Spline-based approach to infer farm vehicle trajectories

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19 May 2016

Overview

1.1 GPS points and Trajectory

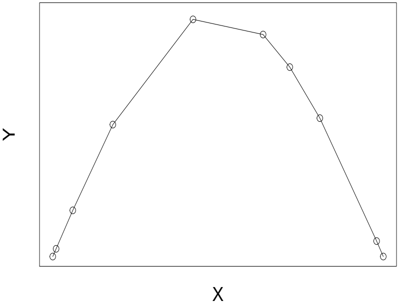
GPS units record time series data of a moving object. These data are in the form of

$$T = \{p_t = [x_t, y_t, v_t, \omega_t, b_t, \cdots] | t \in \mathbb{R}\}. \quad (1)$$

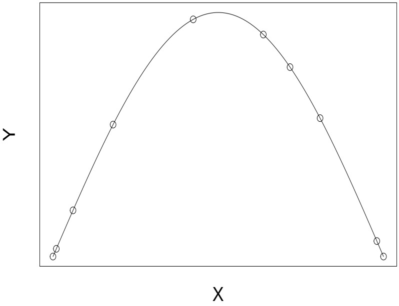
x	longitude
y	latitude
v	velocity
ω	barrier
b	boom status

Trajectory is a connection by a time series successive position recorded by GPS devices.

Line−based Trajectory



Smooth Trajectory



1.2 Current Methods

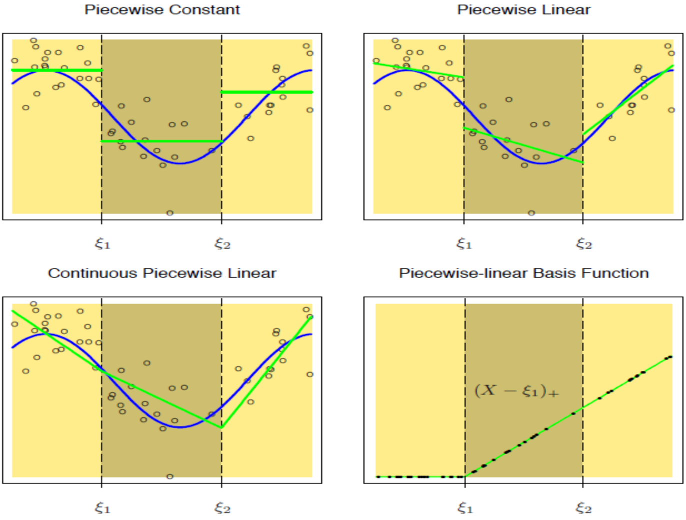
- Curve-based method: $P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ to obtain a spline that passes through any given sequence of points.
- Parametric cubic Bézier curves, if $0 \leq t \leq 1$, then $B(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t) t^2 P_2 + t^3 P_3$.
- B-spline: allows continuity of order two between the curve segments and goes through the points smoothly.
- Hermite interpolation: $p(t) = (2t^3 - 3t^2 + 1) p_0 + (t^3 - 2t^2 + t) m_0 + (-2t^3 + 3t^2) p_1 + (t^3 - t^2) m_1$.

1.3 Smoothing Spline

Denote by $h_m(t) : \mathbb{R}^p \to \mathbb{R}$ the mth transformation of T, $m = 1, ..., M$. We then model

$$f(t) = \sum_{m=1}^{M} \beta_m h_m(t).$$

a linear basis expansion in Y.



[$T.Hastie, R.Tibshirani, J.Friedman, and J.Franklin (2001)$]

Consider the following problem: among all functions $f(t)$ with two continuous derivatives, find one that minimizes the penalized residual sum of squares

$$\text{MSE}(f, \lambda) = \frac{1}{n} \sum_{i=1}^{n} (f(t_i) - y_i)^2 + \lambda \int_0^1 (f''(t))^2 dx \qquad (2)$$

where $\lambda \geq 0$ is a fixed smoothing parameter, (t_i, y_i), $i = 1, \cdots, n$ are observed data and $0 < t_1 < \cdots < t_n < 1$.

$\lambda = 0$: f can be any function that interpolates the data.

$\lambda = \infty$: the simple least squares line fit.

1.4 Cross Validation

The next step is finding best parameter λ. Considering data (t_i, y_i), $i = 1, \cdots, n$. The leave-one-out cross validation error is

$$\mathrm{CV} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{f}^{(-i)}(t_i))^2$$

where $\hat{f}^{(-i)}(t)$ is the smoothing spline without the datapoint (t_i, y_i). So the best parameter λ is the one that could minimize CV-errors.

2.1 A New Objective Function

If we have some knots, such that $a < t_1 < \cdots < t_n < b$, and $z_i = (x_i, y_i)$, $w_i = (u_i, v_i)$, for $i = 1, 2, \cdots, n$, and a positive piecewise parameter $\lambda(t)$, which will control penalty functions, then the equation

$$J[f] = \frac{1}{n}\sum_{i=1}^{n}|f(t_i)-z_i|^2+\frac{\gamma}{n}\sum_{i=1}^{n}|f'(t_i)-w_i|^2+\sum_{i=1}^{n}\lambda_i\int_{t_i}^{t_{i+1}}|f''(t)|^2dt \quad (3)$$

is minimised by a tractor spline, which is linear outside the knots.

Tractor Spline

The solution to objective function (3) is called tractor spline, which on each interior interval (t_i, t_{i+1}), $i = 2, \cdots, n-2$, $f(t)$ is a cubic polynomial, but on interval (t_1, t_2) and (t_{n-1}, t_n) can be a linear function; it fits together at each point t_i in such a way that $f(t)$ itself and its first and second derivatives are continuous at each t_i, $i = 2, \cdots, n-2$.

