

# Vehicle trajectories in a tandem of intersections

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Let  $\dot{x}(t)$  and  $\ddot{x}(t)$  denote the first and second derivative of  $x(t)$  with respect to time  $t$ . Let  $\mathcal{D}[a, b]$  denote the set of valid *trajectories*, which we define as continuously differentiable functions  $\gamma : [a, b] \rightarrow \mathbb{R}$  satisfying the constraints

$$0 \leq \dot{\gamma}(t) \leq 1 \quad \text{and} \quad -\omega \leq \ddot{\gamma}(t) \leq \bar{\omega}, \quad \text{for all } t \in [a, b]. \quad (1)$$

For  $\gamma_1 \in \mathcal{D}[a_1, b_1], \gamma_2 \in \mathcal{D}[a_2, b_2]$ , when we write  $\gamma_1 \leq \gamma_2$  without explicitly mentioning where it applies, we mean  $t \in [a_1, b_1] \cap [a_2, b_2]$ . We also write  $\gamma \leq \min\{\gamma_1, \gamma_2\}$  as a shorthand for  $\gamma \leq \gamma_1$  and  $\gamma \leq \gamma_2$ .

**Definition 1.** *Given some trajectory  $\gamma \in \mathcal{D}[a, b]$  and some time  $\xi \in [a, b]$ , consider the stopping trajectory  $\gamma[\xi]$  that is identical to the original trajectory until  $\xi$ , from where it starts decelerating to a full stop, so that at time  $t \geq \xi$ , the position is given by*

$$\gamma[\xi](t) = \gamma(\xi) + \int_{\xi}^t \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\} d\tau \quad (2a)$$

$$= \gamma(\xi) + \begin{cases} \dot{\gamma}(\xi)(t - \xi) - \omega(t - \xi)^2/2 & \text{for } t \leq \xi + \dot{\gamma}(\xi)/\omega, \\ (\dot{\gamma}(\xi))^2/(2\omega) & \text{for } t \geq \xi + \dot{\gamma}(\xi)/\omega. \end{cases} \quad (2b)$$

The above definition guarantees  $\gamma[\xi] \in \mathcal{D}[a, \infty)$ . Note that a stopping trajectory serves as a lower bound in the sense that, for any  $\mu \in \mathcal{D}[c, d]$  such that  $\gamma = \mu$  on  $[a, \xi] \cap [c, d]$ , we have  $\gamma \leq \mu$  and  $\dot{\gamma} \leq \dot{\mu}$ . Furthermore,  $\gamma[\xi](t)$  is a non-decreasing function in terms of either of its arguments, while fixing the other. To see this for  $\xi$ , fix any  $t$  and consider  $\xi_1 \leq \xi_2$ , then note that  $\gamma[\xi_1](t)$  is a lower bound for  $\gamma[\xi_2](t)$ .

**Property 1.** *Both  $\gamma[\xi](t)$  and  $\dot{\gamma}[\xi](t)$  are continuous when considered as functions of  $(\xi, t)$ .*

*Proof.* Write  $f(\xi, t) := \gamma[\xi](t)$  to emphasize that we are dealing with two variables. Recall that  $\dot{\gamma}$  is continuous by assumption, so the equation  $\tau = \xi + \dot{\gamma}(\xi)/\omega$  defines a separation boundary of the domain of  $f$ . Both cases of (2b) are continuous and they agree at this boundary, so  $f$  is continuous on all of its domain. Since  $x \mapsto \max\{0, x\}$  is continuous, it is easy to see that also  $(\xi, t) \mapsto \dot{\gamma}[\xi](t) = \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\}$  is continuous.  $\square$

Because  $\gamma[\xi](t)$  is continuous and non-decreasing in  $\xi$ , the set

$$X(t_0, x_0) := \{\xi : \gamma[\xi](t_0) = x_0\} \quad (3)$$

is a closed interval (follows from Lemma A.1), so we can consider the maximum

$$\xi(t_0, x_0) := \max X(t_0, x_0). \quad (4)$$

Consider the closed region  $\bar{U} := \{(t, x) : \gamma[a](t) \leq x \leq \gamma[b](t)\}$ . For each  $(t_0, x_0) \in \bar{U}$ , there must be some  $\xi_0$  such that  $\gamma[\xi_0](t_0) = x_0$ , as a consequence of the intermediate value theorem and the above continuity property. Consider  $\bar{U}$  without the points on  $\gamma$ , which we denote by

$$U := \bar{U} \setminus \{(t, x) : \gamma(t) = x\}. \quad (5)$$

Next, we prove that  $\gamma[\xi_0]$  is actually unique if  $(t_0, x_0) \in U$ , so that we may regard  $\xi(t_0, x_0)$  as the canonical representation of this unique trajectory  $\gamma[\xi(t_0, x_0)]$ .

