Vehicle trajectories in a tandem of intersections

Jeroen van Riel

August 2025

Let $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second derivative of $x(t)$ with respect to time t. Let $\mathcal{D}[a,b]$ denote the set of valid *trajectories*, which we define as continuously differentiable functions $\gamma : [a,b] \to \mathbb{R}$ satisfying the constraints

$$0 \leq \dot{\gamma}(t) \leq 1 \quad \text{and} \quad -\omega \leq \ddot{\gamma}(t) \leq \bar{\omega}, \quad \text{for all } t \in [a,b]. \tag{1}$$

For $\gamma_1 \in \mathcal{D}[a_1,b_1], \gamma_2 \in \mathcal{D}[a_2,b_2]$, when we write $\gamma_1 \leq \gamma_2$ without explicitly mentioning where it applies, we mean $t \in [a_1,b_1] \cap [a_2,b_2]$. We also write $\gamma \leq \min\{\gamma_1,\gamma_2\}$ as a shorthand for $\gamma \leq \gamma_1$ and $\gamma \leq \gamma_2$.

Definition 1. *Given some trajectory $\gamma \in \mathcal{D}[a,b]$ and some time $\xi \in [a,b]$, consider the* stopping trajectory $\gamma[\xi]$ *that is identical to the original trajectory until ξ, from where it starts decelerating to a full stop, so that at time $t \geq \xi$, the position is given by*

$$\gamma[\xi](t) = \gamma(\xi) + \int_{\xi}^{t} \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\} d\tau \tag{2a}$$

$$= \gamma(\xi) + \begin{cases} \dot{\gamma}(\xi)(t-\xi) - \omega(t-\xi)^2/2 & \textit{for } t \leq \xi + \dot{\gamma}(\xi)/\omega, \\ (\dot{\gamma}(\xi))^2/(2\omega) & \textit{for } t \geq \xi + \dot{\gamma}(\xi)/\omega. \end{cases} \tag{2b}$$

The above definition guarantees $\gamma[\xi] \in \mathcal{D}[a,\infty)$. Note that a stopping trajectory serves as a lower bound in the sense that, for any $\mu \in \mathcal{D}[c,d]$ such that $\gamma = \mu$ on $[a,\xi] \cap [c,d]$, we have $\gamma \leq \mu$ and $\dot{\gamma} \leq \dot{\mu}$. Furthermore, $\gamma[\xi](t)$ is a non-decreasing function in terms of either of its arguments, while fixing the other. To see this for ξ, fix any t and consider $\xi_1 \leq \xi_2$, then note that $\gamma[\xi_1](t)$ is a lower bound for $\gamma[\xi_2](t)$.

Property 1. *Both $\gamma[\xi](t)$ and $\dot{\gamma}[\xi](t)$ are continuous when considered as functions of (ξ, t).*

Proof. Write $f(\xi,t) := \gamma[\xi](t)$ to emphasize that we are dealing with two variables. Recall that $\dot{\gamma}$ is continuous by assumption, so the equation $\tau = \xi + \dot{\gamma}(\xi)/\omega$ defines a separation boundary of the domain of f. Both cases of (2b) are continuous and they agree at this boundary, so f is continuous on all of its domain. Since $x \mapsto \max\{0,x\}$ is continuous, it is easy to see that also $(\xi,t) \mapsto \dot{\gamma}[\xi](t) = \max\{0, \dot{\gamma}(\xi) - \omega(\tau - \xi)\}$ is continuous. □

Because $\gamma[\xi](t)$ is continuous and non-decreasing in ξ, the set

$$X(t_0,x_0) := \{\xi : \gamma[\xi](t_0) = x_0\} \tag{3}$$

is a closed interval (follows from Lemma 3), so we can consider the maximum

$$\xi(t_0,x_0) := \max X(t_0,x_0). \tag{4}$$

Consider the closed region $\bar{U} := \{(t,x) : \gamma[a](t) \leq x \leq \gamma[b](t)\}$. For each $(t_0,x_0) \in \bar{U}$, there must be some ξ_0 such that $\gamma[\xi_0](t_0) = x_0$, as a consequence of the intermediate value theorem and the above continuity property. Consider $\bar{U}$ without the points on γ, which we denote by

$$U := \bar{U} \setminus \{(t,x) : \gamma(t) = x\}. \tag{5}$$

Next, we prove that $\gamma[\xi_0]$ is actually unique if $(t_0,x_0) \in U$, so that we may regard $\xi(t_0,x_0)$ as the canonical representation of this unique trajectory $\gamma[\xi(t_0,x_0)]$.

