

Vehicle trajectories in a tandem of intersections

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1 Joining two trajectories

Let $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second derivative of $x(t)$ with respect to time t . Let $\mathcal{D}[a, b]$ denote the set of valid *trajectories*, which we define as continuously differentiable functions $\gamma : [a, b] \rightarrow \mathbb{R}$ satisfying the constraints

$$0 \leq \dot{\gamma}(t) \leq 1 \quad \text{and} \quad -\omega \leq \ddot{\gamma}(t) \leq \bar{\omega}, \quad \text{for all } t \in [a, b]. \quad (1)$$

For $\gamma_1 \in \mathcal{D}[a_1, b_1], \gamma_2 \in \mathcal{D}[a_2, b_2]$, when we write $\gamma_1 \leq \gamma_2$ without explicitly mentioning where it applies, we mean $t \in [a_1, b_1] \cap [a_2, b_2]$. We also write $\gamma \leq \min\{\gamma_1, \gamma_2\}$ as a shorthand for $\gamma \leq \gamma_1$ and $\gamma \leq \gamma_2$.

Definition 1. *Given some trajectory $\gamma \in \mathcal{D}[a, b]$ and some time $\xi \in [a, b]$, consider the stopping trajectory $\gamma[\xi]$ that is identical to the original trajectory until ξ , from where it starts decelerating to a full stop, so that at time $t \geq \xi$, the position is given by*

$$\gamma[\xi](t) = \gamma(\xi) + \int_{\xi}^t \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\} d\tau \quad (2a)$$

$$= \gamma(\xi) + \begin{cases} \dot{\gamma}(\xi)(t - \xi) - \omega(t - \xi)^2/2 & \text{for } t \leq \xi + \dot{\gamma}(\xi)/\omega, \\ (\dot{\gamma}(\xi))^2/(2\omega) & \text{for } t \geq \xi + \dot{\gamma}(\xi)/\omega. \end{cases} \quad (2b)$$

The above definition guarantees $\gamma[\xi] \in \mathcal{D}[a, \infty)$. Note that a stopping trajectory serves as a lower bound in the sense that, for any $\mu \in \mathcal{D}[c, d]$ such that $\gamma = \mu$ on $[a, \xi] \cap [c, d]$, we have $\gamma \leq \mu$ and $\dot{\gamma} \leq \dot{\mu}$. Furthermore, $\gamma[\xi](t)$ is a non-decreasing function in terms of either of its arguments, while fixing the other. To see this for ξ , fix any t and consider $\xi_1 \leq \xi_2$, then note that $\gamma[\xi_1](t)$ is a lower bound for $\gamma[\xi_2](t)$.

Property 1. *Both $\gamma[\xi](t)$ and $\dot{\gamma}[\xi](t)$ are continuous when considered as functions of (ξ, t) .*

Proof. Write $f(\xi, t) := \gamma[\xi](t)$ to emphasize that we are dealing with two variables. Recall that $\dot{\gamma}$ is continuous by assumption, so the equation $\tau = \xi + \dot{\gamma}(\xi)/\omega$ defines a separation boundary of the domain of f . Both cases of (2b) are continuous and they agree at this boundary, so f is continuous on all of its domain. Since $x \mapsto \max\{0, x\}$ is continuous, it is easy to see that also $(\xi, t) \mapsto \dot{\gamma}[\xi](t) = \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\}$ is continuous. \square

Because $\gamma[\xi](t)$ is continuous and non-decreasing in ξ , the set

$$X(t_0, x_0) := \{\xi : \gamma[\xi](t_0) = x_0\} \quad (3)$$

is a closed interval (follows from Lemma A.1), so we can consider the maximum

$$\xi(t_0, x_0) := \max X(t_0, x_0). \quad (4)$$

Consider the closed region $\bar{U} := \{(t, x) : \gamma[a](t) \leq x \leq \gamma[b](t)\}$. For each $(t_0, x_0) \in \bar{U}$, there must be some ξ_0 such that $\gamma[\xi_0](t_0) = x_0$, as a consequence of the intermediate value theorem and the above continuity property. Consider \bar{U} without the points on γ , which we denote by

$$U := \bar{U} \setminus \{(t, x) : \gamma(t) = x\}. \quad (5)$$

Next, we prove that $\gamma[\xi_0]$ is actually unique if $(t_0, x_0) \in U$, so that we may regard $\xi(t_0, x_0)$ as the canonical representation of this unique trajectory $\gamma[\xi(t_0, x_0)]$.

