Vehicle trajectory planning in a single lane with minimum following distance and boundary conditions

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Abstract

We study a model for a single-lane road where overtaking is not permitted. Vehicles are modeled as double integrators, i.e., position and speed are governed by acceleration control, subject to bounds on speed and acceleration. Consecutive vehicles must maintain a fixed following distance to avoid collisions. Each vehicle has predetermined times for entry and exit, both of which must occur at full speed. For the objective of minimizing distance to the end of the lane over time, we present an algorithm to compute optimal trajectories. Assuming a fixed minimum lane length, we characterize the feasibility of this trajectory planning problem through a system of linear inequalities that depend only on the schedule times.

1 Optimal control formulation of isolated lane model

We will propose a model for a one-directional single-lane road of finite length where overtaking is not permitted. We will link such lanes together to construct a model representing a network of intersections. The fact that a lane between two intersections is of limited length—unlike the approaching lanes of the isolated intersection model—means that it can be occupied by a limited number of vehicles at the same time. This property makes the characterization of set of feasible trajectories more involved.

Given some lane, consider the set of vehicles that need to travel across this lane as part of their planned route. Suppose that the time of entry to and exit from this lane are fixed for each of these vehicles, then the question is whether there exists a set of trajectories that is safe, i.e. without collisions, and satisfies these *schedule times*. Loosely speaking, we want an easy way to answer this question for any set of schedule times. By requiring vehicles to enter and exit the lane at *full speed*, we will show that this feasibility question is precisely answered by a system of linear inequalities in terms of the schedule times.

Because there is generally not a single feasible set of trajectories, we can consider some performance criterion, for example by measuring some sort of smoothness to model passenger comfort or energy consumption. The resulting optimal control problem is straightforward to solve using a direct transcription method. However, when we seek to minimize each vehicle's distance to the end of the lane at all times, to which we refer as the *haste objective*, we will show that the optimal solution can be computed much more efficiently. We will see that the derivation of this algorithm is a direct byproduct of the feasibility analysis, because the latter will involve the construction of a certain set of upper bounding trajectories, which happen to be optimal under the haste objective.

In the remainder of this introducion, we will precisely establish the notion of a feasible set of trajectories in a lane. After precisely defining the optimal control problem under study, we show some examples of feasible trajectories for the haste objective and for the minimization of some proxy for energy consumption.

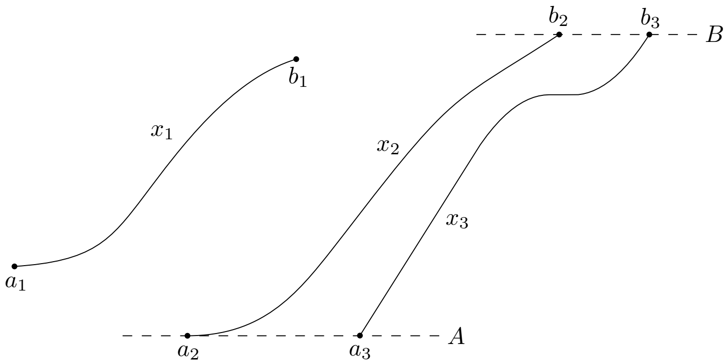

Figure 1: Example trajectories $x_1 \in \mathcal{D}[a_1, b_1]$, $x_2 \in D_1[a_2, b_2]$ and $x_3 \in D_2[a_3, b_3]$ for each the three classes of trajectories that are used throughout this chapter (horizontal axis is time, vertical axis is position).

Vehicle trajectories. We only consider the longitudinal position of vehicles on the lane and we assume that speed and acceleration are bounded. Therefore, let $\mathcal{D}[a, b]$ denote the set of valid *trajectories*, which we define to be all continuously differentiable functions $x : [a, b] \to \mathbb{R}$ satisfying the constraints

$$\dot{x}(t) \in [0, 1] \quad \text{and} \quad \ddot{x}(t) \in [-\omega, \bar{\omega}], \quad \text{for all } t \in [a, b], \tag{1}$$

for some fixed acceleration bounds $\omega, \bar{\omega} > 0$ and with $\dot{x}$ and $\ddot{x}$ denoting the first and second derivative with respect to time t. Note that the unit speed upper bound is not restrictive, since we can always apply an appropriate scaling of time and the acceleration bounds to arrive at this form. We use A and B to denote the start[1] and end position of the lane. Let $D_1[a, b] \subset \mathcal{D}[a, b]$ denote all trajectories x that satisfy the first-order boundary conditions

$$x(a) = A \ \text{ and } \ x(b) = B \tag{2}$$

and additionally satisfy $\dot{x}(a) > 0$ and $\dot{x}(b) > 0$, to avoid the technical difficulties of dealing with vehicles that are waiting at the start or end of the lane. On top of these conditions, let $D_2[a, b] \subset D_1[a, b]$ further induce the second-order boundary conditions

$$\dot{x}(a) = \dot{x}(b) = 1. \tag{3}$$

In words, these boundary conditions require that a vehicle arrives to and departs from the lane at predetermined times a and b and do so at full speed. Figure 1 shows an example for each of these three classes of trajectories.

Trajectory domains. Function domains will play an important role in the analysis of feasible trajectories. Therefore, we introduce some useful notational conventions. First of all, each of the trajectory classes above can be used with the common convention of allowing $a = -\infty$ or $b = \infty$. For instance, we write $\mathcal{D}(-\infty, \infty)$ to denote the set of trajectories defined on the whole real line. Furthermore, we use $\cdot|_{[a,b]}$ to denote function restriction. For example,

$$(t \mapsto t + 1)|_{[\xi,\infty)}$$

denotes some anonymous function with some restricted domain. Furthermore, given two smooth trajectories $\gamma_1 \in \mathcal{D}[a_1, b_1]$ and $\gamma_2 \in \mathcal{D}[a_2, b_2]$, we write inequality $\gamma_1 \preceq \gamma_2$ to mean

$$\gamma_1(t) \leq \gamma_2(t) \ \text{ for all } \ t \in [a_1, b_1] \cap [a_2, b_2].$$

[1] Note that assuming $A \neq 0$ is convenient later when we start piecing together multiple lanes.

Whenever the intersection of domains is empty, we say that the above inequality is *void*. The reason for introducing a dedicated symbol is that $\preceq$ is not transitive. To see this, consider the trajectories in Figure 1, then $x_1 \preceq x_3$ (void) and $x_3 \preceq x_2$, but clearly $x_1 \not\preceq x_2$.

Definition 1. Let $L > 0$ denote the *following distance* between consecutive vehicles. Suppose there are N vehicles scheduled to traverse the lane. For each vehicle i, let a_i and b_i denote the *schedule time* for entry and exit, respectively. Assuming that the schedule times are ordered as $a_1 \leq a_2 \leq \cdots \leq a_N$ and $b_1 \leq b_2 \leq \cdots \leq b_N$, then a *feasible solution* consists of a sequence of trajectories $x_1, \dots, x_N$ such that

$$x_i \in D_2[a_i, b_i] \qquad \text{for each } i \in \{1, \dots, N\}, \tag{4a}$$

$$x_i \preceq x_{i-1} - L \qquad \text{for each } i \in \{2, \dots, N\}. \tag{4b}$$

We will refer to (4b) as the *lead vehicle constraints*. For some performance criterion of trajectories, given as a functional $J(x)$ of trajectory x, the *lane planning problem* is to find a feasible solution that maximizes

$$\max \sum_{i=1}^{N} J(x_i). \tag{5}$$

We emphasize again that (4a) requires vehicles to enter and exit the lane at full speed. The feasibility characterization that we will derive can now be roughly stated as follows. Assuming the system parameters $(\omega, \bar{\omega}, A, B, L)$ to be fixed, with lane length $B - A$ sufficiently large and following distance L sufficiently small, feasibility of the lane planning problem is characterized by a system of linear inequalities in terms of the schedule times a_i and b_i.

Choice of objective. Probably better to introduce these examples earlier on, i.e., in single intersection chapter. We obtain what we call the *haste objective* by choosing

$$J(x_i) = \int_{a_i}^{b_i} x_i(t) \, \mathrm{d}t. \tag{6}$$

Roughly speaking, this objective seeks to keep all vehicles as close to the end of the lane at all times, but it does not capture energy efficiency in any way. Introduce energy objective and present some examples for both for comparison. In the rest of this chapter, we will show that optimal trajectories under the haste objective can be understood as the concatenation of at most four different types of trajectory parts, which we might call *bang-off-bang*. Based on this observation, we present an algorithm to compute optimal trajectories. Extending this algorithm to objectives like [the energy objective], is an interesting topic for further research.

2 Single vehicle with arbitary lead vehicle constraint

Before we analyze the feasibility of the lane planning problem as a whole, we focus on the lead vehicle constraint (4b) for a single vehicle $i \geq 2$. This allows us to lighten the notation slightly by dropping the vehicle index i. Instead of $x_{i-1} - L$, we assume we are given some arbitrary *lead vehicle boundary* u and consider the following problem.

Definition 2. Let $u \in D_1[c, d]$ and assume we are given two schedule times $a, b \in \mathbb{R}$, then the *single vehicle (feasibility) problem* is to find a trajectory $x \in D_2[a, b]$ such that $x \preceq u$.