Property 2. *For* $(t_0, x_0) \in U$*, if* $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0) = x_0$*, then* $\gamma[\xi_1] = \gamma[\xi_2]$.

Proof. Suppose $t_0 < \xi_i$, then $x_0 = \gamma[\xi_i](t_0) = \gamma(t_0)$ contradicts the assumption $(t_0, x_0) \in U$. Therefore, assume $\xi_1 \leq \xi_2 < t_0$, without loss of generality. Since $\gamma[\xi_1] = \gamma[\xi_2]$ on $[a, \xi_1]$, note that we have the lower bounds

$$\gamma[\xi_1] \leq \gamma[\xi_2] \quad \text{and} \quad \dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]. \tag{6}$$

We must have $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, because otherwise $\gamma[\xi_1] > \gamma[\xi_2]$ somewhere in a sufficiently small neighborhood of t_0, which contradicts the first lower bound.

It is clear from Definition 1 that

$$\ddot{\gamma}[\xi_i](t) = \begin{cases} \ddot{\gamma}(t) & \text{for } t < \xi_i, \\ -\omega & \text{for } t \in (\xi_i, \xi_i + \dot{\gamma}(\xi_i)/\omega), \\ 0 & \text{for } t > \xi_i + \dot{\gamma}(\xi_i)/\omega, \end{cases} \tag{7}$$

for both $i \in \{1, 2\}$. Note that $\dot{\gamma}(\xi_1) - \omega(\xi_2 - \xi_1) \leq \dot{\gamma}(\xi_2)$, which can be rewritten as

$$\xi_2 + \dot{\gamma}(\xi_2)/\omega \geq \xi_1 + \dot{\gamma}(\xi_1)/\omega. \tag{8}$$

This shows that $\ddot{\gamma}[\xi_1](t) \geq \ddot{\gamma}[\xi_2](t)$, for every $t \geq \xi_2$. Because $\dot{\gamma}[\xi_1](t_0) = \dot{\gamma}[\xi_2](t_0)$, this in turn ensures that $\dot{\gamma}[\xi_1](t) \geq \dot{\gamma}[\xi_2](t)$ for $t \geq t_0$. Together with the opposite inequality in (6), we conclude that on $[t_0, \infty)$, we have $\dot{\gamma}[\xi_1] = \dot{\gamma}[\xi_2]$ and thus $\gamma[\xi_1] = \gamma[\xi_2]$.

It remains to show that $\gamma[\xi_1] = \gamma[\xi_2]$ on $[\xi_1, t_0]$, so consider the smallest $t^* \in (\xi_1, t_0)$ such that $\gamma[\xi_1](t^*) < \gamma[\xi_2](t^*)$. Since $\dot{\gamma}[\xi_1] \leq \dot{\gamma}[\xi_2]$, this implies that $\gamma[\xi_1](t) < \gamma[\xi_2](t)$ for all $t \geq t^*$, but this contradicts the assumption $\gamma[\xi_1](t_0) = \gamma[\xi_2](t_0)$. ☐

Lemma 1. *Let* $\gamma_1 \in \mathcal{D}[a_1, b_1]$ *and* $\gamma_2 \in \mathcal{D}[a_2, b_2]$ *be two trajectories that are intersecting at exactly one time* t_c *and assume* $\dot{\gamma}_1(t_c) > \dot{\gamma}_2(t_c)$*, then under the conditions*

(C1) $\quad \gamma_2 \geq \gamma_1[a_1]$,

(C2) $\quad b_2 \geq t_c + \dot{\gamma}_1(t_c)/\omega$,

there is a unique trajectory φ *such that*

(i) $\quad \varphi = \gamma_1[\xi]$*, for some* $\xi < t_c$,

(ii) $\quad \varphi(\tau) = \gamma_2(\tau)$ *and* $\dot{\varphi}(\tau) = \dot{\gamma}_2(\tau)$*, for some* $\tau > t_c$,

(iii) $\quad \varphi \leq \gamma_2$.

