(5) Gaussian elimination

Gaussian elimination: Demo

Leads to Row Echelon Form

$$\begin{pmatrix} 2 & 4 & 1 & 5 & 7 & 10 & 11 \\ & 3 & 7 & 9 & 1 & 5 & 5 \\ & & & & 5 & 8 & 9 \\ & & & & & 7 & 4 \\ & & & & & & 3 \end{pmatrix}$$

⊗ Every row in REF is a linear combination of the original rows.

[• Difference to triangular matrix?

• Is the blue line allowed to jump downward by more than one row per column?

If you don't just eliminate downward, but also upward, you get Gauss-Jordan elimination.

Q [What is the result?

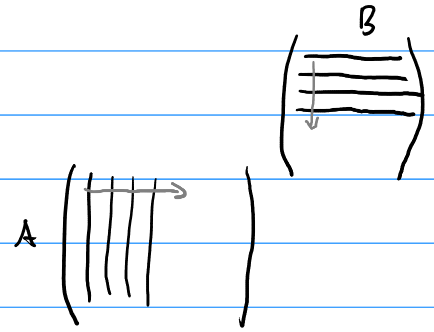
Typical application of Gauss-Jordan:

$$\left(A \mid I \right) \underset{G\text{-}J}{\rightsquigarrow} \left(I \mid A^{-1} \right)$$

Q [Why? / Postpone

Now: Look at Gaussian elimination differently, using ⊗

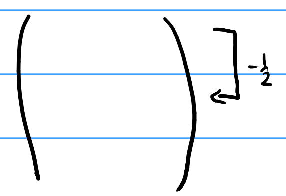
Reconsider matrix-matrix multiplication



Reading 1: Rows of B specify linear comb. of columns of A

Reading 2: Columns of A specify linear combination of rows of B

So:



is representable by matrix-matrix multiplication!

start with an identity

$$M = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ -\frac{1}{2} & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

add a single entry : add the first row $\cdot\left(-\frac{1}{2}\right)$ to the fourth.

So MA has the same result as $\begin{pmatrix} & A & \end{pmatrix}]\, {-\frac{1}{2}}$

[What's M^{-1}?

Has to be $$M^{-1} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ +\frac{1}{2} & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$ to undo what M did.

With enough M's, should be able to arrive at REF

$$M_\ell M_{\ell-1} \cdots M_2 M_1 A = U$$

(upper) row echelon form

Rearrange:

$$A = \underbrace{M_1^{-1} M_2^{-1} \cdots M_{\ell-1}^{-1} M_\ell^{-1}}_{\text{what kind of matrix is this?}} U.$$

$$A = LU$$

Matrices of the form of M are called <u>elimination matrices</u>.

Demo

Dealing with accidents

- Zeros in the way? Just swap them out of the way, using a permutation matrix.

Generally: Just look for largest entry (by abs. value) in current column, swap that up to be the row used for elimination.

("Partial pivoting") The entry is called the pivot.

How do you swap rows? → permutation matrices

In 3×3 case, obtain:

$$M_3 P_2 M_2 M_1 P_1 A = U$$

Want: $PA = LU$ [Can we rewrite to this form? (No.)

Demo Gaussian Elimination with Pivoting

But: Could use $P_2 P_1 A$ (which is now in correct order) and rebuild LU.

Better alternative: Clever bookkeeping. (not here)

→ Demo Why pivoting is a good idea

Computational expense

- What's ℓ?
- What's the (asymptotic) cost of matrix-matrix multiplication?
- What's the (rough) cost of doing GE with el. matrices <u>naively</u>?

It turns out that working with Elimination matrices is actually very cheap.

Demo Behavior of Elim. matrices

→ Same-col. elim. matrices commute.

→ Different-col ones do not.

Overall asymptotic cost of LU?

<inclass-gaussian-el>

How wrong is our result?

Problem: That's not a precise question. ← Fix that.

All the data we work with is approximate. All of it.

Model as: Data = True value + error

$x = x_0 + e$

Two kinds of error measures:

- Absolute $\|e\|$
- Relative $\|e\| / \|x_0\|$

A more precise question:
By what factor is the (rel./abs.) error amplified when I apply operation X?

