ON THE FUNCTIONS COUNTING WALKS WITH SMALL STEPS IN THE QUARTER PLANE

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Abstract. Models of spatially homogeneous walks in the quarter plane $\mathbf{Z}_+^2$ with steps taken from a subset $\mathcal{S}$ of the set of jumps to the eight nearest neighbors are considered. The generating function $(x,y,z)\mapsto Q(x,y;z)$ of the numbers $q(i,j;n)$ of such walks starting at the origin and ending at $(i,j)\in\mathbf{Z}_+^2$ after n steps is studied. For all non-singular models of walks, the functions $x\mapsto Q(x,0;z)$ and $y\mapsto Q(0,y;z)$ are continued as multi-valued functions on $\mathbf{C}$ having infinitely many meromorphic branches, of which the set of poles is identified. The nature of these functions is derived from this result: namely, for all the 51 walks which admit a certain infinite group of birational transformations of $\mathbf{C}^2$, the interval $]0,1/|\mathcal{S}|[$ of variation of z splits into two dense subsets such that the functions $x\mapsto Q(x,0;z)$ and $y\mapsto Q(0,y;z)$ are shown to be holonomic for any z from the one of them and non-holonomic for any z from the other. This entails the non-holonomy of $(x,y,z)\mapsto Q(x,y;z)$, and therefore proves a conjecture of Bousquet-Mélou and Mishna in [5].

Keywords. Walks in the quarter plane; counting generating function; holonomy; group of the walk; Riemann surface; elliptic functions; uniformization; universal covering

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1. Introduction and main results

In the field of enumerative combinatorics, counting walks on the lattice $\mathbf{Z}^2$ is among the most classical topics. While counting problems have been largely resolved for unrestricted walks on $\mathbf{Z}^2$ and for walks staying in a half plane [6], walks confined to the quarter plane $\mathbf{Z}_+^2$ still pose considerable challenges. In recent years, much progress has been made for walks in the quarter plane with small steps, which means that the set $\mathcal{S}$ of possible steps is included in $\{-1,0,1\}^2\setminus\{(0,0)\}$; for examples, see Figures 1 and 10. In [5], Bousquet-Mélou and Mishna constructed a thorough classification of these 2^8 walks. After eliminating trivial cases and exploiting equivalences, they showed that 79 inherently different walks remain to be studied. Let $q(i,j;n)$ denote the number of paths in $\mathbf{Z}_+^2$ having length n, starting from $(0,0)$ and ending at (i,j). Define the counting function (CF) as

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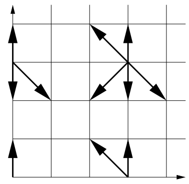


FIGURE 1. Example of model (with an infinite group) considered here—on the boundary, the jumps are the natural ones: those that would take the walk out $\mathbf{Z}_+^2$ are discarded

$$Q(x,y;z)=\sum_{i,j,n\geqslant 0} q(i,j;n)x^iy^jz^n.$$

There are then two key challenges:

(i) Finding an explicit expression for $Q(x,y;z)$;
(ii) Determining the nature of $Q(x,y;z)$: is it holonomic (i.e., see [10, Appendix B.4], is the vector space over $\mathbf{C}(x,y,z)$—the field of rational functions in the three variables x,y,z—spanned by the set of all derivatives of $Q(x,y;z)$ finite dimensional)? And in that event, is it algebraic, or even rational?

The common approach to treat these problems is to start from a functional equation for the CF, which for the walks with small steps takes the form (see [5])

$$K(x,y;z)Q(x,y;z)=K(x,0;z)Q(x,0;z)+K(0,y;z)Q(0,y;z)-K(0,0;z)Q(0,0;z)-xy, \tag{1.1}$$

where

$$K(x,y;z)=xyz[\textstyle\sum_{(i,j)\in\mathcal{S}} x^iy^j-1/z] \tag{1.2}$$

is called the *kernel of the walk*. This equation determines $Q(x,y;z)$ through the boundary functions $Q(x,0;z)$, $Q(0,y;z)$ and $Q(0,0;z)$.

Known results regarding both problems (i) and (ii) highlight the notion of the *group of the walk*, introduced by Malyshev [16, 17, 18]. This is the group

$$\langle\xi,\eta\rangle \tag{1.3}$$

of birational transformations of $\mathbf{C}(x,y)$, generated by

$$\xi(x,y)=\left(x,\frac{1}{y}\frac{\sum_{(i,-1)\in\mathcal{S}}x^i}{\sum_{(i,+1)\in\mathcal{S}}x^i}\right),\qquad \eta(x,y)=\left(\frac{1}{x}\frac{\sum_{(-1,j)\in\mathcal{S}}y^j}{\sum_{(+1,j)\in\mathcal{S}}y^j},y\right). \tag{1.4}$$

Each element of $\langle\xi,\eta\rangle$ leaves invariant the jump function $\sum_{(i,j)\in\mathcal{S}}x^iy^j$. Further, $\xi^2=\eta^2=\mathrm{Id}$, and $\langle\xi,\eta\rangle$ is a dihedral group of order even and larger than or equal to four. It turns out that 23 of the 79 walks have a finite group, while the 56 others admit an infinite group, see [5].

For 22 of the 23 models with finite group, CFs $Q(x,0;z)$, $Q(0,y;z)$ and $Q(0,0;z)$—and hence $Q(x,y;z)$ by (1.1)—have been computed in [5] by means of certain (half-)orbit sums of the functional equation (1.1). For the 23rd model with finite group, known as

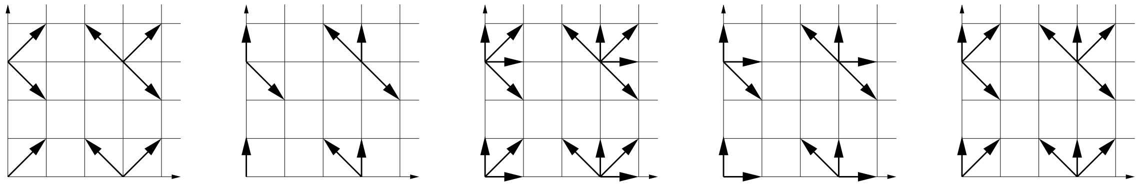


FIGURE 2. The 5 singular walks in the classification of [5]

Gessel's walk (see Figure 10), the CFs have been expressed by radicals in [3] thanks to a guessing-proving method using computer calculations; they were also found in [15] by solving some boundary value problems. For the 2 walks with infinite group on the left in Figure 2, they have been obtained in [20], by exploiting a particular property shared by the 5 models of Figure 2 commonly known as *singular walks*. Finally, in [21], the problem (i) was resolved for all 54 remaining walks—and in fact for all the 79 models. For the 74 non-singular walks, this was done via a unified approach: explicit integral representations were obtained for CFs $Q(x,0;z)$, $Q(0,y;z)$ and $Q(0,0;z)$ in certain domains, by solving boundary value problems of Riemann-Carleman type.

In this article we go further, since both functions $x\mapsto Q(x,0;z)$ and $y\mapsto Q(0,y;z)$ are computed on the whole of $\mathbf{C}$ as *multi-valued* functions with infinitely many meromorphic branches, that are made explicit for all $z\in]0,1/|\mathcal{S}|[$. This result gives not only the most complete continuation of these CFs on their complex planes along all paths, but also permits to establish the nature of these functions, i.e., to solve Problem (ii).

Problem (ii) is actually resolved for only 28 of the 79 walks. All 23 finite group models admit a holonomic CF. Indeed, the nature of $Q(x,y;z)$ was determined in [5] for 22 of these walks: 19 walks turn out to have a holonomic but non-algebraic CF, while for 3 walks $Q(x,y;z)$ is algebraic. As for the 23rd—again, Gessel's model—, the CF is algebraic [3]. Alternative proofs for the nature of the (bivariate) CF for these 23 walks were given in [9]. For the remaining 56 walks with an infinite group, not much is known: in [20] it was shown that for 2 singular walks (namely, the 2 ones on the left in Figure 2), the function $z\mapsto Q(1,1;z)$ has infinitely many poles and, as a consequence [10, Appendix B.4], is non-holonomic. Accordingly [10, Appendix B.4], the trivariate function $Q(x,y;z)$ is non-holonomic as well. It is reasonable to expect that the same approach would lead to the non-holonomy of all 5 singular walks, see [19, 20]. As for the 51 non-singular walks with infinite group (all of them are pictured on Figure 17), Bousquet-Mélou and Mishna [5] conjectured that they also have a non-holonomic CF. In this article we prove the following theorem.

Theorem 1. *For any of the* 51 *non-singular walks with infinite group (1.3), the set* $]0,1/|\mathcal{S}|[$ *splits into subsets* $\mathcal{H}$ *and* $]0,1/|\mathcal{S}|[\setminus\mathcal{H}$ *that are both dense in* $]0,1/|\mathcal{S}|[$ *and such that:*

(i) $x\mapsto Q(x,0;z)$ *and* $y\mapsto Q(0,y;z)$ *are holonomic for any* $z\in\mathcal{H}$*;*
(ii) $x\mapsto Q(x,0;z)$ *and* $y\mapsto Q(0,y;z)$ *are non-holonomic for any* $z\in]0,1/|\mathcal{S}|[\setminus\mathcal{H}$.

Theorem 1 (ii) immediately entails Bousquet-Mélou and Mishna's conjecture: the trivariate function $(x, y, z) \mapsto Q(x, y; z)$ is non-holonomic since the holonomy is stable by specialization of a variable [10, Appendix B.4]. Further, Theorem 1 (i) goes beyond it: it suggests that $Q(x, y; z)$, although being non-holonomic, still stays accessible for further analysis when $z \in \mathcal{H}$, namely by the use of methods developed in [8, Chapter 4], see Remark 16. This important set $\mathcal{H}$ will be characterized in two different ways, see Corollary 15 and Remark 16 below.

The proof of Theorem 1 we shall do here is based on the above-mentioned construction of the CF $x \mapsto Q(x, 0; z)$ (resp. $y \mapsto Q(0, y; z)$) as a multi-valued function, that must now be slightly more detailed. First, we prove in this article that for any $z \in]0, 1/|\mathcal{S}|[$, the integral expression of $x \mapsto Q(x, 0; z)$ given in [21] in a certain domain of $\mathbf{C}$ admits a *direct holomorphic continuation* on $\mathbf{C} \setminus [x_3(z), x_4(z)]$. Points $x_3(z), x_4(z)$ are among four *branch points* $x_1(z), x_2(z), x_3(z), x_4(z)$ of the two-valued algebraic function $x \mapsto Y(x; z)$ defined via the kernel (1.2) by the equation $K(x, Y(x; z); z) = 0$. These branch points are roots of the discriminant (2.2) of the latter equation, which is of the second order. We refer to Section 2 for the numbering and for some properties of these branch points. We prove next that function $x \mapsto Q(x, 0; z)$ does not admit a direct meromorphic continuation on any open domain containing the segment $[x_3(z), x_4(z)]$, but admits a *meromorphic continuation along any path* going once through $[x_3(z), x_4(z)]$. This way, we obtain a second (and different) branch of the function, which admits a direct meromorphic continuation on the whole cut plane $\mathbf{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$. Next, if the function $x \mapsto Q(x, 0; z)$ is continued along a path in $\mathbf{C} \setminus [x_1(z), x_2(z)]$ crossing once again $[x_3(z), x_4(z)]$, we come across its first branch. But its continuation along a path in $\mathbf{C} \setminus [x_3(z), x_4(z)]$ crossing once $[x_1(z), x_2(z)]$ leads to a third branch of this CF, which is meromorphic on $\mathbf{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$. Making loops through $[x_3(z), x_4(z)]$ and $[x_1(z), x_2(z)]$, successively, we construct $x \mapsto Q(x, 0; z)$ as a multi-valued meromorphic function on $\mathbf{C}$ with branch points $x_1(z), x_2(z), x_3(z), x_4(z)$, and with (generically) infinitely many branches. The analogous construction is valid for $y \mapsto Q(0, y; z)$.

In order to prove Theorem 1 (ii), we then show that for any of the 51 non-singular walks with infinite group (1.3), for any $z \in]0, 1/|\mathcal{S}|[\setminus \mathcal{H}$, *the set formed by the poles of all branches of* $x \mapsto Q(x, 0; z)$ *(resp.* $y \mapsto Q(0, y; z)$*) is infinite*—and even dense in certain curves, to be specified in Section 7 (see Figure 11 for their pictures). This is not compatible with holonomy. Indeed, all branches of a holonomic one-dimensional function must verify the *same* linear differential equation with polynomial coefficients. In particular, the poles of all branches are among the zeros of these polynomials, and hence they must be in a finite number.

The rest of our paper is organized as follows. In Section 2 we construct the Riemann surface $\mathbf{T}$ of genus 1 of the two-valued algebraic functions $X(y; z)$ and $Y(x; z)$ defined by

$$K(X(y; z), y; z) = 0, \qquad K(x, Y(x; z); z) = 0.$$

In Section 3 we introduce and study the universal covering of $\mathbf{T}$. It can be viewed as the complex plane $\mathbf{C}$ split into infinitely many parallelograms with edges $\omega_1(z) \in i\mathbf{R}$ and $\omega_2(z) \in \mathbf{R}$ that are uniformization periods. These periods as well as a new important period $\omega_3(z)$ are made explicit in (3.1) and (3.2). In Section 4 we lift CFs $Q(x, 0; z)$

and $Q(0, y; z)$ to some domain of $\mathbf{T}$, and then to a domain on its universal covering. In Section 5, using a proper lifting of the automorphisms ξ and η defined in (1.4) as well as the independence of $K(x, 0; z)Q(x, 0; z)$ and $K(0, y; z)Q(0, y; z)$ w.r.t. y and x, respectively, we continue these functions meromorphically on the whole of the universal covering. All this procedure has been first carried out by Malyshev in the seventies [16, 17, 18], at that time to study the stationary probability generating functions for random walks with small steps in the quarter plane $\mathbf{Z}_+^2$. It is presented in [8, Chapter 3] for the case of ergodic random walks in $\mathbf{Z}_+^2$, and applies directly for our $Q(x, 0; 1/|\mathcal{S}|)$ and $Q(0, y; 1/|\mathcal{S}|)$ if the drift vector $(\sum_{(i,j)\in\mathcal{S}} i, \sum_{(i,j)\in\mathcal{S}} j)$ has not two positive coordinates. In Sections 3, 4 and 5 we carry out this procedure for all $z \in]0, 1/|\mathcal{S}|[$ and all non-singular walks, independently of the drift. Then, going back from the universal covering to the complex plane allows us in Subsection 5.2 to continue $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$ as multi-valued meromorphic functions with infinitely many branches.

For given $z \in]0, 1/|\mathcal{S}|[$, the rationality or irrationality of the ratio $\omega_2(z)/\omega_3(z)$ of the uniformization periods is crucial for the nature of $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$. Namely, Theorem 7 of Subsection 5.3 proves that if $\omega_2(z)/\omega_3(z)$ is rational, these functions are holonomic.

For 23 models of walks with finite group $\langle\xi, \eta\rangle$, the ratio $\omega_2(z)/\omega_3(z)$ turns out to be rational and independent of z, see Lemma 8 below, that implies immediately the holonomy of the generating functions. In Section 6 we gather further results of our approach for the models with finite group concerning the set of branches of the generating functions and their nature. In particular, we recover most of the results of [3, 5, 9, 20].

Section 7 is devoted to 51 models with infinite group $\langle\xi, \eta\rangle$. For all of them, the sets $\mathcal{H} = \{z \in]0, 1/|\mathcal{S}|[: \omega_2(z)/\omega_3(z) \text{ is rational}\}$ and $]0, 1/|\mathcal{S}|[\setminus\mathcal{H} = \{z \in]0, 1/|\mathcal{S}|[: \omega_2(z)/\omega_3(z) \text{ is irrational}\}$ are proved to be dense in $]0, 1/|\mathcal{S}|[$, see Proposition 14. These sets can be also characterized as those where the group $\langle\xi, \eta\rangle$ restricted to the curve $\{(x, y) : K(x, y; z) = 0\}$ is finite and infinite, respectively, see Remark 6. By Theorem 7 mentioned above, $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$ are holonomic for any $z \in \mathcal{H}$, that proves Theorem 1 (i). In Subsections 7.1, 7.2 and 7.3, we analyze in detail the branches of $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$ for any $z \in]0, 1/|\mathcal{S}|[\setminus\mathcal{H}$ and prove the following facts (see Theorem 17):

(i) The only singularities of the first (main) branches of $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$ are two branch points $x_3(z), x_4(z)$ and $y_3(z), y_4(z)$, respectively;
(ii) All (other) branches have only a finite number of poles;
(iii) The set of poles of all these branches is infinite for each of these functions, and is dense on certain curves; these curves are specified in Section 7, and in particular are pictured on Figure 11 for all 51 walks given on Figure 17;
(iv) Poles of branches out of these curves may be only at zeros of $x \mapsto K(x, 0; z)$ or $y \mapsto K(0, y; z)$, respectively.

It follows from (iii) that $x \mapsto Q(x, 0; z)$ and $y \mapsto Q(0, y; z)$ are non-holonomic for any $z \in]0, 1/|\mathcal{S}|[\setminus\mathcal{H}$.

2. Riemann surface $\mathbf{T}$

In the sequel we suppose that $z \in]0, 1/|\mathcal{S}|[$, and we drop the dependence of the different quantities w.r.t. z.

2.1. Kernel $K(x,y)$**.** The kernel $K(x,y)$ defined in (1.2) can be written as

$$xyz[\textstyle\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z] = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = a(x)y^2 + b(x)y + c(x), \tag{2.1}$$

where

$$\begin{array}{lll} \widetilde{a}(y) = zy\sum_{(+1,j)\in\mathcal{S}} y^j, & \widetilde{b}(y) = -y+zy\sum_{(0,j)\in\mathcal{S}} y^j, & \widetilde{c}(y) = zy\sum_{(-1,j)\in\mathcal{S}} y^j, \\ a(x) = zx\sum_{(i,+1)\in\mathcal{S}} x^i, & b(x) = -x+zx\sum_{(i,0)\in\mathcal{S}} x^i, & c(x) = zx\sum_{(i,-1)\in\mathcal{S}} x^i. \end{array}$$

With these notations we define

$$\widetilde{d}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y), \qquad d(x) = b(x)^2 - 4a(x)c(x). \tag{2.2}$$

If the walk is non-singular, then for any $z \in]0, 1/|\mathcal{S}|[$, the polynomial $\widetilde{d}$ (resp. d) has three or four roots, that we call y_ℓ (resp. x_ℓ). They are such that $|y_1| < y_2 < 1 < y_3 < |y_4|$ (resp. $|x_1| < x_2 < 1 < x_3 < |x_4|$), with $y_4 = \infty$ (resp. $x_4 = \infty$) if $\widetilde{d}$ (resp. d) has order three: the arguments given in [8, Part 2.3] for the case $z = 1/|\mathcal{S}|$ indeed also apply for other values of z.

Now we notice that the kernel (1.2) vanishes if and only if $[\widetilde{b}(y) + 2\widetilde{a}(y)x]^2 = \widetilde{d}(y)$ or $[b(x)+2a(x)y]^2 = d(x)$. Consequently [13], the algebraic functions $X(y)$ and $Y(x)$ defined by

$$\textstyle\sum_{(i,j)\in\mathcal{S}} X(y)^i y^j - 1/z = 0, \qquad \sum_{(i,j)\in\mathcal{S}} x^i Y(x)^j - 1/z = 0 \tag{2.3}$$

have two branches, meromorphic on the cut planes $\mathbf{C}\setminus([y_1, y_2]\cup[y_3, y_4])$ and $\mathbf{C}\setminus([x_1, x_2]\cup[x_3, x_4])$, respectively—note that if $y_4 < 0$, $[y_3, y_4]$ stands for $[y_3, \infty[\cup\{\infty\}\cup]-\infty, y_4]$; the same holds for $[x_3, x_4]$.

We fix the notations of the two branches of the algebraic functions $X(y)$ and $Y(x)$ by setting

$$X_0(y) = \frac{-\widetilde{b}(y) + \widetilde{d}(y)^{1/2}}{2\widetilde{a}(y)}, \qquad X_1(y) = \frac{-\widetilde{b}(y) - \widetilde{d}(y)^{1/2}}{2\widetilde{a}(y)}, \tag{2.4}$$

as well as

$$Y_0(x) = \frac{-b(x) + d(x)^{1/2}}{2a(x)}, \qquad Y_1(x) = \frac{-b(x) - d(x)^{1/2}}{2a(x)}. \tag{2.5}$$

The following straightforward result holds.