Proof.

- Identify for which parameters $\xi < t_c < \tau$ we have $\gamma_1[\xi](\tau) = \gamma_2(\tau)$ and $\dot{\gamma}_1[\xi](\tau) = \dot{\gamma}_2(\tau)$.
  - Define the set U and the functions $X(t, x)$ and $\xi(t, x)$ as we did in equations (3)–(5) for γ above, but now for γ_1.
  - For each $\tau > t_c$, observe that $(\tau, \gamma_2(\tau)) \in U$. It follows from Property 2 that $\varphi_{[\tau]} := \gamma_1[\xi(\tau, \gamma_2(\tau))]$ is the unique stopping trajectory such that $\varphi_{[\tau]}(\tau) = \gamma_2(\tau)$. Next, we investigate when this unique trajectory touches γ_2 tangentially. More precisely, consider the set of times

$$T := \{\tau > t_c : \dot{\varphi}_{[\tau]}(\tau) = \dot{\gamma}_2(\tau),\ \xi(\tau, \gamma_2(\tau)) < t_c\}. \tag{9}$$

  - We define the auxiliary function $g(t, x) := \dot{\gamma}_1[\xi(t, x)](t)$, which gives the slope of the unique stopping trajectory through each point $(t, x) \in U$.

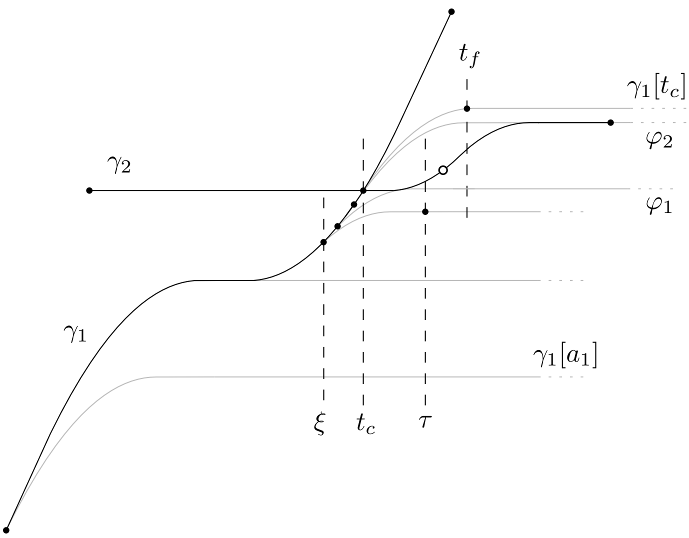

Figure 1: Sketch of some quantities used in the proof of Lemma 1, including some stopping trajectory candidates drawn in grey. The one satisfying the conditions of Lemma 1 is marked as φ_1. The little open dot halfway on γ_2 is refered to in Remark 1.

- Function g is continuous in (t, x). We use the notation $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$.
  - We will write $f_x(\xi, t) = \gamma_1[\xi](t)$, $f_v(\xi, t) = \dot{\gamma}_1[\xi](t)$ and $h_t(\xi) = \gamma_1[\xi](t)$ to emphasize the quantities that we treat as variables. Observe that $h_t^{-1}(x) = X(t, x)$.
  - Let $x_0 = f_x(\xi_0, \tau_0)$ and $v_0 = f_v(\xi_0, \tau_0)$ for some ξ_0 and τ_0 and pick some arbitrary $\varepsilon > 0$. Note that $\xi_0 \in [\xi_1, \xi_2] := h_{\tau_0}^{-1}(x_0)$. We apply the ε-δ definition of continuity to each of these endpoints. Let $i \in \{1, 2\}$, then there exist $\delta_i > 0$ such that

$$\xi \in N_{\delta_i}(\xi_i),\, \tau \in N_{\delta_i}(\tau_0) \implies f_v(\xi, \tau) \in N_\varepsilon(v_0). \tag{10}$$

    Let $\delta = \min\{\delta_1, \delta_2\}$ and define $N_1 := (\xi_1 - \delta, \xi_2 + \delta)$ and $N_2 := N_\delta(\tau_0)$, then

$$\xi \in N_1,\, \tau \in N_2 \implies f_v(\xi, \tau) \in N_\varepsilon(v_0). \tag{11}$$

