Trajectory Optimization of Autonomous Vehicles in Networks of Intersections

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Contents

1 Trajectories in tandem of two intersections

When considering multiple vehicle driving between intersection, we can no longer ignore the issue of road capacity, because the fact that only a limited number of vehicles can drive or wait at the same time on a lane between intersections may cause network-wide effects. The capacity of lanes between intersections is intimately related to the trajectories of vehicles, which we first want to understand better. We have been using an optimal control formulation with the objective that keeps the vehicles as close as possible to the next intersection at all

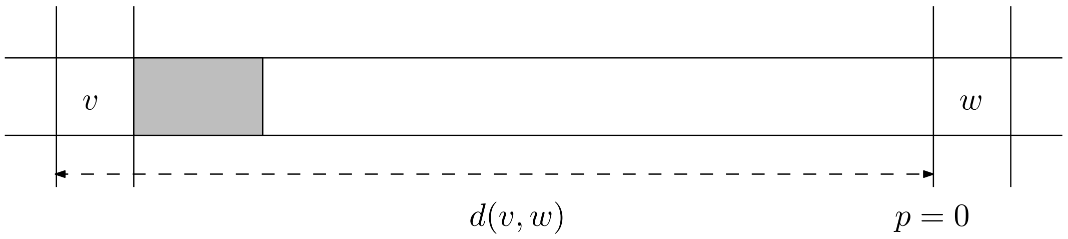

Figure 1: Tandem of two intersections v and w with lane of length $d(v, w)$. The grey rectangle represents some vehicle that just left intersection v. We assume that vehicles must drive at maximum speed as long as they occupy any intersection, so deceleration is only allowed from the shown position onwards.

times (`MotionSynthesize`). This problem can be solved using direct transcription, which works well enough if we just want to simulate the behavior of the system. However, we believe that it is possible to explicitly formulate the optimal controller. We will explain how to compute trajectories corresponding to those obtained by direct transcription, but without using time discretization.

Before we turn to the general case of networks of intersection, we will first investigate the trajectories of vehicles in a tandem of two intersections as depicted in Figure 1. Let v denote the left intersection and w the right intersection and assume that vehicles drive from left to right. Furthermore, we will call the road segment strictly between both intersection areas the *lane*. To facilitate the following discussion, we will use p to denote the position of a vehicle to distinguish it from the state $x = (p, v)$, which also includes the velocity, and we fix position $p = 0$ at the stop-line of intersection w. Let the length and width of a vehicle i be denoted by L_i and W_i, respectively. We measure the position of a vehicle at the front bumper. We will make the following assumption, that allow us to easily derive explicit expressions of these trajectories.

Assumption 1.1. *All vehicles have the same length $L_i = L$ and width $W_i = W$. Lanes are exactly W units wide and are axis-aligned, such that intersection are squares.*

Assumption 1.2. *Vehicles must drive at full speed when entering an intersection and keep driving at full speed as long as they occupy an intersection.*

Now assume that some vehicle is scheduled to exit v at time t_0 and to enter w at some time t_f. Let $p_0 = -d(v,w)+W$ denote the position of the vehicle when it starts to exit v. Let $y(t)$ denote the position of the vehicle that drives in front of the current one, assuming there is one. To avoid collision with this vehicle, the current vehicle must keep an appropriate headway distance of at least L, so we enforce the *safe headway* constraint $p(t) + L \leq y(t)$ at all times. In order to keep the vehicle as close to w as possible at every time, while respecting the double integrator vehicle dynamics $\dot{p} = v$, $\ddot{p} = u$, we can generate a trajectory by solving the optimal control problem

$$\begin{aligned} \max_{u} \quad & \int_{t_0}^{t_f} p(t)dt \\ \text{s.t.} \quad & 0 \leq v(t) \leq v_{\max}, \\ & -a_{\max} \leq u(t) \leq a_{\max}, \\ & p(t) + L \leq y(t), \\ & p(t_0) = p_0, \quad p(t_f) = 0, \\ & v(t_0) = v_{\max}, \ v(t_f) = v_{\max}. \end{aligned} \tag{1}$$

This problem can be solved by using direct transcription. After observing some example solutions, we believe that the optimal controller is of the following form.

Proposition 1. *Optimal control $u(t)$ for problem (1) switches between no acceleration and either full acceleration or full deceleration, i.e., we have a control function $u(t) := \ddot{x}(t)$ that satisfies $u(t) \in \{-a_{\max}, 0, a_{\max}\}$ and some sequence of alternating deceleration and acceleration periods, represented by some sequence of disjoint intervals*

$$(D_0, A_1, D_1, \dots, A_{n-1}, D_{n-1}, A_n),$$

so that the optimal controller is given by

$$u(t) = \begin{cases} -a_{\max} & \text{if } t \in D_k \text{ for some } k, \\ a_{\max} & \text{if } t \in A_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We show that problem (1) is a special case of optimal control problems of the form

$$\begin{aligned} \max \quad & \int_{t=0}^{T} F(x(t),u(t),t)dt \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t),u(t),t), \quad x(0)=x_0, \\ & g(x(t),u(t),t) \geq 0, \\ & h(x(t),t) \geq 0, \\ & a(x(T),T) \geq 0, \\ & b(x(T),T) = 0, \end{aligned} \tag{3}$$

where x is the vector of state variables and u is the control input. Here, the constraints g are called mixed state constraints, because they involve both state and control variables, while constraints h are called pure state constraints. Instead of writing p and v separately, we will now write the state as a vector $x(t) = (x_1(t), x_2(t))$ with position $x_1(t)$ and velocity $x_2(t)$. The control function $u(t)$ corresponds to acceleration. Without loss of generality, we assume that $v_{\max} = 1$. The initial state is $(p_0, 1)$ and the target state is $(0, 1)$. Writing $y(t)$ for the state trajectory of the vehicle in front of the current vehicle, optimal control problem (1) is equivalent to (3) when setting[1]

$$\begin{aligned} F(x,u,t) &= x_1, \\ f(x,u,t) &= (x_2, u), \\ x_0 &= (p_0, 1), \\ g(x,u,t) &= u_{\max}^2 - u^2, \\ h_1(x,t) &= x_2 - x_2^2, \\ h_2(x,t) &= y_1(t) - x_1 - L, \\ b(x,t) &= (x_1, x_2 - 1). \end{aligned} \tag{4}$$