**Property 2.** For  $(t_0, x_0) \in U$ , if  $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0) = x_0$ , then  $\gamma[\xi_1] = \gamma[\xi_2]$ .

*Proof.* Suppose  $t_0 < \xi_i$ , then  $x_0 = \gamma[\xi_i](t_0) = \gamma(t_0)$  contradicts the assumption  $(t_0, x_0) \in U$ . Therefore, assume  $\xi_1 \leq \xi_2 < t_0$ , without loss of generality. Since  $\gamma[\xi_1] = \gamma[\xi_2]$  on  $[a, \xi_1]$ , note that we have the lower bounds

$$\gamma[\xi_1] \leq \gamma[\xi_2] \quad \text{and} \quad \dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]. \quad (6)$$

We must have  $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$ , because otherwise  $\gamma[\xi_1] > \gamma[\xi_2]$  somewhere in a sufficiently small neighborhood of  $t_0$ , which contradicts the first lower bound.

It is clear from Definition 1 that

$$\ddot{\gamma}[\xi_i](t) = \begin{cases} \ddot{\gamma}(t) & \text{for } t < \xi_i, \\ -\omega & \text{for } t \in (\xi_i, \xi_i + \dot{\gamma}(\xi_i)/\omega), \\ 0 & \text{for } t > \xi_i + \dot{\gamma}(\xi_i)/\omega, \end{cases} \quad (7)$$

for both  $i \in \{1, 2\}$ . Note that  $\dot{\gamma}(\xi_1) - \omega(\xi_2 - \xi_1) \leq \dot{\gamma}(\xi_2)$ , which can be rewritten as

$$\xi_2 + \dot{\gamma}(\xi_2)/\omega \geq \xi_1 + \dot{\gamma}(\xi_1)/\omega. \quad (8)$$

This shows that  $\ddot{\gamma}[\xi_1](t) \geq \ddot{\gamma}[\xi_2](t)$ , for every  $t \geq \xi_2$ . Because  $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$ , this in turn ensures that  $\dot{\gamma}[\xi_1](t) \geq \dot{\gamma}[\xi_2](t)$  for  $t \geq t_0$ . Together with the opposite inequality in (6), we conclude that on  $[t_0, \infty)$ , we have  $\dot{\gamma}[\xi_1] = \dot{\gamma}[\xi_2]$  and thus  $\gamma[\xi_1] = \gamma[\xi_2]$ .

It remains to show that  $\gamma[\xi_1] = \gamma[\xi_2]$  on  $[\xi_1, t_0]$ , so consider the smallest  $t^* \in (\xi_1, t_0)$  such that  $\gamma[\xi_1](t^*) < \gamma[\xi_2](t^*)$ . Since  $\dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]$ , this implies that  $\gamma[\xi_1](t) < \gamma[\xi_2](t)$  for all  $t \geq t^*$ , but this contradicts the assumption  $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0)$ .  $\square$

**Lemma 1.** Let  $\gamma_1 \in \mathcal{D}[a_1, b_1]$  and  $\gamma_2 \in \mathcal{D}[a_2, b_2]$  be two trajectories that are intersecting at exactly one time  $t_c$  and assume  $\dot{\gamma}_1(t_c) > \dot{\gamma}_2(t_c)$ , then under the conditions

$$(C1) \quad \gamma_2 \geq \gamma_1[a_1],$$

$$(C2) \quad b_2 \geq t_c + \dot{\gamma}_1(t_c)/\omega,$$

there is a unique trajectory  $\varphi$  such that

- (i)  $\varphi = \gamma_1[\xi]$ , for some  $\xi < t_c$ ,
- (ii)  $\varphi(\tau) = \gamma_2(\tau)$  and  $\dot{\varphi}(\tau) = \dot{\gamma}_2(\tau)$ , for some  $\tau > t_c$ ,
- (iii)  $\varphi \leq \gamma_2$ .

