On the analytical and numerical solution for the Nonlinear Schrödinger equation

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Thursday 22nd February, 2024

1 Introduction

In recent years, one of the most significant challenges that society has faced is the improvement of telecommunications. This progress has become crucial for sharing knowledge, connecting with people, and establishing new alliances. In this endeavor, the quest for a tool to transmit data over long distances with minimal loss has taken precedence. An indispensable component in this evolution has been, and continues to be, fiber optics. This field deals with the transmission of information through light pulses that travel in a extensive fiber (sometimes across intercontinental cables), where the core is typically composed of plastic or glass. This medium allows the reception of data without inflicting substantial harm to the input signal [1].

The main challenge of the use of optical fibers was to control the loss of information along the fiber due to the absorption of impurities that cause the light to attenuate. A breakthrough in this matter was given in 1973 with the discovery that the propagation of such light pulses was governed by the nonlinear Schrödinger equation [1, 9]. The NLS equation has solutions with highly unique attributes known as solitons. A soliton constitutes a wave that travels through nonlinear media at a constant speed [1, 9, 13]. These special properties allowed the development of mechanisms for controlling the attenuation of the light pulse through the fiber.

Although the NLS model has increased fiber understanding and enhancement, much remains unknown. The fundamental reason for this is that the NLS model has few explicit solutions, which has led to the use of numerical methods to study the signal dynamics within the fiber. In this regard, there are primarily two types of numerical methods that have been applied to the Schrödinger equation in the literature: finite difference schemes [8, 3, 17] and spectral methods [8, 6, 18, 19]. The first class of numerical methods involves approximating the derivatives of the equation with finite differences so that the model is turned into a system of equations that can be solved using matrix algebra techniques. On the other hand, the basic idea underlying spectral techniques is to describe the equation solution as a sum of specified basis functions, and then choose the coefficients in the sum to satisfy the model as much as feasible.

In this study, we first present the NLS equation, and its properties, as well as some of the explicit solutios for it. Then on the second part que present the mathematical formulation, properties and requeriments of each method. Then in the third part we present the results of the methods using tables to present the errors thought the mesh, and the visual behaviour of the simulations. Finally we present some conclusions in the performance and comparison of the methods as well as a benchmark presenting the pros and cons of each method/

2 Fiber optics and the NLS

Without a doubt, technological advancements in the sector of telecommunications have made significant progress in recent decades. As a result of all of this effort, today's society is capable of transmitting massive amounts of information from one location to another. In broad terms, every advancement in telecomunications can be traced back to three main factors: the utilization of optical fibers, the mathematical model known as the nonlinear Schrödinger equation (NLS), and solitons.

In 1970, a new type of optical fiber debuted that could transport light signals over considerably larger distances than prior fibers. Glass was used to make the first optical fibers. When light passes between two media with differing refractive indices, there is a critical angle of incidence below which light is totally reflected back to the first medium without passing across the interface. The development of a coating considerably enhanced the design of optical fibers. The sheathed fiber was made of a glass core surrounding by a substance with a slightly lower refractive index than the core [1]. Consequently, light will be unable to escape from the interior of the fiber until it reaches the end (signal/information was transmitted.

Solitons played an important role in ensuring that light pulses flow through the fiber without significant signal distortion due to their exceptional properties: a soliton wave is a traveling wave that moves with constant velocity at all times without changing its profile. These type of waves were discovered in 1834 in a canal in Scotland, John Scott Russell, a Scottish civil engineer, and naval architect, observed a remarkable occurrence while observing a horse-drawn barge traversing the Union Canal near Edinburgh. As the barge came to an abrupt halt, a distinct wave pattern materialized and proceeded to travel along the canal: the soliton [13]. There are two kinds of solitons that are important in optics: bright solitons and dark solitons. Bright solitons are characterized by a localized intensity peak on a homogeneous background, while dark solitons can be described by a localized intensity hole on a continuous wave background [10]. However, the usefulness of these waves in telecommunications begins when researchers realize that they can use these incredible waves to send information (by the use of the NLS equation). Currently, the range of applicability of solitons extends beyond optics to fields such as Bose-Einstein condensate [2], water waves [5], and economics [11], to name a few.