Property 2. *For $(t_0, x_0) \in U$, if $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0) = x_0$, then $\gamma[\xi_1] = \gamma[\xi_2]$.*

Proof. Suppose $t_0 < \xi_i$, then $x_0 = \gamma[\xi_i](t_0) = \gamma(t_0)$ contradicts the assumption $(t_0, x_0) \in U$. Therefore, assume $\xi_1 \leq \xi_2 < t_0$, without loss of generality. Since $\gamma[\xi_1] = \gamma[\xi_2]$ on $[a, \xi_1]$, note that we have the lower bounds

$$\gamma[\xi_1] \leq \gamma[\xi_2] \quad \text{and} \quad \dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]. \quad (6)$$

We must have $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, because otherwise $\gamma[\xi_1] > \gamma[\xi_2]$ somewhere in a sufficiently small neighborhood of t_0 , which contradicts the first lower bound.

It is clear from Definition 1 that

$$\ddot{\gamma}[\xi_i](t) = \begin{cases} \ddot{\gamma}(t) & \text{for } t < \xi_i, \\ -\omega & \text{for } t \in (\xi_i, \xi_i + \dot{\gamma}(\xi_i)/\omega), \\ 0 & \text{for } t > \xi_i + \dot{\gamma}(\xi_i)/\omega, \end{cases}$$

for both $i \in \{1, 2\}$. Note that $\dot{\gamma}(\xi_1) - \omega(\xi_2 - \xi_1) \leq \dot{\gamma}(\xi_2)$, which can be rewritten as

$$\xi_2 + \dot{\gamma}(\xi_2)/\omega \geq \xi_1 + \dot{\gamma}(\xi_1)/\omega.$$

This shows that $\ddot{\gamma}[\xi_1](t) \geq \ddot{\gamma}[\xi_2](t)$, for every $t \geq \xi_2$. Because $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, this in turn ensures that $\dot{\gamma}[\xi_1](t) \geq \dot{\gamma}[\xi_2](t)$ for $t \geq t_0$. Together with the opposite inequality in (6), we conclude that on $[t_0, \infty)$, we have $\dot{\gamma}[\xi_1] = \dot{\gamma}[\xi_2]$ and thus $\gamma[\xi_1] = \gamma[\xi_2]$.

It remains to show that $\gamma[\xi_1] = \gamma[\xi_2]$ on $[\xi_1, t_0]$, so consider the smallest $t^* \in (\xi_1, t_0)$ such that $\gamma[\xi_1](t^*) < \gamma[\xi_2](t^*)$. Since $\dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]$, this implies that $\gamma[\xi_1](t) < \gamma[\xi_2](t)$ for all $t \geq t^*$, but this contradicts the assumption $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0)$. \square

To further support the analysis below, we define the auxiliary function

$$g(t, x) := \dot{\gamma}_1[\xi(t, x)](t), \quad (7)$$

which gives the slope of the unique stopping trajectory through each point $(t, x) \in U$.

Property 3. *Function g is continuous in (t, x) .*

Proof. We write a neighborhood of x as $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$. We will write $f_x(\xi, t) = \gamma_1[\xi](t)$, $f_v(\xi, t) = \dot{\gamma}_1[\xi](t)$ and $h_t(\xi) = \gamma_1[\xi](t)$ to emphasize the quantities that we treat as variables. Observe that $h_t^{-1}(x) = X(t, x)$.

- Let $x_0 = f_x(\xi_0, \tau_0)$ and $v_0 = f_v(\xi_0, \tau_0)$ for some ξ_0 and τ_0 and pick some arbitrary $\varepsilon > 0$. Note that $\xi_0 \in [\xi_1, \xi_2] := h_{\tau_0}^{-1}(x_0)$. We apply the ε - δ definition of continuity to each of these endpoints. Let $i \in \{1, 2\}$, then there exist $\delta_i > 0$ such that

$$\xi \in N_{\delta_i}(\xi_i), \tau \in N_{\delta_i}(\tau_0) \implies f_v(\xi, \tau) \in N_\varepsilon(v_0).$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and define $N_1 := (\xi_1 - \delta, \xi_2 + \delta)$ and $N_2 := N_\delta(\tau_0)$, then

$$\xi \in N_1, \tau \in N_2 \implies f_v(\xi, \tau) \in N_\varepsilon(v_0).$$

This is obvious when ξ is chosen to be in one of $N_{\delta_i}(\xi_i)$. Otherwise, we must have $\xi \in [\xi_1, \xi_2]$, in which case $f_v(\xi, \tau) = f_v(\xi_1, \tau) \in N_\varepsilon(v_0)$.

