Superlinear Convergence of a Remes Algorithm

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April 1981

ABSTRACT

The Remes algorithm is a widely used iterative method for finding polynomial or rational best (minimax) approximations to a given real function on a given interval. But the algorithm is equally effective for other kinds of approximations, not necessarily rational, when provided with a sufficiently close first guess; subsequent convergence is very fast.

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The Remes algorithm in question purports to determine coefficients $a_1, a_2, \ldots, a_n$ in a given formula $F(x, a_1, a_2, \ldots, a_n)$ so as to best approximate a specified real function $f(x)$ on a specified interval $\underline{x} \le x \le \bar{x}$. Also specified is some measure $E(x, a_1, a_2, \ldots, a_n)$ of the error between F and f. For the sake of notational economy let us write $\mathbf{a}$ in place of $a_1, a_2, \ldots, a_n$. Then examples of $E(x, \mathbf{a})$ can be exhibited thus:

$E(x,\mathbf{a}) = f(x) - F(x,\mathbf{a})$ ... absolute error
$E(x,\mathbf{a}) = 1 - F(x,\mathbf{a})/f(x)$ ... relative error
$E(x,\mathbf{a}) = \ln(f(x)/F(x,\mathbf{a}))$ ... relative error
$E(x,\mathbf{a}) = (f(x) - F(x,\mathbf{a}))/W(x,\mathbf{a})$ ... weighted error.

However E may be specified, the best value $\hat{\mathbf{a}}$ of the coefficient vector $\mathbf{a}$ is that which

$$\underset{\mathbf{a}}{\text{minimizes}} \quad \underset{\underline{x} \le x \le \bar{x}}{\text{maximum}} \quad |E(x,\mathbf{a})| .$$

Let that minimized maximum be denoted by

$$|\hat{e}| = \max_{\underline{x} \le x \le \bar{x}} |E(x, \hat{\mathbf{a}})| .$$

Then there will usually exist $n+1$ "alternating extrema" $\hat{x}_i$ which satisfy together a set of interpolating conditions

$$\begin{aligned} &\underline{x} \le \hat{x}_0 < \hat{x}_1 < \cdots < \hat{x}_{n-1} < \hat{x}_n \le \bar{x} && \text{and} \\ &E(\hat{x}_i, \hat{\mathbf{a}}) = (-1)^i \hat{e} && \text{for } i = 0,1,2,\ldots,n-1,n . \end{aligned} \qquad \text{(I)}$$

In other words, the error E will usually achieve its extreme values at $n+1$ consecutive points in the interval $\underline{x} \le x \le \bar{x}$ at which the extrema alternate in sign. Moreover, the internal extrema must occur where $\frac{\partial}{\partial x}E$ changes sign, so

$$(\hat{x}_i - \underline{x})(\hat{x}_i - \bar{x}) \frac{\partial}{\partial x} E(\hat{x}_i, \hat{\mathbf{a}}) = 0 \quad \text{for } i = 0,1,2,\ldots,n. \qquad \text{(M)}.$$

These equations (M) and (I) motivate the Remes algorithm, which is an iterative procedure for solving these $2n+2$ equations for $2n+2$ unknowns $\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_n, \hat{e}$ and the elements $\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n$ of the n-vector $\hat{\mathbf{a}}$. Normally equations (M) and (I) define their solution uniquely, but prudence demands that any solution be tested by examination of the graph of $E(x, \hat{\mathbf{a}})$ plotted against x over $\underline{x} \le x \le \bar{x}$ to be sure that extrema of $|E|$ beyond $|\hat{e}|$ have not been overlooked. This precaution could be incorporated into what follows but will be omitted to simplify the exposition.

The Remes algorithm is an iteration. Given a set of guesses x_i' respectively close enough to $\hat{x}_i$ and ordered similarly, so

$$\underline{x} \le x_1' < x_2' < \cdots < x_{n-1}' < x_n' \le \bar{x} ,$$

the first step is to solve equations like (I), namely

$$(-1)^{i+1} e' + E(x_i', \mathbf{a}') = 0 \quad \text{for } i = 0,1,2,\ldots,n. \qquad \text{(I')}$$

for coefficients e' and $\mathbf{a}'$ presumed to be close to $\hat{e}$ and $\hat{\mathbf{a}}$ respectively. These equations could be solved iteratively via, say, Newton's iteration provided the partial derivative

$$\frac{\partial}{\partial \mathbf{a}} E(x,\mathbf{a}) = \{ \frac{\partial}{\partial a_1} E(x,\mathbf{a}),\ \frac{\partial}{\partial a_2} E(x,\mathbf{a}), \cdots,\ \frac{\partial}{\partial a_n} E(x,\mathbf{a}) \}$$

is computable and slowly varying, provided the Jacobian matrix of first partial derivatives whose i^{th} row is

$$\left[(-1)^{i+1},\ \frac{\partial}{\partial \mathbf{a}} E(x_i,\mathbf{a}') \right] \qquad \text{for } i = 0,1,2,\cdots,n$$

is nonsingular, and provided an initial guess $\mathbf{a}$ close enough to the solution $\mathbf{a}'$ of (I') is available. (Since e' appears linearly in (I') its initial guess is irrelevant.)