    This is obvious when ξ is chosen to be in one of $N_{\delta_i}(\xi_i)$. Otherwise, we must have $\xi \in [\xi_1, \xi_2]$, in which case $f_v(\xi, \tau) = f_v(\xi_1, \tau) \in N_\varepsilon(v_0)$.
  - Because $h_{\tau_0}(\xi)$ is continuous, the image $I := h_{\tau_0}(N_1)$ must be an interval containing x_0, with $\inf I = h_{\tau_0}(\xi_1 - \delta)$ and $\sup I = h_{\tau_0}(\xi_2 + \delta)$. We argue that I contains x_0 in its interior. For sake of contradiction, suppose $x_0 = \max I$, then $h_{\tau_0}(\xi_2 + \delta') = x_0$, for each $\delta' \in (0, \delta)$, because h_{τ_0} is non-decreasing, but this contradicts the definition of ξ_2. Similarly, when $x_0 = \min I$, then $h_{\tau_0}(\xi_1 - \delta') = x_0$, for each $\delta' \in (0, \delta)$, which contradicts the definition of ξ_1.
  - Define $\nu := \min\{x_0 - \inf I, \sup I - x_0\}$ and $N_3 := (x_0 - \nu/2, x_0 + \nu/2)$. Because $h_\tau(\xi)$ is also continuous in τ, there exists a neighborhood $N_2^* \subset N_2$ of τ_0 such that for every $\tau \in N_2^*$, we have

$$\begin{aligned} h_\tau(\xi_1 - \delta) &\leq h_{\tau_0}(\xi_1 - \delta) + \nu/2 = \inf I + \nu/2 < x_0 - \nu/2, \\ h_\tau(\xi_2 + \delta) &\geq h_{\tau_0}(\xi_2 + \delta) - \nu/2 = \sup I - \nu/2 > x_0 + \nu/2, \end{aligned}$$

    which shows that $h_\tau(N_1) \supset N_3$. It follows that $h_\tau^{-1}(N_3) \subset N_1$.
  - Finally, take any $\tau \in N_2^*$ and $x \in N_3$, then there exists some $\xi \in N_1$ such that $h_\tau(\xi) = x$ and $g(\tau, x) = f_v(\max h_\tau^{-1}(x), \tau) = f_v(\xi, \tau) \in N_\varepsilon(v_0)$.

- Function g is non-decreasing and Lipschitz continuous in x.
  - Let $x_1 \leq x_2$ and τ such that $g(\tau, x_1)$ and $g(\tau, x_2)$ are defined. There must be $\xi_1 \leq \xi_2$ such that $h_\tau(\xi_1) = x_1$ and $h_\tau(\xi_2) = x_2$ and we have

$$\begin{aligned} g(\tau, x_1) = \dot\gamma_1[\xi_1](\tau) &= \max\{0, \dot\gamma_1(\xi_1) - \omega(\tau - \xi_1)\} \\ &= \max\{0, \dot\gamma_1(\xi_1) - \omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot\gamma_1(\xi_2) - \omega(\tau - \xi_2)\} = \dot\gamma_1[\xi_2](\tau) = g(\tau, x_2). \end{aligned}$$

  - Furthermore, we have $\dot\gamma_1(\xi_2) \leq \dot\gamma_1(\xi_1) + \bar\omega(\xi_2 - \xi_1)$, so that

$$\begin{aligned} g(\tau, x_2) &= \max\{0, \dot\gamma_1(\xi_2) - \omega(\tau - \xi_2)\} \\ &\leq \max\{0, \dot\gamma_1(\xi_1) + \bar\omega(\xi_2 - \xi_1) - \omega(\tau - \xi_2)\} \\ &= \max\{0, \dot\gamma_1(\xi_1) - \omega(\tau - \xi_1) + (\omega + \bar\omega)(\xi_2 - \xi_1)\} \\ &\leq \max\{0, \dot\gamma_1(\xi_1) - \omega(\tau - \xi_1)\} + (\omega + \bar\omega)(\xi_2 - \xi_1) \\ &= g(\tau, x_1) + (\omega + \bar\omega)(\xi_2 - \xi_1). \end{aligned}$$

    Observe that, together with the above non-decreasing property, this shows that g is Lipschitz continuous in x, with Lipschitz constant $(\omega + \bar\omega)$.

