Trajectory estimation from GPS data using an adaptive smoothing spline

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Overview

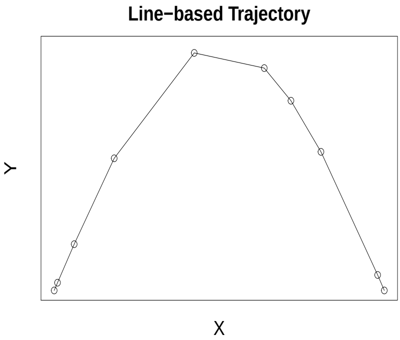
1.1 GPS points and Trajectory

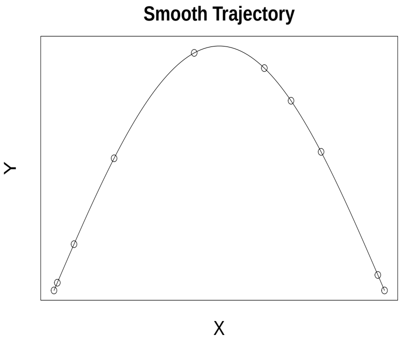
GPS units record time series data of a moving object. These data are in the form of

$$T = \{p_t = [x_t, y_t, v_t, \omega_t, b_t, \cdots]|t \in \mathbb{R}\}. \tag{1}$$

x	longitude
y	latitude
v	velocity
ω	bearing
b	boom status

Trajectory is a connection by a time series successive position recorded by GPS devices.


Line−based Trajectory
Y
X


Smooth Trajectory
Y
X

2.1 A New Objective Function

If we have some knots, such that $a < t_1 < \cdots < t_n < b$, and $z_i = (x_i, y_i)$, $w_i = (u_i, v_i)$, for $i = 1, 2, \cdots, n$, and a positive piecewise parameter $\lambda(t)$, which will control penalty functions, then the equation

$$J[f] = \frac{1}{n}\sum_{i=1}^{n} |f(t_i) - z_i|^2 + \frac{\gamma}{n}\sum_{i=1}^{n} |f'(t_i) - w_i|^2 + \sum_{i=1}^{n} \lambda_i \int_{t_i}^{t_{i+1}} |f''(t)|^2 dt \quad (2)$$

is minimised by a tractor spline, which is linear outside the knots.

2.2 Basis Funtions of Tractor Spline

Hermite Spline basis functions on an arbitrary interval $[t_i, t_{i+1}]$

$$h_{00}^{(i)}(t) = \begin{cases} 2(\frac{t-t_i}{t_{i+1}-t_i})^3 - 3(\frac{t-t_i}{t_{i+1}-t_i})^2 + 1, & t_i \le t \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{10}^{(i)}(t) = \begin{cases} \frac{(t-t_i)^3}{(t_{i+1}-t_i)^2} - 2\frac{(t-t_i)^2}{t_{i+1}-t_i} + (t - t_i), & t_i \le t \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{01}^{(i)}(t) = \begin{cases} -2(\frac{t-t_i}{t_{i+1}-t_i})^3 + 3(\frac{t-t_i}{t_{i+1}-t_i})^2, & t_i \le t \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{11}^{(i)}(t) = \begin{cases} \frac{(t-t_i)^3}{(t_{i+1}-t_i)^2} - \frac{(t-t_i)^2}{t_{i+1}-t_i}, & t_i \le t \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

To construct a tractor spline on the entire interval $[t_1, t_n]$, the new basis functions are defined in such way, that $N_1 = h_{00}^{(1)}$, $N_2 = h_{10}^{(1)}$, and for all $k = 1, 2, \ldots, n-2$,

$$N_{2k+1} = \begin{cases} h_{01}^{(k)} + h_{00}^{(k+1)} & \text{if } t < t_n \\ 2(\frac{t-t_{n-1}}{t_n-t_{n-1}})^3 - 3(\frac{t-t_{n-1}}{t_n-t_{n-1s}})^2 + 1 & \text{if } t = t_n \end{cases}, \tag{3}$$

