EXPLICIT CONSTRUCTION OF NON-LINEAR PSEUDO-ANOSOV MAPS, WITH NONMINIMAL INVARIANT FOLIATIONS

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Abstract. For any pseudo-Anosov map on a genus $g \geqslant 2$ surface, up to considering one of its power, we construct new maps by a very similar procedure Derived from Anosov maps, on the two-torus, are built. This construction is done by perturbing a given pseudo-Anosov map at a fixed point. We first deal with conical fixed points, then with regular fixed points. We then study these transformations, in particular we prove that there exists an invariant measure whose support is a connected hyperbolic attractor for the inverse of the map, with respect to which the map is mixing. The attractor is the unique minimal component of the stable foliation of the map.

1. Introduction

In this article, we give a method to generalized to pseudo-Anosov transformations the construction of Derived from Anosov maps from an Anosov diffeomorphism given by Smale [9]. For a nice and gentle introduction to pseudo-Anosov transformations, we refer to [7]. In particular, we prove that the constructed transformation f is a homeomorphism that fixes the conical points, which are hyperbolic attracting fixed points. Since the dynamic on the basins of attraction is by definition trivial, we focus our attention on the set K, the complement of the union of all basins of attraction, which is invariant by f. We first give some topological properties of K before considering its dynamical properties and the behaviour of f when restricted to K. In particular, by computing explicit stable and unstable vector fields on K we prove that the set K is hyperbolic. By extending the stable vector field to a Lipschitz continuous vector field v^s on $S \setminus \Sigma$ and integrating it into the flow $(h_t)_t$, we prove that the set K is connected and transverse to any unstable leaf of f. Furthermore, we prove that f is topologically transitive with respect to the trace topology on K.

In a second time, we investigate the dependence of v^s with respect to the amplitude parameter of the perturbation passing from a pseudo-Anosov homeomorphism φ to f, and prove that this dependence is continuous. This continuity plays a central role in the proof of the unique ergodicity of the flow $(h_t)_t$. In order to prove this unique ergodicity, we construct a generalization of the decomposition in rectangles

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of a flat surface associated to a pseudo-Anosov map. This construction is then used to produce a Generalized Interval Exchange Transformation (GIET). In order to study this latter, we use a now classical tool in Teichmller Dynamic, namely the Rauzy-Veech induction – see, for example, [12] and [10] for a complete description and [5] for geometric motivations – which makes it possible to semi-conjugate the obtain transformation to one that is known to be uniquely ergodic. From this, it follows that $(h_t)_t$ is minimal in restriction to K and that it is uniquely ergodic as well, of unique invariant measure μ. Furthermore, K is exactly the α-limit and ω-limit set of every infinite orbit of $(h_t)_t$.

Because of the relation of commutativity between the flow $(h_t)_t$ and the map f, it follows that μ is also invariant by f. A simple functional analysis type argument is then enough to prove the mixing of f with respect of μ. In addition, we prove that the support of the measure μ is precisely equal to K.

We also extend this construction of perturbed pseudo-Anosov map by giving a similar construction where the perturbation is done in a neighbourhood of a periodic point of a pseudo-Anosov homeomorphism. The analysis of the resulting transformation f is very similar to the previous construction: there exists a compact, connected set K that is the support of an invariant measure μ with respect to which the map f is mixing.

Finally we discuss about possible properties of this measure μ. In particular, we suggest that μ is the SRB measure of the map f^{-1}, and that the rate of mixing is exponentially fast.

2. Construction of derived from pseudo-Anosov transformations

In this section we give a method to construct a generalization of the Smale derived from Anosov map – which are defined on the two torus – from a perturbation of a given pseudo-Anosov map – on a surface of genus $g \geqslant 2$.

2.1. **Perturbation of a pseudo-Anosov.**

Let φ be a pseudo-Anosov transformation on the Riemann surface S_g of genus g. Up to replacing φ by the unique Teichmller mapping in the isotopy class of φ we assume φ is a Teichmller mapping. Therefore the invariant foliations of φ can be derived from a holomorphic quadratic differential q invariant by φ. Up to consider a cover of order two in most cases, we will assume that the quadratic differential form is Abelian, in other words $q = \omega^2$. Up to multiplying ω by a modulus one complex number, the horizontal and vertical foliations $\{\Re(\omega) = 0\}$ and $\{\Im(\omega) = 0\}$ are the invariant foliations of φ. Let $\lambda > 1$ denote the stretch factor of φ. This stretching corresponds to the vertical measured foliation. The horizontal measured foliation is stretch by a factor λ^{-1}. Let Σ be the set of points where ω vanishes, we call these points *conical points*. We now consider the flat structure induced by ω on $S_g \smallsetminus \Sigma$, that is chart z so that $\omega = \mathrm{d}z$. In the neighborhood of every conical point $\sigma \in \Sigma$, there exist a positive integer n_σ, an open set and a chart z on this set such that $\omega = z^{n_\sigma - 1}\mathrm{d}z$. The angle around σ is then $2\pi n_\sigma$.

Outside of these neighbourhoods of points of Σ, we set f to be equal to φ. We now construct f to be a perturbation of φ around each σ in Σ.

Let σ be a conical point, U a neighborhood of σ and a chart z on U so that $\omega = z^{n_\sigma - 1}\mathrm{d}z$. Let ξ be the branched cover at σ associated to the chart z, $\xi : z \in z^{-1}U \mapsto z^{n_\sigma} \in \xi(z^{-1}(U)) \subset \mathbb{C}$. Let $(W_i)_{1 \leqslant i \leqslant 2n_\sigma}$ be a family of open sets of $\mathbb{C} \smallsetminus \mathbb{R}_+$ such that all $\xi|_{W_i}$ are homeomorphisms. Up to replacing φ by one of its power, we

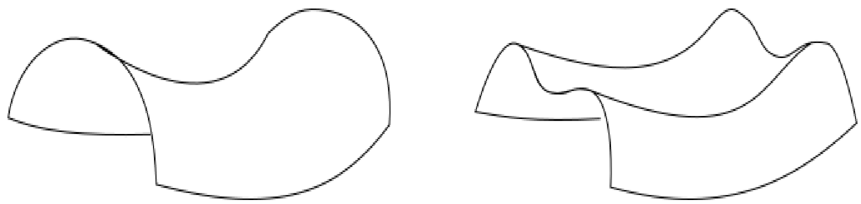


FIGURE 1. Heuristical representations of a saddle and of a perturbed saddle.

assume that every conical point is fixed by φ and that φ respects the leaves of the branched covers : for all i, $\varphi(W_i) \cap U \subset W_i$.

We can define f on the base of the branched cover in the exact same way as Smale does [9]. In order to perform further analysis on the map, we give the following explicit formula that generalized the one used in [1] in the case of the cat map on the two-torus.

For $z = x + iy \in \mathbb{C} \smallsetminus \mathbb{R}_+$, we define f as :

$$f(\xi|_{W_i}^{-1}(z)) \coloneqq \xi|_{W_i}^{-1}\left((\lambda + \beta_\sigma k_\sigma(|z|/\alpha_\sigma))x + i\lambda^{-1}y\right),$$

for some $\alpha_\sigma > 0$, $\beta_\sigma < 1 - \lambda$ and with $|z| \leqslant \alpha_\sigma$ and where $k_\sigma : \mathbb{R} \to \mathbb{R}$ is an even map of class $\mathcal{C}^1$, compactly supported in $[-1, 1]$ with $k'_\sigma < 0$ on $]0, \infty[\cap \{k_\sigma > 0\}$, for example $k_\sigma(r) = (1 - r^2)^2 \mathbb{1}_{[-1,1]}$. We do this perturbation at every conical point. We will see that this defines a function for small enough α_σ.

When such a map f is well defined, we call it Derived from pseudo-Anosov.

We give in figure 1 a representation, when $n_\sigma = 1$ – which corresponds to the case treated by Smale in [9].

3. Study of these derived from pseudo-Anosov maps

In this section, we investigate some classical properties of dynamical systems in the case of Derived from pseudo-Anosov transformations. We first prove that these maps are well defined and are homeomorphisms. Then, we show that for some good choices of parameters, conical points are the only attractive fixed points of the dynamic generated by such a map f. Since the basins of attraction are disjoint open sets and the underlying surface is connected, the complement K of the union of the basins of attraction is not empty. This compact set is of great importance since we will prove that in a certain sense all the chaotic behaviour of the map f is concentrated in this set. More precisely, we show that K is a hyperbolic connected set on which the map f is topologically transitive. In order to prove the hyperbolicity of K we explicitly construct vector fields on K giving invariant directions satisfying the desired property. We managed, at the cost of a few pages of estimates and bounds, to extend the stable vector field to the whole set $S_g \smallsetminus \Sigma$ of regular points, such that the extension is still uniformly contracted by the action of f and so that it is Lipschitz continuous. The flow $(h_t)_t$ obtained by integrating this extended vector field enjoys a nice commutation relation with f (the map f can be used to renormalized this flow). From the properties of $(h_t)_t$, we show that the set K is connected, transverse to any vertical leaf, and that f is transitive with respect to the trace topology on K. Lastly, we investigate the regularity of the stable vector field with respect to the parameter β of f and prove that it is continuous. This regularity property is a central point in the proof of the unique ergodicity of the flow, which implies the mixing of f with respect to the very same measure thanks to the commutation relation between f and $(h_t)_t$.

3.1. **Regularity and first properties.** In order to ensure that the explicit construction of a Derived from pseudo-Anosov makes sense, we need to ensure that the open sets near each conical point do not overlap. This can be easily done by taking the parameter α_σ small enough. We can give a simple bound on their size by geometric considerations.

Let $Syst_{s.c}(S_g) = \inf\{\mathrm{d}(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in \Sigma\}$, where d is the distance for the flat metric on S_g associated to the invariant measured-foliation of φ. Let $Syst(S_g) = \inf\{l(\gamma) \mid \gamma \neq 0 \text{ in } \pi_1(S_g)\}$ be the smallest possible length of any non-trivial loop. Then define $\delta_\Sigma = \min(Syst_{s.c}(S_g), Syst(S_g))$.

Proposition 3.1. *For all $\beta_\sigma \in]-\lambda, 0]$ and all $\alpha_\sigma < \delta_\Sigma/2$, f is a homeomorphism on S_g and is a diffeomorphism on $S_g \setminus \Sigma$.*

Proof. Clearly, f is continuous on S_g and differentiable everywhere except on Σ. The differential on $S_g \setminus \Sigma$ of f is invertible, hence f is a local homeomorphism on $S_g \setminus \Sigma$ and hence $f(S_g \setminus \Sigma)$ is open. In charts around points of Σ, one can see that f is a local homeomorphism in a neighbourhood of Σ. Hence $f(S_g)$ is open. Since S_g is compact, $f(S_g)$ is closed. Hence $f(S_g) = S_g$, because S_g is connected. Therefore, f is a surjective local homeomorphism, hence f is a covering map. Since the pre-image of a point of Σ by f is itself, f is injective. □

By refining the range where the β_σ live, we can turn conical points into attractive fixed points.

Proposition 3.2. *For $\beta_\sigma \in]-\lambda, 1-\lambda[$ and $\alpha_\sigma < \delta_\Sigma/2$, $\sigma \in \Sigma$ is an attractive fixed point for f. Let U_σ be its basin of attraction. Then U_σ is an open set.*

Proof. It is a consequence of the Grobman-Hartman theorem when looking at f through the branched-covering map around σ. We have $U_\sigma = \bigcup_{n\in\mathbb{N}} f^{-n}(B(\sigma, \varepsilon))$, for some small enough $\varepsilon > 0$. □

Since basins of attraction U_σ are disjoint open sets and S_g is connected, these basins are not an open cover. Therefore the complement of the union of basins is not empty. Define $K \coloneqq S_g \setminus \bigsqcup_{\sigma\in\Sigma} U_\sigma$ and $U = \bigsqcup_{\sigma\in\Sigma} U_\sigma$. These sets are clearly invariants by f.