2.2 Basis Funtions of Tractor Spline

Hermite Spline basis functions on an arbitrary interval $[t_i, t_{i+1}]$

$$h_{00}^{(i)}(t) = \begin{cases} 2(\frac{t-t_i}{t_{i+1}-t_i})^3 - 3(\frac{t-t_i}{t_{i+1}-t_i})^2 + 1, & t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{10}^{(i)}(t) = \begin{cases} \frac{(t-t_i)^3}{(t_{i+1}-t_i)^2} - 2\frac{(t-t_i)^2}{t_{i+1}-t_i} + (t - t_i), & t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{01}^{(i)}(t) = \begin{cases} -2(\frac{t-t_i}{t_{i+1}-t_i})^3 + 3(\frac{t-t_i}{t_{i+1}-t_i})^2, & t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$h_{11}^{(i)}(t) = \begin{cases} \frac{(t-t_i)^3}{(t_{i+1}-t_i)^2} - \frac{(t-t_i)^2}{t_{i+1}-t_i}, & t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

We define functions $N_1 = h_{00}^{(1)}$, $N_2 = h_{10}^{(1)}$, $N_{2n-1} = h_{01}^{(n)}$, $N_{2n} = h_{11}^{(n)}$. For all $k = 1, 2, \dots, n-2$ we define N_{2k+1} by

$$N_{2k+1}(t) = \begin{cases} h_{01}^{(k)} + h_{00}^{(k+1)} & t \neq t_{k+1} \\ 1 & t = t_{k+1}. \end{cases}$$

and $N_{2k+2} = h_{11}^{(k)} + h_{10}^{(k+1)}$.
We then have

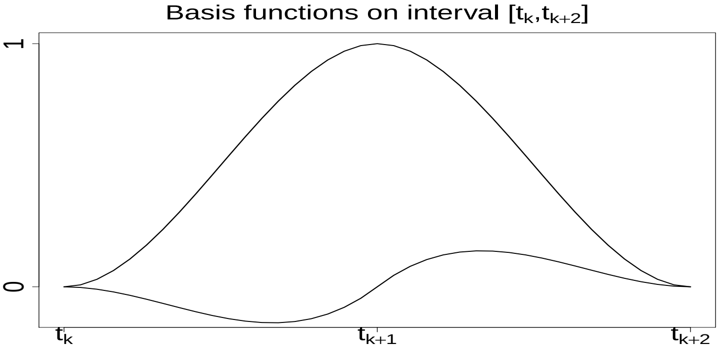
$$N_i(t_j) = \begin{cases} 1 & \text{if } i = 2j-1 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$N_i'(t_j) = \begin{cases} 1 & \text{if } i = 2j \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

The functions $N_1, \ldots, N_{2n}$ *provide a basis for the set of functions on* $[t_1, t_n]$ *which are continuous, have continuous first and second derivatives and which are cubic on each open interval* (t_i, t_{i+1}).



Basis functions on interior interval $[t_k, t_{k+2}]$. The first and second derivatives are continuous

Solution to The New Objective Function

The tractor spline $f(t)$ is a linear combination of basis functions

$$f(t) = \sum_{j=1}^{2n} N_j(t)\theta_j$$

and the objective function reduces to

$$\mathrm{MSE}(\theta, \lambda, \gamma) = (\mathbf{z} - \mathbf{B}\theta)^T(\mathbf{z} - \mathbf{B}\theta) + \gamma(\mathbf{w} - \mathbf{C}\theta)^T(\mathbf{w} - \mathbf{C}\theta) + n\theta^T\mathbf{\Omega}_\lambda\theta,$$

where $\mathbf{z} = \{z(x_i, y_i)\}$ are the knots and $\mathbf{w} = \{(u_i, v_i)\}$ are the tangent at knots.

After solving the above equation, we have

$$\hat{\theta} = (\mathbf{B}^T\mathbf{B} + \gamma \mathbf{C}^T\mathbf{C} + n\boldsymbol{\Omega}_\lambda)^{-1}(\mathbf{B}^T\mathbf{z} + \mathbf{C}^T\mathbf{w}). \tag{4}$$

Then tractor spline could be represented as

$$\hat{f}(t) = \sum_{j=1}^{2n} N_j(t)\hat{\theta}_j \tag{5}$$

2.3 Cross Validation of Tractor Spline

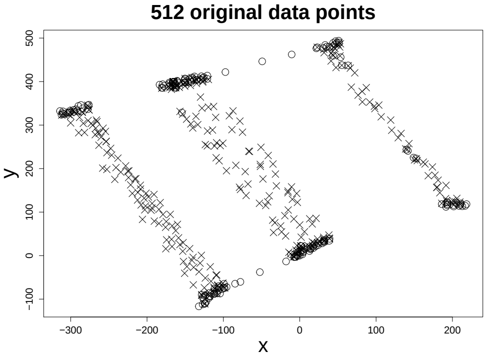
Because $\hat{f}$ and $\hat{f}'$ could be written in the form of

$$\hat{f} = B\hat{\theta} = Sz + \gamma Tw$$
$$\hat{f}' = C\hat{\theta} = Uz + \gamma Vw$$

Then

$$\begin{aligned}
\mathrm{CV} &= \frac{1}{n}\sum(\hat{f}^{(-i)}(t_i) - z_i)^2 \\
&= \frac{1}{n}\sum\left(\frac{\hat{f}(t_i) - z_i + \gamma\frac{T_{ii}}{1-\gamma V_{ii}}(\hat{f}'(t_i) - w_i))}{1 - S_{ii} - \gamma\frac{T_{ii}}{1-\gamma V_{ii}}U_{ii}}\right)^2
\end{aligned}$$

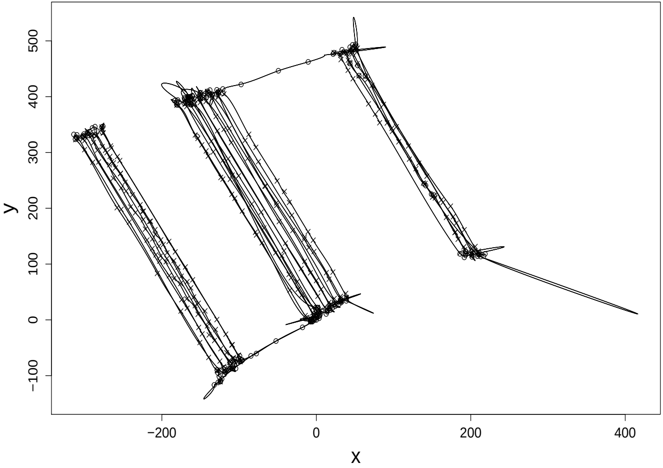
Application



Regarding to the boom status (0 and 1), we divided the penalty function into two parts: Ω_u and Ω_d. That brings two penalty parameters λ_u and λ_d.