The finding that the dynamics of the light pulse through the fiber may be reproduced by the nonlinear Schrödinger equation (NLS) is the beginning point for the theoretical development of this topic. NLS equation is a partial differential equation of the form

$$i\frac{\partial A}{\partial z} + \frac{i\alpha}{2}A - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \gamma|A|^2 A = 0 \tag{1}$$

where A is the pulse envelope, α is the attenuation coefficient [1], β_2 represents dispersion of the group velocity, responsible for the pulse broadening and the dispersion regimen (normal or anomalous) in the pulse evolution, γ is the nonlinear coefficient. The parameter β_2 also determinate the . As previously stated, the NLS model admits bright soliton and dark soliton type solutions, the mathematical formulas for this solitons are

$$u_B(x,t) = \frac{a}{\sqrt{-\beta}} e^{i(vx - \frac{1}{2}(v^2 - a^2)t + \theta_0)} \operatorname{sech}(a(x - vt - x_0)) \quad \text{(bright soliton)} \tag{2}$$

$$u_D(x,t) = \left[\cos\phi \tanh\left(\sqrt{u_0\beta}\cos\phi(x - vt - x_0)\right) + i\sin\phi\right]\sqrt{u_0} e^{i(kx - \omega t + \theta_0)} \quad \text{(dark soliton)} \tag{3}$$

where $u_0 > 0$ is a constant, $\frac{a}{\sqrt{-\beta}}$ is the amplitude, v is the velocity of the soliton, and x_0 and θ_0 are the initial conditions for shift and phase of the soliton at $t = 0$. In these terms, the investigation of the NLS equation provided the initial theoretical foundations for comprehending the propagation of signal through nonlinear media, such as fiber. In fact, within the framework of the NLS model, it is conceivable to comprehend that soliton waves arise from a balance between two opposing phenomena, namely dispersion and non-linearity.

The Schrödinger equation enables us to progress further in the description of the more intricate dynamics of light pulses. However, due to the complexity of this equation, we know that analytical solutions are sometimes extremely difficult or even impossible to find. For this reason, numerical methods are a powerful tool to understand and reproduce the behavior of complex systems. In consequence, the main purpose of this research is to do a detailed numerical analysis of the Schrodinger equation (1) using the Hopscotch and Crank-Nicolson finite difference methods, as well as the Split-step Fourier method and the Fourier pseudospectral method. These approaches are employed separately in the literature, but our study intends to evaluate the merits and drawbacks of each method, allowing future research to have selection criteria based on the problem to be addressed.

3 Numerical methods

The NLS equation (1) can be analyzed using the renormalized version of it

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0 \tag{4}$$

where x represents the space variable and t is the time. In this equation the amplitude is normalized to peak amplitude, distance is normalized to the dispersion length and time is normalized to the pulse duration, $\alpha = 0$ (no attenuation) and $sng(\beta_2) = -1$ (anomalous dispersion). We also assume that the solutions of the equation outside the interval $[-L, L]$ are negligibly small, so they are not considered.

3.1 Split step fourier method

This numerical method is essential in the labor of understanding the NLS, specially the nonlinearities and dispersion effects in the fiber because both are presented separated. It is usually presented as a really fast approach, specially when its compared to the finite difference methods. Let us start with the NLS (4), indicating a focusing equation associated with the initial condition $u(x,0) = u_0(x)$, as shown below

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} - |u|^2 u \tag{5}$$

Then we define the time independent linear operator as $L = -\frac{\partial^2 u(x,t)}{\partial x^2}$, and the nonlinear operator as $N = -|u(x,t)|^2$. Then we present the nonlinear step $i\frac{\partial u}{\partial t} = Nu$, where $N = |u(x,t+\Delta t)|^2 \approx |u(x,t)|^2$, therefore the analytical solution will be given by

$$u(x,t+\Delta t) = \exp(i\Delta t N)u(x,t) = \exp(i\Delta t|u(x,t)|^2 u(x,t)) \tag{6}$$

In the second place we define the linear step $i\frac{\partial u}{\partial t} = Lu$. In where we apply the Fourier transform in both sides collapse the PDE into a ODE, which can be solved as

$$\frac{\partial \hat{u}}{\partial t} = -i\omega^2 \hat{u}$$
$$\hat{u}(\omega, t+\Delta t) = \exp(-i\omega^2 \Delta t)u(\hat{\omega}, t)$$

finally we can write the entire equation as follows

$$u(x,t+\Delta t) = F^{-1}(\exp(-i\omega\Delta t)F(\exp(i\Delta t|u(x,t)|^2)u(x,t))) \tag{7}$$