Proposition 3.3. *If for some $\sigma \in \Sigma$, $\beta_\sigma \in]-\lambda, 1-\lambda[$ and $\alpha_\sigma < \delta_\Sigma/2$, then there exists a fixed hyperbolic point p_i^σ, $1 \leqslant i \leqslant 2n_\sigma$, on each vertical ray starting at σ. We number them by going counter-clockwise around σ. All these points are at the same distance $|p^\sigma|$ from σ. Moreover $B(\sigma, |p^\sigma|) \subset U_\sigma$.*

Proof. Let σ, β_σ and α_σ be as in the proposition. Let $\gamma : [0, \alpha_\sigma] \to S_g$ be a unit speed parametrization of a vertical ray such that $\gamma(0) = \sigma$. Hence, in charts, $f(\gamma(t)) = (\lambda + \beta_\sigma k(t/\alpha_\sigma))t$. Let $h : [0, \alpha_\sigma] \to \mathbb{R}$ be the function $h(t) = (\lambda + \beta_\sigma k(t/\alpha_\sigma))t$. Then $h(0) = 0$, $h(\alpha_\sigma) = \lambda\alpha_\sigma > \alpha_\sigma$, and $h'(0) = \lambda + \beta_\sigma \in]0, 1[$. Hence h has a fixed point in $]0, \alpha_\sigma[$. Call t_0 the smallest fixed point. This value doesn't depend on which vertical ray starting from σ we consider. The point $p = \gamma(t_o)$ is fixed by f and is hyperbolic: in the charts centred at σ, the differential of f at p is

$$\mathrm{d}_p f = \begin{pmatrix} 1+\beta_\sigma t_0 \frac{\partial}{\partial x} k(\frac{d((x,y),\sigma)}{\alpha_\sigma}) & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $\beta_\sigma t_0 \frac{\partial}{\partial x} k(\frac{d((x,y),\sigma)}{\alpha_\sigma}) > 1$.

By definition of t_0, we have $\gamma([0,t_0[) \subset U_\sigma$. Let $z \in B(\sigma, t_0)$. In the appropriate leaf of the branched-cover over σ, we have $z = (x,y)$ in coordinates. Hence,

$$\begin{aligned} \mathrm{d}(f(z),\sigma) &\leqslant (\lambda + \beta_\sigma k(d(z,\sigma)/\alpha_\sigma))^2 x^2 + \lambda^{-2} y^2 \\ &< (\lambda + \beta_\sigma k(t_0/\alpha_\sigma))^2 x^2 + \lambda^{-2} y^2 \leqslant x^2 + \lambda^{-2} y^2 \leqslant \mathrm{d}(z,\sigma). \end{aligned}$$

Hence, the function $z \mapsto \mathrm{d}(f(z),\sigma)/\mathrm{d}(z,\sigma)$ is continuous and strictly bounded from above by 1 on the compact annulus $\{z \in S_g \mid \varepsilon \leqslant \mathrm{d}(z,\sigma) \leqslant t_0 - \varepsilon\}$. Therefore every orbit of point from the ball $B(\sigma, t_0 - \varepsilon)$ ends up entering the ball $B(\sigma, \varepsilon)$. Hence the claim. □

3.2. **Topology and dynamic of the invariant sets.** Here we investigate the topology of the invariant set K. In particular we prove that it can be written as the union of the closure of some stable leaves of the hyperbolic fixed points p_i^σ and that it is a hyperbolic set.

We start by proving that the set U is dense in S_g, or equivalently that K is of empty interior. In order to do this we need the following lemma which is obtained by simply computing the differential of f.

Lemma 3.4. *Define $(q_i^\sigma)_i$ as the $2n_\sigma$ points at distance $|p^\sigma|$ from σ on the* horizontal *rays starting from σ. Then for all $x \in S_g \smallsetminus \bigsqcup_{\sigma\in\Sigma} (B(\sigma, |p^\sigma|) \cup \{q_i^\sigma \mid 1 \leqslant i \leqslant 2n_\sigma\})$, f is a strict dilation in the vertical direction.*

Proposition 3.5. *For all $x \in K$ and all $\varepsilon > 0$, every vertical segment of length ε containing x in its interior crosses U. Hence U is dense and K has empty interior.*

Proof. By contradiction, let $\gamma : [-\varepsilon, \varepsilon] \to S_g$ be a vertical segment parametrized by arc length, containing some $x \in K$ and such that $\gamma([-\varepsilon,\varepsilon]) \cap U = \emptyset$. Without loss of generality, we can assume that $\gamma(0) = x$. Since U is invariant by f, we see that the existence of some $-\varepsilon \leqslant t \leqslant \varepsilon$ such that $f^n(\gamma(t)) \in U$ is impossible. Hence $f^n(\gamma([-\varepsilon,\varepsilon])) \cap U = \emptyset$. By construction of f, the set $f^n(\gamma([-\varepsilon,\varepsilon]))$ is a vertical segment, containing $f^n(x)$ in its interior and of length l_n. Since f is a strict dilation in the vertical direction on the compact set K, there exist $l_* > 1$ such that $l_n \geqslant l_*^n$.

Let $\delta = \inf\{|p^\sigma| \mid \sigma \in \Sigma\}$ and since K is compact and invariant by f let $y \in K$ be a subsequential limit of $(f^n(x))_n$. Let n_k be an increasing sequence of integers such that $f^{n_k}(x)$ converges to y as n_k goes to infinity. We know – see [2, corollary 14.15] – that the vertical leaf containing y is at least infinite in one direction and is dense in S_g. In particular, some sufficiently long section of this leaf, containing y, is $\delta/4$-dense in S_g. Hence, for large enough n_k, the curve $f^n(\gamma([-\varepsilon,\varepsilon]))$ is sufficiently long and sufficiently close to the vertical leaf containing y to be $\delta/2$-dense in S_g. In particular, there exists $-\varepsilon < t < \varepsilon$ such that $\mathrm{d}(f^n(\gamma(t)),\sigma) < \delta$ for some $\sigma \in \Sigma$. This contradicts the fact that $B(\sigma, |p^\sigma|) \subset U$. □

Recall definitions of strong stable and strong unstable leaves of $x \in S_g$ with respect to f as the sets:

$$W^{ss}(x) = \{y \in S_g \mid \mathrm{d}(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\},$$
$$W^{su}(x) = \{y \in S_g \mid \mathrm{d}(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to +\infty\}.$$

If x is fixed by f, then these sets are also fixed by f.

Here, these leaves at hyperbolic fixed points p_i^σ enable to describe precisely the set K. We start by showing that the stable leaves can be seen as the *accessible border* of U – and are obvioluly contained in K. On the other hand, unstable leaves are dense.

Proposition 3.6. *(i) If $x \in U_\sigma$ and $\gamma : [0,1] \to S_g$ is a vertical curve such that $\gamma(0) = x$, $\gamma([0,1[) \subset U_\sigma$ and $\gamma(1) \notin U_\sigma$ then $\gamma(1)$ belongs to $\bigsqcup_{1 \leqslant i \leqslant 2n_\sigma} W^{ss}(p_i^\sigma)$.*

(ii) For all $\sigma \in \Sigma$ and all $1 \leqslant i \leqslant n_\sigma$, the unstable leaf $W^{su}(p_i^\sigma)$ contains a full semi-infinite vertical leaf. Hence $W^{su}(p_i^\sigma)$ is dense in S_g.

Proof. (i) For n large enough, we find that $f^n(x)$ is close to σ. Once close to σ by going upward – or downward, depending on the orientation of $f^n \circ \gamma$) – the first time $f^n \circ \gamma$ intersect K is in $\bigsqcup_{1 \leqslant i \leqslant 2n_\sigma} W^{ss}(p_i^\sigma)$ at $f^n(\gamma(1))$ as a consequence of the Hartman-Grobman theorem

(ii) Let $\gamma : [0, +\infty[\to S_g$ be a unit speed parametrization of the vertical ray starting at $\sigma \in \Sigma$ and containing $p \coloneqq p_i^\sigma$. In particular, $\gamma(0) = \sigma$ and $\gamma(|p^\sigma|) = p$.

By contradiction, assume there exists $t \geqslant |p^\sigma|$ such that $\gamma(t) \notin W^{su}(p)$. Let $t_0 = \inf\{t \geqslant |p^\sigma| \mid \gamma(t) \notin W^{su}(p)\}$.

We now show that $t_0 > |p^\sigma|$. Let $h : t \mapsto (\lambda + \beta_\sigma k(t/\alpha_\sigma))t$. By construction of f, we have the relation $f(\gamma(t)) = \gamma(h(t))$ for every $t \in [0, \alpha_\sigma[$, and hence $f^n(\gamma(t)) = \gamma(h^n(t))$ for all $n \geqslant 0$. Now $(h^{-1})'(|p^\sigma|) < 1$, so for t close to $|p^\sigma|$, $f^n(\gamma(t)) \to p$ as n goes to infinity. Therefore $t_0 > |p^\sigma|$.

We now prove that $\gamma(t_0)$ is a fixed point of f. We know that $f(\gamma(]|p^\sigma|, t_0[) = \gamma(]|p^\sigma|, s[)$ for some s. But $f(\gamma(]|p^\sigma|, t_0[) \subset W^{su}(p)$. Hence $s \leqslant t_0$.

By contradiction, assume there exists $\varepsilon > 0$ such that $s + \varepsilon < t_0$. So $\gamma([|p^\sigma|, s + \varepsilon[) \subset W^{su}(p)$, and so $f^{-1} \circ \gamma([|p^\sigma|, s+\varepsilon[) \subset W^{su}(p)$. However, $f^{-1} \circ \gamma([|p^\sigma|, s+\varepsilon[) \subset W^{su}(p) = \gamma([|p^\sigma|, t_0 + \delta_\varepsilon[)$ for some $\delta_\varepsilon > 0$ since f is strictly preserving vertical orientation. This contradicts the definition of t_0. Therefore $s = t_0$ and $\gamma(t_0)$ is fixed by f.

The point $\gamma(t_0)$ can't be in Σ nor be a p_i^σ, otherwise γ would connect two conical points, which is impossible. By computing the differential of f at $\gamma(t_0)$, we see that $\gamma(t_0)$ is a hyperbolic fixed point of f with a vertical unstable leaf. Therefore there exist points whose iterates by f^{-1} converge to p and to $\gamma(t_0) \neq p$.

□

These properties of stable and unstable leaves yield to the fact that the set K can be written as a finite union of closure of stable leaves. In fact, we have the following slightly stronger result.

Proposition 3.7. *The compact set K can be written as a finite union of closed invariant sets as follow $K = \bigcup_{\sigma \in \Sigma} \bigcup_{i=1}^{n_\sigma} \overline{W^{ss}(p_i^\sigma) \cap W^{su}(p_i^\sigma)}$.*

Proof. Let $x \in K$ and $\varepsilon > 0$. Let $y \in U$ be in the same vertical leaf as x and obtained by *going downward* by a distance less than ε. Since $U = \bigsqcup_\sigma U_\sigma$, there exists $\sigma \in \Sigma$ such that $y \in U_\sigma$. From the Hartman-Grobman theorem, for each $1 \leqslant i \leqslant 2n_\sigma$ there exists a neighbourhood of p_i^σ on which the dynamic of f is the same as the one of the differential of f. Without loss of generality, we assume that these neighbourhoods are rectangles with vertical and horizontal sides and with centers the p_i^σ's. Up to replacing these rectangles by smaller ones, let δ_σ be a common horizontal size for these rectangles.