Thus, we totally have three parameters λ_u, λ_d and γ.

Trajectory Reconstruction



2.4 Adjusted Penalty Term

The crazy curves are caused by tricky points, where there is a break and a long time gap. At these points the velocity $v \approxeq 0$.
To adjust this issue, we bring an adjusted penalty term

$$\lambda(t) = \begin{cases} \frac{\Delta t_i^3}{\Delta d_i^2}\lambda_d, & \text{when } b = 1 \\ \frac{\Delta t_i^3}{\Delta d_i^2}\lambda_u, & \text{when } b = 0 \end{cases}, t_i \leq t < t_{i+1}. \tag{6}$$

Penalty Function

The penalty parameter λ becomes a function $\lambda(t)$ that varying on different domains.

$$\lambda(t) = b\frac{\Delta t^3}{\Delta d^2}\lambda_d + (1-b)\frac{\Delta t^3}{\Delta d^2}\lambda_u, \text{where} \begin{cases} b = 1 & \text{if boom is working} \\ b = 0 & \text{if boom is not working} \end{cases}. \tag{7}$$

A penalty matrix Ω_k on a boom-not-working interval $[t_k, t_{k+1}]$ wil be

$$\Omega_k = \lambda_k \begin{pmatrix} \frac{12}{\Delta t_k^3} & \frac{6}{\Delta t_k^2} & \frac{-12}{\Delta t_k^3} & \frac{6}{\Delta t_k^2} \\ \frac{6}{\Delta t_k^2} & \frac{4}{\Delta t_k} & \frac{-6}{\Delta t_k^2} & \frac{2}{\Delta t_k} \\ \frac{-12}{\Delta t_k^3} & \frac{-6}{\Delta t_k^2} & \frac{12}{\Delta t_k^3} & \frac{-6}{\Delta t_k^2} \\ \frac{6}{\Delta t_k^2} & \frac{2}{\Delta t_k} & \frac{-6}{\Delta t_k^2} & \frac{4}{\Delta t_k} \end{pmatrix} = \frac{\lambda_d}{\Delta d_k^2} \begin{pmatrix} 12 & 6\Delta t_k & -12 & 6\Delta t_k \\ 6\Delta t_k & 4\Delta t_k^2 & -6\Delta t_k & 2\Delta t_k^2 \\ -12 & -6\Delta t_k & 12 & -6\Delta t_k \\ 6\Delta t_k & 2\Delta t_k^2 & -6\Delta t_k & 4\Delta t_k^2 \end{pmatrix}. \tag{8}$$

Matrix (8) can be divided into three sub matrices,

$$\begin{aligned}
\Omega_1^{(k)} &= \frac{1}{\Delta d_k^2}\begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}, \\
\Omega_2^{(k)} &= \frac{\Delta t}{\Delta d_k^2}\begin{pmatrix} 6 & 6 \\ -6 & -6 \end{pmatrix} = \frac{1}{\Delta d\, \hat{v}_k}\begin{pmatrix} 6 & 6 \\ -6 & -6 \end{pmatrix}, \\
\Omega_3^{(k)} &= \frac{\Delta t_k^2}{\Delta d_k^2}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \frac{1}{\hat{v}_k^2}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.
\end{aligned} \tag{9}$$

Matrix $\Omega_1^{(k)}$ shows a correlated relationship between position x_k and x_{k+1}.
Matrix $\Omega_2^{(k)}$ shows a correlated relationship between position x_k, x_{k+1} and velocity v_k, v_{k+1}.
Matrix $\Omega_3^{(k)}$ shows a correlated relationship between velocity v_k and v_{k+1}.

3.1 Gaussian Process Regression

If the observed function $f(t)$ has zero mean, and

$$f(t) \sim N(0, k(t, t'))$$

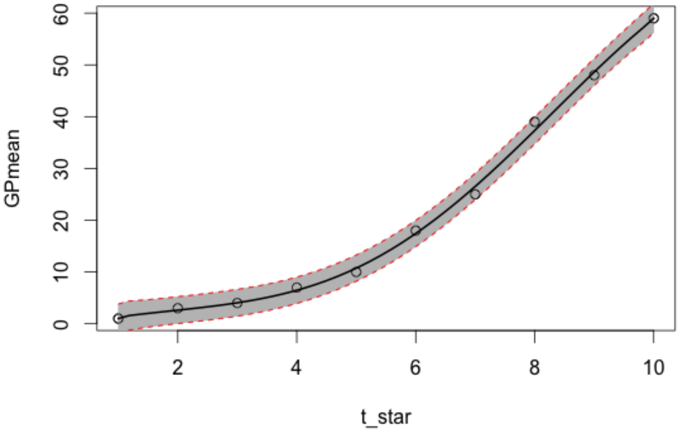
where $k(t, t')$ is covariance matrix, then the prediction mean is

