optimization objective that, roughly speaking, minimizes the distance to the end of the lane at all times, we present an algorithm to compute an optimal set of trajectories. Assuming some minimum lane length, we characterize feasibility of this trajectory optimization problem in terms of a system of linear inequalities involving the schedule times.

1 Lane model

Vehicles are modeled as double integrators with bounded speed and acceleration, which means that we only consider their longitudinal position on the lane. Let $\mathcal{D}[a,b]$ denote the set of valid *trajectories*, which we define as all continuously differentiable functions $x : [a,b] \to \mathbb{R}$ satisfying the constraints

$$0 \leq \dot{x}(t) \leq 1 \quad \text{and} \quad -\omega \leq \ddot{x}(t) \leq \bar{\omega}, \qquad \text{for all } t \in [a,b], \tag{1}$$

for some $\omega, \bar{\omega} > 0$ and we use $\dot{x}$ and $\ddot{x}$ to denote the first and second derivative with respect to time t. When we have a general positive speed upper bound, we can always apply an appropriate scaling of the time axis and the acceleration bounds to obtain the form. Consider positions $A, B \in \mathbb{R}$, such that $B \geq A$, which denote the start[1] and end position of the lane. Let $\bar{D}[a,b] \subset \mathcal{D}[a,b]$ denote the set of trajectories x that satisfy the boundary conditions

$$x(a) = A \;\text{ and }\; x(b) = B. \tag{2}$$

Even further, let $D[a,b] \subset \bar{D}[a,b]$ induce the boundary conditions

$$\dot{x}(a) = \dot{x}(b) = 1. \tag{3}$$

In words, these boundary conditions require that a vehicle arrives to and departs from the lane at predetermined times a and b and do so at full speed.

Let $L > 0$ denote the required *following distance* between consecutive vehicles. Suppose we have N vehicles that are scheduled to traverse the lane. For each vehicle i, let a_i and b_i denote the *schedule time* for entry and exit, respectively. A feasible solution consists of a sequence of trajectories $x_1, \dots, x_N$ such that

$$x_i \in D[a_i, b_i] \qquad \text{for each } i, \tag{4a}$$

$$x_i \leq x_{i-1} - L \qquad \text{for each } i \geq 2, \tag{4b}$$

where we use the shorthand notation $\gamma_1 \leq \gamma_2$ to mean $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in [a_1, b_1] \cap [a_2, b_2]$, given some $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$.

[1] We could have assumed $A = 0$, but we will later piece together multiple lanes to model intersections.

As optimization objective, we will consider

$$\max \sum_{i=1}^{N} \int_{a_i}^{b_i} x_i(t)\,\mathrm{d}t, \tag{5}$$

which, roughly speaking, seeks to keep all vehicles as close to the end of the lane at all times. In particular, we will show that optimal trajectories can be understood as the concatenation of at most four different types of trajectory parts. Based on this observation, we present an algorithm to compute optimal trajectories. Assuming $(\omega, \bar{\omega}, A, B, L)$ to be fixed, with lane length $B - A$ sufficiently large, we will show that feasibility of the trajectory optimization problem is completely characterized by a system of linear inequalities in a_i and b_i.

Note that objective (5) does not capture energy efficiency in any way. Although this is not desirable in practice, it is precisely this assumption that enables the analysis in this section. Deriving a similar characterization of feasibility and optimal trajectories under an objective that does model energy concumption is an interesting topic for further research.

2 Single vehicle problem

We will first consider a somewhat generalized version of the constraints (4) for a single vehicle i. Therefore, we lighten the notation slightly by dropping the vehicle index i and instead of $x_{i-1} - L$, we assume we are given some arbitrary *lead vehicle boundary* $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$, then we consider the optimization problem

$$\max_{x \in D[a,b]} \int_a^b x(t)\,\mathrm{d}t \quad \text{such that} \quad x \leq \bar{x}. \tag{6}$$

to which we will refer as the *single vehicle problem*.

2.1 Necessary conditions

For every trajectory $x \in D[a, b]$, we derive two upper bounding trajectories x^1 and $\hat{x}$ and one lower bounding trajectory $\check{x}$, see Figure 1. Using these bounding trajectories, we will then formulate four necessary conditions for feasibility of the single vehicle problem.

Let the *full speed boundary* x^1 be defined as

$$x^1(t) = A + t - a, \tag{7}$$

for all $t \in [a, b]$, then we clearly have $x \leq x^1$. Next, since deceleration is at most ω, we have $\dot{x}(t) \geq \dot{x}(a) - \omega(t - a) = 1 - \omega(t - a)$, which we combine with the speed constraint $\dot{x} \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \omega(t - a)\}$. Hence, we obtain the lower bound

$$x(t) = x(a) + \int_a^t \dot{x}(\tau)\,\mathrm{d}\tau \geq A + \int_a^t \max\{0, 1 - \omega(\tau - a)\}\,\mathrm{d}\tau =: \check{x}(t). \tag{8}$$

Analogously, we derive an upper bound from the fact that acceleration is at most $\bar{\omega}$. Observe that we have $\dot{x}(t) + \bar{\omega}(b - t) \geq \dot{x}(b) = 1$, which we combine with the speed constraint $\dot{x}(t) \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \bar{\omega}(b - t)\}$. Hence, we obtain the upper bound

$$x(t) = x(b) - \int_t^b \dot{x}(\tau)\,\mathrm{d}\tau \leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b - \tau)\}\,\mathrm{d}\tau =: \hat{x}(t). \tag{9}$$

We refer to $\check{x}$ and $\hat{x}$ as the *entry boundary* and *exit boundary*, respectively.

