Trajectory optimization for vehicles in a lane model with minimum following distance and boundary conditions

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Abstract

This section considers a model of a single-lane road on which overtaking is not allowed. Vehicles are modeled as double integrators with bounds on speed and acceleration. Consecutive vehicles must keep some fixed *following distance* to avoid collisions. It is assumed that vehicles enter and exit the lane at predetermined *schedule times*. Whenever a vehicle enters or exits, it must drive at full speed. For an optimization objective that, roughly speaking, minimizes the distance to the end of the lane at all times, we present an algorithm to compute an optimal set of trajectories. Assuming some minimum lane length, we characterize feasibility of this trajectory optimization problem in terms of a system of linear inequalities involving the schedule times.

1 Lane model

Vehicles are modeled as double integrators with bounded speed and acceleration, which means that we only consider their longitudinal position on the road. Let $\mathcal{D}[a,b]$ denote the set of valid *trajectories*, which we define to be all continuously differentiable functions $x : [a,b] \to \mathbb{R}$ satisfying the constraints

$$0 \leq \dot{x}(t) \leq 1 \quad \text{ and } \quad -\omega \leq \ddot{x}(t) \leq \bar{\omega}, \quad \text{ for all } t \in [a,b], \tag{1}$$

for some fixed acceleration bounds $\omega, \bar{\omega} > 0$. Note that we use $\dot{x}$ and $\ddot{x}$ to denote the first and second derivative with respect to time t. When we have a general positive speed upper bound, we can always apply an appropriate scaling of the time axis and the acceleration bounds to obtain the form. Consider positions $A, B \in \mathbb{R}$, such that $B \geq A$, which denote the start[1] and end position of the lane. Let $\bar{D}[a,b] \subset \mathcal{D}[a,b]$ denote the set of trajectories x that satisfy the boundary conditions

$$x(a) = A \;\text{ and }\; x(b) = B. \tag{2}$$

Even further, let $D[a,b] \subset \bar{D}[a,b]$ induce the boundary conditions

$$\dot{x}(a) = \dot{x}(b) = 1. \tag{3}$$

In words, these boundary conditions require that a vehicle arrives to and departs from the lane at predetermined times a and b and do so at full speed.

Let $L > 0$ denote the required *following distance* between consecutive vehicles. Suppose we have N vehicles that are scheduled to traverse the lane. For each vehicle i, let a_i and b_i denote the *schedule time* for entry and exit, respectively. A feasible solution consists of a sequence of trajectories $x_1, \dots, x_N$ such that

$$\begin{aligned} x_i &\in D[a_i, b_i] && \text{for each } i, && \text{(4a)} \\ x_i &\leq x_{i-1} - L && \text{for each } i \geq 2, && \text{(4b)} \end{aligned}$$

[1]We could have assumed $A = 0$, but we will later piece together multiple lanes to model intersections.

where we use the shorthand notation $\gamma_1 \leq \gamma_2$ to mean $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in [a_1, b_1] \cap [a_2, b_2]$, given some $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$.

As optimization objective, we will consider

$$\max \sum_{i=1}^{N} \int_{a_i}^{b_i} x_i(t)\,\mathrm{d}t, \tag{5}$$

which, roughly speaking, seeks to keep all vehicles as close to the end of the lane at all times. In particular, we will show that optimal trajectories can be understood as the concatenation of at most four different types of trajectory parts. Based on this observation, we present an algorithm to compute optimal trajectories. Assuming $(\omega, \bar{\omega}, A, B, L)$ to be fixed, with lane length $B - A$ sufficiently large, we will show that feasibility of the trajectory optimization problem is completely characterized by a system of linear inequalities in a_i and b_i.

Note that objective (5) does not capture energy efficiency in any way. Although this is not desirable in practice, it is precisely this assumption that enables the analysis in this section. Deriving a similar characterization of feasibility and optimal trajectories under an objective that does model energy concumption is an interesting topic for further research.

2 Single vehicle problem

We will first consider a somewhat generalized version of the constraints (4) for a single vehicle i. Therefore, we lighten the notation slightly by dropping the vehicle index i and instead of $x_{i-1} - L$, we assume we are given some arbitrary *lead vehicle boundary* $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$, then we consider the optimization problem

$$\max_{x \in D[a,b]} \int_a^b x(t)\,\mathrm{d}t \quad \text{ such that } \quad x \leq \bar{x}. \tag{6}$$

to which we will refer as the *single vehicle problem.*

2.1 Necessary conditions

For every trajectory $x \in D[a, b]$, we derive two upper bounding trajectories x^1 and $\hat{x}$ and one lower bounding trajectory $\check{x}$, see Figure 1. Using these bounding trajectories, we will then formulate four necessary conditions for feasibility of the single vehicle problem.