Lemma 2. *For all $y \in \mathbf{C}$, we have $|X_0(y)| \leqslant |X_1(y)|$. Likewise, for all $x \in \mathbf{C}$, we have $|Y_0(x)| \leqslant |Y_1(x)|$.*

Proof. The arguments (via the maximum modulus principle [13]) given in [8, Part 5.3] for $z = 1/|\mathcal{S}|$ also work for $z \in]0, 1/|\mathcal{S}|[$. □

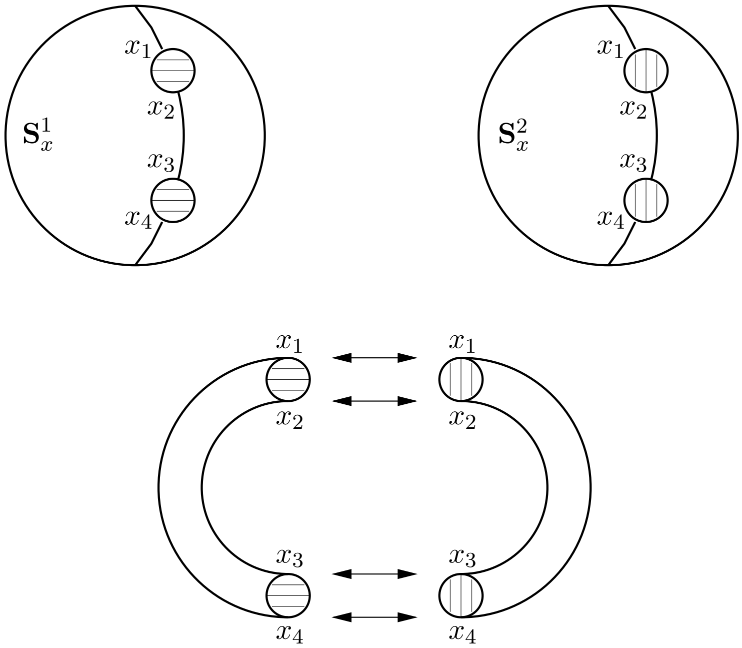


FIGURE 3. Construction of the Riemann surface

2.2. **Riemann surface T.** We now construct the Riemann surface $\mathbf{T}$ of the algebraic function $Y(x)$ introduced in (2.3). For this purpose we take two Riemann spheres $\mathbf{C}\cup\{\infty\}$, say $\mathbf{S}_x^1$ and $\mathbf{S}_x^2$, cut along the segments $[x_1, x_2]$ and $[x_3, x_4]$, and we glue them together along the borders of these cuts, joining the lower border of the segment $[x_1, x_2]$ (resp. $[x_3, x_4]$) on $\mathbf{S}_x^1$ to the upper border of the same segment on $\mathbf{S}_x^2$ and vice versa, see Figure 3. The resulting surface $\mathbf{T}$ is homeomorphic to a torus (i.e., a compact Riemann surface of genus 1) and is projected on the Riemann sphere $\mathbf{S}$ by a canonical covering map $h_x : \mathbf{T} \to \mathbf{S}$.

In a standard way, we can lift the function $Y(x)$ to $\mathbf{T}$, by setting $Y(s) = Y_\ell(h_x(s))$ if $s \in \mathbf{S}_x^\ell \subset \mathbf{T}$, $\ell \in \{1, 2\}$. Thus, $Y(s)$ is single-valued and continuous on $\mathbf{T}$. Furthermore, $K(h_x(s), Y(s)) = 0$ for any $s \in \mathbf{T}$. For this reason, we call $\mathbf{T}$ the Riemann surface of $Y(x)$.

In a similar fashion, one constructs the Riemann surface of the function $X(y)$, by gluing together two copies $\mathbf{S}_y^1$ and $\mathbf{S}_y^2$ of the sphere $\mathbf{S}$ along the segments $[y_1, y_2]$ and $[y_3, y_4]$. It is again homeomorphic to a torus.

Since the Riemann surfaces of $X(y)$ and $Y(x)$ are equivalent, we can work on a single Riemann surface $\mathbf{T}$, but with two different covering maps $h_x, h_y : \mathbf{T} \to \mathbf{S}$. Then, for $s \in \mathbf{T}$, we set $x(s) = h_x(s)$ and $y(s) = h_y(s)$, and we will often represent a point $s \in \mathbf{T}$ by the pair of its *coordinates* $(x(s), y(s))$. These coordinates are of course not independent, because the equation $K(x(s), y(s)) = 0$ is valid for any $s \in \mathbf{T}$.

2.3. **Real points of T.** Let us identify the set Φ of real points of $\mathbf{T}$, that are the points $s \in \mathbf{T}$ where $x(s)$ and $y(s)$ are both real or equal to infinity. Note that for y real, $X(y)$

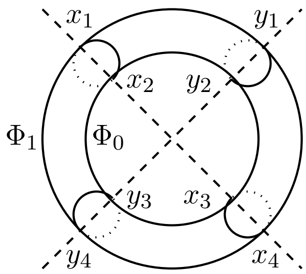


FIGURE 4. Location of the branch points and of the cycles Φ_0 and Φ_1 on the Riemann surface $\mathbf{T}$

is real if $y \in [y_4, y_1]$ or $y \in [y_2, y_3]$, and complex if $y \in]y_1, y_2[$ or $y \in]y_3, y_4[$, see (2.2). Likewise, for real values of x, $Y(x)$ is real if $x \in [x_4, x_1]$ or $x \in [x_2, x_3]$, and complex if $x \in]x_1, x_2[$ or $x \in]x_3, x_4[$. The set Φ therefore consists of two non-intersecting closed analytic curves Φ_0 and Φ_1, equal to (see Figure 4)

$$\Phi_0 = \{s \in \mathbf{T} : x(s) \in [x_2, x_3]\} = \{s \in \mathbf{T} : y(s) \in [y_2, y_3]\}$$

and

$$\Phi_1 = \{s \in \mathbf{T} : x(s) \in [x_4, x_1]\} = \{s \in \mathbf{T} : y(s) \in [y_4, y_1]\},$$

and homologically equivalent to a basic cycle on $\mathbf{T}$—note, however, that the equivalence class containing Φ_0 and Φ_1 is disjoint from that containing the cycle $h_x^{-1}(\{x \in \mathbf{C} : |x| = 1\})$.

2.4. **Galois automorphisms ξ, η.** We continue Section 2 by introducing two Galois automorphisms. Define first, for $\ell \in \{1, 2\}$, the incised spheres

$$\widehat{\mathbf{S}}_x^\ell = \mathbf{S}_x^\ell \setminus ([x_1, x_2] \cup [x_3, x_4]), \qquad \widehat{\mathbf{S}}_y^\ell = \mathbf{S}_y^\ell \setminus ([y_1, y_2] \cup [y_3, y_4]).$$

For any $s \in \mathbf{T}$ such that $x(s)$ is not equal to a branch point x_ℓ, there is a unique $s' \neq s \in \mathbf{T}$ such that $x(s) = x(s')$. Furthermore, if $s \in \widehat{\mathbf{S}}_x^1$ then $s' \in \widehat{\mathbf{S}}_x^2$ and vice versa. On the other hand, whenever $x(s)$ is one of the branch points x_ℓ, $s = s'$. Also, since $K(x(s), y(s)) = 0$, $y(s)$ and $y(s')$ give the two values of function $Y(x)$ at $x = x(s) = x(s')$. By Vieta's theorem and (2.1), $y(s)y(s') = c(x(s))/a(x(s))$.

Similarly, for any $s \in \mathbf{T}$ such that $y(s)$ is different from the branch points y_ℓ, there exists a unique $s'' \neq s \in \mathbf{T}$ such that $y(s) = y(s'')$. If $s \in \widehat{\mathbf{S}}_y^1$ then $s'' \in \widehat{\mathbf{S}}_y^2$ and vice versa. On the other hand, if $y(s)$ is one of the branch points y_ℓ, we have $s = s''$. Moreover, since $K(x(s), y(s)) = 0$, $x(s)$ and $x(s'')$ the two values of function $X(y)$ at $y(s) = y(s'')$. Again, by Vieta's theorem and (2.1), $x(s)x(s'') = \widetilde{c}(y(s))/\widetilde{a}(y(s))$.

Define now the mappings $\xi : \mathbf{T} \to \mathbf{T}$ and $\eta : \mathbf{T} \to \mathbf{T}$ by

$$\left\{ \begin{array}{lll} \xi s = s' & \text{if} & x(s) = x(s'), \\ \eta s = s'' & \text{if} & y(s) = y(s''). \end{array} \right. \tag{2.6}$$

Following [16, 17, 18], we call them *Galois automorphisms* of $\mathbf{T}$. Then $\xi^2 = \eta^2 = \mathrm{Id}$, and

$$y(\xi s) = \frac{c(x(s))}{a(x(s))} \frac{1}{y(s)}, \qquad x(\eta s) = \frac{\widetilde{c}(y(s))}{\widetilde{a}(y(s))} \frac{1}{x(s)}. \tag{2.7}$$

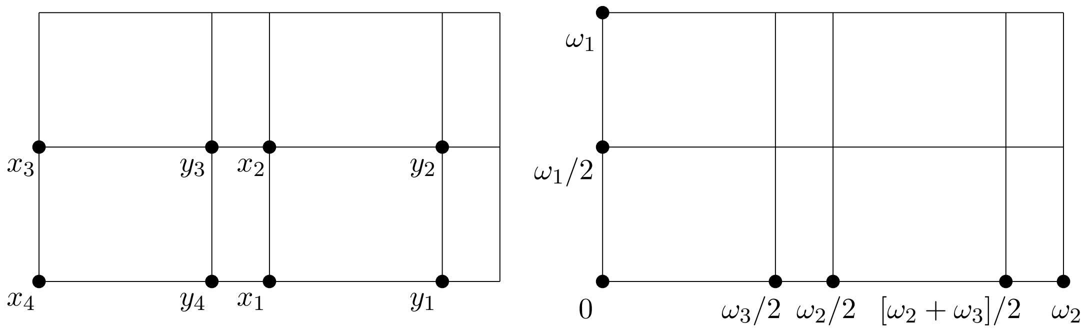


FIGURE 5. The Riemann surface $\mathbf{C}/(\omega_1\mathbf{Z}+\omega_2\mathbf{Z})$ and the location of the branch points

Any $s\in\mathbf{T}$ such that $x(s)=x_\ell$ (resp. $y(s)=y_\ell$) is a fixed point for ξ (resp. η). To illustrate and to get some more intuition, it is helpful to draw on Figure 4 the straight line through the pair of points of Φ_0 where $x(s)=x_2$ and x_3 (resp. $y(s)=y_2$ and y_3); then points s and ξs (resp. s and ηs) can be drawn symmetric about this straight line.

2.5. **The Riemann surface T viewed as a parallelogram whose opposed edges are identified.** Like any compact Riemann surface of genus 1, $\mathbf{T}$ is isomorphic to a certain quotient space

$$\mathbf{C}/(\omega_1\mathbf{Z}+\omega_2\mathbf{Z}), \tag{2.8}$$

where ω_1,ω_2 are complex numbers linearly independent on $\mathbf{R}$, see [13]. The set (2.8) can obviously be thought as the (fundamental) parallelogram $\omega_1[0,1]+\omega_2[0,1]$ whose opposed edges are identified. Up to a unimodular transform, ω_1,ω_2 are unique, see [13]. In our case, suitable ω_1,ω_2 will be found in (3.1).

If we cut the torus on Figure 4 along $[x_1,x_2]$ and Φ_0, it becomes the parallelogram on the left in Figure 5. On the right in the same figure, this parallelogram is translated to the complex plane, and all corresponding important points are expressed in terms of the complex numbers ω_1,ω_2 (see above) and of ω_3 (to be defined below, in (3.2)).

3. Universal covering

3.1. **An informal construction of the universal covering.** The Riemann surface $\mathbf{T}$ can be considered as a parallelogram whose opposite edges identified, see (2.8) and Figure 5. The universal covering of $\mathbf{T}$ can then be viewed as the union of infinitely many such parallelograms glued together, as in Figure 6.

3.2. **Periods and covering map.** We now give a proper construction of the universal covering. The Riemann surface $\mathbf{T}$ being of genus 1, its universal covering has the form $(\mathbf{C},\lambda)$, where $\mathbf{C}$ is the complex plane and $\lambda:\mathbf{C}\to\mathbf{T}$ is a non-branching covering map, see [13]. This way, the surface $\mathbf{T}$ can be considered as the additive group $\mathbf{C}$ factorized by the discrete subgroup $\omega_1\mathbf{Z}+\omega_2\mathbf{Z}$, where the periods ω_1,ω_2 are complex numbers, linearly independent on $\mathbf{R}$. Any segment of length $|\omega_\ell|$ and parallel to ω_ℓ, $\ell\in\{1,2\}$, is projected

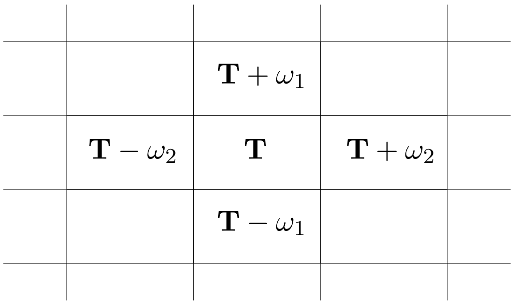


FIGURE 6. Informal construction of the universal covering

onto a closed curve on $\mathbf{T}$ homological to one of the elements of the normal basis on the torus. We choose $\lambda([0,\omega_1])$ to be homological to the cut $[x_1,x_2]$ (and hence also to all other cuts $[x_3,x_4]$, $[y_1,y_2]$ and $[y_3,y_4]$); $\lambda([0,\omega_2])$ is then homological to the cycles of real points Φ_0 and Φ_1; see Figures 5 and 7.

Our aim now is to find the expression of the covering λ. We will do this by finding, for all $\omega\in\mathbf{C}$, the explicit expressions of the pair of coordinates $(x(\lambda\omega),y(\lambda\omega))$, that we have introduced in Section 2. First, the periods ω_1,ω_2 are obtained in [8, Lemma 3.3.2] for $z=1/|\mathcal{S}|$. The reasoning is exactly the same for other values of z, and we obtain that with d as in (2.2),

$$\omega_1 = i\int_{x_1}^{x_2}\frac{\mathrm{d}x}{[-d(x)]^{1/2}},\qquad \omega_2=\int_{x_2}^{x_3}\frac{\mathrm{d}x}{d(x)^{1/2}}. \tag{3.1}$$

We also need to introduce

$$\omega_3=\int_{X(y_1)}^{x_1}\frac{\mathrm{d}x}{d(x)^{1/2}}. \tag{3.2}$$

Further, we define

$$g_x(t)=\left\{\begin{array}{ll} d''(x_4)/6+d'(x_4)/[t-x_4] & \text{if}\quad x_4\neq\infty,\\ d''(0)/6+d'''(0)t/6 & \text{if}\quad x_4=\infty,\end{array}\right.$$

as well as

$$g_y(t)=\left\{\begin{array}{ll} d''(y_4)/6+d'(y_4)/[t-y_4] & \text{if}\quad y_4\neq\infty,\\ d''(0)/6+d'''(0)t/6 & \text{if}\quad y_4=\infty,\end{array}\right.$$

and finally we introduce $\wp(\omega;\omega_1,\omega_2)$, the Weierstrass elliptic function with periods ω_1,ω_2. Throughout, we shall write $\wp(\omega)$ for $\wp(\omega;\omega_1,\omega_2)$. By definition, see [13, 24], we have

$$\wp(\omega)=\frac{1}{\omega^2}+\sum_{(\ell_1,\ell_2)\in\mathbf{Z}^2\setminus\{(0,0)\}}\left[\frac{1}{(\omega-\ell_1\omega_1-\ell_2\omega_2)^2}-\frac{1}{(\ell_1\omega_1+\ell_2\omega_2)^2}\right].$$

Then we have the uniformization [8, Lemma 3.3.1]

$$\left\{\begin{array}{l} x(\lambda\omega)=g_x^{-1}(\wp(\omega)),\\ y(\lambda\omega)=g_y^{-1}(\wp(\omega-\omega_3/2)).\end{array}\right. \tag{3.3}$$

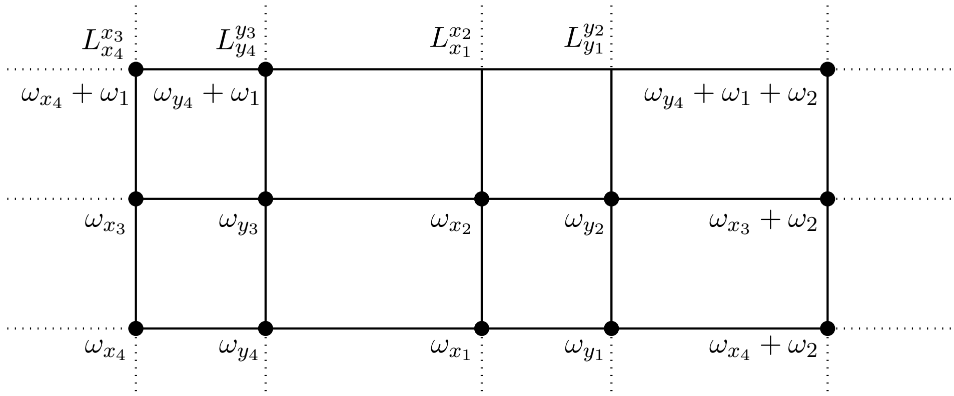


FIGURE 7. Important points and cycles on the universal covering

From now on, whenever no ambiguity can arise, we drop the dependence w.r.t. λ, writing $x(\omega)$ and $y(\omega)$ instead of $x(\lambda\omega)$ and $y(\lambda\omega)$, respectively. The coordinates $x(\omega), y(\omega)$ defined in (3.3) are elliptic:

$$x(\omega+\omega_\ell)=x(\omega), \qquad y(\omega+\omega_\ell)=y(\omega), \qquad \forall \ell\in\{1,2\}, \qquad \forall\omega\in\mathbf{C}. \tag{3.4}$$

Furthermore,

$$\begin{cases} x(0)=x_4 \\ y(0)=Y(x_4) \end{cases}, \quad \begin{cases} x(\omega_1/2)=x_3 \\ y(\omega_1/2)=Y(x_3) \end{cases}, \quad \begin{cases} x(\omega_2/2)=x_1 \\ y(\omega_2/2)=Y(x_1) \end{cases}, \quad \begin{cases} x([\omega_1+\omega_2]/2)=x_2 \\ y([\omega_1+\omega_2]/2)=Y(x_2) \end{cases}.$$

Let us denote the points $0, \omega_1/2, \omega_2/2, [\omega_1+\omega_2]/2$ by $\omega_{x_4}, \omega_{x_3}, \omega_{x_1}, \omega_{x_2}$, respectively, see Figures 5 and 7. Let

$$L_{x_4}^{x_3}=\omega_{x_4}+\omega_1\mathbf{R}, \qquad L_{x_1}^{x_2}=\omega_{x_1}+\omega_1\mathbf{R}.$$

Then $\lambda L_{x_4}^{x_3}$ (resp. $\lambda L_{x_1}^{x_2}$) is the cut of $\mathbf{T}$ where $\mathbf{S}_x^1$ and $\mathbf{S}_x^2$ are glued together, namely, $\{s\in\mathbf{T} : x(s)\in[x_3,x_4]\}$ (resp. $\{s\in\mathbf{T} : x(s)\in[x_1,x_2]\}$).

Moreover, by construction we have (see again Figures 5 and 7)

$$\begin{cases} x(\omega_3/2)=X(y_4) \\ y(\omega_3/2)=y_4 \end{cases}, \quad \begin{cases} x([\omega_1+\omega_3]/2)=X(y_3) \\ y([\omega_1+\omega_3]/2)=y_3 \end{cases}, \quad \begin{cases} x([\omega_2+\omega_3]/2)=X(y_1) \\ y([\omega_2+\omega_3]/2)=y_1 \end{cases},$$

and

$$\begin{cases} x([\omega_1+\omega_2+\omega_3]/2)=X(y_2) \\ y([\omega_1+\omega_2+\omega_3]/2)=y_2 \end{cases}.$$

We denote the points $\omega_3/2, [\omega_1+\omega_3]/2, [\omega_2+\omega_3]/2, [\omega_1+\omega_2+\omega_3]/2$ by $\omega_{y_4}, \omega_{y_3}, \omega_{y_1}, \omega_{y_2}$, respectively. Let

$$L_{y_4}^{y_3}=\omega_{y_4}+\omega_1\mathbf{R}, \qquad L_{y_1}^{y_2}=\omega_{y_1}+\omega_1\mathbf{R}.$$

Then $\lambda L_{y_4}^{y_3}$ (resp. $\lambda L_{y_1}^{y_2}$) is the cut of $\mathbf{T}$ where $\mathbf{S}_y^1$ and $\mathbf{S}_y^2$ are glued together, that is to say $\{s\in\mathbf{T} : y(s)\in[y_3,y_4]\}$ (resp. $\{s\in\mathbf{T} : y(s)\in[y_1,y_2]\}$).

The distance between $L_{x_4}^{x_3}$ and $L_{y_4}^{y_3}$ is the same as between $L_{x_1}^{x_2}$ and $L_{y_1}^{y_2}$; it equals $\omega_3/2$.