2.1 Necessary conditions

Suppose we are given some feasible trajectory x for the single vehicle problem. In addition to the given upper bounding trajectory u, we will derive two upper bounding trajectories x^1

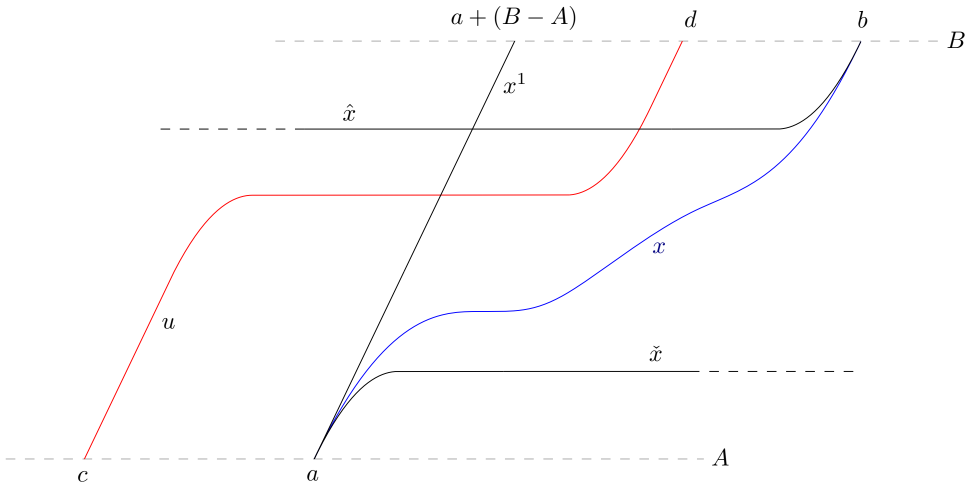

Figure 2: Illustration of the four bounding trajectories $u, x^1, \hat{x}, \check{x}$ that bound feasible trajectories from above and below. We also drew an example of a feasible trajectory x in blue. The horizontal axis represents time and the vertical axis corresponds to the position on the lane, so the vertical dashed grey lines correspond to the start and end of the lane.

and $\hat{x}$ and one lower bounding trajectory $\check{x}$, see Figure 2. Using these bounding trajectories, we will formulate four necessary conditions for the single vehicle problem.

Let the *full speed boundary*, denoted x^1, be defined as

$$x^1(t) = A + t - a, \tag{7}$$

for all $t \in [a, b]$, then we clearly have $x \preceq x^1$. Observe that $x^1(s) = B$ for $s = a + (B - A)$, which can be interpreted as the earliest time of departure from the lane, so we must have $b \geq a + (B - A)$. This is our first necessary condition.

Lemma 1. *If there exists $x \in D_2[a, b]$, then $b - a \geq B - A$.*

Next, since deceleration is at most ω, we have $\dot{x}(t) \geq \dot{x}(a) - \omega(t - a) = 1 - \omega(t - a)$, which we combine with the speed constraint $\dot{x} \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \omega(t - a)\}$. Hence, we obtain the lower bound

$$x(t) = x(a) + \int_a^t \dot{x}(\tau)\,\mathrm{d}\tau \geq A + \int_a^t \max\{0, 1 - \omega(\tau - a)\}\,\mathrm{d}\tau =: \check{x}(t), \tag{8}$$

for all $t \geq a$, so that we have $x \succeq \check{x}$. Analogously, we derive an upper bound from the fact that acceleration is at most $\bar{\omega}$. Observe that we have $\dot{x}(t) + \bar{\omega}(b - t) \geq \dot{x}(b) = 1$, which we combine with the speed constraint $\dot{x}(t) \geq 0$ to derive $\dot{x}(t) \geq \max\{0, 1 - \bar{\omega}(b - t)\}$. Hence, we obtain the upper bound

$$x(t) = x(b) - \int_t^b \dot{x}(\tau)\,\mathrm{d}\tau \leq B - \int_t^b \max\{0, 1 - \bar{\omega}(b - \tau)\}\,\mathrm{d}\tau =: \hat{x}(t), \tag{9}$$

for all $t \leq b$, so we have $x \preceq \hat{x}$. We refer to $\check{x}$ and $\hat{x}$ as the *entry boundary* and *exit boundary*, respectively.

Lemma 2. *Consider some lead boundary $u \in D_1[c, d]$ and assume $[a, b] \cap [c, d] \neq \varnothing$. If there exists a trajectory $x \in D_2[a, b]$ such that $x \preceq u$, then $a \geq c$ and $b \geq d$ and $u \succeq \check{x}$.*

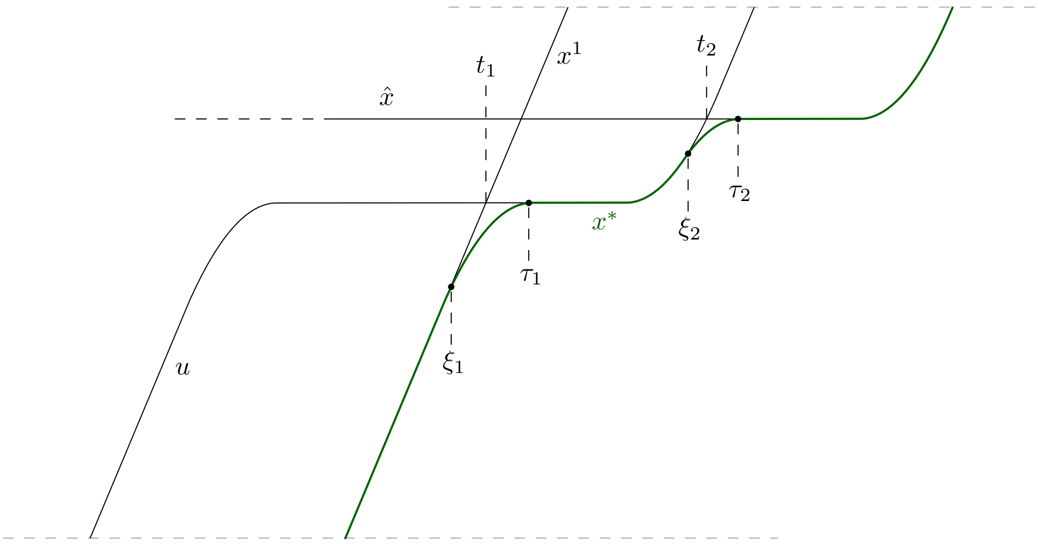

Figure 3: The minimum boundary γ, induced by three upper boundaries u, $\hat{x}$ and x^1, is smoothened around time t_1 and t_2, where the derivative is discontinuous, to obtain the smooth optimal trajectory x^*, drawn in green. The times ξ_i and τ_i correspond to the start and end of the connecting deceleration as defined in Section 2.4.

Proof. Each of these conditions corresponds somehow to one of the bounding trajectories defined above. Suppose $a < c$, then because the domains intersect, we must have $b > c$, but then clearly no x can satisfy $x \preceq u$. When $b < d$, then it is a consequence of $\dot{u}(b) > 0$ that any x will violate $x \preceq u$. To see that the third condition must hold, suppose that $u(\tau) < \check{x}(\tau)$ for some time τ. Since $c \leq a$, this means that u must intersect $\check{a}$ from above. Therefore, any trajectory that satisfies $x \preceq u$ must also intersect $\check{a}$ from above, which contradicts the assumption $x \in D_2[a, b]$. $\square$

Remark 1. The assumption of non-empty domains is required in the previous lemma, because otherwise we include the situation in which x lies completely to the left of u, in which case the stated conditions are obviously not necessary anymore.

We note that the boundaries $\hat{x}$ and $\check{x}$ can be combined to yield yet another necessary condition. It is straightforward to verify from equations (8) and (9) that $\hat{x}(t) \geq B - 1/(2\bar{\omega})$ and $\check{x}(t) \leq A + 1/(2\omega)$. Therefore, whenever $B - A < 1/(2\bar{\omega}) + 1/(2\omega)$, these boundaries intersect for certain values of a and b. Because the exact condition is somewhat cumbersome to characterize, we avoid this case by simply assuming that the lane length is sufficiently large, to keep the analysis simpler.

Assumption 1. *The length of the lane satisfies $B - A \geq 1/(2\omega) + 1/(2\bar{\omega})$.*

Observe that $1/(2\omega)$ is precisely the distance required to decelerate from full speed to a standstill. Similarly, $1/(2\bar{\omega})$ is the distance required for a full acceleration. Therefore, we may interpret Assumption 1 as requiring enough space in the lane such that there is at least one *waiting position*. We will return to this observation in Section 3.2. (check that we do this)

2.2 Sufficient conditions

The goal of the remainder of this section is to prove the following feasibility characterization.

Theorem 1 (Feasibility characterization of single vehicle problem)**.** *Given some lead vehicle boundary $u \in D_1[c, d]$ and some schedule times $a, b \in \mathbb{R}$ such that $[a, b] \cap [c, d] \neq \varnothing$ and assuming Assumption 1, there exists a solution $x \in D_2[a, b]$ satisfying $x \preceq u$ if and only if*

$$
\begin{array}{llll}
(i) & b - a \geq B - A, & & (travel\ constraint) \\
(ii) & a \geq c, & & (entry\ order\ constraint) \\
(iii) & b \geq d, & & (exit\ order\ constraint) \\
(iv) & u \succeq \check{x}. & & (entry\ space\ constraint)
\end{array}
$$

Note that Lemma 1 and Lemma 2 already showed necessity of these conditions. Therefore, we will show that, under these conditions, we can always construct a solution γ^* for the single vehicle problem, thereby showing that the four conditions are also sufficient. The particular solution that we will construct also happens to be a smooth upper boundary for all other solutions, in the sense that, for any other feasible solution x we have $x \preceq \gamma^*$. The starting point of the construction is the *minimum boundary* $\gamma : [a, b] \to \mathbb{R}$, defined as

$$
\gamma(t) := \min\{u(t), \hat{x}(t), x^1(t)\}. \tag{10}
$$

Obviously, γ is a valid upper boundary for any other feasible solution, but in general, γ may have a discontinuous derivative at some[2] isolated points in time, in which case $\gamma \notin \mathcal{D}[a, b]$.

Definition 3. Let $\mathcal{P}[a, b]$ be the set of functions $\mu : [a, b] \to \mathbb{R}$ for which there is a finite subdivision $a = t_0 < \cdots < t_{n+1} = b$ such that the truncation $\mu|_{[t_i,t_{i+1}]} \in \mathcal{D}[t_i, t_{i+1}]$ is a smooth trajectory, for each $i \in \{0, \ldots, n\}$, and for which the one-sided limits of $\dot{\mu}$ satisfy

$$
\dot{\mu}(t_i^-) := \lim_{t \uparrow t_i} \dot{\mu}(t) > \lim_{t \downarrow t_i} \dot{\mu}(t) =: \dot{\mu}(t_i^+), \tag{11}
$$

for each $i \in \{1, \ldots, n\}$. We refer to such μ as a *piecewise trajectory (with downward bends)*.