- Note that T can also be written as

$$T = \{\tau > t_c : g(\tau, \gamma_2(\tau)) = \dot\gamma_2(\tau),\ \xi(\tau, \gamma_2(\tau)) < t_c\}, \tag{12}$$

  so continuity of g shows that it is a closed set (Lemma 3). It is not necessarily connected (see for example Figure 1), so it is the union of a sequence of disjoint closed intervals $T_1, T_2, \dots, T_n$.

  - Define $\tau_i := \min T_i$ and let $\varphi_i := \varphi_{[\tau_i]}$ denote the unique stopping trajectory through $(\tau_i, \gamma_2(\tau_i))$. For $\tau \in T_i$, we have $\dot\gamma_2(\tau) = g(\tau, \gamma_2(\tau))$ by definition of T_i. Moreover, we have

$$\dot\varphi_i(t) = g(t, \varphi_i(t)), \tag{13}$$

    for every t for which these quantities are defined, so in particular on T_i. This shows that γ_2 and φ_i are both solutions to the initial value problem

$$\begin{cases} \dot x(t) \ = g(t, x(t)) & \text{for } t \in T_i, \\ x(\tau_i) = \gamma_2(\tau_i). \end{cases} \tag{14}$$

    Since $g(t, x)$ is continuous in t and Lipschitz continuous in x, it is a consequence of the (local) existence and uniqueness theorem (Picard-Lindelöf) that $\gamma_2 = \varphi_i$ on T_i. Hence, we have $\varphi_i = \varphi_{[\tau]}$ for any $\tau \in T_i$, so we regard φ_i as being the canonical stopping trajectory for T_i.

- Show that τ_1 and thus φ_1 exists. We write $s(\tau) := g(\tau, \gamma_2(\tau))$ and $t_f := t_c + \dot\gamma_1(t_c)/\omega$. Note that this part relies on conditions (C1) and (C2).
  - Suppose $\gamma_2(t_f) \leq \gamma_1[t_c](t_f)$, then it follows from the fact that g is non-decreasing in x that $g(t_f, \gamma_2(t_f)) \leq g(t_f, \gamma_1[t_c](t_f)) = \dot\gamma_1(t_c) - \omega(t_f - t_c) = 0$, so $s(t_f) = 0$.
  - Otherwise $\gamma_2(t_f) > \gamma_1[t_c](t_f)$, then it follows (from Lemma ...) that γ_2 crosses $\gamma_1[t_c]$ at some time $t_d \in (t_c, t_f)$ with $\dot\gamma_2(t_d) > \gamma_1[t_c](t_d) = s(t_d)$.
  - We have $\gamma_1[a_1](t) \leq \gamma_2(t) \leq \gamma_1[t_c](t)$ for $t \in \{t_f, t_d\}$, so the intermediate value theorem guarantees that $s(t)$ actually exists in both cases, because there is some $a_1 \leq \xi < t_c$ such that $\gamma_2(t) = \gamma_1[\xi](t)$ and thus $s(t) = g(t, \gamma_2(t)) = \dot\gamma_1[\xi](t)$ exists.

- In both cases above, we have $\dot{\gamma}_2(t_c) < \dot{\gamma}_1(t_c) = s(t_c)$ and $\dot{\gamma}_2(t_d) \geq s(t_d)$ for some $t_d \in (t_c, t_f]$. Hence, there must be some smallest $\tau_1 \in (t_c, t_d]$ such that $\dot{\gamma}_2(\tau_1) = s(\tau_1)$, which is a consequence of the intermediate value theorem.

- If $i \geq 2$, then $\varphi_i > \gamma_2$ somewhere.