*Proof.*

- Identify for which parameters  $\xi < t_c < \tau$  we have  $\gamma_1[\xi](\tau) = \gamma_2(\tau)$  and  $\dot{\gamma}_1[\xi](\tau) = \dot{\gamma}_2(\tau)$ .
  - Define the set  $U$  and the functions  $X(t, x)$  and  $\xi(t, x)$  as we did in equations (3)–(5) for  $\gamma$  above, but now for  $\gamma_1$ .
  - For each  $\tau > t_c$ , observe that  $(\tau, \gamma_2(\tau)) \in U$ . It follows from Property 2 that  $\varphi_{[\tau]} := \gamma_1[\xi(\tau, \gamma_2(\tau))]$  is the unique stopping trajectory such that  $\varphi_{[\tau]}(\tau) = \gamma_2(\tau)$ . Next, we investigate when this unique trajectory touches  $\gamma_2$  tangentially. More precisely, consider the set of times

$$T := \{\tau > t_c : \dot{\varphi}_{[\tau]}(\tau) = \dot{\gamma}_2(\tau), \xi(\tau, \gamma_2(\tau)) < t_c\}. \quad (9)$$

- We define the auxiliary function  $g(t, x) := \dot{\gamma}_1[\xi(t, x)](t)$ , which gives the slope of the unique stopping trajectory through each point  $(t, x) \in U$ .

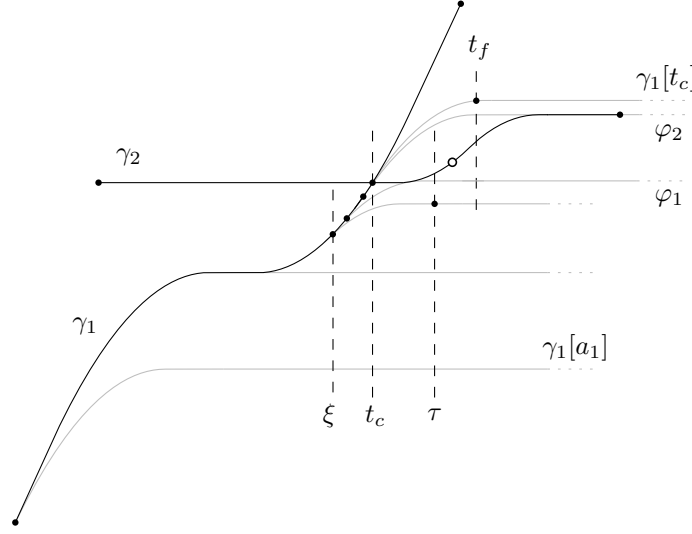


Figure 1: Sketch of some quantities used in the proof of Lemma 1, including some stopping trajectory candidates drawn in grey. The one satisfying the conditions of Lemma 1 is marked as  $\varphi_1$ . The little open dot halfway on  $\gamma_2$  is referred to in Remark 1.

- Function  $g$  is continuous in  $(t, x)$ . We use the notation  $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$ .
- We will write  $f_x(\xi, t) = \gamma_1[\xi](t)$ ,  $f_v(\xi, t) = \dot{\gamma}_1[\xi](t)$  and  $h_t(\xi) = \gamma_1[\xi](t)$  to emphasize the quantities that we treat as variables. Observe that  $h_t^{-1}(x) = X(t, x)$ .
- Let  $x_0 = f_x(\xi_0, \tau_0)$  and  $v_0 = f_v(\xi_0, \tau_0)$  for some  $\xi_0$  and  $\tau_0$  and pick some arbitrary  $\varepsilon > 0$ . Note that  $\xi_0 \in [\xi_1, \xi_2] := h_{\tau_0}^{-1}(x_0)$ . We apply the  $\varepsilon$ - $\delta$  definition of continuity to each of these endpoints. Let  $i \in \{1, 2\}$ , then there exist  $\delta_i > 0$  such that

$$\xi \in N_{\delta_i}(\xi_i), \tau \in N_{\delta_i}(\tau_0) \implies f_v(\xi, \tau) \in N_\varepsilon(v_0). \quad (10)$$

Let  $\delta = \min\{\delta_1, \delta_2\}$  and define  $N_1 := (\xi_1 - \delta, \xi_2 + \delta)$  and  $N_2 := N_\delta(\tau_0)$ , then

$$\xi \in N_1, \tau \in N_2 \implies f_v(\xi, \tau) \in N_\varepsilon(v_0). \quad (11)$$

This is obvious when  $\xi$  is chosen to be in one of  $N_{\delta_i}(\xi_i)$ . Otherwise, we must have  $\xi \in [\xi_1, \xi_2]$ , in which case  $f_v(\xi, \tau) = f_v(\xi_1, \tau) \in N_\varepsilon(v_0)$ .

