Numerical Methods (CS 357)

Worksheet

Problem 1. The SVD and the 2-norm

A matrix A has the Singular Value Decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

What is the largest value that $\|Ax\|_2$ can attain for any x with $\|x\|_2 = 1$?

Problem 2. The SVD and the 2-norm

A matrix A has the Singular Value Decomposition

$$A = \begin{bmatrix} | & | & \cdots & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ \vdots & \vdots & \vdots \\ - & v_n & - \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$.
What is $\|A\|_2$?

(A) σ_1

(B) σ_n

(C) $v_1 \cdot v_1$

(D) $u_1 \cdot v_1$

(E) $u_n \cdot u_n$

Problem 3. Trolley line problem

A matrix X has the Singular Value Decomposition

$$X = \begin{bmatrix} | & | & \cdots & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ \vdots & \vdots & \vdots \\ - & v_n & - \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$.

Let X have the position vectors of n points as its columns. What part of X's SVD gives the direction of the closest trolley line to all points represented in X?

(A) σ_1

(B) v_1

(C) u_1

(D) σ_n

(E) v_n

(F) u_n

Problem 4. Pseudoinverse

A matrix A has the Singular Value Decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

If you were to compute an SVD of the *pseudoinverse* of A, what would be the largest singular value in it?