Periodic Integrals vs. Prohibition of $\tan(\pi/2) = \infty$.

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Abstract: Attempts to compute $\tan(\pi/2)$ or $\tan(90°)$ can cause a calculator to beep or blink, a computer to crash or complain. Prohibition is the American Way; but prohibiting ∞ is as futile as prohibiting whisky, and would complicate derivations of neat formulas for integrals of continuous periodic functions regardless of whether ∞ appeared in those formulas' final forms.

The Dilemma posed by $\tan(n\pi + \pi/2)$.

We can agree that $\tan(\pi/2)$ must be infinite but we cannot agree upon its sign without violating one of the two familiar identities

$$\tan(-\tau) = -\tan(\tau) \quad \text{and} \quad \tan(\tau+\pi) = \tan(\tau)$$

at $\tau = -\pi/2$; they imply $\tan(\pi/2) = \tan(-\pi/2) = -\tan(\pi/2)$. If we wished to do so, we could so define $\tan(\tau)$ that only the second identity would be violated, and that only at $\tau = -\pi/2$; for instance we could set

$$\tan(n\pi + \pi/2) := +\mathrm{signum}(n\pi + \pi/2)\,\infty \quad \text{for all integers } n .$$

We shall do so only for $n = -1$ and 0 . What follows takes signs of infinite values of $\tan(\ldots)$ into account no matter how they are chosen so long as they are fixed in advance and compatible with assignments $\arctan(+\infty) := +\pi/2$ and $\arctan(-\infty) := -\pi/2$.

PHRAC and TINT .

For all real ω define

$$\mathrm{PHRAC}(\omega) := \arctan(\tan(\omega\pi))/\pi \quad \text{and} \quad \mathrm{TINT}(\omega) := \omega - \mathrm{PHRAC}(\omega) .$$

Evidently $\mathrm{TINT}(\omega)$ is the integer nearest ω , identical to the Fortran function $\mathrm{NINT}(\omega)$ except perhaps when $\omega = n + 1/2$ is a half-integer;

$$\begin{aligned} \mathrm{TINT}(n + 1/2) &= n & \text{when } \tan(n\pi + \pi/2) &= +\infty , \\ \ldots &= n+1 & \text{when } \ldots &= -\infty . \end{aligned}$$

The graph of $\mathrm{TINT}(\omega)$ is a staircase ascending from lower left to upper right. The graph of $\mathrm{PHRAC}(\omega)$ is saw-toothed like this:

– – –/////////////– – – $\rightarrow\omega$,

with $-1/2 < \mathrm{PHRAC}(\omega) < 1/2$ except that

$$\mathrm{PHRAC}(n + 1/2) = \mathrm{signum}(\tan(n\pi + \pi/2))/2 .$$

A Procedure for Periodic Integrals.

Given a continuous periodic real function $p(\tau)$ with period π , the customary way to evaluate the indefinite integral

$$J(\tau) := \int p(\tau)\, d\tau$$

symbolically is to substitute $\tau := \arctan \xi$, then compute

$$P(\xi) := \int p(\arctan \xi)\, d\xi/(1 + \xi^2) ,$$

and finally set $J(\tau) := P(\tan \tau) + \gamma$ for some constant γ . This procedure is motivated by the expectation that $P(\ldots)$ will have fewer singularities than $J(\ldots)$ and be therefore easier to recognize. But this procedure can run τ through at most one period, and hence must fail to deliver $J(\ldots)$ correctly when τ ranges over more than one period.

For instance, suppose $p(\tau) = 1$. Then $P(\xi) = \arctan(\xi)$ and we would get $J(\tau) = \arctan(\tan \tau) + \gamma = \pi\,\mathrm{PHRAC}(\tau/\pi) + \gamma$ instead of $J(\tau) = \tau + \gamma$. The error $\tau - \pi\,\mathrm{PHRAC}(\tau/\pi) = \pi\,\mathrm{TINT}(\tau/\pi)$ is piecewise constant, so its derivative vanishes almost everywhere.

For another example take $p(\tau) = 4/(5 + 3\cos 2\tau)$, for which $F(\xi) = \arctan(\xi/2)$ would yield $J(\tau) = \arctan(\tan(\tau)/2) + \gamma$ instead of a correct $J(\tau) = \tau - \arctan(\sin(2\tau)/(3 + \cos 2t)) + \gamma$. The error $\tau - \arctan(\sin(2\tau)/(3 + \cos 2\tau)) - \arctan(\tan(\tau)/2)$ simplifies again to $\pi\,\mathrm{TINT}(\tau/\pi)$, though that is not obvious.