- Because  $h_{\tau_0}(\xi)$  is continuous, the image  $I := h_{\tau_0}(N_1)$  must be an interval containing  $x_0$ , with  $\inf I = h_{\tau_0}(\xi_1 - \delta)$  and  $\sup I = h_{\tau_0}(\xi_2 + \delta)$ . We argue that  $I$  contains  $x_0$  in its interior. For sake of contradiction, suppose  $x_0 = \max I$ , then  $h_{\tau_0}(\xi_2 + \delta') = x_0$ , for each  $\delta' \in (0, \delta)$ , because  $h_{\tau_0}$  is non-decreasing, but this contradicts the definition of  $\xi_2$ . Similarly, when  $x_0 = \min I$ , then  $h_{\tau_0}(\xi_1 - \delta') = x_0$ , for each  $\delta' \in (0, \delta)$ , which contradicts the definition of  $\xi_1$ .
- Define  $\nu := \min\{x_0 - \inf I, \sup I - x_0\}$  and  $N_3 := (x_0 - \nu/2, x_0 + \nu/2)$ . Because  $h_\tau(\xi)$  is also continuous in  $\tau$ , there exists a neighborhood  $N_2^* \subset N_2$  of  $\tau_0$  such that for every  $\tau \in N_2^*$ , we have

$$\begin{aligned} h_\tau(\xi_1 - \delta) &\leq h_{\tau_0}(\xi_1 - \delta) + \nu/2 = \inf I + \nu/2 < x_0 - \nu/2, \\ h_\tau(\xi_2 + \delta) &\geq h_{\tau_0}(\xi_2 + \delta) - \nu/2 = \sup I - \nu/2 > x_0 + \nu/2, \end{aligned}$$

which shows that  $h_\tau(N_1) \supset N_3$ . It follows that  $h_\tau^{-1}(N_3) \subset N_1$ .

- Finally, take any  $\tau \in N_2^*$  and  $x \in N_3$ , then there exists some  $\xi \in N_1$  such that  $h_\tau(\xi) = x$  and  $g(\tau, x) = f_v(\max h_\tau^{-1}(x), \tau) = f_v(\xi, \tau) \in N_\varepsilon(v_0)$ .

- Function  $g$  is non-decreasing and Lipschitz continuous in  $x$ .
- Let  $x_1 \leq x_2$  and  $\tau$  such that  $g(\tau, x_1)$  and  $g(\tau, x_2)$  are defined. There must be  $\xi_1 \leq \xi_2$  such that  $h_\tau(\xi_1) = x_1$  and  $h_\tau(\xi_2) = x_2$  and we have

$$\begin{aligned} g(\tau, x_1) &= \dot{\gamma}_1[\xi_1](\tau) = \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} = \dot{\gamma}_1[\xi_2](\tau) = g(\tau, x_2). \end{aligned}$$

- Furthermore, we have  $\dot{\gamma}_1(\xi_2) \leq \dot{\gamma}_1(\xi_1) + \bar{\omega}(\xi_2 - \xi_1)$ , so that

$$\begin{aligned} g(\tau, x_2) &= \max\{0, \dot{\gamma}_1(\xi_2) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) + \bar{\omega}(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &= \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1) + (\omega + \bar{\omega})(\xi_2 - \xi_1)\} \\ &\leq \max\{0, \dot{\gamma}_1(\xi_1) - \omega(\tau - \xi_1)\} + (\omega + \bar{\omega})(\xi_2 - \xi_1) \\ &= g(\tau, x_1) + (\omega + \bar{\omega})(\xi_2 - \xi_1). \end{aligned}$$

Observe that, together with the above non-decreasing property, this shows that  $g$  is Lipschitz continuous in  $x$ , with Lipschitz constant  $(\omega + \bar{\omega})$ .

- Note that  $T$  can also be written as

$$T = \{\tau > t_c : g(\tau, \gamma_2(\tau)) = \dot{\gamma}_2(\tau), \xi(\tau, \gamma_2(\tau)) < t_c\}, \quad (12)$$

so continuity of  $g$  shows that it is a closed set (Lemma A.1). It is not necessarily connected (see for example Figure 1), so it is the union of a sequence of disjoint closed intervals  $T_1, T_2, \dots, T_n$ .

