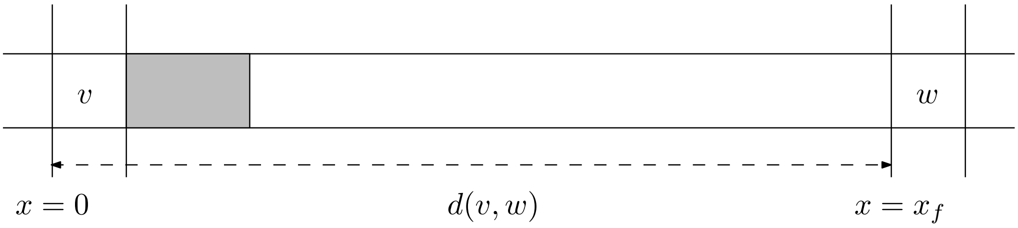


Figure 1: Tandem of two intersections v and w with lane of length $d(v, w)$. The grey rectangle represents some vehicle that just left intersection v, so it has maximum speed and is allowed to decelerate from here on.

1 Tandem of intersections

We need to take into account the fact that a lane between two consecutive intersections has finite capacity for vehicles, in order to model possible blocking effects, also known as *spillback*. The capacity of a lane is intimately related to trajectories of vehicles, which we first want to understand better. We have been using an optimal control formulation with the objective that keeps the vehicles as close as possible to the next intersection at all times (`MotionSynthesize`). This problem can be solved using direct transcription, which works well enough if we just want to simulate the behavior of the system. However, we believe that it is possible to explicitly formulate the optimal control. Below, we will explain how to compute trajectories corresponding to those obtained by direct transcription, but without using time discretization. Our approach may be considered an *ansatz*, because proving that this is indeed the optimal control would need to invoke some kind of sufficiency theorem and, perhaps more importantly, there are still some cases in which the implementation breaks down.

We start with the simplest extension of the single intersection model by considering two intersections in tandem, as illustrated in Figure 1. Let v denote the left intersection and w the right intersection and assume that vehicles drive from left to right. Let the length and width of a vehicle i be denoted by L_i and W_i, respectively. We measure the position of a vehicle at the front bumper and we let position $x = 0$ be at the stopline of intersection v. We denote the position of the stopline at w by $x = x_f = d(v, w)$. To simplify the following discussion, we assume that all vehicles have the same dimensions.

Assumption 1. *All vehicles have the same length $L_i = L$ and width $W_i = W$.*

Now assume that some vehicle is scheduled to cross v at time $t = 0$ and to cross w at some time t_f. Let $y(t)$ denote the trajectory of the predecessor, assuming there is one. In order to keep the vehicle as close to w as possible at every time, we can generate a trajectory by solving the optimal control problem

$$\begin{aligned} \max_{x} \quad & \int_{t=0}^{t_f} x(t)dt \\ \text{s.t.} \quad & 0 \leq \dot{x}(t) \leq v_{\max}, \\ & -a_{\max} \leq \ddot{x}(t) \leq a_{\max}, \\ & y(t) \leq x(t), \\ & x(0) = 0, \qquad x(t_f) = d(v, w), \\ & \dot{x}(0) = v_{\max}, \ \ \dot{x}(t_f) = v_{\max}. \end{aligned} \tag{1}$$

We believe that the optimal control $u(t) := \ddot{x}(t)$ satisfies $u(t) \in \{-a_{\max}, 0, a_{\max}\}$ and there is a sequence of alternating deceleration and acceleration periods, meaning that there is a sequence of disjoint intervals

$$(D_1, A_1, \ldots, D_n, A_n),$$

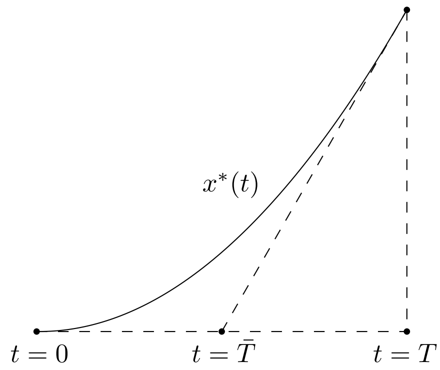


Figure 2: Full acceleration trajectory.

so that the optimal controller is given by

$$u(t) = \begin{cases} -a_{\max} & \text{if } t \in D_k \text{ for some } k, \\ a_{\max} & \text{if } t \in A_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

