Translation of
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How Reliable are Results of Computers?

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With the advent of inexpensive electronic computers there has been a surge in the number of people relying on machine aided computations. Approximately ten years ago when the first pocket calculators appeared on the market these devices were limited to the four basic arithmetic operations. As the cost of producing chips decreased the price of pocket calculators were driven down from their original $ 500 – $ 1000 per unit to a scant $ 10 – $ 50 per unit. Accompanying this price reduction has been an equally dramatic increase in the number and type of operations that can be performed.

In the past 5 years the market has been overwhelmed by a new generation of computers, the "Home" or "Personal" computer. These machines often surpass older mainframes in capability and performance, and as with pocket calculators their prices have been dropping while their capabilities increase. Just five years ago the first table top computers cost between 30 and 40 thousand dollars where as today equipment of similar capability can be purchased for about $ 1000. The dramatic decreases in the price of pocket calculators and desk top computers have brought them within the reach of the general public. In fact, last year alone, a well known manufacturer sold over 200,000 desk top computers to consumers in a single country. This is not a unique case as other manufacturers have had similar results.

Many computer users unquestionably believe the results delivered by a computer and this problem is compounded by the growing number of users. A result produced by a computer is believed to be correct, it is more or less accepted as a mathematical proof. Users are often completely astonished when presented with the fact that in simple numerical computations with only a few operations arbitrarily incorrect results may be produced. They are even more astonished to learn that in floating point computations nothing else can be expected when using todays computation techniques.

The following examples will illustrate the fact (well known to the expert) that with very few operations it is possible to get completely absurd results. This statement is not only true for pocket calculators, it holds true for even the largest of computers. Long calculations compound these problems to the point where one is usually not aware of when, where or even if an error has occurred.

The reader is encouraged to try the following examples on a pocket calculator, desk top computer and/or a large computer.

These sample problems demonstrate some of the pitfalls that every user of the computer should be aware of. Similar examples should be included as part of every introductory computer handbook.

For the specialist it should be noted that it's only been in the past few years that techniques have been developed which not only describe these effects mathematically but permit the elimination of these types of errors in numerical computation. The original papers and further literature describing the new techniques can be found in the references.

1. Compute the value of

$$p^2 - 2q^2$$

for $p = 665{,}857.0$ and $q = 470{,}832.0$.
The correct answer is 1.0.

2. Compute the value of

$$9x^4 - y^4 + 2y^2$$

for $x = 10{,}864.0$ and $y = 18{,}817.0$.
The correct answer is 1.0.

3. Compute the value of

$$a + b - a$$

for $a = 10^{34}$ and $b = -2.0$.
The correct answer is -2.0.

4. Compute the value of

$$p(x) = 170.4 \cdot x^3 - 356.41 \cdot x^2 + 168.97 \cdot x + 18.601$$

for $x = 1.091608$.
The exact value of the polynomial is $8.21248 \cdot 10^{-14}$.

5. Compute the value of

$$p(x) = 8{,}118.0 \cdot x^4 - 11{,}482.0 \cdot x^3 + x^2 + 5{,}741.0 \cdot x - 2{,}030.0$$

for $x = 0.707107$.
The exact value of the polynomial is $-1.91527325270\ldots \cdot 10^{-11}$.

6. Compute the solution of the linear system

$$\begin{aligned} 64{,}919{,}121.0 \cdot x - 159{,}018{,}721.0 \cdot y &= 1.0 \\ 41{,}869{,}520.5 \cdot x - 102{,}558{,}961.0 \cdot y &= 0.0. \end{aligned}$$

Correct formulas for calculation of x and y are

$$\begin{aligned} y &= (41{,}869{,}520.5/64{,}919{,}121.0)/(102{,}558{,}961.0 - 41{,}869{,}520.5 \cdot \\ &\quad \cdot 159{,}018{,}721.0/64{,}919{,}121.0) \\ x &= (102{,}558{,}961.0/41{,}869{,}520.5) \cdot y. \end{aligned}$$

The correct answers are

$$\begin{aligned} x &= 205{,}117{,}922.0 \\ y &= 83{,}739{,}041.0. \end{aligned}$$

7. Compute the solution of the linear system

$$\begin{aligned} -367296.0 \cdot t - 43199.0 \cdot u + 519436.0 \cdot v - 954302.0 \cdot w &= 1 \\ 259718.0 \cdot t - 477151.0 \cdot u - 367295.0 \cdot v - 1043199.0 \cdot w &= 1 \\ 886731.0 \cdot t + 88897.0 \cdot u - 1254026.0 \cdot v - 1132096.0 \cdot w &= 1 \\ 627013.0 \cdot t + 566048.0 \cdot u - 886732.0 \cdot v + 911103.0 \cdot w &= 0 \end{aligned}$$

The correct answers are

$$\begin{aligned} t &= 8.86731088897 \cdot 10^{17} \\ u &= 8.86731088897 \cdot 10^{11} \\ v &= 6.27013566048 \cdot 10^{17} \\ w &= 6.27013566048 \cdot 10^{11}. \end{aligned}$$

8. Does the polynomial, given in example 4, have positive zeros? The correct answer is yes. ($x = 1.091607978\ldots$ and $x = 1.091607981\ldots$).

9. Compute the value of the polynomial

$$\begin{aligned} p(x) = {} & 223{,}200{,}658.0 \cdot x^3 - 1{,}083{,}557{,}822.0 \cdot x^2 + \\ & + 1{,}753{,}426{,}039.0 \cdot x - 945{,}804{,}881.0 \end{aligned}$$

for x between 1.61801916 and 1.61801917 in steps of 10^{-9}, 11 values in total. Is there a zero between these two values?