- Because $h_{\tau_0}(\xi)$ is continuous, the image $I := h_{\tau_0}(N_1)$ must be an interval containing x_0 , with $\inf I = h_{\tau_0}(\xi_1 - \delta)$ and $\sup I = h_{\tau_0}(\xi_2 + \delta)$. We argue that I contains x_0 in its interior. For sake of contradiction, suppose $x_0 = \max I$, then $h_{\tau_0}(\xi_2 + \delta') = x_0$, for each $\delta' \in (0, \delta)$, because h_{τ_0} is non-decreasing, but this contradicts the definition of ξ_2 . Similarly, when $x_0 = \min I$, then $h_{\tau_0}(\xi_1 - \delta') = x_0$, for each $\delta' \in (0, \delta)$, which contradicts the definition of ξ_1 .
- Define $\nu := \min\{x_0 - \inf I, \sup I - x_0\}$ and $N_3 := (x_0 - \nu/2, x_0 + \nu/2)$. Because $h_\tau(\xi)$ is also continuous in τ , there exists a neighborhood $N_2^* \subset N_2$ of τ_0 such that for every $\tau \in N_2^*$, we have

$$\begin{aligned} h_\tau(\xi_1 - \delta) &\leq h_{\tau_0}(\xi_1 - \delta) + \nu/2 = \inf I + \nu/2 < x_0 - \nu/2, \\ h_\tau(\xi_2 + \delta) &\geq h_{\tau_0}(\xi_2 + \delta) - \nu/2 = \sup I - \nu/2 > x_0 + \nu/2, \end{aligned}$$

which shows that $h_\tau(N_1) \supset N_3$. It follows that $h_\tau^{-1}(N_3) \subset N_1$.

- Finally, take any $\tau \in N_2^*$ and $x \in N_3$, then there exists some $\xi \in N_1$ such that $h_\tau(\xi) = x$ and $g(\tau, x) = f_v(\max h_\tau^{-1}(x), \tau) = f_v(\xi, \tau) \in N_\varepsilon(v_0)$. \square

Property 4. *Function g is non-decreasing and Lipschitz continuous in x .*

Proof. Let $x_1 \leq x_2$ and τ such that $g(\tau, x_1)$ and $g(\tau, x_2)$ are defined. There must be $\xi_1 \leq \xi_2$ such that $h_\tau(\xi_1) = x_1$ and $h_\tau(\xi_2) = x_2$ and we have

$$\begin{aligned} g(\tau, x_1) &= \dot{\gamma}_1[\xi_1](\tau) = \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} = \dot{\gamma}_1[\xi_2](\tau) = g(\tau, x_2). \end{aligned}$$

Furthermore, we have $\dot{\gamma}_1(\xi_2) \leq \dot{\gamma}_1(\xi_1) + \bar{\omega}(\xi_2 - \xi_1)$, so that

$$\begin{aligned} g(\tau, x_2) &= \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) + \bar{\omega}(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1) + (\omega + \bar{\omega})(\xi_2 - \xi_1)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} + (\omega + \bar{\omega})(\xi_2 - \xi_1) \\ &= g(\tau, x_1) + (\omega + \bar{\omega})(\xi_2 - \xi_1). \end{aligned}$$

Observe that, together with the above non-decreasing property, this shows that g is Lipschitz continuous in x , with Lipschitz constant $(\omega + \bar{\omega})$. \square

We are now ready to state and prove our central technical result.

Lemma 1. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be two trajectories that are intersecting at exactly one time t_c and assume $\dot{\gamma}_1(t_c) > \dot{\gamma}_2(t_c)$, then under the conditions*

$$(C1) \quad \gamma_2 \geq \gamma_1[a_1],$$

$$(C2) \quad \dot{\gamma}_2(b_2) = 1,$$

there is a unique trajectory φ such that

- (i) $\varphi = \gamma_1[\xi]$, for some $\xi < t_c$,
- (ii) $\varphi(\tau) = \gamma_2(\tau)$ and $\dot{\varphi}(\tau) = \dot{\gamma}_2(\tau)$, for some $\tau > t_c$,
- (iii) $\varphi \leq \gamma_2$.

Proof. Define the set U and the functions $X(t, x)$ and $\xi(t, x)$ as we did in equations (3)–(5) for γ above, but now for γ_1 .