1.1 Single vehicle

We will start by assuming that the system contains a single vehicle, so that we do not have to worry about keeping a safe distance to vehicle in front of it. We will show how the bang-bang intervals, or *bangs* for short, can be obtained in this situation. Without considering the boundary conditions, it follows from the vehicle dynamics that the time it takes to fully accelerate from zero to maximum velocity is given by

$$T = v_{\max}/a_{\max},$$

with corresponding trajectory x^*, given by

$$x^*(t) = a_{\max}t^2/2 \quad \text{for } 0 \leq t \leq T,$$

as illustrated in Figure 2. First, we introduce a transformation that will prove to be helpful. For position x at time t, we define the corresponding *schedule time* by

$$\bar{t}(t, x) := t - x/v_{\max}.$$

This induces equivalence classes in time-position space, corresponding to lines with slope $v_{\max}$. In the following, we will use a bar above a symbol when dealing with schedule time. For example, time duration T translated to schedule time, is given by

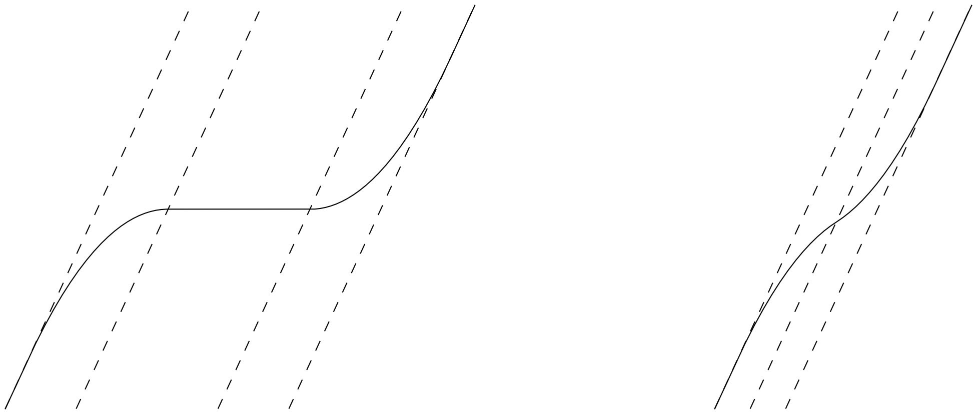
$$\bar{T} = \bar{t}(T, x^*(T)) - \bar{t}(0, 0) = T/2.$$

The crossing time of v and w in schedule time are given by $b = \bar{t}(0, 0)$ and $e = \bar{t}(t_f, x_f)$, respectively. Whenever t_f is sufficiently large, it is clear that we need a full deceleration bang and a full acceleration bang. Therefore, in schedule time, this would take $2\bar{T}$ time. However, for smaller t_f, time of deceleration and acceleration need to decrease equally. Therefore, writing $(x)^+$ for $\max(x, 0)$, the remaining amount of schedule time in which the vehicle is stopped is given by

$$\bar{d}_s = (e - b - 2\bar{T})^+$$

and the length of each bang is

$$\bar{d}_b = (e - b - \bar{d}_s)/2,$$



(a) Positive waiting time $\bar{t}_s > 0$. (b) Vehicle does not stop.

Figure 3: Shape of optimal trajectories $x(t)$ for a single (isolated) vehicle, in case the vehicle stops and waits for some time (a) and in case the vehicle does never come to a full stop (b). In each case, the deceleration bang $\bar{D}$ and acceleration bang $\bar{A}$ in schedule time are indicated by the dashed lines.

so that the bangs are given by

$$\bar{D} = (b, b + \bar{d}_b),$$
$$\bar{A} = (e - \bar{d}_b, e).$$

The corresponding trajectory are shown in Figure 3.