3.2 Fourier pseudo spectral method

The Fourier pseudo spectral method only applies the Fourier transform to the space component, along with the finite difference discretization for the time component. This approach is only applicable for the periodic functions over an interval $x\ in[-P,P]$. Firstly we replace the the temporal first derivative with the following difference scheme

$$\frac{\partial u}{\partial t} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \tag{8}$$

Secondly, we substitute the above relation into the equation (5) to get the following expression

$$u(x,t+\Delta t) = u(x,t) - i\Delta t F^{-1}\left(\omega^2 \frac{\pi^2}{P^2} F(u)\right) + i\Delta t|u(x,t)|^2 u(x,t) \tag{9}$$

The previous equation provides a solution, which is only stable for values of $\Delta t/(\Delta x)^2 < 1/\pi^2$ [8]. For that we need to adjust the equation (9) by using the Fornberg-Whitham principles [17], this leads to an unconditionally stable solution, given by

$$u(x,t+\Delta t) = u(x,t) - i\Delta t F^{-1}\left(\sin\left(\omega^2 \frac{\pi^2}{P^2}\Delta t\right) F(u(x,t))\right) + i\Delta t|u(x,t)|^2 u(x,t) \tag{10}$$

3.3 Crank Nicholson Method

This method is a implicit finite difference scheme that is commonly used to solve the heat equation. It is unconditionally stable with a truncation error of order $O(h^2)+O(\tau^2)$. In this specific scenario, some unstable behaviour may appear due the dominance of the nonlinear term in the dynamics [8]. The CNM can be expressed as follows

$$i\frac{u_n^{m+1} - u_n^m}{\Delta t} = \frac{1}{2(\Delta x)^2}\left[u_{n+1}^m - 2u_n^m + u_{n-1}^m + u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}\right] + (|u|_n^{m+1})^2 u_n^{m+1} + (|u|_n^m)^2 u_n^m \tag{11}$$

The implicit nature of the difference method implies a system of (n-1) equations in the (n-1) unknowns, which we solve using MATLAB tools.

Each method's analysis includes the following elements: error analysis by the comparison of the numerical approximation with the analytical solution, stability study, analysis of the numerical precision and computational cost. Then our objective is to perform a detailed numerical study of the one-dimensional nonlinear Schrödinger equation in order to determine the most effective high-performance numerical techniques for manipulating the NLS model.

We will undertake an exhaustive numerical investigation of the one-dimensional NLS problem, employing both the Hopscotch and Crank-Nicolson finite difference schemes, alongside the Split-step Fourier and Fourier pseudo spectral methods. We will conduct comprehensive comparisons between these proposed approaches and the exact solutions, aiming to discern the merits and demerits of each numerical technique. In this context, our primary objective is to identify high-performance numerical strategies for effectively manipulating the NLS equation. This pursuit aims to establish a robust framework for real-life applications, particularly in fields like fiber optics and water wave dynamics.

4 Justification

The significance of the nonlinear Schrödinger equation (NLS) extends beyond its application in the realm of fiber optics. Indeed, this equation appears as a fundamental model in a variety of physics fields, including fluid dynamics [5], Bose-Einstein condensate [2] and quantum dots [12]. Particularly noteworthy is the fact that recently the NLS equation has been proposed as a model of dissipative turbulence in fluid dynamics [14]. Turbulence is a well-known unsolved classical physics problem. Although the Navier-Stokes equations (NS) naturally dominate the study of fluids, the NLS equation arises as an alternative model since it allows to partially replicate some turbulence properties that are difficult to explain using the NS equations. Therefore, the analytical and numerical study of the nonlinear Schrödinger equation is still very important today, as it promises significant advances in our understanding of the phenomenon of turbulence. As can be seen, many advancements in physics, to name just one branch of knowledge, require a better understanding of the NLS equation. This model is well known for being a genuinely non- linear PDE, which limits its analytical study in many scenarios, and it is therefore preferable to treat this problem using specialist numerical methods. The finite difference schemes and spectral methods are two of the most commonly utilized methods in the literature to approximate the NLS equation due to the practicality and high-accuracy. The goal of this work is to conduct a complete numerical analysis of the NLS model utilizing the Hopscotch and Crank-Nicolson finite difference methods, as well as the Split-step Fourier method and the Fourier pseudospectral method in the context of soliton dynamics.