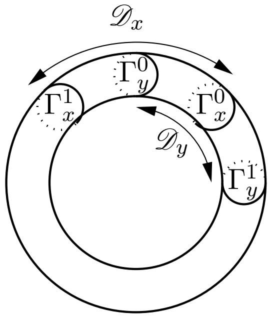


FIGURE 8. Location of the domains $\mathscr{D}_x$ and $\mathscr{D}_y$ on the Riemann surface $\mathbf{T}$

3.3. **Lifted Galois automorphisms $\widehat{\xi}, \widehat{\eta}$.** Any conformal automorphism ζ of the surface $\mathbf{T}$ can be continued as a conformal automorphism $\widehat{\zeta} = \lambda^{-1}\zeta\lambda$ of the universal covering $\mathbf{C}$. This continuation is not unique, but it will be unique if we fix some $\widehat{\zeta}\omega_0 \in \lambda^{-1}\zeta\lambda\omega_0$ for a given point $\omega_0 \in \mathbf{C}$.

According to [8], we define $\widehat{\xi}, \widehat{\eta}$ by choosing their fixed points to be $\omega_{x_2}, \omega_{y_2}$, respectively. Since any conformal automorphism of $\mathbf{C}$ is an affine function of ω [13] and since $\widehat{\xi}^{\,2} = \widehat{\eta}^{\,2} = \mathrm{Id}$, we have

$$\widehat{\xi}\omega = -\omega + 2\omega_{x_2}, \qquad \widehat{\eta}\omega = -\omega + 2\omega_{y_2}. \tag{3.5}$$

It follows that $\widehat{\eta}\widehat{\xi}$ and $\widehat{\xi}\widehat{\eta}$ are just the shifts via the real numbers ω_3 and $-\omega_3$, respectively:

$$\widehat{\eta}\widehat{\xi}\omega = \omega + 2(\omega_{y_2} - \omega_{x_2}) = \omega + \omega_3, \qquad \widehat{\xi}\widehat{\eta}\omega = \omega + 2(\omega_{x_2} - \omega_{y_2}) = \omega - \omega_3. \tag{3.6}$$

By (2.6) and (2.7) we have

$$x(\widehat{\xi}\omega) = x(\omega), \quad y(\widehat{\xi}\omega) = \frac{c(x(\omega))}{a(x(\omega))}\frac{1}{y(\omega)}, \quad x(\widehat{\eta}\omega) = \frac{\widetilde{c}(y(\omega))}{\widetilde{a}(y(\omega))}\frac{1}{x(\omega)}, \quad y(\widehat{\eta}\omega) = y(\omega). \tag{3.7}$$

Finally, $\widehat{\xi}L_{x_1}^{x_2} = L_{x_1}^{x_2}$, $\widehat{\xi}L_{x_4}^{x_3} = L_{x_4}^{x_3} + \omega_2$ and $\widehat{\eta}L_{y_1}^{y_2} = L_{y_1}^{y_2}$, $\widehat{\eta}L_{y_4}^{y_3} = L_{y_4}^{y_3} + \omega_2$.

4. LIFTING OF $x \mapsto Q(x,0)$ AND $y \mapsto Q(0,y)$ TO THE UNIVERSAL COVERING

4.1. **Lifting to the Riemann surface $\mathbf{T}$.** We have seen in Section 2 that for any $z \in]0, 1/|\mathcal{S}|[$, exactly two branch points of $Y(x)$ (namely, x_1 and x_2) are in the unit disc. For this reason, and by construction of the surface $\mathbf{T}$, the set $\{s \in \mathbf{T} : |x(s)| = 1\}$ is composed of two cycles (one belongs to $\mathbf{S}_x^1$ and the other to $\mathbf{S}_x^2$) homological to the cut $\{s \in \mathbf{T} : x(s) \in [x_1, x_2]\}$. The domain $\mathscr{D}_x = \{s \in \mathbf{T} : |x(s)| < 1\}$ is bounded by these two cycles, see Figure 8, and contains the points $s \in \mathbf{T}$ such that $x(s) \in [x_1, x_2]$. Since the function $x \mapsto K(x,0)Q(x,0)$ is holomorphic in the unit disc, we can lift it to $\mathscr{D}_x \subset \mathbf{T}$ as

$$r_x(s) = K(x(s),0)Q(x(s),0), \qquad \forall s \in \mathscr{D}_x.$$

In the same way, the domain $\mathscr{D}_y = \{s \in \mathbf{T} : |y(s)| < 1\}$ is bounded by $\{s \in \mathbf{T} : |y(s)| = 1\}$, which consists in two cycles homological to the cut $\{s \in \mathbf{T} : y(s) \in [y_1, y_2]\}$, see Figure 8, and which contains the latter. We lift the function $y \mapsto K(0,y)Q(0,y)$ to $\mathscr{D}_y \subset \mathbf{T}$ as

$$r_y(s) = K(0,y(s))Q(0,y(s)), \qquad \forall s \in \mathscr{D}_y.$$

It is shown in [21, Lemma 3] that for any $z \in]0, 1/|\mathcal{S}|[$ and any x such that $|x| = 1$, we have $|Y_0(x)| < 1$ and $|Y_1(x)| > 1$. Hence, the cycles that constitute the boundary of $\mathscr{D}_x$ are

$$\Gamma_x^0 = \{s \in \mathbf{T} : |x(s)| = 1,\ |y(s)| < 1\}, \qquad \Gamma_x^1 = \{s \in \mathbf{T} : |x(s)| = 1,\ |y(s)| > 1\}.$$

We thus have $\Gamma_x^0 \in \mathscr{D}_y$ and $\Gamma_x^1 \notin \mathscr{D}_y$, see Figure 8. In the same way, for any $z \in]0, 1/|\mathcal{S}|[$ and any y such that $|y| = 1$, we have $|X_0(y)| < 1$ and $|X_1(y)| > 1$. Therefore, the cycles composing the boundary of $\mathscr{D}_y$ are

$$\Gamma_y^0 = \{s \in \mathbf{T} : |y(s)| = 1,\ |x(s)| < 1\}, \qquad \Gamma_y^1 = \{s \in \mathbf{T} : |y(s)| = 1,\ |x(s)| > 1\}.$$

Furthermore, $\Gamma_y^0 \in \mathscr{D}_x$ and $\Gamma_y^1 \notin \mathscr{D}_x$, see Figure 8.

It follows that $\mathscr{D}_x \cap \mathscr{D}_y = \{s \in \mathbf{T} : |x(s)| < 1,\ |y(s)| < 1\}$ is not empty, simply connected and bounded by Γ_x^0 and Γ_y^0. Since for any $s \in \mathbf{T}$, $K(x(s), y(s)) = 0$, and since the main equation (1.1) is valid on $\{(x, y) \in \mathbf{C}^2 : |x| < 1,\ |y| < 1\}$, we have

$$r_x(s) + r_y(s) - K(0,0)Q(0,0) - x(s)y(s) = 0, \qquad \forall s \in \mathscr{D}_x \cap \mathscr{D}_y. \tag{4.1}$$

4.2. **Lifting to the universal covering C.** The domain $\mathscr{D}$ lifted on the universal covering consists of infinitely many curvilinear strips shifted by ω_2:

$$\lambda^{-1}\mathscr{D}_x = \bigcup_{n\in\mathbf{Z}} \Delta_x^n, \qquad \Delta_x^n \subset \omega_1\mathbf{R} +]n\omega_2, (n+1)\omega_2[,$$

and, likewise,

$$\lambda^{-1}\mathscr{D}_y = \bigcup_{n\in\mathbf{Z}} \Delta_y^n, \qquad \Delta_y^n \subset \omega_1\mathbf{R} + \omega_3/2 +]n\omega_2, (n+1)\omega_2[.$$

Let us consider these strips for $n = 0$, that we rename

$$\Delta_x = \Delta_x^0, \qquad \Delta_y = \Delta_y^0.$$

The first is bounded by $\widehat{\Gamma}_x^1 \subset \lambda^{-1}\Gamma_x^1$ and by $\widehat{\Gamma}_x^0 \subset \lambda^{-1}\Gamma_x^0$, while the second is delimited by $\widehat{\Gamma}_y^0 \subset \lambda^{-1}\Gamma_y^0$ and by $\widehat{\Gamma}_y^1 \subset \lambda^{-1}\Gamma_y^1$.

Further, note that the straight line $L_{x_1}^{x_2}$ (resp. $L_{y_1}^{y_2}$) defined in Section 3 is invariant w.r.t. $\widehat{\xi}$ (resp. $\widehat{\eta}$) and belongs to Δ_x (resp. Δ_y).

Then, by the facts that $\xi\Gamma_x^1 = \Gamma_x^0$ and $\eta\Gamma_y^1 = \Gamma_y^0$, and by our choice (3.5) of the definition of $\widehat{\xi}$ and $\widehat{\eta}$ on the universal covering, we have $\widehat{\xi}\,\widehat{\Gamma}_x^1 = \widehat{\Gamma}_x^0$ and $\widehat{\eta}\,\widehat{\Gamma}_y^1 = \widehat{\Gamma}_y^0$. In addition,

$$\widehat{\xi}\omega \in \Delta_x, \quad \forall \omega \in \Delta_x, \qquad \widehat{\eta}\omega \in \Delta_y, \quad \forall \omega \in \Delta_y. \tag{4.2}$$

Moreover, since $\Gamma_y^0 \in \mathscr{D}_x$, $\Gamma_y^1 \notin \mathscr{D}_x$ and $\Gamma_x^0 \in \mathscr{D}_y$, $\Gamma_x^1 \notin \mathscr{D}_y$, we have $\widehat{\Gamma}_y^0 \in \Delta_x$, $\widehat{\Gamma}_y^1 \notin \Delta_x$ and $\widehat{\Gamma}_x^0 \in \Delta_y$, $\widehat{\Gamma}_x^1 \notin \Delta_y$. It follows that $\Delta_x \cap \Delta_y$ is a non-empty strip bounded by $\widehat{\Gamma}_x^0$ and $\widehat{\Gamma}_y^0$, and that

$$\Delta = \Delta_x \cup \Delta_y$$

is simply connected, as in Figure 9.

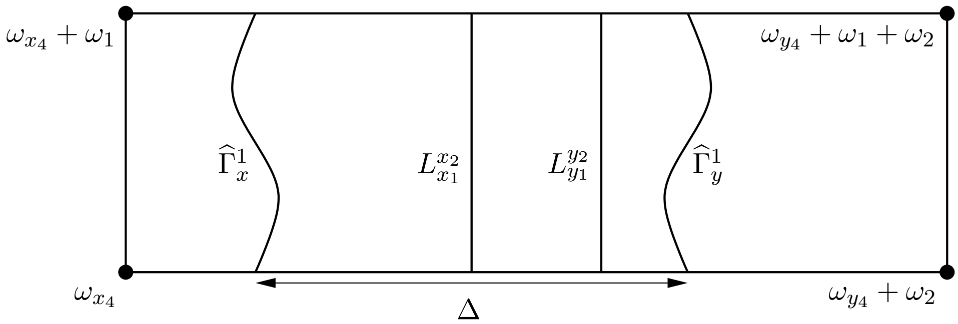


FIGURE 9. Location of $\Delta=\Delta_x\cup\Delta_y$

Let us lift the functions $r_x(s)$ and $r_y(s)$ holomorphically to Δ_x and Δ_y, respectively: we put

$$\begin{cases} r_x(\omega)=r_x(\lambda\omega)=K(x(\omega),0)Q(x(\omega),0), & \forall\omega\in\Delta_x,\\ r_y(\omega)=r_y(\lambda\omega)=K(0,y(\omega))Q(0,y(\omega)), & \forall\omega\in\Delta_y.\end{cases}\tag{4.3}$$

It follows from (4.1) and (4.3) that

$$r_x(\omega)+r_y(\omega)-K(0,0)Q(0,0)-x(\omega)y(\omega)=0,\qquad \forall\omega\in\Delta_x\cap\Delta_y.\tag{4.4}$$

Equation (4.4) allows us to continue functions $r_x(\omega)$ and $r_y(\omega)$ meromorphically on Δ: we put

$$\begin{cases} r_x(\omega)=-r_y(\omega)+K(0,0)Q(0,0)+x(\omega)y(\omega), & \forall\omega\in\Delta_y,\\ r_y(\omega)=-r_x(\omega)+K(0,0)Q(0,0)+x(\omega)y(\omega), & \forall\omega\in\Delta_x.\end{cases}\tag{4.5}$$

Equation (4.4) is then valid on the whole of Δ. We summarize all facts above in the next result.

Theorem 3. *The functions*

$$r_x(\omega)=\begin{cases} K(x(\omega),0)Q(x(\omega),0) & \text{if}\quad \omega\in\Delta_x,\\ -K(0,y(\omega))Q(0,y(\omega))+K(0,0)Q(0,0)+x(\omega)y(\omega) & \text{if}\quad \omega\in\Delta_y,\end{cases}$$

and

$$r_y(\omega)=\begin{cases} K(0,y(\omega))Q(0,y(\omega)) & \text{if}\quad \omega\in\Delta_y,\\ -K(x(\omega),0)Q(x(\omega),0)+K(0,0)Q(0,0)+x(\omega)y(\omega) & \text{if}\quad \omega\in\Delta_x,\end{cases}$$

are meromorphic in Δ. Furthermore,

$$r_x(\omega)+r_y(\omega)-K(0,0)Q(0,0)-x(\omega)y(\omega)=0,\qquad \forall\omega\in\Delta.\tag{4.6}$$

5. MEROMORPHIC CONTINUATION OF $x\mapsto Q(x,0)$ AND $y\mapsto Q(0,y)$ ON THE UNIVERSAL COVERING

5.1. **Meromorphic continuation.** In Theorem 3 we saw that $r_x(\omega)$ and $r_y(\omega)$ are meromorphic on Δ. We now continue these functions meromorphically from Δ to the whole of $\mathbf{C}$.

Theorem 4. *The functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically to the whole of* $\mathbf{C}$. *Further, for any $\omega \in \mathbf{C}$, we have*

$$r_x(\omega-\omega_3) = r_x(\omega) + y(\omega)[x(-\omega+2\omega_{y_2}) - x(\omega)], \tag{5.1}$$

$$r_y(\omega+\omega_3) = r_y(\omega) + x(\omega)[y(-\omega+2\omega_{x_2}) - y(\omega)], \tag{5.2}$$

$$r_x(\omega) + r_y(\omega) - K(0,0)Q(0,0) - x(\omega)y(\omega) = 0, \tag{5.3}$$

$$\begin{cases} r_x(\widehat{\xi}\omega) = r_x(\omega), \\ r_y(\widehat{\eta}\omega) = r_y(\omega), \end{cases} \tag{5.4}$$

$$\begin{cases} r_x(\omega+\omega_1) = r_x(\omega), \\ r_y(\omega+\omega_1) = r_y(\omega). \end{cases} \tag{5.5}$$

For the proof of Theorem 4, we shall need the following lemma.

Lemma 5. *We have*

$$\bigcup_{n\in\mathbf{Z}} (\Delta + n\omega_3) = \mathbf{C}. \tag{5.6}$$

Proof. It has been noticed in Section 4 that $\widehat{\xi}\,\widehat{\Gamma}_x^1 = \widehat{\Gamma}_x^0 \in \Delta_y$. By (4.2), $\widehat{\eta}\,\widehat{\Gamma}_x^0 \in \Delta_y \subset \Delta$, so that, by (3.6),

$$\widehat{\Gamma}_x^1 + \omega_3 = \widehat{\eta}\widehat{\xi}\,\widehat{\Gamma}_x^1 \in \Delta.$$

In the same way, $\widehat{\Gamma}_y^1 - \omega_3 \in \Delta$. It follows that $\Delta \cup (\Delta + \omega_3)$ is a simply connected domain, see Figure 9. Identity (5.6) follows. □

Proof of Theorem 4. For any $\omega \in \Delta$, by Theorem 3 we have

$$r_x(\omega) + r_y(\omega) - K(0,0)Q(0,0) - x(\omega)y(\omega) = 0. \tag{5.7}$$

For any $\omega \in \Delta$ close enough to the cycle $\widehat{\Gamma}_x^1$, we have that $\widehat{\xi}\omega \in \Delta_y$ since $\widehat{\xi}\,\widehat{\Gamma}_x^1 = \widehat{\Gamma}_x^0 \in \Delta_y$. Then $\omega + \omega_3 = \widehat{\eta}\widehat{\xi}\omega \in \Delta_y$ by (4.2). We now compute $r_y(\widehat{\eta}\widehat{\xi}\omega)$ for any such ω. Equation (4.6), which is valid in $\Delta \supset \Delta_y$, gives

$$r_x(\widehat{\xi}\omega) + r_y(\widehat{\xi}\omega) - K(0,0)Q(0,0) - x(\widehat{\xi}\omega)y(\widehat{\xi}\omega) = 0. \tag{5.8}$$

By (3.7), $x(\widehat{\xi}\omega) = x(\omega)$. For our $\omega \in \Delta_x$, by (4.2) we have $\widehat{\xi}\omega \in \Delta_x$, so that Theorem 3 yields

$$r_x(\widehat{\xi}\omega) = K(x(\widehat{\xi}\omega),0)Q(x(\widehat{\xi}\omega),0) = K(x(\omega),0)Q(x(\omega),0) = r_x(\omega).$$

If we now combine the last fact together with Equation (5.7), Equation (5.8) and identity $x(\widehat{\xi}\omega) = x(\omega)$, we obtain that

$$r_y(\widehat{\xi}\omega) = r_y(\omega) + x(\omega)[y(\widehat{\xi}\omega) - y(\omega)].$$

Since $\widehat{\xi}\omega \in \Delta_y$, then by (4.2) we have $\widehat{\eta}\widehat{\xi}\omega \in \Delta_y$. Equation (3.7) and Theorem 3 entail

$$r_y(\widehat{\eta}\widehat{\xi}\omega) = K(0,y(\widehat{\eta}\widehat{\xi}\omega))Q(0,y(\widehat{\eta}\widehat{\xi}\omega)) = K(0,y(\widehat{\xi}\omega))Q(0,y(\widehat{\xi}\omega)) = r_y(\widehat{\xi}\omega).$$

Finally, for all $\omega \in \Delta$ close enough to $\widehat{\Gamma}_x^1$ we have

$$r_y(\widehat{\eta}\widehat{\xi}\omega) = r_y(\omega) + x(\omega)[y(\widehat{\xi}\omega) - y(\omega)].$$

Using (3.6), we obtain exactly Equation (5.2). Thanks to Theorem 3 and Lemma 5, this equation shown for any $\omega \in \Delta$ close enough to $\widehat{\Gamma}_x^1$ allows us to continue r_y meromorphically from Δ to the whole of $\mathbf{C}$. Equation (5.2) therefore stays valid for any $\omega \in \mathbf{C}$. The function $r_y(\widehat{\eta}\omega) = r_y(-\omega + \omega_{y_2})$ is then also meromorphic on $\mathbf{C}$. Since these functions coincide in Δ_y, then by the principle of analytic continuation [13] they do on the whole of $\mathbf{C}$. In the same way, we prove Equation (5.1) for all $\omega \in \Delta_y$ close enough to $\widehat{\Gamma}_y^1$. Together with Theorem 3 and Lemma 5 this allows us to continue $r_x(\omega)$ meromorphically to the whole of $\mathbf{C}$. By the same continuation argument, the identity $r_x(\omega) = r_x(\widehat{\xi}\omega)$ is valid everywhere on $\mathbf{C}$. Consequently Equation (5.3), which a priori is satisfied in Δ, must stay valid on the whole of $\mathbf{C}$. Since $x(\omega)$ and $y(\omega)$ are ω_1-periodic, it follows from Theorem 3 that $r_x(\omega)$ and $r_y(\omega)$ are ω_1-periodic in Δ. The vector ω_3 being real, by (5.1) and (5.2) these functions stay ω_1-periodic on the whole of $\mathbf{C}$. □

5.2. **Branches of $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$.** The restrictions of $r_x(\omega)/K(x(\omega),0)$ on

$$\mathscr{M}_{k,\ell} = \omega_1[\ell, \ell+1[+\omega_2[k/2,(k+1)/2[\tag{5.9}$$

for $k,\ell \in \mathbf{Z}$ provide all branches on $\mathbf{C} \setminus ([x_1,x_2] \cup [x_3,x_4])$ of $Q(x,0)$ as follows:

$$Q(x,0) = \{r_x(\omega)/K(x(\omega),0) : \omega \text{ is the (unique) element of } \mathscr{M}_{k,\ell} \text{ such that } x(\omega) = x\}. \tag{5.10}$$

Due to the ω_1-periodicity of $r_x(\omega)$ and $x(\omega)$, the restrictions of these functions on $\mathscr{M}_{k,\ell}$ do not depend on $\ell \in \mathbf{Z}$, and therefore determine the same branch as on $\mathscr{M}_{k,0}$ for any ℓ. Furthermore, thanks to (5.4), (3.5) and (3.7) the restrictions of $r_x(\omega)/K(x(\omega),0)$ on $\mathscr{M}_{-k+1,0}$ and on $\mathscr{M}_{k,0}$ lead to the same branches for any $k \in \mathbf{Z}$. Hence, the restrictions of $r_x(\omega)/K(x(\omega),0)$ to $\mathscr{M}_{k,0}$ with $k \geqslant 1$ provide all different branches of this function. The analogous statement holds for the restrictions of $r_y(\omega)/K(0,y(\omega))$ on

$$\mathscr{N}_{k,\ell} = \omega_3/2 + \omega_1[\ell,\ell+1[+\omega_2]k/2,(k+1)/2] \tag{5.11}$$

for $k,\ell \in \mathbf{Z}$, namely:

$$Q(0,y) = \{r_y(\omega)/K(0,y(\omega)) : \omega \text{ is the (unique) element of } \mathscr{N}_{k,\ell} \text{ such that } y(\omega) = y\}. \tag{5.12}$$

The restrictions on $\mathscr{N}_{k,\ell}$ for $\ell \in \mathbf{Z}$ give the same branch as on $\mathscr{N}_{k,0}$. For any $k \in \mathbf{Z}_+$ the restrictions on $\mathscr{N}_{-k+1,0}$ and on $\mathscr{N}_{k,0}$ determine the same branches. Hence, the restrictions of $r_y(\omega)/K(0,y(\omega))$ on $\mathscr{N}_{k,0}$ with $k \geqslant 1$ provide all different branches of $y \mapsto Q(0,y)$.

