Bounded lane capacity

Up to this point, we have not taken into account the fact that lanes between intersection have finite capacity. We need to incorporate this aspect in order to develop a model that could be used for practical applications. Under high traffic loads, lanes with finite buffer capacity can give rise to *blocking* of upstream intersections. Therefore, the traffic controller needs to take into account these additional dynamics.

Recall the single intersection scheduling model

$$\begin{aligned}
\min_{y} \quad & \sum_{i \in \mathcal{N}} y_i \\
\text{s.t.} \quad & r_i \leq y_i && \text{for } i \in \mathcal{N}, \\
& y_i + \rho_i \leq y_j && \text{for } (i,j) \in \mathcal{C}, \\
& y_i + \sigma_i \leq y_j \text{ or } y_j + \sigma_j \leq y_i && \text{for } \{i,j\} \in \mathcal{D},
\end{aligned}$$

where we had

$$\begin{aligned}
\mathcal{D} &= \{(i,j) \in \mathcal{N} : l(i) \neq l(j)\}, \\
\mathcal{C} &= \{(i,j) \in \mathcal{N} : l(i) = l(j),\, k(i) + 1 = k(j)\}.
\end{aligned}$$

We start the simplest extension of the single intersection model by considering two intersections in tandem. For each vehicle $i = (l,k)$, we will refer to l as the *vehicle class*, because vehicles are no longer bound to a unique lane. We define a graph (V,E) with *labeled edges* as follows. Let V denote the indices of the intersections. Let E denote the set of ordered triples (l,v,w) for each class l whose vehicles travel from intersection v to w. Let $d(v,w)$ denote the minimum time necessary to travel between intersections v and w. Let $b(v,w)$ denote the maximum number of vehicles that can be on the lane between intersections v and w. Let $\mathcal{N}(l)$ denote all the vehicles of class l. Let $v_0(i)$ denote the first intersection that vehicle i encounters on its route. Let $\mathcal{R}(l)$ denote the set of intersections visited by vehicles from class l. We make the following assumption on vehicles routes.

Assumption 1 (Disjoint Routes)**.** *Every lane (v,w) is visited by at most one vehicle class. Stated formally, $(l_1,v,w) \in E$ and $(l_2,v,w) \in E$ implies $l_1 = l_2$.*

We now show how to obtain schedules in this model by formulating a MILP. Let $y(i,v)$ denote the crossing time of vehicle i at intersection $v \in V$. Let $\mathcal{C}^v$ and $\mathcal{D}^v$ denote the disjunctive and disjunctive pairs, respectively, for each intersection $v \in V$. Writing `conj(...)` and $\texttt{disj}(\dots)$ for the usual conjunctive and disjunctive constraints, we propose the following formulation

$$\min_{y} \quad \sum_{i \in \mathcal{N}} \sum_{v \in \mathcal{R}(l(i))} y(i,v) \tag{1a}$$

$$\begin{aligned}
\text{s.t.} \quad & r_i \leq y(i, v_0(i)) && \text{for } i \in \mathcal{N}, && \text{(1b)}\\
& \texttt{conj}(y(i,v), y(j,v)) && \text{for } (i,j) \in \mathcal{C}^v, v \in V, && \text{(1c)}\\
& \texttt{disj}(y(i,v), y(j,v)) && \text{for } \{i,j\} \in \mathcal{D}^v, v \in V, && \text{(1d)}\\
& y(i,v) + d(v,w) \leq y(i,w) && \text{for } i \in \mathcal{N}(l), (l,v,w) \in E, && \text{(1e)}\\
& y(i,w) + \hat{\rho}_i \leq y(j,v) && \text{for } (i,j,v,w) \in \mathcal{F}, && \text{(1f)}
\end{aligned}$$

where $\mathcal{F}$ is defined as

$$\mathcal{F} = \{(i,j,v,w) : i,j \in \mathcal{N}(l), k(i) + b(v,w) = k(j), (l,v,w) \in E\}.$$

Each $(i,j,v,w) \in \mathcal{F}$ represents a pair of vehicles driving on the same lane (v,w), for which the first vehicle must have left that lane before vehicle j can enter. Under Assumption 1, the constraints (1f) yield the following property of schedules.

Proposition 1. *Let y be a solution to the network scheduling problem* (1) *with Assumption 1. For each $(l,v,w) \in E$, there are always at most $b(v,w)$ vehicles at lane (v,w).*

Proof. Let i be a vehicle that has (v,w) on its route. Define the occupancy interval $D_i = [y(i,v), y(i,w)]$, then we say that i occupies (v,w) at some time t whenever $t \in D_i$. Therefore, the number of vehicles in (v,w) at time t equals the number of such intervals containing t.

Suppose we have a schedule y such that at some time t, there are strictly more than $b(v,w)$ vehicles i such that $t \in D_i$. Let i_1 be such that $y(i_1,v) \leq y(i,v)$ for all i such that $t \in D_i$. From the conjunctive constraints at v follows that there is some n such that

$$y(i_1,v) + \rho_{i_1} \leq y(i_2,v) + \rho_{i_2} \leq \cdots \leq y(i_n,v) + \rho_{i_n},$$

where $i_k = (l(i_1), k(i_1) + k - 1)$. By Assumption 1, the only vehicles that can enter (v,w) after i_1 are precisely $i_2, \ldots, i_n$. By assumption, we have $n \geq b(v,w) + 1$, so $j = (l(i_1), k(i_1) + b(v,w)) \in \{i_2, \ldots, i_n\}$ is such that $t \in D_j$. Hence, $y(j,v) \leq t \leq y(i,w)$, which violates constraint (1f) and thus contradicts the feasibility of y. $\square$

Without Assumption 1, it is less immediate how to formulate a MILP.