The solitons are among the few explicit solutions to the NLS equation, and their importance stems from the fact that their properties are desired in many physical problems, such as the case of the optical fiber, as previously mentioned, and, more recently, the description of the Bose-Einstein condensate (known as the fifth state of matter). The majority of the physical applications framed here necessitate the appropriate selection of a numerical method that allows describing the properties of such a system as faithfully to reality as possible, avoiding the loss of information that occurs in many cases when numerical approximations are used. In this sense, numerical research projects like the one provided here help to generate solid criteria for the most proper choice for the discretization and implementation of numerical simulations for the NLS equation.

Finally, this study aims to serve as a starting point for future numerical research in more complex dynamics, such as the application of these numerical approaches to higher-dimensional issues and their applications to real-world physical problems outside of the contexts discussed here.

5 Scope

In this research, our primary objective is to approximate numerically the one-dimensional nonlinear Schrödinger equation appearing in the context of fiber optics, specifically focusing on the anomalous dispersion regime. For this numerical study, the Hopscotch, Crank-Nicolson, Split-step Fourier method, and the Fourier pseudospectral methods will be used. Particularly, the interest lies in determining the inherent properties of these methods such as: stability, precision, and computational performance.

Because the majority of real-world physical problems correspond to higher-dimensional models (2-3D), this type of research should be extended to more generic models (possibly with additional terms in the equation). Furthermore, the use of spectral methods limits the type of problem that may be addressed, because the essence of these methods requires periodic boundary conditions. In practice, these constraints are ideal for a significant number of problems, but they clearly leave out a huge number of them. Essentially, this study should be seen as a groundwork for future research into the NLS equation and its applications in different fields of knowledge.

6 State of the art

A fiber optic, or optical fiber, is a thin, flexible strand of glass or plastic that is used to transmit information using light signals. It is designed to carry digital or analog data, such as telephone conversations, internet traffic, or television signals, over long distances and at high speeds. The core of a fiber optic cable is surrounded by a cladding material with a lower refractive index, which allows the light signals to be internally reflected and travel down the length of the fiber with minimal loss of signal [1]. Fiber optics have a historical origin in uncladded glass fibers manufactured in the 1920s. However, it was not until the 1950s that a cladding layer was introduced, resulting in improved fiber characteristics. The study of fiber optics experienced significant growth during the 1960s, particularly due to image transmission through a bundle of glass fibers . Unfortunately, these early fibers suffered from considerable losses. This situation changed in 1970 when low-loss silica fibers emerged, revolutionizing optical fiber communications and giving rise to the field of nonlinear fiber optics [1].

The Nonlinear Schrödinger Equation (NLS) can be deduced from various theoretical frameworks [4].Within our study's context, the most suitable derivation stems from Maxwell's equations. Given that optical field propagation in fibers involves electromagnetic waves, it is inherently governed by Maxwell's equations. Consequently, the equation governing pulse propagation and, subsequently, the Nonlinear Schrödinger equation can be derived from this foundational framework [9],[1],[15]. The electromagnetic essence of the NLS engenders several phenomena that have been subject to extensive investigation over the years, encompassing birefringence, four-wave mixing, self-phase modulation, rogue waves, focusing phenomena, modulation stability, among others [14][5][9]. But as one can expect these investigations conclude that despite decades of research, the complex underlying nature of the different phenomena in optical fibers continues to attract significant attention, and in fact that this phenomena remain intricate and demand further exploration [5].

As mentioned earlier, solitons are solitary waves that arise from the cancellation of two destructive phenomena. These solitons maintain a constant envelope over time and retain their original shape when they collide with other solitons [13]. Among their various properties, the sinusoidal nature of solitons and their periodicity resulting from exponential decay as $|x| \to \infty$ stand out as the primary justification for considering them the sole focus of our study [8], [9]. Investigating the true nature of solitons and numerically solving the Nonlinear Schrödinger Equation (NLS) hold significant importance.