After (I') has been solved for e' and $\mathbf{a}'$ the next step is to locate new extrema x_i'', ordered so that $\underline{x} \le x_1'' < x_2'' < \cdots < x_n'' \le \overline{x}$, by solving iteratively an uncoupled set of equations like (M), namely

$$(x_i'' - \underline{x})(x_i'' - \overline{x})\ \frac{\partial}{\partial x} E(x_i'',\mathbf{a}') = 0 \quad \text{for } i = 0,1,2,\cdots,n. \quad \text{(M')}$$

Each such equation can be solved iteratively via, say, the secant iteration starting from, say, x_i' as a first guess. The calculated values x_i'' replace the respective values x_i' and then the Remes iteration is repeated until convergence becomes evident.

But will the Remes iteration converge?

The purpose of this paper is to prove convergence provided three conditions are fulfilled:

Condition 1: $E(x,\mathbf{a})$ is a real analytic function of all its variables in the neighborhoods of each $x = \hat{x}_i$ and $\mathbf{a} = \hat{\mathbf{a}}$, which means that E is infinitely differentiable and represented locally by its Taylor series expansion in every such neighborhood. This condition is usually evident in practice.

Condition 2: The $(n+1)\times(n+1)$ Jacabian matrix of first derivatives $\hat{J} = J(\{\hat{x}_i\},\hat{\mathbf{a}})$ (this notation will be explained later) whose i^{th} row is

$$\left[(-1)^{i+1},\ \frac{\partial}{\partial \mathbf{a}} E(\hat{x}_i,\hat{\mathbf{a}}) \right] \qquad \text{for } i = 0,1,2,\ldots n \quad ,$$

is nonsingular and therefore invertible. This condition can fail in practice, in which case Newton's iteration may fail to converge to a solution $(e',\mathbf{a}')$ of (I'); but such failure is rare and, when it happens, may indicate that $F(x,a_1,a_2,\cdots,a_n)$ is no better than some simpler (fewer coefficients a_j) approximation to $f(x)$. Certainly the failure of this condition undermines confidence in the uniqueness or finiteness of the best approximation's coefficient-vector $\hat{\mathbf{a}}$. Conversely, when the best approximation's coefficient-vector $\hat{\mathbf{a}}$ is known in advance to be unique, perhaps because of a Haar condition or "unisolvence" (Rice [1964]), then condition 2 is most likely to be satisfied.

Condition 3: The minimized maximum $|\hat{e}|$ is strictly positive; and every root $\hat{x}_i$ of (M) is a place where $(-1)^i E(x,\hat{\mathbf{a}})/\hat{e}$ achieves its local maximum 1 as x varies in some neighborhood of $\hat{x}_i$ in $\underline{x} \le x \le \overline{x}$. When $\underline{x} < \hat{x}_i < \overline{x}$ this condition means that $(-1)^i \frac{\partial}{\partial x} E(x,\hat{\mathbf{a}})/\hat{e}$

must change sign from positive to negative as x increases through $\hat{x}_i$, and hence

$$(-1)^i \frac{\partial^2}{\partial x^2} E(x,\hat{\mathbf{a}})/\hat{\varepsilon} \le 0 .$$

And the second derivative could vanish without invalidating our conclusion. Here is where this paper differs from those of previous writers like Veidinger [1960], Werner [1962-3], Ralston [1965] and Dunham [1968], who assumed that the second derivative must not vanish at interior roots $\hat{x}_i$; cf. Meinardus [1967, p. 111 and p. 151].

Conclusion: The foregoing three conditions imply that the Remes iteration will converge superlinearly from any set of starting values $\{x_i\}$ and a close enough to $\{\hat{x}_i\}$ and $\hat{\mathbf{a}}$ respectively. Here "superlinear convergence" means that each iterate's number of correct decimal digits is an ultimately exponentially growing function of the number of iterations; in particular, convergence is quadratic if every

$$\left| \frac{\partial^2}{\partial x^2} E(\hat{x}_i,\hat{\mathbf{a}}) \right| + \left| \frac{\partial}{\partial x} E(\hat{x}_i,\hat{\mathbf{a}}) \right| \ne 0 .$$

This conclusion will be proved below after the proof has been outlined.