Let the *full speed boundary* x^1 be defined as

$$x^1(t) = A + t - a, \tag{7}$$

for all $t \in [a, b]$, then we clearly have $x \leq x^1$. Next, since deceleration is at most ω, we have $\dot{x}(t) \geq \dot{x}(a) - \omega(t - a) = 1 - \omega(t - a)$, which we combine with the speed constraint $\dot{x} \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \omega(t - a)\}$. Hence, we obtain the lower bound

$$x(t) = x(a) + \int_a^t \dot{x}(\tau)\,\mathrm{d}\tau \geq A + \int_a^t \max\{0, 1 - \omega(\tau - a)\}\,\mathrm{d}\tau =: \check{x}(t). \tag{8}$$

Analogously, we derive an upper bound from the fact that acceleration is at most $\bar{\omega}$. Observe that we have $\dot{x}(t) + \bar{\omega}(b - t) \geq \dot{x}(b) = 1$, which we combine with the speed constraint $\dot{x}(t) \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \bar{\omega}(b - t)\}$. Hence, we obtain the upper bound

$$x(t) = x(b) - \int_t^b \dot{x}(\tau)\,\mathrm{d}\tau \leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b - \tau)\}\,\mathrm{d}\tau =: \hat{x}(t). \tag{9}$$

We refer to $\check{x}$ and $\hat{x}$ as the *entry boundary* and *exit boundary*, respectively.

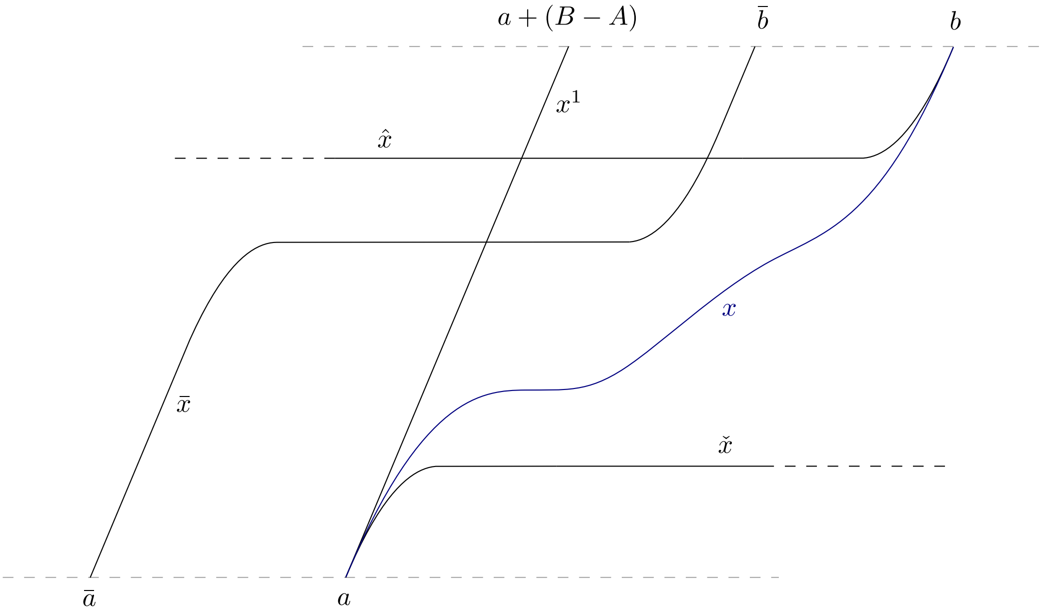

Figure 1: Illustration of the four bounding trajectories $\bar{x}, x^1, \hat{x}, \check{x}$ that bound feasible trajectories from above and below. We also drew an example of a feasible trajectory x in blue. The horizontal axis represents time and the vertical axis corresponds to the position on the lane, so the vertical dashed grey lines correspond to the start and end of the lane.

Lemma 1. *Let $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$ and assume there exists a trajectory $x \in D[a, b]$ such that $x \leq \bar{x}$, then the following conditions must hold*

(i) $b - a \geq B - A,$ *(full speed constraint)*

(ii) $\bar{b} \leq b,$ *(downstream order constraint)*

(iii) $\bar{a} \leq a,$ *(upstream order constraint)*

(iv) $\bar{x} \geq \check{x}.$ *(entry space constraint)*

Proof. Each of the conditions corresponds somehow to one of the four bounding trajectories defined above. Observe that $x^1(t) = B$ for $t = a + (B - A)$, which can be interpreted as the earliest time of departure from the lane. This shows that $b \geq a + (B - A)$, which is equivalent with (i). When either (ii) or (iii) is violated, the constraint $x \leq \bar{x}$ conflicts with one of the boundary conditions $x(a) = A$ or $x(b) = B$. To see that (iv) must hold, suppose that $\bar{x}(\tau) < \check{x}(\tau)$ for some time τ. Since $\bar{a} \leq a$, this means that $\bar{a}$ must intersect $\check{a}$ from above. Therefore, any trajectory that satisfies $x \leq \bar{x}$ must also intersect $\check{a}$ from above, which contradicts the assumption that x was a feasible solution. □

We show that the boundaries $\hat{x}$ and $\check{x}$ together could yield yet another necessary condition. It is straightforward to verify from equations (8) and (9) that $\hat{x}(t) \geq B - 1/(2\bar{\omega})$ and $\check{x}(t) \leq A + 1/(2\omega)$. Therefore, whenever $B - A < 1/(2\bar{\omega}) + 1/(2\omega)$, these boundaries intersect for certain values of a and b. Because the exact condition is somewhat cumbersome to characterize, we avoid this case by simply assuming that the lane length is sufficiently large.