5.3. **Ratio ω_2/ω_3.**

Remark 6. *For any $z \in]0,1/|\mathcal{S}|[$ the value ω_2/ω_3 is rational if and only if the group $\langle \xi, \eta \rangle$ restricted to the curve $\{(x,y) \in \mathbf{C} \cup \{\infty\}^2 : K(x,y) = 0\}$ is finite, see* [8, Section 4.1.2] *and* [21, Proof of Proposition 4].

The rationality or irrationality of the quantity ω_2/ω_3 is crucial for the nature of the functions $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ for a given z. Indeed, the following theorem holds true.

Theorem 7. *For any $z \in]0,1/|\mathcal{S}|[$ such that ω_2/ω_3 is rational, the functions $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ are holonomic.*

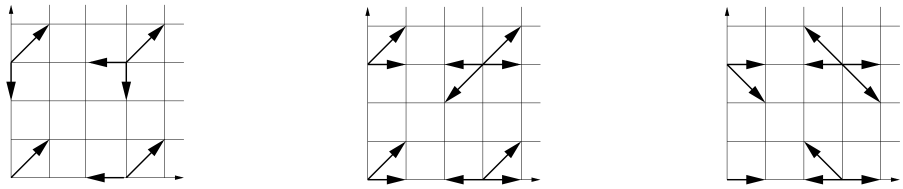


FIGURE 10. Three famous examples, known as Kreweras', Gessel's and Gouyou-Beauchamps' walks, respectively

Proof. The proof of Theorem 7 is completely similar to that of Theorems 1.1 and 1.2 in [9], so here we just recall the main ideas. The proof actually consists in applying [8, Theorem 4.4.1], which entails that if ω_2/ω_3 is rational, the function $Q(x,0)$ can be written as

$$Q(x,0) = w_1(x) + \widetilde{\Phi}(x)\phi(x) + w(x)/r(x),$$

where w_1 and r are rational functions, while ϕ and w are algebraic. Further, in [9, Lemma 2.1] it is shown that $\widetilde{\Phi}$ is holonomic. Accordingly, $Q(x,0)$ is also holonomic. The argument for $Q(0,y)$ is similar. Notice that Theorem 4.4.1 in [8] is proved for $z = 1/|\mathcal{S}|$ only, but in [9] it is observed that this result also holds for $z \in]0, 1/|\mathcal{S}|[$. $\square$

For all 23 models of walks with finite group (1.3), the ratio ω_2/ω_3 is rational and *independent* of z. This fact, which is specified in Lemma 8 below, implies the holonomy of the functions $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ for all $z \in]0, 1/|\mathcal{S}|[$ by Theorem 7, and also leads to some more profound analysis of the models with a finite group. This analysis is the topic of the Section 6.

For all 51 non-singular models of walks with infinite group, ω_2/ω_3 takes rational and irrational values on subsets $\mathcal{H}$ and $]0, 1/|\mathcal{S}|[\setminus\mathcal{H}$, respectively, which are dense on $]0, 1/|\mathcal{S}|[$, as it will be proved in Proposition 14 below. For any $z \in \mathcal{H}$, $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ are holonomic by Theorem 7. For all $z \in]0, 1/|\mathcal{S}|[\setminus\mathcal{H}$, properties of the branches of $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ (in particular, the set of their poles) will be studied in detail in Section 7; the non-holonomy will be derived from this analysis.

6. FINITE GROUP CASE

Define the *covariance* of the model as

$$\textstyle\sum_{(i,j)\in\mathcal{S}} ij - [\sum_{(i,j)\in\mathcal{S}} i][\sum_{(i,j)\in\mathcal{S}} j] = \sum_{(i,j)\in\mathcal{S}} ij. \tag{6.1}$$

The equality above follows from the fact that for each of the 23 models with a finite group, $\sum_{(i,j)\in\mathcal{S}} i = 0$ or $\sum_{(i,j)\in\mathcal{S}} j = 0$, see [5]. Lemma 8 below is proved in [21, Proposition 5].

Lemma 8. *For all 23 models with finite group (1.3), ω_2/ω_3 is rational and independent of z. More precisely:*

- *For the walks with a group of order 4, $\omega_2/\omega_3 = 2$;*
- *For the walks with a group of order 6 and such that the covariance is negative (resp. positive), $\omega_2/\omega_3 = 3$ (resp. $3/2$);*

— *For the walks with a group of order* 8 *and a negative (resp. positive) covariance,* $\omega_2/\omega_3 = 4$ *(resp.* 4/3*).*

In the sequel, we note $\omega_2/\omega_3 = k/\ell$; then, $2k$ is the order of the group. Since $k\omega_3 = \ell\omega_2$, we obviously always have

$$r_x(\omega + \ell\omega_2) - r_x(\omega) = \sum_{1\leqslant m\leqslant k} r_x(\omega + m\omega_3) - r_x(\omega + (m-1)\omega_3).$$

It follows from (5.1) and from properties (3.5), (3.6) and (3.7) of the Galois automorphisms that

$$\begin{aligned} r_x(\omega + \ell\omega_2) - r_x(\omega) &= \sum_{1\leqslant m\leqslant k} (xy)(\omega + m\omega_3) - (xy)(\widehat{\eta}(\omega + m\omega_3)) \\ &= \sum_{1\leqslant m\leqslant k} (xy)((\widehat{\eta}\widehat{\xi})^m\omega) - (xy)(\widehat{\xi}(\widehat{\eta}\widehat{\xi})^{m-1}\omega)) \\ &= \sum_{\theta\in\langle\widehat{\xi},\widehat{\eta}\rangle} (-1)^\theta xy(\theta(\omega)), \end{aligned} \tag{6.2}$$

where $(-1)^\theta$ is the signature of θ; in other words, $(-1)^\theta = (-1)^{\ell(\theta)}$, where $\ell(\theta)$ is the length of θ, i.e., the smallest ℓ such that we can write $\theta = \theta_1 \circ \cdots \circ \theta_\ell$, with $\theta_1, \ldots, \theta_\ell$ equal to $\widehat{\xi}$ or $\widehat{\eta}$. The same identity with the opposite sign holds for r_y. The quantity (6.2) is the *orbit-sum* of the function xy under the group $\langle\widehat{\xi},\widehat{\eta}\rangle$, and is denoted by $\mathscr{O}(\omega)$. It satisfies the property hereunder, which is proved in [5].

Lemma 9. *In the finite group case, the orbit-sum $\mathscr{O}(\omega)$ is identically zero if and only if the covariance* (6.1) *is positive.*

We therefore come to the following corollary.

Corollary 10. *In the finite group case, the functions $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ have a finite number of different branches if and only if the covariance* (6.1) *is positive.*

After the lifting to the universal covering done in Theorem 4, results of [3, 5] concerning the nature of the functions $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ in all finite group cases can now be established by very short reasonings. For the sake of completeness, we show how this works.

Proposition 11 ([3, 5])**.** *For all models with a finite group and a positive covariance* (6.1)*, $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ are algebraic.*

Proposition 12 ([5])**.** *For all models with a finite group and a negative or zero covariance* (6.1)*, $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ are holonomic and non-algebraic.*

Proofs of both of these propositions involve the following lemma.

Lemma 13. *Let $\wp$ be a Weierstrass elliptic function with certain periods $\overline{\omega}, \widehat{\omega}$.*

(P1) *We have*

$$\wp'(\omega)^2 = 4[\wp(\omega) - \wp(\overline{\omega}/2)][\wp(\omega) - \wp([\overline{\omega} + \widehat{\omega}]/2)][\wp(\omega) - \wp(\widehat{\omega}/2)], \qquad \forall \omega \in \mathbf{C}.$$

(P2) *Let p be some positive integer. The Weierstrass elliptic function with periods $\overline{\omega},\widehat{\omega}/p$ can be written in terms of $\wp$ as*

$$\wp(\omega)+\sum_{\ell=1}^{p-1}[\wp(\omega+\ell\widehat{\omega}/p)-\wp(\ell\widehat{\omega}/p)],\qquad \forall\omega\in\mathbf{C}.$$

(P3) *We have the addition theorem:*

$$\wp(\omega+\widetilde{\omega})=-\wp(\omega)-\wp(\widetilde{\omega})+\frac{1}{4}\left[\frac{\wp'(\omega)-\wp'(\widetilde{\omega})}{\wp(\omega)-\wp(\widetilde{\omega})}\right]^2,\qquad \forall\omega,\widetilde{\omega}\in\mathbf{C}.$$

(P4) *For any elliptic function f with periods $\overline{\omega},\widehat{\omega}$, there exist two rational functions R and S such that*

$$f(\omega)=R(\wp(\omega))+\wp'(\omega)S(\wp(\omega)),\qquad \forall\omega\in\mathbf{C}.$$

(P5) *There exists a function Φ which is $\overline{\omega}$-periodic and such that $\Phi(\omega+\widehat{\omega})=\Phi(\omega)-1$, $\forall\omega\in\mathbf{C}$.*

Proof. Properties (P1), (P3) and (P4) are most classical, and can be found, e.g., in [13, 24]. For (P2) we refer to [24, page 456], and for (P5), see [8, Equation (4.3.7)]. Note that the function Φ in (P5) can be constructed via the zeta function of Weierstrass. □

Proof of Proposition 11. If the orbit-sum $\mathscr{O}(\omega)$ is zero, Equation (6.2) implies that $r_x(\omega)$ is $\ell\omega_2$-periodic. In particular, the property (P4) of Lemma 13 entails that there exist two rational functions R and S such that

$$r_x(\omega)=R(\wp(\omega;\omega_1,\ell\omega_2))+\wp'(\omega;\omega_1,\ell\omega_2)S(\wp(\omega;\omega_1,\ell\omega_2)). \tag{6.3}$$

Further, the property (P2) together with the addition formula (P3) of Lemma 13 gives that $\wp(\omega;\omega_1,\ell\omega_2)$ is an algebraic function of $\wp(\omega)$—we recall that $\wp(\omega)$ denotes the Weierstrass function $\wp(\omega;\omega_1,\omega_2)$. Due to Lemma 13 (P1), $\wp'(\omega)$ is an algebraic function of $\wp(\omega)$ too, so that $\wp'(\omega;\omega_1,\ell\omega_2)$ is also an algebraic function of $\wp(\omega)$. Thanks to (6.3), we get that $r_x(\omega)$ is algebraic in $\wp(\omega)$. Since $\wp(\omega)$ is a rational function of $x(\omega)$, see (3.3), we finally obtain that $r_x(\omega)$ is algebraic in $x(\omega)$. Then $q_x(\omega)=r_x(\omega)/K(x(\omega),0)$ is algebraic in $x(\omega)$, and so is $q_y(\omega)$ in $y(\omega)$. □

Proof of Proposition 12. In this proof we have $\ell=1$, see Lemma 8. Thanks to Lemma 13 (P5), there exists a function Φ which is ω_1-periodic and such that $\Phi(\omega+\omega_2)=\Phi(\omega)-1$. In particular, transforming (6.2) we can write

$$r_x(\omega+\omega_2)+\Phi(\omega+\omega_2)\mathscr{O}(\omega+\omega_2)=r_x(\omega)+\Phi(\omega)\mathscr{O}(\omega).$$

This entails that $r_x(\omega)+\Phi(\omega)\mathscr{O}(\omega)$ is elliptic with periods ω_1,ω_2. In particular, for the same reasons as in the proof of Proposition 11, this is an algebraic function of $x(\omega)$. The function $\mathscr{O}(\omega)$ is obviously also algebraic in $x(\omega)$. As for the function $\Phi(\omega)$, it is proved in [8, page 71] that it is a non-algebraic function of $x(\omega)$. Moreover, it is shown in [9, Lemma 2.1] that it is holonomic in $x(\omega)$. Hence $r_x(\omega)$ is holonomic in $x(\omega)$ but not algebraic. The same is true for $q_x(\omega)=r_x(\omega)/K(x(\omega),0)$ in $x(\omega)$ and for $q_y(\omega)$ in $y(\omega)$. □

7. Infinite group case

It has been shown in [21, Part 6.2] that for all 51 models with infinite group, ω_2/ω_3 takes irrational values for infinitely many z. The next proposition states a more complete result.

Proposition 14. *For all* 51 *walks with infinite group, the sets* $\mathcal{H} = \{z \in]0,1/|\mathcal{S}|[: \omega_2/\omega_3$ *is rational*$\}$ *and* $]0,1/|\mathcal{S}|[\setminus\mathcal{H} = \{z \in]0,1/|\mathcal{S}|[: \omega_2/\omega_3$ *is irrational*$\}$ *are dense in* $]0,1/|\mathcal{S}|[$.

Proof. The function ω_2/ω_3 is clearly real continuous function on $]0,1/|\mathcal{S}|[$. In fact, it has been noticed in [21, Part 6.2] that the function ω_2/ω_3 is expandable in power series in a neighborhood of any point of the interval $]0,1/|\mathcal{S}|[$. Thus it suffices to find just one segment within $]0,1/|\mathcal{S}|[$ where this function is not constant.

Proposition 25 below gives the asymptotic of ω_2/ω_3 as $z \to 0$: for any of 51 models there exist some rational $L > 0$ and some $\widetilde{L} \neq 0$ such that as $z > 0$ goes to 0,

$$\omega_2/\omega_3 = L + \widetilde{L}/\ln z + O((1/\ln z)^2).$$

This immediately implies that this function is not constant on a small enough interval in a right neighborhood of 0 and concludes the proof.

Note however that there is another way to conclude the proof that does not need the full power of Proposition 25: it is enough to show (as done in the proof of Proposition 25) that ω_2/ω_3 converges to a rational positive constant L as $z \to 0$ for all 51 models. Indeed, then, since ω_2/ω_3 necessarily takes irrational values for some $z \in]0,1/|\mathcal{S}|[$ (see [21, Part 6.2]), there exists an interval within $]0,1/|\mathcal{S}|[$ where the ratio ω_2/ω_3 is not constant. □

In Subsections 7.1, 7.2 and 7.3 we thoroughly analyze the branches of $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ for z such that ω_2/ω_3 is irrational, and we prove in particular that their set of poles is infinite and dense on the curves given on Figure 11, see Theorem 17. Then the following corollary is immediate.

Corollary 15. *Let* $\mathcal{H} = \{z \in]0,1/|\mathcal{S}|[: \omega_2/\omega_3$ *is rational*$\}$ *and* $]0,1/|\mathcal{S}|[\setminus\mathcal{H} = \{z \in]0,1/|\mathcal{S}|[: \omega_2/\omega_3$ *is irrational*$\}$.

(i) *For all* $z \in \mathcal{H}$, $x \mapsto Q(x,0)$ *and* $y \mapsto Q(0,y)$ *are holonomic;*
(ii) *For all* $z \in]0,1/|\mathcal{S}|[\setminus\mathcal{H}$, $x \mapsto Q(x,0)$ *and* $y \mapsto Q(0,y)$ *are non-holonomic.*

Proof. The statement (i) follows from Theorem 7, and (ii) comes from Theorem 17 (iii) below as explained in the Introduction. □

Remark 16. *It follows from Remark 6 that* $\mathcal{H}$ *can be characterized as the set of* $z \in]0,1/|\mathcal{S}|[$ *such that the group* $\langle\xi,\eta\rangle$ *restricted to the curve* $\{(x,y) \in (\mathbf{C}\cup\{\infty\})^2 : K(x,y;z) = 0\}$ *is finite. Then methods developed in* [8, Chapter 4] *specifically for the finite group case should be efficient for further analysis of* $x \mapsto Q(x,0)$ *and* $y \mapsto Q(0,y)$ *for any fixed* $z \in \mathcal{H}$.

The analysis of the poles being rather technical, we start first with an informal study.

7.1. **Poles of the set of branches of $x \mapsto Q(x,0)$ and $y \mapsto Q(y,0)$ for irrational ω_2/ω_3: an informal study.** Let us fix $z \in]0,1/|\mathcal{S}|[$ such that ω_2/ω_3 is irrational. We first informally explain why the set of poles of all branches of $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ could be dense on certain curves in this case. We shall denote by $\Re\omega$ and $\Im\omega$ the real and imaginary parts of $\omega \in \mathbf{C}$, respectively. Let Π_x and Π_y be the parallelograms defined by

$$\Pi_x = \mathscr{M}_{0,0} \cup \mathscr{M}_{0,1} = \omega_1[0,1[+\omega_2[0,1[, \qquad \Pi_y = \mathscr{N}_{0,0} \cup \mathscr{N}_{0,1} = \omega_3/2 + \omega_1[0,1[+\omega_2]0,1], \tag{7.1}$$

with notations (5.9) and (5.11). Function $r_x(\omega)$ (resp. $r_y(\omega)$) on Π_x (resp. Π_y) defines the first (main) branch of $x \mapsto Q(x,0)$ (resp. $y \mapsto Q(0,y)$) twice via (5.10) (resp. (5.12)).

Denote by $f_y(\omega) = x(\omega)[y(-\omega + 2\omega_{x_1}) - y(\omega)]$ the function used in the meromorphic continuation procedure (5.2). Assume that at some $\omega_0 \in \Pi_y$, $r_y(\omega_0) \neq \infty$ and $f_y(\omega_0) = \infty$. Further, suppose that

$$\nexists \omega \in \Pi_y : \quad \Im\omega = \Im\omega_0, \quad f_y(\omega) = \infty. \tag{7.2}$$

By (5.2), for any $n \geqslant 1$ we have

$$r_y(\omega_0 + n\omega_3) = r_y(\omega_0) + f_y(\omega_0) + \sum_{k=1}^{n-1} f_y(\omega_0 + k\omega_3). \tag{7.3}$$

We have $r_y(\omega_0) + f_y(\omega_0) = \infty$ by our assumptions. If ω_2/ω_3 is irrational, then for any $k \geqslant 1$ there is no $p \in \mathbf{Z}$ such that $\omega_0 + k\omega_3 = \omega_0 + p\omega_2$. Function f_y being ω_2-periodic, it follows from this fact and assumption (7.2) that $f_y(\omega_0 + k\omega_3) \neq \infty$ for any $k \geqslant 1$. Hence by (7.3), $r_y(\omega_0 + n\omega_3) = \infty$ for all $n \geqslant 1$. Due to irrationality of ω_2/ω_3, for any $n \geqslant 1$ there exists a unique $\omega_n(\omega_0) \in \Pi_y$ and $p \in \mathbf{Z}$ such that $\omega_0 + n\omega_3 = \omega_n(\omega_0) + p\omega_2$, and the set $\{\omega_n(\omega_0)\}_{n\geqslant 1}$ is dense on the curve

$$\mathcal{I}_y(\omega_0) = y(\{\omega : \Im\omega = \Im\omega_0,\, \omega \in \Pi_y\}) \subset \mathbf{C} \cup \{\infty\}. \tag{7.4}$$

By definition (5.12), the set of poles of all branches of $y \mapsto K(0,y)Q(0,y)$ is dense on the curve $\mathcal{I}_y(\omega_0)$. The number of zeros of $y \mapsto K(0,y)$ being at most two, the same conclusion holds true for $y \mapsto Q(0,y)$.

Let us now identify the points ω_0 in Π_y where $f_y(\omega_0)$ is infinite. They are (at most) six such points $a_1, a_2, a_3, a_4, b_1, b_2 \in \Pi_y$, which correspond to the following pairs $(x(\omega_0), y(\omega_0))$:

$$a_1 = (x^\star, \infty),\ a_4 = (x^\star, y^\star),\ a_2 = (x^\bigstar, \infty),\ a_3 = (x^\bigstar, y^\bigstar),\ b_1 = (\infty, y^\circ),\ b_2 = (\infty, y^\bullet). \tag{7.5}$$

Here by (2.4) and (2.5)

$$x^\star = \lim_{y\to\infty} \frac{-\widetilde{b}(y) + [\widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y)]^{1/2}}{2\widetilde{a}(y)}, \qquad x^\bigstar = \lim_{y\to\infty} \frac{-\widetilde{b}(y) - [\widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y)]^{1/2}}{2\widetilde{a}(y)},$$

$$y^\star = \lim_{x\to x^\star} \frac{-b(x) + [b(x)^2 - 4a(x)c(x)]^{1/2}}{2a(x)}, \qquad y^\bigstar = \lim_{x\to x^\bigstar} \frac{-b(x) + [b(x)^2 - 4a(x)c(x)]^{1/2}}{2a(x)},$$

$$y^\circ = \lim_{x\to\infty} \frac{-b(x) + [b(x)^2 - 4a(x)c(x)]^{1/2}}{2a(x)}, \qquad y^\bullet = \lim_{x\to\infty} \frac{-b(x) - [b(x)^2 - 4a(x)c(x)]^{1/2}}{2a(x)}.$$

where $a, b, c, \widetilde{a}, \widetilde{b}, \widetilde{c}$ are introduced in (2.1).

