2 Linear functions and matrices

Definition: A linear function f is a function $f: V \to W$ between two vector spaces V and W that respects the vector space operations:

- $f(x+y) = f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

Which of these is linear as $f: \mathbb{R} \to \mathbb{R}$?

- $f(x) = \sin x$ $\quad\to$ Try: ...
- $f(x) = x^2$
- $f(x) = ax+b$
- $f(x) = ax$

Q

Super-restrictive! Most functions are not linear.

Q

- What's $f(0)$ for f linear?

2.1 Linear functions and coordinates

If I have generators that span V: $\quad v_1, v_2, v_3, \dots, v_m \in V$

W: $\quad w_1, w_2, w_3, \dots, w_n \in W$

then I can write down a linear function as a matrix.

Why?

Suppose

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

then

$$f(x) = f(\alpha_1 v_1 + \cdots + \alpha_m v_m)$$

$$= \alpha_1 f(v_1) + \cdots + \alpha_m f(v_m)$$

some combination of $w_1, \ldots, w_n$ → $\beta_{11} w_1 + \cdots + \beta_{n1} w_n$

another combination of $w_1, \ldots, w_n$ → $\beta_{1m} w_1 + \cdots + \beta_{nm} w_n$

$$\begin{array}{c c}
\begin{array}{ccc} \alpha_1 & \cdots & \alpha_m \end{array} & \\
\begin{pmatrix} \beta_{11} & \cdots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & \beta_{nm} \end{pmatrix} & \begin{array}{c} \to \gamma_1 \\ \\ \to \gamma_n \end{array}
\end{array}$$

$$\rightsquigarrow f(x) = \gamma_1 w_1 + \gamma_2 w_2 + \cdots + \gamma_n w_n$$

O [So what does a matrix mean again?

Shorthand:

$$\begin{pmatrix} \beta_{11} & \cdots & \beta_{1m} \\ \vdots & & \\ \beta_{n1} & & \beta_{nm} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

Q [

- Coordinates of image of second gen. vector x_2?
- What does an upper triangular matrix mean?
- Mat·vec as index expression? einsum()?

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- discuss pre-quiz
- whats with the π?
- access to py examples?
- less abstract description of $f: V \to W$
  - $f(v_3) \rightsquigarrow$ why not $(0, 0, \alpha_3)$?
  - get γ from the matrix — where does w come from?

2.2 Chaining linear functions

$f: V \to W$ $\quad$ $g: W \to X$ $\quad$ both linear

represented as $f(x) = Ax$ $\quad$ represented as $g(x) = Bx$

$$g(f(x)) = B \cdot (Ax)$$

Want a representation for $g \circ f$ $\to$ $g \circ f(x) = B(Ax) \overset{!}{=} (BA)x$

Where BA is a new matrix with

$$(BA)_{ij} = \sum_k B_{ik} A_{kj} \qquad \leftarrow \text{matrix multiplication}$$

What if I want to chain three or more linear functions?

$h: X \to Y$ $\leftarrow$ represented as $h(x) = Cx$

$$h(g(f(x))) = C \cdot (B \cdot (Ax)) \overset{?}{=} (CB) \cdot (Ax)$$
$$\overset{?}{=} C \cdot ((BA) x)$$

Matrix multiplication is "associative" $\to$ does not matter

$$C(BA) = (CB)A \; \boxed{= CBA}$$

2.3 What can matrices do?

- Geometry rotation — Demo
- Represent graphs and walks — Demo
- Blur an image — Demo

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reminder: hw1 due in a week

numpy: what if axes are in wrong order? (einsum, broadcast) O[expensive operation?

discuss quiz

Q's: best way to re-learn linear algebra?

get responses first time?

standard basis = preferred linear indep. set?

hw due on cal

codes on hw1 in PDF: only for the sparse matvec and the planes

confused on hw1 p1? → office hours. / maybe extra mat

How do you test for lin. indep?

2.4 Going backwards: Solving and Inverting

If a linear $f(x)$ applies some interesting operation, it would be useful to be able to undo this transformation, generically. (i.e. without thinking! All you'd need is the matrix.)

For an upper triangular matrix: no problem.

$$\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \\ & & & \times \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Can you code it?

Demo Backsubst.

General case of coefficient finding: more problematic

want

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3\end{matrix} \\ \begin{pmatrix}1&1&0\\0&0&0\\0&0&0\\1&0&1\end{pmatrix}\end{array} \quad \begin{matrix}7\\0\\0\\12\end{matrix}$$

have

$(\alpha_i) = (7\ 0\ 5)$? $(\alpha_i) = (0\ 7\ 12)$?

Q [What's the correct answer to this? Does it fit into a numpy array?

Define linear indep. to sidestep this problem.