- Define  $\tau_i := \min T_i$  and let  $\varphi_i := \varphi_{[\tau_i]}$  denote the unique stopping trajectory through  $(\tau_i, \gamma_2(\tau_i))$ . For  $\tau \in T_i$ , we have  $\dot{\gamma}_2(\tau) = g(\tau, \gamma_2(\tau))$  by definition of  $T_i$ . Moreover, we have

$$\dot{\varphi}_i(t) = g(t, \varphi_i(t)), \quad (13)$$

for every  $t$  for which these quantities are defined, so in particular on  $T_i$ . This shows that  $\gamma_2$  and  $\varphi_i$  are both solutions to the initial value problem

$$\begin{cases} \dot{x}(t) = g(t, x(t)) & \text{for } t \in T_i, \\ x(\tau_i) = \gamma_2(\tau_i). \end{cases} \quad (14)$$

Since  $g(t, x)$  is continuous in  $t$  and Lipschitz continuous in  $x$ , it is a consequence of the (local) existence and uniqueness theorem (see Lemma A.2) that  $\gamma_2 = \varphi_i$  on  $T_i$ . Hence, we have  $\varphi_i = \varphi_{[\tau]}$  for any  $\tau \in T_i$ , so we regard  $\varphi_i$  as being the canonical stopping trajectory for  $T_i$ .

- Show that  $\tau_1$  and thus  $\varphi_1$  exists. We write  $s(\tau) := g(\tau, \gamma_2(\tau))$  and  $t_f := t_c + \dot{\gamma}_1(t_c)/\omega$ . Note that this part relies on conditions (C1) and (C2).
- Suppose  $\gamma_2(t_f) \leq \gamma_1[t_c](t_f)$ , then it follows from the fact that  $g$  is non-decreasing in  $x$  that  $g(t_f, \gamma_2(t_f)) \leq g(t_f, \gamma_1[t_c](t_f)) = \dot{\gamma}_1(t_c) - \omega(t_f - t_c) = 0$ , so  $s(t_f) = 0$ .
- Otherwise  $\gamma_2(t_f) > \gamma_1[t_c](t_f)$ , then it follows (from Lemma ...) that  $\gamma_2$  crosses  $\gamma_1[t_c]$  at some time  $t_d \in (t_c, t_f)$  with  $\dot{\gamma}_2(t_d) > \gamma_1[t_c](t_d) = s(t_d)$ .
- We have  $\gamma_1[a_1](t) \leq \gamma_2(t) \leq \gamma_1[t_c](t)$  for  $t \in \{t_f, t_d\}$ , so the intermediate value theorem guarantees that  $s(t)$  actually exists in both cases, because there is some  $a_1 \leq \xi < t_c$  such that  $\gamma_2(t) = \gamma_1[\xi](t)$  and thus  $s(t) = g(t, \gamma_2(t)) = \dot{\gamma}_1[\xi](t)$  exists.

- In both cases above, we have  $\dot{\gamma}_2(t_c) < \dot{\gamma}_1(t_c) = s(t_c)$  and  $\dot{\gamma}_2(t_d) \geq s(t_d)$  for some  $t_d \in (t_c, t_f]$ . Hence, there must be some smallest  $\tau_1 \in (t_c, t_d]$  such that  $\dot{\gamma}_2(\tau_1) = s(\tau_1)$ , which is a consequence of the intermediate value theorem.
- If  $i \geq 2$ , then  $\varphi_i > \gamma_2$  somewhere.
- Let  $i \geq 1$ , we show that  $\varphi_{i+1}(t) > \gamma_2(t)$  for some  $t$ . Recall the lower bound property, so  $\gamma_2(t) \geq \varphi_i(t)$  and  $\dot{\gamma}_2(t) \geq \dot{\varphi}_i(t)$  for  $t \geq \tau_i$ . Define  $\hat{\tau}_i := \max T_i$ , such that  $T_i = [\tau_i, \hat{\tau}_i]$ , then by definition of  $T_i$ , there must be some  $\delta > 0$  such that

$$\gamma_2(\hat{\tau}_i + \delta) > \varphi_i(\hat{\tau}_i + \delta), \quad (15)$$

since otherwise  $\gamma_2 = \varphi_i$  on some open neighborhood of  $\hat{\tau}_i$  and then also