$$\bar{f}_* = K_* K^{-1} y \tag{10}$$

and uncertainty in this estimate is captured in its variance

$$var(f_*) = K_{**} - K_* K^{-1} K_*^T \tag{11}$$

day	1	2	3	4	5	6	7	8	9	10
count	1	3	4	7	10	18	25	39	48	59



3.2 Hilbert Space and Reproducing Kernel for Tractor Spline

For any $f \in \mathbb{H}$, $f = \sum_{i=1}^{2n} \theta_i N_i(t)$. Equipped an inner product

$$(f, g) = (\sum_{i=1}^{2n} a_i N_i(t), \sum_{i=1}^{2n} b_i N_i(t)) = \sum_{i=1}^{2n} a_i b_i, \tag{12}$$

and a kernel function

$$R(s, t) = \sum_{i=1}^{2n} N_i(s) N_i(t), \tag{13}$$

the space $\mathbb{H}$ is a Reproducing Kernel Hilbert Space

$$(f_t(s), R(s, t)) = (\sum_{i=1}^{2n} a_i N_i(s), \sum_{i=1}^{2n} N_i(s) N_i(t)) = \sum_{i=1}^{2n} a_i N_i(s) = f(s).$$

Given the sample points $t_i, i = 1, \cdots, n$ and noting that the space

$$\mathbb{A} = \{f : f = \sum_{i=1}^{n} \alpha_i R(t_i, \cdot) = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{2n} N_j(t_i) N_j(t) \right) \} \tag{14}$$

is a linear subspace of $\mathbb{H}$,

$$\mathbb{B} = \{f : f = \sum_{i=1}^{n} \beta_i \dot{R}(t_i, \cdot) = \sum_{i=1}^{n} \beta_i \left(\sum_{j=1}^{2n} N'_j(t_i) N_j(t) \right) \} \tag{15}$$

is another linear subspace of $\mathbb{H}$. Then $f \in \mathbb{H}$ can be written as

$$f(t) = \sum_{i=1}^{n} c_i R(t_i, t) + \sum_{i=1}^{n} d_i \dot{R}(t_i, t) + \rho(t) \tag{16}$$

where c_i, d_i are coefficients, and $\rho(x) \in \mathbb{H} \ominus \mathbb{A} \ominus \mathbb{B}$.

The objective function can be written as

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}(Y_i-\sum_{j=1}^{n}c_jR(t_j,t_i)-\sum_{j=1}^{n}d_j\dot{R}(t_j,t_i)-\rho(t_i))^2\\
&+\frac{\gamma}{n}\sum_{i=1}^{n}(V_i-\sum_{j=1}^{n}c_jR'(t_j,t_i)-\sum_{j=1}^{n}d_j\dot{R}'(t_j,t_i)-\rho'(t_i))^2\\
&+\int_0^1\lambda(t)(\sum_{j=1}^{n}c_jR''(t_j,t)+\sum_{j=1}^{n}d_j\dot{R}''(t_j,t)+\rho''(t))^2dt
\end{aligned}
\tag{17}
$$

And reduces tow

$$
\begin{aligned}
&(\mathbf{Y} - Q\mathbf{c} - P\mathbf{d})^T(\mathbf{Y} - Q\mathbf{c} - P\mathbf{d}) \\
&+\gamma(\mathbf{V} - \frac{\partial Q}{\partial t}\mathbf{c} - \frac{\partial P}{\partial t}\mathbf{d})^T(\mathbf{V} - \frac{\partial Q}{\partial t}\mathbf{c} - \frac{\partial P}{\partial t}\mathbf{d}) \\
&+n\Omega_\lambda + n\lambda(\rho, \rho).
\end{aligned}
\tag{18}
$$

The last term $\lambda(\rho, \rho)$ is minimized at $\rho = 0$.

Observing $y_i \sim N(f(t_i), \sigma_n^2)$, $v_i \sim N(f'(t_i), \frac{\sigma_n^2}{\gamma})$ and a prior $f \sim N(0, \tau^2)$, the joint distribution of $\mathbf{Y}, \mathbf{V}, f(t)$ and $f'(t)$ is normal with zero mean and covariance matrix

$$\mathrm{cov}(\mathbf{Y}, \mathbf{V}, f, f') = \begin{bmatrix} \tau^2 R(t_i, t_j) + \sigma_n^2 I & \tau^2 R'(t_i, t_j) + \frac{\sigma_n^2}{\sqrt{\gamma}} I & \tau^2 R(t_i, t) & \tau^2 R'(t_i, t) \\ \tau^2 \dot{R}(t_i, t_j) + \frac{\sigma_n^2}{\sqrt{\gamma}} I & \tau^2 \dot{R}'(t_i, t_j) + \frac{\sigma_n^2}{\gamma} I & \tau^2 \dot{R}(t_i, t) & \tau^2 \dot{R}'(t_i, t) \\ \tau^2 R^\top(t_i, t) & \tau^2 \dot{R}^\top(t_i, t) & \tau^2 R(t, t) & \tau^2 R'(t, t) \\ \tau^2 R'^\top(t_i, t) & \tau^2 \dot{R}'^\top(t_i, t) & \tau^2 \dot{R}(t, t) & \tau^2 \dot{R}'(t, t) \end{bmatrix}$$

$$= \begin{bmatrix} \tau^2 Q + \sigma_n^2 I & \tau^2 O + \frac{\sigma_n^2}{\sqrt{\gamma}} I & \tau^2 \xi & \tau^2 \xi' \\ \tau^2 O + \frac{\sigma_n^2}{\sqrt{\gamma}} I & \tau^2 P + \frac{\sigma_n^2}{\gamma} I & \tau^2 \dot{\xi} & \tau^2 \dot{\xi}' \\ \tau^2 \xi^\top & \tau^2 \dot{\xi}^\top & \tau^2 R(t, t) & \tau^2 R'(t, t) \\ \tau^2 \xi'^\top & \tau^2 \dot{\xi}'^\top & \tau^2 \dot{R}(t, t) & \tau^2 \dot{R}'(t, t) \end{bmatrix} \tag{19}$$

