Isolated vehicle

Let the intersection be at position x_f and let the position of vehicle take non-negative real values. The trajectory for a single isolated vehicle (without vehicles in front of it) is given by the solution to the optimal control problem

$$\begin{aligned}
\max_{x} \quad & \int_{t=0}^{t_f} x(t)dt \\
\text{s.t.} \quad & 0 \leq \dot{x}(t) \leq v_{\max}, \\
& -a_{\max} \leq \ddot{x}(t) \leq a_{\max}, \\
& x(0) = 0, \dot{x}(0) = v_{\max}, \\
& x(t_f) = x_f, \dot{x}(t_f) = v_{\max},
\end{aligned}$$

where the objective may be understood in terms of keeping as close as possible to the intersection at all times. We believe that the optimal control is "bang-bang", meaning that every vehicle i has a sequence of disjoint intervals

$$I_i = (D_{i1}, A_{i1}, \dots, D_{in}, A_{in})$$

and that its controller is given by

$$a_i(t) = \begin{cases} -a_{\max} & \text{if } t \in D_{ik} \text{ for some } k, \\ a_{\max} & \text{if } t \in A_{ik} \text{ for some } k. \end{cases}$$

For the single isolated vehicle, these bang-bang intervals, or *bangs* for short, can be easily obtained. First, we will transform the current time scale to better suit our problem. State transformations are common in optimal control. For position x at time t, we define the corresponding *schedule time* by

$$\bar{t}(t, x) = t - x/v_{\max}.$$

Given the vehicle dynamics constraints, without considering the boundary conditions, the time it takes to fully accelerate from zero to maximum velocity takes time

$$T = v_{\max}/a_{\max}$$

and the corresponding trajectory x^* with initial position $x^*(0) = 0$, is given by

$$x^*(t) = a_{\max} t^2/2 \quad \text{for } 0 \leq t \leq T.$$

Time T translated to schedule time, is given by

$$\bar{T} = \bar{t}(T, x^*(T)) - \bar{t}(0, 0) = T/2.$$

What kind of transformation is $\bar{t}$? Lines form some sort of equivalence classes in time-space. Next, we define the start time $b = \bar{t}(0, 0)$ and end time $e = \bar{t}(t_f, x_f)$. Writing $(x)^+$ for $\max(x, 0)$, the amount of schedule time in which we have zero acceleration is given by

$$\bar{t}_n = (e - b - 2\bar{T})^+.$$

and the length of each bang is

$$\bar{t}_b = (e - b - \bar{t}_n)/2$$

so that the bangs are given by

$$\bar{D} = (b, b + \bar{t}_b),$$
$$\bar{A} = (e - \bar{t}_b, e).$$

Back to regular time. We are now left with translating these bangs back to the regular time scale. Let the current velocity be denoted as v_c and suppose $\bar{d}$ denotes the duration of current acceleration bang in schedule time. Consider again the full acceleration trajectory $x^*(t)$ on $0 \leq t \leq T$. Define $t_0 = v_c/a_{\max}$ so that we have $\dot{x}^*(t_0) = v_c$. Next, we find t_1 such that $t_0 \leq t_1 \leq T$ and

$$\bar{t}(t_1, x^*(t_1)) - \bar{t}(t_0, x^*(t_0)) = \bar{d}, \tag{1}$$

such that the duration of the bang in regular time is given by $d = t_1 - t_0$. After some rewriting and substitution of the definitions of x^* and $\bar{t}$ in equation (1), we obtain the quadratic equation

$$-\frac{a_{\max} t_1^2}{2 v_{\max}} + t_1 - t_0 + \frac{a_{\max} t_0^2}{2 v_{\max}} - \bar{d} = 0,$$

for which we are interested in the solution

$$t_1 = T - \sqrt{T^2 - 2T(t_0 + \bar{d}) + t_0^2}.$$

Similarly, for a deceleration bang of length $\bar{d}$ with current velocity v_c, the duration is given by $d = t_1 - t_0$ where $t_0 = (v_{\max} - v_c)/a_{\max}$ and t_1 is the solution to

$$\bar{t}(t_1, -x^*(-t_1)) - \bar{t}(t_0, -x^*(-t_0)) = \bar{d},$$

given by

$$t_1 = -T + \sqrt{T^2 + 2T(t_0 + \bar{d}) + t_0^2}.$$

Using the above formulas, we can translate a sequence of bangs in schedule time to regular time. However, we need to be careful whenever the velocity becomes $v_{\max}$, because the regular time is not unique in this case. Therefore, we specify a *target position* x_t, which fixes the regular time in these cases as follows. Let $D = (b, e)$ be some deceleration bang, then the start of the first deceleration bang in regular time is given by

$$b + (x_t - x^*(T))/v_{max}.$$

The rest of the sequence of bangs in schedule time can now be translated as follows, assuming that the velocity will not reach $v_{\max}$ until the last acceleration bang. We need to argue or prove why this assumption holds. Intuitively, if the assumption does not hold, the whole trajectory could be "shifted up" in the graph, which would decrease the objective. We keep track of the current time t_c and velocity v_c. Each time we process a bang $\bar{A}$ or $\bar{D}$, we update $v_c \leftarrow v_c \pm a_{\max} d$ accordingly, and $t \leftarrow t + d$, where d is the regular bang duration, computed using the formulas from above. This way, we obtain the sequence of bangs in regular time.

Multiple vehicles

Let L be the minimal distance between two consecutive vehicles. From a stationary position, we move to the next stationary position that is exactly L units further on the lane. It is clear that we need equal acceleration and deceleration $\bar{D} = \bar{A}$. By symmetry, the vehicle moves $L/2$ during both acceleration and deceleration. Assume $L/2 < x^*(T)$, then we find $\bar{A}$. Let t_0 be such that $x^*(t_0) = L/2$, then

$$t_0 = \sqrt{L/a_{\max}},$$

with corresponding schedule time

$$\hat{t} = \bar{t}(t_0, L/2) = t_0 - \frac{L}{2v_{\max}}.$$

Therefore, we have

$$\begin{aligned} \bar{D} &= (b, b + \hat{t}), \\ \bar{A} &= (b + \hat{t}, b + 2\hat{t}). \end{aligned}$$

Define

$$\omega = 2(\bar{T} - \hat{t})$$