1.1 Special case: single vehicle

Before we analyze problem (3–4) in the general, we consider the situation in which the headway constraint h_2 can be ignored, which might occur when the preceding vehicle is sufficiently far away, or the current vehicle is the only vehicle in the system. In that case, we can prove Proposition 1 and provide explicit expressions for D_k and A_k of the optimal trajectory. When dealing with optimal control problems, the Pontryagin Maximum Principle (PMP) provides a family of first-order necessary conditions for optimality. We are going to apply such a PMP-style necessary condition that deals with mixed and pure state constraints to characterize the optimal control. A common approach to deal with pure state inequality constraints h is to introduce a multiplier η and append ηh to the Hamiltonian, which is known as the direct adjoining approach. Instead, we use Theorem 5.1 from [1] (see also (4.29) in [2]), which is a so-called indirect adjoining approach, because the ηh^1 is appended to the Hamiltonian, where h^1 is defined as

$$h^1 = \frac{dh}{dt} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t}.$$

Proof of Proposition 1 without safe headway constraint. The first derivate of pure state constraint h_1 is given by

$$h_1^1(x,u,t) = u - 2x_2u,$$

[1]Instead of using quadratic expressions, both the mixed and the pure state constraint could have each been written as two linear constraints, e.g., we could have chosen $g(x,u,t) = (u_{\max} - u, u_{\max} + u)$. However, this does not satisfy the constraint qualification conditions of the theorem that we use in Section 1.1

which shows that h_1 is of first-order, because the control u appears after differentiating once. By adjoining $\mu g + \eta h_1^1$ to the Hamiltonian

$$H(x,u,\lambda,t) = F(x,u,t) + \lambda f(x,u,t) = x_1 + \lambda_1 x_2 + \lambda_2 u,$$

we obtain the Lagrangian in so-called Pontryagin form

$$\begin{aligned} L(x,u,\lambda,\mu,\eta,t) &= H(x,u,\lambda,t) + \mu g(x,u,t) + \eta h_1^1(x,u,t) \\ &= x_1 + \lambda_1 x_2 + \lambda_2 u + \mu(u_{\max}^2 - u^2) + \eta(u - 2x_2 u). \end{aligned}$$

Let $\{x^*(t), u^*(t)\}$ denote an optimal pair for problem (3–4), then this pair must satisfy the necessary conditions in Theorem 5.1 from [1]. The Hamiltonian maximizing condition is

$$H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u, \lambda(t), t)$$

at each $t \in [0,T]$ for all u with $g(x^*(t),u,t) \geq 0$ and satisfying

$$h_1^1(x^*(t),u,t) \geq 0 \text{ whenever } h_1(x^*(t),u,t) = 0.$$

In our case, the latter condition reads

$$u - 2x_2^* u \geq 0 \text{ whenever } x_2^* = \pm 1,$$

which is satisfied by optimal controllers satisfying

$$u^*(t) = \begin{cases} u_{\max} & \text{when } \lambda_2(t) > 0,\ x_2^*(t) < 1, \\ -u_{\max} & \text{when } \lambda_2(t) < 0,\ x_2^*(t) > -1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We now investigate what the costate trajectory should look like. The adjoint equation is given by

$$\dot{\lambda} = -L_x(x^*, u^*, \lambda, \mu, \eta, t) \iff \begin{cases} \dot{\lambda_1} = -1, \\ \dot{\lambda_2} = -\lambda_1 + 2\eta u^*. \end{cases}$$

The Lagrange multipliers $\mu(t)$ and $\eta(t)$ must satisfy the complementary slackness conditions

$$\begin{aligned} \mu(u_{\max}^2 - (u^*)^2) = 0, &\quad \mu \geq 0, \\ \eta(x_2 - x_2^2) = 0, &\quad \eta \geq 0 \text{ and } \dot{\eta} \leq 0. \end{aligned}$$

Whenever $u^* \neq 0$ on any open interval, then it is clear that we must have $x_2 \neq \pm 1$ on that interval, so the second complementary slackness condition requires that we have $\eta = 0$. At isolated points where $u^* \neq 0$, we can simply choose to set $\eta = 0$. Hence, $\eta u^* = 0$ everywhere and we have $\dot{\lambda_2}(t) = -\lambda_1(t) = t + c_1$ for all $t \in [0,T]$, such that $\lambda_2(t) = \frac{1}{2}t^2 + c_1 t + c_2$ for some constants c_1, c_2. In view of (5), this shows that $u^*(t)$ has at most one period of deceleration and at most one period of acceleration. Assume $\lambda_2(t)$ has two zeros, which we denote by t_d and t_a, which give the start of the deceleration period and the start of the acceleration period, respectively.

We now show how to compute $t_a - t_d$. Let D denote the duration of the deceleration, which must of course equal the length of the acceleration, because $x_2(0) = x_2(T)$. Furthermore, D must satisfy both $D \leq t_a - t_d$ and $D \leq 1/u_{\max}$ and can thus be defined as the smallest of these two bounds. From the vehicle dynamics follows that, for all $t \in [t_d, t_d + D]$ in the deceleration period, we must have

$$x_2^*(t) = 1 - u_{\max}(t - t_d)$$

and by integration, we obtain

$$x_1^*(t) = x_1^*(t_d) + (t - t_d) - \frac{u_{\max}}{2}(t - t_d)^2.$$

Similarly, for all $t \in [t_a, t_a + D]$ in the acceleration period, we must have

$$x_2^*(t) = u_{\max} t$$

and by integration, we obtain

$$x_1^*(t) = x_1^*(t_a) + \frac{u_{\max}}{2}(t - t_a)^2.$$

Because the vehicle is stationary in $[t_d + D, t_a]$, we have $x_1^*(t_a) = x_1^*(t_d + D)$. It is now easily verified that we have

$$x_1^*(T) = p_0 + T + t_d - t_a.$$

Recall that $x_1^*(T) = 0$, so we obtain $t_a - t_d = p_0 + T$.