$$N_{2k+2} = \begin{cases} h_{11}^{(k)} + h_{10}^{(k+1)} & \text{if } t < t_n \\ \frac{(t-t_{n-1})^3}{(t_n-t_{n-1})^2} - 2\frac{(t-t_{n-1})^2}{t_n-t_{n-1}} + (t - t_{n-1}) & \text{if } t = t_n \end{cases}, \tag{4}$$

and

$$N_{2n-1} = \begin{cases} h_{01}^{(n-1)} & \text{if } t < t_n \\ -2(\frac{t-t_{n-1}}{t_n-t_{n-1}})^3 + 3(\frac{t-t_{n-1}}{t_n-t_{n-1}})^2 & \text{if } t = t_n \end{cases}, \tag{5}$$

$$N_{2n} = \begin{cases} h_{11}^{(n-1)} & \text{if } t < t_n \\ \frac{(t-t_{n-1})^3}{(t_n-t_{n-1})^2} - \frac{(t-t_{n-1})^2}{t_n-t_{n-1}} & \text{if } t = t_n \end{cases}. \tag{6}$$

Theorem

The functions $N_1, \ldots, N_{2n}$ *provide a basis for the set of functions on* $[t_1, t_n]$ *which are continuous, have continuous first and second derivatives and which are cubic on each open interval* (t_i, t_{i+1}).

Solution to The New Objective Function

The tractor spline $f(t)$ is a linear combination of basis functions

$$f(t) = \sum_{j=1}^{2n} N_j(t)\theta_j$$

and the objective function reduces to

$$\text{MSE}(\theta, \lambda, \gamma) = (\mathbf{z} - \mathbf{B}\theta)^T(\mathbf{z} - \mathbf{B}\theta) + \gamma(\mathbf{w} - \mathbf{C}\theta)^T(\mathbf{w} - \mathbf{C}\theta) + n\theta^T\mathbf{\Omega}_\lambda\theta,$$

where $\mathbf{z} = \{z(x_i, y_i)\}$ are the knots and $\mathbf{w} = \{(u_i, v_i)\}$ are the tangent at knots.

After solving the above equation, we have

$$\hat{\theta} = (\mathbf{B}^T\mathbf{B} + \gamma\mathbf{C}^T\mathbf{C} + n\mathbf{\Omega}_\lambda)^{-1}(\mathbf{B}^T\mathbf{z} + \mathbf{C}^T\mathbf{w}). \tag{7}$$

Then tractor spline could be represented as

$$\hat{f}(t) = \sum_{j=1}^{2n} N_j(t)\hat{\theta}_j \tag{8}$$

2.3 Cross Validation of Tractor Spline

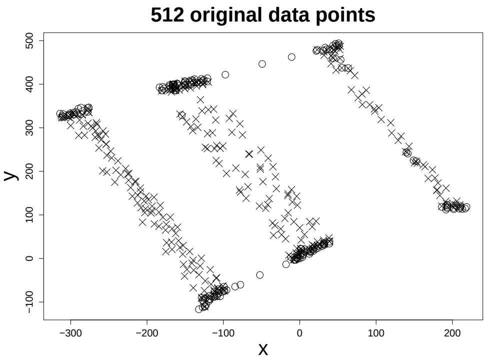
Because $\hat{f}$ and $\hat{f}'$ could be written in the form of

$$\hat{f} = B\hat{\theta} = Sz + \gamma Tw$$
$$\hat{f}' = C\hat{\theta} = Uz + \gamma Vw$$

Then

$$\begin{aligned} \mathrm{CV} &= \frac{1}{n}\sum(\hat{f}^{(-i)}(t_i) - z_i)^2 \\ &= \frac{1}{n}\sum\left(\frac{\hat{f}(t_i) - z_i + \gamma\frac{T_{ii}}{1-\gamma V_{ii}}(\hat{f}'(t_i) - w_i))}{1 - S_{ii} - \gamma\frac{T_{ii}}{1-\gamma V_{ii}}U_{ii}}\right)^2 \end{aligned}$$