Because the control $u(t)$ is not specified in terms of schedule scale, we need to invert these bangs back to regular time. Let v_c denote the current velocity at the start of the bang and let $\bar{d}$ denote the duration of the bang in schedule time. We first consider the acceleration bang. Define $t_0 = v_c/a_{\max}$ so that we have $\dot{x}^*(t_0) = v_c$. Next, we find t_1 such that $t_0 \leq t_1 \leq T$ and

$$\bar{t}(t_1, x^*(t_1)) - \bar{t}(t_0, x^*(t_0)) = \bar{d}, \tag{2}$$

such that the duration of the bang in regular time is given by $d = t_1 - t_0$. After some rewriting and substitution of the definitions of x^* and $\bar{t}$ in equation (2), we obtain the quadratic equation

$$-\frac{a_{\max} t_1^2}{2v_{\max}} + t_1 - t_0 + \frac{a_{\max} t_0^2}{2v_{\max}} - \bar{d} = 0,$$

for which we are interested in the solution

$$t_1 = T - \sqrt{T^2 - 2T(t_0 + \bar{d}) + t_0^2}.$$

Similarly, we derive that the duration of a deceleration bang is given by $d = t_1 - t_0$ where $t_0 = (v_{\max} - v_c)/a_{\max}$ and t_1 is the solution to

$$\bar{t}(t_1, -x^*(-t_1)) - \bar{t}(t_0, -x^*(-t_0)) = \bar{d},$$

given by

$$t_1 = -T + \sqrt{T^2 + 2T(t_0 + \bar{d}) + t_0^2}.$$

We can now calculate the duration of the bangs in schedule time, but this does not yet fix their time in a unique way. Due to the definition of schedule time, we need to be careful whenever the velocity is $v_{\max}$, because the regular time is not unique in this case. Therefore, we specify a *target position* x_t such that the start of the deceleration bang is given by

$$b + (x_t - x^*(T))/v_{max}.$$

This ensures that the vehicle has enough distance to w to accelerate, in case it stops.

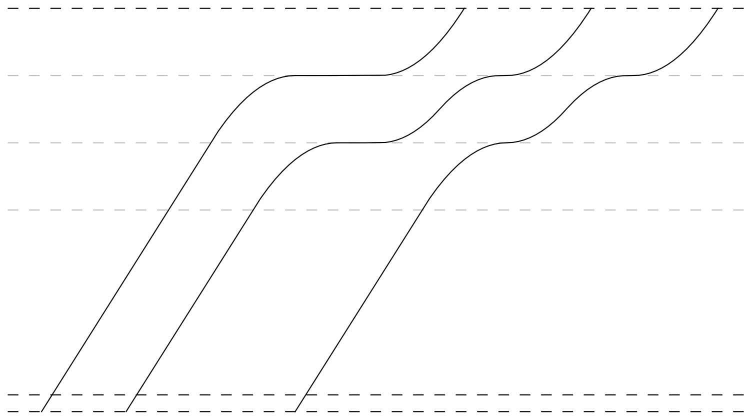


Figure 4: Optimal trajectories displaying the start-stop queueing behavior. The waiting positions are indicated by the grey dashed lines. The bottom two horizontal dashed lines represent intersection v.

1.2 Multiple vehicles

We will now consider the case when the system contains multiple vehicles. Now, we need to take into account the safe following constraints. This also causes additional *buffer constraints* to the crossing time scheduling problem, since the lane only provides space for a limited number of vehicles, which we investigate in Section 1.3. For now, assume we have a feasible crossing time schedule, that satisfies the buffer constraints.

Consider the optimal solution in Figure 4. Observe that all vehicles undergo some start-stop phases. Furthermore, there are some fixed positions where vehicles will come to a full stop, to which we refer as *waiting positions*. Recall that L is the minimal distance between two consecutive vehicles. From a waiting position, we move to the next waiting position that is exactly L units further on the lane. We will now describe such a start-stop trajectory of a single vehicle, without considering safe following constraints, see Figure 5. By symmetry of the control constraints, the vehicle moves the same distance during both acceleration and deceleration. Furthermore, we need zero velocity at the start and end of such trajectory. Hence, it is clear that the duration d_n of acceleration and deceleration need to be the same. Assume for now that full acceleration/deceleration is not required, which is the case whenever we have $L < 2x^*(T)$. Therefore, d_n must satisfy $L = 2x^*(d_n)$, so we obtain

$$d_n = \sqrt{L/a_{\max}},$$

which corresponds to a duration in schedule time of

$$\bar{d_n} = \bar{t}(t_m, L/2) = t_m - \frac{L}{2v_{\max}}.$$

We will now show how to construct the sequence of bangs in schedule time, given the sequence of bangs in schedule time of the preceding vehicle. In general, we observe that the the bangs of the preceding vehicle introduce additional pairs of bangs corresponding to start-stop trajectories, apart from the first deceleration and last acceleration that were also present in the single isolated vehicle case.