We now show that t_a is uniquely determined. From the complementary slackness conditions, we derive that

$$\begin{aligned}
\mu(t) &= 0 \quad \text{for } t \in (0, t_d), \\
\eta(t) &= 0 \quad \text{for } t \in (t_d, t_d + D), \\
\mu(t) &= 0 \quad \text{for } t \in (t_d + D, t_a), \\
\eta(t) &= 0 \quad \text{for } t \in (t_a, t_a + D), \\
\mu(t) &= 0 \quad \text{for } t \in (t_a + D, T).
\end{aligned}$$

Now together with the condition

$$\frac{\partial L}{\partial u}|_{u=u^*(t)} = 0 \iff \lambda_2 - 2\mu u^* - 2\eta x_2 + \eta = 0,$$

this shows that we can choose μ and η such that $\mu(t) \geq 0$ and $\eta(t) \geq 0$ and

$$\begin{aligned}
\lambda_2(t) &= \eta(t) \quad \text{for } t \in (0, t_d), \\
\lambda_2(t) &= -2\mu(t) \quad \text{for } t \in (t_d, t_d + D), \\
\lambda_2(t) &= -\eta(t) \quad \text{for } t \in (t_d + D, t_a), \\
\lambda_2(t) &= 2\mu(t) \quad \text{for } t \in (t_a, t_a + D), \\
\lambda_2(t) &= \eta(t) \quad \text{for } t \in (t_a + D, T).
\end{aligned}$$

Suppose $t_a + D < T$, then for $t \in (t_a + D, T)$, we have $\dot{\eta}(t) = \dot{\lambda}_2(t) > 0$, which contradicts the necessary condition $\dot{\eta}(t) \leq 0$, hence $t_a = T - D$. This fixes a unique costate trajectory $\lambda_2(t)$ and hence a unique optimal controller $u^*(t)$ given by (5). □

1.2 Multiple vehicles

By observing optimal trajectories generated using direct transcription, we observe that there is a lot of structure in how the trajectory of a vehicle influences the trajectories of later vehicles. In this section, we provide a direct way to explicitly calculate trajectories, without proving that these trajectories are always feasible and optimal. Instead, we verify correctness of our method numerically by comparing our explicit trajectories to the the output of direct transcription for various system parameters and crossing times.

We start by investigating the limit on the number of vehicles that can occupy the lane at the same time. Imagine that vehicles enter the lane until it is full and then only after all

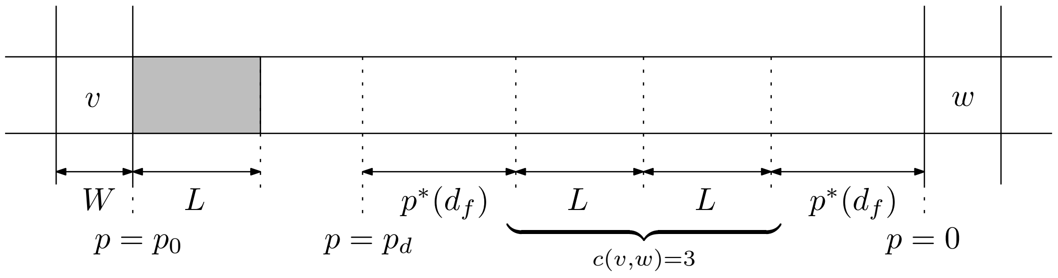

Figure 2: Tandem of intersections with distances used in the determination of the lane capacity and the waiting positions.

vehicles have come to a full stop, they start to leave the lane by crossing w. Without any additional constraints, it follows from the vehicle dynamics that the time it takes to fully accelerate from rest to maximum velocity is given by $d_f = v_{\max}/a_{\max}$ and the corresponding full acceleration trajectory is given by

$$p^*(t) = a_{\max}t^2/2 \quad \text{for } 0 \leq t \leq d_f.$$

To keep the presentation simple, we assume that maximum deceleration is also $a_{\max}$, so it also takes d_f time to fully decelerate from maximum velocity to rest. Suppose that we want to design the tandem network such that at least $c(v, w)$ vehicles can enter and decelerate to some waiting position, from which it is also possible to accelerate again to full speed before crossing w. Vehicles are required to drive at full speed as long as they occupy any intersection. Therefore, a vehicle crossing v can only start decelerating after $p(t) \geq p_0 + L$, so the earliest position where a vehicle can come to a stop is $p = p_0 + L + p^*(d_f)$. Because vehicles need to gain maximum speed before reaching w, the position closest to w where a vehicle can wait is $-p^*(d_f)$. Hence, in order to accomodate for $c(v, w)$ waiting vehicles, the length of the lane must satisfy

$$d(v, w) \geq W + L + 2p^*(d_f) + (c(v, w) - 1)L,$$

as illustrated in Figure 2. Conversely, given the lane length $d(v, w)$, the corresponding lane capacity is given by[2]

$$c(v, w) = \texttt{floor}\left(\frac{d(v, w) - W - 2p^*(d_f)}{L}\right),$$

where $\texttt{floor}(x)$ denotes the largest integer smaller than or equal to x.

It turns out that the fixed locations where vehicles wait in the above scenario are helpful in describing the optimal trajectories, even when vehicles never fully stop. We will denote these fixed *waiting positions* as

$$p_k = -p^*(d_f) - (c(v, w) - k)L,$$

for $k = 1, \ldots, c(v, w)$. Furthermore, let $p_d = p_1 - p^*(d_f)$ denote the position from which vehicles must decelerate in order to stop at the earliest waiting position p_1, which is the farthest from the destination intersection. Now consider a vehicle that moves from p_k to the next waiting position p_{k+1}, so it moves exactly distance L. We consider such a start-stop movement, without considering any safe following constraints. By symmetry of the control constraints, the vehicle moves the same distance during both acceleration and deceleration. Furthermore, the vehicle needs to be at rest at the start and end of such trajectory. Hence,

[2]Without Assumption 1.1, we cannot derive such a simple formula, because it would depend on the specific lengths of those vehicles currently in the system.

it is clear that it takes the same amount of time d_s to accelerate and decelerate. We assume that $d_s < d_f$, which ensures that $v_{\max}$ is never reached during the start-stop movement, which is illustrated in Figure 3. In this case, it is clear that we must have $L = 2p^*(d_s)$, from which we derive that $d_s = \sqrt{L/a_{\max}}$.

1.2.1 Ad-hoc approach

We will now present a method to calculate the trajectory of a vehicle based on its crossing times at v and w and the trajectory of the vehicle ahead of it. We start from a sequence of deceleration and acceleration intervals that are possibly overlapping and then merge them in pairs from left to right until they become disjoint. Specifically, let

$$t_d := t_0 + \frac{p_d - p_0}{v_{\max}}$$

be the start of the initial deceleration interval $D_0 := [t_d, t_d + d_f]$, which is exactly the moment the vehicle needs to start decelerating in order to stop at the first waiting position p_1, so at time t_d the vehicle is at position p_d. Similarly, let $t_a = t_f - d_f$ be the start of the final acceleration interval $A_f := [t_a, t_f]$. Now for every $k = 1, \dots, c(v, w) - 1$, we consider a pair of start-stop intervals $S_k := (A_k, D_k) = ([t_k, t_k + d_s], [t_k + d_s, t_k + 2d_s])$ at some starting time t_k, moving the vehicle from p_k to p_{k+1}. We first show how to merge these intervals such that a sequence of disjoint intervals is obtained. Afterwards, we show how times t_k follow from the trajectory of the vehicle ahead.