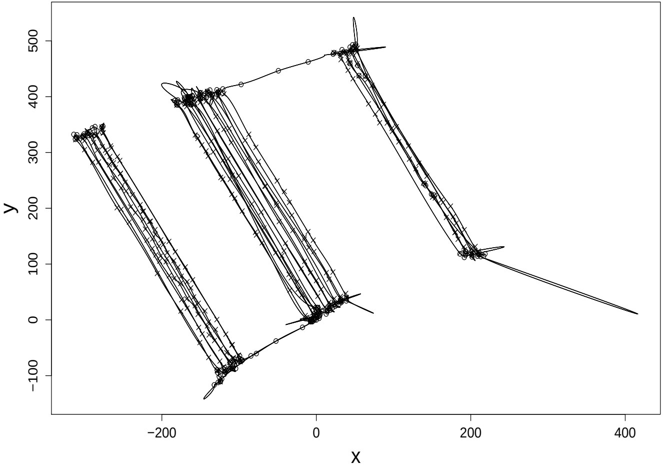
Application



Regarding to the boom status (0 and 1), we divided the penalty function into two parts: Ω_u and Ω_d. That brings two penalty parameters λ_u and λ_d.

Thus, we totally have three parameters λ_u, λ_d and γ.

Trajectory Reconstruction



The crazy curves are caused by tricky points, where there is a break and a long time gap. At these points the velocity $v \approxeq 0$.
To adjust this issue, we bring an adjusted penalty term

$$\lambda(t) = \begin{cases} \frac{\Delta t_i^3}{\Delta d_i^2}\lambda_d, & \text{when } b = 1 \\ \frac{\Delta t_i^3}{\Delta d_i^2}\lambda_u, & \text{when } b = 0 \end{cases}, t_i \leq t < t_{i+1}. \tag{9}$$

Penalty Function

The penalty parameter λ becomes a function $\lambda(t)$ that varying on different domains.

$$\lambda(t) = b\frac{\Delta t^3}{\Delta d^2}\lambda_d + (1-b)\frac{\Delta t^3}{\Delta d^2}\lambda_u, \text{where} \begin{cases} b = 1 & \text{if boom is working} \\ b = 0 & \text{if boom is not working} \end{cases}. \tag{10}$$

3.1 Gaussian Process Regression

If the observed function $f(t)$ has zero mean, and

$$f(t) \sim N(0, k(t, t'))$$

where $k(t, t')$ is covariance matrix, then the prediction mean is

$$\bar{f}_* = K_* K^{-1} y \tag{11}$$

and uncertainty in this estimate is captured in its variance

$$var(f_*) = K_{**} - K_* K^{-1} K_*^T \tag{12}$$

3.2 Hilbert Space and Reproducing Kernel for Tractor Spline

For any $f \in \mathbb{H}$, $f = \sum_{i=1}^{2n} \theta_i N_i(t)$. Building up a new space

$$\mathcal{C}^2_{p.w.}[0,1] = \{f : f, f' \text{ continuous, } f'' \text{ piecewise continuous on } [0,1]\}.$$

Equipped with an appropriate inner product

$$(f,g) = f(0)g(0) + f'(0)g'(0) + \int_0^1 f''g''dx, \tag{13}$$

the space $\mathcal{C}^2_{p.w.}[0,1]$ is made a reproducing kernel Hilbert space.

$f \in \mathcal{C}^2_{p.w.}[0,1]$ can be written as

$$f(x) = d_1 + d_2 x + \sum_{j=1}^{n} c_j R_1(x_j, x) + \sum_{i=j}^{n} b_j \dot{R}_1(x_j, \cdot), \tag{14}$$

where $\mathbf{d}, \mathbf{c}$ and $\mathbf{b}$ are coefficients. $f = \phi^\top \mathbf{d} + \xi^\top \mathbf{c} + \psi^\top \mathbf{b}$, with the coefficients given by