For most of 51 models of walks, assumption (7.2) holds true for none of these points, so that the previous reasoning does not work: some poles of f_y could be compensated in the sum (7.3). Furthermore, it may happen for some of these points that not only $f_y(\omega_0)=\infty$ but also $r_y(\omega_0)=\infty$, and consequently $f_y(\omega)+r_y(\omega)$ may have no pole at $\omega=\omega_0$. For these reasons we need to inspect more closely the location of these six points for each of the 51 models and their contribution to the set of poles via (7.3).

7.2. **Functions $x\mapsto Q(x,0)$ and $y\mapsto Q(0,y)$ for irrational ω_2/ω_3.** In addition to the notation (7.4), define the curve

$$\mathcal{I}_x(\omega_0)=x(\{\omega:\Im\omega=\Im\omega_0,\ \omega\in\Pi_x\})\subset\mathbf{C}\cup\{\infty\}. \tag{7.6}$$

We now formulate the main theorem of this section.

Theorem 17. *For all* 51 *non-singular walks with infinite group* (1.3) *given on Figure 17 and any z such that ω_2/ω_3 is irrational, the following statements hold.*

- (i) *The only singularities on $\mathbf{C}$ of the first branch of $x\mapsto Q(x,0)$ (resp. $y\mapsto Q(0,y)$) are the branch points x_3 and x_4 (resp. y_3 and y_4).*
- (ii) *Each branch of $x\mapsto Q(x,0)$ (resp. $y\mapsto Q(0,y)$) is meromorphic on $\mathbf{C}$ with a finite number of poles.*
- (iii) *The set of poles on $\mathbf{C}$ of all branches of $x\mapsto Q(x,0)$ (resp. $y\mapsto Q(0,y)$) is infinite. With the notations (7.6), (7.4) above and points a_1,b_1 defined in* (7.5)*, it is dense on the following curves (see Figure 11):*
  - (iii.a) *For the walks of Subcase I.A in Figure 17,: $\mathcal{I}_x(a_1)$ and $\mathcal{I}_x(b_1)$ for $x\mapsto Q(x,0)$; $\mathcal{I}_y(a_1)$ and $\mathcal{I}_y(b_1)$ for $y\mapsto Q(0,y)$.*
  - (iii.b) *For the walks of Subcases I.B and I.C in Figure 17: $\mathcal{I}_x(a_1)$ and $\mathbf{R}\backslash]x_1,x_4[$ for $x\mapsto Q(x,0)$; $\mathcal{I}_y(a_1)$ and $[y_4,y_1]$ for $y\mapsto Q(0,y)$.*
  - (iii.c) *For the walks of Subcase II.A in Figure 17: $\mathcal{I}_x(b_1)$ and $[x_4,x_1]$ for $x\mapsto Q(x,0)$; $\mathcal{I}_y(b_1)$ and $\mathbf{R}\backslash]y_1,y_4[$ for $y\mapsto Q(0,y)$.*
  - (iii.d) *For the walks of Subcases II.B, II.C, II.D and Case III in Figure 17: $\mathbf{R}\backslash]x_1,x_4[$ for $x\mapsto Q(x,0)$; $\mathbf{R}\backslash]y_1,y_4[$ for $y\mapsto Q(0,y)$.*
- (iv) *Poles of branches of $x\mapsto Q(x,0)$ and $y\mapsto Q(0,y)$ out of these curves may be only at zeros of $K(x,0)$ and $K(0,y)$, respectively.*

Before giving the proof of Theorem 17, we need to introduce some additional tools. If the value of ω_2/ω_3 is irrational, for any $\omega_0\in\mathbf{C}$ and any $n\in\mathbf{Z}_+$, there exists a unique $\omega_n^y(\omega_0)\in\Pi_y$ (resp. $\omega_n^x(\omega_0)\in\Pi_x$) as well as a unique number $p_y\in\mathbf{Z}$ (resp. $p_x\in\mathbf{Z}$) such that $\omega_0+n\omega_3=p_y\omega_2+\omega_n^y(\omega_0)$ (resp. $\omega_0+n\omega_3=p_x\omega_2+\omega_n^x(\omega_0)$). With these notations we can state the following lemma.

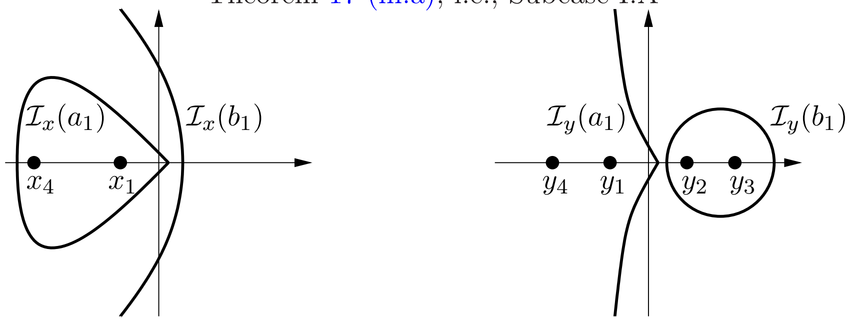
Lemma 18. *Let z be such that ω_2/ω_3 is irrational.*

- (a) *For all $n\neq m$, we have $\omega_n^x(\omega_0)\neq\omega_m^x(\omega_0)$ and $\omega_n^y(\omega_0)\neq\omega_m^y(\omega_0)$.*
- (b) *The set $\{\omega_n^x(\omega_0)\}_{n\in\mathbf{Z}_+}$ (resp. $\{\omega_n^y(\omega_0)\}_{n\in\mathbf{Z}_+}$) is dense on the segment $\{\omega\in\Pi_x:\Im\omega=\Im\omega_0\}$ (resp. $\{\omega\in\Pi_y:\Im\omega=\Im\omega_0\}$).*

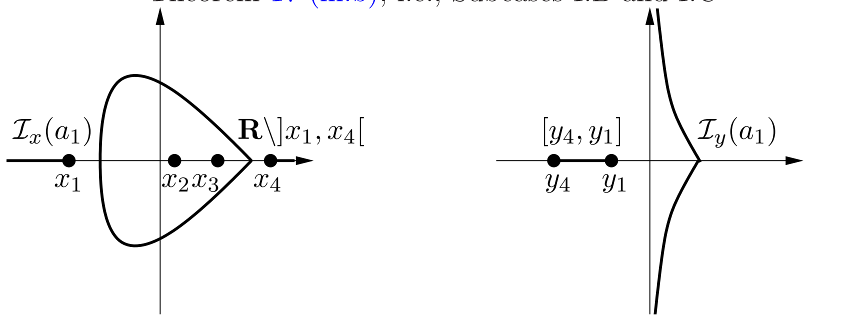
Proof. Both (a) and (b) are direct consequences of the irrationality of ω_2/ω_3. □

In the next definition, we introduce a partial order in Π_y.

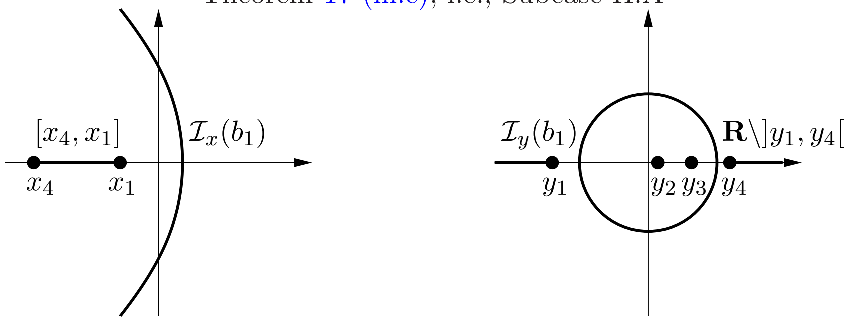
Theorem 17 (iii.a), i.e., Subcase I.A



Theorem 17 (iii.b), i.e., Subcases I.B and I.C



Theorem 17 (iii.c), i.e., Subcase II.A



Definition 19. *For any $\omega, \omega' \in \Pi_y$, we write $\omega \ll \omega'$ if for some $n \in \mathbf{Z}_+$ and some $p \in \mathbf{Z}$, $\omega + n\omega_3 = \omega' + p\omega_2$.*

If $\omega \ll \omega'$ (and if ω_2/ω_3 is irrational), both n and p are unique and sometimes we shall write $\omega \ll_n \omega'$. In particular, for any $\omega \in \Pi_y$, we have $\omega \ll \omega$, since $\omega \ll_0 \omega$.

Definition 20. *If either $\omega \ll \omega'$ or $\omega \ll \omega'$, we say that ω and ω' are ordered, and we write $\omega \sim \omega'$.*

Let us denote by f_x and f_y the (meromorphic) functions used in the meromorphic continuation procedures (5.1) and (5.2), namely, by using (3.5):

$$f_x(\omega) = y(\omega)[x(\widehat{\eta}\omega) - x(\omega)], \qquad f_y(\omega) = x(\omega)[y(\widehat{\xi}\omega) - y(\omega)].$$

The following lemma will be the key tool for the proof of Theorem 17.

Theorem 17 (iii.d), i.e., Subcases II.B, II.C, II.D and Case III

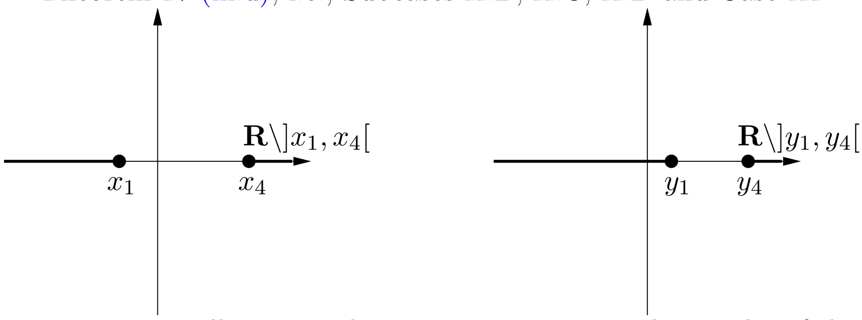


FIGURE 11. For walks pictured on Figure 17, curves where poles of the set of branches of $x \mapsto Q(x, 0)$ and $y \mapsto Q(0, y)$ are dense

Lemma 21. *Let z be such that ω_2/ω_3 is irrational; let $\omega_0 \in \Pi_y$ be such that $r_y(\omega_0) \neq \infty$, and let*

$$\mathcal{A}(\omega_0) = \{\omega \in \Pi_y : \Im\omega = \Im\omega_0,\ f_y(\omega) = \infty\}.$$

Assume that $\omega_0 \in \mathcal{A}(\omega_0)$ and that for some $\omega^1, \ldots, \omega^k \in \mathcal{A}(\omega_0)$:

(A) $\omega_0 \ll_{n_1} \omega^1 \ll_{n_2} \cdots \ll_{n_k} \omega^k$;
(B) $\lim_{\omega \to \omega_0}\{f_y(\omega) + f_y(\omega + n_1\omega_3) + f_y(\omega + n_2\omega_3) + \cdots + f_y(\omega + n_k\omega_k)\} = \infty$;
(C) *there is no other $\omega \in \mathcal{A}(\omega_0)$ such that $\omega_0 \ll \omega$.*

Then the set of poles of all branches of $x \mapsto Q(x, 0)$ (resp. $y \mapsto Q(0, y)$) is dense on the curve $\mathcal{I}_x(\omega_0)$ (resp. $\mathcal{I}_y(\omega_0)$) defined in (7.6) (resp. (7.4)).

Proof. By Equation (5.2) of Theorem 4, we have, for any $n \in \mathbf{Z}_+$ and any $\omega \in \Pi_y$,

$$r_y(\omega + n\omega_3) = r_y(\omega) + f_y(\omega) + f_y(\omega + \omega_3) + f_y(\omega + 2\omega_3) + \cdots + f_y(\omega_0 + (n-1)\omega_3). \tag{7.7}$$

Let ω_0 be as in the statement of Lemma 21. Due to assumption (C), Lemma 18 (a) and the ω_2-periodicity of f_y, the set $\{\omega_0 + n\omega_3\}_{n > n_1 + \cdots + n_k}$ does not contain any point ω where $f_y(\omega) = \infty$. Further, by the assumptions (A) and (C), Lemma 18 (a) and also by the ω_2-periodicity of f_y, the set $\{\omega_0 + n\omega_3\}_{0 \leqslant n \leqslant n_1 + \cdots + n_k}$ contains exactly $k+1$ poles of f_y that are $\omega_0, \omega_0 + n_1\omega_3, \ldots, \omega_0 + n_k\omega_3$. Then, by (7.7), assumption (B) and the fact that $r_y(\omega_0) \neq \infty$, we reach the conclusion that for any $n > n_1 + \cdots + n_k$, the point $\omega_0 + n\omega_3$ is a pole of $r_y(\omega)$.

Due to Equation (5.3), any ω pole of r_y such that $x(\omega)y(\omega) \neq \infty$ is also a pole of r_x. Define now $\mathcal{B}$, the set of (at most twelve) points in Π_y where either $x(\omega) = \infty$, $y(\omega) = \infty$, $K(x(\omega), 0) = 0$ or $K(0, y(\omega)) = 0$. Introduce also $M = \max\{m \geqslant 0 : \omega_0 \ll_m \omega \text{ for some } \omega \in \mathcal{B}\}$—with the usual convention $M = -\infty$ if $\omega_0 \ll \omega$ for none $\omega \in \mathcal{B}$. If $n > \max(M, n_1 + \cdots + n_k)$, the points $\omega_0 + n\omega_3$ are poles of r_x as well, and both $K(x(\omega_0 + n\omega_3), 0)$ and $K(0, y(\omega_0 + n\omega_3))$ are non-zero. By Lemma 18 (b) and definitions (5.10) and (5.12), Lemma 21 follows. □

We are now ready to give the proof of Theorem 17.

Proof of Theorem 17. Functions $f_y(\omega)$ and $y(\omega)$ being ω_2-periodic, it follows that both of them have no pole at any ω with $0 \leqslant \Im\omega < \omega_1$ and $\Im\omega \notin \{\Im a_1, \Im a_2, \Im a_3, \Im a_4, \Im b_1, \Im b_2\}$.

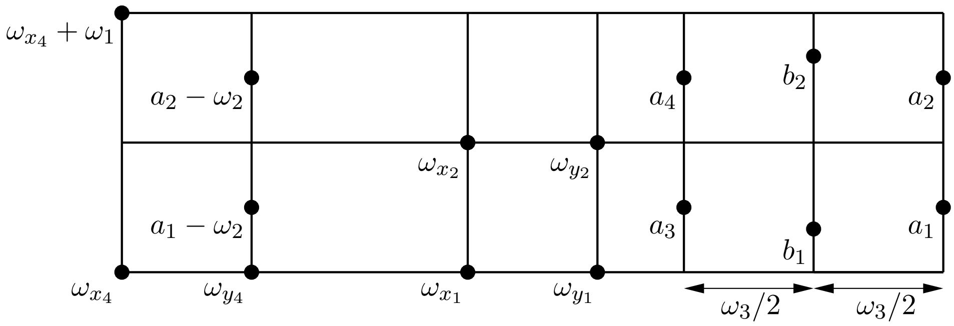


FIGURE 12. Location of $a_1, a_2, a_3, a_4, b_1, b_2$ if $y_4 < 0$ and $x_4 < 0$, i.e., Subcase I.A

Then, by (5.2),

$$\forall \omega \in \bigcup_{k=0}^{\infty} \mathscr{N}_{k,0} \text{ with } \Im\omega \neq \{\Im a_1, \Im a_2, \Im a_3, \Im a_4, \Im b_1, \Im b_2\}, \quad r_y(\omega) \neq \infty. \tag{7.8}$$

Function $x(\omega)$ being ω_2-periodic, it has no pole at any ω such that $0 \leqslant \Im\omega < \omega_1$ and $\Im\omega \notin \{\Im b_1, \Im b_2\}$. Then Equation (5.3) and the fact that $\cup_{k=1}^{\infty}\mathscr{M}_{k,0} \subset \cup_{k=0}^{\infty}\mathscr{N}_{k,0}$ imply that

$$\forall \omega \in \bigcup_{k=1}^{\infty} \mathscr{M}_{k,0} \text{ with } \Im\omega \neq \{\Im a_1, \Im a_2, \Im a_3, \Im a_4, \Im b_1, \Im b_2\}, \quad r_x(\omega) \neq \infty. \tag{7.9}$$

In order to prove Theorem 17 (i), we shall prove the following proposition.

Proposition 22. *For all 51 models, for any $\omega \in \mathscr{M}_{1,0}$ (resp. $\omega \in \mathscr{N}_{1,0}$) with $x(\omega) \neq \infty$ (resp. $y(\omega) \neq \infty$) and $\Im\omega \in \{\Im a_1, \Im a_2, \Im a_3, \Im a_4, \Im b_1, \Im b_2\}$, we have $r_x(\omega) \neq \infty$ (resp. $r_y(\omega) \neq \infty$).*

The proof of this proposition is postponed to the next subsection. By this proposition, (7.8) and (7.9), the only singularities of the first branches of $K(x,0)Q(x,0)$ (resp. $K(0,y)Q(0,y)$) may be only among the branch points x_1, x_2, x_3, x_4 (resp. y_1, y_2, y_3, y_4). Let us recall that the function $x \mapsto Q(x,0)$ is initially defined as a series $\sum_{i,n\geqslant 0} q(i,0;n)x^i z^n$. The elementary estimate $\sum_{i\geqslant 0} q(i,0;n) \leqslant |\mathcal{S}|^n$ implies that for any $z \in]0, 1/|\mathcal{S}|[$ and $x \in \mathbf{C}$ with $|x| \leqslant 1$ this series is absolutely is convergent. Since $|x_1| < 1$, $|x_2| < 1$, and since also $K(x,0)$ is a polynomial with (at most two) roots that are smaller or equal to 1 by absolute value, the only singularities of the first branch of $x \mapsto Q(x,0)$ are the branch points x_3 and x_4, that are out of the unit disc. By the same arguments the analogous statement holds true for $y \mapsto Q(0,y)$. This finishes the proof of Theorem 17 (i).

Since for any $p \in \mathbf{Z}_+$, there exist only finitely many $\omega \in \cup_{k=1}^{p}\mathscr{M}_{k,0}$ (resp. $\omega \in \cup_{\ell=1}^{p}\mathscr{N}_{0,\ell}$) where $f_x(\omega) = \infty$ (resp. $f_y(\omega) = \infty$), Theorem 17 (ii) immediately follows from the meromorphic continuation procedure of r_x and r_y done in Section 5, namely Equations (5.1) and (5.2) as well as the definitions (5.10) and (5.12).

The following proposition proves Theorem 17 (iii).

Proposition 23. *The poles of $x \mapsto Q(x,0)$ (resp. $y \mapsto Q(0,y)$) are dense on the six curves $\mathcal{I}_x(\omega_0)$ (resp. $\mathcal{I}_y(\omega_0)$) with $\omega_0 \in \{a_1,a_2,a_3,a_4,b_1,b_2\}$. For any of 51 models, the set of these curves coincide with the one claimed in Theorem 17 (iii).*

The proof of this proposition is postponed to the next subsection as well, it will be based on Lemma 21 with ω_0 appropriately chosen among a_1,a_2,a_3,a_4,b_1,b_2.

The last statement (iv) of the theorem follows immediately from (7.8) and (7.9), Proposition 23 and definitions (5.10) and (5.12). □

7.3. **Proof of Propositions 22 and 23.** To start with the proofs of Propositions 22 and 23, we need to study closer the location of points (7.5) a_1,a_2,a_3,a_4,b_1,b_2 on Π_y. It depends heavily on the signs of x_4 and y_4, see Figures 12, 13, 14, 15 and 16. Let us recall that $0<x_3<\infty$ and $0<y_3<\infty$, see Section 2.