First, we show how to merge D_0 and S_1. When we have $t_d + d_f \leq t_1$, it is clear that both parts are already completely disjoint, so we do not have to do anything further. When t_d and t_1 are closer than d_f, we need to merge D_0 and A_1. In this case, we observe that D_0 gets shorter at the end by some ϵ and the acceleration part of S_1 gets shorter at the beginning, also by the same amount ϵ, because the velocities must match. More precisely, we construct two new consecutive intervals

$$\begin{aligned}
D_0' &= [t_1 + 2\epsilon - d_f,\ t_1 + \epsilon], \\
A_1' &= [t_1 + \epsilon,\ t_1 + d_s], \\
D_1 &= [t_1 + d_s,\ t_1 + 2d_s].
\end{aligned}$$

We now determine ϵ as a function of $t_1 - t_d$. Because D_0' and A_1' are both ϵ shorter, the total distance that is traversed is now $2p^*(\epsilon)$ shorter. This means that the start of D_0' should be $2p^*(\epsilon)/v_{\max}$ later than t_d, which gives equation

$$t_d + \frac{2p^*(\epsilon)}{v_{\max}} = t_1 + 2\epsilon - d_f,$$

which is quadratic in ϵ. Because we must have $\epsilon < d_f$, we need the smallest solution

$$\epsilon = d_f - \sqrt{d_f(t_1 - t_d)}.$$

When we keep increasing t_d while keeping t_1 fixed, we observe that eventually part A_1 will completely disappear and D_0 and D_1 will become a single interval, which is easily seen to happen when $\epsilon \geq d_s$. Equivalently, this happens when t_d is such that $t_d \geq t_1 - d_f + 2d_s - 2p^*(d_s)/v_{\max}$.

We now show how two start-stop parts have to be merged if they overlap. Consider two start-stop parts A_k, D_k and A_{k+1}, D_{k+1} and suppose that $t_k + 2d_s < t_{k+1}$ such that both parts have to be merged. The parts D_k and A_{k+1} merge by removing ϵ on both sides, similarly as above. However, this causes the D_k, A_{k+1}, D_{k+1} parts to shift up. Therefore, A_k and D_k each need to be lengthened at the side where they meet by some δ to match

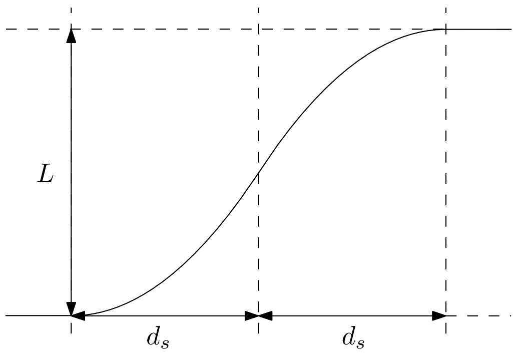

Figure 3: Shape of some start-stop trajectory of a single isolate vehicle moving forward from some current waiting position p_k to the next p_{k+1}.

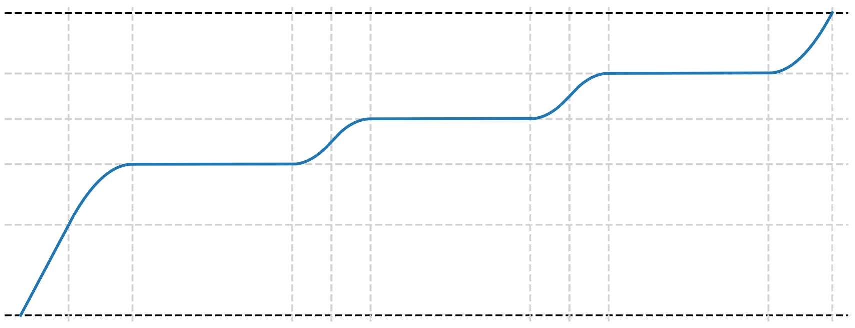

Figure 4: Sketch of vehicle trajectory in tandem with all the full start-stop parts unmerged.

this. Hence, it turns out we have to use the intervals

$$\begin{aligned} A'_k &= [t_k, t_k + d_s + \delta], \\ D'_k &= [t_k + d_s + \delta, t_{k+1} + \epsilon], \\ A'_{k+1} &= [t_{k+1} + \epsilon, t_{k+1} + d_s], \\ D_{k+1} &= [t_{k+1}, t_{k+1} + d_s]. \end{aligned}$$

We have that ϵ and δ need to satisfy

$$\begin{cases} 2\delta + 2\epsilon = t_{k+1} - t_k, \\ 2p^*(d_s + \delta) - 2p^*(d_s - \epsilon) = L. \end{cases}$$

Solving this system of equations yields

$$\begin{aligned} \delta &= \frac{L/a_{\max}}{t_{k+1} - t_k} - d_s + \frac{t_{k+1} - t_k}{4}, \\ \epsilon &= (t_{k+1} - t_k)/2 - \delta. \end{aligned}$$

Finally, we consider the merge with the final acceleration bang.

Note that the above types of merging are enough to process the whole sequence of bangs, because when the first merge of D_0 and A_1, D_1 results in a single D'_1 and there is a next A_2, D_2, we are again in the first situation.

We now show how t_k follow from the trajectory of the preceding vehicle.

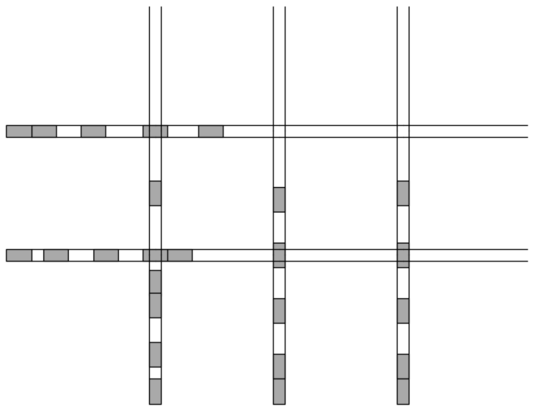
Figure 5: Illustration of some grid-like network of intersections with vehicles drawn as grey rectangles. There are five vehicle routes: two from east to west and three from south to north. Turning at intersections is not allowed.