Under the conditions of Theorem 1, it is not difficult to see from Figure 2 that γ satisfies the above definition, so $\gamma \in \mathcal{P}[a, b]$. In other words, γ consists of a number of pieces that are smooth and satisfy the vehicle dynamics, with possibly some sharp bend downwards where these pieces come together. Next, we present a simple procedure to smoothen out this kind of discontinuity by decelerating from the original trajectory somewhat before some t_i, as illustrated in Figure 3. We will argue that this procedure can be repeated as many times as necessary to smoothen out every discontinuity.

In Section 2.3, we first define a parameterized family of functions to model the deceleration part that we introduce for the smoothing procedure, which is described in Section 2.4. We apply this procedure to γ to obtain γ^*, after which it is relatively straightforward to show that γ^* is an upper bound for all other feasible solutions, which is done in Section 2.5.

2.3 Deceleration boundary

Recall the derivation of $\check{x}$ in equation (8) and the discussion preceding it, which we will now generalize a bit. Let $x \in \mathcal{D}[a, b]$ be some smooth trajectory, then observe that $\dot{x}(t) \geq \dot{x}(\xi) - \omega(t - \xi)$ for all $t \in [a, b]$. Combining this with the constraint $\dot{x}(t) \in [0, 1]$, this yields

$$
\dot{x}(t) \geq \max\{0, \min\{1, \dot{x}(\xi) - \omega(t - \xi)\}\} =: \{\dot{x}(\xi) - \omega(t - \xi)\}_{[0,1]}, \tag{12}
$$

where use $\{\cdot\}_{[0,1]}$ as a shorthand for this clipping operation. Hence, for any $t \in [a, b]$, we obtain the following lower bound

$$
x(t) = x(\xi) + \int_\xi^t \dot{x}(\tau)\, \mathrm{d}\tau \geq x(\xi) + \int_\xi^t \{\dot{x}(\xi) - \omega(\tau - \xi)\}_{[0,1]}\, \mathrm{d}\tau =: x[\xi](t), \tag{13}
$$

where we will refer to the right-hand side as the *deceleration boundary* of x at ξ. Observe that this definition indeed generalizes the definition of $\check{x}$, because we have $\check{x} = (x[a])|_{[a,b]}$.

[2] In fact, it can be shown that, under the necessary conditions, there are at most two of such discontinuities.

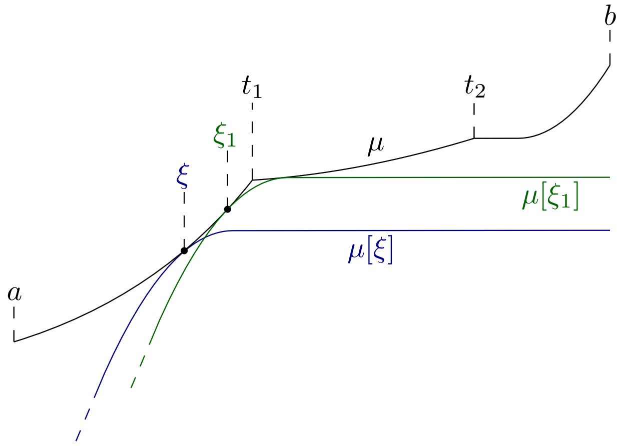

Figure 4: Illustration of some piecewise trajectory $\mu \in \mathcal{P}[a, b]$ with a discontinuous derivative at times t_1 and t_2. Furthermore, the figure shows some arbitrary deceleration boundary $\mu[\xi]$ at time ξ in blue and the unique connecting deceleration $\mu[\xi_1]$ to the cover discontinuity at t_1 in green. We truncated the start of both deceleration boundaries for a more compact figure. The careful observer may notice that μ cannot occur as the minimum boundary defined in (10), but please note that the class of piecewise trajectories $\mathcal{P}[a, b]$ is just slightly more general than necessary for our current purposes.

Note that $x[\xi]$ depends on x only through the two real numbers $x(\xi)$ and $\dot{x}(\xi)$. It will be convenient later to rewrite the right-hand side of (13) as

$$x^-[p, v, \xi](t) := p + \int_\xi^t \{v - \omega(\tau - \xi)\}_{[0,1]} \, \mathrm{d}\tau, \tag{14}$$

such that $x[\xi](t) = x^-[x(\xi), \dot{x}(\xi), \xi](t)$. We can expand the integral in this expression further by carefully handling the clipping operation. Observe that the expression within the clipping operation reaches the bounds 1 and 0 for $\delta_1 := \xi - (1 - v)/\omega$ and $\delta_0 := \xi + v/\omega$, respectively. Using this notation, a straightforward calculation shows that

$$x^-[p, v, \xi](t) = p + \begin{cases} (1 - v)^2/(2\omega) + (t - \xi) & \text{for } t \leq \delta_1, \\ v(t - \xi) - \omega(t - \xi)^2/2 & \text{for } t \in [\delta_1, \delta_0], \\ v^2/(2\omega) & \text{for } t \geq \delta_0. \end{cases} \tag{15}$$

It is easily verified that the three cases above coincide at $t \in \{\delta_1, \delta_0\}$, which justifies the overlaps in the case distinction. Furthermore, since x and $\dot{x}$ are continuous by assumption, it follows that $x[\xi](t) = x^-[x(\xi), \dot{x}(\xi), \xi](t)$ is continuous as a function of either of its arguments.[3] Assuming $0 \leq v \leq 1$, it can be verified that for every $t \in \mathbb{R}$, we have $\ddot{x}^-[p, v, \xi](t) \in \{-\omega, 0\}$ and $\dot{x}^-[p, v, \xi](t) \in [0, 1]$ due to the clipping operation, so that $x^-[p, v, \xi] \in \mathcal{D}(-\infty, \infty)$.

Piecewise trajectories. Let $\mu \in \mathcal{P}[a, b]$ be some piecewise trajectory with corresponding subdivision $a = t_0 < \cdots < t_{n+1} = b$ as defined in Definition 3. It is straightforward to generalize the definition of a deceleration boundary to μ. Whenever $\xi \in [a, b] \setminus \{t_1, \ldots, t_n\}$, we just define $\mu[\xi] := x^-[\mu(\xi), \dot{\mu}(\xi), \xi]$, exactly like we did for x. However, at the points of discontinuity $\xi \in \{t_1, \ldots, t_n\}$, the derivative $\dot{\mu}(\xi)$ is not defined, so we choose to use the left-sided limit instead, by defining $\mu[\xi] := x^-[\mu(\xi), \dot{\mu}(\xi^-), \xi]$.

[3] Even more, it can be shown that $x[\xi](t)$ is continuous as a function of (ξ, t).

Remark 2. Please note that we cannot just replace x with μ in inequality (13) to obtain a similar bound for μ on the its full interval $[a, b]$. Instead, we get the following *piecewise lower bounding* property. Consider some interval $I \in \{[a, t_1], (t_1, t_2], \dots, (t_n, b]\}$, then what remains true is that $\xi \in I$ implies $\mu(t) \geq \mu[\xi](t)$ for every $t \in I$.

2.4 Smoothing procedure

Let $\mu \in \mathcal{P}[a, b]$ be some piecewise trajectory and let $a = t_0 < \cdots < t_{n+1} = b$ denote the subdivision as in Definition 3. We first show how to smoothen the discontinuity at t_1 and then argue how to repeat this process for the remaining times t_i. Our aim is to choose some time $\xi \in [a, t_1]$ from which the vehicle starts fully decelerating, such that $\mu[\xi] \preceq \mu$ and such that $\mu[\xi]$ touches μ at some time $\tau \in [t_1, b]$ tangentially. We will show there is a unique trajectory $\mu[\xi]$ that satisfies these requirements and refer to it as the *connecting deceleration*, see Figure 4 for an example. The construction relies on the following technical assumption.

Assumption 2. *Throughout the following discussion, we assume $\mu \succeq \mu[a]$ and $\mu \succeq \mu[b]$.*

Touching. Recall Remark 2, which asserts that we have $\mu[\xi](t) \leq \mu(t)$ for every $t \in [a, t_1]$ for any $\xi \in [a, t_1]$. After the discontinuity, so for every $t \in [t_1, b]$, we want $\mu[\xi](t) \leq \mu(t)$ and equality at least somewhere, so we measure the relative position of $\mu[\xi]$ with respect to μ here, by considering

$$d(\xi) := \min_{t \in [t_1, b]} \mu(t) - \mu[\xi](t). \tag{16}$$

Since $\mu(t)$ and $\mu[\xi](t)$ are both continuous as a function of t on the interval $[t_1, b]$, this minimum actually exists (extreme value theorem). Furthermore, since d is the minimum of a continuous function over a closed interval, it is continuous as well (see Lemma A.1). Observe that $d(a) \geq 0$, because $\mu \succeq \mu[a]$ by Assumption 2. By definition of t_1, we have $\dot{\mu}(t_1^-) > \dot{\mu}(t_1^+)$, from which it follows that $\mu(t) < \mu[t_1](t)$ for $t \in (t_1, t_1 + \epsilon)$ for some small $\epsilon > 0$, which shows that $d(t_1) < 0$. By the intermediate value theorem, there is $\xi_1 \in [a, t_1)$ such that $d(\xi_1) = 0$. This shows that $\mu[\xi_1]$ touches μ at some time $\tau_1 \in [t_1, b]$.

Uniqueness. It turns out that ξ_1 itself is not necessarily unique, which we explain below. Instead, we are going to show that the connecting deceleration $\mu[\xi_1]$ is unique. More precisely, given any other $\xi \in [a, t_1)$ such that $d(\xi) = 0$, we will show that $\mu[\xi] = \mu[\xi_1]$.

