Introducing continued fractions

Donald Knuth offers a start to this rich subject in *Seminumerial Algorithms*. The general form of a continued fraction is

$$ \frac {b_1} {a_1 + \frac {b_2} {a_2 + \frac {b_3} {a_3}}} = b_1 / (a_1 + b_2 / (a_2 + b_3 / a3)) \ \ . $$

We'll look at continued fractions in which all the $ b $'s are one. A convenient notation is

$$ \newcommand{\sslash}{\mathbin{/\mkern-6mu/}} \sslash x_1, x_2, x_3, \ldots, x_n \sslash = \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{ \cdots \frac{\rule{0in}{.1in}}{x_{n-1} + \frac{1}{x_n}}}}} \ \ . $$

Sampler

One reason to study continued fractions is that they are beautiful expressions. This sampler is from Knuth and *The Handbook of Mathematical Functions*, usually known by its authors, Abramowitz & Stegun.

$$ \phi = \frac{1 + \sqrt{5}}{2} = 1 + \sslash 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \sslash $$
$$ e = 2 + \sslash 1, 2, 1, 1, 41, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots \sslash $$
$$ \pi = 3 + \sslash 7, 15, 1, 192, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, \ldots \sslash $$
$$ tan(1) = \sslash 1, -3, 5, -7, 9, -11, 13, \ldots \sslash $$
$$ ln(1 + z) = \frac {z} {1 + \frac {z} {2 + \frac {z} {3 + \frac {4 z} {4 + \frac {4 z} {5 + \frac {9 z} {6 + \cdots}}}}}} $$
$$ e^z = \frac {1} {1 - \frac {z} {1 + \frac {z} {2 - \frac {z} {3 + \frac {z} {2 - \frac {z} {5 + \frac {z} {2 - \cdots}}}}}}} $$

Continuants

Euler and others investigated the useful *continuant polynomials*:

\[K_{n}(x_1, x_2, \ldots, x_n) = \left\{ \begin{array}{ll} 1 & \mbox{if $n = 0$} \\ x_1 & \mbox{if $n = 1$} \\ {x_1 K_{n-1}(x_2, x_3, \ldots, x_n) + K_{n-2}(x_3, x_4, \ldots, x_n)} & \mbox{if $n > 1$} \end{array} \right. \]

Here are the first several:

\[\begin{array}{ll} K_0 & 1 \\ K_1(x_1) & x_1 \\ K_2(x_1, x_2) & x_1 x_2 + 1 \\ K_3(x_1, x_2, x_3) & x_1 x_2 x_3 + x_1 + x_3 \\ K_4(x_1, x_2, x_3, x_4) & x_1 x_2 x_3 x_4 + x_1 x_2 + x_1 x_4 + x_2 x _3 + 1 \\ K_5(x_1, x_2, x_3, x_4, x_5) & \begin{array}{l} x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 + x_1 x_2 x_5 + x_1 x _4 x_5 + \\ \ \ \ x_3 x_4 x_5 + x_1 + x_3 + x_5 \end{array} \end{array} \]

Fun facts about continuants:

- \(K_n(x_1, \ldots, x_n) \) is the sum of all terms starting with \(x_1 x_2 \ldots x_n \) and then deleting nonoverlapping pairs of consecutive variables \(x_j x_{j+1} \).
- Just the \(K_{2k} \) have a term \(1 \).
- \(K_n(x_1, x_2, \ldots, x_n) = K_n(x_n, x_{n-1}, \ldots, x_1) \).
- The number of terms in \(K_n(x_1, \ldots, x_n)\) is \(F_{n+1} \) from the Fibonacci sequence \(0, 1, 1, 2, 3, 5, \ldots \).

Continued fractions are quotients of K-polynomials: \[\sslash x_1, x_2, \ldots, x_n \sslash = \frac{K_{n-1}(x_2, x_3, \ldots, x_n)} {K_n(x_1, x_2, \ldots, x_n)} = \frac{1} {\frac{K_n(x_1, \ldots, x_n)} {K_{n-1}(x_2, \ldots, x_n)}} \] To see this, expand the denominator: \[\frac{x_1 K_{n-1}(x_2, \ldots, x_n) + K_{n-2}(x_3, \ldots, x_n)} {K_{n-1}(x_2, \ldots, x_n)} \] The right-hand side of the formula above is thus \[\frac{1} {x_1 + \frac{K_{n-2}(x_3, \ldots, x_n)} {K_{n-1}(x_2, \ldots, x_n)}} \] which leads by induction to \[\frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{ \cdots \frac{\rule{0in}{.1in}}{x_{n-1} + \frac{1}{x_n}}}}} = \sslash x_1, x_2, \ldots, x_n \sslash \ \ . \]