If $y_4<0$ (resp. $x_4<0$), the point $y=\infty$ (resp. $x=\infty$) obviously belongs to the real cycle $]y_3,\infty[\cup\{\infty\}\cup]\infty,y_4[$ (resp. $]x_3,\infty[\cup\{\infty\}\cup]\infty,x_4[$) of the complex sphere $\mathbf{S}$. By construction of the Riemann surface $\mathbf{T}$ and of its universal covering, the points a_1,a_2 (resp. b_1,b_2) then lie on the open interval $\{\omega:\omega\in L_{y_4}^{y_3}+\omega_2,\ 0<\Im\omega<\omega_1\}$ (resp. $\{\omega:\omega\in L_{x_4}^{x_3}+\omega_2,\ 0<\Im\omega<\omega_1\}$). These points are symmetric w.r.t. the center of the interval, namely $\omega_1/2+\omega_2+\omega_3/2$ (resp. $\omega_1/2+\omega_2$). The points ω corresponding to a_3 and a_4 are on the open interval $\{\omega:\omega\in L_{y_4}^{y_3}+\omega_2-\omega_3,\ 0<\Im\omega<\omega_1\}$ and are symmetric w.r.t. the center $\omega_1/2+\omega_2-\omega_3/2$ as well. Furthermore, $a_4+\omega_3=a_2$ and $a_3+\omega_3=a_1$, so that $a_4\lll_1 a_2$ and $a_3\lll_1 a_1$, see Figure 12. Finally, we have $\Im a_4=\Im a_2\neq\Im a_1=\Im_3$, hence for any $a\in\{a_2,a_4\}$ and any $a'\in\{a_1,a_3\}$, $a\not\sim a'$ (in the sense of Definition 20).

If $y_4>0$ or $y_4=\infty$ (resp. $x_4>0$ or $x_4=\infty$), the point $y=\infty$ (resp. $x=\infty$) is on $]y_4,\infty]\cup\{\infty\}\cup]\infty,y_1[$ (resp. $]x_4,\infty]\cup\{\infty\}\cup]\infty,x_1[$). Accordingly, the points a_1,a_2 (resp. b_1,b_2) and also a_3,a_4 are on the segment $]\omega_3/2,\omega_2+\omega_3/2]$.Their location on this segment will be specified latter.

Therefore, Propositions 22 and 23 must be proved separately for eight subclasses of 51 models according to the signs of x_4 and y_4: these are those of the walks pictured on Figure 17, Subcases I.A, I.B, I.C, II.A, II.B, II.C, II.D and Case III.

The following remark gives a geometric interpretation of this classification.

Remark 24. *Let $\mathbf{1}_{(i,j)}$ be 1 if $(i,j)\in\mathcal{S}$, otherwise 0. Then $x_4>0$ (resp. <0, $=\infty$) if and only if $\mathbf{1}_{(1,0)}^2-4\mathbf{1}_{(1,1)}\mathbf{1}_{(1,-1)}>0$ (resp. <0, $=0$), see Equation (2.2). A symmetric statement holds for y_4.*

As an example, Remark 24 implies that $x_4<0$ if and only if $(1,1)\in\mathcal{S}$ and $(1,-1)\in\mathcal{S}$.

Case I*: $y_4<0$, Subcase* I.A*: $x_4<0$.* This assumption yields $x^\star\neq x^\bigstar$; $x^\star,x^\bigstar\neq\infty$; $y\neq y^\bullet$; $y^\circ,y^\bullet\neq\infty$; $y^\star,y^\bigstar\neq\infty$. The location of the six points a_1,a_2,a_3,a_4,b_1,b_2 is already described above and is pictured on Figure 12.

We first show that for all $\omega\in\{\omega:\Im\omega=\Im a_3,\ \omega_{y_1}\leqslant\Re\omega<\omega_{y_4}+\omega_2\}$, we have $r_y(\omega)\neq\infty$. The proof consists in three steps.

Step 1. Let us first prove that $r_y(a_3)\neq\infty$ and $r_y(a_4)\neq\infty$. If $|y^\bigstar|<1$, then $a_3\in\Delta_y$ (see Section 4 for the definition of Δ_y) and it is immediate from Theorem 3 that $r_y(a_3)\neq\infty$. If

$|y^\star| \geqslant 1$, then by Lemma 5 there exists $n \in \mathbf{Z}_+$ such that $a_3 - n\omega_3 \in \Delta$, and by Equation (5.2) of Theorem 4,

$$r_y(a_3) = r_y(a_3 - n\omega_3) + \sum_{k=n}^{1} f_y(a_3 - k\omega_3). \tag{7.10}$$

Introducing, for any ω_0, the set

$$\mathscr{O}^\Delta(\omega_0) = \{\omega_0 - \omega_3, \omega_0 - 2\omega_3, \ldots, \omega_0 - n_{\omega_0}\omega_3\}, \quad n_{\omega_0} = \inf\{\ell \geqslant 0 : \omega_0 - \ell\omega_3 \in \Delta\}, \tag{7.11}$$

we can rewrite (7.10) as

$$r_y(a_3) = r_y(a_3 - n_{a_3}\omega_3) + \sum_{\omega \in \mathscr{O}^\Delta(a_3)} f_y(\omega). \tag{7.12}$$

In (7.12), the quantity $r_y(a_3 - n_{a_3}\omega_3)$ is defined thanks to Theorem 3. It may be infinite, but only if $y(a_3 - n_{a_3}\omega_3) = \infty$. In this case we must have $a_3 - n_{a_3}\omega_3 = a_1 - \omega_2$. But since $a_3 + \omega_3 = a_1$, we then have $(n_{a_3} + 1)\omega_3 = \omega_2$ which is impossible, due to irrationality of ω_2/ω_3. Hence $r_y(a_3 - n_{a_3}\omega_3) \neq \infty$. Further, we immediately have (see indeed Figure 12) that $a_2, a_4, a_2 - \omega_2, a_4 - \omega_2 \notin \mathscr{O}^\Delta(a_3)$. Moreover, since either $\Im b_1 \neq \Im a_3$, or $\Im b_1 = \Im a_3$ but then $a_3 + \omega_3/2 = b_1$ (see again Figure 12), we also have that $b_1, b_2, b_1 - \omega_2, b_2 - \omega_2 \notin \mathscr{O}^\Delta(a_3)$. Finally $a_1 = a_3 + \omega_3 \notin \mathscr{O}^\Delta(a_3)$ and $a_1 - \omega_2 = a_3 + \omega_3 - \omega_2 \notin \mathscr{O}^\Delta(a_3)$, since ω_2/ω_3 is irrational. Thus $f_y(a_3 - k\omega_3) \neq \infty$ for any $k \in \{1, \ldots, n_{a_3}\}$. Accordingly, $r_y(a_3) \neq \infty$ and by the same arguments, $r_y(a_4) \neq \infty$.

Step 2. We now show that $r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) \neq \infty$ and $r_y(a_2 - \omega_2) + f_y(a_2 - \omega_2) \neq \infty$. By Equation (5.3), $r_y(a_1 - \omega_2) = -r_x(a_1 - \omega_2) + K(0,0)Q(0,0) + x(a_1 - \omega_2)y(a_1 - \omega_2)$ and $f_y(a_1 - \omega_2) = x(a_1 - \omega_2)[y(a_4) - y(a_1 - \omega_2)]$; hence

$$r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) = -r_x(a_1 - \omega_2) + K(0,0)Q(0,0) + x(a_1 - \omega_2)y(a_4). \tag{7.13}$$

It follows from Equation (5.3) and from the first step that $r_x(a_4) \neq \infty$, since $x(a_4)y(a_4) = x^\star y^\star \neq \infty$. Then, by (3.5) and (5.4) we get that $r_x(a_1 - \omega_2) = r_x(\widehat{\xi a_4}) = r_x(a_4) \neq \infty$. Furthermore, $x(a_1 - \omega_2)y(a_4) = x^\star y^\star \neq \infty$. Finally, thanks to (7.13), $r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) \neq \infty$ and by the same arguments, $r_y(a_2 - \omega_2) + f_y(a_2 - \omega_2) \neq \infty$.

Step 3. Let us now take any ω_0 in $\{\omega : \Im\omega = \Im a_3, \omega_{y_1} \leqslant \Re\omega < \omega_{y_4} + \omega_2\}$. If $\omega_0 \in \Delta$, then $\omega_0 \in \Delta_y$. Indeed, it is proved in Section 4 that the domain Δ_y (resp. Δ_x), which is bounded by $\widehat{\Gamma}_y^0$ and $\widehat{\Gamma}_y^1$ (resp. $\widehat{\Gamma}_x^0$ and $\widehat{\Gamma}_x^1$), is centered around $L_{y_1}^{y_2}$ (resp. $L_{x_1}^{x_2}$). Furthermore, $\widehat{\Gamma}_y^0 \in \Delta_x$ and $\widehat{\Gamma}_y^1 \notin \Delta_x$ (resp. $\widehat{\Gamma}_x^0 \in \Delta_y$ and $\widehat{\Gamma}_x^1 \notin \Delta_y$). It follows that for any $\omega_0 \in \Delta \setminus \Delta_y$, $\Re\omega_0 < \omega_{y_1}$. Then, by Theorem 3, $r_y(\omega_0) \neq \infty$. If $\omega_0 \notin \Delta$, with (7.11) and (7.12) we have

$$r_y(\omega_0) = r_y(\omega_0 - n_{\omega_0}\omega_3) + \sum_{\omega \in \mathscr{O}^\Delta(\omega_0)} f_y(\omega).$$

For the same reasons as in the first step, we have that $a_2, a_4, a_2 - \omega_2, a_4 - \omega_2 \notin \mathscr{O}^\Delta(\omega_0)$. If $\Re\omega_0 < \Re a_3$, for obvious reasons $\mathscr{O}^\Delta(\omega_0)$ cannot contain a_3. If $\Re a_3 \leqslant \Re\omega_0 < \omega_{y_4} + \omega_2$, it can neither contain a_3, since $\omega_{y_4} + \omega_2 - \Re a_3 = \omega_3$, and hence $\Re\omega_0 - \omega_3 < \Re a_3$. If $\Im b_1 \neq \Im a_3$, or

$\Im b_1 = \Im a_3$ and $\Re\omega_0 < \Re b_1$, it cannot contain b_1. If $\Im b_1 = \Im a_3$ and $\Re b_1 \leqslant \Re\omega_0 < \omega_{y_4} + \omega_2$, then $\Re\omega_0 - \Re b_1 \leqslant \omega_{y_4} + \omega_2 - \Re b_1 = \omega_3/2 < \omega_3$, and $b_1 \notin \mathscr{O}^{\Delta}(\omega_0)$.

If $a_1 - \omega_2 \notin \mathscr{O}^{\Delta}(\omega_0)$, then we have $r_y(\omega_0 - n_{\omega_0}\omega_3) \neq \infty$ and $f_y(\omega_0 - k\omega_3) \neq \infty$ for all $k \in \{1, \ldots, n_{\omega_0}\}$, so that $r_y(\omega_0) \neq \infty$ by (7.12).

If $a_1 - \omega_2 \in \mathscr{O}^{\Delta}(\omega_0)$, then for some $j \in \{1, \ldots n_{\omega_0}\}$, we have $\omega_0 - j\omega_3 = a_1 - \omega_2$. Then $r_y(\omega_0 - n_{\omega_0}\omega_3) + \sum_{k=n}^{j+1} f_y(\omega_0 - k\omega_3) = r_y(a_1 - \omega_2)$ and thus by (7.12),

$$r_y(\omega_0) = r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) + \sum_{k=j-1}^{1} f_y(\omega_0 - k\omega_3).$$

The first term here is finite by the second step and $f_y(\omega_0 - k\omega_3) \neq \infty$ for $k \in \{1, \ldots, j-1\}$ by all properties said above, so that $r_y(\omega_0) \neq \infty$.

So far we have proved that for all $\omega \in \{\omega : \Im\omega = \Im a_3,\, \omega_{y_1} \leqslant \Re\omega < \omega_{y_4} + \omega_2\}$, $r_y(\omega) \neq \infty$. In the same way, we obtain that $r_y(\omega) \neq \infty$ for $\omega \in \{\omega : \Im\omega = \Im a_4,\, \omega_{y_1} \leqslant \Re\omega < \omega_{y_4} + \omega_2\}$.

Since by (3.5),

$$\widehat{\eta}\{\omega : \Im\omega = \Im a_3,\, \omega_{y_1} \leqslant \Re\omega < \omega_{y_4} + \omega_2\} = \{\omega : \Im\omega = \Im a_4,\, \omega_{y_4} < \Re\omega \leqslant \omega_{y_1}\},$$
$$\widehat{\eta}\{\omega : \Im\omega = \Im a_4,\, \omega_{y_1} \leqslant \Re\omega < \omega_{y_4} + \omega_2\} = \{\omega : \Im\omega = \Im a_3,\, \omega_{y_4} < \Re\omega \leqslant \omega_{y_1}\},$$

Equation (5.4) implies that $r_y(\omega) \neq \infty$ on the segments $\{\omega \in \Pi_y : \Im\omega = a_3, a_4\}$, except for their ends a_1, a_2. The segments $\{\omega : \Im\omega = a_3, a_4,\, \omega_{x_1} \leqslant \Re\omega \leqslant \omega_{x_4} + \omega_2\}$ do not contain any point where $y(\omega) = \infty$. It follows from Equation (5.3) that $r_x(\omega) \neq \infty$ on these segments except for points where $x(\omega) = \infty$ if they exist. This last fact happens if and only if $\Im b_1 = \Im a_3$ and only at the ends b_1, b_2 of the segments.

If $\Im b_1 \neq \Im a_3$, we can show exactly in the same way that $r_y(\omega) \neq \infty$ on the two segments $\{\omega \in \Pi_y : \Im\omega = b_1, b_2\}$ and that $r_x(\omega) \neq \infty$ on the segments $\{\omega : \Im\omega = b_1, b_2,\, \omega_{x_1} \leqslant \Re\omega \leqslant \omega_{x_4} + \omega_2\}$, except for their ends b_1, b_2. This concludes the proof of Proposition 22.

We proceed with the proof of Proposition 23. Let us verify the assumptions of Lemma 21 for $\omega_0 = a_3, a_4, b_1, b_2$. We have proved that $r_y(a_3), r_y(a_4), r_y(b_1), r_y(b_2) \neq \infty$, $a_3 \ll_1 a_1$, $a_4 \ll_1 a_2$ and that the pairs $\{a_1, a_3\}$ and $\{a_2, a_4\}$ are not ordered. Let us now show that for any $k \in \{3, 4\}$ and $\ell \in \{1, 2\}$, it is impossible to have $a_k \sim b_\ell$. If $\Im b_\ell \neq \Im a_3, \Im a_4$, this is obvious. If $\Im b_\ell = \Im a_3$, then it is enough to note that $b_\ell - a_3 = \omega_3/2$ and $a_1 - b_1 = \omega_3/2$ (see Figure 12). From the irrationality of ω_2/ω_3, it follows that $b_\ell \not\sim a_1, a_3$ and in the same way $b_\ell \not\sim a_2, a_4$. Then there is no other $\omega \in \Pi_y$ except for a_1 (resp. a_2) such that $a_3 \ll \omega$ (resp. $a_4 \ll \omega$) and $f_y(\omega) = \infty$. There is no $\omega \in \Pi_y$ such that $b_\ell \ll \omega$ and $f_y(\omega) = \infty$, $\ell = 1, 2$. Hence, Lemma 21 could be applied to any of four points $\omega_0 = a_3, a_4, b_1, b_2$ if the assumption (B) of this lemma is satisfied for these points. It is then immediate that $\lim_{\omega \to b_\ell} f_y(\omega) = \lim_{\omega \to b_\ell} x(\omega)[y(\widehat{\xi}\omega) - y(\omega)] = \infty$, $\ell \in \{1, 2\}$, since $x(\omega) \to \infty$ and the other term converges to $\pm[y^\circ - y^\bullet] \neq 0$. Let us verify that $\lim_{\omega \to a_3}\{f_y(\omega) + f_y(\omega + \omega_3)\} = \infty$.

We have

$$\begin{aligned}\lim_{\omega\to a_3}\{f_y(\omega)+f_y(\omega+\omega_3)\} &= \lim_{\omega\to a_3}\{x(\omega)[y(\widehat{\xi}\omega)-y(\omega)]+x(\widehat{\eta}\widehat{\xi}\omega)[y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)-y(\widehat{\eta}\widehat{\xi}\omega)]\}\\ &= \lim_{\omega\to a_3}\{x(\widehat{\eta}\widehat{\xi}\omega)y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)-x(\omega)y(\omega)\}\\ &+ \lim_{\omega\to a_3}\{x(\omega)y(\widehat{\xi}\omega)-x(\widehat{\eta}\widehat{\xi}\omega)y(\widehat{\eta}\widehat{\xi}\omega)\}.\end{aligned}$$

The first term above converges to $x^\star y^\star - x^\bigstar y^\bigstar$. By (3.7) the second term equals the limit of the product $y(\widehat{\xi}\omega)[x(\widehat{\xi}\omega)-x(\widehat{\eta}\widehat{\xi}\omega)]$. If $\omega\to a_3$, then $\widehat{\xi}\omega\to a_2-\omega_2$ so that the first term in the product converges to $y(a_2-\omega_2)=y(a_2)=\infty$. The second term of this product converges to $x(a_2-\omega_2)-x(a_1)=x^\bigstar-x^\star$ which is different from 0 as $x^\bigstar\neq x^\star$. Then assumption (B) is satisfied for $\omega_0=a_3$ and in the same way for $\omega_0=a_4$. Lemma 21 applies to any of the four points $\omega_0=a_3,a_4,b_1,b_2$. But by (3.7), $\mathcal{I}_x(a_3)=\mathcal{I}_x(a_4)=\mathcal{I}_x(a_1)=\mathcal{I}_x(a_2)$, $\mathcal{I}_y(a_3)=\mathcal{I}_y(a_4)=\mathcal{I}_y(a_1)=\mathcal{I}_y(a_2)$, $\mathcal{I}_x(b_1)=\mathcal{I}_x(b_2)$, $\mathcal{I}_y(b_1)=\mathcal{I}_y(b_2)$ so that poles of $x\mapsto Q(x,0)$ are dense on the curves $\mathcal{I}_x(a_1)$ and $\mathcal{I}_x(b_1)$ and those of $y\mapsto Q(0,y)$ are dense on the curves $\mathcal{I}_y(a_1)$ and $\mathcal{I}_y(b_1)$. Proposition 23 is proved.

Case I*:* $y_4<0$*, Subcase* I.B*:* $x_4=\infty$. This assumption implies that $x^\star\neq x^\bigstar$; $x^\star,x^\bigstar\neq\infty$; $y^\circ=y^\bullet\neq\infty$; $y^\star,y^\bigstar\neq\infty$.

The points a_1,a_2,a_3,a_4 are located as in the previous case, see Figure 12. Consequently we have the following facts: $r_y(\omega)\neq\infty$ on the segments $\{\omega\in\Pi_y:\Im\omega=a_3,a_4\}$, except for their ends $\omega=a_1,a_2$; $r_x(\omega)\neq\infty$ on the segments $\{\omega:\Im\omega=a_3,a_4,\ \omega_{x_1}\leqslant\Re\omega\leqslant\omega_{x_4}+\omega_2\}$. Lemma 21 applies to $\omega_0=a_3,a_4$ as in the previous case, as $x^\star\neq x^\bigstar$. Then the set of poles of all branches of $x\mapsto Q(x,0)$ (resp. $y\mapsto Q(0,y)$) is dense on $\mathcal{I}_x(a_1)$ (resp. $\mathcal{I}_y(a_1)$) where $\mathcal{I}_x(a_1)=\mathcal{I}_x(a_2)=\mathcal{I}_x(a_3)=\mathcal{I}_x(a_4)$ (resp. $\mathcal{I}_y(a_1)=\mathcal{I}_y(a_2)=\mathcal{I}_y(a_3)=\mathcal{I}_y(a_4)$).

Since $x_4=\infty$, we have that $b_1=b_2=\omega_{x_4}+\omega_2$. Take any ω_0 with $\Im\omega_0=0$ such that $\omega_{y_1}\leqslant\Re\omega_0\leqslant\omega_{y_4}+\omega_2$. Then $y(\omega_0)\neq\infty$. Let us show that $r_y(\omega_0)\neq\infty$. If $\omega_0\in\Delta$, then by the same reasons as in Subcase I.A $\omega_0\in\Delta_y$ and $r_y(\omega_0)\neq\infty$. If $\omega_0\notin\Delta$, consider the set $\mathscr{O}^\Delta(\omega_0)$ defined as in (7.11) and (7.12). Clearly $b_1-\omega_2=\omega_{x_4}\notin\mathscr{O}^\Delta(\omega_0)$. Since $b_1+\omega_3/2=\omega_{y_4}+\omega_2$, we have $\omega_0-\omega_3\leqslant\omega_{y_4}+\omega_2-\omega_3<b_1$ and then $b_1\notin\mathscr{O}^\Delta(\omega_0)$. Hence $\mathscr{O}^\Delta(\omega_0)$ does not contain any point where $y(\omega)$ or $f_y(\omega)$ is infinite. Thus by (7.12), $r_y(\omega_0)\neq\infty$.

We have $\widehat{\eta}\{\omega:\Im\omega=0,\ \omega_{y_1}\leqslant\Re\omega_0\leqslant\omega_{y_4}+\omega_2\}=\{\omega:\Im\omega=\omega_1,\ \omega_{y_4}\leqslant\Re\omega_0\leqslant\omega_{y_1}\}$. Then by (5.4) and (5.5), we get that $r_y(\omega)\neq\infty$ for all $\omega\in\Pi_y$ with $\Im\omega=0$. The segment $\{\omega:\Im\omega=0,\ \omega_{x_1}\leqslant\omega\leqslant\omega_{x_4}+\omega_2\}$ does not contain any point with $y(\omega)=\infty$. By (5.3) this gives $r_x(\omega)\neq\infty$ for all ω on this segment except for the points where $x(\omega)=\infty$ (that is only at $\omega=\omega_{x_4}+\omega_2=b_1$), and this concludes the proof of Proposition 22.