$$\dot{\gamma}_2(t) = \dot{\varphi}_i(t) \stackrel{(13)}{=} g(t, \varphi_i(t)) = g(t, \gamma_2(t)), \quad (16)$$

which contradicts the definition of  $\hat{\tau}_i$ . Therefore, we have  $\gamma_2(t) > \varphi_i(t)$  for all  $t \geq \hat{\tau}_i + \delta$ . For  $t = \tau_{i+1}$ , in particular, it follows that  $\varphi_{i+1}(\tau_{i+1}) = \gamma_2(\tau_{i+1}) > \varphi_i(\tau_{i+1})$ , which shows that  $\varphi_{i+1} > \varphi_i$  on  $(\xi_i, \infty)$ , due to Property 2, but this means that  $\varphi_{i+1}(\tau_i) > \varphi_i(\tau_i) = \gamma_2(\tau_i)$ .

- If  $\varphi_i > \gamma_2$  somewhere, then  $i \geq 2$ .
- Suppose  $\varphi_i(t_x) > \gamma_2(t_x)$  for some  $t_x \in (t_c, \tau_i)$ , then there must be some  $\tau_0 \in (t_c, t_x)$  such that  $\gamma_2(\tau_0) = \varphi_i(\tau_0)$  and  $\dot{\gamma}_2(\tau_0) < \dot{\varphi}_i(\tau_0)$ . Note that this crossing must happen because we require  $\xi_i < t_c$ .
- Since  $g(t, x)$  is non-decreasing in  $x$ , we have

$$s(t) = g(t, \gamma_2(t)) \leq g(t, \varphi_i(t)) = \dot{\varphi}_i(t), \quad (17)$$

for every  $t \in [\tau_0, \tau_i]$  and at the endpoints, we have

$$s(\tau_0) = \varphi_i(\tau_0), \quad s(\tau_i) = \varphi_i(\tau_i). \quad (18)$$

Furthermore, observe that  $\gamma_2(\tau_0) = \varphi_i(\tau_0)$  and  $\gamma_2(\tau_i) = \varphi_i(\tau_i)$  require that

$$\int_{\tau_0}^{\tau_i} \dot{\gamma}_2(t) dt = \int_{\tau_0}^{\tau_i} \dot{\varphi}_i(t) dt. \quad (19)$$

- Since  $\dot{\gamma}_2(\tau_0) < \dot{\varphi}_i(\tau_0)$ , it follows from (19) that there must be some  $t \in (\tau_0, \tau_i)$  such that  $\dot{\gamma}_2(t) > \dot{\varphi}_i(t)$ . Together with  $s(\tau_0) = \dot{\varphi}_i(\tau_0) > \dot{\gamma}_2(\tau_0)$  and  $s(t) \leq \dot{\varphi}_i(t)$  for  $t \in [\tau_0, \tau_i]$ , this means there is some  $\tau^*$  such that  $\dot{\gamma}_2(\tau^*) = s(\tau^*)$ , again as a consequence of the intermediate value theorem. Therefore,  $\tau^* \in T_j$  for some  $j < i$ , which shows that  $i \geq 2$ .
- The above two points establish that  $\varphi_i \leq \gamma_2$  if and only if  $i = 1$ . To conclude, we have shown that  $\varphi := \varphi_1$  exists and is the unique trajectory satisfying the stated requirements with  $\tau = \tau_i$  and  $\xi = \xi(\tau_i, \gamma_2(\tau_i))$ .  $\square$

**Remark 1.** It is easy to see that condition (C1) in Lemma 1 is necessary. Suppose there is some  $t_x \in (t_c, \infty)$  such that  $\gamma_1[a_1](t_x) > \gamma_2(t_x)$ , then for any other  $\xi \in (a_1, t_c)$ , we have  $\gamma_1[\xi](t_x) > \gamma_2(t_x)$  as well, due to the lower bound property of stopping trajectories, so requirement (iii) is violated. Condition (C2) is not necessary, which can be seen from stopping trajectory  $\varphi_1$  in Figure 1, which satisfies the conditions, but would also have been valid if  $\gamma_2$  ended somewhat earlier than  $t_f$ , for example until the open dot.