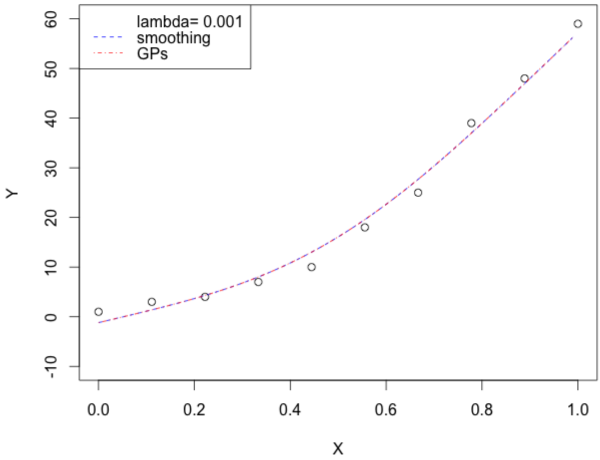
$$\mathbf{d} = (T^\top M^{-1} T)^{-1} T^\top M^{-1} \begin{bmatrix} Y \\ V \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} = (M^{-1} - M^{-1} T (T^\top M^{-1} T)^{-1} T^\top M^{-1}) \begin{bmatrix} Y \\ V \end{bmatrix},$$

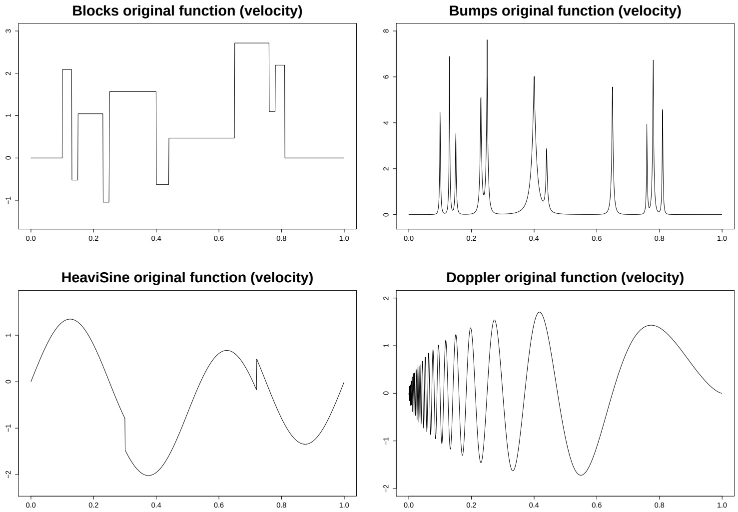
where $T = \begin{bmatrix} S \\ S' \end{bmatrix}$ and $M = \begin{bmatrix} Q + n\lambda I & P \\ Q' & P' + \frac{n\lambda}{\gamma} I \end{bmatrix}$.

Numeric Testing

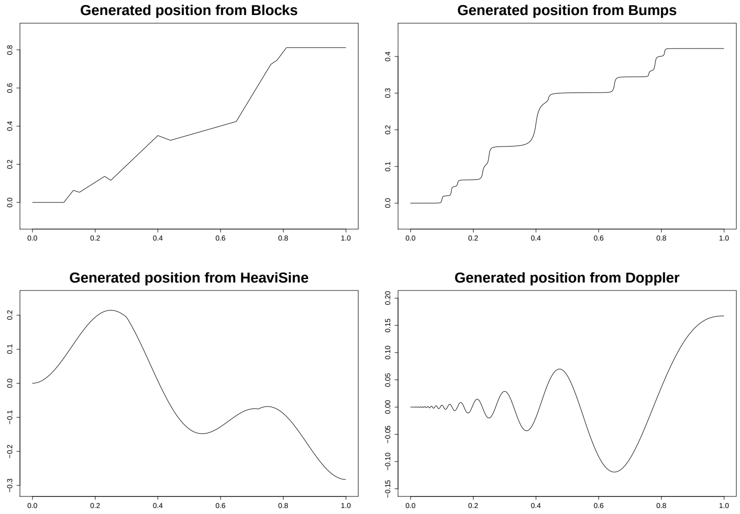
When λ is chosen 0.001, smoothing spline and GPs return the same results.