Assumption 1. *The length of the lane satisfies $B - A \geq 1/(2\omega) + 1/(2\bar{\omega})$.*

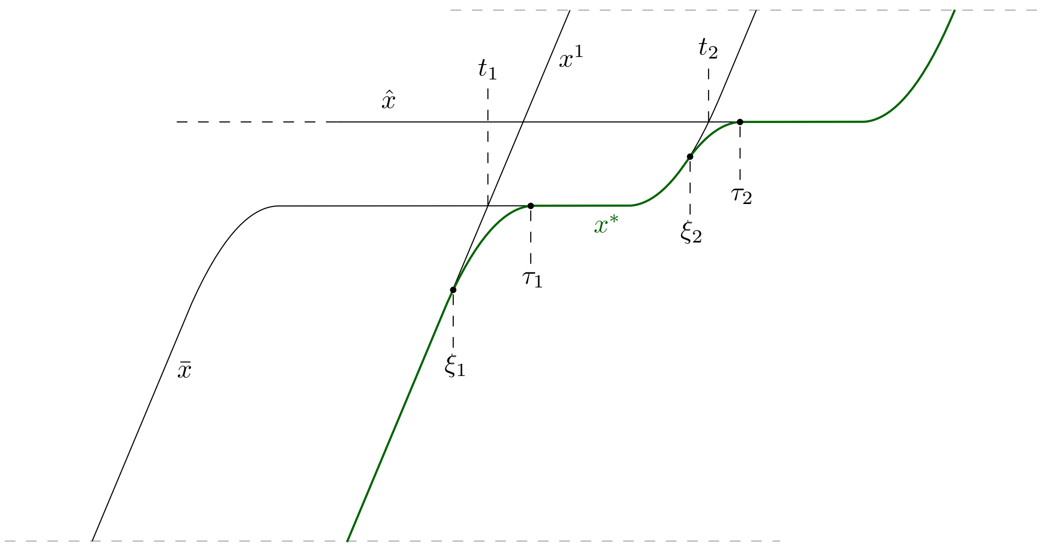

Figure 2: The minimum boundary γ, induced by three upper boundaries $\bar{x}$, $\hat{x}$ and x^1, is smoothened around time t_1 and t_2, where the derivative is discontinuous, to obtain the smooth optimal trajectory x^*, drawn in green. The times ξ_i and τ_i correspond to the start and end of the connecting deceleration boundary as defined in Section 2.2.2.

2.2 Constructing the optimal trajectory

Assuming the four conditions of Lemma 1 hold, we will construct an optimal solution x^* for the single vehicle problem, thereby showing that these conditions are thus also sufficient for feasibility. First, we construct x^* by combining the upper boundaries $\bar{x}$, $\hat{x}$ and x^1 in a certain way to obtain a smooth trajectory satisfying $x^* \in D[a, b]$. We show that x^* is still an upper boundary for any other feasible solution, which shows that it is optimal.

The starting point of the construction is the *minimum boundary* $\gamma : [a, b] \to \mathbb{R}$ induced by the upper boundaries, defined as

$$\gamma(t) := \min\{\bar{x}(t), \hat{x}(t), x^1(t)\}. \tag{10}$$

Obviously, γ is still a valid upper boundary for any feasible solution, but in general, γ may have a discontinuous derivative at some[2] isolated points in time.

Definition 1. *Let $\mathcal{P}[a, b]$ be the set of functions $\mu : [a, b] \to \mathbb{R}$ for which there is a finite subdivision $a = t_0 < \cdots < t_{n+1} = b$ such that the truncation $\mu|_{[t_i, t_{i+1}]} \in \mathcal{D}[t_i, t_{i+1}]$ is a smooth trajectory, for each $i \in \{0, \ldots, n\}$, and for which the one-sided limits of $\dot{\mu}$ satisfy*

$$\dot{\mu}(t_i^-) := \lim_{t \uparrow t_i} \dot{\mu}(t) > \lim_{t \downarrow t_i} \dot{\mu}(t) =: \dot{\mu}(t_i^+), \tag{11}$$

for each $i \in \{1, \ldots, n\}$. We refer to such μ as a piecewise trajectory (with downward bends).

It is not difficult to see from Figure 1 that, under the necessary conditions, γ satisfies the above definition, so $\gamma \in \mathcal{P}[a, b]$. In other words, γ consists of a number of pieces that are smooth and satisfy the vehicle dynamics, with possibly some sharp bend downwards where these pieces come together. Next, we present a simple procedure to smoothen out this kind of discontinuity by decelerating from the original trajectory somewhat before some t_i, as illustrated in Figure 2. We show that this procedure can be repeated at every point of discontinuity.

[2] In fact, it can be shown that, under the necessary conditions, there are at most two of such discontinuities.