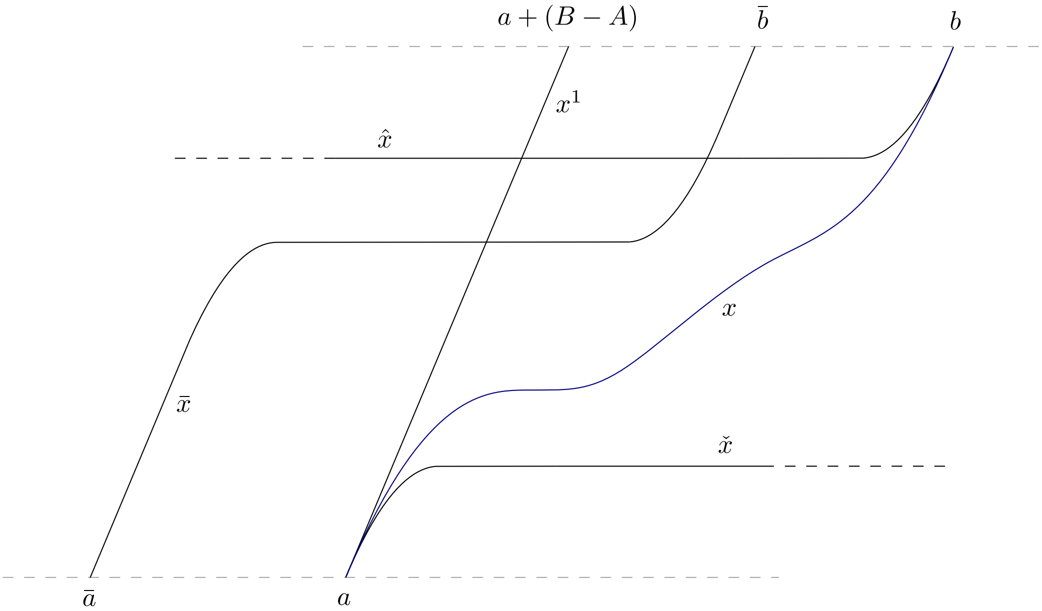

Figure 1: Illustration of the four bounding trajectories $\bar{x}, x^1, \hat{x}, \check{x}$ that bound feasible trajectories from above and below. We also drew an example of a feasible trajectory x in blue. The horizontal axis represents time and the vertical axis corresponds to the position on the lane, so the vertical dashed grey lines correspond to the start and end of the lane.

Lemma 1. *Let $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$ and assume there exists a trajectory $x \in D[a, b]$ such that $x \leq \bar{x}$, then the following conditions must hold*

(i)	$b - a \geq B - A,$	*(full speed constraint)*
(ii)	$\bar{b} \leq b,$	*(downstream order constraint)*
(iii)	$\bar{a} \leq a,$	*(upstream order constraint)*
(iv)	$\bar{x} \geq \check{x}.$	*(entry space constraint)*

Proof. Each of the conditions corresponds somehow to one of the four bounding trajectories defined above. Observe that $x^1(t) = B$ for $t = a + (B - A)$, which can be interpreted as the earliest time of departure from the lane. This shows that $b \geq a + (B - A)$, which is equivalent with (i). When either (ii) or (iii) is violated, the constraint $x \leq \bar{x}$ conflicts with one of the boundary conditions $x(a) = A$ or $x(b) = B$. To see that (iv) must hold, suppose that $\bar{x}(\tau) < \check{x}(\tau)$ for some time τ. Since $\bar{a} \leq a$, this means that $\bar{a}$ must intersect $\check{a}$ from above. Therefore, any trajectory that satisfies $x \leq \bar{x}$ must also intersect $\check{a}$ from above, which contradicts the assumption that x was a feasible solution. □

We show that the boundaries $\hat{x}$ and $\check{x}$ together could yield yet another necessary condition. It is straightforward to verify from equations (8) and (9) that $\hat{x}(t) \geq B - 1/(2\bar{\omega})$ and $\check{x}(t) \leq A + 1/(2\omega)$. Therefore, whenever $B - A < 1/(2\bar{\omega}) + 1/(2\omega)$, these boundaries intersect for certain values of a and b. Because the exact condition is somewhat cumbersome to characterize, we avoid this case by simply assuming that the lane length is sufficiently large.

Assumption 1. *The length of the lane satisfies $B - A \geq 1/(2\omega) + 1/(2\bar{\omega})$.*

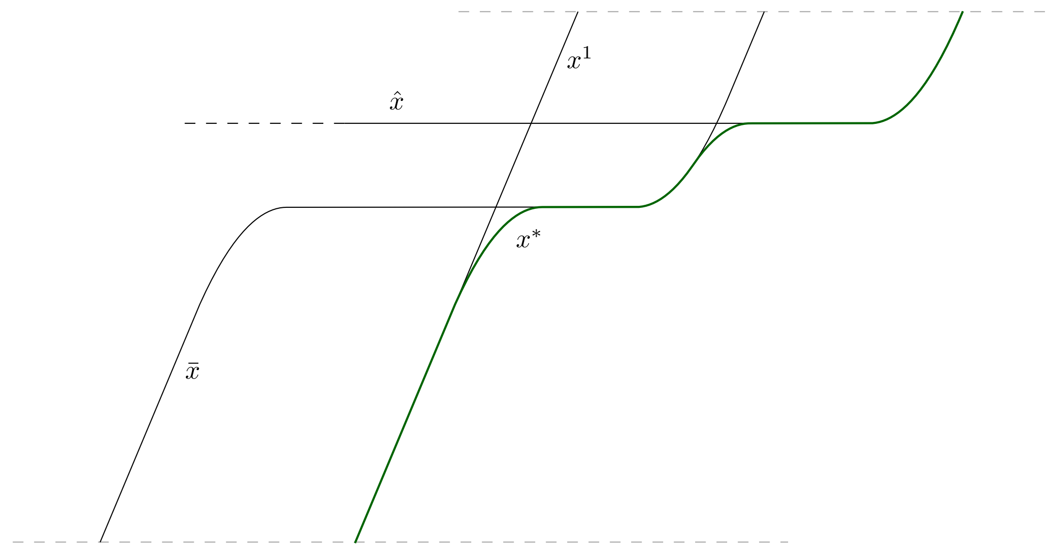

Figure 2: The minimum boundary γ, induced by three upper boundaries $\bar{x}$, $\hat{x}$ and x^1, is smoothened around the two times where its derivative is discontinuous to obtain the smooth optimal trajectory x^*, drawn in green.

2.2 Optimal trajectories

Assuming the four conditions of Lemma 1 hold, we will construct an optimal solution x^* for the single vehicle problem, thereby showing that these conditions are thus also sufficient for feasibility. First, we construct x^* by combining the upper boundaries $\bar{x}$, $\hat{x}$ and x^1 in a certain way to obtain a smooth trajectory satisfying $x^* \in D[a, b]$. We show that x^* is still an upper boundary for any other feasible solution, which shows that it is optimal.