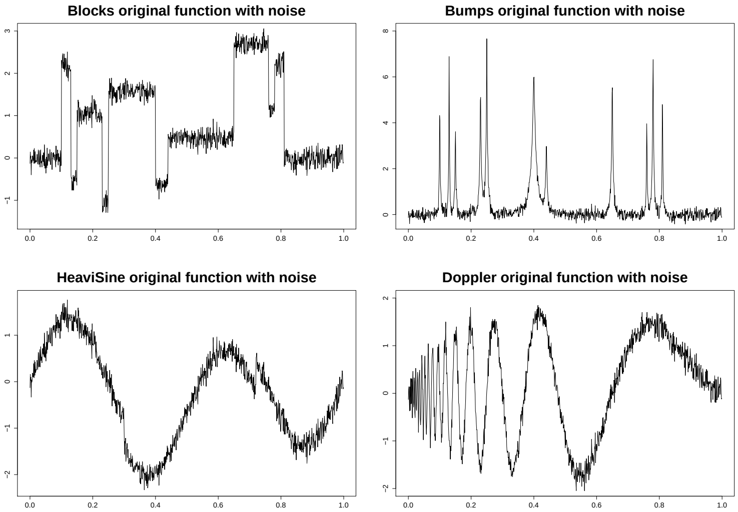
4.1 Simulation



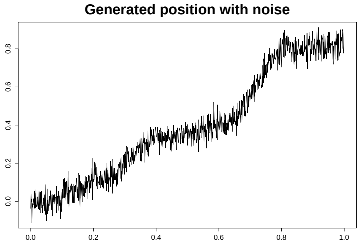
By setting initial $y_0 = 0$ and $a_0 = 0$, and $y_{i+1} = y_i + (v_i + v_{i+1})\frac{t_{i+1} - ti}{2}$ to calculate positions.