2.2.1 Deceleration boundary

In order to formalize the smoothing procedure, we will first define some parameterized family of functions to model the deceleration part. Recall the derivation of $\check{x}$ in equation (8) and the discussion preceding it, which we will now generalize a bit. Let $x \in \mathcal{D}[a, b]$ be some smooth trajectory, then observe that $\dot{x}(t) \geq \dot{x}(\xi) - \omega(t - \xi)$ for all $t \in [a, b]$. Combining this with the constraint $\dot{x}(t) \in [0, 1]$, this yields

$$\dot{x}(t) \geq \max\{0, \, \min\{1, \, \dot{x}(\xi) - \omega(t - \xi)\}\} =: \{\dot{x}(\xi) - \omega(t - \xi)\}_{[0,1]}, \tag{12}$$

where use $\{\cdot\}_{[0,1]}$ as a shorthand for this clipping operation. Hence, for any $t \in [a, b]$, we obtain the following lower bound

$$x(t) = x(\xi) + \int_{\xi}^{t} \dot{x}(\tau) \, \mathrm{d}\tau \geq x(\xi) + \int_{\xi}^{t} \{\dot{x}(\xi) - \omega(\tau - \xi)\}_{[0,1]} \, \mathrm{d}\tau =: x[\xi](t), \tag{13}$$

where we will refer to the right-hand side as the *deceleration boundary* of x at ξ. Observe that this definition indeed generalizes the definition of $\check{x}$, because we have $\check{x} = (x[a])|_{[a,b]}$, so $x[a]$ restricted to the interval $[a, b]$.

Note that $x[\xi]$ depends on x only through the two real numbers $x(\xi)$ and $\dot{x}(\xi)$. It will be convenient later to rewrite the right-hand side of (13) as

$$x^{-}[p, v, \xi](t) = p + \int_{\xi}^{t} \{v - \omega(\tau - \xi)\}_{[0,1]} \, \mathrm{d}\tau, \tag{14}$$

such that $x[\xi](t) = x^{-}[x(\xi), \dot{x}(\xi), \xi](t)$. We can expand the integral in this expression further by carefully handling the clipping operation. Observe that the expression within the clipping operation reaches the bounds 1 and 0 for $\delta_1 := \xi - (1 - v)/\omega$ and $\delta_0 := \xi + v/\omega$, respectively. Using this notation, a simple calculation shows that

$$x^{-}[p, v, \xi](t) = p + \begin{cases} (1 - v)^2/(2\omega) + (t - \xi) & \text{for } t \leq \delta_1, \\ v(t - \xi) - \omega(t - \xi)^2/2 & \text{for } t \in [\delta_1, \delta_0], \\ v^2/(2\omega) & \text{for } t \geq \delta_0. \end{cases} \tag{15}$$

Assuming $0 \leq v \leq 1$, it can be verified that for every $t \in \mathbb{R}$, we have $\ddot{x}^{-}[p, v, \xi](t) \in \{-\omega, 0\}$ and $\dot{x}^{-}[p, v, \xi](t) \in [0, 1]$ due to the clipping operation, so that $x^{-}[p, v, \xi] \in \mathcal{D}(-\infty, \infty)$.

Let $\mu \in \mathcal{P}[a, b]$ be some piecewise trajectory and let $a = t_0 < \cdots < t_{n+1} = b$ denote the corresponding subdivision as in Definition 1, then we generalize the definition of a deceleration boundary to μ. Whenever $\xi \in [a, b] \setminus \{t_1, \ldots, t_n\}$, we just define $\mu[\xi] := x^{-}[\mu(\xi), \dot{\mu}(\xi), \xi]$. However, when $\xi \in \{t_1, \ldots, t_n\}$, the derivative $\dot{\mu}(\xi)$ is not defined, so we to use the left-sided limit instead, by defining $\mu[\xi] := x^{-}[\mu(\xi), \dot{\mu}(\xi^{-}), \xi]$.

Finally, please note that we cannot just replace x with μ in inequality (13) to obtain a similar bound for the whole interval $[a, b]$. Instead, consider some interval $I \in \{[a, t_1], (t_1, t_2], \ldots, (t_n, b]\}$, then what remains true is that $\xi \in I$ implies $\mu(t) \geq \mu[\xi](t)$ for every $t \in I$.

2.2.2 Smoothing procedure

Let $\mu \in \mathcal{P}[a, b]$ be some piecewise trajectory and let $a = t_0 < \cdots < t_{n+1} = b$ again denote the subdivision as in Definition 1. We first show how to smoothen the discontinuity at t_1 and then argue how to repeat this process for the remaining times t_i.