For $n \geqslant 0$ large enough, the point y lies in $B(\sigma, \delta_\sigma/4)$. By construction and by the previous lemma, we know that by going *upward* from y we cross some $W^{ss}(p_i^\sigma)$, for some $1 \leqslant i \leqslant 2n_\sigma$. Therefore, by going *upward* from $f^{-n}(y)$ we cross the rectangle of linearisation associated with p_i^σ, and hence the stable leaf $W^{ss}(p_i^\sigma)$ at some point y^u.

Let δ be the modulus of absolute continuity of f^{-n} associated with ε. By density of the unstable leaf of p_i^σ, we can chose a point z such that $\mathrm{d}(f^n(z), f^n(y)) < \min(\delta, \delta_\sigma/4)$ so that by going *upward* from $f^n(z)$ we cross $W^{ss}(p_i^\sigma)$ at some point z^u, at distance less than δ from y^u. Finally, the point $f^{-n}(z^u) \in W^{ss}(p_i^\sigma) \cap W^{su}(p_i^\sigma)$ is at distance less than 3ε from x. □

Finally, we explicit stable and unstable foliations such that the set K is hyperbolic with respect to f. To do this, we compute a vector field that is uniformly contracted by the differential of f.

Theorem 3.8. *The set K is hyperbolic.*

Proof. We will explicit the stable and the unstable directions of the splitting of the tangent space. Write the differential of f at $x \in S_g \setminus \Sigma$ in the basis (e_h, e_v):

$$\mathrm{d}_x f = \begin{pmatrix} a(x) & b(x) \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Therefore, for every positive integer n, we have the following,

$$\begin{aligned} \mathrm{d}_x(f^n) &= \mathrm{d}_{f^{n-1}(x)} f \cdots \mathrm{d}_{f(x)} f \, \mathrm{d}_x f, \\ &= \begin{pmatrix} a(f^{n-1}(x)) & b(f^{n-1}(x)) \\ 0 & \lambda^{-1} \end{pmatrix} \cdots \begin{pmatrix} a(f(x)) & b(f(x)) \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a(x) & b(x) \\ 0 & \lambda^{-1} \end{pmatrix}, \\ &= \begin{pmatrix} A_n(x) & B_n(x) \\ 0 & \lambda^{-n} \end{pmatrix}. \end{aligned}$$

We have that $A_n(x) = \prod_{i=0}^{n-1} a(f^i(x))$. We can compute B_n explicitly. This sequence satisfies a recurrence formula, which can be solved:

$$\begin{aligned} B_{n+1} - a(f^n) B_n &= \lambda^{-n} b(f^n), \\ B_{n+1}/A_{n+1} - B_n/A_n &= \lambda^{-n} b(f^n)/A_{n+1}, \\ B_n/A_n &= \sum_{i=0}^{n-1} \lambda^{-i} b(f^i)/A_{i+1}. \end{aligned}$$

Finally we get:

$$\frac{B_n}{A_n}(x) = \sum_{i=0}^{n-1} \lambda^{-i} b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))}.$$

We can now explicit the eigenvectors of $\mathrm{d}_x(f^n)$. The obvious one, associated with the eigenvalue $A_n(x)$, is e_v. The other one is

$$v_n(x) = \begin{pmatrix} -B_n(x)/A_n(x) \\ 1-\lambda^{-n}/A_n(x) \end{pmatrix} = \begin{pmatrix} -\sum\limits_{i=0}^{n-1} \lambda^{-i} b(f^i(x)) \prod\limits_{j=0}^{i} \frac{1}{a(f^j(x))} \\ 1-\prod\limits_{i=0}^{n-1} \frac{1}{\lambda a(f^i(x))} \end{pmatrix}.$$

We now study the convergence of the v_n's as n goes to infinity. First, as K is compact and $a > 1$, b are continuous functions over K, there exist constants a^* and C such that $a > a^* > 1$ and $|b| < C$. Therefore, the second coordinate converges to 1 as n goes to infinity. For the first coordinate, we have the uniform bound over K

$$\sum_{i=0}^{n-1} \left| \lambda^{-i} b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))} \right| \leqslant C \sum_{i=0}^{n-1} (\lambda a^*)^{-i} \leqslant C \frac{\lambda a^*}{\lambda a^* - 1}.$$

Hence, the series of continuous functions converges uniformly over K to a continuous function. Call v^s the limit of v_n as n goes to infinity.

A short computation shows that for all x in K, v^s satisfies $\mathrm{d}_x f \cdot v^s(x) = \lambda^{-1} v^s(f(x))$. Finally, we get the following splitting of the tangent space at each x in K, $T_x S_g = \mathbb{R} v^s(x) \oplus \mathbb{R} e_v$. This splitting makes K a hyperbolic set. $\square$

3.3. **Construction of an useful open cover of S_g.** In the next section, we prove that the formula giving the vector field v^s on K still converges on $S_g \setminus \Sigma$ and gives rise to a Lipschitz vector field. To do this, we first need to construct an open cover of $S_g \setminus \Sigma$ such that f satisfies some nice estimates on elements of the cover. This is done in the following proposition.

Proposition 3.9. *For some $\varepsilon > 0$ small enough, there exist $\eta > 0$, $\delta > 0$, and an open cover $S_g = A_\eta \cup \bigsqcup_{\sigma\in\Sigma} B_{\sigma,\delta}$ such that $a > 1+\eta$ on A_η and $\mathrm{d}(f(x),\sigma) < (1-\delta)\mathrm{d}(x,\sigma)$ on $B_{\sigma,\delta} \setminus \{\sigma\}$.*

Proof. By continuity of f, there exists an $\varepsilon > 0$ such that

$$\{x \in V_\sigma \mid \mathrm{d}(f(x),\sigma) < \mathrm{d}(x,\sigma)\} \supset B(\sigma, |p^\sigma|) \cup \bigcup_{i=1}^{2n_\sigma} B(q_i^\sigma, \varepsilon) =: B_\sigma^\varepsilon,$$

for all σ, where $V = \bigsqcup_{\sigma\in\Sigma} V_\sigma$ is the open neighbourhood of Σ on which $f \not\equiv \varphi$.

Since $S_g \setminus \bigsqcup_{\sigma\in\Sigma} B_\sigma^\varepsilon$ is compact and $a > 1$ on it, there exists $\eta > 0$ such that $a > 1+2\eta$ on this compact set. Call $A_\eta = \{x \in S_g \mid a > 1+\eta\}$. By construction, $S_g = A_\eta \cup \bigcup_{\sigma\in\Sigma} B_\sigma^\varepsilon$.

Since all B_σ^ε are open sets, radial and centred on σ, we have $B_\sigma^\varepsilon = \bigcup_{n\geqslant 1} \left(1-\frac{1}{n}\right) B_\sigma^\varepsilon$. Now, by compactness of S_g, there exists n_0 such that:

$$S_g = A_\eta \cup \bigcup_{\sigma\in\Sigma} \left(1-\tfrac{1}{n_0}\right) B_\sigma^\varepsilon.$$

On a small open neighbourhood W_σ of σ, by construction of f we have that $\mathrm{d}(f(x),\sigma)/\mathrm{d}(x,\sigma) < C^{st} < 1$. Now, on the compact set $\overline{(1-\frac{1}{2n_0})B_\sigma^\varepsilon} \setminus W_\sigma$, the continuous function $\mathrm{d}(f(x),\sigma)/\mathrm{d}(x,\sigma)$ is positive and strictly bounded from above by 1. Hence, there exists $\delta > 0$, independent of σ, such that for all x in $(1-\frac{1}{n_0})B^\varepsilon_\sigma \smallsetminus \{\sigma\}$, $\mathrm{d}(f(x),\sigma) < (1-\delta)\mathrm{d}(x,\sigma)$. We then call $B_{\sigma,\delta} = (1-\frac{1}{n_0})B^\varepsilon_\sigma$. □

3.4. **Lipschitz extension of v^s to $S_g \smallsetminus \Sigma$ and associated flow $(h_t)_t$.** Here we prove that the infinte sum in the definition of the vector field v^s on K does converge on all $S_g \smallsetminus \Sigma$. This way we can define v^s on $S_g \smallsetminus \Sigma$. Furthermore, we prove that this extended vector field is Lipschitz continuous.

We start by showing that v^s is bounded and continuous on $S_g \smallsetminus \Sigma$. To do this, we need several lemmas which follow directly from computation of $\mathrm{d}f$.

Lemma 3.10. *The partial derivative $b = \langle \mathrm{d}f(e_v), e_h\rangle$ of f is locally Lipschitz in some neighbourhood of Σ. Furthermore, by continuity we can set $b(\sigma) = 0$ for each $\sigma \in \Sigma$.*

Lemma 3.11. *On each U_σ, the partial derivative $a = \langle \mathrm{d}f(e_h), e_h\rangle$ of f is bounded from below by $\lambda + \beta_\sigma$.*

Theorem 3.12. *If $\beta_\sigma \in]-\lambda+\lambda^{-2}, -\lambda+1[$ for all σ in Σ, then the infinite sum defining v^s is bounded on $S_g \smallsetminus \Sigma$. Furthermore, it is continuous on $S_g \smallsetminus \Sigma$. By construction, the formula $\mathrm{d}f(v^s) = \lambda^{-1} v^s \circ f$ holds on $S_g \smallsetminus \Sigma$.*

Proof. Call $s_i = \lambda^{-i}\, b \circ f^i \prod_{j=1}^{i} \frac{1}{a\circ f^j}$. Let V be a neighbourhood of some σ such that b is Lipschitz on it and f contracts by a factor $\max(\lambda^{-1}, \lambda+\beta_\sigma+\delta_\sigma) < 1$. Without loss of generality, we assume that V is a ball centred at σ of radius ε and that $f(V) \subset V$. Since $U_\sigma = \bigcup_{N\geqslant 0} f^{-N}V$, for all $x \in U_\sigma$ there exist some $N = N(x)$ and an integer n_V which only depends on V, such that for all $n \geqslant N$, $f^n(x) \in V$, at most n_V points of the orbits fall into $B_{\sigma,\delta} \smallsetminus V$ and the rest lives in A_η.

Let $x \in U_\sigma$, $x \neq \sigma$. Since $U_\sigma = \bigcup_{n\geqslant 0} f^{-n}V$, let N be the smallest integer such that $f^N(x) \in V$. We distinguish three cases :

- $i \leqslant N - n_V$. Therefore $|s_i(x)| \leqslant \lambda^{-i}\left(\frac{1}{1+\eta}\right)^{i+1} \sup|b|$.
- $N - n_V < i \leqslant N$. Hence $|s_i(x)| \leqslant \lambda^{-i}\left(\frac{1}{1+\eta}\right)^{N-n_V}\left(\frac{1}{\lambda+\beta}\right)^{i-(N-n_V)} \sup|b|$.
- $i = j+N > N$. We get $|s_i(x)| \leqslant \lambda^{-(j+N)}\left(\frac{1}{\lambda+\beta}\right)^{j+N} \mathrm{Lip}(b)\varepsilon \max(\lambda^{-1}, \lambda + \beta_\sigma+\delta_\sigma)^j$.

Therefore, if $\lambda^{-2} < \lambda+\beta_\sigma$, then

$$\sum_{i\geqslant 0} |s_i(x)| \leqslant \sup|b| \frac{\lambda(1+\eta)}{\lambda(1+\eta)-1}\left(1+\sum_{i=0}^{n_V}\left(\frac{1}{\lambda+\beta}\right)^i\right) + \frac{\mathrm{Lip}(b)\varepsilon}{1-\frac{\max(\lambda^{-1},\lambda+\beta_\sigma+\delta_\sigma)}{\lambda(\lambda+\beta_\sigma)}},$$

which is uniform in x on U_σ. Hence, the convergence is uniform on the compact subsets of $U_\sigma \smallsetminus \{\sigma\}$ and $\sum s_i$ is continuous on $U_\sigma \smallsetminus \{\sigma\}$, for all $\sigma \in \Sigma$.