  - Let $i \geq 1$, we show that $\varphi_{i+1}(t) > \gamma_2(t)$ for some t. Recall the lower bound property, so $\gamma_2(t) \geq \varphi_i(t)$ and $\dot{\gamma}_2(t) \geq \dot{\varphi}_i(t)$ for $t \geq \tau_i$. Define $\hat{\tau}_i := \max T_i$, such that $T_i = [\tau_i, \hat{\tau}_i]$, then by definition of T_i, there must be some $\delta > 0$ such that

$$\gamma_2(\hat{\tau}_i + \delta) > \varphi_i(\hat{\tau}_i + \delta), \tag{15}$$

  since otherwise $\gamma_2 = \varphi_i$ on some open neighborhood of $\hat{\tau}_i$ and then also

$$\dot{\gamma}_2(t) = \dot{\varphi}_i(t) \overset{(13)}{=} g(t, \varphi_i(t)) = g(t, \gamma_2(t)), \tag{16}$$

  which contradicts the definition of $\hat{\tau}_i$. Therefore, we have $\gamma_2(t) > \varphi_i(t)$ for all $t \geq \hat{\tau}_i + \delta$. For $t = \tau_{i+1}$, in particular, it follows that $\varphi_{i+1}(\tau_{i+1}) = \gamma_2(\tau_{i+1}) > \varphi_i(\tau_{i+1})$, which shows that $\varphi_{i+1} > \varphi_i$ on (ξ_i, ∞), due to Property 2, but this means that $\varphi_{i+1}(\tau_i) > \varphi_i(\tau_i) = \gamma_2(\tau_i)$.

- If $\varphi_i > \gamma_2$ somewhere, then $i \geq 2$.

  - Suppose $\varphi_i(t_x) > \gamma_2(t_x)$ for some $t_x \in (t_c, \tau_i)$, then there must be some $\tau_0 \in (t_c, t_x)$ such that $\gamma_2(\tau_0) = \varphi_i(\tau_0)$ and $\dot{\gamma}_2(\tau_0) < \dot{\varphi}(\tau_0)$. Note that this crossing must happen because we require $\xi_i < t_c$.
  - Since $g(t, x)$ is non-decreasing in x, we have

$$s(t) = g(t, \gamma_2(t)) \leq g(t, \varphi_i(t)) = \dot{\varphi}_i(t), \tag{17}$$

  for every $t \in [\tau_0, \tau_i]$ and at the endpoints, we have

$$s(\tau_0) = \varphi_i(\tau_0), \quad s(\tau_i) = \varphi_i(\tau_i). \tag{18}$$

  Furthermore, observe that $\gamma_2(\tau_0) = \varphi_i(\tau_0)$ and $\gamma_2(\tau_i) = \varphi_i(\tau_i)$ require that

$$\int_{\tau_0}^{\tau_i} \dot{\gamma}_2(t)dt = \int_{\tau_0}^{\tau_i} \dot{\varphi}_i(t)dt. \tag{19}$$

  - Since $\dot{\gamma}_2(\tau_0) < \dot{\varphi}_i(\tau_0)$, it follows from (19) that there must be some $t \in (\tau_0, \tau_i)$ such that $\dot{\gamma}_2(t) > \dot{\varphi}_i(t)$. Together with $s(\tau_0) = \dot{\varphi}_i(\tau_0) > \dot{\gamma}_2(\tau_0)$ and $s(t) \leq \dot{\varphi}_i(t)$ for $t \in [\tau_0, \tau_i]$, this means there is some τ^* such that $\dot{\gamma}_2(\tau^*) = s(\tau^*)$, again as a consequence of the intermediate value theorem. Therefore, $\tau^* \in T_j$ for some $j < i$, which shows that $i \geq 2$.

- The above two points establish that $\varphi_i \leq \gamma_2$ if and only if $i = 1$. To conclude, we have shown that $\varphi := \varphi_1$ exists and is the unique trajectory satisfying the stated requirements with $\tau = \tau_i$ and $\xi = \xi(\tau_i, \gamma_2(\tau_i))$. □

Remark 1. It is easy to see that condition (C1) in Lemma 1 is necessary. Suppose there is some $t_x \in (t_c, \infty)$ such that $\gamma_1[a_1](t_x) > \gamma_2(t_x)$, then for any other $\xi \in (a_1, t_c)$, we have $\gamma_1[\xi](t_x) > \gamma_2(t_x)$ as well, due to the lower bound property of stopping trajectories, so requirement *(iii)* is violoated. Condition (C2) is not necessary, which can for example be seen from the stopping trajectory in Figure 1 satisfying the conditions (in grey), which would have been valid even if γ_2 ended somewhat earlier than t_f, for example until the open dot.