Assumption 2. *Throughout the following discussion, we assume $\mu \geq \mu[a]$ and $\mu \geq \mu[b]$.*

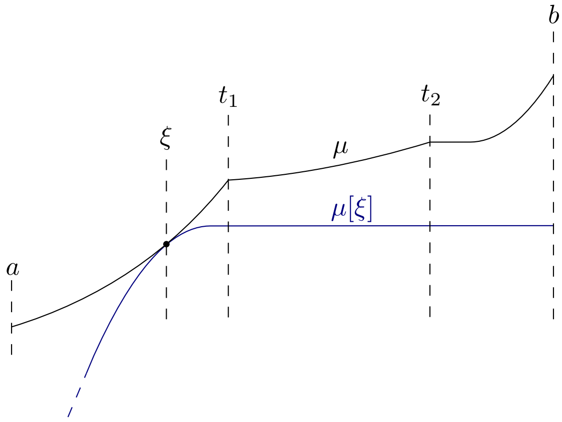

Figure 3: Illustration of a deceleration boundary $\mu[\xi]$ of some piecewise trajectory $\mu \in \mathcal{P}[a, b]$ at time ξ. We truncated $\mu[\xi]$ for a more compact figure. This particular trajectory μ has a discontinuous derivative at times t_1 and t_2. The careful reader may notice that this μ cannot occur as the minimum boundary defined in (10). This only means that the class of piecewise trajectories $\mathcal{P}[a, b]$ is just slightly more general than necessary for our current purposes.

We want to pick some $\xi \in [a, t_1)$, such that $\mu[\xi] \leq \mu$ and such that $\mu[\xi]$ touches μ at some time $\tau \in [t_1, b]$ tangentially. Therefore, we measure the relative position of $\mu[\xi]$ with respect to μ on this interval, by considering

$$d(\xi) := \min_{t \in [t_1, b]} \mu(t) - \mu[\xi](t). \tag{16}$$

Since $\mu(t)$ and $\mu[\xi](t)$ are both continuous in t, this minimum actually exists due to the extreme value theorem (Weierstrass). Furthermore, since the minimum is taken of a continuous function over a closed interval, d is a continuous function as well (see Lemma A.2).

Existence. Observe that $d(a) \geq 0$, because $\mu \geq \mu[a]$ by assumption. By definition of t_1, we have $\dot{\mu}(t_1^-) > \dot{\mu}(t_1^+)$, from which it follows that $\mu < \mu[t_1]$ on $(t_1, t_1 + \epsilon)$ for some small $\epsilon > 0$, which shows that $d(t_1) < 0$. Therefore, the intermediate value theorem ensures that there is some $\xi_1 \in [a, t_1)$ such that $d(\xi_1) = 0$.

Uniqueness. We will see that ξ_1 itself is not necessarily unique. Instead, we are going to show that the connecting deceleration boundary $\mu[\xi_1]$ is unique. More precisely, for any $\xi \in [a, t_1)$ such that $d(\xi) = 0$, we will show that $\mu[\xi] = \mu[\xi_1]$.

The first step is to establish that the level set

$$X := \{\xi \in [a, t_1) : d(\xi) = 0\} \tag{17}$$

is a closed interval. To this end, we show that d is non-increasing on $[a, t_1)$, which together with continuity implies the desired result (see Lemma A.1). To show that d is non-increasing, it suffices to show that $\mu[\xi](t)$ is non-decreasing as a function of ξ, for every $t \in [t_1, b]$. We can do this by computing the partial derivative of $\mu[\xi]$ with respect to ξ and verifying it is non-negativity. Recall the definition of $\mu[\xi]$, based on x^- in equation (15). Using the same notation, we write $\delta_1 = \xi - (1 - \dot{\mu}(\xi))/\omega$ and $\delta_0 = \xi + \dot{\mu}(\xi)/\omega$ and compute

$$\frac{\partial}{\partial \xi}\mu[\xi](t) = \dot{\mu}(\xi) + \begin{cases} \ddot{\mu}(\xi)(\dot{\mu}(\xi) - 1)/\omega - 1 & \text{for } t \leq \delta_1, \\ \ddot{\mu}(\xi)(t - \xi) - \dot{\mu}(\xi) + \omega(t - \xi) & \text{for } t \in [\delta_1, \delta_0], \\ \ddot{\mu}(\xi)\dot{\mu}(\xi)/\omega & \text{for } t \geq \delta_0. \end{cases} \tag{18}$$

It is easily verified that these three cases match at δ_1 and δ_0, which justifies the overlapping cases for t. Consider any $\xi \in [a, t_1)$ and $t \in [t_1, b]$, then we have $\delta_1 \leq \xi \leq t$, so we verify the second and third case. In the second case, so $t \in [\delta_1, \delta_0]$, we have

$$\frac{\partial}{\partial \xi}\mu[\xi](t) = (\ddot{\mu}(\xi) + \omega)(t - \xi) \geq 0. \tag{19}$$

In the third case, so $t \geq \delta_0$, we have

$$\frac{\partial}{\partial \xi}\mu[\xi](t) \geq \dot{\mu}(\xi) + (-\omega)\dot{\mu}(\xi)/\omega = 0. \tag{20}$$

This concludes the argument for X being a closed interval.