The first step is to establish that the level set

$$X := \{\xi \in [a, t_1) : d(\xi) = 0\} \tag{17}$$

is a closed interval. To this end, we show that d is non-increasing on $[a, t_1)$, which together with continuity implies the desired result (see Lemma A.2). To show that d is non-increasing, it suffices to show that $\mu[\xi](t)$ is non-decreasing as a function of ξ, for every $t \in [t_1, b]$. We can do this by computing the partial derivative of $\mu[\xi]$ with respect to ξ and verifying it is non-negativity. Recall the definition of $\mu[\xi]$, based on x^- in equation (15). Using similar notation, we write $\delta_1(\xi) = \xi - (1 - \dot{\mu}(\xi))/\omega$ and $\delta_0(\xi) = \xi + \dot{\mu}(\xi)/\omega$ and compute

$$\frac{\partial}{\partial \xi} \mu[\xi](t) = \dot{\mu}(\xi) + \begin{cases} \ddot{\mu}(\xi)(\dot{\mu}(\xi) - 1)/\omega - 1 & \text{for } t \leq \delta_1(\xi), \\ \ddot{\mu}(\xi)(t - \xi) - \dot{\mu}(\xi) + \omega(t - \xi) & \text{for } t \in [\delta_1(\xi), \delta_0(\xi)], \\ \ddot{\mu}(\xi)\dot{\mu}(\xi)/\omega & \text{for } t \geq \delta_0(\xi). \end{cases} \tag{18}$$

It is easily verified that the cases match at $t \in \{\delta_1(\xi), \delta_0(\xi)\}$, which justifies the overlaps there. Consider any $\xi \in [a, t_1)$ and $t \in [t_1, b]$, then we always have $\delta_1(\xi) \leq \xi \leq t$, so we only have to verify the second and third case:

$$\frac{\partial}{\partial \xi}\mu[\xi](t) = (\ddot{\mu}(\xi) + \omega)(t - \xi) \geq 0 \qquad \text{for } t \in [\delta_1(\xi), \delta_0(\xi)], \tag{19a}$$

$$\frac{\partial}{\partial \xi}\mu[\xi](t) \geq \dot{\mu}(\xi) + (-\omega)\dot{\mu}(\xi)/\omega = 0 \qquad \text{for } t \geq \delta_0(\xi). \tag{19b}$$

This concludes the argument for X being a closed interval.

Assuming ξ to be fixed, observe that there is equality in (19a) for some $t \in [\delta_1(\xi), \delta_0(\xi)]$ if and only if there is equality in (19b) for some other $t' \geq \delta_0(\xi)$. Note that this happens precisely when $\ddot{\mu}(\xi) = -\omega$. Therefore, whenever μ is fully deceleration, so $\dot{\mu}(t) = -\omega$ on some open interval $U \subset (a, t_1)$, we have $(\partial/\partial\xi)\mu[\xi](t) = 0$ for all $t \geq \delta_1(\xi)$. This essentially means that any choice of $\xi \in U$ produces the same trajectory $\mu[\xi]$. Please see Figure 5 for an example of this case. This observation is key to the remaining uniqueness argument.

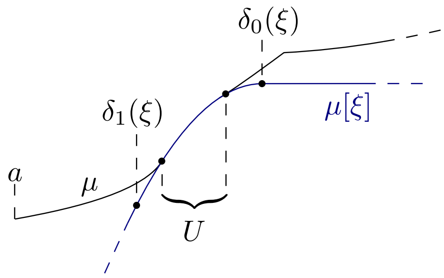

Figure 5: Example of a piecewise trajectory μ with a part of full deceleration over some interval U such that any choice of $\xi \in U$ produces the same deceleration boundary $\mu[\xi]$, which naturally coincides with μ on U.

Since X is a closed interval, we may define $\xi_0 = \min X$. Consider any $\xi' \in X$ with $\xi' > \xi_0$, then we show $\mu[\xi'](t) = \mu[\xi_0](t)$ for all $t \in [\xi_0, b]$. For sake of contradiction, suppose there is some $t' \in [\xi_0, b]$ such that $\mu[\xi'](t') > \mu[\xi_0](t')$, then there must be some open interval $U \subset (\xi_0, \xi')$ such that

$$\frac{\partial}{\partial \xi}\mu[\xi](t') > 0 \text{ for all } \xi \in U. \tag{20}$$

However, we argued in the previous paragraph that this actually holds for any $t' \geq \delta_1(\xi)$. In particular, let $t^* \in [t_1, b]$ be such that $\mu(t^*) = \mu[\xi_0](t^*)$, then $t^* \geq t_1 \geq \xi \geq \delta_1(\xi)$, so (20) yields $\mu[\xi'](t^*) > \mu[\xi_0](t^*)$, but then $d(\xi') > d(\xi_0) = 0$, so $\xi' \notin X$, a contradiction.

Touching tangentially. It remains to show that μ and $\mu[\xi_0]$ touch tangentially somewhere on $[t_1, b]$. Let $\tau_1 \in [t_1, b]$ be the smallest time such that $\mu(\tau_1) - \mu[\xi_0](\tau_1) = d(\xi_0) = 0$ and consider the following three cases.

First of all, note that $\tau_1 = t_1$ is not possible, because this would require

$$\dot{\mu}(t_1^+) > \dot{\mu}[\xi_0](t_1^+) = \dot{\mu}[\xi_0](t_1), \tag{21}$$

but since μ is a piecewise trajectory, we must have $\dot{\mu}(t_1^-) > \dot{\mu}(t_1^+) > \dot{\mu}[\xi_0](t_1)$. This shows that $\mu(t_1 - \epsilon) < \mu[\xi_0](t_1 - \epsilon)$, for some small $\epsilon > 0$, which contradicts $\mu[\xi_0] \preceq \mu$.

Suppose $\tau_1 \in (t_1, b)$, then recall the definition of $d(\xi_0)$ and observe that the usual first-order necessary condition (derivative zero) for local minima requires $\dot{\mu}(\tau_1) = \dot{\mu}[\xi_0](\tau_1)$.

Finally, consider $\tau_1 = b$. Observe that $\dot{\mu}(b) > \dot{\mu}[\xi_0](b)$, would contradict minimality of $\tau_1 = b$. Therefore, suppose $\dot{\mu}(b) < \dot{\mu}[\xi_0](b)$, then $\dot{\mu}[b](b) = \dot{\mu}(b) < \dot{\mu}[\xi_0](b)$, so

$$\dot{\mu}[b](t) \leq \dot{\mu}[\xi_0](t) \text{ for } t \leq b, \tag{22}$$

but then $\mu[b](t) > \mu[\xi_0](t)$ for $t < b$. In particular, for $t = \xi_0$, this shows $\mu[b](\xi_0) > \mu[\xi_0](\xi_0) = \mu(\xi_0)$, which contradicts part $\mu[b] \preceq \mu$ of Assumption 2.

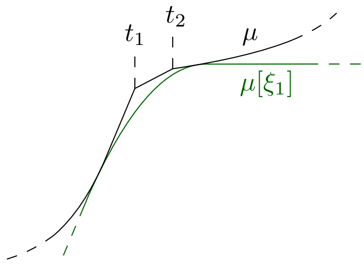

Figure 6: Part of a piecewise trajectory μ on which a single connecting deceleration covers the two discontinuities at t_1 and t_2 at once.

Repeat for remaining discontinuities. Let us summarize what we have established so far. The times $\xi_0 \in [a, t_1)$ and $\tau_1 \in (t_1, b]$ have been chosen such that

$$\mu[\xi_0](t) \leq \mu(t) \quad \text{for} \quad t \in [\xi_0, \tau_1], \tag{23a}$$

$$\dot{\mu}[\xi_0](\xi_0) = \dot{\mu}(\xi_0) \quad \text{and} \quad \dot{\mu}[\xi_0](\tau_1) = \dot{\mu}(\tau_1). \tag{23b}$$

Instead of ξ_0, it will be convenient later to choose $\xi_1 := \max X$ as the representative of the unique connecting deceleration. We can now use $(\mu[\xi_1])|_{[\xi_1,\tau_1]}$ to replace μ at $[\xi_1, \tau_1]$ to obtain a trajectory without the discontinuity at t_1. More precisely, we define

$$\mu_1(t) = \begin{cases} \mu(t) & \text{for } t \in [a, \xi_1] \cup [\tau_1, b], \\ \mu[\xi_1](t) & \text{for } t \in [\xi_1, \tau_1]. \end{cases} \tag{24}$$

From the way we constructed $\mu[\xi_1]$, it follows from (23) that we have $\mu_1 \in \mathcal{P}[a, b]$, but without the discontinuity at t_1. Observe that a single connecting deceleration may cover more than one discontinuity, as illustrated in Figure 6. Note that we must have $\dot{\mu}_1(a) = \dot{\mu}(a)$ and $\dot{\mu}_1(b) = \dot{\mu}(b)$ by construction. Hence, it is not difficult to see that μ_1 must still satisfy Assumption 2, so that we can keep repeating the exact same process, obtaining connecting decelerations $(\xi_2, \tau_2), (\xi_3, \tau_3), \ldots$ and the corresponding piecewise trajectories $\mu_2, \mu_3, \ldots$ to remove any remaining discontinuities until we end up with a smooth trajectory $\mu^* \in \mathcal{D}[a, b]$. We emphasize again that $\dot{\mu}^*(a) = \dot{\mu}(a)$ and $\dot{\mu}^*(b) = \dot{\mu}(b)$.

Proof of Theorem 1. Let us now return to the minimum boundary γ defined in (10). From Figure 2 and the conditions of Theorem 1, it is clear that γ must satisfy $\gamma(a) = A$, $\gamma(b) = B$ and $\dot{\gamma}(a) = \dot{\gamma}(b) = 1$, so whenever we have $\gamma \in \mathcal{D}[a, b]$, i.e., γ does not contain discontinuities, we automatically have $\gamma \in D_2[a, b]$ so that γ itself is already a feasible solution. Explain why Assumption 2 holds. Otherwise, we perform the smoothing procedure presented above to obtain the smoothed trajectory $\gamma^* \in \mathcal{D}[a, b]$. This completes the proof of Theorem 1.

2.5 Upper boundary solution

As a byproduct of the above analysis, the next lemma shows that the solution γ^* is also an upper boundary for any other feasible trajectory.