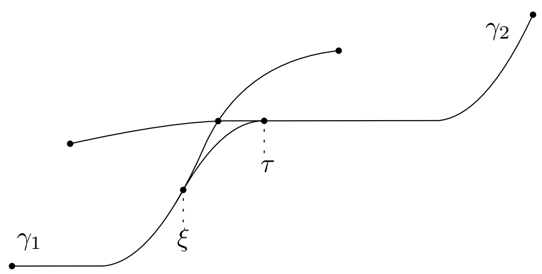

Figure 2: Two intersecting trajectories joined together by a part of a stopping trajectory.

Suppose we have two trajectories that cross each other exactly once. Lemma 1 gives conditions under which, roughly speaking, these trajectories can be glued together to form a smooth trajectory by introducing a stopping trajectory in between, as illustrated in Figure 2. Suppose we have two trajectories $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ that are intersecting at exactly a single time t_c. We are looking for some trajectory ψ that satisfies $\psi \in \mathcal{D}[a_1, b_2]$ and $\psi \leq \min\{\gamma_1, \gamma_2\}$. When the two trajectories intersect tangentially, i.e., with equal derivatives at t_c, it is clear that $\min\{\gamma_1, \gamma_2\}$ is the unique trajectory satisfying these requirements. When the intersection is not tangentially, it follows from Lemma 1 that ψ is given by

$$\psi(t) = \begin{cases} \gamma_1(t) & \text{for } t < \tau, \\ \varphi(t) & \text{for } t \in [\tau, \xi], \\ \gamma_2(t) & \text{for } t > \xi, \end{cases} \tag{20}$$

where φ and (τ, ξ) are as given by Lemma 1. To conclude, the above discussion motivates and justifies the following definition.

Definition 2. *Let $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ and suppose they intersect at exactly a single time t_c. We write $\gamma_1 * \gamma_2$ to denote the unique trajectory that satisfies $\gamma_1 * \gamma_2 \in \mathcal{D}[a_1, b_2]$ and $\gamma_1 * \gamma_2 \leq \min\{\gamma_1, \gamma_2\}$.*

Lemma 2. *Let $\gamma_1 \in \mathcal{D}[a_1, b_2]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$ be such that $\gamma_1 * \gamma_2$ exists. All trajectories $\gamma \in \mathcal{D}[a, b]$ that are such that $\gamma \leq \min\{\gamma_1, \gamma_2\}$, must satisfy $\gamma \leq \gamma_1 * \gamma_2$.*

Proof. Write $\psi := \gamma_1 * \gamma_2$ as a shorthand. We obviously have $\gamma \leq \psi$ on $[a_1, \xi] \cup [\tau, b_2]$, so consider the interval (ξ, τ) of the joining deceleration part. Suppose there exists some $t_d \in (\xi, \tau)$ such that $\gamma(t_d) > \psi(t_d)$. Because $\gamma(\xi) \leq \psi(\xi)$, this means that γ must intersect ψ at least once in $[\xi, t_d)$, so let $t_c := \sup\{t \in [\xi, t_d) : \gamma(t) = \psi(t)\}$ be the latest time of intersection such that $\gamma \geq \psi$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{\gamma}(t_v) > \dot{\psi}(t_v)$, otherwise

$$\gamma(t_d) = \gamma(t_c) + \int_{t_c}^{t_d} \dot{\gamma}(t) dt \leq \psi(t_c) + \int_{t_c}^{t_d} \dot{\psi}(t) dt = \psi(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{\gamma}(t) \geq \dot{\gamma}(t_v) - \omega(t - t_v) > \dot{\psi}(t_v) - \omega(t - t_v) = \dot{\psi}(t).$$

It follows that $\gamma(\tau) > \psi(\tau)$, which contradicts $\gamma \leq \gamma_2$. □