2 Trajectories in networks

We now extend the single intersection model to a network of intersections without turning routes, illustrated in Figure 5. We define a directed graph $(\bar{V}, E)$ with nodes $\bar{V}$ and arcs E, representing the possible paths that vehicles can follow. Nodes with only outgoing arcs are *entrypoints* and nodes with only incoming arcs are *exitpoints*. Let V be the set of *intersections*, which are all the nodes with in-degree at least two. Let $d(v, w)$ denote the distance between nodes v and w. For each route index $r \in \mathcal{R}$, we let

$$\bar{V}_r = (v_r(0), v_r(1), \dots, v_r(m_r), v_r(m_{r+1}))$$

be the path that vehicles $i \in \mathcal{N}_r$ follow through the network. We require that the first node $v_r(0)$ is an entrypoint and that the last node $v_r(m_{r+1})$ is an exitpoint and we write

$$V_r = \bar{V}_r \setminus \{v_r(0),\, v_r(m_{r+1})\}$$

to denote the path restricted to intersections. We say that some $(v, w) \in E$ is on path V_r whenever v and w are two consecutive nodes on the path and we write E_r to denote the set of all these edges. We require that routes can only overlap at nodes by making the following assumption.

Assumption 2.1. *Every arc $(v, w) \in E$ is part of at most one route V_r.*

We start by considering networks in which all roads are axis-aligned such that intersections always involve perpendicular lanes and where routes are such that no turning is required. For each $v \in V_r$ define the conflict zone $\mathcal{E}_r(v) = (b_r(v), e_r(v))$ and consider the union

$$\mathcal{E}_r = \bigcup_{v \in V_r} \mathcal{E}_r(v)$$

corresponding to the positions of vehicles $i \in \mathcal{N}_r$ for which it occupies an intersection on its path V_r. By reading $\mathcal{E}_i \equiv \mathcal{E}_r$ for $r(i) = r$, the single intersection problem naturally extends to the network case. Like before, the resulting problem can be numerically solved by a direct transcription method.

2.1 General decomposition

The general two-stage decomposition for the single intersection extends rather naturally to the present model. Let for each pair (i, v) of some vehicle $i \in \mathcal{N}$ and an intersection $v \in V_{r(i)}$ along its route, let

$$\inf\{t : x_i(t) \in \mathcal{E}_r(v)\} \quad \text{and} \quad \sup\{t : x_i(t) \in \mathcal{E}_r(v)\}$$

be the crossing time and exit time, which we denote by $y(i, v)$ and $y(i, v) + \sigma(i, v)$, respectively. Instead of a single set of conflicts, we now define for each intersection $v \in V$ in the network the set of conflict pairs

$$\mathcal{D}^v = \{\{i, j\} \subset \mathcal{N} : r(i) \neq r(j), v \in V_{r(i)} \cap V_{r(j)}\}.$$

Now the two-stage approach is to solve

$$\begin{aligned} \min_{y,\sigma} \quad & \sum_{r \in \mathcal{R}} F(y_r, \sigma_r) & \\ \text{s.t.} \quad & y(i, v) + \sigma(i, v) \leq y(j, v) \text{ or} & \\ & y(j, v) + \sigma(j, v) \leq y(i, v), & \text{for all } \{i, j\} \in \mathcal{D}^v \text{ and } v \in V, \\ & (y_r, \sigma_r) \in \mathcal{S}_r, & \text{for all } r \in \mathcal{R}, \end{aligned}$$

where $F(y_r, \sigma_r)$ and $\mathcal{S}_r$ are the value function and set of feasible parameters, respectively, of the parametric trajectory optimization problems

$$\begin{aligned} F(y_r, \sigma_r) = \min_{x_r} \quad & \sum_{r \in \mathcal{R}} J(x_i) & \\ \text{s.t.} \quad & x_i(t) \in D_i(s_{i,0}), & \text{for } i \in \mathcal{N}_r, \\ & x_i(y(i, v)) = b_r(v), & \text{for } v \in V_r, i \in \mathcal{N}_r, \\ & x_i(y(i, v) + \sigma(i, v)) = e_r(v), & \text{for } v \in V_r, i \in \mathcal{N}_r, \\ & x_i(t) - x_j(t) \geq L, & \text{for } (i, j) \in \mathcal{C} \cap \mathcal{N}_r, \end{aligned}$$

where we again use subscript r to group variables according to their associated route.

2.2 Decomposition for delay objective

Suppose we use use the crossing at the last intersection as performance measure, by defining the objective function as

$$J(x_i) = \inf\{t : x_i(t) \in \mathcal{E}_r(v_r(m_r))\}.$$

We show how to reduce the resulting problem to a scheduling problem, like we did in the single intersection case. We will again assume Assumption 1.1 and Assumption 1.2, so vehicles will always cross intersections at full speed, and all vehicles share the same geometry. Hence, the occupation time $\sigma \equiv \sigma(i, v)$ is the same for all vehicles and intersections. For this reason, we will write the shorthand $y_r \in \mathcal{S}_r$, because σ_r is no longer a free variable.

As a consequence of Assumption 1.1 and Assumption 1.2, each lower-level trajectory optimization problem for a given route $r \in \mathcal{R}$ decomposes into a sequence of problems, each corresponding to two consecutive intersection along V_r. This means that $y_r \in \mathcal{S}_r$ is equivalent to $y_{(v,w)} \in \mathcal{S}_{(v,w)}$ for each $(v, w) \in E_r$, where $y_{(v,w)}$ denotes the vector of all variables $y(i, v)$ and $y(i, w)$ for all $i \in \mathcal{N}_r$ and $\mathcal{S}_{(v,w)}$ denotes the set of values of $y_{(v,w)}$ for which a feasible trajectory part can be found. Hence, we will now focus on a tandem of two intersections and investigate the trajectories of vehicles in this with the goal of stating sufficient conditions for $y_{(v,w)} \in \mathcal{S}_{(v,w)}$.

2.3 Crossing time scheduling as extension of job-shop scheduling

The Job-Shop Scheduling Problem (JSSP) is a widely studied problem in which a set of n jobs must be assigned to non-overlapping time slots on a set of m machines. Each job i has a set of n_i operations $O_{i1}, \dots, O_{in_i}$ that need to be executed in this order. Each operation O_{ij} requires p_{ij} processing time on machine M_{ij}. Each machine can process at most one operation and early preemption is not allowed. The task of the scheduler is to determine a valid schedule of start times y_{ij} for each operation, while minimizing some objective function. Let $C_{ij} = y_{ij} + p_{ij}$ denote the *completion time* of operation O_{ij}, then the *makespan* objective is given by the latest completion time $\max_{i,j} C_{ij}$. Another objective is the *total completion time*, given by

$$\sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij},$$

which may be intuitively be thought of as representing some sort of total cost of inventory, assuming we need to physically store jobs somewhere.