Lemma 3. *Let $\mu \in \mathcal{P}[a, b]$ be a piecewise trajectory and let $\mu^* \in \mathcal{D}[a, b]$ denote the result after smoothing. All trajectories $x \in \mathcal{D}[a, b]$ that are such that $x \preceq \mu$, must satisfy $x \preceq \mu^*$.*

Proof. Consider some interval (ξ, τ) where we introduced some connecting deceleration boundary. Suppose there exists some $t_d \in (\xi, \tau)$ such that $x(t_d) > \mu(t_d)$. Because $x(\xi) \leq \mu(\xi)$, this means that x must intersect μ at least once in $[\xi, t_d)$, so let $t_c := \sup\{t \in [\xi, t_d) : x(t) = \mu(t)\}$ be the latest time of intersection such that $x(t) \geq \mu(t)$ for all $t \in [t_c, t_d]$. There must be some $t_v \in [t_c, t_d]$ such that $\dot{x}(t_v) > \dot{\mu}(t_v)$, otherwise

$$x(t_d) = x(t_c) + \int_{t_c}^{t_d} \dot{x}(t)\,\mathrm{d}t \leq \mu(t_c) + \int_{t_c}^{t_d} \dot{\mu}(t)\,\mathrm{d}t = \mu(d_t),$$

which contradicts our choice of t_d. Hence, for every $t \in [t_v, \tau]$, we have

$$\dot{x}(t) \geq \dot{x}(t_v) - \omega(t - t_v) > \dot{\mu}(t_v) - \omega(t - t_v) = \dot{\mu}(t).$$

It follows that $x(\tau) > \mu(\tau)$, which contradicts the assumption $x \preceq \mu$. □

Remark 3. The above upper boundary property has the following interesting consequence if we extend the single vehicle problem to an optimal control problem by considering maximizing the haste criterion $J(x)$ defined in (6) as optimization objective. In particular, observe that it follows from the above lemma that any other $x \in D_2[a, b]$ satisfying $x \preceq u$ must also satisfy

$$\int_a^b x(t)\,\mathrm{d}t \leq \int_a^b \gamma^*(t)\,\mathrm{d}t \tag{25}$$

Consequently, γ^* is an optimal solution to the single vehicle optimal control problem

$$\max_{x \in D_2[a,b]} J(x) \;\text{ such that }\; x \preceq u. \tag{26}$$

We might need this later…

Acceleration boundary. Before we present the decomposition, we first define an auxiliary upper boundary. Similar to how we generalized the entry boundary $\check{x}$ to the deceleration boundary in Section 2.3, we now generalize the exit boundary $\hat{x}$ to obtain the *acceleration boundary*. Because the derivation is completely analogous, we will only present the resulting expressions. Let $x \in \mathcal{D}[a, b]$ be some smooth trajectory, then the acceleration boundary $x^+[\xi]$ of x at some $\xi \in [a, b]$ is defined as the right-hand side of the inequality

$$x(t) \leq x(\xi) + \int_{\xi}^{t} \{\dot{x}(\xi) + \bar{\omega}(\tau - \xi)\}_{[0,1]} \, \mathrm{d}\tau =: x^+[\xi](t), \tag{27}$$

which holds for every $t \in [a, b]$. Observe that the exit boundary can now be written as the restricted acceleration boundary $\hat{x} = (x^+[b])|_{[a,b]}$. Similar to definition (14), we define

$$x^+[p, v, \xi](t) := p + \int_{\xi}^{t} \{v + \bar{\omega}(\tau - \xi)\}_{[0,1]} \, \mathrm{d}\tau, \tag{28}$$

such that $x^+[\xi](t) = x^+[x(\xi), \dot{x}(\xi), \xi](t)$ and similar to (15), we calculate

$$x^+[p, v, \xi](t) = p + \begin{cases} \dots & \text{for } t \leq \bar{\delta}_0, \\ \dots & \text{for } t \in [\bar{\delta}_0, \bar{\delta}_1], \\ \dots & \text{for } t \geq \bar{\delta}_1, \end{cases} \tag{29}$$

with $\bar{\delta}_0 :=$ and $\bar{\delta}_1 :=$.

3 Lane planning feasibility

We will now return to the feasibility of the lane planning problem and show how it decomposes in terms of a sequence of single vehicle feasibility problems. The general idea is to repeat the construction of the previous section for each vehicle to obtain a solution x_i, while using each constructed trajectory as the boudary $u = x_i$ for the next problem of finding $x_{i+1} \preceq u$. We will show that feasibility is equivalent to a_i and b_i satisfying a certain system of inequalities.

Assumption 3 (Waiting position)**.** *Assume that $L < B - A$.*

For some $y \in \mathcal{D}[\alpha, \beta]$, we define the inverse at some position p in its range to be

$$y^{-1}(p) = \inf\{t \in [\alpha, \beta] : y(t) = p\}. \tag{30}$$

Given some feasible trajectory $u \in D[c, d]$, we define its *downshift*

$$\bar{u}(t) = \begin{cases} u(t) - L & \text{for } t \in [u^{-1}(A + L), d], \\ B - L + t - d & \text{for } t \in [d, d + L]. \end{cases} \tag{31}$$

For ease of reference, we denote the endpoints of its domain as $\bar{a} := u^{-1}(A+L)$ and $\bar{b} := d+L$.

Lemma 4 (Boundary extension)**.** *Consider some trajectory $u \in \mathcal{D}[c, d]$ such that $u(d) \geq A$. If $x \in \mathcal{D}[a, b]$ is such that $x(a) = A$ and $x \preceq u$, then it satisfies $x \preceq (u(d) + t - d)|_{[d,\infty)}$, which may be interpreted as extending the upper boundary u to the right at full speed.*

Proof. If $b < d$, then $x \preceq (\cdot)|_{[d,\infty)}$ is always void and the statement is trivially true. Assume $b \geq d$ and consider an arbitrary $t \geq d$. Suppose $a \leq d$, then we have $x(t) \leq x(d) + t - d \leq u(d) + t - d$. Suppose $a > d$, then we have $x(t) \leq x(a) + t - a = A + t - a \leq u(d) + t - d$. □

Why do we assume $a \geq c$ and $b \geq d$ in the next lemma? Because we assume $a_1 \leq a_2 \leq \cdots \leq a_N$ and similarly for b_i.

Lemma 5 (Downshift boundary equivalence)**.** *For each $u \in D_2[c, d]$, the downshift trajectory satisfies $\bar{u} \in D_1[\bar{a}, \bar{b}]$. For each $x \in D[a, b]$ such that $a \geq c$ and $b \geq d$, we have $x \preceq u - L$ if and only if $x \preceq \bar{u}$.*

Proof. The two cases in the definition of $\bar{u}$ coincide, so that $\bar{u} \in \mathcal{D}$. Furthermore, it is easily verified that $\bar{u}(\bar{a}) = A$ and $\bar{u}(\bar{b}) = B$, so the first claim follows.

Suppose $x \preceq u - L$, and suppose there exists some $t \in [a, b] \cap [c, d]$. If $t \in [\bar{a}, d]$, then $x(t) \leq u(t) - L = \bar{u}(t)$ by definition. If $t \in [d, \bar{b}]$, then apply Lemma 4 to $u - L$ (using Assumption 3 for $u(d) - L \geq A$) to obtain $x \leq (\tau \mapsto u(d) - L + \tau - d)|_{[d,\infty)} = (\tau \mapsto B - L + \tau - d)|_{[d,\infty)}$, so that $x(t) \leq B - L + t - d = \bar{u}(t)$.

For the other direction, suppose $x \preceq \bar{u}$. First of all, since $u(c) = A$ and $u(\bar{a}) = A + L > A$ and u is non-decreasing, we have $c < \bar{a}$. Suppose $c \leq a < \bar{a}$, then since $b \geq d \geq \bar{a}$ and $\dot{x}(a) = 1$, we must have $x(\bar{a}) > x(a) = A = \bar{u}(\bar{a})$, contradicting the initial assumption. Hence, $a \geq \bar{a}$, so any $t \in [a, b] \cap [c, d]$ satisfies $t \in [\bar{a}, d]$, but then $x(t) \leq \bar{u}(t) = u(t) - L$ by definition. □

For brevity, we will write $x \in D_2^N[a, b]$ to denote the vector $x = (x_1, \ldots, x_N)$ of N trajectories $x_i \in D_2[a_i, b_i]$, while assuming that the schedule times $a = (a_1, a_2, \ldots, a_N)$ and $b = (b_1, b_2, \ldots, b_N)$ are ordered as $a_1 \leq a_2 \leq \cdots \leq a_N$ and $b_1 \leq b_2 \leq \cdots \leq b_N$.

Lemma 6. *The following five statements are equivalent:*

(C0) The lane planning problem is feasible.

(C1) There exists $x \in D_2^N[a, b]$ such that $x_i \leq x_{i-1} - L$ for all $i \in \{2, \ldots, N\}$.

(C2) There exists $x \in D_2^N[a,b]$ *such that* $x_i \leq \bar{x}_{i-1}$ *for all* $i \in \{2, \ldots, N\}$.

(C3) There exists $x \in D_2^N[a,b]$ *such that* $b_i - a_i \geq B - A$ *for all* $i \in \{1, \ldots, N\}$*; and*

(i) $b_i \geq \bar{b}_{i-1}$, *(exit order constraint)*

(ii) $a_i \geq \bar{a}_{i-1}$, *(entry order constraint)*

(iii) $\bar{x}_i \geq \check{x}_i$. *(entry space constraint)*

(C4) We have $b_i - a_i \geq B - A$ *for all* $i \in \{1, \ldots, N\}$*; and*

(ii)* $b_i \geq b_{i-1} + L$,

(iii)* $a_i \geq a_{i-1} + L$,

for all $i \in \{2, \ldots, N\}$*; and*

(c)* $a_i \geq a^*$, *(capacity constraint)*

for all $i \in \{n, \ldots, N\}$.

Proof. Of course, (C0) and (C1) are equivalent by definition. Note that equivalence of (C1) and (C2) is handled by Lemma 5.

We show "(C2) $\implies$ (C3)". (simple)

We show "(C3) $\implies$ (C2)". (induction)

We show "(C?) $\implies$ (C4)". (necessity)

We show "(C4) $\implies$ (C?)". (sufficiency) □

Special treatment of first vehicle.