Numerous efforts have been directed toward understanding solitons and exploring different approaches to their analysis. Previous reasearch have work into finite difference solutions, including methods like the Crank-Nicolson method (CNM), the Hopscotch method (HSM), and the semi-implicit finite difference (SIFD) technique. Despite these finite difference approximations displaying reasonable accuracy, their outcomes often deviate from our expectations [3], [8].

Due to the limitations of finite difference schemes, current research has shifted focus to the notable advantages offered by pseudo-spectral Fourier methods. The inherent periodicity of optical fields in waveguides and fibers aligns well with spectral methodologies, enabling spatial derivatives to be translated into simple multiplicative operations. This, in turn, enhances both computational efficiency and accuracy [3], [8], particularly in the context of split-step Fourier methods and their variations. The underlying concept of these methods involves splitting the equation into linear and nonlinear components, solving them in Fourier space, reintegrating them, advancing in Fourier space, and finally transitioning back to the physical domain. Previous studies have demonstrated that this approach is indeed the most effective way to address the NLS [8], [19], [18]. Even when compared with other pseudo-spectral methods like the time-space pseudo-spectral method, which combines finite difference methods for space and Fourier methods for time, the Split-step method stands out as superior [8], [6], exhibiting a faster response time.

7 Proposed methodology

The Nonlinear Schrödinger equation (NLS) given by Equation (4), as previously discussed, presents a significant challenge due to the limited number of known solutions and the underlying theoretical complexity of the equations. For that we propose a comprehensive numerical study of the equation to identify the most effective high-performance numerical techniques for simulating the NLS model. In this context, we consider two primary approaches: finite difference methods and pseudospectral methods.

Firstly, we dive into finite difference methods, where we seek an approximation u_n^m to the original function $u(x,t)$ at discrete points x_n, t_m on a rectangular grid in the x, t plane. Here, $x_n = hn$ and $t_m = km$, with h representing the increment in x and k denoting the increment in t. These characteristics give rise to algebraic relations between grid points, making this method a preferred choice due to its simplicity and effectiveness. For our NLS, we propose the Crank-Nicolson method and the Hopscotch method, described as follows:

- HSM : $i\dfrac{u_n^{m+1}-u_n^m}{\Delta t} =\dfrac{u_{n+1}^{m+1}-2u_n^{m+1}+u_{n-1}^{m+1}}{(\Delta)^2}+\left[(|u_{n-1}^{m+1}|)^2u_{n-1}^{m+1}+(|u_{n+1}^{m+1}|)^2u_{n+1}^{m+1}\right] \quad (12)$
- CNM : $i\dfrac{u_n^{m+1}-u_n^m}{\Delta t} =\dfrac{1}{2(\Delta x)^2}\left[u_{n+1}^m-2u_n^m+u_{n-1}^m+u_{n+1}^{m+1}-2u_n^{m+1}+u_{n-1}^{m+1}\right]+ (|u|_n^{m+1})^2u_n^{m+1}+(|u|_n^m)^2u_n^m \quad (13)$

One can observe that the Crank-Nicolson method employs an implicit scheme, requiring the solution of a system of $n-1$ equations and $n-1$ variables at each step. This system can be solved using methods such as Newton. However, due to this implicit nature, the method carries the drawback of computational expense, demanding additional calculations. Despite this cost, we know from previous study that the truncation errors are of order $O((\Delta t)^2)+O((\Delta x)^2)$ for both methods, which is highly satisfactory [17]. Additionally, the study reveals that both methods are unconditionally stable [17] meaning that the scheme do not become unstable, regardless of the step size (time or spatial discretization) used in the computation. This unconditional stability property is highly desirable in numerical methods, enabling accurate solutions even with minimal computational cost.