Then

$$
\begin{aligned}
E\left[\begin{matrix} f \\ f' \end{matrix}\middle| \begin{matrix} \mathbf{Y} \\ \mathbf{V} \end{matrix}\right] &= \begin{bmatrix} \xi^\top & \dot{\xi}^\top \\ \xi'^\top & \dot{\xi}'^\top \end{bmatrix} \begin{bmatrix} Q + n\lambda I & O + \frac{n\lambda}{\sqrt{\gamma}} I \\ O + \frac{n\lambda}{\sqrt{\gamma}} I & P + \frac{n\lambda}{\gamma} I \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Y} \\ \gamma \mathbf{V} \end{bmatrix} \\
&\triangleq \begin{bmatrix} \xi^\top & \dot{\xi}^\top \\ \xi'^\top & \dot{\xi}'^\top \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \gamma \mathbf{V} \end{bmatrix} \\
&= \begin{bmatrix} \xi^\top (A\mathbf{Y} + B\gamma\mathbf{V}) + \dot{\xi}^\top (C\mathbf{Y} + D\gamma\mathbf{V}) \\ \xi'^\top (A\mathbf{Y} + B\gamma\mathbf{V}) + \dot{\xi}'^\top (C\mathbf{Y} + D\gamma\mathbf{V}) \end{bmatrix}
\end{aligned}
\tag{20}
$$

where $n\lambda = \sigma_n^2/\tau^2$.

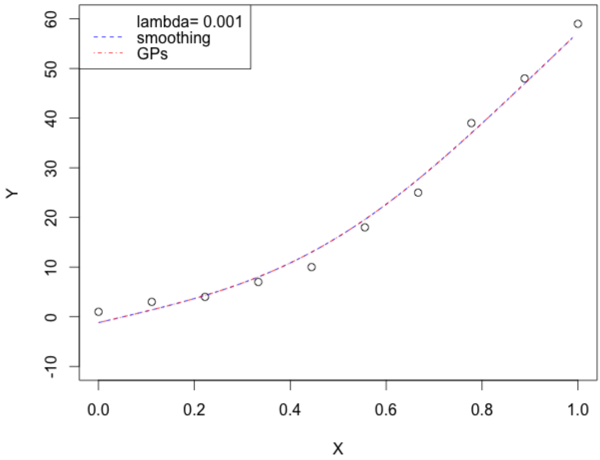
The posterior mean $E(f|\mathbf{Y},\mathbf{V})$ is a linear combination of basis functions $N_i(t)$, and both ξ and $\dot{\xi}$ contain $N_i(t)$, thus the posterior mean is of the form $\xi^\top \mathbf{c} + \dot{\xi}^\top \mathbf{d}$. Similarly, $E(f'|\mathbf{Y},\mathbf{V})$ is of the form $\xi'^\top \mathbf{c} + \dot{\xi}'^\top \mathbf{d}$, with the same coefficients given by

$$\mathbf{c} = A\mathbf{Y} + B\gamma\mathbf{V} \tag{21}$$

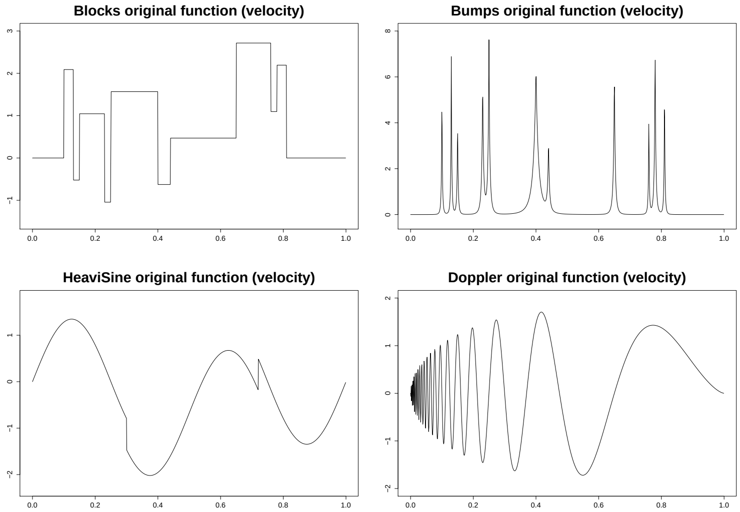
$$\mathbf{d} = C\mathbf{Y} + D\gamma\mathbf{V} \tag{22}$$

Numeric Testing

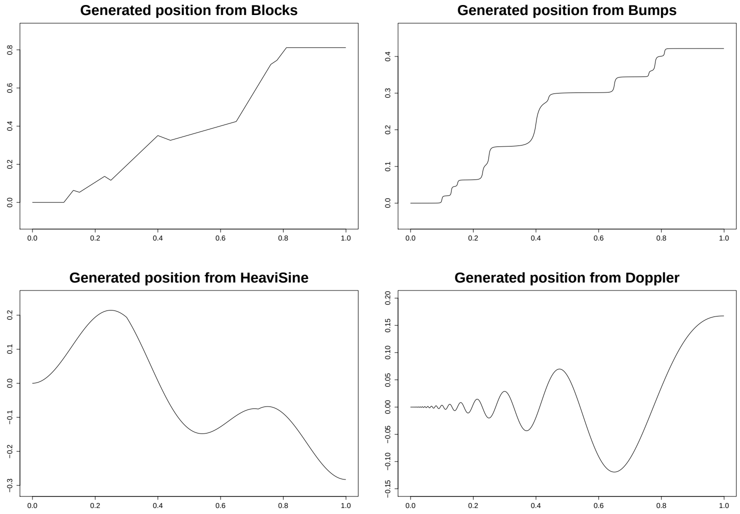
When λ is chosen 0.001, smoothing spline and GPs return the same results.