3.1 Decomposition

Let the optimal solution of the single vehicle problem be denoted by

$$x^*(\alpha, \beta, \bar{x}) := \operatorname*{arg\,max}_{x \in D[\alpha,\beta]} \int_\alpha^\beta x(t) \quad \text{such that } x \leq \bar{x} \tag{32}$$

and let $F_1(\alpha, \beta, \bar{x})$ denote the corresponding objective value. We now restate the lane planning problem here for easy reference. Assuming the system parameters $(\omega, \bar{\omega}, A, B, L)$ to be fixed, given some vector of arrival times $a = (a_1, \dots, a_N)$ and departure times $b = (b_1, \dots, b_N)$, the goal is to maximize

$$\begin{aligned}
F(a, b) := \max \quad & \sum_{i=1}^{N} \int_{a_i}^{b_i} x_i(t)\, \mathrm{d}t, && \text{(33a)}\\
\text{s.t.} \quad & x_i \in D[a_i, b_i] \quad \text{for each } i \in \{1, \dots, N\}, && \text{(33b)}\\
& x_i \leq x_{i-1} - L \quad \text{for each } i \in \{2, \dots, N\}. && \text{(33c)}
\end{aligned}$$

Now consider the safe following constraints (33c). We show how to model these by defining a boundary $\bar{x}_i \in \bar{D}[\bar{a}_i, \bar{b}_i]$ for each $i \in \{2, \dots, N\}$, such that the single vehicle problem can be applied. It becomes clear from Figure 7 that inequality (33c) only applies on some subinterval $I_i \subset [a_{i-1}, b_{i-1}]$. More specifically, by defining the inverse function

$$x_i^{-1}(p) := \inf\{t : x_i(t) = p\}, \tag{34}$$

it is easily seen that $I_i = [x_{i-1}^{-1}(A + L), b_{i-1}]$. However, since $\dot{x}_{i-1}(b_{i-1}) = 1$, we can extend the boundary until $b_{i-1} + L$ without any problems. More precisely, we define $\bar{a}_i := x_{i-1}^{-1}(A + L)$ and $\bar{b}_i := b_{i-1} + L$ such that the boundary $\bar{x}_i \in \bar{D}[\bar{a}_i, \bar{b}_i]$ is given by

$$\bar{x}_i := \begin{cases} x_{i-1}(t) - L & \text{for } t \in [\bar{a}_i, b_{i-1}], \\ t - b_{i-1} + B - L & \text{for } t \in [b_{i-1}, \bar{b}_i]. \end{cases} \tag{35}$$

Now, it is clear that the optimal trajectories x_i are recursively given by

$$\begin{aligned}
x_1 &= x^*(a_1, b_1, \varnothing), && \text{(36a)}\\
x_i &= x^*(a_i, b_i, \bar{x}_i), \quad \text{for } i \geq 2, && \text{(36b)}
\end{aligned}$$

where we use the notation $\bar{x}_1 = \varnothing$ to denote that the single vehicle problem for the first vehicle $i = 1$ does not have an active boundary constraint.[4] The corresponding total objective for the lane planning problem is simply given by

$$F(a, b) = F_1(a_1, b_1, \varnothing) + \sum_{i=2}^{N} F_1(a_i, b_i, \bar{x}_i). \tag{37}$$

[4] Alternatively, we could think about this as having some $\bar{x} \in \bar{D}[\bar{a}, \bar{b}]$ with very small $\bar{a} \ll a$ and $\bar{b} \ll b$.

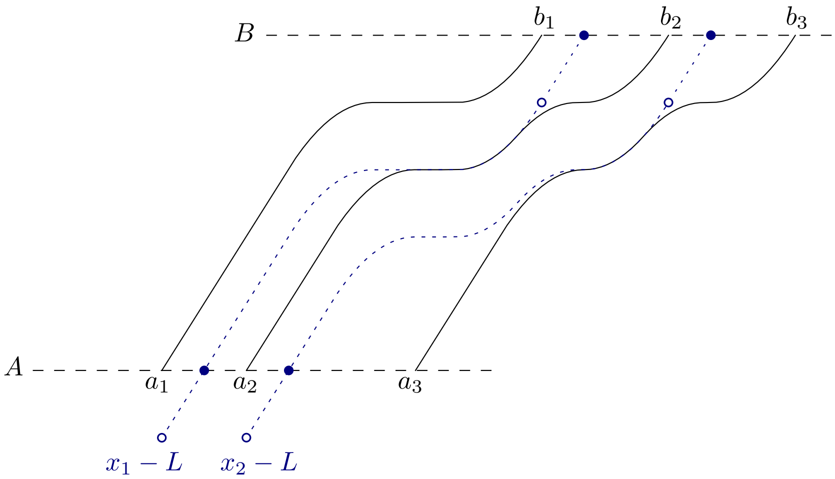

Figure 7: Optimal trajectories x_i for three vehicles. The dotted blue trajectories between the little open circles illustrates the safe following constraints (33c). The dotted blue trajectories between the solid dots are the following boundaries $\bar{x}_2 \in \bar{D}[\bar{a}_2, \bar{b}_2]$ and $\bar{x}_3 \in \bar{D}[\bar{a}_3, \bar{b}_3]$.

3.2 Feasibility as system of linear inequalities

We showed above how to decompose the lane planning problem into individual single vehicle problems by a simple construction of the boundary $\bar{x}_i$, based on the trajectory x_{i-1} of the preceding vehicle. Recall Lemma 2, which states the necessary conditions for feasibility of the single vehicle problem. We showed in Section 2.2 that these conditions are also sufficient. We will restate them here for easy reference, applied to each single vehicle problem belonging to the lane planning problem. By the construction at the beginning of this section, it follows that the lane planning problem is feasible if and only if each vehicle $i \in \{1, \dots, N\}$ satisfies the conditions:

(i) $b_i - a_i \geq B - A$, (full speed constraint)

(ii) $\bar{b}_i \leq b_i$, (exit order constraint)

(iii) $\bar{a}_i \leq a_i$, (entry order constraint)

(iv) $\bar{x}_i \geq \check{x}_i$. (entry space constraint)

Note that conditions (ii), (iii) and (iv) are of course void for the first vehicle $i = 1$, because $\bar{x}_1 = \varnothing$. The goal of this section is to turn these conditions into a system of linear inequalities in terms of $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$. Observe that condition (i) is already of the desired type. Furthermore, condition (ii) is also easy to rewrite in this form, because we defined $\bar{b}_i = b_{i-1} + L$. For the remaining two conditions, some additional analysis is required.

Earliest arrival. We show that the entry order constraint $\bar{a}_i \leq a_i$ and the entry space constraint $\bar{x}_i \geq \check{x}_i$ together are equivalent to the two constraints

$$a_i \geq a_{i-1} + L, \tag{38a}$$
$$a_i \geq \check{a}_i(a, b), \tag{38b}$$

where $\check{a}_i(a, b)$ denotes some expression in terms of schedule times. Consequently, $\max\{a_{i-1} + L, \check{a}_i(a, b)\}$ can be interpreted as the earliest possible arrival time for vehicle i. Recall that $\bar{a}_i = x_{i-1}^{-1}(A + L)$.

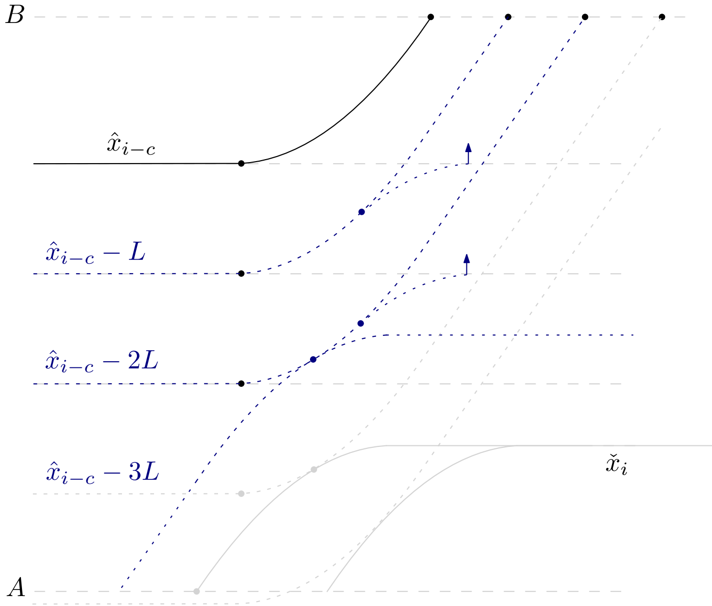

Figure 8: Earliest arrival due to entry order constraint and entry space constraint.

Recall the definition of $\hat{x}$ in equation (9). By carefully handling the $\max\{\cdot\}$, we can expand this expression as

$$\hat{x}(t) = \begin{cases} B - b + t + \bar{\omega}(b-t)^2/2 & \text{for } t \geq b - 1/\bar{\omega}, \\ B - 1/(2\bar{\omega}) & \text{for } t \leq b - 1/\bar{\omega}. \end{cases} \tag{39}$$

Waiting capacity is given by

$$C = \left\lfloor \frac{B - A - 1/(2\bar{\omega}) - 1/(2\omega)}{L} \right\rfloor + 1, \tag{40}$$

where the brackets indicate the floor function.