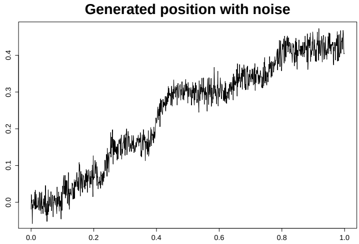


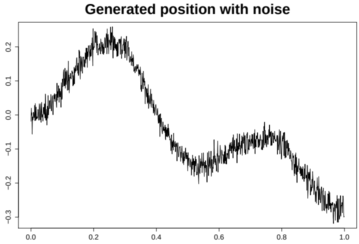
Original Functions with Noises

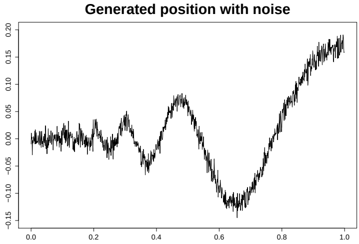


Generated Position Functions with Noises



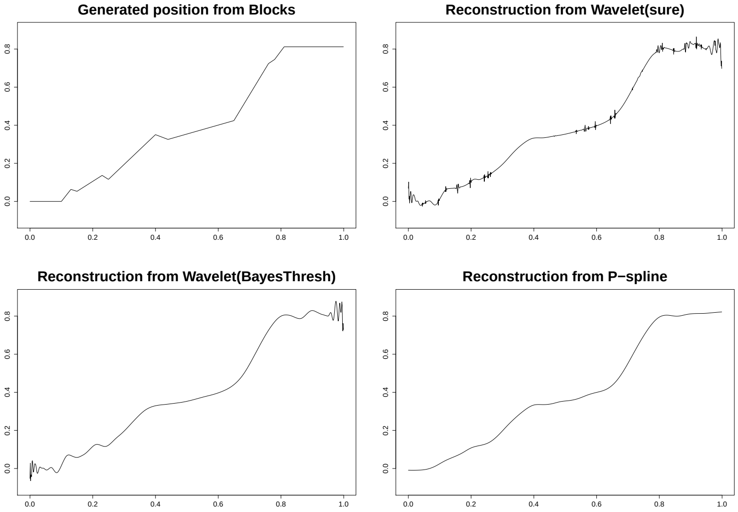




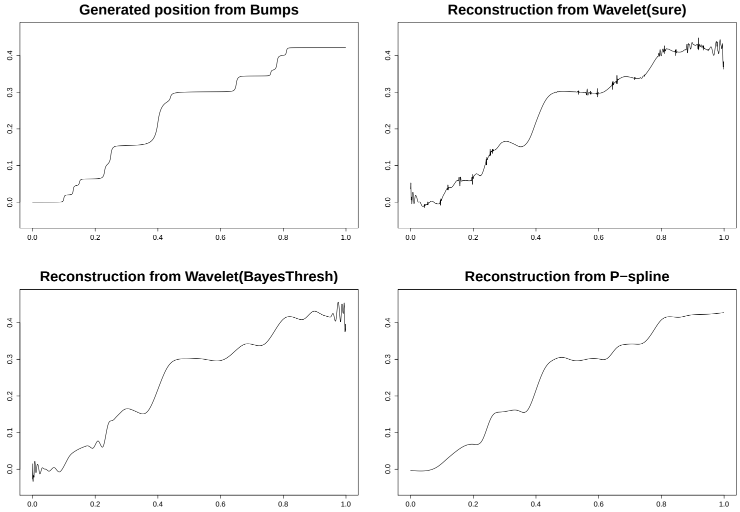


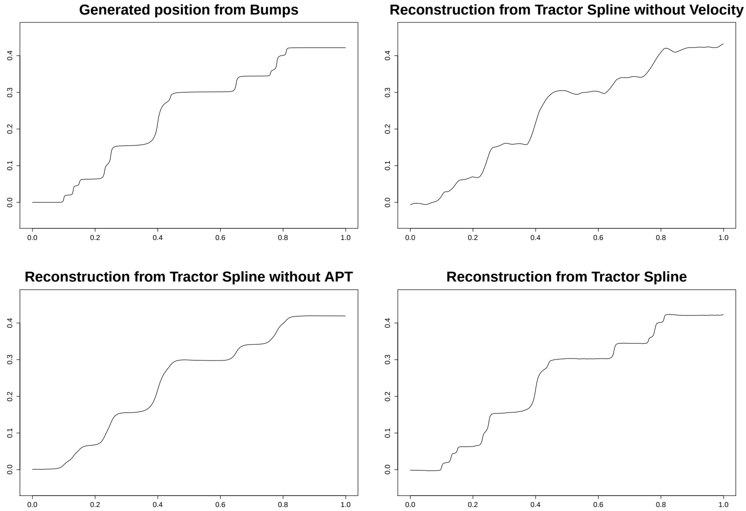
Comparing with Other Methods

- wavelet(sure) in *Library("WaveThresh")* of *R*
- wavelet(Bayesian)
- P-spline (penalized B spline)
- Tractor Spline without velocity term in MSE ($\gamma = 0$)
- Tractor Spline without adjusted penalty term ($\frac{\Delta t^0}{\Delta d^0}$)
- Tractor Spline

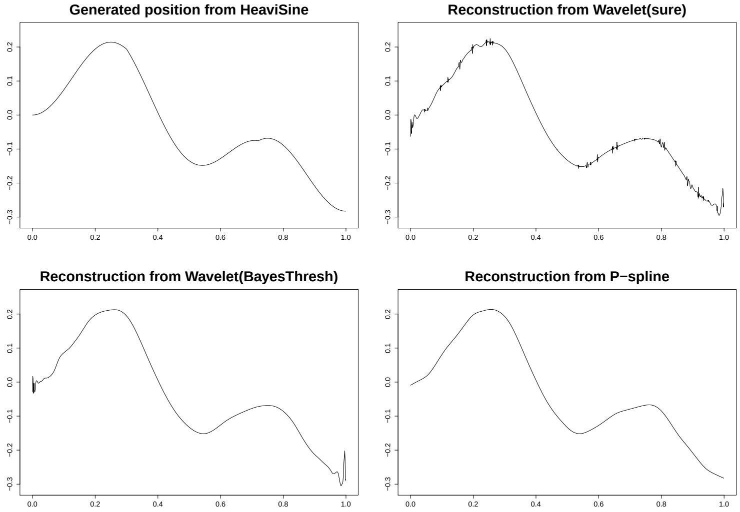

Generated position from Blocks
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
0.0
0.2
0.4
0.6
0.8
1.0