We now show that this function defined on $U = \sqcup U_\sigma$ can be extended by continuity on K. Call $u(x)$ the vector of coordinates $(-\sum s_i(x), 1)$, based at $x \in S_g \smallsetminus \Sigma$.

Let $x \in K$ and, by density of U in S_g, $(x_n)_n \in U^{\mathbb{N}}$ such that $x_n \to x$ as n goes to infinity. Since $(u(x_n))_n$ is bounded, up to extracting, the sequence converges to some u_0. Furthermore, by a diagonal argument and up to extracting, $u(f^k(x_n)) \to u_k$ for all $k \in \mathbb{Z}$ as n goes to infinity. Now, by construction of u, $\mathrm{d}f(u) = \lambda^{-1} u \circ$

f. Hence, by continuity of f and $\mathrm{d}f$, $\mathrm{d}_x(f^k)(u_0) = \lambda^{-k}u_k$. We now show that $u_0 = v^s(x)$. By hyperbolicity of K, there exist real numbers x_s, x_u such that $u_0 = x_s v^s(x) + x_u e_v$. Therefore, by hyperbolicity of K,

$$\begin{aligned}
|x_u| &= ||x_u e_v||,\\
&= ||\mathrm{d}_{f^k(x)}f^{-k}\mathrm{d}_x f^k x_u e_v||,\\
&\leqslant C^{st}(\tfrac{1}{a_*})^k||\mathrm{d}_x f^k x_u e_v||,\\
&= C^{st}(\tfrac{1}{a_*})^k||\mathrm{d}_x f^k(u_0 - x_s v^s(x))||,\\
&\leqslant C^{st}(\tfrac{1}{a_*})^k\lambda^{-k}(\sup||u|| + x_s\sup||v^s||),
\end{aligned}$$

which goes to zero as k goes to infinity. Hence $u_0 = x_s v^s(x)$. Now both u_0 and $v^s(x)$ have the same non-zero coordinate along e_v in the base (e_v, e_h). Hence $u_0 = v^s(x)$. Finally, u extends continuously on K by v^s. We call v^s this vector field on $S_g \setminus \Sigma$. □

We can now present the proof of the Lipschitz continuity of v^s on $S_g \setminus \Sigma$. To this end, we need a few more estimates on the differential of f and on its coefficients.

Lemma 3.13. *For all $x \in S_g \setminus \Sigma$, we have the following estimate $\frac{||\,\mathrm{d}_x f^n||}{A_n(x)} \leqslant 2\max\left(1, \frac{|B_n|(x)+\lambda^{-n}}{A_n(x)}\right)$. In particular, $||\,\mathrm{d}f^n||/A_n$ is bounded on $\bigcup\limits_{i=0}^{n} f^{-i}A_\eta$. Furthermore, the bound B can be chosen independently of n.*

Proof. By a direct computation,

$$\begin{aligned}
||\,\mathrm{d}_x f^n(u,v)||^2 &= (A_n(x)u + B_n(x)v)^2 + (\lambda^{-1}v)^2,\\
&\leqslant 4A_n(x)^2u^2 + (4B_n(x)^2 + \lambda^{-2n})v^2,\\
&\leqslant 4\max(A_n(x)^2, B_n(x)^2 + \lambda^{-2n})||(u,v)||^2.
\end{aligned}$$

For $x \in \bigcup\limits_{i=0}^{n} f^{-i}A_\eta$, we know that $\lambda^{-k}/A_k(x) < (\lambda(1+\eta))^{-k}$ and that $-B_n/A_n$ is the partial sum of $\sum s_i$, hence uniformly bounded. □

Lemma 3.14. *The functions a and $\frac{1}{a}$ are Lipschitz on S_g.*

Theorem 3.15. *If $\beta_\sigma \in]-\lambda+\lambda^{-2}, -\lambda+1[$ for all σ in Σ, then the vector field v^s is Lipschitz continuous on $S_g \setminus \Sigma$.*

Proof. Since all the partial sums of $\sum s_i$ are Lipschitz continuous, we give summable estimates of local Lipschitz constants. Let $x \in U_\sigma$. Let V, $N = N(x)$ and n_V be as in the proof of theorem 3.12. Therefore $U_\sigma = \bigcup_{n\geqslant 0} f^{-n}V$. We use the notation $\mathrm{Lip}_x(g)$ to indicate the local Lipschitz constant of a function g in at least one neighbourhood of x.

Let $\varepsilon > 0$. On a small enough neighbourhood of x, we have that $\mathrm{Lip}_x(f^j) \leqslant (1+\varepsilon)||\,\mathrm{d}_x f^j||$ and $\sup\frac{1}{A_j} \leqslant (1+\varepsilon)\frac{1}{A_j(x)}$ for all $j \leqslant i$. We distinguish the three following cases:

- $i \leqslant N - n_V$. We have directly that,

$$\begin{aligned}
\mathrm{Lip}_x(s_i) &\leqslant \lambda^{-i}\left(\mathrm{Lip}(b)\ \mathrm{Lip}(f^i)\sup\tfrac{1}{A_i} + \sup(b\circ f^i)\ \mathrm{Lip}\tfrac{1}{a}\sum_{j=0}^{i}\mathrm{Lip}(f^j)\sup\tfrac{1}{A_{j-1}}\sup\tfrac{A_j}{A_i}\right),\\
&\leqslant \lambda^{-i}B(1+\varepsilon)^2\left(\mathrm{Lip}(b) + \sup|b|\ \mathrm{Lip}\tfrac{1}{a}\sup(a)\sum_{j=0}^{i}\left(\frac{1}{1+\eta}\right)^j\right),\\
&\leqslant C^{st}_{\perp\!\!\perp i,N,x}\,\lambda^{-i}.
\end{aligned}$$

- $N - n_V \leqslant i < N$. Up to multiplying some part of the above estimate by $(\frac{1}{\lambda+\beta_\sigma})^{n_V}$, we have:

$$\mathrm{Lip}_x(s_i) \leqslant C^{st}_{\perp\!\!\perp i,N,x}\,\lambda^{-i}.$$

- $i = l + N \geqslant N$. In this case, the following estimates hold:

$$\begin{aligned}
\mathrm{Lip}_x(b\circ f^{l+N})\sup\frac{1}{A_{l+N}} &\leqslant \mathrm{Lip}(b)\ \mathrm{Lip}(f^N)\sup\frac{1}{A_N}\ \mathrm{Lip}_{f^N(x)}(f^l)\sup\frac{A_N}{A_{l+N}},\\
&\leqslant Lip(b)(1+\varepsilon)^2||\,\mathrm{d}_x f^N||/A_N(x)\max(\lambda^{-1},\lambda+\beta_\sigma+\delta_\sigma)^l\left(\frac{1}{\lambda+\beta_\sigma}\right)^l,\\
&\leqslant C^{st}_{\perp\!\!\perp x,i,N}\ \max\left(\frac{\lambda^{-1}}{\lambda+\beta_\sigma}, 1+\frac{\delta_\sigma}{\lambda+\beta_\sigma}\right)^l.
\end{aligned}$$

$$\begin{aligned}
\sup(b\circ f^{l+N})\ \mathrm{Lip}_x\frac{1}{A_{l+N}} &\leqslant \varepsilon\max(\lambda^{-1},\lambda+\beta_\sigma+\delta_\sigma)^l\ \mathrm{Lip}\tfrac{1}{a}\left(\sum_{j=0}^{N-1} Lip(f^j)\sup\frac{1}{A_{j-1}}\sup\frac{A_j}{A_{l+N}}\right.\\
&\qquad\left. + \sum_{j=0}^{l} Lip(f^N)\ \mathrm{Lip}_{f^N(x)}(f^l)\sup\frac{1}{A_N}\sup(a)\sup\frac{A_N}{A_{l+N}}\right),\\
&\leqslant \varepsilon\max(\lambda^{-1},\lambda+\beta_\sigma+\delta_\sigma)^l\ \mathrm{Lip}\tfrac{1}{a}C^{st}_{\perp\!\!\perp x,i}\left(\frac{1}{\eta} + n_V\left(\frac{1}{\lambda+\beta_\sigma}\right)^{n_V}\right.\\
&\qquad\left. + \sup(a)\sum_{j=0}^{l}\max\left(\frac{\lambda^{-1}}{\lambda+\beta_\sigma}, 1+\frac{\delta_\sigma}{\lambda+\beta_\sigma}\right)^j\right).
\end{aligned}$$

These two bounds are independent of N, hence of x.

By setting $\beta_\sigma \in]-\lambda+\lambda^{-2}, -\lambda-1[$, all the bounds on $\mathrm{Lip}_x(s_i)$ decay geometrically. Hence all partial sums of $\sum s_i$ share a common Lipschitz constant near each point of U, independent of the base-point.

We give now some estimates when $x \in K$. Therefore $f^n(x) \in A_\eta$ for all n. The following estimate holds:

$$\begin{aligned}
\operatorname{Lip}_x(s_i) &\leqslant \lambda^{-i}\left(\operatorname{Lip}(b)\ \operatorname{Lip}(f^i)\sup\frac{1}{A_i} + \sup|b|\ \operatorname{Lip}\left(\frac{1}{a}\right)\sum_{j=0}^{i}\operatorname{Lip}(f^j)\sup\frac{1}{A_{j-1}}\sup\frac{A_j}{A_i}\right),\\
&\leqslant \lambda^{-i}(1+\varepsilon)^2 B\left(\operatorname{Lip}(b) + \sup|b|\sup(a)\ \operatorname{Lip}\left(\frac{1}{a}\right)\sum_{j=0}^{i}\left(\frac{1}{1+\eta}\right)^j\right),\\
&\leqslant C^{st}_{\perp\!\!\!\perp x,i}\,\lambda^{-i}.
\end{aligned}$$

Finally, every partial some of $\sum s_i$ shares a common Lipschitz constant on $S_g \smallsetminus \Sigma$. Therefore v^s is Lipschitz continuous on $S_g \smallsetminus \Sigma$. □

Since v^s is Lipschitz, we can integrate it. Let $(h_t)_t$ be the flow generated by v^s. Some trajectories are not well defined for all times: they start or end at a conical point.

Proposition 3.16. *For all $x \in S_g \smallsetminus \Sigma$ and t for which $h_t(f(x))$ is well defined, f and $(h_t)_t$ satisfy the relation,*

$$f \circ h_{\lambda t}(x) = h_t \circ f(x).$$

Proof. Since $\mathrm{d}_x f(v^s(x)) = \lambda^{-1} v^s(f(x))$, for $x \in S_g \smallsetminus \Sigma$, remark that,

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ h_{\lambda t}(x)) = \mathrm{d}_{h_{\lambda t}(x)} f\left(\frac{\mathrm{d}}{\mathrm{d}t} h_{\lambda t}(x)\right) = \mathrm{d}_{h_{\lambda t}(x)} f(\lambda v^s(h_{\lambda t}(x))) = v^s(f \circ h_{\lambda t}(x)).$$

Therefore the two functions $t \to f(h_{\lambda t}(x))$ and $t \to h_t(f(x))$ solve the same differential problem. Hence $f \circ h_{\lambda t} = h_t \circ f$ for all t where the solution is defined. □

This commutation relation between f and $(h_t)_t$ is a central argument throughout this article. First, it is used to prove that the set K is invariant by the flow. To this end, we have to verify that the flow is complete on K.

Let $\mathcal{F} \coloneqq S_g \smallsetminus (\Sigma \cup \{x \in S_g \smallsetminus \Sigma \mid \forall t \in \mathbb{R},\, h_t(x) \text{ exists}\})$ be the set of points whose trajectory are not well defined for all time. We can fully caracterise this set, this is the subject of the following lemma.