By studying the Taylor expansion of $E(x,\hat{\mathbf{a}})$ in powers of $x-\hat{x}_i$, we are able to cope with the possibility that the graph of $E(x,\hat{\mathbf{a}})$ may be very nearly flat near those interior stationary points $\hat{x}_i$ where both $\frac{\partial}{\partial x}E$ and $\frac{\partial^2}{\partial x^2}E$ vanish. Because E is real analytic, the difference between $E(x,\hat{\mathbf{a}})$ and its extreme value $\pm\hat{\varepsilon}$ is approximately proportional, when $|x_i - \hat{x}_i|$ is small enough, to $(x_i - \hat{x}_i)^{k_i}$ for some $k_i \ge 2$.

A similar study of the Taylor expansion of $E(x,\mathbf{a})$ in powers of $x-\hat{x}_i$ and $\mathbf{a}-\hat{\mathbf{a}}$ implies that solving (I') will produce an approximation $\mathbf{a}'$ to $\hat{\mathbf{a}}$ with $\mathbf{a}'-\hat{\mathbf{a}}$ very nearly a linear combination of powers $(x_i'-\hat{x}_i)^{k_i}$ where $k_i \ge 2$ was introduced above. Then solving (M') produces new estimates x_i'' for which $(x_i'' - \hat{x}_i)^{k_i-1}$ is very nearly a linear functional of $\mathbf{a}' - \hat{\mathbf{a}}$. Consequently each $(x_i'' - \hat{x}_i)^{k_i-1}$ is nearly a linear combination of the powers $(x_j' - \hat{x}_j)^{k_j}$, whence follows superlinear convergence.

Now for the details.

Condition 3, which says that $(-1)^i E(x,\hat{a})/\hat{e}$ achieves a local maximum 1 at $x = \hat{x}_i$, says something about the Taylor series

$$E(x,\hat{a}) = E(\hat{x}_i,\hat{a}) + \sum_1^\infty (x-\hat{x}_i)^k \left(\frac{\partial}{\partial x}\right)^k E(\hat{x}_i,\hat{a}) / k!$$

$$= (-1)^i \hat{e} - v_i(x) \qquad \text{where}$$

$$v_i(x) = -\sum_{k \ge k_i} (x-\hat{x}_i)^k \left(\frac{\partial}{\partial x}\right)^k E(\hat{x}_i,\hat{a}) / k!$$

$$= (x-\hat{x}_i)^{k_i} (\hat{v}_i + 0(x-\hat{x}_i)) \quad \text{as } x \to \hat{x}_i \ .$$

Here the positive integer k_i and the nonzero coefficient $\hat{v}_i = -(\frac{\partial}{\partial x})^{k_i} E(\hat{x}_i,\hat{a}) / k_i!$ must be constrained as follows to ensure that the local maximum is achieved at $x = \hat{x}_i$:

If $\hat{x}_i = \underline{x}$ then $(-1)^i \hat{v}_i / \hat{e} > 0$ and $k_i \ge 1$ and $i = 0$.

If $\underline{x} < \hat{x}_i < \overline{x}$ then $(-1)^i \hat{v}_i / \hat{e} > 0$ and k_i is even (so $k_i \ge 2$).

If $\hat{x}_i = \overline{x}$ then $(-1)^{i+k_i} \hat{v}_i / \hat{e} > 0$ and $k_i \ge 1$ and $i = n$.

By differentiating the Taylor series we find that

$$\frac{\partial}{\partial x} E(x,\hat{a}) = \sum_{k \ge k_i} (x-\hat{x}_i)^{k-1} \left(\frac{\partial}{\partial x}\right)^k E(\hat{x}_i,\hat{a}) / (k-1)!$$

$$= -k_i (x-\hat{x}_i)^{k_i-1} (\hat{v}_i + 0(x-\hat{x}_i)) \quad \text{as } x \to \hat{x}_i \ ,$$

which will be useful when the solution of (M') is analyzed.