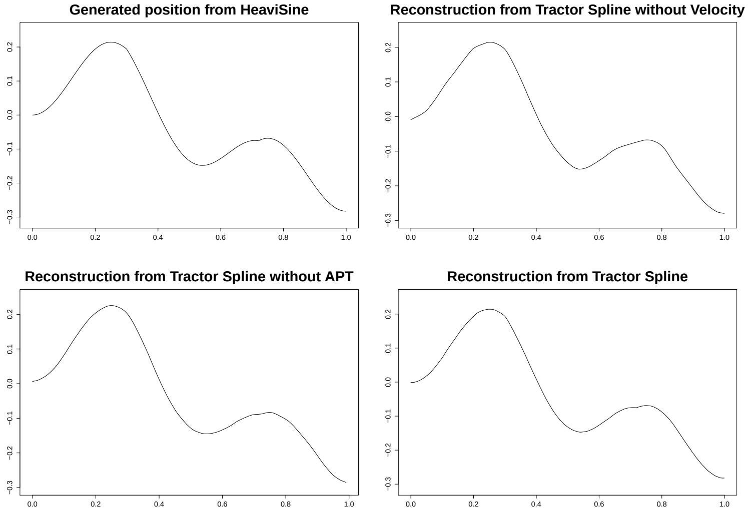

Generated position from Blocks
Reconstruction from Tractor Spline without Velocity
Reconstruction from Tractor Spline without APT
Reconstruction from Tractor Spline
0.0
0.2
0.4
0.6
0.8
1.0

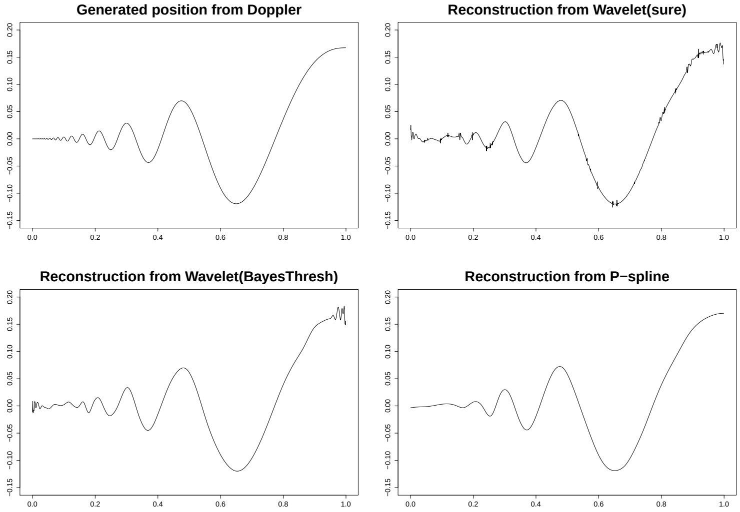

Generated position from Bumps
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
0.0
0.1
0.2
0.3
0.4
0.6
0.8
1.0

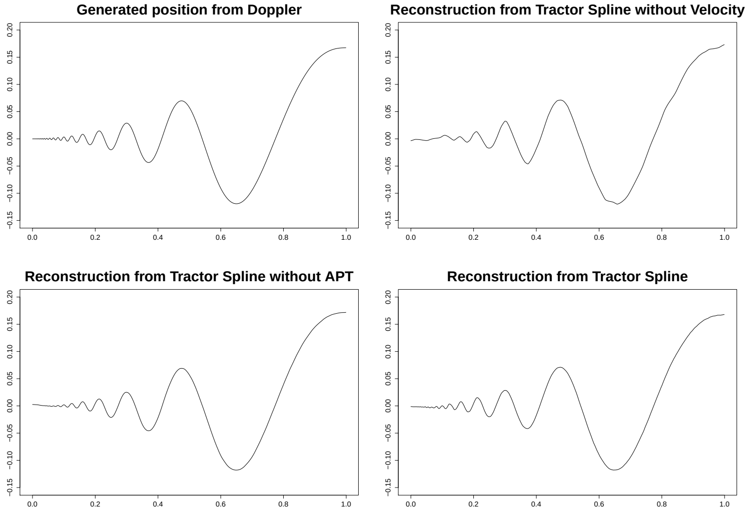


Generated position from Bumps
Reconstruction from Tractor Spline without Velocity
Reconstruction from Tractor Spline without APT
Reconstruction from Tractor Spline
0.0 0.1 0.2 0.3 0.4
0.0 0.2 0.4 0.6 0.8 1.0