We only need the start of the acceleration bangs of the preceding vehicle in schedule time, which we denote by $\bar{t}_1, \ldots, \bar{t}_n$. For every $\bar{t}_i$, we introduce a pair of start-stop bangs at time $\bar{t}_i + L/v_{\max}$. Furthermore, we have an initial deceleration at $\bar{D}_0 = [a, a + \bar{T}]$ and the final acceleration at $\bar{A}_f = [y - \bar{T}, y]$. This results in some sequence of bangs in schedule time

$$(\bar{D}_0, \bar{A}_1, \bar{D}_1, \ldots, \bar{A}_n, \bar{D}_n, \bar{A}_f).$$

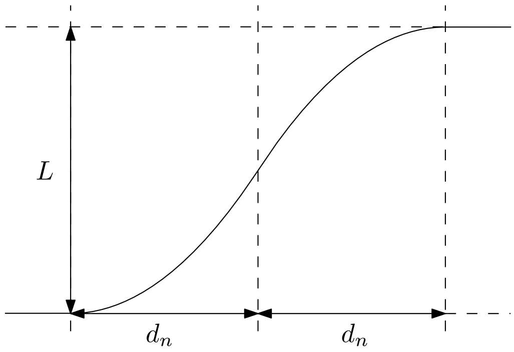


Figure 5: Shape of the start-stop trajectory of a single isolate vehicle moving forward from the current waiting position to the next waiting position.

Note that these bangs are not necessarily disjoint, so we *merge* them in the following simple way. Merging two intervals $[t_0, t_1]$ and $[t_2, t_3]$ with $t_1 > t_2$, results in the intervals $[t_0, t_m]$ and $[t_m, t_3]$ with $t_m = (t_1 + t_2)$.

Now that we have a sequence of disjoint bangs, we still need to translate them back to the regular time scale, which can be done using the formulas from Section 1.1. We assume that the velocity never reaches $v_{\max}$. If this assumption does not hold, the whole trajectory could be "shifted up" in the graph, which would decrease the objective, so the trajectory would not be optimal. We keep track of the current time t_c and velocity v_c. For each next bang $\bar{A}$ or $\bar{D}$ in the sequence, we compute the duration d in regular time and update $v_c \leftarrow v_c \pm a_{\max} d$ accordingly and set $t \leftarrow t + d$. This way, we obtain the sequence of bangs in regular time, which completely defines the controller.

1.3 Lane capacity

Suppose that we want to design the tandem network such that at least $c(v, w)$ vehicles can enter and decelerate to some waiting position, from which it is also possible to accelerate again to full speed before crossing w. Vehicles are required to drive at full speed $v = v_{\max}$ as long as they occupy any intersection. Therefore, a vehicle crossing v can only start decelerating after $x(t) \geq L + W$, so the earliest position where a vehicle can come to a stop is $x = L + W + x^*(T)$. Because vehicles need to gain maximum speed $v = v_{\max}$ before reaching w, the latest position where a vehicle can wait is $x_f - x^*(T)$. Hence, in order to accomodate for $c(v, w)$ waiting vehicles, the length of the lane must satisfy

$$d(v, w) \geq L + W + 2x^*(T) + (c(v, w) - 1)L, \tag{3}$$

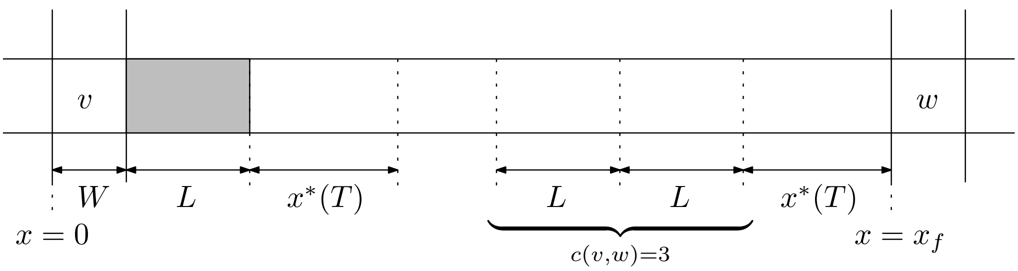


Figure 6: Tandem of intersections with indicated distances used in the capacity calculation.

as illustrated in Figure 6. Conversely, given the lane length $d(v, w)$, the corresponding capacity is given by

$$c(v, w) = \texttt{floor}\left(\frac{d(v, w) - W - 2x^*(T)}{L}\right), \tag{4}$$

where $\texttt{floor}(x)$ denotes the largest integer smaller than or equal to x.