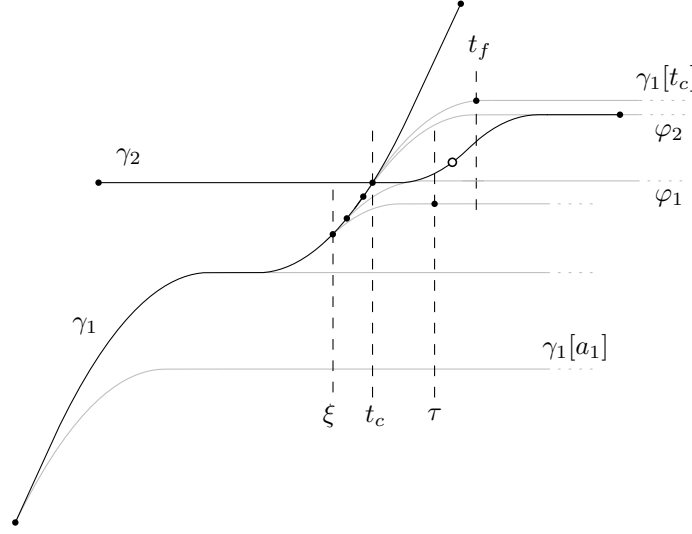


Figure 1: Sketch of some quantities used in the proof of Lemma 1, including some stopping trajectory candidates drawn in grey. The one satisfying the conditions of Lemma 1 is marked as φ_1 . The little open dot halfway on γ_2 is referred to in Remark 1.

- Identify for which parameters $\xi < t_c < \tau$ we have $\gamma_1[\xi](\tau) = \gamma_2(\tau)$ and $\dot{\gamma}_1[\xi](\tau) = \dot{\gamma}_2(\tau)$.
- For each $\tau > t_c$, observe that $(\tau, \gamma_2(\tau)) \in U$. It follows from Property 2 that $\varphi_{[\tau]} := \gamma_1[\xi(\tau, \gamma_2(\tau))]$ is the unique stopping trajectory such that $\varphi_{[\tau]}(\tau) = \gamma_2(\tau)$. Next, we investigate when this unique trajectory touches γ_2 tangentially. More precisely, consider the set of times

$$T := \{\tau > t_c : \dot{\varphi}_{[\tau]}(\tau) = \dot{\gamma}_2(\tau), \xi(\tau, \gamma_2(\tau)) < t_c\}, \quad (8)$$

which can also be written as

$$T = \{\tau > t_c : g(\tau, \gamma_2(\tau)) = \dot{\gamma}_2(\tau), \xi(\tau, \gamma_2(\tau)) < t_c\}, \quad (9)$$

so continuity of g shows that it is a closed set (Lemma A.1). It is not necessarily connected (see for example Figure 1, in which φ_1 and φ_2 touch γ_2 tangentially on separated intervals). In general, T is the union of a sequence of disjoint closed intervals T_1, T_2, \dots, T_n .

- Define $\tau_i := \min T_i$ and let $\varphi_i := \varphi_{[\tau_i]}$ denote the unique stopping trajectory through $(\tau_i, \gamma_2(\tau_i))$. For $\tau \in T_i$, we have $\dot{\gamma}_2(\tau) = g(\tau, \gamma_2(\tau))$ by definition of T_i . Moreover, we have

$$\dot{\varphi}_i(t) = g(t, \varphi_i(t)), \quad (10)$$

for every t for which these quantities are defined, so in particular on T_i . This shows that γ_2 and φ_i are both solutions to the initial value problem

$$\begin{cases} \dot{x}(t) = g(t, x(t)) & \text{for } t \in T_i, \\ x(\tau_i) = \gamma_2(\tau_i). \end{cases} \quad (11)$$

Since $g(t, x)$ is continuous in t and Lipschitz continuous in x , it is a consequence of the (local) existence and uniqueness theorem (Lemma A.2) that $\gamma_2 = \varphi_i$ on T_i . Hence, we have $\varphi_i = \varphi_{[\tau]}$ for any $\tau \in T_i$, so we regard φ_i as being the canonical stopping trajectory for T_i .

- Next, we show that τ_1 and thus φ_1 exists. This part relies on conditions (C1) and (C2). Note that $s(\tau) := g(\tau, \gamma_2(\tau))$ is a continuous function. We have $\gamma_1[a_1](t) \leq \gamma_2(t) \leq \gamma_1[t_c](t)$ for $t \in \{t_c, b_2\}$, so the intermediate value theorem guarantees that $s(t)$ actually exists in both cases, because there is some $a_1 \leq \xi < t_c$ such that $\gamma_2(t) = \gamma_1[\xi](t)$ and thus $s(t) = g(t, \gamma_2(t)) = \dot{\gamma}_1[\xi](t)$ exists. We have $\dot{\gamma}_2(t_c) < \dot{\gamma}_1(t_c) = s(t_c)$ and $\dot{\gamma}_2(b_2) = 1 \geq s(b_2)$, so there must be some smallest $\tau_1 \in (t_c, b_2]$ such that $\dot{\gamma}_1(\tau_1) = s(\tau_1)$, as a consequence of the intermediate value theorem.