The starting point of the construction is the *minimum boundary* $\gamma : [a, b] \to \mathbb{R}$ induced by the upper boundaries, defined as

$$\gamma(t) := \min\{\bar{x}(t), \hat{x}(t), x^1(t)\}. \tag{10}$$

Obviously, γ is still a valid upper boundary for any feasible solution, but in general, γ may have a discontinuous derivative at some[2] isolated points in time.

Definition 1. *We say $\mu : [a, b] \to \mathbb{R}$ is a* piecewise trajectory (with downward bends) *if and only if there exists a finite subdivision $a = t_0 < \cdots < t_{n+1} = b$ such that $\mu|_{[t_i, t_{i+1}]} \in \mathcal{D}[t_i, t_{i+1}]$, for $i \in \{0, \dots, n\}$ and the one-sided limits satisfy $\dot{\mu}(t_i^-) > \dot{\mu}(t_i^+)$, for $i \in \{1, \dots, n\}$.*

It is not difficult to see from Figure 1 that, under the necessary conditions, γ satisfies the above definition. In other words, γ consists of a number of pieces that are smooth and satisfy the vehicle dynamics, with possibly some sharp bend downwards where these pieces connect. Next, we present a simple procedure to smoothen out this kind of discontinuity by decelerating from the original trajectory somewhat before t_c, as illustrated in Figure 2. We show that this procedure can be repeated at every point of discontinuity.

2.2.1 Stopping trajectory.

In order to formalize the smoothing procedure, we will first define some parameterized family of functions to model the deceleration part. Recall the derivation of $\check{x}$ in equation (8) and the discussion preceding it, which we will now generalize a bit. Let $x \in \mathcal{D}[a, b]$ be some smooth

[2] In fact, it can be shown that, under the necessary conditions, there are at most two of such discontinuities.

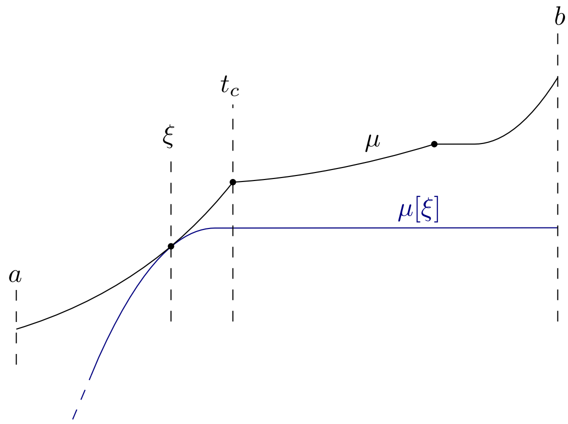

Figure 3: Illustration of a stopping trajectory of μ at ξ. This particular trajectory μ has a discontinuous first derivative at two points, of which we identify the first one as t_c. The careful reader may notice that this μ cannot occur as the minimum boundary as defined in equation (10), but we stress again that this is not required for the smoothing procedure.

trajectory, then observe that $\dot{x}(t) \geq \dot{x}(\xi) - \omega(t - \xi)$ for all $t \in [a, b]$. Combining this with the constraint $\dot{x}(t) \in [0, 1]$, this yields

$$\dot{x}(t) \geq \max\{0, \min\{1, \dot{x}(\xi) - \omega(t - \xi)\}\} =: \{\dot{x}(\xi) - \omega(t - \xi)\}_{[0,1]}, \tag{11}$$

where use $\{\cdot\}_{[0,1]}$ as a shorthand for this clipping operation. Hence, for any $t \in [a, b]$, we obtain the following lower bound

$$x(t) = x(\xi) + \int_{\xi}^{t} \dot{x}(\tau)\,\mathrm{d}\tau \geq x(\xi) + \int_{\xi}^{t} \{\dot{x}(\xi) - \omega(\tau - \xi)\}_{[0,1]}\,\mathrm{d}\tau =: x[\xi](t), \tag{12}$$

where we will refer to the right-hand side as the *stopping trajectory* of x at ξ. We can expand the integral further by carefully evaluating the clipping operation. Observe that the expression within the clipping operation reaches the bounds 1 and 0 for $\delta_1 := \xi - (1 - \dot{x}(\xi))/\omega$ and $\delta_0 := \xi + \dot{x}(\xi)/\omega$, respectively. Using this notation, a simple calculation shows that

$$x[\xi](t) = x(\xi) + \begin{cases} (1 - \dot{x}(\xi))^2/(2\omega) + t - \xi & \text{for } t \leq \delta_1, \\ \dot{x}(\xi)(t - \xi) - \omega(t - \xi)^2/2 & \text{for } t \in [\delta_1, \delta_0], \\ (\dot{x}(\xi))^2/(2\omega) & \text{for } t \geq \delta_0. \end{cases} \tag{13}$$

2.2.2 Smoothing procedure

Let μ be some piecewise trajectory defined on $[a, b]$ and let $a = t_0 < \cdots < t_{n=1} = b$ as in Definition 1. First of all, notice that we can consider the stopping trajectory of μ at any $\xi \in [a, b]$ without any problems. This is not true, because the definition is ambiguous at every t_i, since $\dot{\mu}(t_i)$ does not exist. We can choose to pick $\dot{\mu}(t_i^-)$. However, please note that the lower bound (12) is no longer guaranteed to be valid over any interval $[t_i, t_{i+1}]$ that does not contain ξ. From here on, we assume that $\mu \geq \mu[a]$ and $\mu \geq \mu[b]$.

We first show how to smoothen the first discontinuity at t_1. Pick some $\xi \in [a, t_1)$, then we consider the distance between $\mu[\xi](t)$ and $\mu(t)$ for all times $t \geq t_1$, by defining

$$d(\xi) := \min_{t \in [t_1, b]} \mu(t) - \mu[\xi](t). \tag{14}$$

Since μ and $\mu[\xi]$ are both continuous, this minimum actually exists due to the extreme value theorem. Furthermore, since $\mu(t) - \mu[\xi](t)$ is continuous as a function of t and the minimum is taken over a closed interval, d must be continuous as well (see Lemma A.3).

Show that d is non-increasing. This follows from the uniqueness property of stopping trajectories.

Now observe that $d(a) \geq 0$, because $\mu \geq \mu[a]$ by assumption. It follows from the fact that $\dot{\mu}(t_1^-) > \dot{\mu}(t_1^+)$ that $\mu < \mu[t_1]$ on $(t_1, t_1 + \epsilon)$ for some small $\epsilon > 0$. This shows that $d(t_1) < 0$. It follows from the intermediate value theorem that there must exist some ξ_1 such that $d(\xi_1) = 0$. In fact, since d is continuous and non-increasing, the level set $X := \{\xi : d(\xi) = 0\}$ is a closed interval.