The effect of varying the second argument a in $E(x,\mathbf{a})$ can be summarized by writing

$$E(x,\mathbf{b}) = E(x,\hat{\mathbf{a}}) + E'(x,\mathbf{b}) \cdot (\mathbf{b} - \hat{\mathbf{a}})$$

for all $\mathbf{b}$, where

$$E'(x,\mathbf{b}) = \int_0^1 \frac{\partial}{\partial \mathbf{a}} E(x,\hat{\mathbf{a}} + t(\mathbf{b} - \hat{\mathbf{a}}))dt$$

$$= \frac{\partial}{\partial \mathbf{a}} E(x,\hat{\mathbf{a}}) + 0(\mathbf{b} - \hat{\mathbf{a}}) \quad \text{as } \mathbf{b} \to \hat{\mathbf{a}} \ .$$

If $\mathbf{a}, \hat{\mathbf{a}}$ and $\mathbf{b}$ are regarded as column vectors then $E'(x,\mathbf{b})$ is a row vector, all with n elements. For any set $\{x_i\}$ of $n+1$ values satisfying $\underline{x} \le x_0 < x_1 < \cdots < x_{n-1} < x_n \le \overline{x}$, and for any n-vector $\mathbf{a}$ there exists an $(n+1) \times (n+1)$ matrix

$$J(\{x_i\} \ , \ \mathbf{a})$$

whose i^{th} row is

$$\left[(-1)^{i+1}, \quad E'(x_i,\mathbf{a})\right] .$$

Recalling conditon 2 above we observe that

$$J(\{x_i\},\mathbf{a}) \to \hat{J} = J(\{\hat{x}_i\},\hat{\mathbf{a}}) \quad \text{as all } x_i \to \hat{x}_i \text{ and } \mathbf{a} \to \hat{\mathbf{a}} \ .$$

and since $\hat{J}^{-1}$ exists the same must be true for

$$J(\{x_i\},\mathbf{a})^{-1} \to \hat{J}^{-1} \text{ as all } x_i \to \hat{x}_i \text{ and } \mathbf{a} \to \hat{\mathbf{a}} .$$

This fact will be useful when we analyze the solution of (I').

Now let $\{x_i'\}$ be a given set of close approximations to $\{\hat{x}_i\}$, so close as to ensure the truth of all subsequent statements known to be true when all $|x_i' - \hat{x}_i|$ are sufficiently small. In particular, assume $\underline{x} \le x_0' < x_1' < \cdots < x_{n-1}' < x_n' \le \bar{x}$. By substituting the foregoing definitions of $v_i(x)$ and J into (I') we obtain

$$0 = (-1)^{i+1}e' + E(x_i',\mathbf{a}') = (-1)^{i+1}\hat{e} + E(\hat{x}_i,\hat{\mathbf{a}})$$

$$= (-1)^{i+1}(e' - \hat{e}) + E(x_i',\mathbf{a}') - E(x_i',\hat{\mathbf{a}}) + E(x_i',\hat{\mathbf{a}}) - E(\hat{x}_i,\hat{\mathbf{a}})$$

$$= (-1)^{i+1}(e' - \hat{e}) + E'(x_i',\mathbf{a}') \cdot (\mathbf{a}' - \hat{\mathbf{a}}) - v_i(x_i')$$

so

$$\begin{bmatrix} e' - \hat{e} \\ \mathbf{a}' - \hat{\mathbf{a}} \end{bmatrix} = J(\{x_i'\},\mathbf{a}')^{-1}\, \mathbf{v}(\{x_i'\}) \qquad \text{(A)}$$

where $\mathbf{v}(\{x_i'\})$ is the $(n+1)$-vector whose i^{th} component is $v_i(x_i')$. Although the last equation appears to express e' and $\mathbf{a}'$ in terms of $\{x_i'\}$, it has $\mathbf{a}'$ on both sides. However, the implicit function theorem can be invoked here to ensure that equation (I') has a unique solution $\mathbf{a}'$ in a neighborhood of $\hat{\mathbf{a}}$ for every $\{x_i'\}$ in a small enough neighborhood of $\{\hat{x}_i\}$, because equations (I) and (I') are continuous perturbations of each other and the derivative matrix $\hat{J}$ is invertible. Consequently $\mathbf{a}' - \hat{\mathbf{a}} = 0(\{x_i' - \hat{x}_i\})$ as all $x_i' \to \hat{x}_i$ by virtue of the implicit function theorem alone, and then from (A) above we deduce

$$\begin{bmatrix} e' - \hat{e} \\ \mathbf{a}' - \hat{\mathbf{a}} \end{bmatrix} = \left(\hat{J}^{-1} + 0(\{x_i' - \hat{x}_i\})\right)\hat{\mathbf{v}}(\{x_i'\}) \quad \text{as all } x_i' \to \hat{x}_i$$

where $\hat{\mathbf{v}}(\{x_i'\})$ is the $(n+1)$-vector whose i^{th} component is

$$\hat{v}_i(x_i') = (x_i' - \hat{x}_i)^{k_i}\, \hat{v}_i = v_i(x_i')(1 + 0(x_i' - \hat{x}_i)) .$$

Having computed $\begin{bmatrix} e' \\ \mathbf{a}' \end{bmatrix}$ we turn to the calculation of a new set of extrema $\{x_i''\}$ satisfying (M') and close to $\{\hat{x}_i\}$ respectively. Do such extrema x_i'' exist? Yes, and they can be found as follows:

Provided all $|x_j' - \hat{x}_j|$ are tiny enough, in which case $\mathbf{a}' - \hat{\mathbf{a}}$ will be tiny too, all $(-1)^i E(x_i',\mathbf{a}') = e'$ must be as close as one likes to $(-1)^i E(\hat{x}_i,\hat{\mathbf{a}}) = \hat{e}$, so close that $e' \neq 0$ too; then $E(x,\mathbf{a}')/e'$ must reverse sign as x increases from x_{i-1}' to x_i'. Therefore for each $i = 1,2,\ldots,n$ some y_i' can be found such that $x_{i-1}' < y_i' < x_i'$ and $(-1)^i E(x,\mathbf{a}')/e'$ changes sign from negative (its sign when $x = x_{i-1}'$) through zero at $x = y_i'$ to positive (its sign when $x = x_i'$) as x increases through y_i'. Let $y_0' = \underline{x}$ and $y_{n+1}' = \bar{x}$ to simplify the next sentence. Then for each $i = 0,1,2,\ldots,n$ at least one place $x = x_i''$ can be found in $y_i' \le x \le y_{i+1}'$ where $(-1)^i E(x,\mathbf{a}')/e'$ is locally maximum, and these places x_i'' lie among the roots of equation (M') as well as satisfying $\underline{x} \le x_0'' < x_1'' < \cdots < x_{n-1}'' < x_n'' \le \bar{x}$. Whatever iterative method might be used to solve (M') for x_i'', it could proceed from x_i' as a first guess.

The foregoing procedure might not determine each x_i'' uniquely, but this does not matter. The following analysis will exhibit for each $i = 0,1,2,\ldots,n$ a radius $\rho_i > 0$, independent of x_i' and $\mathbf{a}'$, such that whenever all $|x_i' - \hat{x}_i| / \rho_i$ are tiny enough then every root x_i'' of (M') that lies in $|x_i'' - \hat{x}_i| < \rho_i$ must lie much closer to $\hat{x}_i$ than within ρ_i . In other words, spurious roots x_i'' cause no confusion because, if every x_i' is close enough to $\hat{x}_i$ respectively, each root x_i''

close enough to x_i' is even closer to $\hat{x}_i$, as we shall see.

To simplify the exposition, temporarily drop the double prime on x_i'' and drop the subscript i everywhere. This simplifies the equation (M') satisfied by x thus:

$$(x - \underline{x})(x - \bar{x}) \frac{\partial}{\partial x} E(x, \mathbf{a}') = 0 .$$

And to further economize on subscripts, let us introduce certain norms for row and column vectors: let $\|\mathbf{r}\|$ stand for the sum of the magnitudes of the components of any row n-vector $\mathbf{r}$, and let $\|\mathbf{c}\|$ stand for the largest of the magnitudes of the components of any column n-vector $\mathbf{c}$. Then the scalar product $\mathbf{rc}$ is bounded thus: $|\mathbf{rc}| \le \|\mathbf{r}\| \cdot \|\mathbf{c}\|$.

The third factor of the simplified version of (M') in the previous paragraph can be rewritten

$$\frac{\partial}{\partial x} E(x, \mathbf{a}') = \frac{\partial}{\partial x} E(x, \hat{\mathbf{a}}) + \frac{\partial}{\partial x} E'(x, \mathbf{a}') \cdot (\mathbf{a}' - \hat{\mathbf{a}}) ,$$

in which $\frac{\partial}{\partial x} E'$ is a row vector and, as $x \to \hat{x}$ and $\mathbf{a}' \to \hat{\mathbf{a}}$,

$$\frac{\partial}{\partial x} E'(x, \mathbf{a}') \to \frac{\partial}{\partial x} E'(\hat{x}, \hat{\mathbf{a}}) = \frac{\partial^2}{\partial x \, \partial \mathbf{a}} E(\hat{x}, \hat{\mathbf{a}}) .$$

However, $\frac{\partial}{\partial x} E$ is a scalar which, as we have seen earlier, can be written

$$\frac{\partial}{\partial x} E(x, \hat{\mathbf{a}}) = -k(x - \hat{x})^{k-1} (\hat{v} + 0(x - \hat{x})) \quad \text{as } x \to \hat{x}$$

for some integer $k \ge 1$ and nonzero constant

$$\hat{v} = -\left(\frac{\partial}{\partial x}\right)^k E(\hat{x}, \hat{\mathbf{a}}) / k! .$$