- Suppose $\gamma_2(t_f) \leq \gamma_1[t_c](t_f)$, then it follows from the fact that g is non-decreasing in x that $g(t_f, \gamma_2(t_f)) \leq g(t_f, \gamma_1[t_c](t_f)) = \dot{\gamma}_1(t_c) - \omega(t_f - t_c) = 0$, so $s(t_f) = 0$.
- Otherwise $\gamma_2(t_f) > \gamma_1[t_c](t_f)$, then it follows (from Lemma ...) that γ_2 crosses $\gamma_1[t_c]$ at some time $t_d \in (t_c, t_f)$ with $\dot{\gamma}_2(t_d) > \gamma_1[t_c](t_d) = s(t_d)$.
- We have $\gamma_1[a_1](t) \leq \gamma_2(t) \leq \gamma_1[t_c](t)$ for $t \in \{t_f, t_d\}$, so the intermediate value theorem guarantees that $s(t)$ actually exists in both cases, because there is some $a_1 \leq \xi < t_c$ such that $\gamma_2(t) = \gamma_1[\xi](t)$ and thus $s(t) = g(t, \gamma_2(t)) = \dot{\gamma}_1[\xi](t)$ exists.
- In both cases above, we have $\dot{\gamma}_2(t_c) < \dot{\gamma}_1(t_c) = s(t_c)$ and $\dot{\gamma}_2(t_d) \geq s(t_d)$ for some $t_d \in (t_c, t_f]$. Hence, there must be some smallest $\tau_1 \in (t_c, t_d]$ such that $\dot{\gamma}_2(\tau_1) = s(\tau_1)$, which is a consequence of the intermediate value theorem.

- If $i \geq 2$, then $\varphi_i > \gamma_2$ somewhere.

- Let $i \geq 1$, we show that $\varphi_{i+1}(t) > \gamma_2(t)$ for some t . Recall the lower bound property, so $\gamma_2(t) \geq \varphi_i(t)$ and $\dot{\gamma}_2(t) \geq \dot{\varphi}_i(t)$ for $t \geq \tau_i$. Define $\hat{\tau}_i := \max T_i$, such that $T_i = [\tau_i, \hat{\tau}_i]$, then by definition of T_i , there must be some $\delta > 0$ such that

$$\gamma_2(\hat{\tau}_i + \delta) > \varphi_i(\hat{\tau}_i + \delta), \quad (12)$$

since otherwise $\gamma_2 = \varphi_i$ on some open neighborhood of $\hat{\tau}_i$ and then also

$$\dot{\gamma}_2(t) = \dot{\varphi}_i(t) \stackrel{(10)}{=} g(t, \varphi_i(t)) = g(t, \gamma_2(t)), \quad (13)$$

which contradicts the definition of $\hat{\tau}_i$. Therefore, we have $\gamma_2(t) > \varphi_i(t)$ for all $t \geq \hat{\tau}_i + \delta$. For $t = \tau_{i+1}$, in particular, it follows that $\varphi_{i+1}(\tau_{i+1}) = \gamma_2(\tau_{i+1}) > \varphi_i(\tau_{i+1})$, which shows that $\varphi_{i+1} > \varphi_i$ on (ξ_i, ∞) , due to Property 2, but this means that $\varphi_{i+1}(\tau_i) > \varphi_i(\tau_i) = \gamma_2(\tau_i)$.

- If $\varphi_i > \gamma_2$ somewhere, then $i \geq 2$.
- Suppose $\varphi_i(t_x) > \gamma_2(t_x)$ for some $t_x \in (t_c, \tau_i)$, then there must be some $\tau_0 \in (t_c, t_x)$ such that $\gamma_2(\tau_0) = \varphi_i(\tau_0)$ and $\dot{\gamma}_2(\tau_0) < \dot{\varphi}_i(\tau_0)$. Note that this crossing must happen because we require $\xi_i < t_c$.
- Since $g(t, x)$ is non-decreasing in x , we have

$$s(t) = g(t, \gamma_2(t)) \leq g(t, \varphi_i(t)) = \dot{\varphi}_i(t), \quad (14)$$

for every $t \in [\tau_0, \tau_i]$ and at the endpoints, we have

$$s(\tau_0) = \varphi_i(\tau_0), \quad s(\tau_i) = \varphi_i(\tau_i). \quad (15)$$