Full system of inequalities. In conclusion, feasibility of the lane planning problem is expressed through the system of linear inequalities

$$\begin{aligned} b_i - a_i &\geq B - A && \text{for all } i \in \{1, \dots, N\}, && \text{(41a)} \\ b_i &\geq b_{i-1} + L && \text{for all } i \in \{2, \dots, N\}, && \text{(41b)} \\ a_i &\geq a_{i-1} + L && \text{for all } i \in \{2, \dots, N\}, && \text{(41c)} \\ a_i &\leq \check{a}_i(a, b) && \text{for all } i \in \{2, \dots, N\}. && \text{(41d)} \end{aligned}$$

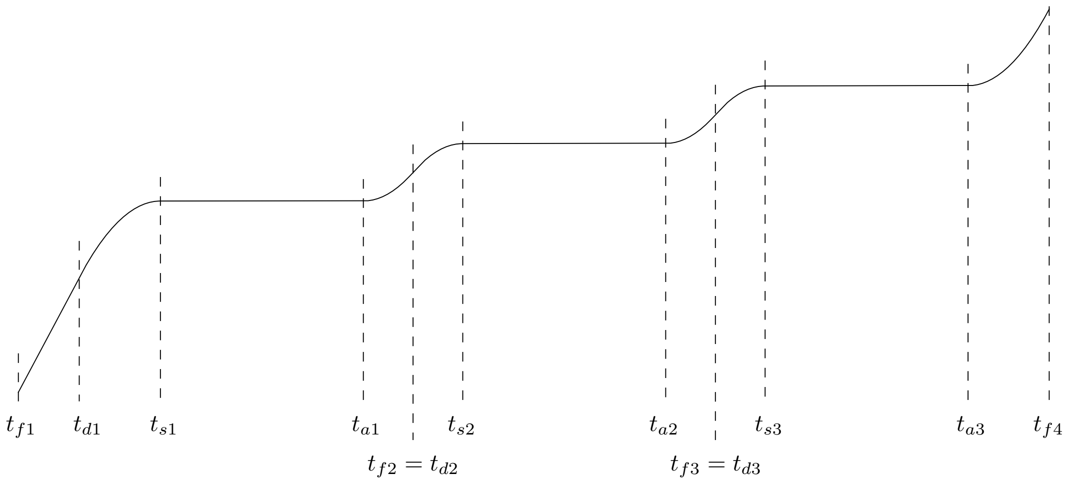

Figure 9: Example of an alternating vehicle trajectory with its defining time intervals. The particular shape of this trajectory is due to two leading vehicles, which causes the two start-stop *bumps* around the times where these leading vehicles depart from the lane.

4 Optimal solution for haste objective

Due to the recursive nature of the problem, we will see that optimal trajectories possess a particularly simple structure, which enables a simple computation.

Definition 4. Let a trajectory $\gamma \in \mathcal{D}[a,b]$ be called *alternating* if for all $t \in [a,b]$, we have

$$\ddot{\gamma}(t) \in \{-\omega, 0, \bar{\omega}\} \quad \text{and} \quad \ddot{\gamma}(t) = 0 \implies \dot{\gamma}(t) \in \{0, 1\}. \tag{42}$$

We now argue that each vehicle's optimal trajectory x_i is alternating. First, consider $x_1 = x^*(a_1, b_1, \varnothing)$, which is constructed by joining $x^1[x_1]$ and $\hat{x}[x_1]$ together by smoothing. Observe that both boundaries are alternating by definition. Let $\gamma_1(t) = \min\{x^1[x_1](t), \hat{x}[x_1](t)\}$ be the minimum boundary, then it is clear that the smoothened $x_1 = \gamma_1^*$ must also be alternating, because we only added a part of deceleration at some interval $[\xi, \tau]$, which clearly satisfies $\ddot{\gamma}_1^*(t) = -\omega$ for $t \in [\xi, \tau]$. Assume that x_{i-1} is alternating, we can similarly argue that x_i is alternating. Again, let $\gamma_i(t) = \min\{\bar{x}[x_{i-1}], \hat{x}[x_i](t), x^1[x_i](t)\}$ be the minimum boundary. After adding the required decelerations for smoothing, it is clear that $x_i = \gamma_i^*$ must also be alternating.

Observe that an alternating trajectory $\gamma \in \mathcal{D}[a,b]$ can be described as a sequence of four types of consecutive repeating phases, see Figure 9 for an example. In general, there exists a partition of $[a,b]$, denoted by

$$a = t_{f1} \leq t_{d1} \leq t_{s1} \leq t_{a1} \leq t_{f2} \leq t_{d2} \leq t_{s2} \leq t_{a2} \leq \cdots \leq t_{f,n+1} = b,$$

such that we have the consecutive *alternating intervals*

$$\begin{aligned} F_i &:= [t_{f,i}, t_{d,i}] \quad &&\text{(full speed)}, \quad & S_i &:= [t_{s,i}, t_{a,i}] \quad &&\text{(stopped)}, \\ D_i &:= [t_{d,i}, t_{s,i}] \quad &&\text{(deceleration)}, \quad & A_i &:= [t_{a,i}, t_{f,i+1}] \quad &&\text{(acceleration)}, \end{aligned}$$

such that on these intervals, γ satisfies

$$\begin{aligned} \dot{\gamma}(t) &= 1 \quad &&\text{for } t \in F_i, \quad & \dot{\gamma}(t) &= 0 \quad &&\text{for } t \in S_i, \\ \ddot{\gamma}(t) &= -\omega \quad &&\text{for } t \in D_i, \quad & \ddot{\gamma}(t) &= \bar{\omega} \quad &&\text{for } t \in A_i. \end{aligned}$$

Partial trajectories. Next, we will define parameterized functions x^f, x^d, x^s, x^a to describe alternating trajectory γ on each of these alternating intervals. Given some initial position $p \in [A, B]$, velocity $v \in [0, 1]$, start and end times a and b such that $a \leq b$ and $v + \bar{\omega}(b-a) \leq 1$, we define the acceleration trajectory $x^a[p, v, a, b] : [a, b] \to \mathbb{R}$ by setting

$$x^a[p, v, a, b](\tau) := p + v(\tau - a) + \bar{\omega}(\tau - a)^2/2, \tag{43a}$$

$$\dot{x}^a[p, v, a, b](\tau) := v + \bar{\omega}(\tau - a). \tag{43b}$$

Similarly, for p, v, a, b satisfying $a \leq b$ and $v - \omega(b - a) \geq 0$, let the deceleration trajectory $x^d[p, v, a, b] : [a, b] \to \mathbb{R}$ be defined as

$$x^d[p, v, a, b](\tau) := p + v(\tau - a) - \omega(\tau - a)^2/2, \tag{44a}$$

$$\dot{x}^d[p, v, a, b](\tau) := v - \omega(\tau - a). \tag{44b}$$

One may notice that x^d is essentially the same as the deceleration boundary x^-, which we defined in Section 2.3. However, note that the condition $v - \omega(b - a) \geq 0$ restricts the domain such that we do not need the clipping operation. Furthermore, the parameterization of x^d will be more convenient in the next section.

We use the following notation for trajectories with constant minimum or maximum speed. We write $x^s[p, a, b](\tau) \equiv p$, with domain $[a, b]$, to model a stopped vehicle and let $x^f[p, a, b](\tau) = (p + \tau - a, 1)$ model a vehicle that drives at full speed, also with domain $[a, b]$.

4.1 Connecting partial trajectories

It can be shown that the smoothing procedure introduces a part of deceleration only between the four pairs of partial trajectories

$$x^a \to x^a, \qquad x^a \to x^s, \qquad x^f \to x^a, \qquad x^f \to x^s.$$

We will use these results to characterize optimal trajectories for our optimal control problem.

Lemma 7 ($x^f \to x^s$)**.** *Let $x^f[p, a, b]$ and $x^s[q, c, d]$ be two trajectories. Considering τ_1 and τ_2 as variables in the equation*

$$x^d[x^f[p, a, b](\tau_1), \tau_1, \tau_2](\tau_2) = x^s[q, c, d](\tau_2),$$

it has solution $\tau_2 = q - p + a + 1/2\omega$ and $\tau_1 = \tau_2 - 1/\omega$, whenever $\tau_1 \in [a, b]$ and $\tau_2 \in [c, d]$.

Proof. The expanded system of state equations is given by

$$\begin{cases} p + \tau_1 - a + (\tau_2 - \tau_1) - \omega(\tau_2 - \tau_1)^2/2 = q, \\ 1 - \omega(\tau_2 - \tau_1) = 0. \end{cases}$$

The second equation yields $\tau_2 - \tau_1 = 1/\omega$, which after substituting back in the first equation yields $p - a + \tau_2 - 1/2\omega - q = 0$, from which the stated solution follows. □

To keep the expressions for the case of joining $x^f \to x^a$ a little bit simpler, we first consider a full line joining to a acceleration trajectory of full length $1/\omega$.

Lemma 8. *Consider some full acceleration trajectory $x^a[(p, 0), a, a + 1/\omega]$ and the line through $(\lambda, 0)$ with slope 1. Whenever λ, which can be interpreted as a time epoch, satisfies $\lambda \in [a - p - 1/2\omega, a - p + 1/2\omega]$, then the equation*

$$x^+[(p, 0), a, a + 1/\omega](\tau) = x^d[(q, 1), q + \lambda, q + \lambda + 1/\omega](\tau),$$

with τ and q considered as variables, has a unique solution

$$\tau = a + 1/\omega - \sqrt{\frac{a - p + 1/2\omega - \lambda}{\omega}},$$

$$q = 2\tau - a - 1/\omega - \lambda,$$

so the joining deceleration is given by $x^d[(q, 1), q + \lambda, \tau]$

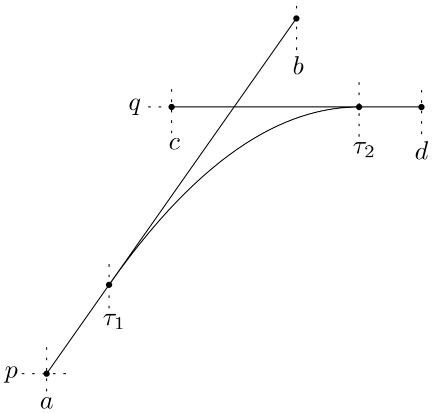

Figure 10: $x^f \to x^s$

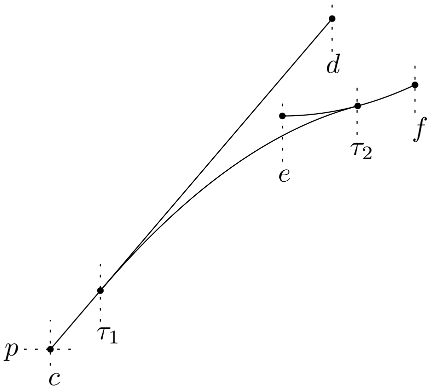

Figure 11: $x^f \to x^a$

Proof. First of all, the expanded system of state equations is given by

$$\begin{cases} p + \omega(\tau - a)^2/2 = q + (\tau - q - \lambda) - \omega(\tau - q - \lambda)^2/2, \\ \omega(\tau - a) = 1 - \omega(\tau - q - \lambda). \end{cases}$$