Lemma 3.17. *If $x \in \mathcal{F}$, then there exist $\sigma \in \Sigma$ and $t_0 \in \mathbb{R}$ such that $h_t(x) \to \sigma$ as t tends to t_0*

Proof. By compactness of S_g, up to taking a sub-sequence $(t_n)_n$ that converges to t_0, the limit of $(h_{t_n}(x))_n$ exists. If this limit doesn't belong to Σ, we can extend the solution past t_0. □

Proposition 3.18. *The orbit $\{h_t(x)\}$ of any point x in K is well defined for all time t. Furthermore, for all $t \in \mathbb{R}$, $h_t(K) = K$.*

Proof. We prove that $\mathcal{F} \subset U$. By contradiction, let $x \in \mathcal{F} \cap K$. Let t_0 and σ be as in lemma 3.17. Hence, the smooth curves $f^n \circ h_t(x) : t \in [0, t_o] \to S_g$ join K to Σ and are of length less than $\lambda^{-n} t_0 ||v^s||_\infty$. This contradicts the fact that $\mathrm{d}(K, \Sigma) > 0$.

Since $\mathcal{F} \cap K = \emptyset$, $h_t(x)$ is well defined for all $x \in K$ and all time t. Let $x \in K$. By contradiction, assume there exists t_1 such that $h' t_1(x) \in U$. Therefore $f^n(h_{t_1}(x))$ converges to some σ as n goes to infinity and the curves $f^n \circ h_t(x) : t \in [0, t_1] \to S_g$

joins K to some arbitrarily close point to σ for n large enough. Since such a curve is of length at most $\lambda^{-n}t||v^s||_\infty$, it contradicts $\mathrm{d}(K,\Sigma)>0$. □

3.5. **Some topological properties of K.** From the relationship between f and $(h_t)_t$, we can deduce further topological properties about stable leaves and the set K. We first prove that for each fixed hyperbolic point p_i^σ, its stable leaf coincides with the orbit by $(h_t)_t$ of this point. From this fact and by proposition 3.7, we deduce that K is transverse to any vertical leaf. We then show that K is in fact equal to the closure of the stable leaf of any hyperbolic fixed point p_i^σ, hence is connected. Finally, we prove that f is topologically transitive with respect to the trace topology of S_g on K.

Proposition 3.19. *For all p_i^σ, we have the equality of sets $W^{ss}(p_i^\sigma)=h_{\mathbb{R}}(p_i^\sigma)$.*

Proof. Let $t\in\mathbb{R}$. Hence $f^n(h_t(p_i^\sigma))=h_{\lambda^{-n}t}(p_i^\sigma)$ converges to p_i^σ as n goes to infinity. Hence $h_{\mathbb{R}}(p_i^\sigma)\subset W^{ss}(p_i^\sigma)$. By the relation between f and $(h_t)_t$, we get that $h_{\mathbb{R}}(p_i^\sigma)$ is invariant by f. In the linearisation near p_i^σ given by the Hartman-Grobman theorem, the only invariant part by f corresponds to a small piece γ of the stable leaf of p_i^σ. By invariance of $h_{\mathbb{R}}(p_i^\sigma)$ by f, we get $\gamma\subset h_{\mathbb{R}}(p_i^\sigma)$. Finally, since $W^{ss}(p_i^\sigma)=\bigcup_{n\geqslant 0}f^{-n}(\gamma)$, we get $h_{\mathbb{R}}(p_i^\sigma)=W^{ss}(p_i^\sigma)$. □

Corollary 3.20. *The set K is transverse to any vertical leaf.*

Proof. Since the convergence of the infinite sum defining v^s is uniform on K, the vertical component of the vector field v^s is continuous, hence bounded. Therefore, all the stable leaves $W^{ss}(p_i^\sigma)$ are transverse to any vertical leaf. The result holds by taking the closure since slopes are bounded and by proposition 3.7. □

Theorem 3.21. *The set K is connected and it can be written as $K=\overline{W^{ss}(p_i^\sigma)}$, for any $\sigma\in\Sigma$ and any $1\leqslant i\leqslant 2n_\sigma$.*

Proof. Let $\sigma_1,\sigma_2\in\Sigma$ and i_1,i_2 be two integers. For simplicity, call $p_1=p_{i_1}^{\sigma_1}$ and $p_2=p_{i_2}^{\sigma_2}$. Let W_2 be the open set given by the Hartman-Grobman theorem - without loss of generality we assume it is a rectangle with horizontal and vertical sides. Since $W^{su}(p_2)$ contains a dense vertical leaf, and $W^{ss}(p_1)$ is transverse with all vertical leaves, the intersection $W^{su}(p_2)\cap W^{ss}(p_1)$ is non-empty. Let $x\in W^{su}(p_2)\cap W^{ss}(p_1)$ and let γ be a small connected piece of $W^{ss}(p_1)$ containing x in its interior. Then, for large enough $n\geqslant 0$, we see that $f^{-n}(\gamma)\cap W_2$ accumulates on $W^{ss}(p_2)\cap W_2$. Therefore, $W^{ss}(p_2)\cap W_2\subset\overline{W^{su}(p_2)\cap W^{ss}(p_1)}\subset\overline{W^{ss}(p_1)}$. Since $\overline{W^{ss}(p_1)}$ is invariant by the action of f and $W^{ss}(p_2)=\bigcup\limits_{n\geqslant 0}f^{-n}(W^{ss}(p_2)\cap W_2)$, we get the inclusion $\overline{W^{ss}(p_2)}\subset\overline{W^{ss}(p_1)}$. Since the choice of p_1 and p_2 is arbitrary, the result follows from 3.7. □

Theorem 3.22. *The function $f:K\to K$ is transitive with respect to the trace topology of S_g on K.*

Proof. Let U_1 and U_2 be open sets in S_g that intersect with K. Let $p_1=p_{i_1}^{\sigma_1}$ and $p_2=p_{i_2}^{\sigma_2}$ with $\sigma_1,\sigma_2\in\Sigma$ such that $U_i\cap(W^{ss}(p_i)\cap W^{su}(p_i))\neq\emptyset$ for $i=1,2$. Since $W^{ss}(p_2)$ is transverse with all the vertical leaves, we can find a rectangle contained in U_2 whose sides are vertical and horizontal, such that $W^{ss}(p_2)$ crosses V_2 *from side to side.*

By density of $W^{su}(p_1)$, there exists $x_2 \in V_2 \cap W^{su}(p_1)$. Let W_1 be the open set of linearisation near p_1 - without loss of generality, we can assume W_1 to be a rectangle with horizontal and vertical sides. For large enough $n \geqslant 0$, the set $f^{-n}(V_2)$ crosses *horizontally* W_1.

Let $x_1 \in U_1 \cap W^{ss}(p_1)$ and $\varepsilon > 0$ be such that the vertical segment γ of length ε, containing x_1 in its interior, is contained in U_1. For all large enough $m \geqslant 0$, the line $f^m(\gamma)$ crosses *vertically* W_1. Hence $f^m(U_1) \cap f^{-n}(U_2) \neq \emptyset$. ☐

3.6. **Regularity of v^s with respect to β.** In the next section we prove that $(h_t)_t$ is uniquely ergodic and that f is mixing with respect to the invariant measure of $(h_t)_t$. To do so, we first prove that the family of vector fields v^s is smooth with respect to the amplitude parameter β in the definition of f.

We will use the following notations. For all $\beta = (\beta_\sigma)_{\sigma\in\Sigma}$, write f_β the function f with the amplitude parameter β, and v^s_β its corresponding vector field.

In this section, we only consider the case where $\#\Sigma = 1$, hence the vector β has only one component. The general case leads to very similar computations.

More precisely, we prove the following theorem.

Theorem 3.23. *The function $(\beta, x) \mapsto v^s_\beta(x)$ is continuous on $]-\lambda+\lambda^{-2}, 0] \times (S_g \setminus \Sigma)$.*

To show this continuity, we split the domain into three subset. First we need the following lemma to be true.

Lemma 3.24. *For all β in $]-\lambda+\lambda^{-2}, 0]$, the eigenspace of $(f_\beta)_* := (\mathrm{d}f_\beta)^{-1}\, U_{f_\beta}$ associated with the eigenvalue λ is of dimension one when acting on the space of bounded and continuous vector fields of $S_g \setminus \Sigma$, where U_f stands for the Koopman operator of f.*

Proof. Let $\beta \in]-\lambda+\lambda^{-2}, 0]$. Let w be a vector field in the eigenspace of $(f_\beta)_*$ associated with the eigenvalue λ. In other words, w is such that $\mathrm{d}_x f_\beta(w(x)) = \lambda^{-1} w(f_\beta(x))$, for all x. Now, since v^s is continuous, non vanishing and transverse to e_v, there exist two functions w_1 and w_2 uniquely determined such that $w(x) = w_1(x) v^s(x) + w_2(x) e_v$ for all x. These two functions are bounded and continuous. Hence, we have,

$$\begin{aligned} \mathrm{d}_x f_\beta(w_2(x)e_v) = a(x)w_2(x)e_v &= \mathrm{d}_x f_\beta(w(x) - w_1(x)v^s(x)), \\ &= \lambda^{-1}(w(f_\beta(x)) - w_1(x)v^s(f_\beta(x))), \\ w(f_\beta(x)) &= w_1(x)v^s(f_\beta(x)) + \lambda a(x) w_2(x) e_v. \end{aligned}$$

Therefore, w_1 is invariant by f_β and for all $i > 0$,

$$w_2(x) = \prod_{j=0}^{i-1} \frac{1}{\lambda a(f^j_\beta(x))} w_2(f^i_\beta(x)).$$

By continuity of w_2 and compactness of S_g, w_2 is bounded. Now, we distinguish two cases in order to prove that $w_2 = 0$.

For $\beta_\sigma < 1-\lambda$, there exists a fixed point p^σ_i, in K, whose unstable leaf is dense. Since at this point $a(p^\sigma_i) > 1$, by continuity of a, we get that $a > 1$ in a neighbourhood of p^σ_i, hence $1/(\lambda a) < \lambda^{-1} < 1$ and $w_2 = 0$.

For $1-\lambda \leqslant \beta_\sigma \leqslant 0$, we know that the unstable leaf of σ is dense in S_g. By continuity on every leaf of the branched cover at σ, we can set $a(\sigma) = \lambda + \beta_\sigma \geqslant 1$. Hence, in a neighbourhood of σ, we get $1/(\lambda a) \leqslant \lambda^{-1} < 1$, hence $w_2 = 0$.

In order to prove that w_1 is constant, we also distinguish two cases.

For $\beta_\sigma < 1-\lambda$, the unstable leaf of each p_i^σ is dense. Hence $w_1(x) = w_1(p_i^\sigma)$ for all x. Hence the claim in this case.

For $1-\lambda \leqslant \beta_\sigma \leqslant 0$, the unstable leaf of σ is dense. Therefore, $w_1(x) = w_1(\sigma)$ for all x. Hence the claim. $\square$

Proposition 3.25. *For all $\beta_0 \in]-\lambda+\lambda^{-2}, 1-\lambda[$, we have*

$$||v_\beta^s - v_{\beta_0}^s||_\infty \xrightarrow[\beta\to\beta_0]{} 0$$

Hence, the function $(x,\beta) \mapsto v_\beta^s(x)$ is continuous on $]-\lambda+\lambda^{-2}, 1-\lambda[\times (S_g \smallsetminus \Sigma)$.