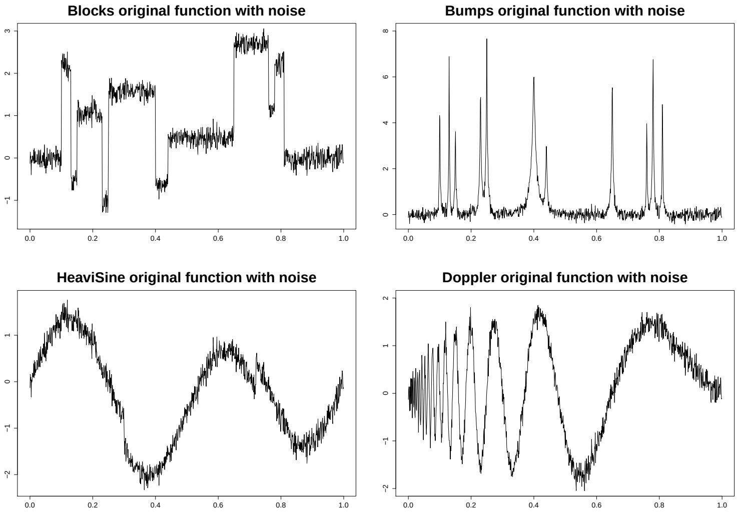
4.1 Simulation



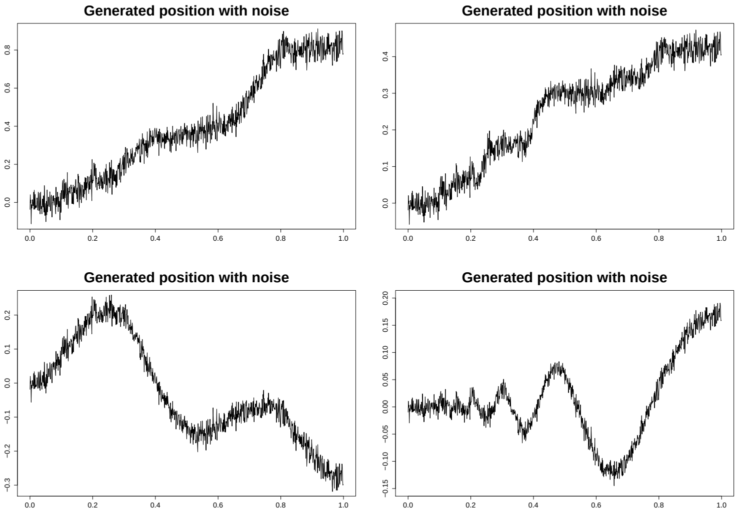
By setting initial $y_0 = 0$ and $a_0 = 0$, and $y_{i+1} = y_i + (v_i + v_{i+1})\frac{t_{i+1} - ti}{2}$ to calculate positions.



Original Functions with Noises

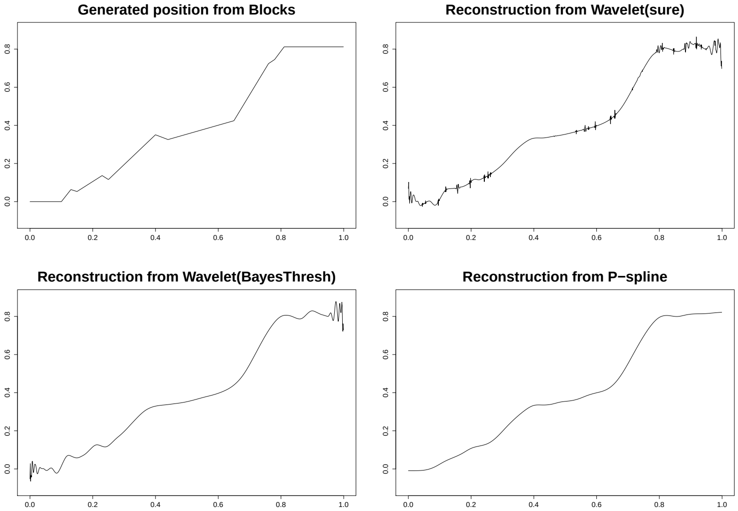


Generated Position Functions with Noises

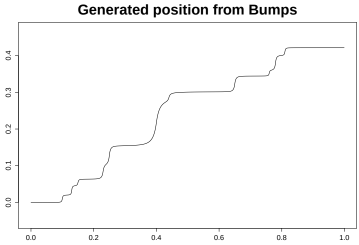


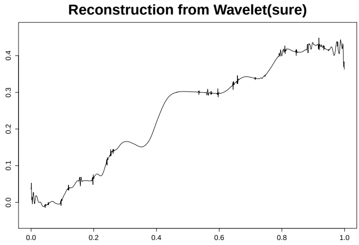
Comparing with Other Methods

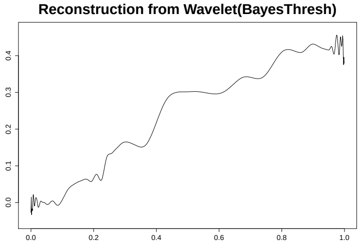
- wavelet(sure) in *Library("WaveThresh")* of *R*
- wavelet(Bayesian)
- P-spline (penalized B spline)
- Tractor Spline without velocity term in MSE ($\gamma = 0$)
- Tractor Spline without adjusted penalty term ($\frac{\Delta t^0}{\Delta d^0}$)
- Tractor Spline

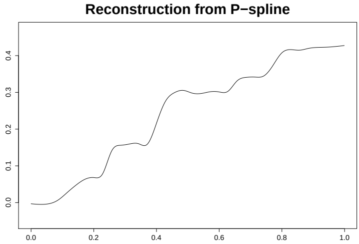

Generated position from Blocks
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
0.0
0.2
0.4
0.6
0.8
1.0