A commonly used representation of JSSP instances is the *disjunctive graph* $G = (\mathcal{O}, \mathcal{C}, \mathcal{D})$, with vertices $\mathcal{O} = \{O_{ik} : 1 \leq i \leq n, 1 \leq k \leq n_i\}$ corresponding all the operations. The set of *conjunctive arcs* $\mathcal{C}$ encodes all the precedence constraints $O_{i,k} \rightarrow O_{i,k+1}$ among each job's operations. The set of *disjunctive edges* $\mathcal{D}$ consists of undirected edges between each pair of operations from distinct jobs that need to be processed on the same machine, effectively encoding all such *conflicts*. Each valid schedule induces an ordering of operations on machines that is encoded by fixing the direction of each disjunctive edge such that we obtain a direct acyclic graph.

- Introduce general job-shop problem and disjunctive graph, see Figure 6.
- Introduce travel constraints and its disjunctive graph arcs.
- Introduce buffer constraints and its disjunctive graph arcs.
- Formulate MILP problem and investigate how the solving time scales with network size in terms of number of intersections and number of vehicles in the network. Do the single intersection cutting planes still hold? Are there any obvious cutting planes?

Let $y(i, v)$ denote the crossing time of vehicle i at intersection $v \in V$. Let $\mathcal{C}$ be the conjunctive pairs and let $\mathcal{D}^v$ denote the disjunctive pairs at intersection $v \in V$. Writing `conj(...)` and `disj(...)` for the usual conjunctive and disjunctive constraints, we propose to solve the optimization problem

$$\begin{aligned}
\min_{y} \quad & \sum_{i \in \mathcal{N}} \sum_{v \in \mathcal{R}(l(i))} y(i, v) && && \text{(6a)} \\
\text{s.t.} \quad & r_i \leq y(i, v_0(l(i))) && \text{for } i \in \mathcal{N}, && \text{(6b)} \\
& \texttt{conj}(y(i, v), y(j, v)) && \text{for } (i, j) \in \mathcal{C}, v \in V, && \text{(6c)} \\
& \texttt{disj}(y(i, v), y(j, v)) && \text{for } \{i, j\} \in \mathcal{D}^v, v \in V, && \text{(6d)} \\
& y(i, v) + d(v, w) \leq y(i, w) && \text{for } i \in \mathcal{N}(l), (l, v, w) \in E, && \text{(6e)} \\
& y(i, w) + \rho(v, w) \leq y(j, v) && \text{for } (i, j, v, w) \in \mathcal{F}, && \text{(6f)}
\end{aligned}$$

where $\mathcal{F}$ is defined as

$$\mathcal{F} = \{(i, j, v, w) : i, j \in \mathcal{N}(l), k(i) + c(v, w) = k(j), (l, v, w) \in E\}$$

and we have $\rho(v, w) = c(v, w)L - d(v, w)$. Each $(i, j, v, w) \in \mathcal{F}$ represents a pair of vehicles driving on the same lane (v, w), for which the first vehicle must have made enough space in the lane before vehicle j can enter.

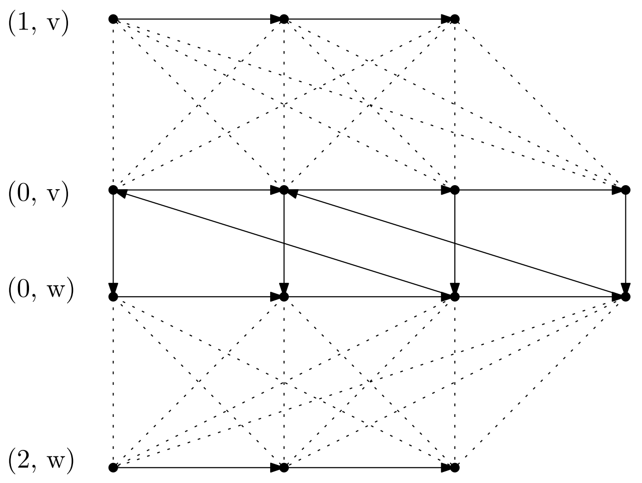

Figure 6: Empty disjunctive graph belonging to a tandem of two intersections, labeled v and w. There are three routes $\mathcal{R} = \{0, 1, 2\}$. On route 0, four vehicle are arriving, the other two routes have 3 arrivals each. Conjunctive arcs are drawn as from left to right, travel arcs are drawn from top to bottom and disjunctive arcs are drawn as dotted lines. Two buffer constraints are drawn as diagonal arcs, corresponding to a capacity of 2 vehicles for the lane between both intersections.

Lower bounds $\beta(i, v)$ on the crossing times of vehicles, where $i = (r, k) \in \mathcal{N}$ is some vehicle index and $v \in V_r$.

There are two natural choices for the objective to optimize in this scheduling setting. We minimize the time each vehicle is in the system, which is equivalent to minimizing the crossing time at the last intersection of each vehicle's route, written as

$$\text{obj}_1(y) = \min \sum_{i \in \mathcal{N}} y(i, v_r(m_r)).$$

Alternatively, it also makes sense to minimize the crossing time at every intersection along each vehicle's route, so we can also minimize

$$\text{obj}_2(y) = \min \sum_{i \in \mathcal{N}} \sum_{v \in V_{r(i)}} y(i, v).$$

3 Constructive heuristics

Like in the single intersection case, we are going to model optimal solutions using autoregressive models. In the single intersection case, we argued that each schedule is uniquely defined by its route order η. Generalizing this to the case of multiple intersections, we see that a schedule is uniquely defined by the set of route orders η^v at each intersection $v \in V$. Instead of working with such a set of sequences, we will intertwine the sequences to obtain a single *crossing sequence* η, which is a sequence of pairs (r, v) of a route $r \in \mathcal{R}$ at some intersection $v \in V_r$ and we will refer to such a pair as a *crossing.* Of course, this crossing sequence can be constructed in many different ways, because it does not matter in which order the intersections are considered, which is illustrated in Figure 7. Given some problem instance s, we will consider autoregressive models of the form

$$p(\eta|s) = \prod_{t=1}^{N} p(\eta_t|s, \eta_{1:t-1}). \tag{7}$$

These models can also be understood in terms of a step-by-step schedule construction process that transitions from a partial schedule state $s_t = (s, \eta_{1:t})$ to the next state s_{t+1} by selecting some *action* $\eta_t = (r_t, v_t)$. We say a crossing is pending when it still has unscheduled vehicles. Similarly, we say an intersection is pending when some of its crossings are still pending. Therefore, the set of *valid actions* $\mathcal{A}(s_t)$ at some intermediate state s_t is exactly the set of pending crossings. We again emphasize that multiple sequences of actions lead to the same schedule, because the order in which intersections are considered does not matter for the final schedule, which is illustrated in Figure 7. The models that we study can be understood as being parameterized as some function of the disjunctive graph G_t of partial schedule s_t, so an alternative way of writing (7) that emphasizes this is

$$p(\eta = ((r_1, v_1), \dots, (r_N, v_N)) \mid s) = \prod_{t=1}^{N} p(r_t, v_t|G_{t-1}). \tag{8}$$