Proof. From proofs of theorems 3.12 and 3.15, we can see that on a small enough neighbouhood B_0 of β_0, the vector fields v_β^s are uniformly bounded, as well as their Lipschitz constants. By the Arzela-Ascoli theorem, the set $\{v_\beta^s \mid \beta \in B_0\}$ is relatively compact. Take a sequence of $(\beta_n)_n$ converging to β_0, then every subsequential limit w of $(v_{\beta_n}^s)_n$ must satisfies $(f_{\beta_0})_* w = \lambda w$. By lemma 3.24, the space of such vector fields is one dimensional, hence there exists a constant c such that $w = c v_{\beta_0}^s$. Since in the basis (e_h, e_v) all the component of v_β^s along e_h is 1, we get that $c = 1$. Hence $v_{\beta_n}^s$ converges uniformly to $v_{\beta_0}^s$, and so for all sequence $(\beta_n)_n$. The rest of the claim follows directly by the triangle inequality and Lipschitz continuity. $\square$

Proposition 3.26. *For all $\beta_0 \in [1-\lambda, 0]$, we have*

$$||v_\beta^s - v_{\beta_0}^s||_\infty \xrightarrow[\substack{\beta\to\beta_0 \\ \beta\in[1-\lambda,0]}]{} 0$$

Hence, the function $(x,\beta) \mapsto v_\beta^s(x)$ is continuous on $[1-\lambda, 0] \times (S_g \smallsetminus \Sigma)$.

Proof. The same argument as in the previous proof holds. Indeed, for all $\beta \in [1-\lambda, 0]$ we get

$$\sum_{i\geqslant 0} \left| \lambda^{-i} b_\beta \circ f_\beta^i \prod_{j=0}^{i} \frac{1}{a_\beta \circ f_\beta^j} \right| \leqslant \frac{1}{1-\lambda^{-1}} ||b_\beta||_\infty .$$

Hence v_β^s is uniformly bounded for β in a neighbourhood of β_0. Similarly, the following estimate on the Lipschitz constant holds for all $\varepsilon > 0$

$$\sum_{i\geqslant 0} \mathrm{Lip}_x \left(\lambda^{-i} b_\beta \circ f_\beta^i \prod_{j=0}^{i} \frac{1}{a_\beta \circ f_\beta^j} \right) \leqslant (1+\varepsilon)^2 ||v_\beta^s||_\infty \sum_{i\geqslant 0} \lambda^{-i} \Bigg(\mathrm{Lip}(b_\beta) + i \, \mathrm{Lip}\left(\frac{1}{a_\beta}\right) ||a_\beta||_\infty ||b_\beta||_\infty \Bigg)$$

$\square$

Proposition 3.27. *The function* $(x,\beta) \mapsto v_\beta^s(x)$ *is continuous at each point of* $\{1-\lambda\} \times (S_g \setminus \Sigma)$.

Proof. Recall notations from proposition 3.9 and let V be a neighbourhood of some $\sigma \in \Sigma$ as in the proof of theorem 3.12. Let $x \in f^{-N}(V) \cap U_\sigma$ and let $n(x)$ be the number of points in the orbit of x that belong to $B_{\sigma,\delta} \setminus V$. Then $N - n(x) \geqslant 0$ and we have the following estimates depending on i:

- $i \leqslant N - n(x)$. Therefore $|s_i(x)| \leqslant \lambda^{-i} \left(\frac{1}{1+\eta}\right)^{i+1} \sup |b|$.
- $N - n(x) < i \leqslant N$. Hence $|s_i(x)| \leqslant \lambda^{-i} \left(\frac{1}{1+\eta}\right)^{N-n(x)} \left(\frac{1}{\lambda+\beta}\right)^{i-(N-n(x))} \sup |b|$

so that $|s_i(x)| \leqslant \sup |b| \lambda^{-i} \left(\frac{1}{\lambda+\beta}\right)^{i}$.

- $i = j + N > N$. We have $|s_i(x)| \leqslant \lambda^{-(j+N)} \left(\frac{1}{\lambda+\beta}\right)^{j+N} \mathrm{Lip}(b) \varepsilon \max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma)^j$.

Therefore, $\sum_{i \geqslant 0} |s_i(x)| \leqslant ||b||_\infty \left(\frac{\lambda}{\lambda-1} + \frac{\lambda(\lambda+\beta)}{\lambda(\lambda+\beta)-1} + \varepsilon \frac{\mathrm{Lip}(b)}{1 - \frac{\max(\lambda^{-1}, \lambda+\beta_\sigma+\delta_\sigma)}{\lambda(\lambda+\beta)}} \right)$, and so for all $\varepsilon > 0$.

Hence, the family of vector fields $(v_\beta^s)_\beta$ is uniformly bounded on S_g and the bound can be chosen uniformly in β for $\beta \in [1-\lambda-\varepsilon, 1-\lambda]$. However, the estimates we had on the Lipschitz constants are no longer good enough to apply the same argument as in previous proofs.

Let $x \in S_g$ and $(x_n, \beta_n)_n$ be a sequence converging to $(x, 1-\lambda)$ and such that $\beta_n < 1-\lambda$ for all n. For n large enough, the sequence $(v_{\beta_n}^s(x_n))_n$ is bounded and let $w(x)$ be a sub-sequential limit. Since for all $k \geqslant 0$, the sequence $(v_{\beta_n}^s(f_{\beta_n}^k(x_n)))_n$ is bounded, by a diagonal argument we can assume up to extracting that the sequences converge to some vectors $w(f_{1-\lambda}^k(x))$. By continuity of $\mathrm{d}f_\beta$ in β, we get that $\mathrm{d}_x f_{1-\lambda}^k w(x) = \lambda^{-k} w(f_{1-\lambda}^k(x))$ for all k. By expressing vectors $w(f_{1-\lambda}^k(x))$ in the basis $(v_{1-\lambda}^s(x), e_v)$, we see that $w(x) \in \mathbb{R} v_{1-\lambda}^s(x)$. Since each vector of the form $v_{\beta_n}^s(f_{\beta_n}^k(x_n))$ has a component equal to 1 along e_v in the basis (e_h, e_v), we get $w(x) = v_{1-\lambda}^s(x)$. Hence the continuity at $(x, 1-\lambda)$. $\square$

3.7. **Mixing of f on K and unique ergodicity of $(h_t)_t$.** In this part we prove that there exists a measure with respect to which f is mixing. To do so, we first claim that the unique ergodicity of $(h_t)_t$ is a sufficient condition and that f is mixing with respect to the unique invariant measure of the flow. We then prove that $(h_t)_t$ is indeed uniquely ergodic, with the support of its unique invariant measure being exactly K.

Theorem 3.28. *If* $(h_t)_t$ *is uniquely ergodic, of invariant measure* μ*, and the support of* μ *is included in* K*, then* μ *is invariant by* f *and* f *is mixing with respect to* μ.

Proof. Since K is invariant by f and by the flow $(h_t)_t$ and since $(h_t)_t$ is well defined for all t on K, we have

$$f_*\mu = f_*((h_t)_*\mu) = (f \circ h_t)_*\mu = (h_{\lambda^{-1}t})_*(f_*\mu).$$

Therefore the measure $f_*\mu$ is invariant by the flow $(h_t)_t$. By unique ergodicity of the flow, we must have $f_*\mu = \mu$.

Let $F \in \mathrm{L}^2(\mu)$ be such that $\mu(F) = 0$. We now prove that the sequence $(F \circ f^n)_n$ weakly converges to zero. By invariance of the measure, the sequence is bounded in the $\mathrm{L}^2(\mu)$ norm. By the Banach-Alaoglu-Bourbaki theorem, this sequence lives in a weakly compact set. Let $\bar{F}$ be a sub-sequential weak limit of $(F \circ f^n)_n$ and let $(n_k)_k$ be a strictly increasing sequence of integers such that $F \circ f^{n_k} \xrightarrow[k\to\infty]{} \bar{F}$.

$$\begin{aligned}||F \circ f^{n_k} \circ h_t - F \circ f^{n_k}||_{L^2} &= ||F \circ h_{\lambda^{-n_k} t} \circ f^{n_k} - F \circ f^{n_k}||_{L^2},\\ &= ||F \circ h_{\lambda^{-n_k} t} - F||_{L^2},\\ &\xrightarrow[k\to\infty]{} 0,\end{aligned}$$

where the final limit follows from the density of continuous functions in $\mathrm{L}^2(\mu)$. Now, $F \circ f^{n_k} \circ h_t - F \circ f^{n_k}$ converges weakly to $\bar{F} \circ h_t - \bar{F}$. The identification of the strong limit with the weak limit gives $\bar{F} \circ h_t - \bar{F} = 0$. By unique ergodicity of $(h_t)_t$, $\bar{F}$ is constant. By integration, this constant is zero. Hence all the sub-sequential weak limit of $(F \circ f^n)_n$ are 0, which proves the mixing. □

Theorem 3.29. *The flow $(h_t)_t$ is uniquely ergodic. Furthermore the support of the invariant measure is K.*

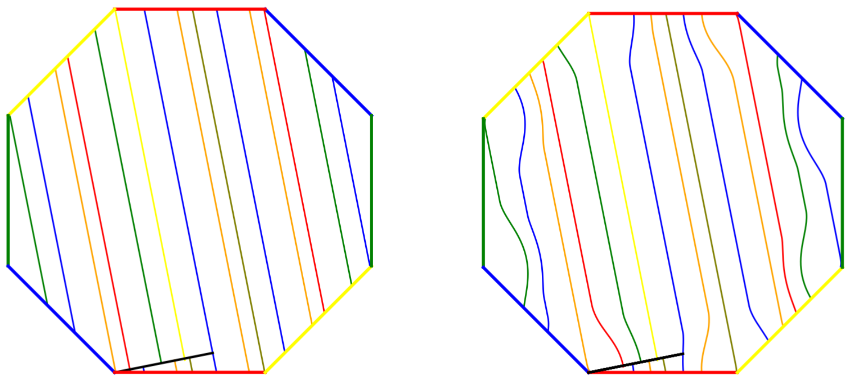
In order to prove this result, we heavily rely on the semi-conjugacy theorem from [10], more precisely if an Interval Exchange Transformation (IET) and a Generalized Interval Exchange Transformation (GIET) have the same combinatorial datum and follow the same full path in the Rauzy diagram when renormalized by the Rauzy-Veech algorithm, then there exists a continuous, increasing and surjective function that semi-conjugates the two transformations.

3.7.1. *Construction of a GIET and $(h_t)_t$ as its suspension.* Recall some notation from 2.1. Let φ be the pseudo-Anosov map that we perturbed in order to get f. By construction, φ fixes each conical point and each separatrix. Let $\sigma \in \Sigma$ be a conical point and γ_0 be a segment of a vertical separatrix starting at σ such that $\sigma \in \partial\gamma_0$. From a general property of the pseudo-Anosov maps (see [6, proposition 5.3.4]), there exists a decomposition in rectangles $\mathcal{R}_0 = (R_1^0, \dots, R_{|\sigma|}^0)$ of S such that (up to shortening γ_0) the *bases* of these rectangles form a partition of γ_0.

Denote by $\partial_v \mathcal{R}_0$ (resp. $\partial_h \mathcal{R}_0$) the vertical (resp. horizontal) components of $\bigcup \partial R_i$. By construction, $\partial_h \mathcal{R}_0 = \gamma$. Now, $\partial_v \mathcal{R}_0$ is made of portions of trajectories for the horizontal flow associate to φ that connect a conical point to γ_0, but don't intersect γ_0 at some other previous time.