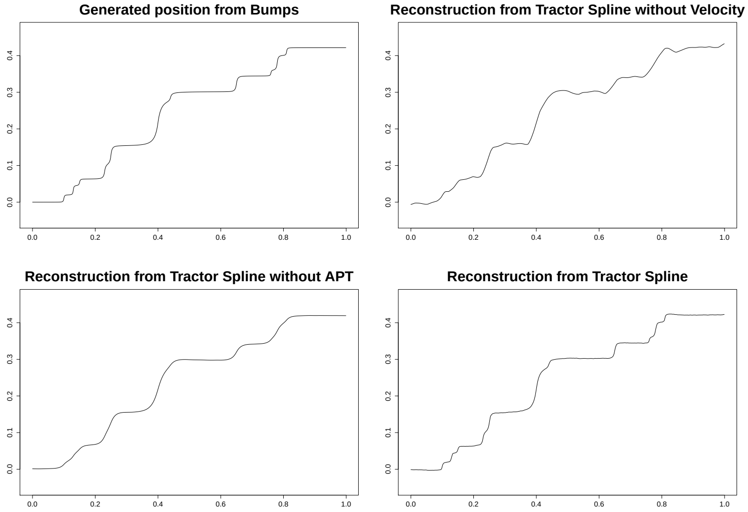

Generated position from Blocks
Reconstruction from Tractor Spline without Velocity
Reconstruction from Tractor Spline without APT
Reconstruction from Tractor Spline
0.0
0.2
0.4
0.6
0.8
1.0

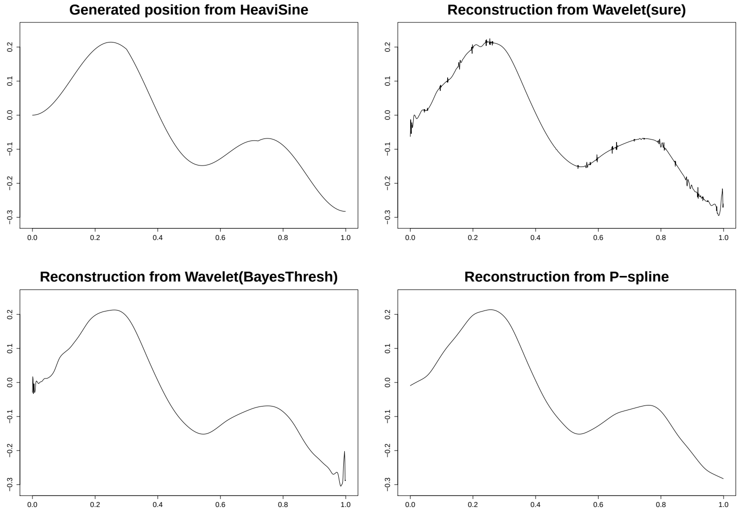

Generated position from Bumps
0.0
0.1
0.2
0.3
0.4
0.0
0.2
0.4
0.6
0.8
1.0


Reconstruction from Wavelet(sure)
0.0
0.1
0.2
0.3
0.4
0.0
0.2
0.4
0.6
0.8
1.0

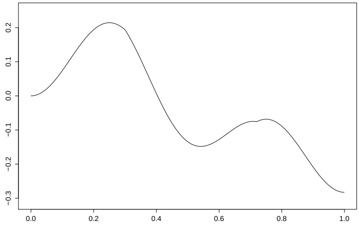

Reconstruction from Wavelet(BayesThresh)
0.0
0.1
0.2
0.3
0.4
0.0
0.2
0.4
0.6
0.8
1.0


Reconstruction from P−spline
0.0
0.1
0.2
0.3
0.4
0.0
0.2
0.4
0.6
0.8
1.0

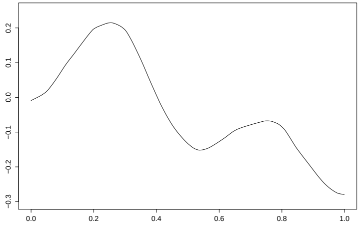

Generated position from Bumps
Reconstruction from Tractor Spline without Velocity
Reconstruction from Tractor Spline without APT
Reconstruction from Tractor Spline
0.0
0.1
0.2
0.3
0.4
0.0
0.2
0.4
0.6
0.8
1.0


Generated position from HeaviSine
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
0.0
0.2
0.4
0.6
0.8
1.0
−0.3
−0.2
−0.1
0.0
0.1
0.2

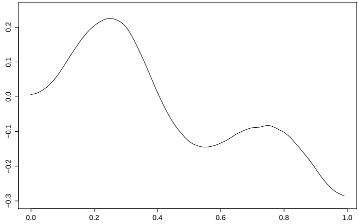
Generated position from HeaviSine



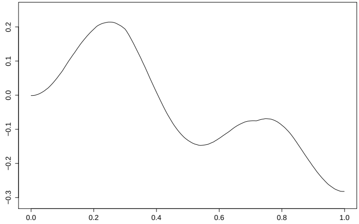
Reconstruction from Tractor Spline without Velocity