Furthermore, observe that $\gamma_2(\tau_0) = \varphi_i(\tau_0)$ and $\gamma_2(\tau_i) = \varphi_i(\tau_i)$ require that

$$\int_{\tau_0}^{\tau_i} \dot{\gamma}_2(t) dt = \int_{\tau_0}^{\tau_i} \dot{\varphi}_i(t) dt. \quad (16)$$

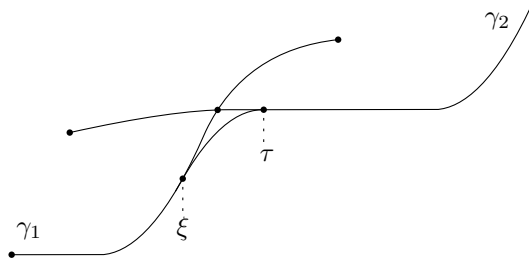


Figure 2: Two intersecting trajectories joined together by a part of a stopping trajectory.

- Since $\dot{\gamma}_2(\tau_0) < \dot{\varphi}_i(\tau_0)$, it follows from (16) that there must be some $t \in (\tau_0, \tau_i)$ such that $\dot{\gamma}_2(t) > \dot{\varphi}_i(t)$. Together with $s(\tau_0) = \dot{\varphi}_i(\tau_0) > \dot{\gamma}_2(\tau_0)$ and $s(t) \leq \dot{\varphi}_i(t)$ for $t \in [\tau_0, \tau_i]$, this means there is some τ^* such that $\dot{\gamma}_2(\tau^*) = s(\tau^*)$, again as a consequence of the intermediate value theorem. Therefore, $\tau^* \in T_j$ for some $j < i$, which shows that $i \geq 2$.
- The above two points establish that $\varphi_i \leq \gamma_2$ if and only if $i = 1$. To conclude, we have shown that $\varphi := \varphi_1$ exists and is the unique trajectory satisfying the stated requirements with $\tau = \tau_i$ and $\xi = \xi(\tau_i, \gamma_2(\tau_i))$. \square

Remark 1. It is easy to see that condition (C1) in Lemma 1 is necessary. Suppose there is some $t_x \in (t_c, \infty)$ such that $\gamma_1[a_1](t_x) > \gamma_2(t_x)$, then for any other $\xi \in (a_1, t_c)$, we have $\gamma_1[\xi](t_x) > \gamma_2(t_x)$ as well, due to the lower bound property of stopping trajectories, so requirement (iii) is violated. Condition (C2) is not necessary, which can be seen from stopping trajectory φ_1 in Figure 1, which satisfies the conditions, but would also have been valid if γ_2 ended somewhat earlier than t_f , for example until the open dot.

Suppose we have two trajectories that cross each other exactly once. Lemma 1 gives conditions under which, roughly speaking, these trajectories can be glued together to form a smooth trajectory by introducing a stopping trajectory in between, as illustrated in Figure 2. The above discussion motivates and justifies the following definition.

Definition 2. Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ and suppose they intersect at exactly a single time t_c . We write $\gamma_1 * \gamma_2$ to denote the unique trajectory

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t < \tau, \\ \gamma_1[\xi](t) & \text{for } t \in [\tau, \xi], \\ \gamma_2(t) & \text{for } t > \xi, \end{cases} \quad (17)$$

satisfying $\gamma_1 * \gamma_2 \in \mathcal{D}[a_1, b_2]$, where τ and ξ are as given by Lemma 1. If γ_1 and γ_2 are intersecting tangentially, so $\dot{\gamma}_1(t_c) = \dot{\gamma}_2(t_c)$, then we define $\tau = \xi = t_c$.

Our main interest in $\gamma_1 * \gamma_2$ is due to the following two results.

Lemma 2. Let $\gamma_1 \in \mathcal{D}[a_1, b_2]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be such that $\gamma_1 * \gamma_2$ exists. All trajectories $\mu \in \mathcal{D}[a, b]$ that are such that $\mu \leq \min\{\gamma_1, \gamma_2\}$, must satisfy $\mu \leq \gamma_1 * \gamma_2$.

