(6) Orthogonality

Definition An inner product $(\cdot,\cdot)$ is a function from $V \times V \to$ scalar (where V is a vector space) with the following properties:

- $(\alpha x, y) = \alpha (x,y)$ for all $x,y \in V$ } linearity in first arg
- $(x+y, z) = (x,z) + (y,z)$ for all $x,y,z \in V$ } linearity in first arg
- $(x,y) = (y,x)$ } symmetry
- $(x,x) \geq 0$ for all $x \in V$ } pos. definiteness
  with $(x,x) = 0 \iff x = 0$

0 [linear in second argument? (yes: flip, use linearity in first, flip again

Example $(x,y) = x_1 y_1 + \cdots + x_n y_n$
for coordinate vectors x and y
↑ specifically this IP is called the dot product: $x \cdot y$

(lec 12) review inner prod. props — hw2 due in a week
review quiz

Facts:

- x and y called "orthogonal" iff $(x,y) = 0$. "$x \perp y$"
  ↑ aka perpendicular
- Inner products make norms: $\|x\| := \sqrt{|(x,x)|}$

  [Abs value needed?
  Dot product makes which norm?

- Pythagorean theorem
  $x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

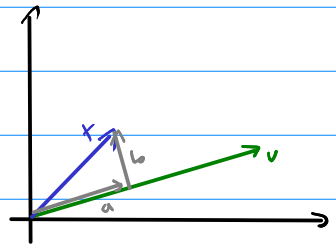
- For $x, v \in V$, there exist a, b s.t.

$$x = \underbrace{a}_{x^{\parallel v}} + \underbrace{b}_{x^{\perp v}}$$

with

$a = \alpha v$

$b \perp v$



$$\rightsquigarrow \quad (b, v) = 0$$

$b = x - a$ $\Leftrightarrow (x - a, v) \stackrel{!}{=} 0$

$\Leftrightarrow (x - \alpha v, v) \stackrel{!}{=} 0$

$\Leftrightarrow (x, v) - \alpha (v, v) \stackrel{!}{=} 0$

$\Leftrightarrow \alpha = \dfrac{(x, v)}{(v, v)}$

$$\rightsquigarrow x^{\perp v} = x - \frac{(x, v)}{(v, v)} v$$

$$x^{\parallel v} = x - x^{\perp v}$$ Done

What happens to this if $\|v\| = 1$? denominator drops out

What is the closest point to x on the line βv (for all β)? a

Why? (Pyth.) How can we compute it? (see above)

- Can define a line/ plane/ hyperplane this way.

$x = a + \alpha b$ (a: in general $n-1$ direction vectors (in nD space))

↑ Example: line

$\rightsquigarrow$ find n with $(n, b) = 0$ [How? Nullsp. finding

$\rightsquigarrow (n, x) = (n, a) + \alpha (n, b) = 0$

$\rightsquigarrow (n, x) = (n, a)$ defines the plane

$\rightsquigarrow$ If $\|n\| = 1$, $(n, x) - (n, a)$ computes distance from plane [Why? (Plug $a - x$ into norm.)

n called the normal

- Orthogonal basis $(b_i) \subset V$: $b_i \perp b_j$ if $i \neq j$
  ("pairwise $\perp$")

  (and needs to be a basis)

- Orthonormal basis (ONB): orthogonal basis and $\|b_i\| = 1$.

How do I find coordinates with respect to an ONB? Very easy!
$\rightsquigarrow$ $x = (x, b_1)b_1 + (x, b_2)b_2 + \cdots + (x, b_n)b_n$.

[How would I build a matrix that finds these coordinates?
(assuming the dot product as the inner product) ↓

- Orthogonal matrix: orthonormal basis (of entire space) for columns

[Suppose Q is orth. $QQ^T = ?$ (Identity)

[Is Q square? (yes, because of "basis" requirement)

[Is Q^T orthogonal as well? (i.e. is $Q^TQ = I$?)

[$Q^{-1} = ?$ (Q^T)

What if the orthonormal vectors in Q are not a full basis?

$Q =$  $\rightsquigarrow QQ^T =$  $=: P$

$P^2 = QQ^TQQ^T = Q$  $Q^T = P$

↑ Identity restricted to colspace(Q).

A function with $f(f(x)) = f(x)$ is called a projection.

QQ^T is an orthogonal projection onto colspace (Q).

Demo Orthogonal projection

<in-class-orthogonality>

lec 13

CF comments:
- multiple sessions: fixed
- too much clicking: will fix
- hw grace period: will fix
- code area resizable: will fix
- email-based log-in...
- don't combine poll & quiz

Review quiz
Outline!
Show G-S movie
Mod G-S

Orthogonalization / Gram-Schmidt

Orthonormal vectors seem useful. How do we make many of them?