Generated position from HeaviSine
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
0.0
0.2
0.4
0.6
0.8
1.0
−0.3
−0.2
−0.1
0.0
0.1
0.2


Generated position from HeaviSine
Reconstruction from Tractor Spline without Velocity
Reconstruction from Tractor Spline without APT
Reconstruction from Tractor Spline


Generated position from Doppler
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
0.0 0.2 0.4 0.6 0.8 1.0
−0.15 −0.10 −0.05 0.00 0.05 0.10 0.15 0.20


Generated position from Doppler
Reconstruction from Tractor Spline without Velocity
Reconstruction from Tractor Spline without APT
Reconstruction from Tractor Spline
−0.15
−0.10
−0.05
0.00
0.05
0.10
0.15
0.20
0.0
0.2
0.4
0.6
0.8
1.0

$$\mathrm{MSE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{f}_{\lambda,\gamma}(t_i))^2, \tag{15}$$

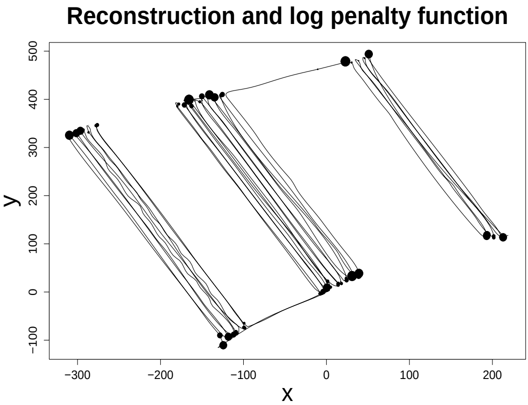
MSE (10^{-4})	SNR	Tractor Spline	TS $\gamma = 0$	TS without APT	P-spline	Wavelet(sure)	Wavelet(Bayes)
Blocks	7	16.53	15.99	16.69	16.14	*15.39	16.68
Blocks	3	89.79	*87.64	89.94	88.27	98.35	90.24
Bumps	7	4.40	4.19	4.55	4.33	*4.18	4.59
Bumps	3	23.93	*23.19	24.10	23.55	26.23	23.74
HeaviSine	7	4.16	4.01	4.16	4.02	*3.79	4.19
HeaviSine	3	22.63	*22.19	22.65	22.02	23.53	22.07
Doppler	7	1.15	*1.07	1.10	1.15	*1.07	1.13
Doppler	3	6.27	*5.94	6.28	6.05	6.85	6.29

$$\mathrm{TMSE} = \frac{1}{n}\sum_{i=1}^{n}(f(t_i) - \hat{f}_{\lambda,\gamma}(t_i))^2. \tag{16}$$

TMSE(10^{-6})	SNR	Tractor Spline	TS $\gamma = 0$	TS without APT	P-spline	Wavelet(sure)	Wavelet(Bayes)
Blocks	7	*1.75	54.25	28.68	54.76	201.02	182.12
Blocks	3	*16.44	152.5	30.76	171.59	1138.08	712.36
Bumps	7	*1.64	23.44	21.10	24.21	71.71	69.26
Bumps	3	*8.51	77.78	37.12	77.52	330.77	238.79
HeaviSine	7	*1.53	7.80	1.56	9.54	55.37	44.88
HeaviSine	3	*8.21	33.56	8.49	34.26	240.72	110.49
Doppler	7	1.51	6.67	*1.08	8.26	14.87	12.01
Doppler	3	*8.10	22.14	8.25	19.95	81.48	50.33

4.2 Application

Black dots represent the value of $\log(\lambda(t))$. The bigger penalty values, the larger dots.



5.1 Future Work

- Moving from batch inference to online inference
- Deducing high-level description of farm vehicle motion

5.2 Acknowledgment

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  - Grant UOOX 1208
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Ministry of Business, Innovation & Employment

TRACMAP AGRICULTURE

6 References

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