Remark 1. *Without Assumption 1, we cannot derive such a simple expression fo the capacity, because it would depend on the specific combination of lengths of the vehicles that arrived to the system.*

In order to guarantee feasible trajectories, we need to add capacity constraints to the crossing time scheduling problem. We are almost sure that it is *sufficient* to add constraints of the type illustrated in Figure 7. Recall that $\rho = L/v_{\max}$ denotes the minimum time between two crossing times of vehicles on the same lane. For every pair of vehicles $i, j \in \mathcal{N}(l)$ such that $k(i) + c(v, w) = k(j)$, we include the constraint

$$\bar{t}(y(i, w), d(v, w)) + c(v, w) \leq \bar{t}(y(j, v), 0),$$

which after substituting the definitions yields

$$y(i, w) - \frac{d(v, w)}{v_{\max}} + c(v, w)\rho \leq y(j, v).$$

However, we are not sure whether these constraints are *necessary*, i.e., there might be situations in which vehicle j is able to cross v slightly earlier due to the shape of the trajectories.

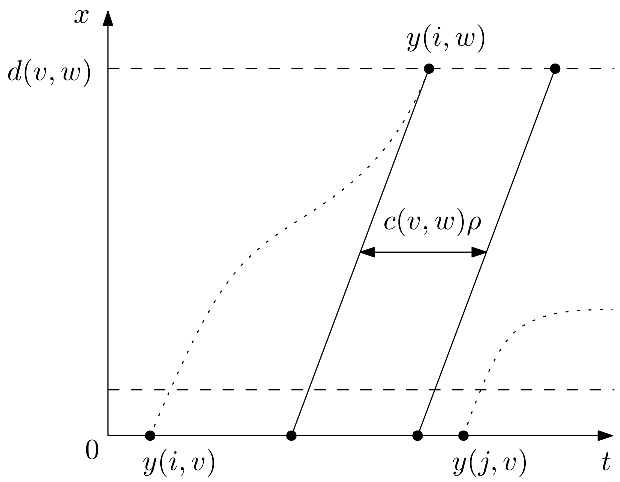


Figure 7: Illustration of the buffer constraint between $y(i, w)$ and $y(j, v)$. The slope of the two parallel lines is $v_{\max}$, so they correspond to the schedule times. The two dotted lines are examples of trajectories $x_i(t)$ and $x_j(t)$.

2 Optimal control with state constraints

The general form of an optimal control problem with mixed and pure state inequality constraints is given by

$$
\begin{aligned}
\max \quad & \int_{t=0}^{T} F(x(t), u(t), t) dt \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \\
& g(x(t), u(t), t) \geq 0, \\
& h(x(t), t) \geq 0, \\
& a(x(T), T) \geq 0, \\
& b(x(T), T) = 0.
\end{aligned} \tag{5}
$$

We will redefine problem (1) to fit this notation. In this section, $x(t) = (x_1(t), x_2(t))$ is the *state*, which consists of both the position $x_1(t)$ and velocity $x_2(t)$. The *control* is some function $u(t)$, which corresponds to acceleration in our particular problem. Without loss of generality, we assume that $v_{\max} = 1$. It will be more convenient to consider the start position $p_0 = -d(v, w)$, so that we have initial state $(p_0, 1)$ and target state $(0, 1)$. Writing $y(t)$ for the state trajectory of the vehicle in front of the current vehicle, optimal control problem (1) is equivalent to (5) when setting

$$
\begin{aligned}
F(x, u, t) &= x_1, \\
f(x, u, t) &= (x_2, u), \\
x_0 &= (p_0, 1), \\
g(x, u, t) &= u_{\max}^2 - u^2, \\
h_1(x, t) &= x_2 - x_2^2, \\
h_2(x, t) &= y_1(t) - x_1, \\
b(x, t) &= (x_1, x_2 - 1).
\end{aligned} \tag{6}
$$

Instead of using quadratic expressions, both the mixed and the pure state constraint could have each been written as two linear constraints, e.g., we could have chosen $g(x, u, t) = (u_{\max} - u, u_{\max} + u)$. However, this does not satisfy the constraint qualification conditions of the theorem that we would like to apply.