Now let $t^* \in [t_1, b]$ be such that the minimum in the definition of $d(\xi_1) = 0$ is attained, i.e., $\mu(t^*) = \mu[\xi_1](t^*)$. If t^* is a local minimum in (t_1, b), then it is necessary that $\dot{\mu}(t^*) = \dot{\mu}[\xi_1](t^*)$. Note that $t^* = t_1$ cannot happen. When $t^* = b$, argue that the derivatives must match as a consequence of assumption $\mu \geq \mu[b]$.

2.2.3 Optimality after smoothing

Observe that the necessary conditions require $\gamma(a) = A$, $\gamma(b) = B$ and $\dot{\gamma}(a) = \dot{\gamma}(b) = 1$, so whenever we have $\gamma \in \mathcal{D}[a, b]$, we automatically have $\gamma \in D[a, b]$ so that γ is already an optimal solution.

Lemma 2. *Let $\mu \in \mathcal{P}[a, b]$ be a piecewise trajectory and let $\mu^* \in \mathcal{D}[a, b]$ denote the result after smoothing. All trajectories $x \in \mathcal{D}[a, b]$ that are such that $x \leq \mu$, must satisfy $x \leq \mu^*$.*

Proof. Consider the interval (ξ, τ) of a joining deceleration part. Suppose there exists some $t_d \in (\xi, \tau)$ such that $x(t_d) > \mu(t_d)$. Because $x(\xi) \leq \mu(\xi)$, this means that x must intersect μ at least once in $[\xi, t_d)$, so let $t_c := \sup \{t \in [\xi, t_d) : x(t) = \mu(t)\}$ be the latest time of intersection such that $x \geq \mu$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{x}(t_v) > \dot{\mu}(t_v)$, otherwise

$$x(t_d) = x(t_c) + \int_{t_c}^{t_d} \dot{x}(t) \, \mathrm{d}t \leq \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t) \, \mathrm{d}t = \mu(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{x}(t) \geq \dot{x}(t_v) - \omega(t - t_v) > \dot{\mu}(t_v) - \omega(t - t_v) = \dot{\mu}(t).$$

It follows that $x(\tau) > \mu(\tau)$, which contradicts the assumption. □

Definition 2. *Consider some* $\mu : [a, b] \to \mathbb{R}$*, then we say* μ *is a* piecewise trajectory *if and only if* μ *is continuous on* $[a, b]$ *and there exists a finite subdivision* $\{t_0, \dots, t_n\}$ *of* $[a, b]$ *such that* $\mu \in \mathcal{D}(t_i, t_{i+1})$ *for every* $i \in \{0, \dots, n-1\}$*.*

Given some piecewise trajectory μ, we are looking for $\gamma^* \in \mathcal{D}[a, b]$ such that $\gamma \leq \gamma^*$ for all $\gamma \in \mathcal{D}[a, b]$ such that $\gamma \leq \mu$.

Consider two arbitrary trajectories $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$, then we define their *minimum concatenation* to be

$$\min\{\gamma_1, \gamma_2\}(t) := \min\{\gamma_1(t), \gamma_2(t)\} \quad \text{for } t \in [a_1, b_1] \cap [a_2, b_2]. \tag{15}$$

Suppose γ_1 and γ_2 intersect at precisely a single time $t_c \in [a_1, b_1] \cap [a_2, b_2]$. When $\dot{\gamma}_1(t_c) = \dot{\gamma}_2(t_c)$, it is easily seen that the minimum concatenation satisfies $\min\{\gamma_1, \gamma_2\} \in \mathcal{D}[a, b]$, where $[a, b] := [a_1, b_1] \cap [a_2, b_2]$. When $\dot{\gamma}_1(t_c) > \dot{\gamma}_2(t_c)$, we consider the prototypical concatenation $(\gamma_1 * \gamma_2)[\xi, \tau]$, by defining

$$(\gamma_1 * \gamma_2)[\xi, \tau](t) := \begin{cases} \gamma_1(t) & \text{for } t < \xi, \\ \gamma_1[\xi](t) & \text{for } t \in [\xi, \tau], \\ \gamma_2(t) & \text{for } t > \tau. \end{cases} \tag{16}$$

Lemma 3. *If there exist an $\varphi := (\gamma_1 * \gamma_2)[\xi, \tau]$ such that*

(i) $\varphi(\tau) = \gamma_2(\tau)$ *and* $\dot{\varphi}(\tau) = \dot{\gamma}_2(\tau)$

(ii) $\varphi \leq \gamma_2$,

then φ is unique.

Definition 3. *Given some trajectory* $x \in \mathcal{D}[a,b]$ *and some time* $\xi \in [a,b]$*, consider the* stopping trajectory $x[\xi]$ *that is identical to the original trajectory until* ξ*, from where it starts decelerating to a full stop, so that at time* $t \geq \xi$*, the position is given by*

$$x[\xi](t) = x(\xi) + \int_{\xi}^{t} \max\{0, \dot{x}(\xi) - \omega(\tau - \xi)\}\,\mathrm{d}\tau \tag{17a}$$

$$= x(\xi) + \begin{cases} \dot{x}(\xi)(t-\xi) - \omega(t-\xi)^2/2 & \textit{for } t \leq \xi + \dot{x}(\xi)/\omega, \\ (\dot{x}(\xi))^2/(2\omega) & \textit{for } t \geq \xi + \dot{x}(\xi)/\omega. \end{cases} \tag{17b}$$

This definition guarantees $x[\xi] \in \mathcal{D}[a,\infty)$. Note that a stopping trajectory serves as a lower bound in the sense that, for any $z \in \mathcal{D}[c,d]$ such that $x = z$ on $[a,\xi] \cap [c,d]$, we have $x \leq z$ and $\dot{x} \leq \dot{z}$.

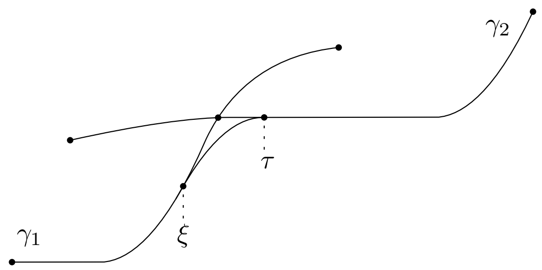

Figure 4: Two intersecting trajectories joined together by a part of a stopping trajectory.