For the above two cases, observe that we have equality in (19) if and only if there is equality in (20), which happens when $\ddot{\mu}(\xi) = -\omega$. This observation is the key for the remaining argument. Next, let $\xi \in X$ such that $d(\xi) = 0$. Let $t_1^* \in [t_1, b]$ be such that

$$\mu(t) - \mu[\xi_1](t_1^*) = 0 \tag{21}$$

Matching derivatives. It remains to show that $\mu[\xi_1]$ touches μ tangentially.

- Let $t^* \in [t_1, b]$ be a minimizer of the minimum $d(\xi_1) = 0$, such that $\mu(t^*) = \mu[\xi_1](t^*)$. If $\tau_1 \in (t_1, b)$, then obviously $\dot{\mu}(\tau_1) = \dot{\mu}[\xi_1](\tau_1)$ is a necessary condition of a local minimum. Note that $\tau_1 = t_1$ cannot happen.
- When $\tau_1 = b$, the derivatives must match as a consequence of assumption $\mu \geq \mu[b]$.

Repeat for remaining discontinuities.

- Conclude that we have found a ξ, τ such that the boundary $\mu[\xi]_{[\xi,\tau]}$ can be added to μ to obtain $\mu' \in \mathcal{P}[a, b]$ with one less discontinuity. For μ', we can repeat the exact same process as above. Eventually, we end up with a trajectory $\mu^* \in \mathcal{D}[a, b]$.
- Stress again that the smoothing procedure leaves $\dot{\mu}^*(a) = \dot{\mu}(a)$ and $\dot{\mu}^*(b) = \dot{\mu}(b)$ untouched.

2.2.3 Optimality after smoothing

Observe that the necessary conditions require $\gamma(a) = A$, $\gamma(b) = B$ and $\dot{\gamma}(a) = \dot{\gamma}(b) = 1$, so whenever we have $\gamma \in \mathcal{D}[a, b]$, we automatically have $\gamma \in D[a, b]$ so that γ is already an optimal solution.

Lemma 2. *Let $\mu \in \mathcal{P}[a, b]$ be a piecewise trajectory and let $\mu^* \in \mathcal{D}[a, b]$ denote the result after smoothing. All trajectories $x \in \mathcal{D}[a, b]$ that are such that $x \leq \mu$, must satisfy $x \leq \mu^*$.*

Proof. Consider the interval (ξ, τ) of a joining deceleration part. Suppose there exists some $t_d \in (\xi, \tau)$ such that $x(t_d) > \mu(t_d)$. Because $x(\xi) \leq \mu(\xi)$, this means that x must intersect μ at least once in $[\xi, t_d)$, so let $t_c := \sup\{t \in [\xi, t_d) : x(t) = \mu(t)\}$ be the latest time of intersection such that $x \geq \mu$ on $[t_c, t_d]$. There must be some $t_c \in [t_c, t_d]$ such that $\dot{x}(t_v) > \dot{\mu}(t_v)$, otherwise

$$x(t_d) = x(t_c) + \int_{t_c}^{t_d} \dot{x}(t)\,\mathrm{d}t \leq \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t)\,\mathrm{d}t = \mu(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{x}(t) \geq \dot{x}(t_v) - \omega(t - t_v) > \dot{\mu}(t_v) - \omega(t - t_v) = \dot{\mu}(t).$$

It follows that $x(\tau) > \mu(\tau)$, which contradicts the assumption. $\square$

3 Computing optimal trajectories

Recall the original trajectory optimization problem

$$
\begin{aligned}
G(a,b) := \max \quad & \sum_{i=1}^{N} \int_{a_i}^{b_i} x_i(t)\,\mathrm{d}t, && (22a)\\
\text{s.t.} \quad & x_i \in D[a_i,b_i] \quad \text{for each } i \in \{1,\dots,N\}, && (22b)\\
& x_i \le x_{i-1} - L \quad \text{for each } i \in \{2,\dots,N\}, && (22c)
\end{aligned}
$$

where we use a and b to denote the vectors of arrival and departure times. It is straightforward to decompose this problem into a sequence of instances of the single vehicle problems as follows. Let the optimal solution of the single vehicle problem be denoted as

$$
x^*(\alpha,\beta,\bar{x}) := \operatorname*{arg\,max}_{x\in D[\alpha,\beta]} \int_{\alpha}^{\beta} x(t) \quad \text{such that } x \le \bar{x} \tag{23}
$$

and let $F(\alpha,\beta,\bar{x})$ denote the corresponding objective value. It is clear that the optimal trajectories x_i^* can be recursively computed as

$$
\begin{aligned}
x_1^* &= x^*(a_1,b_1,\varnothing), && (24a)\\
x_i^* &= x^*(a_i,b_i,x_{i-1}^*), \quad \text{for } i \ge 2. && (24b)
\end{aligned}
$$

We use the notation $\bar{x} = \varnothing$ to denote the single vehicle problem without the boundary constraint. Alternatively, we could think about this as having some $\bar{x} \in \bar{D}[\bar{a},\bar{b}]$ with very small $\bar{a} \ll a$ and $\bar{b} \ll b$. The corresponding objective value is simply given by

$$
G(a,b) = F(a_1,b_1,\varnothing) + \sum_{i=2}^{N} F(a_i,b_i,x_{i-1}^*). \tag{25}
$$

3.1 Alternating trajectories

Due to the recursive nature of the problem, we will see that optimal trajectories possess a particularly simple structure, which enables a very simple computation.