2.1 No headway constraint: indirect approach

Before we analyze the general form of optimal controls of problem (5–6), we consider the situation in which the headway constraint h_2 can be ignored, which might occur when the preceding vehicle is sufficiently far away, or the current vehicle is the only vehicle in the system.

When dealing with optimal control problems, the Pontryagin Maximum Principle (PMP) provides a family of first-order necessary conditions for optimality. We are going to apply such a PMP-style necessary condition that deals with mixed and pure state constraints to characterize the optimal control. A common approach to deal with pure state inequality constraints h is to introduce a multiplier η and append ηh to the Hamiltonian, which is known as the direct adjoining approach. Instead, we use Theorem 5.1 from [2] (see also (4.29) in [1]), which is a so-called indirect adjoining approach, because the ηh^1 is appended to the Hamiltonian, where h^1 is defined as

$$
h^1 = \frac{dh}{dt} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t}.
$$

For our specific problem, we have

$$
h_1^1(x, u, t) = u - 2x_2 u,
$$

which shows that pure constraint h_1 is of first-order, because the control u appears in this expression. By adjoining $\mu g + \eta h_1^1$ to the Hamiltonian

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t) = x_1 + \lambda_1 x_2 + \lambda_2 u,$$

we obtain the Lagrangian in so-called Pontryagin form

$$\begin{aligned} L(x, u, \lambda, \mu, \eta, t) &= H(x, u, \lambda, t) + \mu g(x, u, t) + \eta h_1^1(x, u, t) \\ &= x_1 + \lambda_1 x_2 + \lambda_2 u + \mu(u_{\max}^2 - u^2) + \eta(u - 2x_2 u). \end{aligned}$$

Let $\{x^*(t), u^*(t)\}$ denote an optimal pair for problem (5–6), then this pair must satisfy the necessary conditions in Theorem 5.1 from [2]. The Hamiltonian maximizing condition is

$$H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u, \lambda(t), t)$$

at each $t \in [0, T]$ for all u with $g(x^*(t), u, t) \geq 0$ and satisfying

$$h_1^1(x^*(t), u, t) \geq 0 \quad \text{whenever} \quad h_1(x^*(t), u, t) = 0.$$

In our case, the latter condition reads

$$u - 2x_2^* u \geq 0 \quad \text{whenever} \quad x_2^* = \pm 1,$$

which is satisfied by optimal controllers satisfying

$$u^*(t) = \begin{cases} u_{\max} & \text{when } \lambda_2(t) > 0,\ x_2^*(t) < 1, \\ -u_{\max} & \text{when } \lambda_2(t) < 0,\ x_2^*(t) > -1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We now investigate what the costate trajectory should look like. The adjoint equation is given by

$$\dot{\lambda} = -L_x(x^*, u^*, \lambda, \mu, \eta, t) \iff \begin{cases} \dot{\lambda}_1 = -1, \\ \dot{\lambda}_2 = -\lambda_1 + 2\eta u^*. \end{cases}$$

The Lagrange multipliers $\mu(t)$ and $\eta(t)$ must satisfy the complementary slackness conditions

$$\begin{aligned} \mu(u_{\max}^2 - (u^*)^2) &= 0, \quad \mu \geq 0, \\ \eta(x_2 - x_2^2) &= 0, \quad \eta \geq 0 \ \text{ and } \ \dot{\eta} \leq 0. \end{aligned}$$

Whenever $u^* \neq 0$ on any open interval, then it is clear that we must have $x_2 \neq \pm 1$ on that interval, so the second complementary slackness condition requires that we have $\eta = 0$. At isolated points where $u^* \neq 0$, we can simply choose to set $\eta = 0$. Hence, $\eta u^* = 0$ everywhere and we have $\dot{\lambda}_2(t) = -\lambda_1(t) = t + c_1$ for all $t \in [0, T]$, such that $\lambda_2(t) = \frac{1}{2}t^2 + c_1 t + c_2$ for some constants c_1, c_2. In view of (7), this shows that $u^*(t)$ has at most one period of deceleration and at most one period of acceleration. Assume $\lambda_2(t)$ has two zeros, which we denote by t_d and t_a, which give the start of the deceleration period and the start of the acceleration period, respectively.