Not all routes are valid at every intersection, so instead of modeling the joint probability distribution $p(r_t, v_t|G_{t-1})$ over crossings, we apply the chain rule to factorize this as

$$p(r_t, v_t|G_{t-1}) = p(r_t|v_t, G_{t-1})p(v_t|G_{t-1}). \tag{9}$$

In this case, the model $p(r_t|v_t, G_{t-1})$ can be thought of as predicting a set of actions, one for each intersection.

In the general class of autoregressive models for inputs and outputs with sets, of which our model above is a special case (we have a set of sequences as output), it has been noted before that the order in which inputs or outputs are presented to the model during training has a considerable impact on the eventual model fit [3]. For our model, we also expect to find such differences. Specifically, we will investigate the impact of the order in which intersections are visited, which is determined by $p(v_t|G_{t-1})$. We now propose some heuristic ways to define $p(v_t|G_{t-1})$. Later, we will consider a neural network parameterization. First of all, the simple *random* strategy would be to sample some intersection with pending crossings at each step. In the *boundary* strategy, we keep visiting the same intersection until it is done (when it has no pending crossings anymore), then move to some next intersection. When the network of intersections is free of cycles, we could for example follow some topological order. We use the term "boundary" because this strategy produces trajectories along the boundary of the grid in Figure 7. In the *alternating* strategy, we keep alternating between intersection to keep the number of scheduled vehicles balanced among them. This produces trajectories that can be understood as being close to the "diagonal" of the grid in Figure 7. Again, the order in which we alternate between intersections may again be based on some topological order.

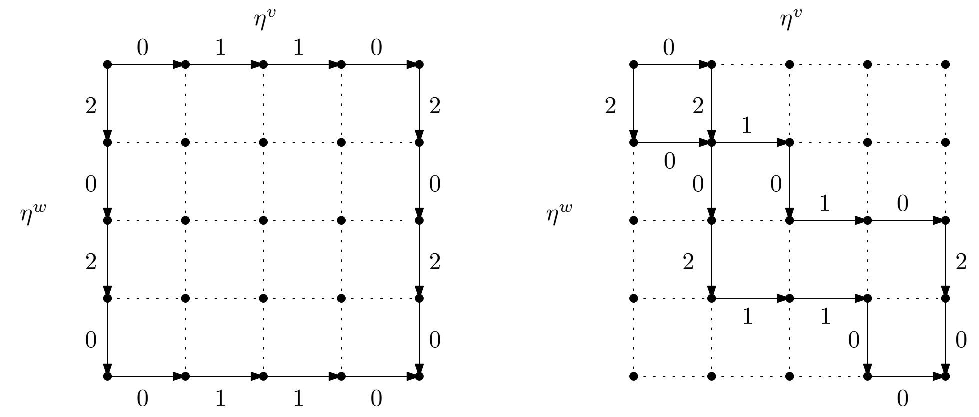

Figure 7: When each route order is locally fixed at every intersection, the global crossing sequence is not uniquely determined, because these local sequences may be merged in any order. Suppose we have a tandem of two intersections and a horizontal arrows correspond to taking the next local action at intersection v, a vertical arrows correspond to taking next local action at intersection w. Each valid crossing sequence corresponds to some path from the top-left to the bottom-right corner. Although any such crossing sequence produces the same schedule, it might be that our autoregressive model fits better on some sequences than others. For example, we might expect that sequences on the boundary of the grid, shown in the left grid, are harder to learn from data than sequences that stay closer to the diagonal, like in the right grid. The intuition is that we need to "look into the future" more to learn the former, while in the latter trajectories, progress in the two intersections is more balanced.

Threshold heuristics. It is straightforward to extend the threshold rule to networks of intersections, when assuming a fixed intersection order. Each time some next intersection is visited, we apply the single intersection threshold rule to pick the next route. This is straightforward to do, because we can just consider the disjunctive subgraph induced by the nodes belonging to that intersection to arrive at the single intersection case. Furthermore, the definition of the threshold rule itself does not depend on the network of intersections. This is a desirable property, because it allows us to tune the threshold on small networks and then apply it on larger ones.

3.1 Neural constructive heuristic

We will now propose a neural network parameterization of $p(r_t|v_t, G_{t-1})$ and train it based on optimal schedules in a supervised learning setting. The model can be best understood as solving a so-called multi-label classification problem, because it needs to provide a distribution over routes at every intersection. The training data set $\mathcal{X}$ consists of pairs $(G_{t-1}, (r_t, v_t))$, to which we refer to as *state-action* pairs to draw the parallel with the terminology used in the reinforcement learning literature. To obtain these pairs, we sample a collection of problem instances, which are solved to optimality using a MILP solver. For each optimal schedule, we compute the corresponding optimal route order η^v for each intersection. From these, we can construct $\mathcal{X}$ in different ways. For example, for each solved instance, we can randomly select some intersection order μ, which fixes the crossing order η. We can then replay this sequence of actions step-by-step to obtain the corresponding sequence of state-action pairs. The model might become more robust when training on multiple samples of intersections orders per instance. Alternatively, we can consider one of the fixed intersection orders described at the start of this section (random, boundary, alternating). Furthermore, combined with one of the above strategies, we can also employ some kind of fixed lookahead procedure, as illustrated in Figure 8. At inference time, we use the same intersection order

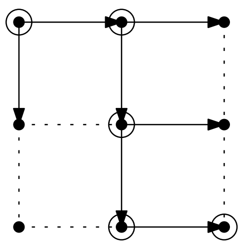

Figure 8: Possible strategy for collecting state-action pairs: one action look-ahead. At every state s_t with a set of possible actions $\mathcal{A}(s_t)$, we add all state-action pairs $\{(s,a) : a \in \mathcal{A}(s)\}$, but we pick a single action a^* to move to the next state. In the figure, the state-action pairs are depicted as solid arrows. The encircled dots are the states that are actually visited. Observe that this procedure can be naturally generalized to a look-ahead of arbitrary depth.

as during training and at each step we greedily select r_t.