Consider two crossing trajectories γ_1 and γ_2 like in Figure 4. Roughly speaking, we want to construct a new trajectory μ by finding some times ξ and τ such that $\mu = \gamma_1$ until ξ and $\mu = \gamma_2$ after τ and such that full deceleration during the interval $[\xi, \tau]$ takes us from γ_1 onto γ_2, with matching tangents. After precisely defining the kind of trajectories we allow, we show that such a *joined trajectory* is unique, when it exists. Furthermore, we provide a condition for existence and show a certain upper bounding property, which we will apply in the next section to characterize optimal trajectories.

Furthermore, $\gamma[\xi](t)$ is a non-decreasing function in terms of either of its arguments, while fixing the other. To see this for ξ, fix any t and consider $\xi_1 \leq \xi_2$, then note that $\gamma[\xi_1](t)$ is a lower bound for $\gamma[\xi_2](t)$.

Property 1. *Both $\gamma[\xi](t)$ and $\dot{\gamma}[\xi](t)$ are continuous when considered as functions of (ξ, t).*

Proof. Write $f(\xi, t) := \gamma[\xi](t)$ to emphasize that we are dealing with two variables. Recall that $\dot{\gamma}$ is continuous by assumption, so the equation $\tau = \xi + \dot{\gamma}(\xi)/\omega$ defines a separation boundary of the domain of f. Both cases of (17b) are continuous and they agree at this boundary, so f is continuous on all of its domain. Since $x \mapsto \max\{0, x\}$ is continuous, it is easy to see that also $(\xi, t) \mapsto \dot{\gamma}[\xi](t) = \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\}$ is continuous. $\square$

Because $\gamma[\xi](t)$ is continuous and non-decreasing in ξ, the set

$$X(t_0, x_0) := \{\xi : \gamma[\xi](t_0) = x_0\} \tag{18}$$

is a closed interval (follows from Lemma A.1), so we can consider the maximum

$$\xi(t_0, x_0) := \max X(t_0, x_0). \tag{19}$$

Consider the closed region $\bar{U} := \{(t, x) : \gamma[a](t) \leq x \leq \gamma[b](t)\}$. For each $(t_0, x_0) \in \bar{U}$, there must be some ξ_0 such that $\gamma[\xi_0](t_0) = x_0$, as a consequence of the intermediate value theorem and the above continuity property. Consider $\bar{U}$ without the points on γ, which we denote by

$$U := \bar{U} \setminus \{(t, x) : \gamma(t) = x\}. \tag{20}$$

Next, we prove that $\gamma[\xi_0]$ is actually unique if $(t_0, x_0) \in U$, so that we may regard $\xi(t_0, x_0)$ as the canonical representation of this unique trajectory $\gamma[\xi(t_0, x_0)]$.

Property 2. *For $(t_0, x_0) \in U$, if $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0) = x_0$, then $\gamma[\xi_1] = \gamma[\xi_2]$.*

Proof. Suppose $t_0 < \xi_i$, then $x_0 = \gamma[\xi_i](t_0) = \gamma(t_0)$ contradicts the assumption $(t_0, x_0) \in U$. Therefore, assume $\xi_1 \leq \xi_2 < t_0$, without loss of generality. Since $\gamma[\xi_1] = \gamma[\xi_2]$ on $[a, \xi_1]$, note that we have the lower bounds

$$\gamma[\xi_1] \leq \gamma[\xi_2] \quad \text{and} \quad \dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]. \tag{21}$$

We must have $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, because otherwise $\gamma[\xi_1] > \gamma[\xi_2]$ somewhere in a sufficiently small neighborhood of t_0, which contradicts the first lower bound.

It is clear from Definition 3 that

$$\ddot{\gamma}[\xi_i](t) = \begin{cases} \ddot{\gamma}(t) & \text{for } t < \xi_i, \\ -\omega & \text{for } t \in (\xi_i, \xi_i + \dot{\gamma}(\xi_i)/\omega), \\ 0 & \text{for } t > \xi_i + \dot{\gamma}(\xi_i)/\omega, \end{cases}$$

for both $i \in \{1, 2\}$. Note that $\dot{\gamma}(\xi_1) - \omega(\xi_2 - \xi_1) \leq \dot{\gamma}(\xi_2)$, which can be rewritten as

$$\xi_2 + \dot{\gamma}(\xi_2)/\omega \geq \xi_1 + \dot{\gamma}(\xi_1)/\omega.$$

This shows that $\ddot{\gamma}[\xi_1](t) \geq \ddot{\gamma}[\xi_2](t)$, for every $t \geq \xi_2$. Because $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, this in turn ensures that $\dot{\gamma}[\xi_1](t) \geq \dot{\gamma}[\xi_2](t)$ for $t \geq t_0$. Together with the opposite inequality in (21), we conclude that on $[t_0, \infty)$, we have $\dot{\gamma}[\xi_1] = \dot{\gamma}[\xi_2]$ and thus $\gamma[\xi_1] = \gamma[\xi_2]$.

It remains to show that $\gamma[\xi_1] = \gamma[\xi_2]$ on $[\xi_1, t_0]$, so consider the smallest $t^* \in (\xi_1, t_0)$ such that $\gamma[\xi_1](t^*) < \gamma[\xi_2](t^*)$. Since $\dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]$, this implies that $\gamma[\xi_1](t) < \gamma[\xi_2](t)$ for all $t \geq t^*$, but this contradicts the assumption $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0)$. □

To further support the analysis below, we define the auxiliary function

$$g(t, x) := \dot{\gamma}_1[\xi(t, x)](t), \tag{22}$$

which gives the slope of the unique stopping trajectory through each point $(t, x) \in U$.

Property 3. *Function g is continuous in (t, x).*

Proof. We write a neighborhood of x as $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$. We will write $f_x(\xi, t) = \gamma_1[\xi](t)$, $f_v(\xi, t) = \dot{\gamma}_1[\xi](t)$ and $h_t(\xi) = \gamma_1[\xi](t)$ to emphasize the quantities that we treat as variables. Observe that $h_t^{-1}(x) = X(t, x)$.