Proof. Write $\gamma := \gamma_1 * \gamma_2$ as a shorthand. We obviously have $\mu \leq \gamma$ on $[a_1, \xi] \cup [\tau, b_2]$, so consider the interval (ξ, τ) of the joining deceleration part. Suppose there exists some $t_d \in (\xi, \tau)$ such that $\mu(t_d) > \gamma(t_d)$. Because $\mu(\xi) \leq \gamma(\xi)$, this means that μ must intersect γ at least once in $[\xi, t_d]$, so let $t_c := \sup\{t \in [\xi, t_d] : \mu(t) = \gamma(t)\}$ be the latest time of intersection such that $\mu \geq \gamma$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{\mu}(t_v) > \dot{\gamma}(t_v)$, otherwise

$$\mu(t_d) = \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t) dt \leq \gamma(t_c) + \int_{t_c}^{t_d} \dot{\gamma}(t) dt = \gamma(t_d),$$

which contradicts our choice of t_d . Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{\mu}(t) \geq \dot{\mu}(t_v) - \omega(t - t_v) > \dot{\gamma}(t_v) - \omega(t - t_v) = \dot{\gamma}(t).$$

It follows that $\mu(\tau) > \gamma(\tau)$, which contradicts $\mu \leq \gamma_2$. \square

Lemma 3. *Given $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$, if $\gamma_1 * \gamma_2$ does not exist, then there is no $\mu \in \mathcal{D}[a_1, b_2]$ such that*

- $\mu \leq \min\{\gamma_1, \gamma_2\},$
- $\mu(a_1) = \gamma_1(a_1), \quad \mu(b_2) = \gamma_2(b_2),$
- $\dot{\mu}(a_1) = \dot{\gamma}_1(a_1), \quad \dot{\mu}(b_2) = \dot{\gamma}_2(b_2).$

Proof. Suppose μ satisfying all the above requirements exists, so there are some ξ and τ such that $\mu = \gamma_1, \dot{\mu} = \dot{\gamma}_1$ on $[a_1, \xi]$ and $\mu = \gamma_2, \dot{\mu} = \dot{\gamma}_2$ on $[\tau, b_2]$, then we show that $\gamma_1 * \gamma_2$ must exist.

Suppose that $\dot{\mu} = -\omega$ not on the whole of $[\xi, \tau]$, otherwise $\mu = \gamma_1 * \gamma_2$. \square

2 Optimal trajectories as joined upper bounds

Consider the set $D[a, b] \subset \mathcal{D}[a, b]$ of trajectories γ that satisfy the following additional constraints

$$\gamma(a) = A, \quad \gamma(b) = B, \quad \dot{\gamma}(a) = \dot{\gamma}(b) = 1, \quad (18)$$

for some fixed A, B such that

$$B - A \geq 1/(2\omega) + 1/(2\bar{\omega}). \quad (19)$$

For every such trajectory $\gamma \in D[a, b]$, we derive two upper bounding trajectories. First of all, let $x^1 \in D(-\infty, \infty)$ be defined as $x^1(t) = A + t - a$, then it is clearly an upper bound for any trajectory $\gamma \in D[a, b]$. Furthermore, note that we have $\dot{\gamma}(t) + \bar{\omega}(b - t) \geq \dot{\gamma}(b) = 1$, which can be rewritten to $\dot{\gamma}(t) \geq 1 - \bar{\omega}(b - t)$. Combined with $\dot{\gamma}(t) \geq 0$, this gives

$$\dot{\gamma}(t) \geq \max\{0, 1 - \bar{\omega}(b - t)\}. \quad (20)$$

Hence, we derive the upper bound $\hat{x} \in D(-\infty, b]$ from the inequality

$$\gamma(t) = \gamma(b) - \int_t^b \dot{\gamma}(\tau) d\tau \quad (21a)$$

$$\leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b - \tau)\} d\tau =: \hat{x}(t). \quad (21b)$$