Definition 2. *Let a trajectory $\gamma \in \mathcal{D}[a,b]$ be called* alternating *if for all $t \in [a,b]$, we have*

$$
\ddot{\gamma}(t) \in \{-\omega, 0, \bar{\omega}\} \quad \textit{and} \quad \ddot{\gamma}(t) = 0 \implies \dot{\gamma}(t) \in \{0,1\}. \tag{26}
$$

We now argue that each vehicle's optimal trajectory x_i^* is alternating. First, consider $x_1^* = x^*(a_1,b_1,\varnothing)$, which is constructed by joining x^1 and $\hat{x}$ together by smoothing. Observe that x^1 and $\hat{x}$ are both alternating. Let $\gamma_1(t) = \min\{x_1^1(t), \hat{x}_1(t)\}$ be the minimum boundary, then it is clear that the smoothened $x_1^* = \gamma_1^*$ must also be alternating, because we only added a part of deceleration at some interval $[\xi,\tau]$, which clearly satisfies $\ddot{\gamma}_1^*(t) = -\omega$ for $t \in [\xi,\tau]$. Assume that x_{i-1}^* is alternating, we can similarly argue that x_i^* is alternating. Again, let $\gamma_i(t) = \min\{\bar{x}_{i-1}^*, \hat{x}_i(t), x_i^1(t)\}$ be the minimum boundary. After adding the required decelerations for smoothing, it is clear that $x_i^* = \gamma_i^*$ must also be alternating.

Observe that an alternating trajectory $\gamma \in \mathcal{D}[a,b]$ can be described as a sequence of four types of consecutive repeating phases, see Figure 4 for an example. In general, there exists a partition of $[a,b]$, denoted by

$$
a = t_{f1} \le t_{d1} \le t_{s1} \le t_{a1} \le t_{f2} \le t_{d2} \le t_{s2} \le t_{a2} \le \cdots \le t_{f,n+1} = b,
$$

such that we have the consecutive intervals

$$
\begin{aligned}
F_i &:= [t_{f,i}, t_{d,i}] \quad \text{(full speed)}, & S_i &:= [t_{s,i}, t_{a,i}] \quad \text{(stopped)},\\
D_i &:= [t_{d,i}, t_{s,i}] \quad \text{(deceleration)}, & A_i &:= [t_{a,i}, t_{f,i+1}] \quad \text{(acceleration)},
\end{aligned}
$$

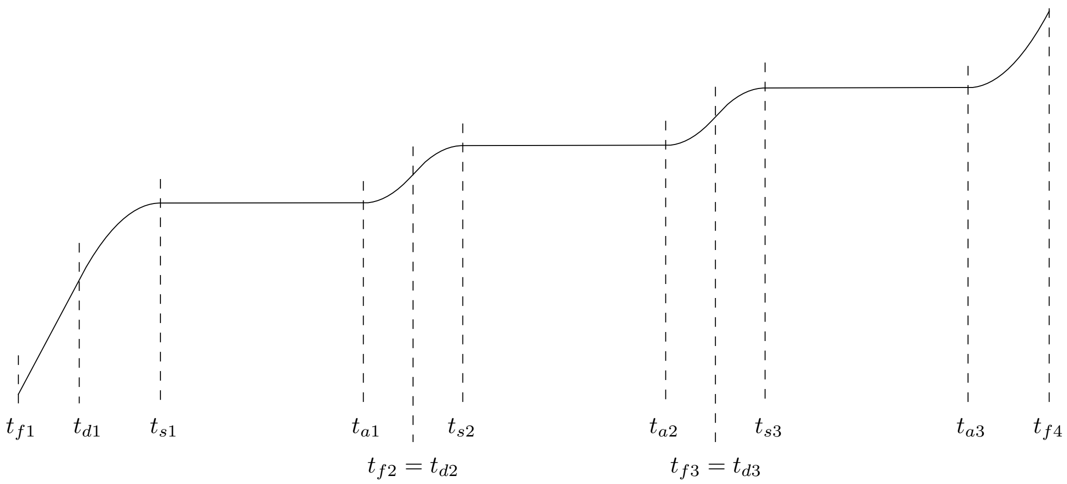

Figure 4: Some example of an alternating vehicle trajectory. The particular shape of this trajectory is due to two preceding vehicles, which causes the two *bumps* at the times where these vehicles depart from the lane.

such that on these intervals, γ satisfies

$$\begin{aligned} \dot{\gamma}(t) &= 1 && \text{for } t \in F_i, && \dot{\gamma}(t) = 0 && \text{for } t \in S_i, \\ \ddot{\gamma}(t) &= -\omega && \text{for } t \in D_i, && \ddot{\gamma}(t) = \bar{\omega} && \text{for } t \in A_i. \end{aligned}$$

We will define parameterized functions x^1, x^-, x^0, x^+ to describe γ on each of these four types of intervals. In the next section, we will show that this makes the smoothing procedure particularly simple.

