Uniqueness Theory of Meromorphic Functions Sharing Weighted Values with its k-th derivative

Zhanglong CAO

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- References

Self introduction

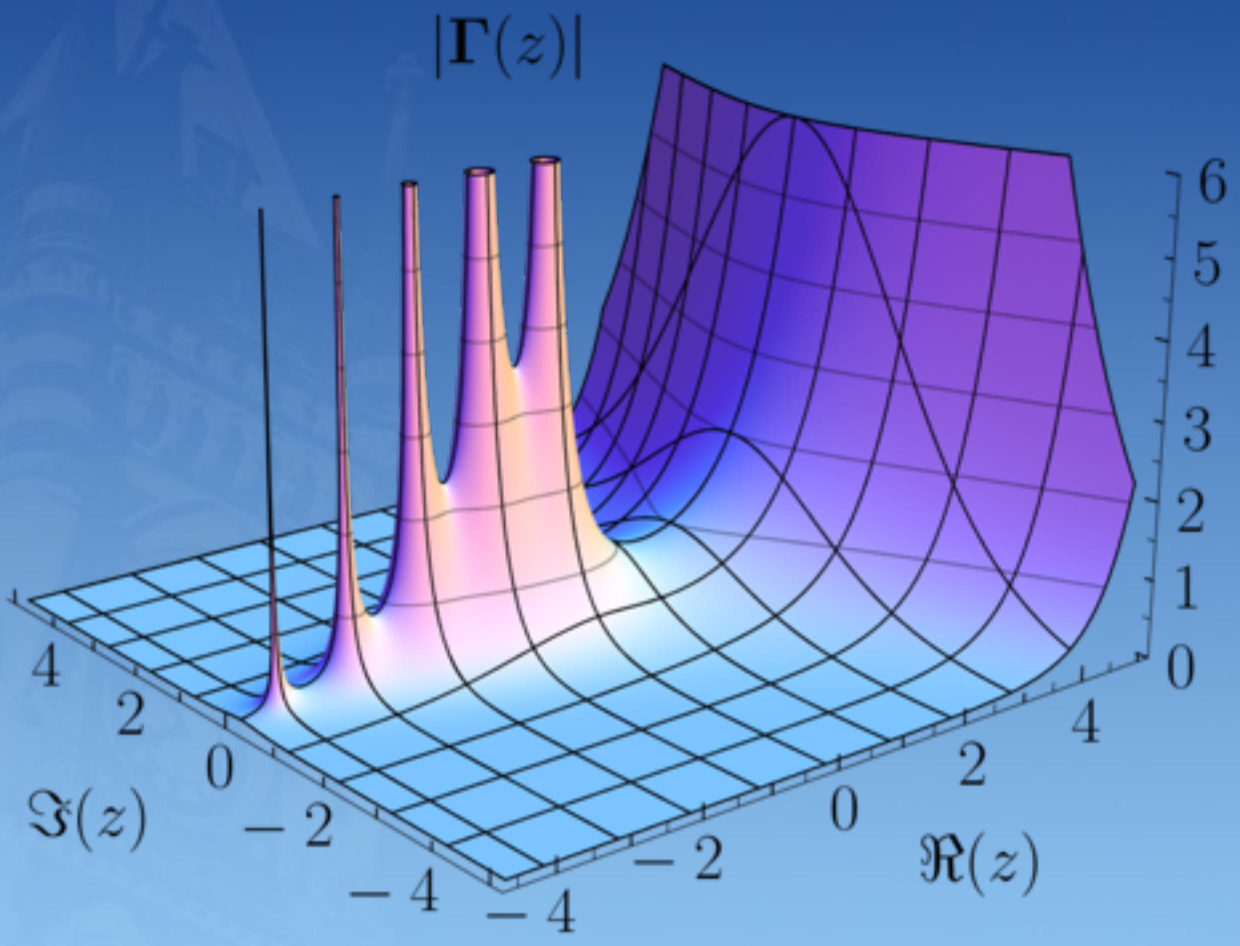
- Zhanglong CAO, known as Jerome
- 1st year PhD student in statistics, University of Otago
- Worked in Beijing as Data analyst
- Masters in Complex Analysis, USST
- Bachelors in Mathematics, JSNU

- # Preliminaries

- ## Meromorphic Functions

In the mathematical field of complex analysis, a meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all D except a set of isolated points (the poles of the function), at each of which the function must have a Laurent series.

The gamma function is meromorphic in the whole complex plane.



- **Sharing Values**

Let f and g be two non-constant meromorphic functions and let a be a finite complex number.

We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities.

Similarly, we say that f and g share the value a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities.

- **Weighted Values**

Let k be a nonnegative integer or infinity. For any $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a–points of f, where an a-point of multiplicity m is counted m times if $m \leq k$, and $k+1$ times if $m > k$.

If $E_k(a, f) = E_k(a, g)$, we say that f and g share the value a with weight k.

We write f , g share (a,k) to mean that f , g share the value a with weight k .

We also note that f , g share a value a IM or CM if and only if f , g share $(a,0)$ or (a,∞) , respectively.

- **Background**

- In 1980, Gundersen proved the following theory.

- If f is a non-constant meromorphic function, $b(\neq 0)$ is an finite value. If f and f' share 0 and b CM, then $f \equiv f'$.

- In 1986, Frank-Ohlenroth proved the following,
- f is a non-constant meromorphic function. If f and $f^{(k)}$ share two distinct non-zero finite complex values a and b CM, then $f \equiv f^{(k)}$.
- In the same year, Frank-Weissenborn proved that, for any finite distinct complex values a and b , if f and $f^{(k)}$ share them CM, then $f \equiv f^{(k)}$.

Here comes a conjecture,

Will the equation $f \equiv f^{(k)}$ still hold if the condition f and $f^{(k)}$ share b CM, changes to f and $f^{(k)}$ share b IM?

Unfortunately, the conjecture is false.

Luckily, I studied in this area and was interested in this topic.
So the theorem is as following,

Let $f(z)$ be a non-constant meromorphic function, k is a positive integer and the multiplicity of poles $\geq 2k+4$. If $f(z)$ and $f^{(k)}(z)$share (a,∞), $(b,1)$,where a and b are distinct finite complex numbers, then we have

$$f(z) \equiv f^{(k)}(z)$$

Proof

$$\text{S1}: ab \neq 0 \xrightarrow{\text{the multiplicity of } f \geq 2k+4} \text{contradiction}$$

$$\text{S2}: ab = 0 \xrightarrow[b=1]{a=0} \begin{cases} \phi = \dfrac{f'(f - f^{(k)})}{f(f-1)} \\[2ex] \chi = \dfrac{f^{(k+1)}(f - f^{(k)})}{f^{(k)}(f^{(k)}-1)} \end{cases} \longrightarrow \begin{cases} T(r,\phi) \leq \dfrac{k+1}{2k+4} T(r,f) + S(r,f) \\[2ex] T(r,\chi) \leq \dfrac{k+1}{2k+4} T(r,f) + S(r,f) \end{cases}$$

$$\longrightarrow H_{(m_0,n_0)} = m_1\phi - n_1\chi \longrightarrow \begin{cases} \text{S2.1 } H_{(m_0,n_0)} \equiv 0 \longrightarrow \text{contradiction} \\[2ex] \text{S2.2 } H_{(m_0,n_0)} \not\equiv 0 \xrightarrow{\text{2nd Ne.T}} \text{contradiction} \end{cases}$$

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Thanks to Xinli Wang,
lead me on the path of research.

Thanks to Matthew Parry,
the supervisor gave chance to start my PhD study.

Thank You!