Secondly, there are the pseudospectral methods that exploit the fact that the behavior of certain functions can be accurately represented in the spectral domain, such as in our case, through Fourier space. In a pseudospectral method, the primary idea is to transform the partial differential equation (PDE) from the spatial or temporal domain to the Fourier domain, where differentiation operations become algebraic operations. This transformation is achieved through the Fast Fourier Transform (FFT) in our case. For our NLS (4), we propose the Split-Step Fourier method and the Fourier pseudospectral method, which are described as follows

- SSFM : $u(x,t+\Delta t) =F^{-1}(exp(-i\omega\Delta t)F(exp(i\Delta t|u(x,t)|^2)u(x,t))) \quad (14)$
- FPSM : $u(x,t+\Delta t) =u(x,t)-i\Delta tF^{-1}\left(\sin\left(\omega^2\dfrac{\pi^2}{P^2}\Delta t\right)F(u(x,t))\right)+ i\Delta t|u(x,t)|^2u(x,t) \quad (15)$

Pseudospectral methods are particularly effective when dealing with problems that exhibit periodic behavior, as the spectral representation inherently captures periodic phenomena [7, 8, 16]. Assuming periodicity is indeed an strong assumption, but because of this we obtain a spectral precision, this is because spectral methods can achieve high accuracy since they are well-suited for functions that have smooth spectral representations, which is our case.

To lead the validation and verification of these methods, we will undertake a comparative analysis. Our approaches will be compared against analytical solutions of the NLS equation. In this context, we shall focus on theoretical solitons outlined in preceding sections. Subsequently, our assessment will make a comparison of accuracy measures for varying grid sizes across the different methods. This will point out differences in absolute errors and computational time between methods. Ultimately, these evaluations will converge toward a conclusive determination of which method yields better results.

For these implementations, we have opted for MATLAB as our chosen software. This selection is driven by MATLAB's amalgamation of high performance and code efficiency. Its native Fast Fourier Transform (FFT) implementation and built-in function for solving systems of equations (Fsolve) are particularly advantageous for our purposes.

8 Results

Various numerical methods are use in order to approximate the NLS equation (4), namely: (i) The Split-step fourier method, (ii) The Fourier pseudo spectral method, (iii) Crank-Nicholson implicit scheme. We obtain a comparison between the Split-Step fourier methods and other utilized schemes. Our approach for comparison is the L_∞ error between the exact solution an a numerical scheme, for different computations of dx and fixing $dt = 0.01$ beginning at t=0 and ending at t=T. For performing this tests, we select the bright soliton solution given by (2). Then we show the time response of the solitons for each method to visualize the nature of the accuracy, finally we present a table that contains the table with L_∞ error. Having all this results we made some conclusions about the nature of each method for this problem, to finally make a decision of which one is the best. The bright soliton time response with $T = 10$, are presented in the next figure

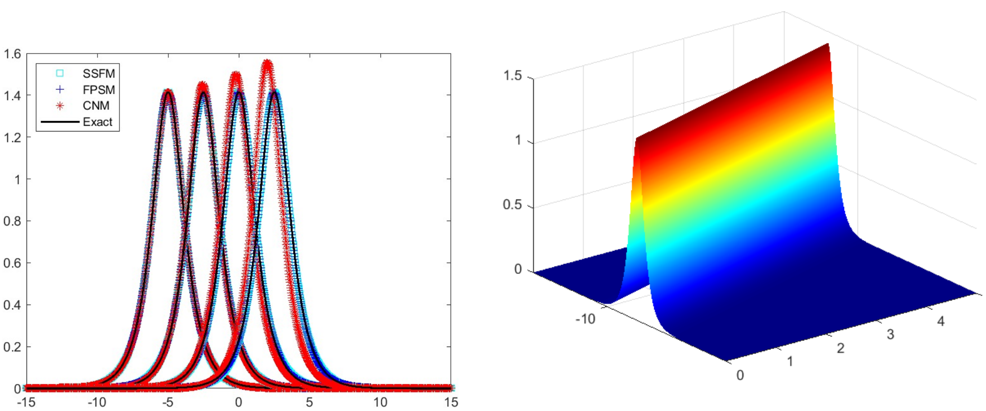

Figure 1: Soliton time response

As one can see in the past figures there it is a very precise response for the 3 presented methods. But there are certain cases in which the simulations present some notable errors, and those are for the CNM when the simulation time progress. For that now we present the errors of the methods through time, therefore we compute the average squares of errors through time for all the simulations presented before in the figures