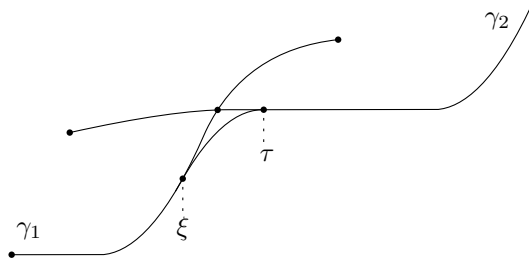


Figure 2: Two intersecting trajectories joined together by a part of a stopping trajectory.

Suppose we have two trajectories that cross each other exactly once. Lemma 1 gives conditions under which, roughly speaking, these trajectories can be glued together to form a smooth trajectory by introducing a stopping trajectory in between, as illustrated in Figure 2. The above discussion motivates and justifies the following definition.

**Definition 2.** Let  $\gamma_1 \in \mathcal{D}[a_1, b_1]$  and  $\gamma_2 \in \mathcal{D}[a_2, b_2]$  and suppose they intersect at exactly a single time  $t_c$ . We write  $\gamma_1 * \gamma_2$  to denote the unique trajectory

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t < \tau, \\ \gamma_1[\xi](t) & \text{for } t \in [\tau, \xi], \\ \gamma_2(t) & \text{for } t > \xi, \end{cases} \quad (20)$$

satisfying  $\gamma_1 * \gamma_2 \in \mathcal{D}[a_1, b_2]$ , where  $\tau$  and  $\xi$  are as given by Lemma 1. If  $\gamma_1$  and  $\gamma_2$  are intersecting tangentially, so  $\dot{\gamma}_1(t_c) = \dot{\gamma}_2(t_c)$ , then we define  $\tau = \xi = t_c$ .

Our main interest in  $\gamma_1 * \gamma_2$  is due to the following property.

**Lemma 2.** Let  $\gamma_1 \in \mathcal{D}[a_1, b_2]$  and  $\gamma_2 \in \mathcal{D}[a_2, b_2]$  be such that  $\gamma_1 * \gamma_2$  exists. All trajectories  $\mu \in \mathcal{D}[a, b]$  that are such that  $\mu \leq \min\{\gamma_1, \gamma_2\}$ , must satisfy  $\mu \leq \gamma_1 * \gamma_2$ .

*Proof.* Write  $\gamma := \gamma_1 * \gamma_2$  as a shorthand. We obviously have  $\mu \leq \gamma$  on  $[a_1, \xi] \cup [\tau, b_2]$ , so consider the interval  $(\xi, \tau)$  of the joining deceleration part. Suppose there exists some  $t_d \in (\xi, \tau)$  such that  $\mu(t_d) > \gamma(t_d)$ . Because  $\mu(\xi) \leq \gamma(\xi)$ , this means that  $\mu$  must intersect  $\gamma$  at least once in  $[\xi, t_d]$ , so let  $t_c := \sup\{t \in [\xi, t_d] : \mu(t) = \gamma(t)\}$  be the latest time of intersection such that  $\mu \geq \gamma$  on  $[t_c, t_d]$ . There must be some  $t_v \in [t_c, t_d]$  such that  $\dot{\mu}(t_v) > \dot{\gamma}(t_v)$ , otherwise

$$\mu(t_d) = \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t) dt \leq \gamma(t_c) + \int_{t_c}^{t_d} \dot{\gamma}(t) dt = \gamma(t_d),$$

which contradicts our choice of  $t_d$ . Hence, for every  $t \in [t_v, \tau]$ , we have

$$\dot{\mu}(t) \geq \dot{\mu}(t_v) - \omega(t - t_v) > \dot{\gamma}(t_v) - \omega(t - t_v) = \dot{\gamma}(t).$$

It follows that  $\mu(\tau) > \gamma(\tau)$ , which contradicts  $\mu \leq \gamma_2$ . □