The correct answers are

$$\begin{aligned} P(1.618019160) &= -0.17081105112320 && D-11 \\ P(1.618019161) &= -0.89804011575510 \ldots && D-12 \\ P(1.618019162) &= -0.34596536943057 \ldots && D-12 \\ P(1.618019163) &= -0.51884933054474 && D-13 \\ P(1.618019164) &= -0.15797467422848 && D-13 \end{aligned}$$

$$\begin{aligned}
P(1.618019165) &= -0.23770163333175 & & D-12\\
P(1.618019166) &= -0.71759609157723 & \ldots\ & D-12\\
P(1.618019167) &= -0.14554795029553 & \ldots\ & D-11\\
P(1.618019168) &= -0.24513505282621 & \ldots\ & D-11\\
P(1.618019169) &= -0.37052078282936 & \ldots\ & D-11\\
P(1.618019170) &= -0.52170500638460 & & D-11.
\end{aligned}$$

The first, fourth, fifth, sixth and eleventh value of the polynomial are exact.

10. Let three pairs (x, y) be given:

x	5,201,477.0	5,201,478.0	5,201,479.0
y	99,999.0	100,000.0	100,001.0

It is obvious that a straight line fits through these 3 points. For optimal curve fitting of $L(x) = mx + b$ with respect to least square approximation we use the formulas

$$m = \frac{x_1y_1 + x_2y_2 + x_3y_3 - \frac{1}{3}(x_1 + x_2 + x_3)(y_1 + y_2 + y_3)}{x_1^2 + x_2^2 + x_3^2 - \frac{1}{3}(x_1 + x_2 + x_3)^2} \quad \text{and}$$

$$b = \frac{1}{3}(y_1 + y_2 + y_3) - \frac{m}{3}(x_1 + x_2 + x_3).$$

Calculate the values for m, b and $L(5,201,480.0)$.

The correct answers are $m = 1$, $b = -5,101,478.0$ and $L(5,201,480.0) = 100,002.0$.

11. Which positive real zero(s) exist for the polynomial

$$2,124,476,931.0 \cdot x^4 - 1,226,567,328.0 \cdot x^3 - \\ -708,158,977.0 \cdot x^2 + 408,855,776.0 \cdot x + 1E-27?$$

The correct answer is: There are no positive real zeros for this polynomial.

12. The elements of the Hilbert-matrix are defined by

$$H_{ij} := 1/(i + j - 1).$$

To obtain integers for a Hilbert-matrix with n lines, we multiply the elements with $lcm(1, 2, \ldots, 2n-1)$. The largest matrix H^*, still representable by a binary floating point number with a 24 bit mantissa, has 9 rows. It is called $H_9{}^*$.

Compute the first element of the first line of the inverse of $H_9{}^*$.

The correct answer is $6.611036022800 \ldots \cdot 10^{-6}$.

13. The largest matrix H^*, still representable by a floating point number with a 54 bit mantissa, has 21 lines.

Compute the first element of the first line of the inverse of $H_{21}{}^*$.

The correct answer is $2.013145339298 \ldots \cdot 10^{-15}$.

14. Compute the solution of the linear system

$$\begin{aligned}
64,079.0 \cdot x + 57,314.0 \cdot y &= 2.0\\
51,860.0 \cdot x + 46,385.0 \cdot y &= 305.0.
\end{aligned}$$

The correct answer is

$$\begin{aligned}
x &= -46,368.0\\
y &= 51,841.0.
\end{aligned}$$

15. Let the following rational function be given:

$$f(t) := \frac{4970.0 \cdot t - 4923.0}{4970.0 \cdot t^2 - 9799.0 \cdot t + 4830.0}.$$

An approximation for $f''(t)$ is

$$f''(t) \approx \frac{f(t-h) - 2 \cdot f(t) + f(t+h)}{h^2}.$$

Compute an approximation for $f''(1)$, using the above equation, for $h = 10^{-4}$, $h = 10^{-5}$ and $h = 10^{-8}$.

The correct answers are

approximation with $h = 10^{-4}$: 70.78819...
approximation with $h = 10^{-5}$: 93.76790...
approximation with $h = 10^{-8}$: 94.00000...

The exact value for the second derivative is $f''(1) = 94.0$.

16. Compute the value of

$$83,521.0 \cdot y^8 + 578.0 \cdot x^2y^4 - 2.0 \cdot x^4 + 2.0 \cdot x^6 - x^8$$

for $x = 9,478,657.0$ and $y = 2,298,912.0$.

The correct value of the expression is $-179,689,877,047,297.0$.

What is the result obtained on a pocket calculator, on a large computer?

17. Calculate the scalar product

$$a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5$$

for

$$\begin{array}{ll} a_1 = 2.718281828 & b_1 = 1{,}486.2497 \\ a_2 = -3.141592654 & b_2 = 878{,}366.9879 \\ a_3 = 1.414213562 & b_3 = -22.37492 \\ a_4 = 0.5772156649 & b_4 = 4{,}773{,}714.647 \\ a_5 = 0.3010299957 & b_5 = 0.000185049 \end{array}$$

The correct value for the scalar product is $-1.00657107 \cdot 10^{-11}$

18. Compute the value of the expression

$$(1682.0 \cdot xy^4 + 3.0 \cdot x^3 + 29.0 \cdot xy^2 - 2.0 \cdot x^5 + 832.0)/107751.0$$

for $x = 192{,}119{,}201.0$ and $y = 35{,}675{,}640.0$.

The correct answer is the year when the mathematician L. Euler died.

What is the result obtained on a pocket calculator, on a large computer?

References

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