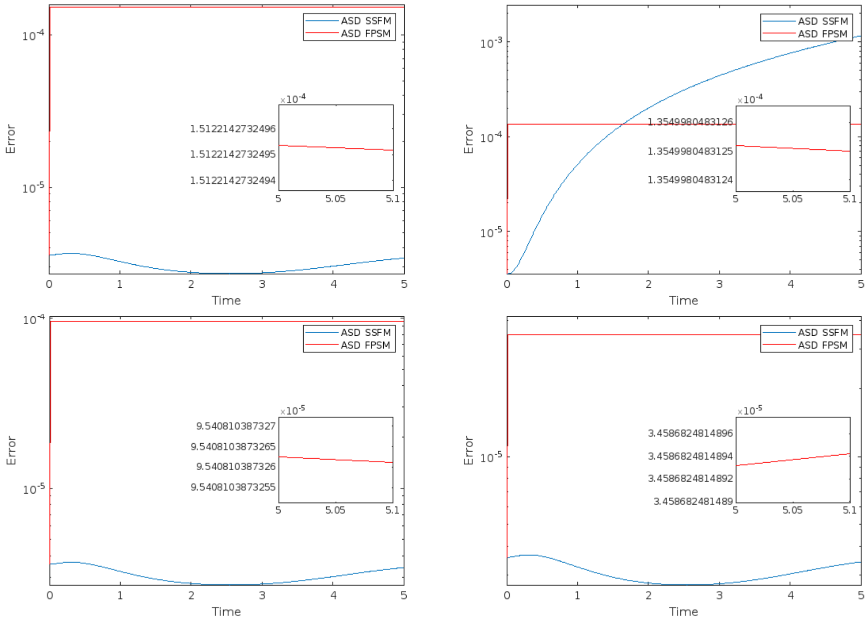

Figure 2: Average square of differences, spectral methods with fixed $dt = 0.01$, $dx = 0.2, 0.1, 0.05, 0.025$.

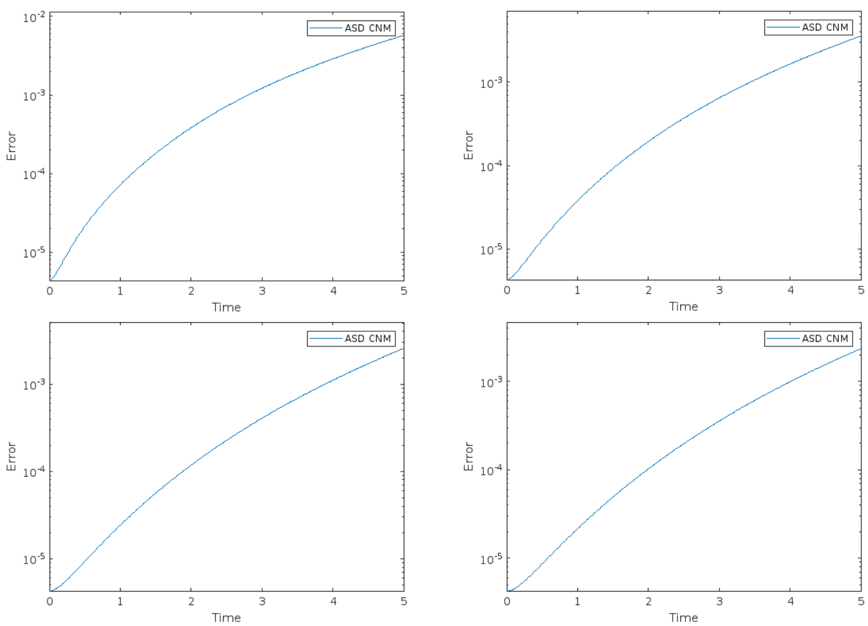

Figure 3: Average square of differences, CNM with fixed $dt = 0.01$, $dx = 0.2, 0.1, 0.05, 0.025$.

Some notable insights are that the Spectral methods present a very decent approximation for the theoretical soliton in both cases, being the Split step fourier method, the best one. It also shows how the error decreases as we refine the mesh. In the case of the Crank-Nicolson, one can see not a good scenario, having a very poor result for some cases, and some acceptable for others, but finally making a good performance in the task of simulating the soliton.