Not much → "well-conditioned"
A lot → "poorly conditioned"

⇝ call the ampl. factor the rel./abs. "condition number"

- **Example:** Solving a linear system

$$A x = b$$

output (x), input (b)

$$A(x+\Delta x) = (b + \Delta b) \qquad \|\cdot\| : \text{some norm (doesn't matter which)}$$

$$\frac{\|\Delta x\|}{\|x\|} \Big/ \frac{\|\Delta b\|}{\|b\|} = \frac{\|\Delta x\| \cdot \|b\|}{\|x\| \cdot \|\Delta b\|}$$

$$= \frac{\|b\|}{\|x\|} \cdot \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$= \frac{\|Ax\|}{\|x\|} \cdot \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$= \frac{\|Ax\|}{\|x\|} \cdot \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|}$$

$\|Ax\| \leq \|A\| \|x\|$

$$\leq \|A\| \cdot \|A^{-1}\|$$

Useful number! Called condition number of the matrix A (relative to the norm $\|\cdot\|$.)

Example:
$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \|A\|_2 = 3 \qquad \|A^{-1}\| = 1$$

$$\rightsquigarrow \kappa_2 = \|A\| \cdot \|A^{-1}\| = 3.$$

Demo

lec10

feedback debrief

review prequiz

hw2

hw1 stats?

Q's: practice exam set

matrix in Q5 well- or ill-conditioned?

finding norms (and of inv?)

matrix norm inequality

Is κ always $\|A\|\|A^{-1}\|$?

teach before hw?

Today:
- Condition number
- Applications of LU/GE
- Orthogonality

What does the condition number mean in terms of "losing digits"?

$\kappa(A^{-1}) = ?$

What is the cond # if "operation" is $Ax=b$? (input $\to b$, output $\to x$)

<demo condition nr.>

Applications of LU/GE:

The swiss army chainsaw of computational linear algebra.

What remains the same in $PA=LU$?

- ☑ rowspace (PA) = rowspace (U)
- ☑ rank (PA) = rank(U)
- ☒ colspace $(PA) \neq$ colspace (U)
- ☑ $N(PA) = N(U)$

- Find a basis of a span [How?

- Solve linear equations: $Ax = b$

$PA = LU \quad \rightsquigarrow \quad \Leftrightarrow PAx = Pb$

$\Leftrightarrow L\underbrace{Ux}_{\text{"}y\text{"}} = Pb$

how does that help?

(1) solve $Ly = Pb$ (fwd substitution)

(2) solve $Ux = y$ (back substitution)

[Why so complicated? expensive?

- Solving a matrix equation $AX = B$

[How?

[Expense?

[Can I use this to compute an inverse?

- Find determinant $\det(P)\det(A) = \det(L)\det(U)$

- Find rank (but...)

Lec 11

seems easier to use GE and
free variables → yes, for people

Point of Nullspace finding?
one reason: see how
another:

Set · matrix SA
$N(AM)$ and $N(MA)$
Inverses of Permutations

Equation manipulations w/ matrices
• transpose • left/right multiply

Liked practice exam:
will add review q's to pre-quiz

review quiz

• Find basis of the nullspace.

How? $PA = LU$ $N(U) = ?$ not easy to find!

→ Try something different:

$$\bar P A^T = \bar L \bar U \iff A \bar P^T = \bar U^T \bar L^T$$
$$\iff A = \bar U^T \bar L^T \bar P$$

echelon form

$N(\bar U^T) = ?$ $(\quad)^T = (\quad)$

$\dim N(\bar U^T) = ?$ # columns - rank ✓

Have: $\bar U^T \overbrace{N(\bar U^T)}^{\text{pick } x \text{ in here}} = 0$

Want: $\bar U^T \bar L^T \bar P \underbrace{N(A)}_{\text{solve for } y \text{ in here}} = 0$

$\bar L^T \bar P y = x$ ← linear system

$y = \bar P^{-1} (\bar L^T)^{-1} x$ ← solve

$\rightsquigarrow N(A) = \bar P^{-1} \bar L \, N(\bar U^T)$

Demo

Demo traffic Flow

<inclass-gausston-el-app-2>