3.2 Calculating smoothing times

Derive x^+ similarly to how we derived x^- when we introduced the deceleration boundary.

It can be shown that smoothing introduces a part of deceleration x^- only between the four pairs of partial trajectories

$$x^+ \to x^+, \qquad x^+ \to x^0, \qquad x^1 \to x^+, \qquad x^1 \to x^0.$$

4 Feasibility as system of linear inequalities

Show that the follow constraint $a_i \geq \bar{a}_i$ can be written in terms of a_{i-1}.

We need to express the entry space constraint, condition (iv) in Lemma 1, in terms of the schedule times. Recall that this conditions requires that

$$\bar{x}_i \geq \check{x}_i. \tag{27}$$

We will show that this condition can be rewritten in the form

$$a_i \geq \check{a}_i(a), \tag{28}$$

where $\check{a}_i(a, b)$ denotes some expression of the schedule times $a_1, \dots, a_n$.

In conclusion, feasibility is expressed through the system of linear inequalities

$$\begin{aligned} b_i - a_i &\geq B - A && \text{for all } i \in \{1, \dots, N\}, && \text{(29a)} \\ a_i &\geq a_{i-1} - 1/\omega && \text{for all } i \in \{2, \dots, N\}, && \text{(29b)} \\ b_i &\geq b_{i-1} - 1/\omega && \text{for all } i \in \{2, \dots, N\}, && \text{(29c)} \\ a_i &\leq \check{a}_i(a) && \text{for all } i \in \{2, \dots, N\}. && \text{(29d)} \end{aligned}$$

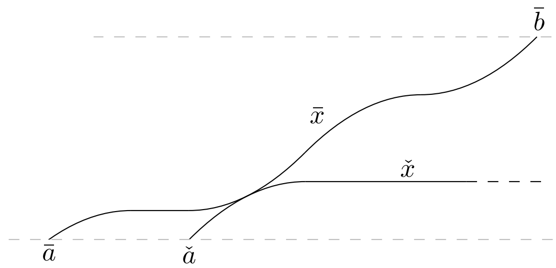

Figure 5: Illustration of entry space constraint and the induced minimum entry time $\check{a}$.

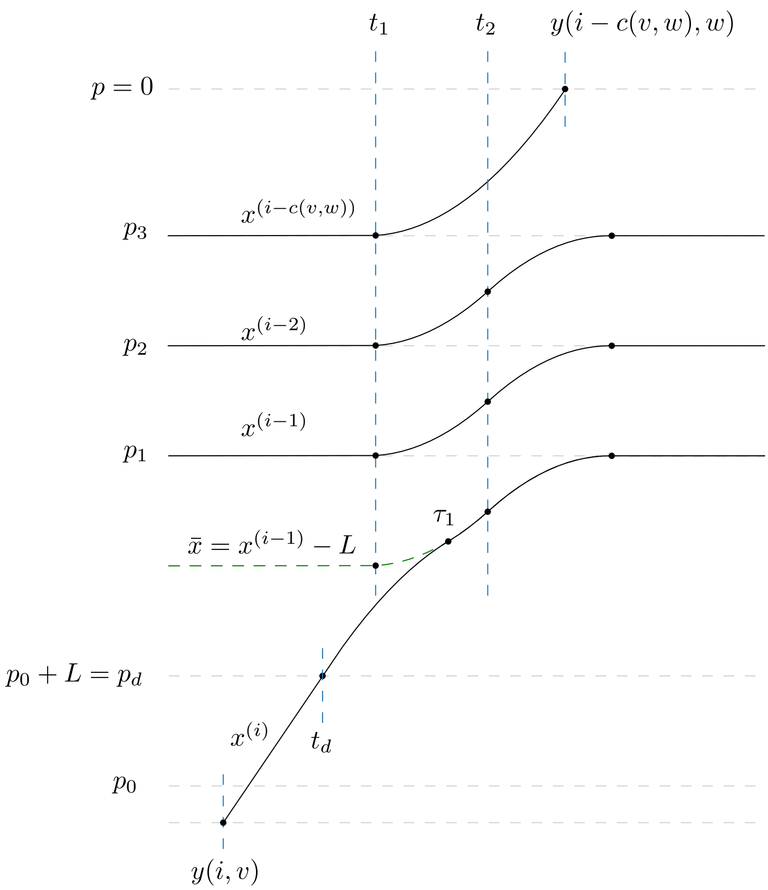

Figure 6: We include this old sketch here temporarily, which is related to derive entry space constraint in terms of schedule times.

A Miscellaneous

Lemma A.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $y \in \mathbb{R}^m$, then the level set $N := f^{-1}(\{y\})$ is a closed subset of $\mathbb{R}^n$.*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. $\square$

Lemma A.2. *Let $f : X \times Y \to \mathbb{R}$ be some continuous function. If Y is compact, then the function $g : X \to \mathbb{R}$, defined as $g(x) = \inf\{f(x, y) : y \in Y\}$, is also continuous.*