(8) Eigenvalues

$A \in \mathbb{R}^{n \times n}$

x is an eigenvector of A :

$x \neq 0$, there is a λ so that $Ax = \lambda x$

[eigenvectors unique?

No.

λ eigenvalue of A

$\Leftrightarrow$ $(A - \lambda I)x = 0$ has a solution $x \neq 0$

$\Leftrightarrow$ $N(A - \lambda I) \neq \{0\}$

[must A be square?

Yes.

$$\begin{pmatrix} 1 & * & \times & \times & \times \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 7 \end{pmatrix}$$

For ∇ matrices, eigenvalues appear on the diagonal.

Because eigenvalues satisfy a polynomial equation of degree n, and because there is no formula to solve such equations, num. methods for eigenvalues must be approximate!

What matrix operations do to eigenvalues

Suppose $Ax = \lambda x$.

- "Scaling" βA $\rightsquigarrow$ $(\beta A)x = \beta\lambda x$
- "Shift" $A - \sigma I$ $\rightsquigarrow$ $(A - \sigma I)x = Ax - \sigma x = (\lambda - \sigma)x$
- Power A^k $\rightsquigarrow$ $A^k x = \lambda^k x$
- Inverse A^{-1} $\rightsquigarrow$ $Ax = \lambda x \rightsquigarrow \frac{1}{\lambda}x = A^{-1}x$
- Similarity $T^{-1}AT$ $\rightsquigarrow$ $y := T^{-1}x$

$$\begin{aligned}(T^{-1}AT)\,y &= T^{-1}Ax \\ &= \lambda T^{-1}x \\ &= \lambda y\end{aligned}$$

$\rightsquigarrow$ Similarity transforms preserve eigenvalues ⚠

Will also say: A, B are similar if there exists a matrix T such that $B = T^{-1}AT$.

If A is similar to a diagonal matrix, it's called diagonalizable.

Let $X = (||||)$ be a matrix full of linearly independent eigenvectors. Then

$$AX = X\underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D} \rightsquigarrow X^{-1}AX = D.$$

So that would do it (= diagonalize a matrix).

algebraic multiplicity $\geq$ geometric multiplicity

(multiplicity of λ as a root of the characteristic polynomial) (# of lin. indep. eigenvectors corresponding to λ)

Defective matrix has eigenvalue with

AM > GM.

Defective matrices are not diagonalizable.

Diagonalizable matrices are not defective.

Tool: Schur form

Every matrix is orthonormally similar to an upper triangular matrix.

$$A = QUQ^T$$

If we knew how to compute this, why would it be helpful for knowing eigenvalues? on the diagonal of U.

Power iteration

Eigenvalues of $A^{20,000}$?

$|\lambda_1| > |\lambda_2| > \dots$

$\uparrow \quad \uparrow$

$x_1 \quad x_2$

Assume $\|x_i\| = 1$

$$y = A^{20,000}\underbrace{(\alpha x_1 + \beta x_2)}_{x} = \alpha\lambda_1^{20,000} x_1 + \beta\lambda_2^{20,000} x_2$$

$$\frac{y}{\lambda_1^{20,000}} = \alpha x_1 + \underbrace{\beta \Big(\underbrace{\frac{\lambda_2}{\lambda_1}}_{<1}\Big)^{20,000}}_{\ll 1}$$

Possible problems:

- starting vector has no component for λ_1 ← no problem because of rounding
- $\lambda_1 = \lambda_2$

Demo: Power iteration and variants

Power it: $x_{k+1} = A x_k$

Normalized power it. $x_{k+1} = \dfrac{A x_k}{\|A x_k\|}$ ← fixes overflow issue

Inverse iteration: $x_{k+1} = \dfrac{A^{-1} x_k}{\|A^{-1} x_k\|}$ ← finds eigvec of smallest eigenvalue (by magnitude)

Inverse iteration w/shift $x_{k+1} = \dfrac{(A-\sigma I)^{-1} x_k}{\|(A-\sigma I)^{-1} x_k\|}$ ← finds eigenvalues near σ

Rayleigh quotient iteration: set σ to the Rayleigh quotient in inv. it. w/shift.

The Rayleigh quotient is an estimate of the eigenvalue for an approx. eigenvector x:

$$\frac{x \cdot Ax}{x \cdot x}$$ ← what if $Ax = \lambda x$ exactly?

[Downside of all these methods? One eigenvector at a time.

Idea: Just iterate multiple vectors at a time.

→ "Simultaneous iteration"

0 [Why doesn't that work? → all vectors converge to the eigenvector for λ_1 (the largest by magnitude)

Idea: Keep them linearly independent. Even better: keep them orthogonal!

Orthogonal iteration

$X_0 \in \mathbb{R}^{n \times p}$ $(p \leq n)$ arbitrary (ideally w/ full rank)

$Q_1 R_1 = X_0$

$X_1 = A Q_1$

$Q_2 R_2 = X_1$

$X_2 = A Q_2$

$\vdots$

Suppose this "converges" so that $X_{k+1} \approx X_k$.

$\rightsquigarrow \quad Q_{k+1} R_{k+1} = X_k \approx X_{k+1}$

$\rightsquigarrow \quad X_{k+1} = A Q_{k+1}$

$\rightsquigarrow \quad Q_{k+1} R_{k+1} = A Q_{k+1}$

$\rightsquigarrow \quad Q_{k+1} R_{k+1} Q_{k+1}^T = A \longleftarrow$ [? Schur form!