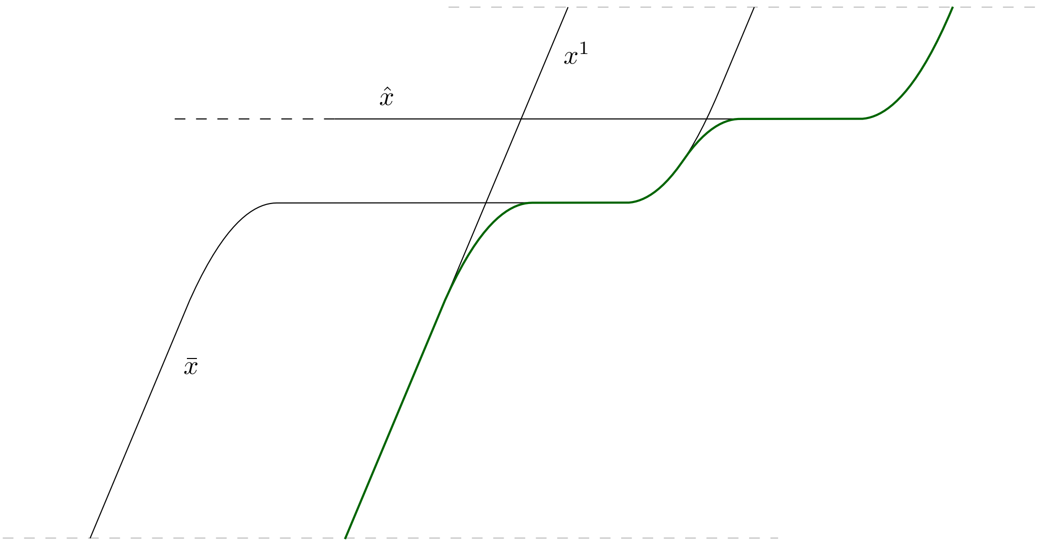

Figure 3: Sketch of how the three boundaries are joined to form the optimal trajectory.

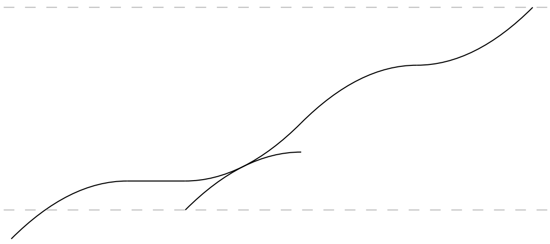

Figure 4: Illustration of "buffer constraint".

Next, consider the set $D[a,b] \subset \mathcal{D}[a,b]$ of trajectories γ that satisfy the following additional constraints

$$\gamma(a) = A, \quad \gamma(b) = B, \quad \dot{\gamma}(a) = \dot{\gamma}(b) = 1, \tag{21}$$

for some $A \leq B$ such that $B - A \geq (\omega + \bar{\omega})/2$.

For every such trajectory $\gamma \in D[a,b]$, we have $\dot{\gamma}(t) + \bar{\omega}(b-t) \geq \dot{\gamma}(b) = 1$, which can be rewritten to $\dot{\gamma}(t) \geq 1 - \bar{\omega}(b-t)$. Combined with $\dot{\gamma}(t) \geq 0$, this gives

$$\dot{\gamma}(t) \geq \max\{0, 1 - \bar{\omega}(b-t)\}. \tag{22}$$

Hence, we derive the upper bound

$$\gamma(t) = \gamma(b) - \int_t^b \dot{\gamma}(\tau) d\tau \tag{23a}$$

$$\leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b-\tau)\} d\tau =: \hat{x}(t), \tag{23b}$$

and observe that $\hat{x} \in D(-\infty, b]$. Furthermore, let $x^1 \in D(-\infty, \infty)$ be defined as $x^1(t) = A + t - a$, then it clearly an upper bound for any trajectory $\gamma \in D[a,b]$.

A Miscellaneous

Lemma 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $y \in \mathbb{R}^m$, then the level set $N := f^{-1}(\{y\})$ is a closed subset of $\mathbb{R}^n$.*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. □

Lemma 4. *Let $f : D \to \mathbb{R}^n$ be a function that is continuous in t and globally Lipschitz continuous in x. If there exists some closed rectangle $D \subseteq \mathbb{R} \times \mathbb{R}^n$ such that $(t_0, x_0) \in \operatorname{int} D$, then there exists some $\varepsilon > 0$ such that the initial value problem*

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{24}$$

has a unique solution $x(t)$ on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

The above existence and uniqueness theorem is also known as the Picard-Lindelöf or Cauchy-Lipschitz theorem. The above statement is based on the Wikipedia page on this theorem, so we still need a slightly better reference.