- Let $x_0 = f_x(\xi_0, \tau_0)$ and $v_0 = f_v(\xi_0, \tau_0)$ for some ξ_0 and τ_0 and pick some arbitrary $\varepsilon > 0$. Note that $\xi_0 \in [\xi_1, \xi_2] := h_{\tau_0}^{-1}(x_0)$. We apply the ε-δ definition of continuity to each of these endpoints. Let $i \in \{1, 2\}$, then there exist $\delta_i > 0$ such that

  $$\xi \in N_{\delta_i}(\xi_i),\ \tau \in N_{\delta_i}(\tau_0) \implies f_v(\xi, \tau) \in N_\varepsilon(v_0).$$

  Let $\delta = \min\{\delta_1, \delta_2\}$ and define $N_1 := (\xi_1 - \delta, \xi_2 + \delta)$ and $N_2 := N_\delta(\tau_0)$, then

  $$\xi \in N_1,\ \tau \in N_2 \implies f_v(\xi, \tau) \in N_\varepsilon(v_0).$$

  This is obvious when ξ is chosen to be in one of $N_{\delta_i}(\xi_i)$. Otherwise, we must have $\xi \in [\xi_1, \xi_2]$, in which case $f_v(\xi, \tau) = f_v(\xi_1, \tau) \in N_\varepsilon(v_0)$.

- Because $h_{\tau_0}(\xi)$ is continuous, the image $I := h_{\tau_0}(N_1)$ must be an interval containing x_0, with $\inf I = h_{\tau_0}(\xi_1 - \delta)$ and $\sup I = h_{\tau_0}(\xi_2 + \delta)$. We argue that I contains x_0 in its interior. For sake of contradiction, suppose $x_0 = \max I$, then $h_{\tau_0}(\xi_2 + \delta') = x_0$, for each $\delta' \in (0, \delta)$, because h_{τ_0} is non-decreasing, but this contradicts the definition of ξ_2. Similarly, when $x_0 = \min I$, then $h_{\tau_0}(\xi_1 - \delta') = x_0$, for each $\delta' \in (0, \delta)$, which contradicts the definition of ξ_1.

- Define $\nu := \min\{x_0 - \inf I, \sup I - x_0\}$ and $N_3 := (x_0 - \nu/2, x_0 + \nu/2)$. Because $h_\tau(\xi)$ is also continuous in τ, there exists a neighborhood $N_2^* \subset N_2$ of τ_0 such that for every $\tau \in N_2^*$, we have

  $$\begin{aligned} h_\tau(\xi_1 - \delta) &\leq h_{\tau_0}(\xi_1 - \delta) + \nu/2 = \inf I + \nu/2 < x_0 - \nu/2, \\ h_\tau(\xi_2 + \delta) &\geq h_{\tau_0}(\xi_2 + \delta) - \nu/2 = \sup I - \nu/2 > x_0 + \nu/2, \end{aligned}$$

  which shows that $h_\tau(N_1) \supset N_3$. It follows that $h_\tau^{-1}(N_3) \subset N_1$.

- Finally, take any $\tau \in N_2^*$ and $x \in N_3$, then there exists some $\xi \in N_1$ such that $h_\tau(\xi) = x$ and $g(\tau, x) = f_v(\max h_\tau^{-1}(x), \tau) = f_v(\xi, \tau) \in N_\varepsilon(v_0)$. □

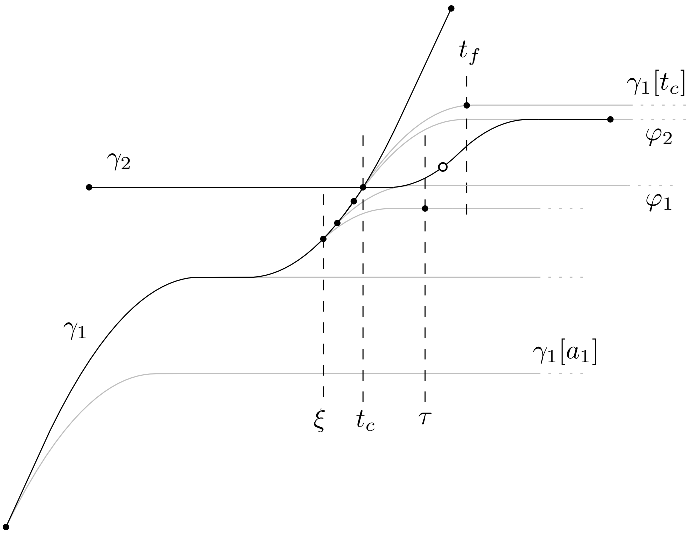

Figure 5: Sketch of some quantities used in the proof of Lemma 4, including some stopping trajectory candidates drawn in grey. The unique stopping trajectory satisfying the requirements of Lemma 4 is marked as φ_1.

Property 4. *Function g is non-decreasing and Lipschitz continuous in x.*

Proof. Let $x_1 \leq x_2$ and τ such that $g(\tau, x_1)$ and $g(\tau, x_2)$ are defined. There must be $\xi_1 \leq \xi_2$ such that $h_\tau(\xi_1) = x_1$ and $h_\tau(\xi_2) = x_2$ and we have

$$\begin{aligned} g(\tau, x_1) = \dot{\gamma}_1[\xi_1](\tau) &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} = \dot{\gamma}_1[\xi_2](\tau) = g(\tau, x_2). \end{aligned}$$

Furthermore, we have $\dot{\gamma}_1(\xi_2) \leq \dot{\gamma}_1(\xi_1) + \bar{\omega}(\xi_2 - \xi_1)$, so that

$$\begin{aligned} g(\tau, x_2) &= \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) + \bar{\omega}(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1) + (\omega + \bar{\omega})(\xi_2 - \xi_1)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} + (\omega + \bar{\omega})(\xi_2 - \xi_1) \\ &= g(\tau, x_1) + (\omega + \bar{\omega})(\xi_2 - \xi_1). \end{aligned}$$

Observe that, together with the above non-decreasing property, this shows that g is Lipschitz continuous in x, with Lipschitz constant $(\omega + \bar{\omega})$. ∎

We are now ready to state and prove uniqueness of joined trajectories.