(The current version, 1.63, of DERIVE gets a formally correct $J(\tau) = \tau - \arctan(\sin(2\tau)/(1 + \cos 2\tau)) + \arctan(\sin(2\tau)/(2 + 2\cos 2\tau))$ which is unsatisfactory when $2\tau/\pi$ is an odd integer because then the two arctans' arguments become $0/0$, requiring that limits be taken to ascertain that both arguments approach the same ∞ .)

The Secular Term $\tau\,\Delta F/\pi$.
Almost two centuries ago, astronomers and geologists coined the phrase *secular term* to represent the non-periodic part of the integral $J(\tau) = \int p(\tau)\,d\tau$ of a periodic function $p(\tau)$. If p has period π then the secular term is $\tau \int_{-\pi/2}^{\pi/2} p(\theta)\,d\theta/\pi + \text{const.}$ The coefficient of τ is the average of $p(\tau)$ over one period. After the substitution $\tau = \arctan\xi$ changes $\int p(\tau)\,d\tau$ into $F(\xi) = \int p(\arctan\xi)\,d\xi/(1 + \xi^2)$, that coefficient turns into

$$\Delta F/\pi := (F(+\infty) - F(-\infty))/\pi \ .$$

This is the place from which ∞ cannot easily be removed.

Subtracting the secular term from $J(\tau)$ turns it into a periodic continuous function

$$\begin{aligned} J(\tau) - \tau\,\Delta F/\pi &= \int (p(\tau) - \Delta F/\pi)\,d\tau \\ &= \int (p(\arctan\xi) - \Delta F/\pi)\,d\xi/(1 + \xi^2) \\ &= F(\xi) - (\Delta F/\pi)\arctan(\xi) + \gamma \\ &= F(\tan\tau) - (\Delta F/\pi)\arctan(\tan\tau) + \gamma \end{aligned}$$

for some constant γ . Consequently, a correct derivation of J from F yields a formula involving $\Delta F/\pi$ thus:

$$\begin{aligned} J(\tau) &= F(\tan\tau) + (\Delta F/\pi)(\tau - \arctan(\tan\tau)) + \gamma \\ &= F(\tan\tau) + (\Delta F/\pi)\,\mathrm{TINT}(\tau/\pi) + \gamma \ . \end{aligned}$$

This is about as simple a formula as can be expected in general.

For example, when $p(\tau) = |\cos\tau|$ we find that

$$F(\xi) = \int d\xi/(1 + \xi^2)^{3/2} = \xi/\sqrt{1 + \xi^2} = \mathrm{signum}(\xi)/\sqrt{1 + 1/\xi^2}$$

so $\Delta F/\pi = 2/\pi$ and hence $\int p(\tau)\,d\tau$ turns out to be

$$J(\tau) = \mathrm{signum}(\tan\tau)\,|\sin\tau| + (2/\pi)(\tau - \arctan(\tan\tau)) + \gamma \ .$$

(DERIVE 1.63 gets a less satisfactory result

$$J(\tau) = \mathrm{sign}(\cos\tau)\sin\tau + (2/\pi)(\tau - \arctan(\tan\tau)) + \gamma$$

which becomes problematical when $\cos\tau = 0$.)

Simplifying $J(\tau)$.
In the special case that $p(\tau)$ is a continuous rational function of $\sin 2\tau$ and $\cos 2\tau$, substituting $\tau = \arctan\xi$ transforms its integral into an integral $F(\xi)$ of a smooth rational function $p(\tan\xi)/(1 + \xi^2)$ with no poles on the real axis, not even at ∞ . Consequently $F(\xi)$ can have no real poles either; it must be a linear combination of rational functions of ξ and arctangents of rational functions of ξ . Therefore the continuous periodic function $J(\tau) - \tau(\Delta F/\pi) = F(\xi) - (\Delta F/\pi)\arctan(\xi) + \gamma$ consists of a linear combination of bounded rational functions of ξ and arctangents of possibly unbounded rational functions of ξ . If any arctangents jump when their arguments pass from $+\infty$ to $-\infty$, their jumps must collectively cancel. Therefore, some way must exist to consolidate them into arctangents with no jumps.