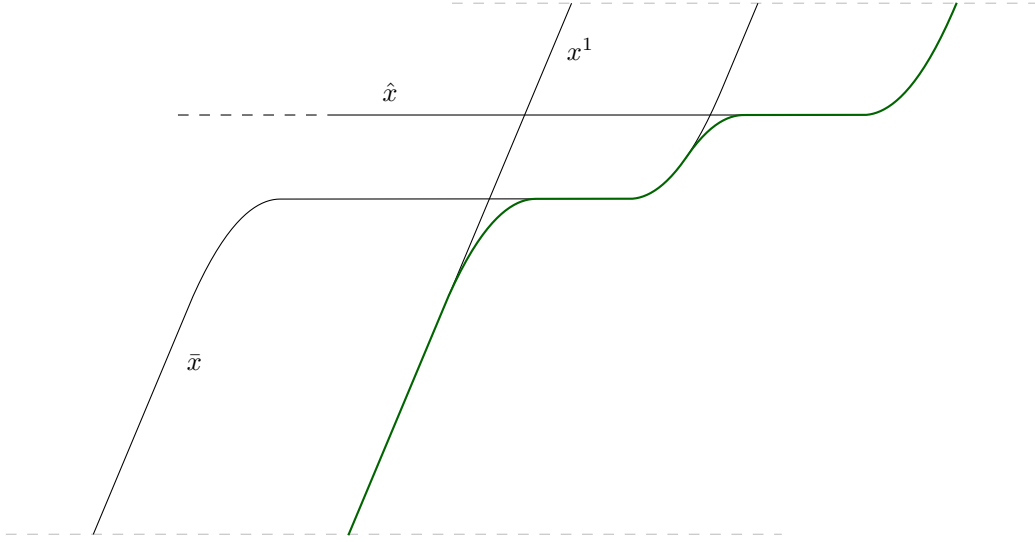


Figure 3: Sketch of how the three boundaries are joined to form the optimal trajectory.

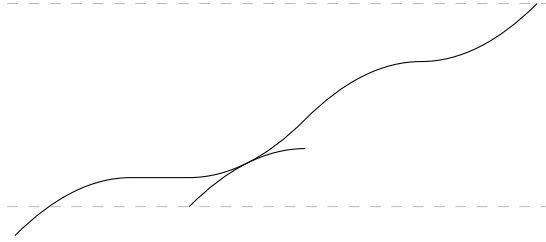


Figure 4: Illustration of “buffer constraint”.

Next, consider the set  $D[a, b] \subset \mathcal{D}[a, b]$  of trajectories  $\gamma$  that satisfy the following additional constraints

$$\gamma(a) = A, \quad \gamma(b) = B, \quad \dot{\gamma}(a) = \dot{\gamma}(b) = 1, \quad (21)$$

for some fixed  $A, B$  such that  $B - A \geq (\omega + \bar{\omega})/2$ .

For every such trajectory  $\gamma \in D[a, b]$ , we have  $\dot{\gamma}(t) + \bar{\omega}(b - t) \geq \dot{\gamma}(b) = 1$ , which can be rewritten to  $\dot{\gamma}(t) \geq 1 - \bar{\omega}(b - t)$ . Combined with  $\dot{\gamma}(t) \geq 0$ , this gives

$$\dot{\gamma}(t) \geq \max\{0, 1 - \bar{\omega}(b - t)\}. \quad (22)$$

Hence, we derive the upper bound

$$\gamma(t) = \gamma(b) - \int_t^b \dot{\gamma}(\tau) d\tau \quad (23a)$$

$$\leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b - \tau)\} d\tau =: \hat{x}(t), \quad (23b)$$

and note that  $\hat{x} \in D(-\infty, b]$ . Furthermore, let  $x^1 \in D(-\infty, \infty)$  be defined as  $x^1(t) = A + t - a$ , then it is clearly an upper bound for any trajectory  $\gamma \in D[a, b]$ .

## A Miscellaneous

**Lemma A.1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be continuous and  $y \in \mathbb{R}^m$ , then the level set  $N := f^{-1}(\{y\})$  is a closed subset of  $\mathbb{R}^n$ .*

*Proof.* For any  $y' \neq y$ , there exists an open neighborhood  $M(y')$  such that  $y \notin M(y')$ . The preimage  $f^{-1}(M(y'))$  is open by continuity. Therefore, the complement  $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$  is open.  $\square$

**Lemma A.2.** *Let  $D \subseteq \mathbb{R} \times \mathbb{R}^n$  be some closed rectangle such that  $(t_0, x_0) \in \text{int } D$ . Let  $f : D \rightarrow \mathbb{R}^n$  be a function that is continuous in  $t$  and globally Lipschitz continuous in  $x$ , then there exists some  $\varepsilon > 0$  such that the initial value problem*

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{24}$$

*has a unique solution  $x(t)$  on the interval  $[t_0 - \varepsilon, t_0 + \varepsilon]$ .*

The above existence and uniqueness theorem is also known as the Picard-Lindelöf or Cauchy-Lipschitz theorem. The above statement is based on the [Wikipedia page on this theorem](#), so we still need a slightly better reference.