Lemma 4. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be two trajectories that are intersecting at exactly one time t_c and assume $\dot{\gamma}_1(t_c) > \dot{\gamma}_2(t_c)$. If there exists a trajectory φ such that*

(i) $\varphi = \gamma_1[\xi]$, *for some* $\xi < t_c$,

(ii) $\varphi(\tau) = \gamma_2(\tau)$ *and* $\dot{\varphi}(\tau) = \dot{\gamma}_2(\tau)$, *for some* $\tau > t_c$,

(iii) $\varphi \leq \gamma_2$,

then φ is unique.

Proof. Define the set U and the functions $X(t,x)$ and $\xi(t,x)$ as we did in equations (18)–(20) for γ above, but now for γ_1.

- Identify for which parameters $\xi < t_c < \tau$ we have $\gamma_1[\xi](\tau) = \gamma_2(\tau)$ and $\dot{\gamma}_1[\xi](\tau) = \dot{\gamma}_2(\tau)$.
  - For each $\tau > t_c$, observe that $(\tau, \gamma_2(\tau)) \in U$. It follows from Property 2 that $\varphi_{[\tau]} := \gamma_1[\xi(\tau, \gamma_2(\tau))]$ is the unique stopping trajectory such that $\varphi_{[\tau]}(\tau) = \gamma_2(\tau)$. Next, we investigate when this unique trajectory touches γ_2 tangentially. More precisely, consider the set of times
$$T := \{\tau > t_c : \dot{\varphi}_{[\tau]}(\tau) = \dot{\gamma}_2(\tau),\ \xi(\tau, \gamma_2(\tau)) < t_c\}, \tag{23}$$
which can also be written as
$$T = \{\tau > t_c : g(\tau, \gamma_2(\tau)) = \dot{\gamma}_2(\tau),\ \xi(\tau, \gamma_2(\tau)) < t_c\}, \tag{24}$$
so continuity of g shows that it is a closed set (Lemma A.1). It is not necessarily connected (see for example Figure 5, in which φ_1 and φ_2 touch γ_2 tangentially on separated intervals). In general, T is the union of a sequence of disjoint closed intervals $T_1, T_2, \ldots, T_n$.
  - Define $\tau_i := \min T_i$ and let $\varphi_i := \varphi_{[\tau_i]}$ denote the unique stopping trajectory through $(\tau_i, \gamma_2(\tau_i))$. For $\tau \in T_i$, we have $\dot{\gamma}_2(\tau) = g(\tau, \gamma_2(\tau))$ by definition of T_i. Moreover, we have
$$\dot{\varphi}_i(t) = g(t, \varphi_i(t)), \tag{25}$$
for every t for which these quantities are defined, so in particular on T_i. This shows that γ_2 and φ_i are both solutions to the initial value problem
$$\begin{cases} \dot{x}(t) = g(t, x(t)) & \text{for } t \in T_i, \\ x(\tau_i) = \gamma_2(\tau_i). \end{cases} \tag{26}$$
Since $g(t,x)$ is continuous in t and Lipschitz continuous in x, it is a consequence of the (local) existence and uniqueness theorem (Lemma A.2) that $\gamma_2 = \varphi_i$ on T_i. Hence, we have $\varphi_i = \varphi_{[\tau]}$ for any $\tau \in T_i$, so we regard φ_i as being the canonical stopping trajectory for T_i.

- If $i \geq 2$, then $\varphi_i > \gamma_2$ somewhere.
  - Let $i \geq 1$, we show that $\varphi_{i+1}(t) > \gamma_2(t)$ for some t. Recall the lower bound property, so $\gamma_2(t) \geq \varphi_i(t)$ and $\dot{\gamma}_2(t) \geq \dot{\varphi}_i(t)$ for $t \geq \tau_i$. Define $\hat{\tau}_i := \max T_i$, such that $T_i = [\tau_i, \hat{\tau}_i]$, then by definition of T_i, there must be some $\delta > 0$ such that
$$\gamma_2(\hat{\tau}_i + \delta) > \varphi_i(\hat{\tau}_i + \delta), \tag{27}$$
since otherwise $\gamma_2 = \varphi_i$ on some open neighborhood of $\hat{\tau}_i$ and then also
$$\dot{\gamma}_2(t) = \dot{\varphi}_i(t) \overset{(25)}{=} g(t, \varphi_i(t)) = g(t, \gamma_2(t)), \tag{28}$$
which contradicts the definition of $\hat{\tau}_i$. Therefore, we have $\gamma_2(t) > \varphi_i(t)$ for all $t \geq \hat{\tau}_i + \delta$. For $t = \tau_{i+1}$, in particular, it follows that $\varphi_{i+1}(\tau_{i+1}) = \gamma_2(\tau_{i+1}) > \varphi_i(\tau_{i+1})$, which shows that $\varphi_{i+1} > \varphi_i$ on (ξ_i, ∞), due to Property 2, but this means that $\varphi_{i+1}(\tau_i) > \varphi_i(\tau_i) = \gamma_2(\tau_i)$.

- If $\varphi_i > \gamma_2$ somewhere, then $i \geq 2$.

- Suppose $\varphi_i(t_x) > \gamma_2(t_x)$ for some $t_x \in (t_c, \tau_i)$, then there must be some $\tau_0 \in (t_c, t_x)$ such that $\gamma_2(\tau_0) = \varphi_i(\tau_0)$ and $\dot\gamma_2(\tau_0) < \dot\varphi(\tau_0)$. Note that this crossing must happen because we require $\xi_i < t_c$.
- Since $g(t, x)$ is non-decreasing in x, we have

$$s(t) = g(t, \gamma_2(t)) \leq g(t, \varphi_i(t)) = \dot\varphi_i(t), \tag{29}$$

for every $t \in [\tau_0, \tau_i]$ and at the endpoints, we have

$$s(\tau_0) = \varphi_i(\tau_0), \quad s(\tau_i) = \varphi_i(\tau_i). \tag{30}$$

Furthermore, observe that $\gamma_2(\tau_0) = \varphi_i(\tau_0)$ and $\gamma_2(\tau_i) = \varphi_i(\tau_i)$ require that

$$\int_{\tau_0}^{\tau_i} \dot\gamma_2(t)\,\mathrm{d}t = \int_{\tau_0}^{\tau_i} \dot\varphi_i(t)\,\mathrm{d}t. \tag{31}$$

- Since $\dot\gamma_2(\tau_0) < \dot\varphi_i(\tau_0)$, it follows from (31) that there must be some $t \in (\tau_0, \tau_i)$ such that $\dot\gamma_2(t) > \dot\varphi_i(t)$. Together with $s(\tau_0) = \dot\varphi_i(\tau_0) > \dot\gamma_2(\tau_0)$ and $s(t) \leq \dot\varphi_i(t)$ for $t \in [\tau_0, \tau_i]$, this means there is some τ^* such that $\dot\gamma_2(\tau^*) = s(\tau^*)$, again as a consequence of the intermediate value theorem. Therefore, $\tau^* \in T_j$ for some $j < i$, which shows that $i \geq 2$.