We have proved in particular that $r_y(\omega_0)\neq\infty$ for $\omega_0=b_1$. Furthermore, there is no $\omega\in\Pi_y$ such that $b_1\ll\omega$ and $f_y(\omega)=\infty$. Finally

(7.14)
$$\lim_{\omega\to b_1}f_y(\omega)=\lim_{\omega\to b_1}x(\omega)[y(\widehat{\xi}\omega)-y(\omega)]=\lim_{\omega\to b_1}x(\omega)\frac{[b(x(\omega))^2-4a(x(\omega))c(x(\omega))]^{1/2}}{a(x(\omega))},$$

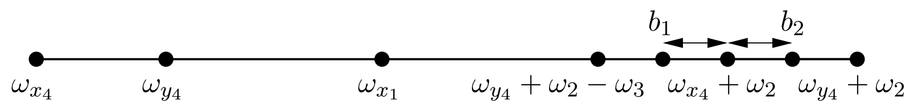


FIGURE 13. Location of b_1, b_2 if $x_4 > 0$, Subcases I.C and II.C

where $x(\omega) \to \infty$ as $x \to b_1$. For all models in Subcase I.B $\deg a(x) = 2$, $\deg b(x) = 1$ and $\deg c(x) = 1$, so that (7.14) is of the order $O(|x(\omega)|^{1/2})$. Thus $\lim_{\omega\to b_1} f_y(\omega) = \infty$. By Lemma 21 with $\omega_0 = b_1$, the poles of $x \mapsto Q(x, 0)$ and those of $y \mapsto Q(0, y)$ are dense on $\mathcal{I}_x(b_1) = \mathcal{I}_x(b_2)$ and $\mathcal{I}_y(b_1) = \mathcal{I}_y(b_2)$, respectively. They are the intervals of the real line claimed in Theorem 17 (iii). Proposition 23 is proved.

Case I*:* $y_4 < 0$*, Subcase* I.C*:* $x_4 > 0$. The statements and results about a_1, a_2, a_3, a_4 are the same as in Subcases I.A and I.B, see Figure 12 for their location.

We now locate b_1, b_2. By definition (see Section 2), the values y_1, y_2, y_3, y_4 are the roots of

$$\widetilde{d}(y) = (\widetilde{b}(y) - 2[\widetilde{a}(y)\widetilde{c}(y)]^{1/2})(\widetilde{b}(y) + 2[\widetilde{a}(y)\widetilde{c}(y)]^{1/2}) = 0.$$

Hence, for two of these roots $\widetilde{b}(y) = -2[\widetilde{a}(y)\widetilde{c}(y)]^{1/2}$ and then $X(y) \geqslant 0$ (see (2.4)), and for the two others $\widetilde{b}(y) = 2[\widetilde{a}(y)\widetilde{c}(y)]^{1/2}$ and then $X(y) \leqslant 0$. But $X(y_2)$ and $X(y_3)$ are on the segment $[x_2, x_3] \subset]0, \infty[$. Thus $X(y_1) \leqslant 0$ and $X(y_4) \leqslant 0$. Since $x(b_1) = x(b_1 - \omega_2) = \infty$, $x_4 = x(\omega_{x_4}) > 0$ and $X(y_4) = x(\omega_{y_4}) < 0$, it follows that $b_1 - \omega_2 \in]\omega_{x_4}, \omega_{y_4}[$, in such a way that $b_1 \in]\omega_{x_4} + \omega_2, \omega_{y_4} + \omega_2[$. Also, $b_2 = \widehat{\xi}(b_1 - \omega_2) - \omega_1 = 2(\omega_{x_4} + \omega_2) - b_1$ is symmetric to b_1 w.r.t. $\omega_{x_4} + \omega_2$. Since $x(\omega_{y_1}) = X(y_1) \leqslant 0$ and $x(\omega_{x_4} + \omega_2) = x_4 > 0$, it follows that $\omega_{y_1} < b_2 < \omega_{x_4} + \omega_2$, see Figure 13.

Now we show that for any ω_0 with $\Im\omega_0 = 0$ and $\omega_{y_1} \leqslant \Re\omega_0 \leqslant \omega_{y_4} + \omega_2$, $r_y(\omega_0) \neq \infty$. Note that $y(\omega_0) \neq \infty$. If $\omega_0 \in \Delta$, by the same arguments as in Subcase I.A, $\omega_0 \in \Delta_y$ and $r_y(\omega_0) \neq \infty$. If $\omega_0 \notin \Delta$, then consider $\mathscr{O}^\Delta(\omega_0)$ with the notation (7.11).

Note that $b_1 - \omega_2 \notin \Delta$. For this, it is enough to prove that $b_1 - \omega_2 \notin \Delta_x$ and that $b_1 - \omega_2 \notin \Delta_y$. First, $b_1 - \omega_2 \notin \Delta_x$, since $x(b_1) = x(b_1 - \omega_2) = \infty$. Furthermore, Δ_y is centered w.r.t. $L_{y_1}^{y_2}$, and $\omega_{y_4} \notin \Delta_y$ (since $|y_4| > 1$). Hence the point $b_1 - \omega_2 < \omega_{y_4}$ cannot be in Δ_y.

Since $\Delta \cap \{\omega \in \mathbf{C} : \Im\omega = 0\}$ is an open interval containing ω_{y_1}, and since $b_1 - \omega_2 < \omega_{y_1} \leqslant \omega_0$, it follows that $b_1 - \omega_2 < \omega_0 - n_{\omega_0}\omega_3$ (see (7.11)), so that $b_1 - \omega_2 \notin \mathscr{O}^\Delta(\omega_0)$. Obviously $b_2 - \omega_2 \notin \mathscr{O}^\Delta(\omega_0)$. Furthermore, since $\omega_{y_4} + \omega_2 - b_2 = \omega_3/2 + \omega_{x_4} + \omega_2 - b_2 < \omega_3/2 + \omega_3/2 = \omega_3$, it follows that $\omega_0 - \omega_3 < b_2 < b_1$ for any such ω_0. Hence $b_1, b_2 \notin \mathscr{O}^\Delta(\omega_0)$. Thus for any $\omega \in \mathscr{O}^\Delta(\omega_0)$, $y(\omega) \neq \infty$ and $f_y(\omega) \neq \infty$. By (7.12), $r_y(\omega_0) \neq \infty$. This implies, exactly as in Subcase I.B—by (5.4) and (5.5)—, that $r_y(\omega) \neq \infty$ for all $\omega \in \Pi_y$ such that $\Im\omega = 0$. By (5.3) this gives $r_x(\omega) \neq \infty$ for all ω with $\Im\omega = 0$ and $\omega_{x_1} \leqslant \Im\omega \leqslant \omega_{x_4} + \omega_2$, except for points ω where $x(\omega) = \infty$ (that happens for $\omega = b_2$ only), and this concludes the proof of Proposition 22.

In particular, we proved that $r_y(b_1) \neq \infty$ and also $r_y(b_2) \neq \infty$. Since $y^\circ \neq y^\bullet$, we have $\lim_{\omega\to b_1} f_y(\omega) = \infty$ and also $\lim_{\omega\to b_2} f_y(\omega) = \infty$ by the same arguments as in Subcase I.A. If b_1 and b_2 are not ordered, Lemma 21 applies to both of these points. If $b_1 \ll b_2$

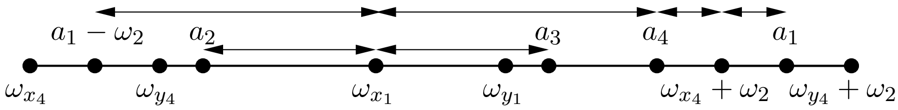


FIGURE 14. Location of a_1, a_2, a_3, a_4 if $y_4 > 0$, $(x_4, Y(x_4)) \neq (\infty, \infty)$, Subcases II.A, II.B and II.C

(resp. $b_2 \ll b_1$), then there is no $\omega \in \Pi_y$ such that $\omega \neq b_2$ (resp. $\omega \neq b_1$), $f_y(\omega) = \infty$ and $b_2 \ll \omega$ (resp. $b_1 \ll \omega$). Hence Lemma 21 applies to $\omega_0 = b_2$ (resp. $\omega_0 = b_1$). Thus the set of poles of all branches of $x \mapsto Q(x, 0)$ (resp. $y \mapsto Q(0, y)$) is dense on the curves $\mathcal{I}_x(b_2)$ and $\mathcal{I}_y(b_2)$ (resp. $\mathcal{I}_x(b_1)$ and $\mathcal{I}_y(b_1)$). We conclude the proof of Proposition 23 with the observation that $\mathcal{I}_x(b_2) = \mathcal{I}_x(b_1)$, while $\mathcal{I}_y(b_2) = \mathcal{I}_y(b_1)$ are intervals of the real line as claimed in Theorem 17 (iii).

Case II*: $y_4 > 0$, location of a_1, a_2, a_3, a_4.* We first exclude Subcase II.D where $x_4 = \infty$ and $Y(x_4) = \infty$, and we locate the points a_1, a_2, a_3, a_4. In this case we have $x^\star \neq x^\bigstar$. Note that x_1, x_2, x_3, x_4 are the roots of the equation

$$d(x) = (b(x) - [a(x)c(x)]^{1/2})(b(x) + [a(x)c(x)]^{1/2}) = 0.$$

Hence for two of these roots $b(x) = -2[a(x)c(x)]^{1/2}$ and then $Y(x) \geqslant 0$ (see (2.5)), and for two others $b(x) = 2[a(x)c(x)]^{1/2}$ and then $Y(x) \leqslant 0$. But $Y(x_2)$ and $Y(x_3)$ are on the segment $[y_2, y_3] \subset]0, \infty[$. Hence $Y(x_1) \leqslant 0$ and $Y(x_4) \leqslant 0$. If in addition $x_4 = \infty$, then $Y(x_4)$ equals 0 or ∞; note also that if $Y(x_4) = \infty$ then necessarily $x_4 = \infty$. But the case when $Y(x_4) = \infty$ and $x_4 = \infty$ is excluded from our consideration at this moment. It follows that $\infty \in [y_4, Y(x_4)[$, and in fact $\infty \in]y_4, Y(x_4)[$, since the case $y_4 = \infty$ is excluded from Case II.

It follows from the above considerations that $a_1 \in]\omega_{x_4} + \omega_2, \omega_{y_4} + \omega_2[$, see Figure 14. In particular, we have $a_1 - \omega_2 \in]\omega_{x_4}, \omega_{y_4}[$ and $a_2 = \widehat{\eta} a_1 - \omega_1 = -(a_1 - \omega_2) + 2\omega_{y_4}$. This means that $a_1 - \omega_2$ and a_2 are symmetric w.r.t. ω_{y_4}. Furthermore $a_2 \in]\omega_{y_4}, \omega_{y_1}[$, but since $y_4 > 0$ and $Y(x_1) \leqslant 0$, we have $a_2 \in]\omega_{y_4}, \omega_{x_1}[$. We must put $a_3 = \widehat{\xi} a_2 - \omega_1 = -a_2 + 2\omega_{x_1}$, in such a way that the points a_2 and a_3 are symmetric w.r.t. ω_{x_1}. Finally, $a_4 = \widehat{\xi}(a_1 - \omega_2) - \omega_1 = -(a_1 - \omega_2) + 2\omega_{x_1} = a_3 + a_2 - (a_1 - \omega_2)$. Note that $\omega_{x_4} + \omega_2 - a_4 = a_1 - \omega_2 - \omega_{x_4} > 0$. Furthermore, $a_1 - a_3 = \omega_3$ and $a_2 + \omega_2 - a_4 = \omega_3$, so that $a_3 \ll a_1$ and $a_4 \ll a_2$.

Case II*: $y_4 > 0$, Subcase* II.A*: $x_4 < 0$.* In this case we have $y^\circ \neq y^\bullet$; $y^\circ, y^\bullet \neq \infty$; $x^\star, x^\bigstar \neq \infty$; $y^\star, y^\bigstar \neq \infty$. The points b_1, b_2 are located as in Subcase I.A, see Figure 12. Further, we can show as in Subcase I.A that $r_y(\omega) \neq \infty$ on the segments $\{\omega : \Im\omega = b_1, b_2, \omega_{y_1} \leqslant \Re\omega \leqslant \omega_{y_4} + \omega_2\}$, and we deduce that $r_x(\omega) \neq \infty$ on $\{\omega : \Im\omega = b_1, b_2, \omega_{x_1} \leqslant \Re\omega \leqslant \omega_{x_4} + \omega_2\}$ except for their ends b_1, b_2. Consequently, the set of poles of all branches of $x \mapsto Q(x, 0)$ (resp. $y \mapsto Q(0, y)$) is dense on the curve $\mathcal{I}_x(b_1) = \mathcal{I}_x(b_2)$ (resp. $\mathcal{I}_y(b_1) = \mathcal{I}_y(b_2)$), as claimed in Proposition 23.

Consider now $r_x(\omega)$ and $r_y(\omega)$ for ω with $\Im\omega = 0$. See Figure 14 for the location of the points a_1, a_2, a_3, a_4.

We first prove that for

$$r_y(\omega_0) \neq \infty, \quad \forall \omega_0 \in \{\omega : \Im\omega = 0,\, \omega_{y_1} \leqslant \omega_0 \leqslant \omega_{y_4} + \omega_2\} \setminus \{a_1\}. \tag{7.15}$$

The proof consists in three steps.

Step 1. We prove that $r_y(a_3) \neq \infty$ and $r_y(a_4) \neq \infty$.

If $a_3 \in \Delta$, then necessarily $r_y(a_3) \neq \infty$, since $x^\bigstar, y^\bigstar \neq \infty$. Otherwise $|x^\bigstar|, |y^\bigstar| \geqslant 1$, and then $a_2 = \widehat{\xi} a_3 - \omega_1 \notin \Delta$. Since $a_2 < \omega_{x_1} < a_3$, $\omega_{x_1} \in \Delta$ and $a_2, a_3 \notin \Delta$, it follows that any $\omega \in \mathscr{O}^\Delta(a_3)$ must be in $]a_2, a_3[$, hence $f_y(\omega) \neq \infty$, $x(\omega) \neq \infty$ and $y(\omega) \neq \infty$. Thus by (7.12), $r_y(a_3) \neq \infty$.

We now show that $r_y(a_4) \neq \infty$. Suppose first that $a_1 - \omega_2 \in \Delta$. Then, since $a_1 - \omega_2 \notin \Pi_y$, we have $a_1 - \omega_2 \in \Delta_x$, so that $r_x(a_1 - \omega_2) \neq \infty$ by Theorem 3. Then by Equations (5.4) and (5.5), $r_x(a_4) = r_x(\widehat{\xi}(a_1 - \omega_2) - \omega_1) = r_x(a_1) \neq \infty$. Since $x^\star, y^\star \neq \infty$, we also have $r_y(a_4) \neq \infty$ by (5.3). Assume now that $a_1 - \omega_2 \notin \Delta$. Since $a_1 - \omega_2 < \omega_{x_1} < a_4$, then $a_1 - \omega_2 \notin \mathscr{O}^\Delta(a_4)$. Furthermore $a_2 \notin \mathscr{O}^\Delta(a_4)$ as $a_4 + \omega_3 = a_2 + \omega_2$ and ω_3/ω_2 is irrational. Finally $a_4 - a_3 < a_1 - a_3 = \omega_3$, so that $a_3 \notin \mathscr{O}^\Delta(a_4)$. It follows that for any $\omega \in \mathscr{O}^\Delta(a_4)$, we have $f_y(\omega) \neq \infty$ and $y(\omega) \neq \infty$. Hence $r_y(a_4) \neq \infty$.

Step 2. We prove that $r_y(a_2) + f_y(a_2) \neq \infty$ and that $r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) \neq \infty$. By Equation (5.3)

$$\begin{aligned} r_y(a_2) + f_y(a_2) &= -r_x(a_2) + K(0,0)Q(0,0) + x(a_2)y(a_2) + x(a_2)[y(\widehat{\xi} a_2) - y(a_2)] \\ &= -r_x(a_2) + K(0,0)Q(0,0) + x(a_2)y(\widehat{\xi} a_2). \end{aligned} \tag{7.16}$$

Since $r_y(a_3) \neq \infty$ and since $x^\bigstar, y^\bigstar \neq \infty$, it follows from Equation (5.3) that $r_x(a_3) \neq \infty$. Then by (5.4) and (5.5), $r_x(a_2) = r_x(\widehat{\xi} a_3 - \omega_1) = r_x(a_3) \neq \infty$. Since $x(a_2) = x^\bigstar \neq \infty$ and $y(\widehat{\xi} a_2) = y^\bigstar \neq \infty$, (7.16) is finite. By completely analogous arguments we obtain that $r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) \neq \infty$.

Step 3. Let us now show (7.15). If $\omega_0 \in \Delta$, then by the same arguments as in Subcase I.A $\omega_0 \in \Delta_y$ and then $r_y(\omega_0) \neq \infty$ by Theorem 3. Otherwise, consider $\mathscr{O}^\Delta(\omega_0)$ defined in (7.11). Note that

$$\omega \notin \mathscr{O}^\Delta(\omega_0), \quad \forall \omega \in]\omega_{x_4} + \omega_2 - \omega_3/2, \omega_{y_4} + \omega_2], \tag{7.17}$$

since in this case $\omega_0 - \omega_3 < \omega$. In particular, $a_4 \notin \mathscr{O}^\Delta(\omega_0)$. Furthermore, $a_3 \in \mathscr{O}^\Delta(\omega_0)$ implies that $\omega_0 = a_3 + \ell\omega_3$ for some $\ell \geqslant 1$. But $a_3 + \omega_3 = a_1 \neq \omega_0$ and $a_3 + \ell\omega_3 > \omega_{y_4} + \omega_2$ for any $\ell \geqslant 2$. Hence $a_3 \notin \mathscr{O}^\Delta(\omega_0)$. Since $a_2 - (a_1 - \omega_2) < \omega_3$, it is impossible that both a_2 and $a_1 - \omega_2$ belong to $\mathscr{O}^\Delta(\omega_0)$. If none of them belongs to $\mathscr{O}^\Delta(\omega_0)$, then $y(\omega) \neq \infty$ and $f_y(\omega) \neq \infty$ for any $\mathscr{O}^\Delta(\omega_0)$, and then $r_y(\omega_0) \neq \infty$. Suppose, e.g., that $a_2 \in \mathscr{O}^\Delta(\omega_0)$. Then for some $\ell \geqslant 1$, $\omega_0 = a_2 + \ell\omega_3$, and by (7.12), $r_y(\omega_0) = r_y(a_2) + f_y(a_2) + \sum_{k=\ell-1}^{1} f_y(\omega_0 - k\omega_3)$. But $r_y(a_2) + f_y(a_2) \neq \infty$ by the second step, and obviously $\sum_{k=\ell-1}^{1} f(\omega_0 - k\omega_3) \neq \infty$ by all facts said above, so that $r_y(\omega_0) \neq \infty$. The reasoning is the same if $a_1 - \omega_2 \in \mathscr{O}^\Delta(\omega_0)$. This concludes the proof of (7.15).

Applying (5.4) and (5.5) exactly as in Subcase I.B, we now reach the conclusion that $r_y(\omega_0) \neq \infty$ for all $\omega_0 \neq a_2 = \widehat{\eta} a_1$ with $\Im\omega_0 = 0$ and $\omega_{y_4} \leqslant \Re\omega_0 \leqslant \omega_{y_1}$ as well. Next, exactly as in Subcase I.B, thanks to (5.3), we derive that $r_x(\omega) \neq \infty$ for all ω_0 such that $\Im\omega = 0$ and $\omega_{x_1} \leqslant \Re\omega \leqslant \omega_{x_4} + \omega_2$ except possibly for points where $x(\omega) = \infty$. But these points are absent on this segment in this case. This concludes the proof of Proposition 22.

For the same reason as in Subcase I.A, the fact that $x^\star \neq x^\bigstar$ gives $\lim_{\omega \to a_4}\{f_y(\omega) + f_y(\omega+\omega_3)\} = \infty$ and $\lim_{\omega\to a_3}\{f_y(\omega)+f_y(\omega+\omega_3)\} = \infty$. Further, $a_3 \ll_1 a_1$ and $a_4 \ll_1 a_2$. If in addition $a_3 \ll a_4$, then due to the fact that $a_3+\omega_3 = a_1$ we have $a_3 \ll_1 a_1 \ll a_4 \ll_1 a_2$. There is no point $\omega \in \Pi_y \setminus \{a_2\}$ such that $a_4 \ll \omega$ and $f_y(\omega) = \infty$. Lemma 21 applies to $\omega_0 = a_4$. If $a_4 \ll a_3$, then also $a_4 \ll_1 a_2 \ll a_3 \ll_1 a_1$ and this lemma applies to $\omega_0 = a_3$. If $a_3 \not\sim a_4$, Lemma 21 can be applied to both a_3 and a_4. Since $\mathcal{I}_x(a_3) = \mathcal{I}_x(a_4) = \mathcal{I}_x(a_1) = \mathcal{I}_x(a_2) = [x_1, x_4]$ and $\mathcal{I}_y(a_3) = \mathcal{I}_y(a_4) = \mathcal{I}_y(a_1) = \mathcal{I}_y(a_2) = \mathbf{R}\setminus]y_4, y_1[$, the set of poles of $x \mapsto Q(x,0)$ (resp. $y \mapsto Q(0,y)$) is dense on the announced intervals and Proposition 23 is proved.

