FIVE FRIGHTENING FACTS
ABOUT
FLOATING-POINT ARITHMETIC

W. Kahan May 3, 1988

1. WHAT YOU SEE IS NOT NECESSARILY WHAT YOU GET.

2. WHAT YOU GET IS NOT NECESSARILY WHAT YOU EXPECT.

3. MANY A DISCREPANCY BETWEEN WHAT YOU GET AND WHAT YOU EXPECT IS MOST UNLIKELY EVER TO BE UNDERSTOOD, MUCH LESS CORRECTED, EVEN THOUGH IN PRINCIPLE EVERY DISCREPANCY CAN BE CORRECTED.

4. SIGNIFICANT DISCREPANCIES ARE VERY RARE, TOO RARE TO WORRY ABOUT ALL THE TIME, YET NOT RARE ENOUGH TO IGNORE.

5. THE DISCREPANCIES THAT ALREADY AFFLICT FLOATING-POINT ARITHMETIC DO **NOT** GRANT THE SYSTEM DESIGNER

CARTE BLANCHE

TO ADD A FEW MORE OF HIS OWN.

---------- Borland Turbo Basic on an IBM PC ----------

```
+---------------------------- Edit -----------------------------+
|   D:DISNT2.BAS                                                |
|                                                               |
|   CLS                                                         |
|   p = 2 :  for i=1 to 6 :  p = p*p :  next i                  |
|   pp1 = p + 1 :   pm1 = p - 1                                 |
|   d = pp1 - pm1                                               |
|   Print "  We expect d = 2 ,  but actually  d = "; d ; " ,"   |
|   Print "  although  (p+1) - (p-1) = "; (p+1) - (p-1) ; " . !"|
|   Print                                                       |
|   Print "  What you get isn't necessarily what you expected."|
|   End                                                         |
+---------------------------------------------------------------+
+---------------------------- Run ------------------------------+
|                                                               |
|     We expect d = 2 ,  but actually  d =  0  ,                |
|     although  (p+1) - (p-1) =  1  . !                         |
|                                                               |
|     What you get isn't necessarily what you expected.         |
|                                                               |
+-----------------+-------------------------+-----------------+
                  +-------------------------+
```

WHAT YOU GET ISN'T NECESSARILY WHAT YOU EXPECTED.

```
+-------------------------------- Edit ---------------------------------+
|     D:SIXTH.BAS                                                       |
|                                                                       |
|     CLS                                                               |
|     pOK  =  ((((2.0^2)^2)^2)^2)^2                                     |
|     pBAD = (((((2.0^2)^2)^2)^2)^2)^2 ' <<< Error 5: Illegal function call |
|     Print                                                             |
|     Print "This  Error  is really caused by misuse of the  8087's stack." |
|     End                                                               |
+-----------------------------------------------------------------------+
+-------------------------------- Run ----------------------------------+
|                                                                       |
|                +-------------- Message --------------+                |
|                |  Error search:  SIXTH                |                |
|                |  Time: 00:00                         |                |
|                |  Line:    1  Stmt:    1  Free:  190k |                |
|                +                                      +                |
|                +--------------------------------------+                |
```

YOU ARE UNLIKELY EVER TO UNDERSTAND, MUCH LESS TO REMEDY EVERY ANOMALY.

---------- Borland Turbo Basic on an IBM PC ----------

```
+------------------------- Edit -------------------------------+
|     D:WYSINWYG.BAS                                           |
|                                                              |
|     CLS                                                      |
|     q = 3.0/7.0                                              |
|     Print "  The value of  q = "; q                         |
|     Print "     but  3.0/7.0 = "; 3.0/7.0                    |
|     Print                                                    |
|     Print "  What You See Is Not Necessarily What You Get." |
|     End                                                      |
+--------------------------------------------------------------+
+------------------------- Run --------------------------------+
|                                                              |
|     The value of  q =  .4285714328289032                     |
|        but  3.0/7.0 =  .4285714285714286                     |
|                                                              |
|     What You See Is Not Necessarily What You Get.            |
|                                                              |
+---------------+-----------------------------+----------------+
                +-----------------------------+
```

```
+------------------------- Edit -------------------------------+
|     D:WYS1NWYG.BAS                                           |
|                                                              |
|     CLS                                                      |
|     z = 0          :  Print Using  "   z = ###.###"; z       |
|     y = 0.000123   :  Print Using  "   y = ###.###"; y       |
|     x = y/100      :  Print Using  "   x = ###.###"; x       |
|                       Print Using  " y/x = ###.###"; y/x     |
|     Print                                                    |
|     Print "  What You See Is Not Necessarily What You Get." |
|     End                                                      |
+--------------------------------------------------------------+
+------------------------- Run --------------------------------+
|                                                              |
|     z =   0.000                                              |
|     y =   0.000                                              |
|     x =   0.000                                              |
|   y/x = 100.000                                              |
|                                                              |
|   What You See Is Not Necessarily What You Get.              |
+---------------+-----------------------------+----------------+
                +-----------------------------+
```

WHAT YOU SEE IS NOT NECESSARILY WHAT YOU GET.

---------- Borland Turbo Basic on an IBM PC ----------

```
+------------------------ Edit -------------------------------+
|  D:2J&2K.BAS                                                 |
|                                                              |
|  for i = 1 to 8000     :     tj = 1     '          j  k   d  |
|  for j = 0 to 15       :     tk = 1     '          ~~ ~~  ~~ |
|     for k = 0 to j                      '          0  0   2  |
|        d = tj + tk  ' ... = 2^j + 2^k              1  0   3  |
|        q = i/d   :   x = q*d    '       <----<<    1  1   4  |
|        IF NOT( x=i ) THEN       '       <----<<    2  0   5  |
|           print "x = ";x;" NOT = ";i '<----<<    2  1   6  |
|           STOP :  END IF        '       <----<<    2  2   8  |
|        tk = tk+tk ' ... = 2^(k+1)                  3  0   9  |
|        next k                   '                  3  1  10  |
|     tj = tj+tj '     ... = 2^(j+1)                 3  2  12  |
|     next j  :   next i          '                  3  3  16  |
|  print  "  x = i  ALWAYS !"  '... and  d = 17, 18, 20, 24, 32,|
|  end                          '...       33, 34, 36, 40, 48, ...|
+--------------------------------------------------------------+

+------------------------ Run --------------------------------+
|                                                              |
|    x = i  ALWAYS !                                           |
|                                                              |
+------------------+--------------------+--------------------+
                   +--------------------+
```

A PECULIAR PROPERTY OF

DIVISION AND MULTIPLICATION

WHEN ROUNDED ACCORDING TO

IEEE STANDARD 754

This very simple program is certain to stop prematurely if the computer's floating-point arithmetic uses any other radix than 2 (binary), and almost certain to stop prematurely if division or multiplication is not rounded according to the IEEE standard. It stops prematurely on IBM 370s, DEC VAXs, CDC Cybers, CRAYs, all decimal calculators, ... But it says " x = i ALWAYS ! " on IBM PCs that use an 80x87 math. coprocessor, on all Apples that use SANE, on all SUN IIIs, on the ELXSI 6400, ...

CAN YOU EXPLAIN THIS ?