Since the family of vector fields $(x, \beta) \mapsto v^s_\beta(x)$ is continuous, we can deform by some homotopy $\mathcal{R}_0$ to $\mathcal{R}_\beta = (R_0^\beta, \dots, R_{|\Sigma|}^\beta)$ while preserving the vertical direction, where β is the amplitude of the perturbations in the construction of f. In more details, the homotopy sends the portions of trajectories of the horizontal flow that connect conical points to γ_0, to the portions of trajectories of (h_t) which contain a conical point. Since the vector field (v^s_β) has its horizontal component constant equal to 1, these latter trajectories are the ones connecting conical points to γ, where γ is a slightly longer or shorter version of γ_0. Since any two trajectories do not intersect, these portions of trajectories of (h_t) are still the shortest ones that connect conical points to γ

Call T (resp. T_0) the Poincar first return map to γ (resp γ_0) of the flow $(h_t)_t$ (resp. of the horizontal flow associate to φ). It is clear from the construction that



Rectangle decomposition $\mathcal{R}_0$ Perturbed rectangle decomposition $\mathcal{R}_\beta$

FIGURE 2. Rectangle decompositions in the case of a flat genus two surface. The pseudo-Anosov transformation on this surface is explicited in the appendix of [8] as the composition of an upper triangular matrix with its transpose matrix.

T_0 is an IET and that T is a GIET. Since for all β, the horizontal component of (v^s_β) is equal to one, both T and T_0 have the same suspension data. Furthermore, by construction, T and T_0 have the same path in the Rauzy-graph – otherwise for some parameter β^* the GIET T_{β^*} induced by $\mathcal{R}_{\beta^*}$ would have a connection, which corresponds geometrically to a side of a rectangle of $\mathcal{R}_{\beta^*}$ connecting a conical point to another one: this is impossible since f_{β^*} would contract this curve.

Since foliations associated to a pseudo-Anosov have no closed leaf (see [2]), it follows that T_0 has no connection, hence, by [10], the path of T_0 in the Rauzy graph is full and so T_0 and T are semi-conjugated by a continuous, increasing and surjective function. Also, since T_0 has no connection, it is minimal.

We summarize all this in the following proposition.

Proposition 3.30. *If σ is a conical point, there exist two portions of a same separatrix (both containing σ) γ_0 and γ, and maps $T : \gamma \to \gamma$, $T_0 : \gamma_0 \to \gamma_0$ such that:*

- *T_0 is an IET and T is a GIET,*
- *T_0 is the Poincar first return map of the horizontal flow associated to φ,*
- *T is the Poincar first return map of the flow (h_t) associated to f,*
- *T_0 and T have the same suspension data, and the same path in the Rauzy-graph,*
- *there exists a continuous, increasing and surjective function h such that $h \circ T = T_0 \circ h$.*

3.7.2. *Minimality of the flow on K.* In this part, using the fact $(h_t)_t$ can be recovered as a suspension of the map T and the analysis carried out in [10], we prove that $(h_t)_t$ is minimal on K by proving that T is minimal on the intersection of its domain with K. From this property, it will easily follows that the support of the unique invariant measure of $(h_t)_t$ is K. In fact, we prove a slightly stronger result: the set K is an attractor for $(h_t)_t$ for positive and negative times.

As in [10], define $S(\infty)$ as the union of the forward orbit of the discontinuity points of T^{-1} and the backward orbits of the discontinuity points of T. Simarly, define $S_0(\infty)$ from the discontinuities of T_0 and T_0^{-1}. By construction, h is an increasing bijection from $S(\infty)$ to $S_0(\infty)$.

Define Ω as the set of non-isolated points of $\overline{S(\infty)}$. Clearly, Ω is a closed set. In order to prove that T is minimal on Ω, we first need the following lemma.

Lemma 3.31. *There exists a decomposition of Ω in closed sets $\Omega = \Omega_+ \cup \Omega_-$ such that $T(\Omega_+) \subset \Omega_+$ and $T^{-1}(\Omega_-) \subset \Omega_-$.*

Proof. Let $S(\infty)_+$ be the forward orbits by T of the discontinuity points of T^{-1} and similarly $S(\infty)_-$ be the set of the backward orbits by T of the discontinuity points of T. By definition of $S(\infty)$, $\overline{S(\infty)} = \overline{S(\infty)_+} \cup \overline{S(\infty)_+}$. Define $\Omega_\pm$ as the set of non-isolated points of $\overline{S(\infty)_\pm}$. These sets satisfy the conclusion of the lemma. □

Theorem 3.32. *When restricted to the set Ω, T is minimal.*

Proof. Let x be a point of Ω. Up to considering its backward orbit, we assume that $x \in \Omega_+$. We want to prove that $(T^n(x))_{n\geqslant 0}$ is dense in Ω. By contraction, let U be an open set such that $U \cap \Omega \neq \emptyset$ and $T^n(x) \notin U \cap \Omega$ for all n. Since Ω_+ is stable by the action of T, we can relax the last condition by $T^n(x) \notin U$ for all n.

Since $U \cap \Omega \neq \emptyset$, $U \cap \Omega$ contains at least two different points of $S(\infty)$, therefore h is not constant on U. Hence $h(U)$ has a non-empty interior. Finally, since the sequence $h \circ T^n(x) = T_0^n(h(x))$ avoids an open set and T_0 is minimal, we get a contradiction. □

In order to prove that Ω is an attractor for both T and T^{-1}, we need the following three technical lemmas.

Lemma 3.33. *The function h such that $h \circ T = T_0 \circ h$ is constant on the connected components of $\gamma \smallsetminus \overline{S(\infty)}$.*

Proof. By contradiction, let $]j_-, j_+[$ be a connected component of $\gamma \smallsetminus \overline{S(\infty)}$ on which h is not constant. Therefore $h(j_-) < h(j_+)$. By density of $S_0(\infty)$ in γ_0, there exist infinitely many points of $S_0(\infty)$ in the middle third segment of $[h(j_-), h(j_+)]$. Since $h : S(\infty) \to S_0(\infty)$ is a bijection, the image by h^{-1} of all these points of $S_0(\infty)$ is relatively compact in $]j_-, j_+[$. Hence, there exist accumulation points of $S(\infty)$ in $]j_-, j_+[$, which is a contradiction. □

Lemma 3.34. *The connected component of $\gamma \smallsetminus \overline{S(\infty)}$ are permuted without cycle by T.*

Proof. By construction of $S(\infty)$, T and T^{-1} are continuous on each connected componant of $\gamma \smallsetminus \overline{S(\infty)}$. If J is a connected componant of $\gamma \smallsetminus \overline{S(\infty)}$, then it is easy to see that $T(J)$ is a subset of a connected componant of $\gamma \smallsetminus \overline{S(\infty)}$. The same argument applied with T^{-1} proves that the connected componants are permuted by the action of T.

By contradiction, let J be a connected component of $\gamma \smallsetminus \overline{S(\infty)}$ and $n > 0$ be such that $T^n J = J$. Therefore $h \circ T^n(J) = h(J) = \{x\}$ by the lemma 3.33. Now $h \circ T^n(J) = T_0^n(h(J))$. Therefore x is a periodic point for T_0, which is a contradiction by minimality of T_0. □

Lemma 3.35. *The isolated points of $\overline{S(\infty)}$ are wandering points.*

Proof. Let x be an isolated point of $\overline{S(\infty)}$. Therefore there exists an open set U such that $U \cap \overline{S(\infty)} = \{x\}$. Hence $U \smallsetminus \{x\} = U_1 \sqcup U_2$ is included in the union of two connected components of $\gamma \smallsetminus \overline{S(\infty)}$, which are wandering sets by lemma 3.34. Therefore, $T^n(U \smallsetminus \{x\}) \cap U \neq \emptyset$ for only finitely many values of n. Now, if $T^n(x) \in U$ then $T^n(x) = x$ and therefore $h(x)$ is a periodic point of T_0 which is impossible. Finally, we proved that $T^n U \cap U \neq \emptyset$ for only finitely many values of n, in other words x is a wandering point. □

Theorem 3.36. *For every point $x \in \gamma$ whose forward orbit is infinite, then the ω-limit set of x satisfies $\omega(x) = \Omega$. The counterpart is true infinite backward orbits and α-limit sets. In other words, Ω is an attractor for the transformation T, and similarly for T^{-1}.*

Proof. We prove both inclusions. We start by showing that $\Omega \subset \omega(x)$. By contradiction, let $y \in \Omega$ such that $y \notin \omega(x)$. Since $\omega(x)$ is a closed set, there exists an open set U containing y such that $U \cap \Omega \neq \emptyset$ and $U \cap \omega(x) = \emptyset$. Therefore $T^n(x) \notin U$ for large enough n. Since $U \cap \Omega \neq \emptyset$, U contains at least two distinct points of $S(\infty)$. Since h is one-to-one on $S(\infty)$ and continuous on γ, the set $h(U)$ has a non-empty interior. Therefore the sequence $T_0^n(h(x)) = h \circ T^n(x)$ is dense in γ_0 (by minimality of T_0) and avoids the set of non-empty interior $h(U)$, hence a contradiction.

We now prove that $\Omega^c \subset \omega(x)^c$. Let y be in Ω^c. There are two cases. If $y \in \gamma \smallsetminus \overline{S(\infty)}$, then by lemma 3.34 y is contained in a wandering interval: y can not be obtain as a limit point of an orbit by T, hence $y \notin \omega(x)$. Otherwise, y is an isolated point of $\overline{S(\infty)}$. By contradiction, $y \in \omega(x)$ implies that y is a non-wandering point, which contradicts lemma 3.35. □

Theorem 3.37. *The set Ω coincide with the set $\Omega(T)$ of non-wandering points for T.*

Proof. By minimality of T when restricted to Ω, we get $\Omega \subset \Omega(T)$. Since T permutes the connected components of $\gamma \smallsetminus \overline{S(\infty)}$, all points of $\gamma \smallsetminus \overline{S(\infty)}$ are wandering points. Therefore $\Omega(T) \subset \overline{S(\infty)}$. Finally, by the lemma 3.35 we can refined this last inclusion by $\Omega(T) \subset \Omega$. □

Proposition 3.38. *The sets Ω and K are related by $\Omega = \gamma \cap K$.*

Proof. Let $p = p_i^\sigma$ be in $\gamma \cap K$. We know that $h_{\mathbb{R}}(p)$ is dense in K, therefore $(T^n(p))_n$ is dense in $\gamma \cap K$. However, $\omega_T(x) = \Omega$ for all $x \in \gamma$, in particular for $x = p$. Hence $\Omega = \gamma \cap K$. □

Theorem 3.39. *When restricted to K, the flow $(h_t)_t$ is minimal. Furthermore, the set K is an attractor for the flow $(h_t)_t$, for positive and negative times.*

Proof. Let $u : \gamma \to \mathbb{R}$ be the function giving the first return time in γ. This function is bounded by some constant C. Clearly, we have the equality $h_{\mathbb{R}}(\Omega) = h_{[0,C]}(\Omega)$ and the left hand side is a closed set containing the orbit of $p = p_i^\sigma \in \gamma \cap K = \Omega$, hence $h_{[0,C]}(\Omega) = K$. This last equality proves the minimality of $(h_t)_t$ when restricted to K.

From $h_{[0,C]}(\Omega) = K$ and 3.36, we obtain that every infinite forward trajectory of $(h_t)_t$ accumulates on K. Similarly, every infinite backward trajectory of $(h_t)_t$ accumulates on K. □

3.7.3. *Proof of the unique ergodicity of* $(h_t)_t$.