Lemma 4. *Optimal control problem*

$$\max_{x \in D[a, b]} \int_a^b x(t) dt \quad (22)$$

has a solution if and only if

$$b - a \geq B - A, \quad (23)$$

*and the optimal solution is given by $x^1 * \hat{x}$.*

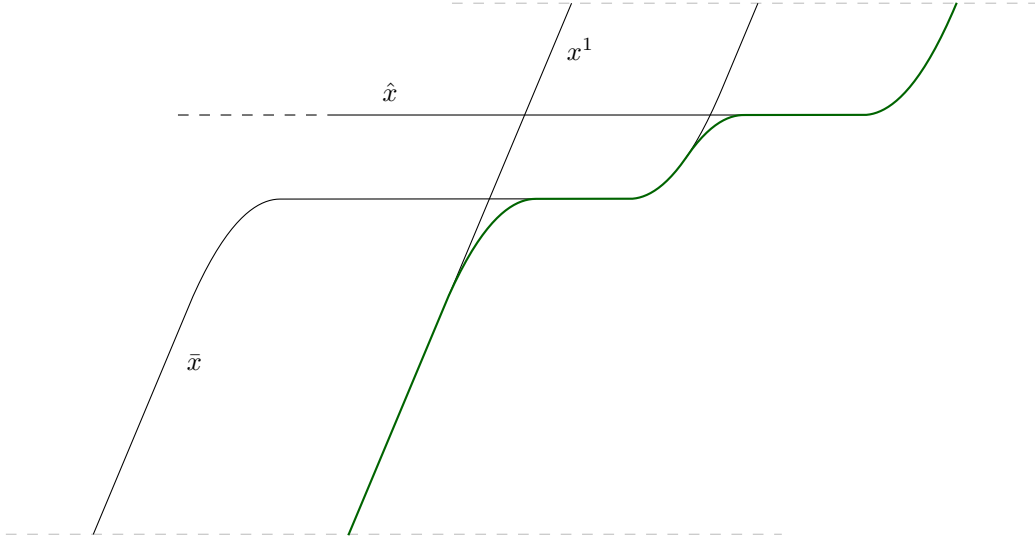


Figure 3: Sketch of how the three upper bounding trajectories are joined to form the optimal trajectory $x^1 * (\bar{x} * \hat{x})$ in Lemma 5.

Proof. Let $\gamma_1 = x^1$ and $\gamma_2 = \hat{x}$, then we show that the conditions of Lemma 1 are satisfied. Note that condition (C2) is always satisfied by definition of \hat{x} . It follows from the definition of x^1 and Definition 1 that $x^1[a] \leq A + 1/(2\omega)$. Similarly, the definition of \hat{x} ensures that $\hat{x} \geq B - 1/(2\bar{\omega})$. Therefore, by assumption (19) and (23), we have $\gamma_2 = \hat{x} \geq x^1[a] = \gamma_1[a]$, which shows that condition (C1) is satisfied. Since condition (C1) is also necessary (Remark 1), we may conclude that a solution exists if and only if $b - a \geq B - A$. It follows immediately from Lemma 2 that $x^1 * \hat{x}$ is an optimal solution if the problem is feasible, because any feasible solution x must satisfy $x \leq \min\{x^1, \hat{x}\}$. \square

We extend the previous lemma by considering an additional upper bound \bar{x} for trajectories, to model a *lead vehicle* driving in front of the current vehicle.

Lemma 5. *Let $\bar{x} \in \mathcal{D}[\bar{a}, \bar{b}]$, then optimal control problem*

$$\max_{x \in \mathcal{D}[a, b]} \int_a^b x(t) dt \quad \text{such that } x \leq \bar{x}. \quad (24)$$

*has a solution if and only if the joined trajectory $x^1 * (\bar{x} * \hat{x})$ exists. When a solution exists, this joined trajectory is also an optimal solution.*

Proof. Note that x^1 , \hat{x} and \bar{x} are all upper bounds for any feasible trajectory, so optimality of $x^1 * (\bar{x} * \hat{x})$ follows immediately from Lemma 2. \square

Definition 3. *Let $\gamma \in \mathcal{D}[a, b]$ be called alternating if there exists*

Lemma 6. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be both alternating, then when $\gamma_1 * \gamma_2$ exists, it is also alternating.*

Proof. \square

Remark 2. From the proof above, it also becomes clear that a connecting deceleration can only happen between the following four pairs of partial trajectories:

$$x^+ \rightarrow x^+, \quad x^+ \rightarrow x^0, \quad x^1 \rightarrow x^+, \quad x^1 \rightarrow x^0.$$

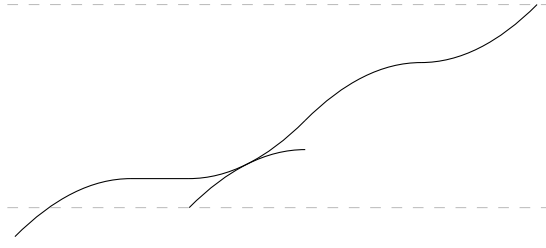


Figure 4: Illustration of buffer constraint.

A Miscellaneous

Lemma A.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be continuous and $y \in \mathbb{R}^m$, then the level set $N := f^{-1}(\{y\})$ is a closed subset of \mathbb{R}^n .*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. \square

Lemma A.2. *Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be some closed rectangle such that $(t_0, x_0) \in \text{int } D$. Let $f : D \rightarrow \mathbb{R}^n$ be a function that is continuous in t and globally Lipschitz continuous in x , then there exists some $\varepsilon > 0$ such that the initial value problem*

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{25}$$

has a unique solution $x(t)$ on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

The above existence and uniqueness theorem is also known as the Picard-Lindelöf or Cauchy-Lipschitz theorem. The above statement is based on the Wikipedia page on this theorem, so we still need a slightly better reference.