We now show how to compute $t_a - t_d$. Let D denote the duration of the deceleration, which must of course equal the length of the acceleration, because $x_2(0) = x_2(T)$. Furthermore, D must satisfy both $D \leq t_a - t_d$ and $D \leq 1/u_{\max}$ and can thus be defined as the smallest of these two bounds. From the vehicle dynamics follows that, for all $t \in [t_d, t_d + D]$ in the deceleration period, we must have

$$x_2^*(t) = 1 - u_{\max}(t - t_d)$$

and by integration, we obtain

$$x_1^*(t) = x_1^*(t_d) + (t - t_d) - \frac{u_{\max}}{2}(t - t_d)^2.$$

Similarly, for all $t \in [t_a, t_a + D]$ in the acceleration period, we must have

$$x_2^*(t) = u_{\max} t$$

and by integration, we obtain

$$x_1^*(t) = x_1^*(t_a) + \frac{u_{\max}}{2}(t - t_a)^2.$$

Because the vehicle is stationary in $[t_d + D, t_a]$, we have $x_1^*(t_a) = x_1^*(t_d + D)$. It is now easily verified that we have

$$x_1^*(T) = p_0 + T + t_d - t_a.$$

Recall that $x_1^*(T) = 0$, so we obtain $t_a - t_d = p_0 + T$.

We now show that t_a is uniquely determined. From the complementary slackness conditions, we derive that

$$\begin{aligned}
\mu(t) &= 0 \quad \text{for } t \in (0, t_d), \\
\eta(t) &= 0 \quad \text{for } t \in (t_d, t_d + D), \\
\mu(t) &= 0 \quad \text{for } t \in (t_d + D, t_a), \\
\eta(t) &= 0 \quad \text{for } t \in (t_a, t_a + D), \\
\mu(t) &= 0 \quad \text{for } t \in (t_a + D, T).
\end{aligned}$$

Now together with the condition

$$\frac{\partial L}{\partial u}|_{u=u^*(t)} = 0 \iff \lambda_2 - 2\mu u^* - 2\eta x_2 + \eta = 0,$$

this shows that we can choose μ and η such that $\mu(t) \geq 0$ and $\eta(t) \geq 0$ and

$$\begin{aligned}
\lambda_2(t) &= \eta(t) \quad \text{for } t \in (0, t_d), \\
\lambda_2(t) &= -2\mu(t) \quad \text{for } t \in (t_d, t_d + D), \\
\lambda_2(t) &= -\eta(t) \quad \text{for } t \in (t_d + D, t_a), \\
\lambda_2(t) &= 2\mu(t) \quad \text{for } t \in (t_a, t_a + D), \\
\lambda_2(t) &= \eta(t) \quad \text{for } t \in (t_a + D, T).
\end{aligned}$$

Suppose $t_a + D < T$, then for $t \in (t_a + D, T)$, we have $\dot{\eta}(t) = \dot{\lambda}_2(t) > 0$, which contradicts the necessary condition $\dot{\eta}(t) \leq 0$, hence $t_a = T - D$. This fixes a unique costate trajectory $\lambda_2(t)$ and hence a unique optimal controller $u^*(t)$ given by (7).

2.2 No headway constraint: direct approach

Before we proceed with including the headway constraint, we also perform the direct approach.

For t in boundary interval (so where $h_1 = 0$), we have

$$\begin{aligned}
\eta^d(t) &= -\dot{\eta}(t), \\
\lambda^d(t) &= \lambda(t) + \eta(t) h_x(x^*(t), t).
\end{aligned}$$

For $(0, t_d)$, we have $x_2^* = 1$ and $\eta = \lambda_2$, so

$$\begin{aligned}
\eta^d(t) &= -\dot{\lambda}_2(t) = \lambda_1(t), \\
\lambda^d(t) &= \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \lambda_2(t) \begin{pmatrix} 0 \\ 1 - 2x_2^* \end{pmatrix} = \begin{pmatrix} \lambda_1(t) \\ 0 \end{pmatrix}.
\end{aligned}$$