Therefore some positive radius ρ must exist such that

$$\left| \frac{\partial}{\partial x} E(x, \hat{\mathbf{a}}) \right| \ge k \, |x - \hat{x}|^{k-1} \, |\hat{v}| / 2 \quad \text{as long as } |x - \hat{x}| \le \rho ,$$

provided $\underline{x} \le x \le \bar{x}$ too of course; if $(\hat{x} - \underline{x})(\hat{x} - \bar{x}) \ne 0$ we shall choose $\rho < \min\{\hat{x} - \underline{x} , \bar{x} - \hat{x}\}$ too. Within that radius ρ of $\hat{x}$ we may define the row vector function

$$\mathbf{r}(x, \mathbf{a}) = -(x - \hat{x})^{k-1} \frac{\partial}{\partial x} E'(x, \mathbf{a}) / \frac{\partial}{\partial x} E(x, \hat{\mathbf{a}}) \qquad \text{if } 0 < |x - \hat{x}| \le \rho$$

$$= -(k-1)! \frac{\partial}{\partial x} E'(\hat{x}, \mathbf{a}) / \left(\frac{\partial}{\partial x}\right)^k E(\hat{x}, \hat{\mathbf{a}}) \qquad \text{if } x = \hat{x} .$$

This row vector is a continuous function of x, and therefore a real analytic function of x and $\mathbf{a}$ in the neighborhood of $x = \hat{x}$ and $\mathbf{a} = \hat{\mathbf{a}}$ at which it takes the finite value

$$\mathbf{r}(\hat{x}, \hat{\mathbf{a}}) = -(k-1)! \frac{\partial^2}{\partial x \, \partial \mathbf{a}} E(\hat{x}, \hat{\mathbf{a}}) / \left(\frac{\partial}{\partial x}\right)^k E(\hat{x}, \hat{\mathbf{a}}) .$$

This row vector figures in an equation

$$(x - \underline{x})(x - \bar{x}) \left[(x - \hat{x})^{k-1} - \mathbf{r}(x, \mathbf{a}') \cdot (\mathbf{a}' - \hat{\mathbf{a}}) \right] = 0 \qquad (\tilde{\mathrm{M}})$$

which is the form that the simplified equation (M') takes after it is multiplied by a factor $(x - \hat{x})^{k-1} / \frac{\partial}{\partial x} E(x, \hat{\mathbf{a}})$ that is finite and nonzero provided

$0 < | x - \hat{x} | \le \rho$. Therefore equations $(\tilde{\mathrm{M}})$ and (M') simplified have the same roots x lying within the distance ρ of $\hat{x}$. We wish to show that as $\mathbf{a}' \to \hat{\mathbf{a}}$ at least one root $x \to \hat{x}$ but the rest of the roots stay away from $\hat{x}$ at least that positive distance ρ independent of $\mathbf{a}'$.

Because $\mathbf{r}(x,\mathbf{a})$ is continuous for $| x - \hat{x} | \le \rho$ and all $\mathbf{a}$ close enough to $\hat{\mathbf{a}}$, we can define for all sufficiently small positive α the function

$$R(\alpha) = \max \|\mathbf{r}(x,\mathbf{a})\| \text{ for } | x - \hat{x} | < \rho, \ \underline{x} \le x \le \bar{x} \text{ and } \|\mathbf{a} - \hat{\mathbf{a}}\| \le \alpha.$$

Evidently $R(\alpha)$ is a monotonic non-decreasing function of α, and therefore some positive α can be chosen to satisfy

$$\alpha\, R(\alpha) < \rho^{k-1} \quad .$$

We shall assume $\|\mathbf{a}' - \hat{\mathbf{a}}\| \le \alpha$ in what follows.

Every root x of $(\tilde{\mathrm{M}})$ must cause at least one of the three factors of that equation to vanish. There are two cases to consider, according as $(\hat{x} - \underline{x})(\hat{x} - \bar{x})$ vanishes or not.

If $\underline{x} < \hat{x} < \bar{x}$ then every root x of $(\tilde{\mathrm{M}})$ within $| x - \bar{x} | \le \rho$ must cause only the third factor of $(\tilde{\mathrm{M}})$ to vanish because we chose $\rho < \min \{ \hat{x} - \underline{x}\ ,\ \bar{x} - \hat{x} \}$. And that third factor

$$(x - \hat{x})^{k-1} - \mathbf{r}(x,\mathbf{a}')\cdot(\mathbf{a}' - \hat{\mathbf{a}})$$

must have at least one root x with $| x - \hat{x} | \le \rho$ as long as $\|\mathbf{a}' - \hat{\mathbf{a}}\| \le \alpha$ for two reasons. First, $k-1$ is odd because $\underline{x} < \hat{x} < \bar{x}$, so $(x - \hat{x})^{k-1}$ reverses sign as x runs from $\hat{x} - \rho$ to $\hat{x} + \rho$. Second,