In the endeavour of making a objective criteria based comparison between the methods, we present the table that contains the L_∞ error for the methods

h	L_∞ (SSFM)	L_∞ (FPSM)	L_∞ (CNM)
0.2	1.8535e-07	1.4933e-05	0.002737
0.1	1.8619e-07	1.3742e-05	0.002676
0.05	1.8618e-07	9.397e-06	0.002663
0.025	1.8226e-07	1.3090e-06	0.00265978

Table 1: L_∞ norm through mesh refinement

From the table we can draw some conclusions, in the first place one can note that the SSFM present the best performance above all methods, being the most precise one in the vast majority of scenarios. In the second place, the CNM doesn't represent a good alternative to face this problem, from the table we can draw that it is indeed the poorest alternative in all scenarios, besides the most slow one also. Finally the FPSM its a good alternative, but having the SSFM which is easier to implement, and faster, the FPSM can not compete with the SSFM.

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```

% Spectral methods

dx = vdx(w);
L = 200; % Intervalo de espacio
N = L/dx; % Numero de intervalos en x

n = (-N/2: 1 : N/2 - 1)';
x = n * dx; % Espacio
k = 2 * n * pi / L; % Wavenumbers


M = tf/dt;
m = (1 : 1 : M)';
t = m * dt;

x0=0; % Posicion inicial
psi=1;
eta=1; % Amplitud
v=1; % Velocidad

u1 = zeros(length(x),length(t)); % Split-step resultados
u2 = zeros(length(x),length(t)); % Fourier pseudospectral
    resultados
ut = zeros(length(x),length(t));

f=@(x,t) (2*eta)^(0.5)*exp(1i*(0.5*v*x+(eta-0.25*v^2)*t+psi))...
            .*sech(eta^(0.5)*(x-v*t-x0));


u1(:,1)=f(x,0);                u2(:,1)=f(x,0);
u1(:,end)=f(x,t(end));         u2(:,end)=f(x,t(end));
u1(1,:)=f(-L,t);               u2(1,:)=f(-L,t);
u1(end,:)=f(L,t);              u2(end,:)=f(L,t);

q=u1(:,1);

```

```
    ut(:,1) = f(x,t(1));
    for m = 2:length(t)
        %SSFM
        q = ifft(fftshift(fftshift(fft(exp(dt * 1i * (abs(q).*abs(q)
            )) .* q)).*exp(-dt * 1i * k.*k)));
        u1(2:end-1,m) = q(2:end-1);

        %FPSM
        qn = f(x,t(m-1)) + 2*1i*ifft(sin(k.^2*(pi^2/L^2)*dt).*fft(f(
            x,t(m))))-4*1i*dt*(abs(f(x,t(m))) .* abs(f(x,t(m)))) .* f
            (x,t(m));
        u2(2:end-1,m) = qn(2:end-1);

        ut(:,m) = f(x,t(m));
    end
```

```

    % Crank -  Nicholson method
    dx = vdx(w);
    L = 100;
    N = floor(L/dx);
    n = (-N-1 : 1 : N)';
    x = n * dx;

    M = floor(tf/dt);

    m = (1 : 1 : M)';
    t = m * dt;

    x0=-5;
    psi=0;
    xi=1;
    eta=1;
    c=1;


    lambda=1/(2*(dx^2));

    u = zeros(2*N+2,M);
    ut = zeros(2*N+2, M);

    q = @(x,t) (2*eta)^(0.5)*exp(1i*(0.5*c*x+(eta-0.25*c^2)*t+psi))
        .*sech(eta^(0.5)*(x-c*t-x0));

    u(:,1) = q(x,t(1));
    ut(:,1) = q(x,t(1));
```

```


    A = zeros(2*N+2,2*N+2);
    B = zeros(2*N+2,2*N+2);


    for i=1:2*N+2
        A(i,i) = (1i/dt - 1/(dx^2));
        B(i,i) = (1i/dt + 1/(dx^2));
        if i>1
            A(i-1,i)= lambda; A(i,i-1)=  lambda;
            B(i-1,i)= -lambda; B(i,i-1)=  -lambda;
        end
    end
    F = -(abs(u(:,1)).^2).*u(:,1)+B*u(:,1)-(abs(u(:,1)).^2).*u(:,1);
    u(:,2) = A\F;
    ut(:,2) = q(x,t(2));


    for j=2:M-1
        F = -(abs(u(:,j+1)).^2).*u(:,j+1)+B*u(:,j)-(abs(u(:,j)).^2).*u(:,j);
        u(:,j+1)= A\F;
        ut(:,j+1) = q(x,t(j+1));
    end
```