For $(t_d, t_d + D)$, we have $\eta^d(t) = \eta(t) = 0$.
For $(t_d + D, t_a)$, we have $x_2^* = 0$ and $\eta = -\lambda_2$, so

$$\begin{aligned} \eta^d(t) &= \dot{\lambda}_2(t) = -\lambda_1(t), \\ \lambda^d(t) &= \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} - \lambda_2(t) \begin{pmatrix} 0 \\ 1 - 2x_2^* \end{pmatrix} = \begin{pmatrix} \lambda_1(t) \\ 0 \end{pmatrix}. \end{aligned}$$

For $(t_a, t_a + D)$, we have $\eta^d(t) = \eta(t) = 0$.
Using the same Hamiltonian as before, we write the Lagrangian

$$\begin{aligned} L(x, u, \lambda^d, \mu, \eta^d, t) &= H(x, u, \lambda^d, t) + \mu g(x, u, t) + \eta^d h_1(x, t) \\ &= x_1 + \lambda_1^d x_2 + \lambda_2^d u + \mu(u_{\max}^2 - u^2) + \eta^d (x_2 - x_2^2). \end{aligned}$$

The adjoint equation is given by

$$\begin{cases} \dot{\lambda}_1^d = -1 \\ \dot{\lambda}_2^d = -\lambda_1^d - \eta^d + 2\eta^d x_2 \end{cases}$$

$$\frac{\partial L}{\partial u}|_{u=u^*(t)} = 0 \iff \lambda_2^d = 2\mu u^*$$

Complementary slackness conditions

$$\begin{aligned} \mu(u_{\max}^2 - (u^*)^2) = 0, &\quad \mu \geq 0, \\ \eta^d (x_2^* - (x_2^*)^2) = 0, &\quad \eta^d \geq 0. \end{aligned}$$

The Hamiltonian maximizing condition is

$$H(x^*(t), u^*(t), \lambda^d(t), t) \geq H(x^*(t), u, \lambda^d(t), t)$$

at each $t \in [0, T]$ for all u with $g(x^*(t), u, t) \geq 0$.

2.3 With headway constraint

We now return to the full problem (5–6) by considering the headway constraint h_2. This constraint is of second order, because the control u only appears in the second derivative, given by

$$h_2^2(x,t) = -u.$$

Because we have not found a clear indirect approach, we try to use the direct approach.

Using the same Hamiltonian as before

$$H(x,u,\lambda,t) = F(x,u,t) + \lambda f(x,u,t) = x_1 + \lambda_1 x_2 + \lambda_2 u$$

we write the Lagrangian

$$\begin{aligned} L(x,u,\lambda,\mu,\nu,t) &= H(x,u,\lambda,t) + \mu g(x,u,t) + \eta h(x,t) \\ &= x_1 + \lambda_1 x_2 + \lambda_2 u + \mu(u_{\max}^2 - u^2) + \eta_1(x_2 - x_2^2) + \eta_2(y_1(t) - x_1) \end{aligned}$$

with multipliers μ and η_1, η_2.

The adjoint equation

$$\dot{\lambda} = -L_x(x^*, u^*, \lambda, \mu, \eta, t)$$

for our problem is given by

$$\begin{cases} \dot{\lambda}_1 = -1 + \eta_2 \\ \dot{\lambda}_2 = -\lambda_1 - \eta_1 + 2\eta_1 x_2 \end{cases}$$

$$\frac{\partial L}{\partial u}|_{u=u^*(t)} = 0 \iff \lambda_2 = 2\mu u^*$$

Complementary slackness conditions

$$\begin{aligned} \mu(u_{\max}^2 - (u^*)^2) = 0, &\quad \mu \geq 0, \\ \eta_1(x_2^* - (x_2^*)^2) = 0, &\quad \eta_1 \geq 0, \\ \eta_2(y_1 - x_1^*) = 0, &\quad \eta_2 \geq 0. \end{aligned}$$

The Hamiltonian maximizing condition is

$$H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u, \lambda(t), t)$$

at each $t \in [0,T]$ for all u with $g(x^*(t), u, t) \geq 0$.

See Exercise 4.12 in Sethi's "Optimal Control Theory"

References

[1] S. P. Sethi, *Optimal Control Theory: Applications to Management Science and Economics.* Cham: Springer International Publishing, 2019.

[2] R. F. Hartl, S. P. Sethi, and R. G. Vickson, “A Survey of the Maximum Principles for Optimal Control Problems with State Constraints,” *SIAM Review*, vol. 37, no. 2, pp. 181–218, 1995.