Lemma 3.40. *In the coordinates of the suspension, every $(h_t)_t$-invariant measure μ must be of the form $\mathrm{d}\mu(x,t) = C\,\mathrm{d}\nu(x)\,\mathrm{d}Leb(t)$, for $x \in \gamma$, $0 \leqslant t < u(x)$ and some constant $C > 0$, where $u(x)$ is the time of first return to γ of x and Leb is the Lebesgue mesure.*

Proof. Let $\tilde{\pi} : \gamma \times \mathbb{R} \to \mathcal{R}$ be a covering map. The lift of $(h_t)_t$ is simply the unit speed translation flow along the second coordinate. Let μ be an invariant measure for this flow. Let $\tilde{\mu}$ be a lift of μ to $\gamma \times \mathbb{R}$. Therefore $\tilde{\mu}$ is invariant by translation along the second coordinate. Hence $\tilde{\mu} = C\nu \otimes Leb$, where Leb is the Lebesgue measure and $\nu(S) := \tilde{\mu}(S \times [0,\varepsilon])$ is a measure on γ, for some $\varepsilon > 0$. Taking back the projection by $\tilde{\pi}$, we get $\mathrm{d}\mu(x,t) = C\,\mathrm{d}\nu(x)\,\mathrm{d}Leb(t)$, as long as $\varepsilon < \inf_x u(x)$. □

We can now prove the unique ergodicity of $(h_t)_t$.

Proof of theorem 3.29. Let μ be a measure invariant by the flow $(h_t)_t$. By the lemma 3.40, we can find a constant C and a measure ν on γ such that $\mathrm{d}\mu(x,t) = C\,\mathrm{d}\nu(x)\,\mathrm{d}Leb(t)$. By applying Fubini's theorem on sufficiently small rectangles, we obtain that ν is invariant by T.

Since the horizontal foliation associated to a pseudo-Anosov map is uniquely ergodic – see [3, Expos 12] – it follows that T_0 is uniquely ergodic.

Now, T and T_0 have the same path in the Rauzy-graph. By [10], T is semi-conjugated to T_0 by some continuous monotonic function h. This function h is bijective when restricted, up to a countable set of points, to the set of non-wandering points of T. Therefore T is also uniquely ergodic, of invariant measure ν.

Hence (h_t) is uniquely ergodic, of invariant measure μ.

We now prove that the support of μ is K. First, since $supp(\nu)$ is included in the set of non-wandering points of T, which is Ω, and $supp(\nu)$ is a closed set, by minimality of T we get that $supp(\nu) = \Omega$. Now, by the factorization of μ and the fact that $h_{\mathbb{R}}(\Omega) = K$, we get $supp(\mu) = K$. □

As a final remark for this section, we can perform a similar analysis by pertubing a pseudo-Anosov only at some conical points $\Sigma_0 \subsetneq \Sigma$. The proofs are mostly the same by replacing Σ by Σ_0.

We give in figure 3 a graphical representation of the set K in the case of the fully explit example outlined in the description of figure 2.

4. Perturbation at a regular periodic point

Because of the following general property concerning pseudo-Anosov maps – see for example [2] – we can consider periodic points that are not conical points – they are regular points.

Proposition 4.1. *If $\varphi : S_g \to S_g$, then the set of periodic points of φ is dense in S_g*

Let $\theta \in S_g \setminus \Sigma$ be a periodic point of φ that is not a conical point. Up to considering a power of φ, we assume that θ is a fixed point.

In this part, we present that a very similar analysis can be done when a pseudo-Anosov map is perturbed at a periodic point that is regular instead of conical.

4.1. **Definitions, regularity and first properties.** We can proceed to the same type of perturbation as described in sections above at the point θ, except that it is much easier to define since θ is not a conical point and we do not have to deal with branched cover.

Write $\varphi(x+iy) = \lambda x + i\lambda^{-1}y$ in some local chart centred at θ. In these coordinates, define

$$f(x+iy) \coloneqq \left(\lambda + \beta k\left(\frac{x^2+y^2}{\alpha}\right)\right)x + i\lambda^{-1}y,$$

for some $\beta \in]-\lambda, -\lambda+1[$, $0 < \alpha < \min\left(\frac{1}{2}Syst(S_g), \inf\{\mathrm{d}(\theta,\sigma) \mid \sigma \in \Sigma\}\right)$ and $k : \mathbb{R} \to \mathbb{R}$ is an even map of class $\mathcal{C}^1$, compactly supported in $[-1,1]$ with $k' < 0$ on $]0,\infty[\cap\{k>0\}$, for example $k(r) = (1-r^2)^2\mathbb{1}_{[-1,1]}$. Set $f = \varphi$ elsewhere. With these parameters, f is regular in the following way.

Proposition 4.2. *If $\beta \in]-\lambda, 0]$ and $0 < \alpha < \min\left(\frac{1}{2}Syst(S_g), \inf\{\mathrm{d}(\theta,\sigma) \mid \sigma \in \Sigma\}\right)$, then f is a homeomorphism on S_g and is a diffeomorphism on $S_g \smallsetminus \Sigma$.*

Also, for a refined condition on β, we get

Proposition 4.3. *If $\beta \in]-\lambda, -\lambda+1[$ and $0 < \alpha < \min\left(\frac{1}{2}Syst(S_g), \inf\{\mathrm{d}(\theta,\sigma) \mid \sigma \in \Sigma\}\right)$, then θ is an attractive fixed point for f. Call U_θ its basin of attraction. Moreover U_θ is an open set.*

Definition 4.4. *Call $K \coloneqq S_g \smallsetminus U_\theta$ the complement of the basin of attraction of θ. Clearly, K is a compact subset, invariant by f.*

Our goal is to understand the dynamical behaviour of f on K – and near it. First we need to give some more topological properties about the set K. The next property also shows that K is not the empty set.

Proposition 4.5. *If $\beta \in]-\lambda, 1-\lambda[$ and $\alpha < \delta_\Sigma/2$, then there exist fixed hyperbolic points p_i, $i \in \{1,2\}$, one on each vertical ray starting at θ. These two points are at the same distance $|p|$ from θ. We have $B(\theta,|p|) \subset U_\theta$.*

All the proofs of these properties are essentially the same as in subsection 3.1. In fact all the following properties are proved by very similar arguments – if not the same – as their counterparts in the previous case of a perturbation at a conical point.

Proposition 4.6.
- *the open set U_θ is dense in S_g.*
- *The set K is hyperbolic. The stable foliation v^s has exactly the same expression as in the case of section 3.*
- *The formula giving v^s on K still makes sense on $S_g \setminus \Sigma$, and gives a bounded Lipschitz continuous vector field, still noted v^s, when $\beta \in]-\lambda+\lambda^{-2}, 0]$.*
- *The flow $(h_t)_t$ generated by v^s satisfies $f \circ h_{\lambda t} = h_t \circ f$ whenever both sides are well defined.*
- *The set K is invariant by $(h_t)_t$.*
- *The set K is the closure of the trajectory of p_i, $i \in \{1,2\}$, for $(h_t)_t$, in fact $K = \overline{W^{ss}(p_i) \cap W^{su}(p_i)}$. Furthermore $W^{ss}(p_i) = h_{\mathbb{R}}(p_i)$, in particular K is connected.*

4.2. **Finer properties about dynamics of f and $(h_t)_t$.** Again, with almost the same proof, we can prove that

Proposition 4.7. *The vector field $v^s = v^s_\beta$ depends on β since $f = f_\beta$ does. Furthermore, the map $(x, \beta) \mapsto v^s_\beta(x)$ is continuous on $(S_g \setminus \Sigma) \times]-\lambda+\lambda^{-2}, 0]$.*

From this last property, we can construct a rectangle decomposition of S_g in a similar manner as previously. This time, the segment γ will start at θ. In order to construct such a decomposition $\mathcal{R}$, we start from a decomposition $\mathcal{R}_0$ – with true rectangles this time – associated to the horizontal flow and to a segment γ_0 starting at θ and included in a vertical leaf. To get $\mathcal{R}$ we deform $\mathcal{R}_0$ as described in 3.7.1. These decompositions lead to the following proposition.

Proposition 4.8.
- *The flow $(h_t)_t$ induces a map $T : \gamma \to \gamma$, which is the Poincar first return map of this flow. The map T is a GIET. By construction, $(h_t)_t$ can be recovered by taking a suspension flow over T.*
- *The horizontal (unit speed) flow induces a map $T_0 : \gamma_0 \to \gamma_0$, which is the Poincar first return map of this flow. The map T_0 is an IET. By construction, the horizontal flow can be recovered by taking a suspension flow over T_0.*
- *The maps T and T_0 have the same path in the Rauzy diagram. Furthermore this path is full. Hence T is semi-conjugated to T_0.*

In a very similar fashion as in subsections 3.7.2 and 3.7.3, since T_0 is minimal and uniquely ergodic, we can prove the unique ergodicity of $(h_t)_t$ and its minimality when restricted to K. We sum up these results in the following theorem.

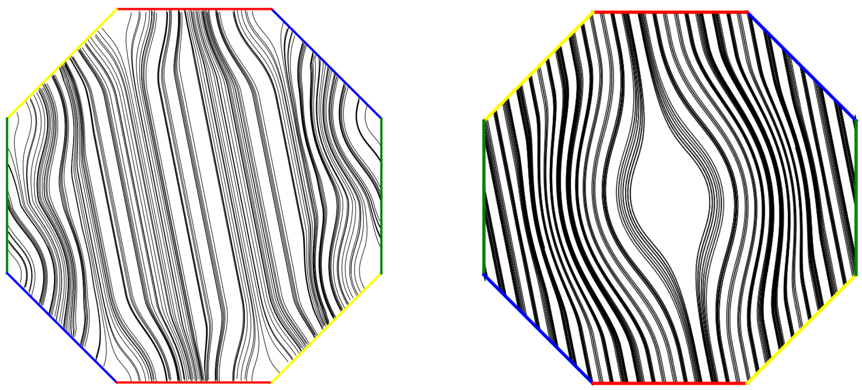


FIGURE 3. Numerical representation of the set K for a perturbation of a pseudo-Anosov homeomorphism on a genus two surface. RIGHT: pertubation at the unique conical point. LEFT: perturbation at a regular fixed point.

Theorem 4.9.
- *For every x in $S_g \setminus \Sigma$ such that its forward trajectory by $(h_t)_t$ is defined for all times, its ω-limit set coincide with K, $\omega(x) = K$. The same goes for backward trajectories and α-limit sets.*
- *The flow $(h_t)_t$ is uniquely ergodic, of unique invariant measure noted μ. By uniqueness and the commutation property between $(h_t)_t$ and f, μ is also invariant by f.*
- *The map f is mixing with respect to μ.*
- *The support of μ is exactly K.*

5. Open questions about the invariant measure μ

In this last part, we share some insights and possible continuations in the study of these derived from pseudo-Anosov maps. In particular, we suspect the measure μ to be the SRB measure of the system, and also the mixing with respect to the measure μ to be exponentially fast.

5.1. **Rate of mixing.** Proving that a map is mixing already tells quite a lot about the chaotic behaviour of such a map. However, it is possible to be more quantitative about this property by considering the quantity

$$C_{fg}(n) \coloneqq \int f \circ T^n g \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu \int g \, \mathrm{d}\mu,$$

where μ is an invariant measure of T and f and g are observables. This quantity $C_{fg}(n)$ is called the correlation function of (T, μ) for the observables f and g. The rate of convergence to zero of the correlation function depends on the choice of observables. For some observables, this convergence might be extremely slow.

In [4], Faure, Gouzel and Lanneau proved that for any pseudo-Anosov map, the correlation function decreases exponentially fast for some class of observable functions.

Since, until now, most of the properties of the pseudo-Anosov maps were found to be true for the perturbed pseudo-Anosov maps, it seems plausible that perturbed pseudo-Anosov transformations also have the property of exponentially fast mixing.

5.2. **SRB measure for f.** Among the many equivalent definitions of a SRB measure, it seems that the invariant measure μ for the perturbed pseudo-Anosov transformation f is the SRB measure of f^{-1}.

In [11], Young defines the SRB measure in the case of a $\mathcal{C}^2$ diffeomorphism with an Axiom A attractor as a measure having absolutely continuous conditional measures on unstable manifolds. This property is satisfied by μ, however f is only a diffeomorphism of class $\mathcal{C}^1$ outside of the singularities.

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