We use the second equation to express q in terms of τ, which yields

$$q = 2\tau - 1/\omega - a - \lambda,$$

which we substitute in the first equation to derive the equation

$$\omega\tau^2 - 2(1 + \omega a)\tau + \omega a^2 + a + p + \lambda + 1/2\omega = 0.$$

This is a quadratic equation in τ, with solutions

$$\tau = a + 1/\omega \pm \sqrt{\frac{a - p + 1/2\omega - \lambda}{\omega}},$$

of which only the smallest one is valid, because $\tau \leq a + 1/\omega$. Furthermore, we see that τ is defined as a real number when

$$a - p + 1/2\omega - \lambda \geq 0 \iff \lambda \leq a - p + 1/2\omega.$$

The other requirement is that $\tau \geq a$, which is equivalent to

$$1/\omega \geq \sqrt{\frac{a - p + 1/2\omega - \lambda}{\omega}} \iff \lambda \geq a - p - 1/2\omega.$$

□

Lemma 9 ($x^f \to x^a$)**.** *Consider partial trajectories $x^f[p, c, d]$ and $x^a[x, e, f]$.*

Proof. First of all, observe that $x^f[p, c, d]$ lies on the line with slope 1 through $(\lambda, 0) := (c-p, 0)$ and $x^a[x, e, f]$ lies on the full acceleration curve $x^a[(x_1 - x_2^2/(2\omega), 0), e - x_2/\omega, e - x_2/\omega + 1/\omega]$, see Figure 11. Now apply Lemma 8 to $p = x_1 - x_2^2/(2\omega)$, $a = e - x_2/\omega$ and $\lambda = c - p$ yields some solutions τ and q. Let $\tau_2 := \tau$ and let τ_1 denote the time where the line and x^a join, given by $\tau_1 = \lambda + q$. Now we simply check whether this solution is also feasible for the smaller trajectories. We must have $\tau_1 \in [c, d]$ and $\tau_2 \in [e, f]$. □

Lemma 10. *Consider the acceleration trajectory $x^a[(p, 0), a, b]$ and the horizontal line through $(0, q)$. Let $\tau_1 = a + \sqrt{(q - p)/\omega}$ and $\tau_2 = a + 2\sqrt{(q - p)/\omega}$. If τ_1 satisfies $\tau_1 \in [a, b]$, then both trajectories are joined by deceleration trajectory $x^d[x^a[(p, 0), a, b](\tau_1), \tau_1, \tau_2]$*

Proof. Consider the following equation

$$x^d[x^a[(p,0),a,b](\tau_1),\tau_1,\tau_2](\tau_2) = (q,0).$$

The expanded system of state equations is given by

$$\begin{cases} p + \omega(\tau_1 - a)^2/2 + (\omega(\tau_1 - a))(\tau_2 - \tau_1) - \omega(\tau_2 - \tau_1)^2/2 = q, \\ \omega(\tau_1 - a) - \omega(\tau_2 - \tau_1) = 0. \end{cases}$$

From the second equation, we derive $\tau_1 - a = \tau_2 - \tau_1$. Plugging this back in the first equation yields the quadratic equation $p + \omega(\tau_1 - a)^2 = q$ with solutions $\tau_1 = a \pm \sqrt{(q-p)/\omega}$, of which only the larger one is valid. Finally, the second equation gives $\tau_2 = 2\tau_1 - a$. ☐

Lemma 11 ($x^a \to x^s$)**.** *Consider partial trajectories* $x^a[x,c,d]$ *and* $x^s[q,e,f]$*.*

Proof. Observe that $x^a[x,c,d]$ lies on the full acceleration curve $x^a[(x_1 - x_2^2/(2\omega), 0), c - x_2/\omega, c - x_2/\omega + 1/\omega]$. Hence, we can apply Lemma 10 with $p = x_1 - x_2^2/(2\omega)$, $a = c - x_2/\omega$, which yields some solutions τ_1 and τ_2, which are feasible solutions if $\tau_1 \in [c,d]$ and $\tau_2 \in [e,f]$. ☐

Lemma 12. *Consider full acceleration trajectories* $x^a[(p,0),a,b]$ *and* $x^a[(q,0),c,d]$*.*

Proof. Consider the equation

$$x^d[x^a[(p,0),a,b](\tau_1),\tau_1,\tau_2](\tau_2) = x^a[(q,0),c,d](\tau_2),$$

expanded to the system of equations

$$\begin{cases} p + \omega(\tau_1 - a)^2/2 + \omega(\tau_1 - a)(\tau_2 - \tau_1) - \omega(\tau_2 - \tau_1)^2/2 = q + \omega(\tau_2 - c)^2/2, \\ \omega(\tau_1 - a) + \omega(\tau_2 - \tau_1) = \omega(\tau_2 - c). \end{cases}$$

☐

Lemma 13 ($x^a \to x^a$)**.** *Consider partial trajectories* $x^a[x,a,b]$ *and* $x^a[y,c,d]$*.*

Proof. ☐

4.2 Algorithm

Put everything together into pseudocode.

Algorithm 1 Computing connecting deceleration for alternating trajectories.

Let i such that I_i is the latest such that $t_1 < I_i$.
Let j such that I_j is the earliest such that $t_1 > I_j$.

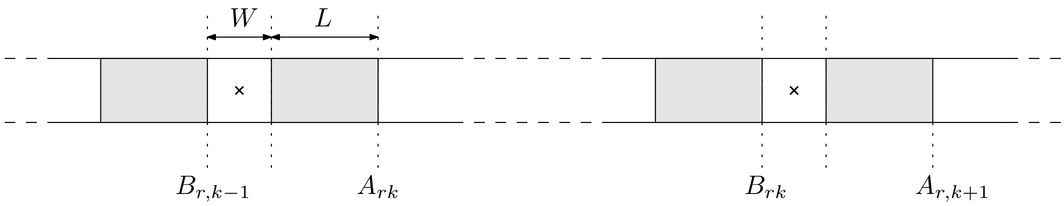

Figure 12: Illustration of how the individual lane models are connected to form a route with intersections, marked with a little cross. The four shaded rectangles illustrate four possible vehicle positions. The length of the intersection is W. The longitudinal positions A_{rk} and B_{rk} denote the start and end, respectively, of the kth lane on route r.

5 Network planning problem

Due to the second-order boundary conditions $\dot{x}_i(a_i) = \dot{x}_i(b_i) = 1$, the feasibility of the lane planning problem is completely characterized in terms of its schedule times. We will show that it is precisely this property that us to construct a network consisting of individual lanes, connected at intersections. The fact that intersections are shared among multiple lanes naturally gives rise to some collision-avoidance constraints. We will see that the feasibility of finding collision-free trajectories can be stated completely in terms of schedule times, which essentially means that we do not need to worry about vehicle dynamics at all.

Network topology. We will use the lane model to build a simple network model. The network model is based on a directed graph $(\bar{V}, E)$ with nodes $\bar{V}$ and arcs E, which we will use to encode the possible routes. Nodes with no incoming arcs are *entrypoints* and nodes with no outgoing arcs are *exitpoints*. We use V to denote the set of *intersections*, which are nodes with in-degree at least two. Let $\mathcal{R}$ denote the index set for routes, then each $r \in \mathcal{R}$ corresponds to the route

$$\bar{V}_r = (v_r(0), v_r(1), \dots, v_r(m_r), v_r(m_r + 1)),$$

where we require $v_r(0)$ to be an entrypoint and $v_r(m_r + 1)$ to be an exitpoint. Furthermore, we use $V_r = \bar{V}_r \setminus \{v_r(0),\, v_r(m_r + 1)\}$ to denote the intersections on this route. Let $E_r \subset E$ denote the set of edges that make up V_r. We require that routes are *edge-disjoint*, which is made precise in the following assumption.

Assumption 4. *For every distinct routes $p, q \in \mathcal{R}$ such that $p \neq q$, we assume $E_p \neq E_q$.*

This assumption ensures that each route $\bar{V}_r$ can be modeled by connecting a sequence of lanes together, with some *intersection areas* of some fixed size W in between them, see Figure 12. Hence, we set the longitudinal start and end position of each lane model as follows. Let $d(v, w)$ denote the length of edge $(v, w) \in E_r$, then we recursively define

$$\begin{aligned}
A_{r1} &= 0, && \text{(45a)}\\
A_{rk} &= B_{r,k-1} + W + L, && \text{(45b)}\\
B_{rk} &= A_{rk} + d(v_r(k-1), v_r(k)), && \text{(45c)}
\end{aligned}$$

for each $k \in \{1, \dots, m_r + 1\}$.

Network scheduling. Introduce the global trajectory planning problem by defining the collision-avoidance constraints. Mention that this problem can be solved at once, by using direct transcription. Show that the bilevel formulation decomposes into a (combinatorial) scheduling problem.

Assumption 4 ensures that the order of vehicles on each lane is completely determined by the order of vehicles on the corresponding lane.

Instead of schedule times a_i and b_i, we are now going to use crossing times y_i.
Occupancy time slot scheduling

6 Discussion

There are different ways of relaxing the second-order constraints that could be studied. The interesting question is whether feasibility can still be easily characterized. Instead of fixing the speed to be maximal at entry and exit, we could require, for example, that the speed is bounded from below, i.e.,

$$\eta \leq \dot{x}(a) \leq 1 \quad \text{and} \quad \eta \leq \dot{x}(b) \leq 1, \tag{46}$$

for some $\eta > 0$.

A Miscellaneous

Lemma A.1. *Let $f : X \times Y \to \mathbb{R}$ be some continuous function. If Y is compact, then the function $g : X \to \mathbb{R}$, defined as $g(x) = \inf\{f(x, y) : y \in Y\}$, is also continuous.*

Lemma A.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $y \in \mathbb{R}^m$, then the level set $N := f^{-1}(\{y\})$ is a closed subset of $\mathbb{R}^n$.*

Proof. For any $y' \neq y$, there exists an open neighborhood $M(y')$ such that $y \notin M(y')$. The preimage $f^{-1}(M(y'))$ is open by continuity. Therefore, the complement $N^c = \{x : f(x) \neq y\} = \cup_{y' \neq y} f^{-1}(\{y'\}) = \cup_{y' \neq y} f^{-1}(M(y'))$ is open. □