$$| \mathbf{r}(x,\mathbf{a}')\cdot(\mathbf{a}' - \hat{\mathbf{a}}) | \le \|\mathbf{r}(x,\mathbf{a}')\|\cdot\|\mathbf{a}' - \hat{\mathbf{a}}\|$$

$$\le R(\alpha)\cdot\alpha \ \ < \rho^{k-1}$$

so the third factor must reverse sign too at least once as x runs from $\hat{x} - \rho$ to $\hat{x} + \rho$. Therefore every root x of $(\tilde{\mathrm{M}})$ with $| x - \hat{x} | \le \rho$ satisfies

$$| x - \hat{x} |^{k-1} = | \mathbf{r}(x,\mathbf{a}')\cdot(\mathbf{a}' - \hat{\mathbf{a}}) | \le R(\alpha)\cdot\|\mathbf{a}' - \hat{\mathbf{a}}\|$$
$$\to 0 \quad \text{as } \mathbf{a}' \to \hat{\mathbf{a}},$$

as claimed.

On the other hand, if $\hat{x} = \underline{x}$ then $x = \hat{x}$ is one root of $(\tilde{\mathrm{M}})$ and any other root with $| x - \hat{x} | \le \rho$, if such a root exists, must make the third factor vanish. But now $k-1$ could be even, perhaps zero, so the third factor need not vanish anywhere in $| x - \hat{x} | \le \rho$, although if $k > 1$ and the third factor does vanish then the conclusion at the end of the previous paragraph remains valid. A similar argument dispatches the case $x = \bar{x}$.

Let us now restore the subscript i and the double prime to x_i'', and summarize what is known so far. For every $i = 0,1,2,\ldots,n$ there exist positive constants ρ_i, α_i and $R_i \le \rho_i^{k_i-1} / \alpha_i$ about which the following may be said. Provided all $| x_i' - \hat{x}_i | / \rho_i$ are tiny enough, equation (A) and its immediate consequences ensure that equation (I') will possess just one solution $\mathbf{a}'$ so near $\hat{\mathbf{a}}$ that every $\alpha_i > \|\mathbf{a}' - \hat{\mathbf{a}}\|$. Then every root x_i'' of (M') within $| x_i'' - \hat{x}_i | \le \rho_i$ must satisfy either $x_i'' = \hat{x}_i$ when $k_i = 1$ or $| x_i'' - \hat{x}_i |^{k_i-1} \le R_i \|\mathbf{a}' - \hat{\mathbf{a}}\|$ when $k_i > 1$. Since equation (A) and its consequences imply the existence of some positive constant s depending upon $\hat{J}^{-1}$ and the coefficients $\hat{v}_i$ such that $\|\mathbf{a}' - \hat{\mathbf{a}}\| \le s \cdot \max_j \{ | x_j' - \hat{x}_j |^{k_j} \}$ provided all $| x_i' - \hat{x}_i | / \rho_i$ are tiny enough, we can now relate the new set of differences $\{ x_i'' - \hat{x}_i \}$ to the old $\{ x_j' - \hat{x}_j \}$:

Provided all $|x_i' - \hat{x}_i| / \rho_i$ be tiny enough, either

$$k_i > 1 \quad \text{and} \quad |x_i'' - \hat{x}_i|^{k_i - 1} \le R_i \cdot s \cdot \max_j \{ |x_j' - \hat{x}_j|^{k_j} \} \ ,$$

or

$$k_i = 1 \quad \text{and} \quad x_i'' = \hat{x}_i \ .$$

All that remains is to show why the last inequality implies superlinear convergence of the Remes iteration.

After every iterate x_i' has approached its respective limiting value $\hat{x}_i$ closely enough, all subsequent iterates x_i'' will stick at $x_i'' = \hat{x}_i$ for each $k_i = 1$. Therefore restrict the index i henceforth to those of the integers $0, 1, 2, \ldots, n$ for which $k_i > 1$ and define

$$\lambda = \max_i (R_i \cdot s)^{k_i} \quad \text{and} \quad k = \max_i k_i \ .$$