Consolidation requires perhaps repeated application of an identity

$$\arctan(x) - \arctan(y) = \arctan((x-y)/(1+xy)) \quad \text{if} \quad xy \geq -1 ,$$
$$= \arctan((x-y)/(1+xy)) + \pi \,\mathrm{signum}(x-y)$$

otherwise. (In conformity with IEEE standards 754/854 for floating-point arithmetic, we take $1+xy$ to be $+0$, not -0, when it vanishes.) The term $\pi \,\mathrm{signum}(x-y)$ will affect only the constant γ, so it can be ignored, *if it really is a constant.*

For example, when $p(\tau) = 4/(5 + 3\cos 2\tau)$ we obtain

$$\begin{aligned} J(\tau) - \tau &= P(\xi) - \arctan\xi + \gamma \qquad (\Delta P/\pi = 1) \\ &= \arctan(\xi/2) - \arctan\xi + \gamma \\ &= \arctan(-\xi/(2+\xi^2)) + \gamma \\ &= -\arctan(\sin(2\tau)/(3+\cos 2\tau)) + \gamma , \end{aligned}$$

as was claimed above.

Another example,

$$\begin{aligned} p(\tau) &= -3\tan^2(\tau)/(1 - \tan^2(\tau) + \tan^4(\tau)) \\ &= (3\cos(4\tau) - 3)/(3\cos(4\tau) + 5) , \end{aligned}$$

will be integrated in three ways. First set $\xi = \tan\tau$ to get

$$\begin{aligned} P(\xi) &= \int p(\arctan\xi)\, d\xi/(1+\xi^2) \\ &= \int -3\xi^2\, d\xi/((1-\xi^2+\xi^4)(1+\xi^2)) \\ &= \arctan\xi - \arctan(2\xi+\sqrt{3}) - \arctan(2\xi-\sqrt{3}) . \end{aligned}$$

(DERIVE 1.63 was unable to produce this expression.) Now that $\Delta P/\pi = (P(+\infty) - P(-\infty))/\pi = -1$, we shall have to consolidate

$$\begin{aligned} J(\tau) + \tau &= P(\xi) + \arctan\xi + \gamma \qquad (\gamma \text{ is any constant}) \\ &= 2\arctan\xi - \arctan(2\xi+\sqrt{3}) - \arctan(2\xi-\sqrt{3}) + \gamma \\ &= \arctan\left(\frac{\sqrt{3}-\xi}{2\xi^2-\xi\sqrt{3}+1}\right) - \arctan\left(\frac{\sqrt{3}+\xi}{2\xi^2+\xi\sqrt{3}+1}\right) + \gamma . \end{aligned}$$

Neither denominator can change sign, so no terms jump any more, so consolidation can stop here; and to avoid trouble when $\tan\tau$ is infinite we introduce continued fractions to obtain a result

$$\begin{aligned} J(\tau) = &\arctan(1/(2\tan\tau + \sqrt{3} + 4/(\tan\tau - \sqrt{3}))) \\ &- \arctan(1/(2\tan\tau - \sqrt{3} + 4/(\tan\tau + \sqrt{3}))) - \tau + \gamma . \end{aligned}$$

Alternatively, consolidation can continue without encountering a denominator whose sign changes, and produce ultimately

$$J(\tau) = \arctan(-1/(\tan\tau + 1/(\tan\tau - 2/(\tan\tau + \cot\tau)))) - \tau + \gamma .$$

The same result can be obtained a second way starting from

$$\begin{aligned} P(\xi) &= \int p(\arctan\xi)\, d\xi/(1+\xi^2) \\ &= \int -3\xi^2\, d\xi/((1-\xi^2+\xi^4)(1+\xi^2)) \\ &= \int -3\xi^2\, d\xi/(1+\xi^6) = -\arctan\xi^3 . \end{aligned}$$

Once again $\Delta P/\pi = -1$, so the expression to be consolidated is

$$\begin{aligned} J(\tau) + \tau &= P(\tau) + \arctan\xi + \gamma \\ &= \arctan\xi - \arctan\xi^3 + \gamma \\ &= \arctan((\xi - \xi^3)/(1+\xi^4)) + \gamma \quad \text{etc. as before.} \end{aligned}$$

