PROBLEM for CS 179 : by Prof. W. Kahan

Exhibit a program that starts from any three given floating-point numbers x, y and z, and computes $p := x \cdot y \cdot z$ in some order that avoids undeserved over/underflow. Do likewise for $q := x \cdot y / z$.

SOLUTIONS: The proofs that these programs work correctly depend upon the properties of three *Environmental Constants* associated with the floating-point formats in which x, y, z, p and q are represented, regardless of whether those constants appear in the programs. The *Overflow threshold* Ω is the biggest finite number in that format; the *Underflow threshold* η is the smallest *normalized* positive number. The magnitudes of x, y and z are presumed to lie between Ω and $\varepsilon\eta$ inclusive where $\varepsilon\eta$ is the smallest *nonzero* magnitude and may be far tinier than η if underflow is *gradual*; on machines that underflow abruptly to zero $\varepsilon\eta = \eta$ except for CDC Cyber 17x's. $\varepsilon\eta = 2\eta$ for these Cybers to cope with "partially underflowed" numbers between η and $\varepsilon\eta$ that behave normally in *add*, *subtract* and *compare* but behave like zero in *multiply* and *divide*. Little is presumed about the product $\eta\Omega$, which lies very far from 1 on some machines.

An obvious program to compute p and q would first obtain their magnitudes using logarithms; $|p| = \exp(\ln|x| + \ln|y| + \ln|z|)$ and $|q| = \exp(\ln|x| + \ln|y| - \ln|z|)$. But these formulas lose accuracy badly when the data are very big or very small; the loss is caused by rounding each logarithm to working precision, and can be observed by comparing the computed values of $\exp(\ln|x|)$ and $|x|$ when it lies near Ω or η. And computing logarithms and exponentials wastes time. Our programs waste neither accuracy nor time.

Both programs start by Sorting $|x|, |y|$ and $|z|$ and continue thus:

Program for p :

Assume now that sorted $|x| \le |y| \le |z|$. Compute $x \cdot z$ first and then $p := (x \cdot z) \cdot y$ except on a machine with gradual underflow; on such a machine if $(x \cdot z)$ underflows recompute $p := (z \cdot y) \cdot x$.

Proof that p is correct.

If $x \cdot z$ overflowed, then $1 < |x| \le |y| \le \Omega < |x \cdot z| < |(x \cdot z) \cdot y|$ so p deserves to overflow too (except perhaps on a CRAY, which can overflow in certain cases when a product lies between $\Omega/2$ and Ω; but that is too perverse to consider here). Similarly if $x \cdot z$ underflowed on a machine that underflows abruptly to zero, then

$$1 > |z| \ge |y| \ge |x| \ge \eta > |x \cdot z| > |(x \cdot z) \cdot y|$$

so p must underflow too. On a machine that underflows gradually conformity with IEEE standards 754/854 requires also the ability to detect underflow, and this should be exploited if any of the data can be subnormal (i.e., between $\varepsilon\eta$ and η in magnitude). Then $x \cdot z$ underflows only when $1/\varepsilon \ge |z| \ge |y| \ge |x| \ge \varepsilon\eta$ and $\eta > |x \cdot z|$; since $\Omega > 1/\varepsilon^2$ on those machines, $\Omega > z \cdot y$ so $z \cdot y$ cannot overflow and if it underflows too then either $|z| > 1$ and then $|x \cdot y \cdot z| = |(x \cdot z)(z \cdot y)/z| < \eta^2/|z| < \eta$, or else $|z| \le 1$ and then $|x \cdot y \cdot z| < |x|\,\eta \le \eta$, and p deserves to underflow either way.

Programs for q :
If we could treat q as a product $x \cdot y \cdot (1/z)$, we could compute it safely using the program for p; but the risk that $1/z$ may over/underflow precludes that option. A safe and simple program works on machines that allow programs to branch on over/underflow:

First swap x and y if necessary to establish $|x| \leq |y|$;
  next compute $p := x \cdot y$; subsequently
    if (p overflowed and $|z| > 1$) then $q := (y/z) \cdot x$
      else if (p underflowed and $|z| < 1$) then $q := (((x/\varepsilon)/z) \cdot y) \cdot \varepsilon$
        else $q := p/z$. (For Cybers use $\varepsilon = 1$ here, not 2.)

The validity of this program is easy to establish provided we may presume that $\sqrt{(\eta)}/\varepsilon^2 < \eta\Omega < \sqrt{\Omega}$, as appears to be true for all machines I know. But the ability to test for over/underflow and continue is not so common; what if over/underflow is silent? In the absence of a (portable) way to branch on over/underflow, we must produce a spaghetti-like code with branches that preclude spurious over/underflows. Such a program follows.

Two constants are needed. One is λ, the smallest power of the machine's radix no smaller than $\max\{1, 1/(\varepsilon\eta\Omega)\}$. The other is μ, the biggest power of the radix not exceeding $\min\{1, 1/(\eta\Omega)\}$. Multiplication by λ or μ is exact, so it cannot cause underflow on a machine that conforms to IEEE 754/854.

First sort $|x|, |y|$ and $|z|$, keeping track of z. This reduces the situation to one of three cases, depending upon whether $|z|$ is minimal, maximal, or neither:

In case $|z|$ is > minimal, say $|z| \leq |x| \leq |y|$, test $|y|$;
  if $|y| > 1$ then $q := (x/z) \cdot y$
  else $q := (x/(\lambda z)) \cdot (\lambda y)$.
In case $|z|$ is maximal, say $|z| \leq |y| \leq |x|$, test $|x|$;
  if $|x| < 1$ then $q := (y/z) \cdot x$
  else $q := (y/(\mu z)) \cdot (\mu x)$.
In case $|z|$ is neither, say $|x| \leq |z| \leq |y|$, test both;
  if $|x| > 1$ then $q := (y/z) \cdot x$
  else if $|y| < 1$ then $q := (x/z) \cdot y$
    else $q := (x \cdot y)/z$.

The proof that this program is correct is a tedious exercise in elementary inequalities, and is left to the reader.