We will now describe how the model is parameterized based on recurrent embeddings of the sequences of crossing time lower bounds at each crossing, in a somewhat similarly to how we used the horizons in the single intersection model. Let $k(r,v)$ denote the number of scheduled vehicles at crossing (r,v) and let n_r denote the total number of vehicles on route r. For each crossing (r,v), consider the earliest crossing time of the next unscheduled vehicle, which we denote as

$$T(r,v) = \beta(r, k(r,v)+1, v).$$

We define the *horizon* of crossing (r,v) to be the sequence of relative lower bounds

$$h(r,v) = (\beta(r, k(r,v)+2, v) - T, \dots, \beta(r, n_r, v) - T).$$

Each such horizon is embedded using an Elman RNN. To produce a logit for each crossing, these embeddings are fed through a feedforward network, consisting of two hidden layers of size 128 and 64 neurons with relu activation and batch normalization layers in between. See Figure 9 for a schematic overview of the architecture. Next: instead of $h(r,v)$, feed $(T(r,v), h(r,v))$ to the FF.

To train the model, we use the Adam optimizer with learning rate 10^{-4} and a batch size of 40 state-action pairs. We experienced a case of the exploding gradients problem when we did not use the batch normalization layers. To allow a fair comparison of methods accross instances of different sizes, both in terms of the number of vehicles as the size of the network, we adapt both objective as follows. For the first variant, we define

$$\text{obj}_1^*(y) = \frac{1}{|\mathcal{N}|} \min \sum_{i \in \mathcal{N}} y(i, v_r(m_r)) - a(i, v_r(m_r)).$$

Given some problem instance, let N denote the length of a crossing sequence, so it is also the total number of vehicle-intersection pairs (i,v) that need to be scheduled. For the second objective variant, we consider the average delay per vehicle-intersection pair, as given by

$$\text{obj}_2^*(y) = \frac{1}{N} \sum_{i \in \mathcal{N}} \sum_{v \in V_{r(i)}} y(i,v) - a(i,v).$$

For the current comparison we use obj_2^*, the results are listed in Table 1.

3.2 Reinforcement learning

Instead of using a fixed training set $\mathcal{X}$ of state-action pairs during training, we can fit our model from the perspective of a reinforcement learning problem, which we already alluded to

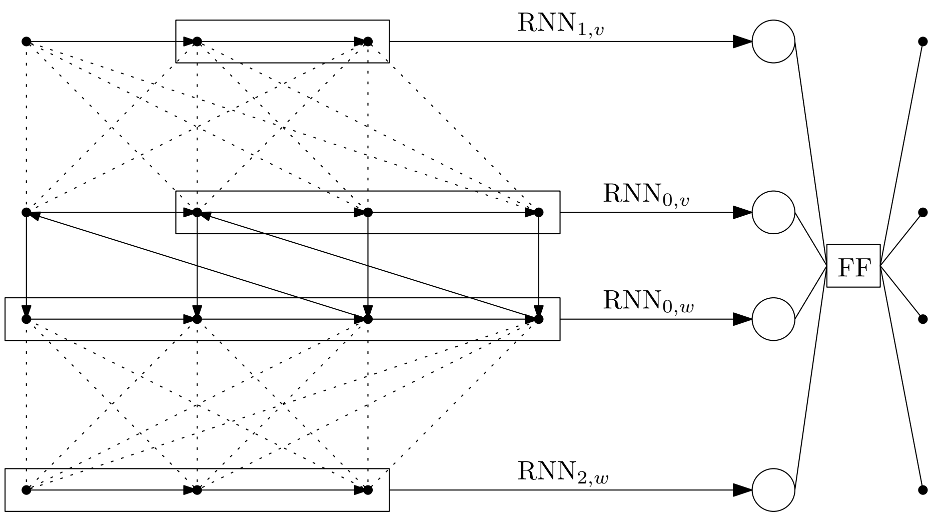

Figure 9: Illustration of model with RNN encoding of horizon at each crossing. The disjunctive graph nodes whose lower bounds are part of the current horizons are indicated with rectangles. The RNN embeddings (open circles) are fed through a final feedforward network to produce a logit (dots) for each crossing.

Table 1: Average scaled optimal objective value computed using MILP and using the threshold heuristic with threshold $\tau = 0$. Each class of problem instances is identified by the number n of vehicle arrivals per route and the grid network size as cols x rows.

n	size	MILP	time	$\tau = 0$ (gap)	random (gap)	boundary (gap)	alternate (gap)
5	2x1	57.27	0.06	65.27 (11.45%)	58.96 (2.95%)	58.72 (2.52%)	58.23 (1.66%)
5	3x1	57.67	0.12	68.34 (15.44%)	59.77 (3.65%)	59.82 (3.74%)	58.72 (1.83%)
5	3x2	57.35	1.38	69.17 (18.32%)	60.88 (6.16%)	60.36 (5.25%)	58.82 (2.56%)

in Section 3.1. More precisely, given some scheduling problem instance s, we are dealing with a Deterministic Markov Decision Process (DMDP), where partial disjunctive graphs serve as states and actions correspond to crossings. The potential benefit of using reinforcement learning is that we do not have to fix an intersection order: our hope is that the training procedure will automatically converge to some good intersection order.

The threshold heuristic ($\tau = 0$) can provide a *baseline* for reinforcement learning, reducing the variance of the REINFORCE estimator.

3.3 Network-agnostic architecture

We now focus on models that generalize across network sizes. In particular, we are interested in models that can be learned on relatively small networks (for which exact solutions might be tractable) and can then be applied to larger networks.

We use a GIN to compute an embedding for each node, which is then fed through an MLP and softmax to produce a probability over nodes. The GNN computes node embeddings, which are mapped to a score for each node. We compute the softmax over the scores of the nodes and then compute the negative log likelihood loss for backpropagation. Like in Zhang et al., each action corresponds to a unique node, encoding the operations that is dispatched next. However, we only really need to provide a route-intersection pair, but how to exploit this in the policy model? At this point, during the collection of observation-action pairs, we copy the whole disjunctive graph for each state. Alternatively, we could use some sort of masking for non-final states.

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