The third way starts from the observation that $p(\tau)$ actually has period $\pi/2$, so the substitution $\xi = \tan 2\tau$ can be tried. Now

$$\begin{aligned} P(\xi) &= \int p(\arctan(\xi)/2)\, d\xi/(2+2\xi^2) \\ &= -(3/2)\int \xi^2\, d\xi/((4+\xi^2)(1+\xi^2)) \\ &= \arctan(\xi)/2 - \arctan(\xi/2) , \end{aligned}$$

and $\Delta P/(\pi/2) = -1$, so now

$$\begin{aligned} J(\tau) + \tau &= P(\xi) + \arctan(\xi)/2 + \gamma \\ &= \arctan\xi - \arctan(\xi/2) + \gamma \quad \ldots \text{ consolidate it} \\ &= \arctan(\xi/(2+\xi^2)) + \gamma \\ &= \arctan(\tan(2\tau)/(2+\tan^2 2\tau)) + \gamma \\ &= \arctan(\sin(4\tau)/(3+\cos 4\tau)) + \gamma . \end{aligned}$$

This yields the same $J(\tau)$ as before, though not obviously so.

Two Errors to Avoid.
The process described above is not foolproof. First, it does not relieve the analyst of an obligation to detect improper integrals.

For instance, $\int \cot^2\tau\, d\tau = -\cot\tau - \tau + \gamma$ has a secular term, rather than $-\cot\tau - \arctan(\tan\tau) + \gamma'$ with spurious jumps at half-integer multiples of π ; but the integral must be infinite across any interval that includes an integer multiple of π . And $\int \sec^2\tau\, d\tau = \tan\tau + \gamma$ has no secular term depite that $\Delta P = +\infty$; this integral is infinite across any interval containing a half-integer multiple of π . And $2\int \csc(2\tau)\, d\tau = \ln|\tan\tau| + \gamma$ has an indeterminate $\Delta P = \ln(\infty) - \ln(\infty)$ but no secular term.

In short, an improper integral may include a secular term to cancel out spurious finite jumps without protecting against the effects of infinite jumps (poles).

The second error to avoid is overlooking the term $\pi\,\mathrm{signum}(x-y)$ in the identity for $\arctan(x) - \arctan(y)$ while consolidating arctans in $P(\xi) - (\Delta P/\pi)\arctan\xi$. One way to avoid this error is to start consolidation from a linear combination of arctans of linear functions of ξ , checking each denominator $(1+xy)$ in a new consolidation to ensure that it cannot reverse sign. If not checked, consolidation may introduce spurious jumps.

For example. $P(\xi) = \int (\xi^4 - 3\xi^2 + 6)\, d\xi/(\xi^6 - 5\xi^4 + 5\xi^2 + 4)$ is a linear combination of arctans of linear functions of ξ with complicated coefficients containing surds and cube roots. When it is "simplified" by consolidation ignoring $\pi\,\mathrm{signum}(x-y)$, the result is $Q(\xi) = \arctan((\xi^3 - 3\xi)/(\xi^2 - 2))$, whose derivative matches that of $P(\xi)$ except for two spikes at $\xi = \pm\sqrt{2}$. Were those spikes ignored, the coefficient of τ in the the secular term would be taken to be $\Delta Q/\pi = 1$. This is wrong. Correctly,

$$P(\xi) = Q(\xi) + \arctan((\xi^2-2)/\xi) + \arctan(\xi/(\xi^2-2)) + \arctan(1/\xi) + \arctan(\xi) ,$$

and $\Delta P/\pi = 3$. (The four added arctans have a derivative that vanishes everywhere but at $\xi = \pm\sqrt{2}$.) Now to further consolidate $P(\xi) - 3\arctan\xi$ is no simple matter; the final result

$$P(\xi) - 3\arctan\xi = \arctan((2\xi^2 + 1)(\xi^2 - 3)\xi/(\xi^6 - 3\xi^4 + 2\xi^2 + 2))$$

has no jumps, not even at $\xi = \infty$.

This result, free from surds, comes as a surprise. In general, to find a continuous expression $P(\xi) - (\Delta P/\pi)\arctan\xi$, whose coefficients are free from unnecessarily complicated algebraic numbers, and without excessively many arctans, is still an open problem. The best algorithm published so far, by D. Lazard and R. Rioboo in *J. Symbolic Computation* 9 (1990) 113-5, does not avoid spurious jumps. The problem is explored a little more in my note "The Persistence of Irrationals in Some Integrals."