③ Basis and dimension

Suppose we have a vector space $V \subseteq \mathbb{R}^4$

with generators $\underbrace{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{v_3}$.

0 [Anything wrong with that?

0 [How can we tell if there's a vector too many?

If we can get one as a linear combination of the others.

$\rightarrow$ rearrange

<u>Definition:</u> A set of vectors $v_1, \dots, v_n$ is called <u>linearly dependent</u> if and only iff

$$0 = \alpha_1 v + \dots + \alpha_n v_n \quad \text{with at least one } \alpha_i \neq 0.$$

<u>Opposite:</u> linearly independent $\leftarrow$ usually good / desirable

0 [In case above, multiple α_i had to be non zero to get 0.
Can it happen that exactly one $\alpha_i \neq 0$ already yields 0?

<u>Definition:</u> Linearly independent generators of a vector space V are called a <u>basis of V</u>.

0 [Rephrase: $v_1, v_2, \dots, v_n$ are a basis of $V \iff$ (1) ... and (2) ...

Facts:

- Subsets of linearly independent vectors are also linearly independent. (Why? By contradiction)

- Can always reduce a finite set of vectors to be linearly independent.

Q [~~What's the basis of the vector space $\{0\}$?~~ (in quiz)

0 [Can we always grow a set of vectors into a basis by adding vectors one by one?

- Let V be a vector space,
  $S = \{v_1, v_2, \dots, v_m\}$ a set of generators of V (i.e. they span V)
  and $B = \{b_1, b_2, \dots, b_n\}$ a set of linearly indep. vectors in V.

  Then $m \geq n$.

  (Klein book: "Morphing lemma" for smuggling B into S)

- Consequence: (How? Suppose you have two bases...)

  All bases of a given vector space have the same size.

  $\to$ Dimension of the vector space: $\dim V$

0 [How many dimensions in $\mathbb{R}^2$? $\mathbb{R}^3$?
~~a 100x100 image?~~ a bread recipe?
(in quiz)

- What does $V \subseteq W$ mean for $\dim V \_\_ \dim W$?

- What if $V \subseteq W$ and $\dim V = \dim W$?

- Consequence:
  In a finite-dimensional vector space, you can always 'grow' a basis.

<inclass-lin-dep-basis>

lec 7

norms — quiz

computationally, too — rem hw1 due

kick reading

align quiz + prelec? — hw1 p1 – will improve test

0 So linear (in)dependence seems useful to know about.
How do we test for it?

↳ Def. said $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \\ \end{array}$$
$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

0 still don't know how to solve that, but matrix POV provides another way of thinking about the problem.

e.g. (v_i) linearly indep $\iff$ ...?

Definition: Nullspace of a linear function: $N(f) = \{x : f(x) = 0\}$

sometimes "kernel" or "ker" — a vector space (why?)

Also: Nullspace of a matrix: $N(A) = \{x : Ax = 0\}$

$\dim N(A) \longleftarrow$ sometimes "nullity"

Seems sensible to look at linear independence through matrix goggles.

$\longrightarrow$ same stuff, new words.

$$A = \begin{pmatrix} | & | & & | \\ \beta_1 & \beta_2 & \cdots & \beta_m \\ | & | & & | \end{pmatrix} \rightsquigarrow \operatorname{span}(\beta_1, \ldots, \beta_m) = \text{column space } (A)$$

$$\dim \text{ column space } (A) = \text{col rank}(A)$$

O [$\operatorname{rank}\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right) = ?$

O [relation to m?

Could also say:

$$A = \begin{pmatrix} — & \overline{\beta}_1 & — \\ — & \overline{\beta}_2 & — \\ & \vdots & \\ — & \overline{\beta}_n & — \end{pmatrix} \rightsquigarrow \operatorname{span}(\overline{\beta}_1, \ldots, \overline{\beta}_n) = \text{row space}(A)$$

$$\dim \text{ row space } (A) = \text{row rank}(A)$$

O [in example above?

Massive'ly surprising!

Facts: • col rank (A) = row rank (A) = rank (A)

O
- What does this mean?
- Rank of an outer product?
- Rank of a sum of outer products?
- Revisit image compression demo
- Can you see the rank?
- So what can't I do in that demo?

Massively surprising!

Facts, cont'd:

- For a matrix with n columns:

  $$\operatorname{rank} A + \dim \mathcal{N}(A) = n$$

  - What does this mean?
  - If a matrix has full rank, ...
  - If a matrix has rank zero, ...
  - Do these count dimensions in the same vector space? (input/output?)
  - Suppose I have a matrix with a zero column. (rank/nullsp conseq.?)
    What can't I do by adding a lin. dep. column vector?

lec8

review rank/nullspace

review quiz

reminders:

- hw1 due
- mt1 in a week

IEF

Q's: videos shorter & to point → fewer views

what's on the test?

matrix square for $\exists$ LU decomp?

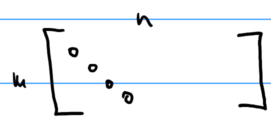
overview of HW1 f/c

~/plane-patch.py

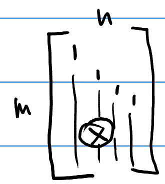
more rank examples?

readings

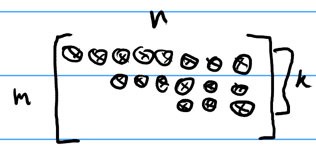
Rank / Nullspace examples:

- 

rank? dim $N(A)$?

- 

rank? dim $N(A)$?

- 

rank? dim $N(A)$?

O Why is rank(U) = rank(A) in $A = LU$?

Does the permutation matrix change anything?

Properties of linear functions (in terms of rank, dim $N(A)$)

$f: V \to W$ linear $\quad \dim(V) = n \quad \dim(W) = m \quad$ matrix A repr. f

- one-to-one?

  $f(x) = f(y) \Rightarrow x = y \quad \leftarrow$ Def 1-to-1

  Linearity: $f(x) = f(y) \Rightarrow f(x-y) = 0$

  If $\underline{\dim N(A) = 0}$, then $f(x-y) = 0 \Rightarrow x - y = 0$.

- onto?

  $f(V) = W \quad \leftarrow$ Def onto

  $\Leftarrow \dim f(V) = \dim(W) \Leftarrow \operatorname{rank}(A) = m$

- invertible?

  must satisfy $\overbrace{\operatorname{rank}(A) = m}^{\text{onto}}, \overbrace{\dim N(A) = 0}^{\text{1-to-1}}$

  fact (above): $\underbrace{\operatorname{rank}(A)}_{m} + \underbrace{\dim N(A)}_{0} = n \Rightarrow m = n \Rightarrow A$ square

  name for matrix representing f^{-1}: A^{-1}!

Rank and linear independence, computationally

- If I draw two random vectors, what's the probability that they'll be linearly dependent?

- If I draw two random vectors, what's the probability they'll point the same way?

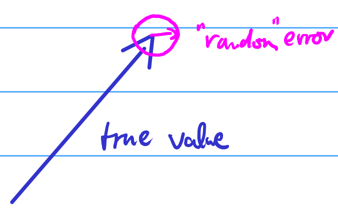
⚠ Keep this in mind — will be super useful later!

Computers don't represent numbers exactly. (in floating point)

Every FP number: $\overbrace{3.14159}^{\text{good digits}}\underbrace{777777}_{\text{junk}}$

Model as: true answer + small "random" error

In a vector:



What does this mean for a computational test of linear dependence?

Demo