This identity

$$ K_n(x_1, \ldots, x_n) K_n(x_2, \ldots, x_{n+1}) - K_{n+1}(x_1, \ldots, x_{n+1}) K_{n-1}(x_2, \ldots, x_n) = (-1)^{n} $$

for $n \geq 1$ is very useful. To verify it by induction, advance to step $n+1$. First expand the left-hand term to

$$ (x_1 K_n(x_2, \ldots, x_{n+1}) + K_{n-1}(x_3, \ldots, x_{n+1})) K_{n+1}(x_2, \ldots x_{n+2}) \ \ . $$

Then expand the right-hand term to

$$ (x_1 K_{n+1}(x_2, \ldots, x_{n+2}) + K_{n}(x_3, \ldots, x_{n+2})) K_{n}(x_2, \ldots x_{n+1}) \ \ . $$

The terms with factor x_1 cancel, leaving

$$ K_{n-1}(x_3, \ldots, x_{n+1}) K_{n+1}(x_2, \ldots x_{n+2}) - K_{n}(x_3, \ldots, x_{n+2}) K_{n}(x_2, \ldots x_{n+1}) = (-1)^{n+1} $$

by the assumption for step n.

Continued fractions and continuants

We can now make the remarkable connection between the K-polynomials and continued fractions:

$$ \sslash x_1, x_2, \ldots, x_n \sslash = \frac{1}{q_0 q_1} -\frac{1}{q_1 q_2} + \frac{1}{q_2 q_3} + \cdots + \frac{(-1)^{n-1}}{q_{n-1}q_{n}} $$

where $q_k = K_k(x_1, \ldots, x_k)$.

It's just a bit more K-polynomial algebra. Start with the continued fraction as a quotient of continuants

$$ \sslash x_1, \ldots, x_n \sslash = \frac{K_{n-1}(x_2, \ldots, x_n) K_{n-1}(x_1, \ldots, x_{n-1}) }{K_n(x_1, \ldots, x_n) K_{n-1}(x_1, \ldots, x_{n-1})} $$

with the extra terms chosen in order to exploit the identity of the previous section.

Rewriting the numerator leads to

$$ \frac{ (-1)^{n-1} + K_{n}(x_1, \ldots, x_{n}) K_{n-2}(x_2, \ldots, x_{n-1}) }{K_n(x_1, \ldots, x_n) K_{n-1}(x_1, \ldots, x_{n-1})} = \frac{K_{n-2}(x_2, \ldots, x_{n-1})}{K_{n-1}(x_1, \ldots, x_{n-1})} + \frac{(-1)^{n-1}}{q_{n-1}q_{n}} $$

which is the inductive step.

Regular continued fractions

Every real number X with $0 \leq X < 1$ has a *regular continued fraction* defined by this process. Set $X = X_0$, and for every $n \geq 0$, if $X_n \neq 0$

$$ A_{n+1} = \lfloor 1 / X_n \rfloor \ , \ \ \ \ X_{n+1} = 1 / X_n - A_{n+1} $$

If $X_n = 0$ the process stops and $X = \sslash A_1, A_2, \ldots, A_n \sslash$.

If $X_n \neq 0$ then $0 \leq X_{n+1} < 1$, so all the A's are positive integers. The definiton above exapnds to $$X = X_0 = \frac{1}{A_1 + X_1} = \frac{1}{A_1 + \frac{1}{A_2 + X_2}} = \cdots$$ so $$X = \sslash A_1, A_2, \ldots, A_n + X_n \sslash$$ for all $n \geq 1$, whenever X_n is defined.

Because $K_n(A_1, \ldots, A_{n-1}, A_{n} + Y)$ is monotoinc in Y, X lies between $\sslash A_1, \ldots, A_n \sslash$ and $\sslash A_1, \ldots, A_n + 1 \sslash$. The alternating signs in the identity of the last section suggest that successive approximations approach X from above and below, depending on whether n is odd or even.

The A's are called the *partial quotients* of X.

The accuracy of approximatioin

Regular continued fractions approach their target quickly. To see this, consider $$\begin{align} | X - \sslash A_1, \ldots, A_n \sslash | & = | \sslash A_1, \ldots, A_n + X_n \sslash - \sslash A_1, \ldots, A_n \sslash | \\ & = | \sslash A_1, \ldots, A_n, 1 / X_n \sslash - \sslash A_1, \ldots, A_n \sslash | \\ & = 1 / (K_n(A_1, \ldots, A_n) K_{n+1}(A_1, \ldots, A_n, 1 / X_n)) \\ & \leq 1 / (K_n(A_1, \ldots, A_n) K_{n+1}(A_1, \ldots, A_n, A_{n+1})) \end{align}$$ with the usual algebra applied in order to achieve a numerator of ± 1 and the common denominator shown. The inequality arises because $A_{n+1} = \lfloor 1 / X_n \rfloor$ and K is monotonic in each of its parameters.

Next (lib_cf_fractions.html)

Home (aaacontinued_fractions_toc.html)

Previous (aaacontinued_fractions_toc.html)