- The above two points establish that $\varphi_i \leq \gamma_2$ if and only if $i = 1$. To conclude, we have shown that if $\varphi := \varphi_1$ exists, it is the unique trajectory satisfying the stated requirements with $\tau = \tau_i$ and $\xi = \xi(\tau_i, \gamma_2(\tau_i))$. □

The uniqueness result justifies the following definition of joined trajectories.

Definition 4. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ and suppose they intersect at exactly a single time t_c. We write $\gamma_1 * \gamma_2$ to denote the unique* joined trajectory

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t < \xi, \\ \gamma_1[\xi](t) & \text{for } t \in [\xi, \tau], \\ \gamma_2(t) & \text{for } t > \tau, \end{cases} \tag{32}$$

satisfying $\gamma_1 * \gamma_2 \in \mathcal{D}[a_1, b_2]$, when it exists. If $\dot\gamma_1(t_c) = \dot\gamma_2(t_c)$, then we define $\tau = \xi = t_c$.

Next, we discuss existence of joined trajectories.

Lemma 5. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$. If there is some trajectory $\mu \in \mathcal{D}[a_1, b_2]$ such that $\mu \leq \min\{\gamma_1, \gamma_2\}$ and*

$$\begin{aligned} \mu(a_1) &= \gamma_1(a_1), & \mu(b_2) &= \gamma_2(b_2), \\ \dot\mu(a_1) &= \dot\gamma_1(a_1), & \dot\mu(b_2) &= \dot\gamma_2(b_2). \end{aligned}$$

then $\gamma_1 * \gamma_2$ exists.

Proof. Still todo, but I can adapt an earlier version of this argument, which was previously part of Lemma 1. □

Our main interest in $\gamma_1 * \gamma_2$ is due to the following upper bounding property.

Lemma 6. *Let $\gamma_1 \in \mathcal{D}[a_1, b_2]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be such that $\gamma_1 * \gamma_2$ exists. All trajectories $\mu \in \mathcal{D}[a, b]$ that are such that $\mu \leq \min\{\gamma_1, \gamma_2\}$, must satisfy $\mu \leq \gamma_1 * \gamma_2$.*

Proof. Write $\gamma := \gamma_1 * \gamma_2$ as a shorthand. We obviously have $\mu \leq \gamma$ on $[a_1, \xi] \cup [\tau, b_2]$, so consider the interval (ξ, τ) of the joining deceleration part. Suppose there exists some $t_d \in (\xi, \tau)$ such that $\mu(t_d) > \gamma(t_d)$. Because $\mu(\xi) \leq \gamma(\xi)$, this means that μ must intersect γ at least once in $[\xi, t_d)$, so let $t_c := \sup \{t \in [\xi, t_d) : \mu(t) = \gamma(t)\}$ be the latest time of intersection such that $\mu \geq \gamma$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{\mu}(t_v) > \dot{\gamma}(t_v)$, otherwise

$$\mu(t_d) = \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t)\,\mathrm{d}t \leq \gamma(t_c) + \int_{t_c}^{t_d} \dot{\gamma}(t)\,\mathrm{d}t = \gamma(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{\mu}(t) \geq \dot{\mu}(t_v) - \omega(t - t_v) > \dot{\gamma}(t_v) - \omega(t - t_v) = \dot{\gamma}(t).$$

It follows that $\mu(\tau) > \gamma(\tau)$, which contradicts $\mu \leq \gamma_2$. $\square$

3 Computing optimal trajectories

Definition 5. *Let* $\gamma \in \mathcal{D}[a, b]$ *be called* alternating *if for all* $t \in [a, b]$*, we have*

$$\ddot{\gamma}(t) \in \{-\omega, 0, \bar{\omega}\} \quad \text{and} \quad \ddot{\gamma}(t) = 0 \implies \dot{\gamma}(t) \in \{0, 1\}. \tag{33}$$

Observe that we can distinguish four consecutive phases of an alternating trajectory: full speed $\dot{\gamma} = 1$, full deceleration $\ddot{\gamma} = -\omega$, full stop $\dot{\gamma} = 0$ and full acceleration $\ddot{\gamma} = \bar{\omega}$.

Lemma 7. *Let* $\gamma_1 \in \mathcal{D}[a_1, b_1]$ *and* $\gamma_2 \in \mathcal{D}[a_2, b_2]$ *be both alternating, then when* $\gamma_1 * \gamma_2$ *exists, it is also alternating.*

Remark 1. From the proof above, it also becomes clear that a connecting deceleration can only happen between the following four pairs of partial trajectories:

$$x^+ \to x^+, \qquad x^+ \to x^0, \qquad x^1 \to x^+, \qquad x^1 \to x^0.$$

4 Feasibility as system of linear inequalities

A Miscellaneous

Lemma A.1. *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be continuous and* $y \in \mathbb{R}^m$*, then the level set* $N := f^{-1}(\{y\})$ *is a closed subset of* $\mathbb{R}^n$*.*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. $\square$

Lemma A.2. *Let* $D \subseteq \mathbb{R} \times \mathbb{R}^n$ *be some closed rectangle such that* $(t_0, x_0) \in \operatorname{int} D$*. Let* $f : D \to \mathbb{R}^n$ *be a function that is continuous in* t *and globally Lipschitz continuous in* x*, then there exists some* $\varepsilon > 0$ *such that the initial value problem*

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{34}$$

has a unique solution $x(t)$ *on the interval* $[t_0 - \varepsilon, t_0 + \varepsilon]$*.*

The above existence and uniqueness theorem is also known as the Picard-Lindelöf or Cauchy-Lipschitz theorem. The above statement is based on the Wikipedia page on this theorem, so we still need a slightly better reference.

Lemma A.3. *Let* $f : X \times Y \to \mathbb{R}$ *be some continuous function. If* Y *is compact, then the function* $g : X \to \mathbb{R}$*, defined as* $g(x) = \inf\{f(x, y) : y \in Y\}$*, is also continuous.*