We assume $k > 1$ since convergence must be very sudden otherwise. We shall use

$$\mu' = \lambda \cdot \max_j |x_j' - \hat{x}_j|^{k_j} \quad \text{and} \quad \mu'' = \lambda \cdot \max_i |x_i'' - \hat{x}_i|^{k_i}$$

to measure the error in the respective sets $\{x_j'\}$ and $\{x_i''\}$ of iterates. The previous paragraph's inequality implies:

$$\begin{aligned} \lambda\, |x_i'' - \hat{x}_i|^{k_i} &\le \lambda (R_i \cdot s \cdot \mu' / \lambda)^{k_i/(k_i - 1)} \\ &= ((R_i \cdot s)^{k_i} / \lambda)^{1/(k_i - 1)} \ (\mu')^{k_i/(k_i-1)} \\ &\le (\mu')^{k_i/(k_i - 1)} \quad , \text{ so} \\ \mu'' \quad &\le (\mu')^{k/(k-1)} \end{aligned}$$

provided μ' be tiny enough. Similarly, if μ''' measures the error in the set $\{x_i'''\}$ of iterates that follow $\{x_i''\}$,

$$\mu''' \le (\mu'')^{k/(k-1)} \le (\mu')^{(k/(k-1))^2} \ .$$

More generally, the m^{th} set of iterates after $\{x_j'\}$ can have its error measure no bigger than

$$(\mu')^{(k/(k-1))^m}$$

which converges to zero superlinearly as $m \to \infty$, as claimed.

The foregoing proof is not altogether reassuring to the Numerical Analyst because it includes the possibility that k might be quite large, in which case the equation (M') to be solved for the roots $\{x_i''\}$ might be very nearly an equation some of whose roots have multiplicities as large as $k - 1$. Therefore some roots x_i'' might be very ill-conditioned, in effect practically impossible to determine accurately. Closer scrutiny of the proof will show that that ill-condition afflicts only the nearly multiple roots x_i'' ; the coefficients $\hat{\mathrm{a}}$ and $\hat{e}$ are not degraded by the inaccuracy to which $\hat{x}_i$ can be calculated. This is so because the error $x_i'' - \hat{x}_i$ influences the next iteration, where it has been renamed $x_i' - \hat{x}_i$, mainly by its appearance in equation (A)'s right-hand side terms

$$v_i(x_i') = (-1)^i \hat{e} - E(x_i', \hat{\mathrm{a}}) = (x_i' - \hat{x}_i)^{k_i} (\hat{v}_i + 0(x_i' - \hat{x}_i)) \ .$$

The proof above shows that these terms converge to zero superlinearly, which means in practice that these terms soon become small comparable to the roundoff that accrues during the evaluation of $E(x_i', \mathrm{a}')$. Consequently, the implementor of the Remes iteration should not expect successive sets $\{x_i'\}$ to settle down numerically, but should test some other variables to decide when to stop.

A better way to monitor convergence is via comparison of successive values

$$|e'| = |E(x_i', \mathbf{a}')| \quad \text{for } i = 0,1,2,\ldots,n$$

(obtained when (I') is solved) with corresponding values

$$|d'| = \max |E(x_i'', \mathbf{a}')| \quad \text{over } i = 0,1,2,\ldots n$$

obtainable after (M') has been solved. Provided precautions have been taken to enhance the prospect that $x = x_i''$ is where $(-1)^i E(x, \mathbf{a}')/e'$ achieves its maximum value for $x_{i-1}' \le x \le x_{i+1}'$ (with the understanding that $x_{-1}' = \underline{x}$ and $x_{n+1}' = \bar{x}$) there is every reason to expect $|d'| > |e'|$. Normally successive iterations will produce convergent sequences of values $|d'|$ decreasing towards $|\hat{e}|$ while $|e'|$ increases towards $|\hat{e}|$. Indeed, this monotonic behavior is guaranteed if the family of functions $E(x, \mathbf{a})$ possess a local Haar property (Meinardus, [1967, p. 142]). Therefore a reasonable time to stop the Remes iteration is when $|d'|$ and $|e'|$ approach each other so closely that $|\hat{e}|$, which probably lies between them, is determined accurately enough for practical purposes.

References

C. B. Dunham [1968] "Transformed Rational Chebyshev Approximation II" Numer. Math. **12**, 8-10.

G. Meinardus [1967] "Approximation of Functions: Theory and Numerical Methods" translated by L. L. Schumaker, volume 13 of "Springer Tracts in Natural Philosophy", Springer-Verlag, NY.

A. Ralston [1965] "Rational Chebyshev approximation by Remes' algorithms" Numer. Math. 7, 322-330.

J. R. Rice [1964] "The Approximation of Functions" volume 1, and volume 2 in 1969, Addison-Wesley, Reading, MA.

L. Veidinger [1960] "On the numerical determination of the best approximations in the Chebyshev sense" Numer. Math. **2**, 99-105.

H. Werner [1962-3] Three papers (in German) on best rational approximation in Arch. Rational Mech. Anal. **10**, 205-219 (1962), **11**, 368-384 (1962) and **13**, 330-347 (1963).