- Know how to make two vectors orthogonal
- Need pairwise orthogonality beyond that

Demo — Orthogonalizing three vectors

Movie — Gram-Schmidt

Demo — Gram-Schmidt and Modified Gram-Schmidt

Demo — Keeping track of the coefficients → QR factorization

Application of QR: Solving linear systems [How?

$$Ax = b \rightsquigarrow QRx = b \rightsquigarrow Qy = b \rightsquigarrow Rx = y$$

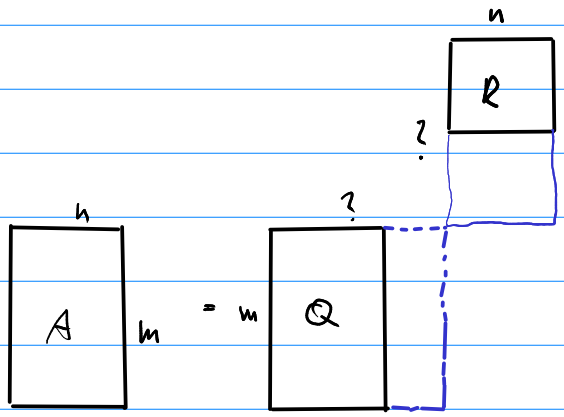
(y)

[How? $Q^T b = y$

[How? back-subst

[Asymptotic cost? $O(n^3)$, but practically more expensive than LU

What about QR of non-square matrices? (→ hw)



A: $m \times n$, $m \geq n$

Either:

Q $m \times m$ square
R: $m \times n$

} "full"

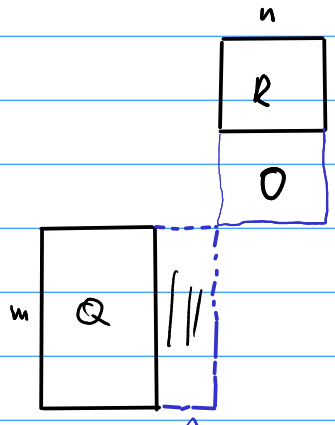
Or:

Q: $m \times n$
R: $n \times n$ square

} "thin"

[What does Gram-Schmidt produce? "thin"

How can we turn thin into full?



↑ some more orthogonal vectors
(can start G-S from random ones: why?)

Demo

What happens to 2-norms before/after an orthogonal matrix is applied?

$$\|Qx\|_2^2 = (Qx)\cdot(Qx) = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|_2^2 !$$

In other words: Orthogonal matrices preserve the 2-norm.

[Can Q be non-square here? No.

[Which step breaks if Q is non-square?

"Solving" overdetermined linear systems

$Ax=b$ with $A \to$  tall and skinny

⇝ bad news:
- many more equations than unknowns
- won't find an x that works exactly unless we're very lucky
- best hope in general: minimize residual $\|Ax-b\|$

$$\min_x \|Ax-b\|_2^2 = (Ax-b)_1^2 + (Ax-b)_2^2 + \cdots + (Ax-b)_n^2$$

"least-squares problem"

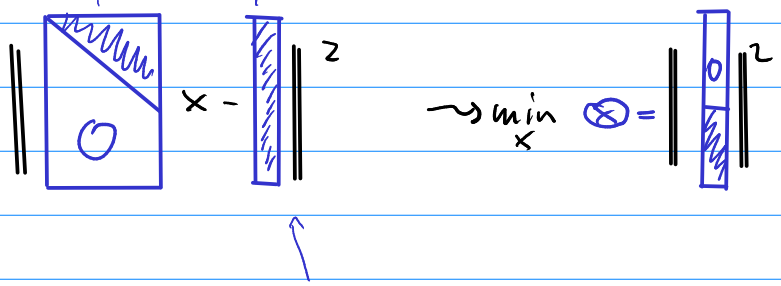
How do we solve least squares problems?

Just learned QR — maybe that can help.

$\otimes = \|QRx - b\|_2^2 \overset{\text{why?}}{=} \|Q^T(QRx - b)\|_2^2$ [Full QR? Reduced QR?

$= \|\underbrace{Q^TQ}_{Id}Rx - Q^Tb\|_2^2$

$= \|Rx - Q^Tb\|_2^2$



[How do we minimize that? By solving $Rx = (Q^Tb)_{(upper)}$.

lec 14

review QR for LSQ
review questions below
review data fitting
review quiz
outline

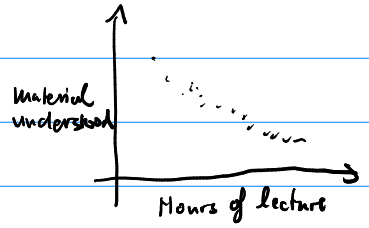
[What's the norm of the residual going to be? $\|(Q^Tb)_{(lower)}\|$

[What's the relationship between the residual and the columnspace of A?

Data Fitting

Have: Data points

$(x_i, y_i)_i$



Have: Idea for a Model → How does y depend on x?

$$y = f(x) = a + bx$$

for example

a, b: don't know

Want: Values for a, b that lead to small ~~error~~ 2-norm residual

$$a + bx_1 \overset{\approx}{\neq} y_1$$
$$\vdots$$
$$a + bx_n \overset{\approx}{\neq} y_n$$

↝ to matrix form

$$\begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \overset{\approx}{\neq} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Demo: Data Fitting with Least Squares

General form of model that Least squares can handle:

$$f(x) = a_1 \cdot \varphi_1(x) + a_2 \varphi_2(x) + \cdots + a_n \varphi_n(x)$$

$\sum$ coefficient · function

coefficient: unknown; function: given

<in class - least - squares - 2>

lec 15

outline
review quiz
hw3 out later today

questions!
SVD numerical example
focus on coding aspects?
what's the svd good for?
how do you actually compute it?

Another application of least squares:

Machine learning Demo: Predicting Breast Cancer