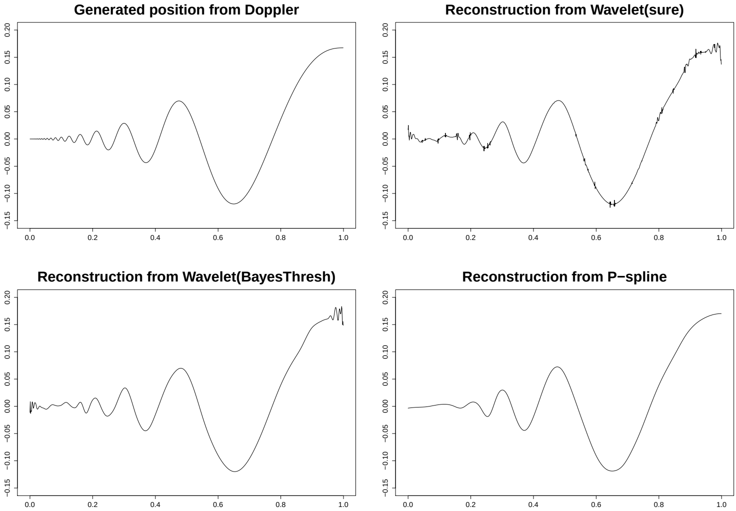


Reconstruction from Tractor Spline without APT

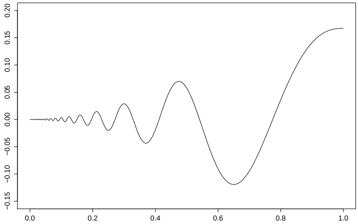


Reconstruction from Tractor Spline

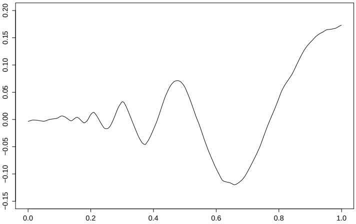



Generated position from Doppler
Reconstruction from Wavelet(sure)
Reconstruction from Wavelet(BayesThresh)
Reconstruction from P−spline
−0.15
−0.10
−0.05
0.00
0.05
0.10
0.15
0.20
0.0
0.2
0.4
0.6
0.8
1.0

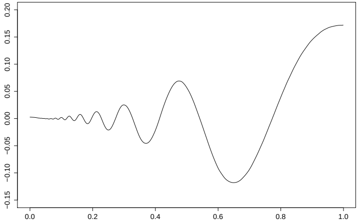
Generated position from Doppler



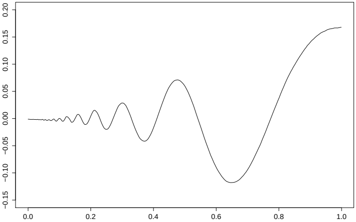
Reconstruction from Tractor Spline without Velocity



Reconstruction from Tractor Spline without APT



Reconstruction from Tractor Spline



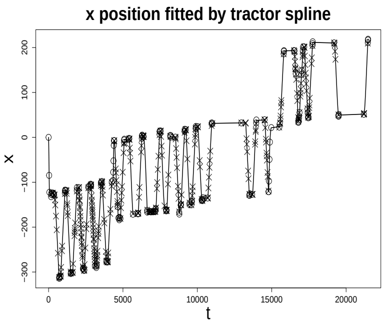
$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{f}_{\lambda,\gamma}(t_i))^2, \tag{23}$$

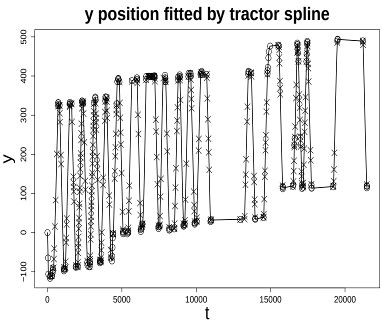
MSE (10^{-4})	SNR	Tractor Spline	TS $\gamma = 0$	TS without APT	P-spline	Wavelet(sure)	Wavelet(Bayes)
Blocks	7	16.53	15.99	16.69	16.14	*15.39	16.68
Blocks	3	89.79	*87.64	89.94	88.27	98.35	90.24
Bumps	7	4.40	4.19	4.55	4.33	*4.18	4.59
Bumps	3	23.93	*23.19	24.10	23.55	26.23	23.74
HeaviSine	7	4.16	4.01	4.16	4.02	*3.79	4.19
HeaviSine	3	22.63	*22.19	22.65	22.02	23.53	22.07
Doppler	7	1.15	*1.07	1.10	1.15	*1.07	1.13
Doppler	3	6.27	*5.94	6.28	6.05	6.85	6.29

$$\mathrm{TMSE} = \frac{1}{n}\sum_{i=1}^{n}(f(t_i) - \hat{f}_{\lambda,\gamma}(t_i))^2. \tag{24}$$

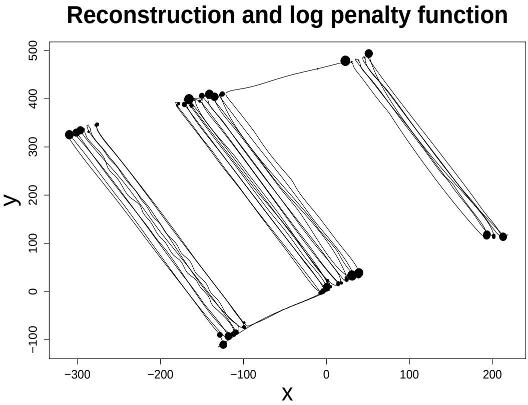
TMSE(10^{-6})	SNR	Tractor Spline	TS $\gamma = 0$	TS without APT	P-spline	Wavelet(sure)	Wavelet(Bayes)
Blocks	7	*1.75	54.25	28.68	54.76	201.02	182.12
Blocks	3	*16.44	152.5	30.76	171.59	1138.08	712.36
Bumps	7	*1.64	23.44	21.10	24.21	71.71	69.26
Bumps	3	*8.51	77.78	37.12	77.52	330.77	238.79
HeaviSine	7	*1.53	7.80	1.56	9.54	55.37	44.88
HeaviSine	3	*8.21	33.56	8.49	34.26	240.72	110.49
Doppler	7	1.51	6.67	*1.08	8.26	14.87	12.01
Doppler	3	*8.10	22.14	8.25	19.95	81.48	50.33

4.2 Application





Black dots represent the value of $\log(\lambda(t))$. The bigger penalty values, the larger dots.



A video at :`https://www.youtube.com/watch?v=lQOSt8HrYRU`

5.1 Future Work

- Solving issues in the current tractor spline.
- Moving from batch inference to online inference
- Deducing high-level description of farm vehicle motion

5.2 Acknowledgment

- Ministry of Business, Innovation & Employment
  – Grant UOOX 1208
- TracMap
- Dr Matthew Parry
- Professor David Bryant

Ministry of Business, Innovation & Employment

TRACMAP AGRICULTURE

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