Case II*:* $y_4 > 0$*, Subcase* II.B*:* $x_4 > 0$ *and exactly one of* $y^\circ, y^\bullet$ *is* ∞*.* Assume, e.g., that $y^\circ = \infty$ and $y^\bullet \neq \infty$. Then $x^\star = \infty$; $y^\star = y^\bullet \neq \infty$; $x^\bigstar \neq x^\star = \infty$; $y^\bigstar \neq \infty$. It follows that $b_1 = a_1$ and $b_2 = \widehat{\xi} b_1 - \omega_1 - \omega_2 = a_4$, while a_1, a_2, a_3, a_4 are pictured as previously, in Subcase II.A, see Figure 14.

We first derive (7.15). By the same reasoning as in Subcase II.A, we reach the conclusion that $r_y(a_3) \neq \infty$. Let us note that in this case $a_1 - \omega_2 \notin \Delta$, as $x(a_1-\omega_2), y(a_1-\omega_2) = \infty$. Next, we derive as in Subcase II.A that $r_y(a_4) \neq \infty$ and that $r_y(a_2) + f_y(a_2) \neq \infty$, since $x^\bigstar y^\bigstar \neq \infty$. Finally, again by the same arguments as in Subcase II.A, we conclude that for any $\omega_0 \neq a_1$ with $\Im\omega_0 = 0$ and $\omega_{y_1} \leqslant \Re\omega_0 \leqslant \omega_{y_4} + \omega_2$, the orbit $\mathscr{O}^\Delta(\omega_0)$ does not contain a_3 and a_4. The orbit can neither contain $a_1 - \omega_2$, since $a_1 - \omega_2 \notin \Delta$ and $a_1 - \omega_2 < \omega_{y_1} < \omega_0$, where $\omega_{y_1} \in \Delta$. Since $r_y(a_2) + f_y(a_2) \neq \infty$, then as in Subcase II.A we have $r_y(\omega_0) \neq \infty$.

It follows from (5.4) and (5.5) that $r_y(\omega_0) \neq \infty$ for any ω_0 such that $\Im\omega_0 = 0$ and $\omega_{y_4} \leqslant \omega_0 \leqslant \omega_{y_4} + \omega_2$, except for a_1 and $\widehat{\eta} a_1 - \omega_1 = a_2$. By (5.3), $r_x(\omega_0) \neq \infty$ for any ω_0 such that $\Im\omega_0 = 0$ and $\omega_{x_1} \leqslant \omega_0 \leqslant \omega_{x_4} + \omega_2$, except for points ω_0 where $x(\omega_0) = \infty$ (this is $\omega_0 = a_4$ in this case). This finishes the proof of Proposition 22 in this case.

Since $x^\star \neq x^\bigstar$, the same reasoning as in Subcase I.A gives $\lim_{\omega\to a_4}\{f_y(\omega)+f_y(\omega+\omega_3)\} = \infty$ and $\lim_{\omega\to a_3}\{f_y(\omega) + f_y(\omega+\omega_3)\} = \infty$. The rest of the proof of Proposition 23 via the use of Lemma 21 with $\omega_0 = a_3$ if $a_4 \ll a_3$ or with $\omega_0 = a_4$ if $a_3 \ll a_4$, or with indifferent choice of a_3 or a_4 if $a_3 \not\sim a_4$, is the same as in Subcase II.A.

Case II*:* $y_4 > 0$*, Subcase* II.C*:* $x_4 > 0$ *and* $y^\circ, y^\bullet \neq \infty$*, or* $x_4 = \infty$ *and* $Y(x_4) \neq \infty$*.* In this case, we have $y^\circ, y^\bullet \neq \infty$; $x^\star, x^\bigstar \neq \infty$; $x^\star \neq x^\bigstar$, $y^\star, y^\bigstar \neq \infty$.

The points a_1, a_2, a_3, a_4 are pictured as in Subcases II.A and II.B, see Figure 14, while b_1, b_2 are pictured as in Subcase I.B (where $b_1 = b_2 = \omega_{x_4+\omega_2}$) or I.C, see Figure 13. They are such that $b_1 \neq a_1$ and $b_2 \neq a_4$. In particular, $b_1, b_2 \in]\omega_{x_4} + \omega_2 - \omega_3, \omega_{x_4} + \omega_2[$ and are symmetric w.r.t. $\omega_{x_4} + \omega_2$; $b_1 = b_2$ is in the middle of this interval if and only if $x_4 = \infty$. Hence for any ω_0 with $\Im\omega_0 = 0$ and $\omega_{y_1} \leqslant \Re\omega_0 \leqslant \omega_{y_4} + \omega_2$, we have $\omega_0 - \omega_3 < b_2 < b_1$, so that $b_1, b_2 \notin \mathscr{O}^\Delta(\omega_0)$. Furthermore, by the same arguments as in Subcase I.C, $b_1 - \omega_2 \notin \mathscr{O}^\Delta(\omega_0)$. Hence (7.15) proved in Subcase II.A stays valid in this case and by (5.4) and (5.5), $r_y(\omega) \neq \infty$ for all ω with $\Im\omega = 0$ and $\omega_{y_4} \leqslant \Re\omega \leqslant \omega_{y_4} + \omega_2$, except for $\omega = a_1, a_2$. By the identity (5.3), $r_x(\omega) \neq \infty$ for all ω with $\Im\omega = 0$ and

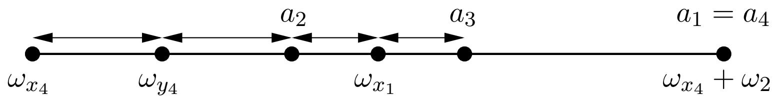


FIGURE 15. Location of a_1, a_2, a_3, a_4 if $y_4 > 0$, $x_4 = \infty$, $Y(x_4) = \infty$, i.e., Subcase II.D

$\omega_{x_1} \leqslant \Re\omega \leqslant \omega_{x_4} + \omega_2$, except for points ω where $x(\omega) = \infty$, namely $\omega = b_2$. This concludes the proof of Proposition 22 and proves in particular that $r_y(\omega_0) \neq \infty$ for any $\omega_0 \in \{a_3, a_4, b_1, b_2\}$.

Using $x^\star \neq x^\bigstar$, we verify as in Subcase I.A that $\lim_{\omega\to a_4}\{f_y(\omega) + f_y(\omega+\omega_3)\} = \infty$ and $\lim_{\omega\to a_3}\{f_y(\omega)+f_y(\omega+\omega_3)\} = \infty$. If $x_4 > 0$, since $y^\circ \neq y^\bullet$, we verify as in Subcase I.A that $\lim_{\omega\to b_1} f_y(\omega) = \infty$ and $\lim_{\omega\to b_2} f_y(\omega) = \infty$. If $x_4 = \infty$, then $b_1 = b_2$ and we verify as in Subcase I.B that $\lim_{\omega\to b_1} f_y(\omega) = \infty$. If a_3, a_4, b_1, b_2 are ordered (e.g., $a_3 \ll b_1 \ll a_4 \ll b_2$, then immediately $a_3 \ll_1 a_1 \ll b_1 \ll a_4 \ll_1 a_2 \ll b_2$), there is a *maximal* point *in the sense of this order*. If the maximal element is b_ℓ for some $\ell \in \{1, 2\}$, then there is no $\omega \in \Pi_y$ with $f_y(\omega) = \infty$ such that $b_\ell \ll \omega$. If the maximal element is a_3 (resp. a_4), then there is no $\omega \in \Pi_y$ except for a_1 (resp. a_2) with $f_y(\omega) = \infty$ such that $a_3 \ll \omega$ (resp. $a_4 \ll \omega$). Lemma 21 applies with ω_0 equal this maximal element since all assumptions (A), (B) and (C) are satisfied. If a_3, a_4, b_1, b_2 are not all ordered, then it is enough to apply Lemma 21 to the maximal element of any ordered subset. Finally $\mathcal{I}_x(\omega_0) = \mathbf{R}\setminus]x_1, x_4[$ and $\mathcal{I}_y(\omega_0) = \mathbf{R}\setminus]y_4, y_1[$ for any $\omega_0 \in \{a_3, a_4, b_1, b_2\}$, hence the set of poles of $x \mapsto Q(x, 0)$ (resp. $y \mapsto Q(0, y)$) is dense on the announced intervals. Proposition 23 is proved.

Case II*:* $y_4 > 0$*, Subcase* II.D*:* $x_4 = \infty$ *and* $Y(x_4) = \infty$*.* In this case $x^\star = \infty$; $y^\star = \infty$; $x^\bigstar, y^\bigstar \neq \infty$.

We have $a_1 = \omega_{x_4} + \omega_2$, and $a_2 = \widehat{\eta}a_1 - \omega_1 = \omega_{x_4} + \omega_3$ is symmetric to $a_1 - \omega_2$ w.r.t. ω_{y_4}. Further, since $y_4 > 0$ and $Y(x_1) \leqslant 0$, $a_2 \in]\omega_{y_4}, \omega_{x_1}[$. Then $a_3 = \widehat{\xi}a_2 - \omega_1 = \omega_{x_4} + \omega_2 - \omega_3$ is symmetric to a_2 w.r.t. ω_{x_1}. Finally, $a_4 = \omega_{x_4} + \omega_2 = a_1$, $b_1 = b_2 = a_1$, $a_3 + \omega_3 = a_1$ and $a_1 + \omega_3 = a_2 + \omega_2$, so that $a_3 \ll_1 a_1 \ll_1 a_2$, see Figure 15.

We prove (7.15). For this purpose, we first show that $r_y(a_3) \neq \infty$. If $a_3 \in \Delta$, $x^\bigstar, y^\bigstar \neq \infty$, then by Theorem 3, $r_y(a_3) \neq \infty$ If $a_3 \notin \Delta$, consider $\mathscr{O}^\Delta(\omega_0)$. Since $a_3+2\omega_3 = a_2+\omega_2$, $a_2 \notin \mathscr{O}^\Delta(\omega_0)$ by the irrationality of ω_2/ω_3. Obviously $a_1-\omega_2 = \omega_{x_4} \notin \Delta$ and then it is not in $\mathscr{O}^\Delta(\omega_0)$. Hence $r_y(a_3) \neq \infty$. Next we prove that $r_y(a_2)+f_y(a_2)$ exactly as in Subcase II.A by using that $x^\bigstar y^\bigstar \neq \infty$.

For any $\omega_0 \neq a_1$ with $\Im\omega_0 = 0$ and $\omega_{y_1} \leqslant \omega_0 \leqslant \omega_{y_4}+\omega_2$, the orbit $\mathscr{O}^\Delta(\omega_0)$ cannot contain a_3, since $a_3+\omega_3 = a_1$ and $a_3+2\omega_3 > \omega_0$. It can neither contain a_1, since $\omega_0-\omega_3 < a_1$, nor obviously $a_1 - \omega_2 = \omega_{x_4}$. If it does not contain a_2, then by (7.12), $r_y(\omega_0) \neq \infty$. If it does, then exactly as in Subcase II.A, using $r_y(a_2) + f_y(a_2) \neq \infty$, we prove that $r_y(\omega_0) \neq \infty$ as well. This finishes the proof of (7.15).

By Equations (5.4), (5.5) and (5.3), we derive as in Subcase II.A that $r_x(\omega_0) \neq \infty$ for all $\omega_0 \in \{\omega : \Im\omega = 0,\ \omega_{x_1} \leqslant \omega_0 \leqslant \omega_{x_4} + \omega_2\}$ except for ω_0 where $x(\omega_0) = \infty$, that is for a_1. This finishes the proof of Proposition 22 in this case.

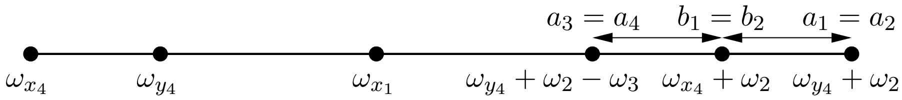


FIGURE 16. Location of $a_1, a_2, a_3, a_4, b_1, b_2$ if $y_4 = \infty$, Case III

To prove Proposition 23, we would like to apply Lemma 21 with $\omega_0 = a_3$. We have shown that $r_y(a_3) \neq \infty$, $a_3 \ll_1 a_1 = a_4 = b_1 = b_2 \ll a_2$, so that there is no $\omega \in \Pi_y \setminus \{a_1, a_2\}$ such that $a_3 \ll \omega$ and $f_y(\omega) = \infty$. It remains to verify assumption (B) of Lemma 21 for $\omega_0 = a_3$, that is that $f_y(\omega) + f_y(\omega+\omega_3) + f_y(\omega+2\omega_3)$ converges to infinity if $\omega \to a_3$. The last quantity is the sum of

$$x(\omega)[y(\widehat{\xi}\omega) - y(\omega)] + x(\widehat{\eta}\widehat{\xi}\omega)[y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega) - y(\widehat{\eta}\widehat{\xi}\omega)] + x(\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)[y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega) - y(\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)],$$

which equals

$$x(\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega) - x(\omega)y(\omega) + x(\widehat{\eta}\widehat{\xi}\omega)[y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega) - y(\widehat{\eta}\widehat{\xi}\omega)] + x(\omega)y(\widehat{\xi}\omega) - x(\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega) \tag{7.18}$$

where we used (3.7). If $\omega \to a_3$, then the first term in this sum converges to $x^\star y^\star - x^\star y^\star = 0$. Next, $\widehat{\eta}\widehat{\xi}\omega \to a_1$, so that $x(\widehat{\eta}\widehat{\xi}\omega) \to \infty$. We can also compute the values of $y(\widehat{\xi}\omega)$ and $y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)$ as $(-b(x) \pm [b(x)^2 - 4a(x)c(x)]^{1/2})/(2a(x))$ with $x = x(\widehat{\eta}\widehat{\xi}\omega)$. Since for all of the 9 models composing Subcase II.D, $\deg a = \deg b = 1$ and $\deg c = 2$, then $y(\widehat{\xi}\omega)$ and $y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)$ are of order $O(|x(\widehat{\eta}\widehat{\xi}\omega)|^{1/2})$, and their difference $|y(\widehat{\xi}\omega) - y(\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega)|$ is not smaller than $O(|x(\widehat{\eta}\widehat{\xi}\omega)|^{1/2})$ as $\omega \to a_3$. Finally, $x(\omega), x(\widehat{\eta}\widehat{\xi}\widehat{\eta}\widehat{\xi}\omega) \to x^\star \neq \infty$ as $\omega \to a_3$. Then as $\omega \to a_3$ in the sum (7.18) the second term is of the order not smaller than $O(|x(\widehat{\eta}\widehat{\xi}\omega)|^{3/2})$ while the first vanishes and the third has the order $O(|x(\widehat{\eta}\widehat{\xi}\omega)|^{1/2})$. This proves the assumption (B) of Lemma 21 for $\omega_0 = a_3$. By this lemma the poles of $x \mapsto Q(x,0)$ and $y \mapsto Q(0,y)$ are dense on the intervals of the real line, as announced in the proposition.

Case III*:* $y_4 = \infty$*.* It remains here exactly one case to study, see Figure 17. It is such that $y_4 = \infty$, $x_4 = \infty$ and $X(y_4) \neq \infty$. Then $x^\star = x^\bigstar \neq \infty$; $y^\circ = y^\bullet \neq \infty$; $y^\star = y^\bigstar \neq \infty$.

The points $b_1 = b_2 = \omega_{x_4} + \omega_2$ are located as in Subcase I.B, $a_1 = a_2 = \omega_{y_4} + \omega_2$ and $a_3 = a_4 = \omega_{y_4} + \omega_2 - \omega_3$. In particular, $a_3 + \omega_3/2 = b_1$, $b_1 + \omega_3/2 = a_1$, see Figure 16.

We start by showing that $r_y(a_3) \neq \infty$. If $a_3 \in \Delta$, this is true thanks to (5.3) and since $x^\star, y^\star \neq \infty$. If $a_3 \notin \Delta$, consider the orbit $\mathscr{O}^\Delta(a_3)$. It cannot contain $a_1 - \omega_2$ since $a_3 + \omega_3 = a_1$, neither b_1, nor $b_1 - \omega_2 = \omega_{x_4}$. It follows that $r_y(a_3) \neq \infty$.

Since $x^\star, y^\star \neq \infty$, it follows from Equations (5.3), (5.4) and (5.5) that $r_x(a_1 - \omega_2) = r_x(\widehat{\xi}(a_1 - \omega_2)) = r_x(\widehat{\xi}(a_1 - \omega_2) - \omega_1) = r_x(a_3) \neq \infty$. Then, by (5.3), $r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) = -r_x(a_1 - \omega_2) + K(0,0)Q(0,0) + x(a_1 - \omega_2)y(\widehat{\xi}(a_1 - \omega_2)) \neq \infty$.

Take any ω_0 with $\Im\omega_0 = 0$ and $\omega_{y_1} \leqslant \omega_0 < \omega_{y_4} + \omega_2$. If $\omega_0 \in \Delta$, then by the same arguments as in Subcase I.A, $\omega_0 \in \Delta_y$, so that $r_y(\omega_0) \neq \infty$. Otherwise, we notice that $\omega_0 - \omega_3 < a_3$, so that no point—and in particular b_1—of $[\omega_{y_4} + \omega_2 - \omega_3, \omega_{y_4} + \omega_2[$ belongs to $\mathscr{O}^\Delta(\omega_0)$. Clearly $b_1 - \omega_2 = \omega_{x_4} \notin \mathscr{O}^\Delta(\omega_0)$. Since either $a_1 - \omega_2 \notin \mathscr{O}^\Delta(\omega_0)$ or $a_1 - \omega_2 \in \mathscr{O}^\Delta(\omega_0)$ but $r_y(a_1 - \omega_2) + f_y(a_1 - \omega_2) \neq \infty$, and by the same reasoning as in

Subcase I.A, we derive that $r_y(\omega_0) \neq \infty$. The rest of the proof of Proposition 22 in this case goes along the same lines as in Case II.

Now note that b_1 is not ordered with a_1 and a_3. Indeed, since $b_1 + \omega_3/2 = a_1$ and $b_1 - \omega_3/2 = a_3$, this would contradict the irrationality of ω_2/ω_3. Then there is no $\omega \in \Pi_y$ such that $b_1 \ll \omega$ and $f_y(\omega) = \infty$. We also have $f_y(\omega) = x(\omega)[y(\widehat{\xi}\omega) - y(\omega)] = x(\omega)[b^2(x(\omega)) - 4a(x(\omega))c(x(\omega))]^{1/2}/a(x(\omega))$. If $\omega \to b_1$, then $x(\omega) \to \infty$ and since $\deg a = 2$ and $\deg b = \deg c = 1$, we have $f_y(\omega) \to \infty$. Lemma 21 applies with $\omega_0 = b_1$ and proves Proposition 23.

7.4. **Asymptotic of $\omega_2(\omega)/\omega_3(\omega)$ as $z \to 0$.** It remains to prove the following result announced at the beginning of Section 7.

Proposition 25. *For any of* 51 *non-singular walks having an infinite group, there exist a rational constant $L > 0$ and a constant $\widetilde{L} \neq 0$ such that*

$$\omega_2/\omega_3 = L + \widetilde{L}/\ln(z) + O(1/\ln(z))^2. \tag{7.19}$$

Proof. In order to prove (7.19), we shall use expressions of the periods ω_2 and ω_3 different from that given in (3.1) and (3.2). To that purpose, define the complete and incomplete elliptic integrals of the first kind by, respectively,

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{[1-t^2]^{1/2}[1-k^2t^2]^{1/2}}, \tag{7.20}$$

$$F(w,k) = \int_0^w \frac{\mathrm{d}t}{[1-t^2]^{1/2}[1-k^2t^2]^{1/2}}. \tag{7.21}$$

Then the new expressions of ω_2 and ω_3 are

$$\omega_2 = M\Omega_2, \qquad \omega_3 = M\Omega_3, \tag{7.22}$$

where

$$\Omega_2 = K\left(\sqrt{\frac{(x_4-x_1)(x_3-x_2)}{(x_4-x_2)(x_3-x_1)}}\right), \tag{7.23}$$

$$\Omega_3 = F\left(\sqrt{\frac{(x_4-x_2)(x_1-X(y_1))}{(x_4-x_1)(x_2-X(y_1))}}, \sqrt{\frac{(x_4-x_1)(x_3-x_2)}{(x_4-x_2)(x_3-x_1)}}\right), \tag{7.24}$$

and where (below, $\mathbf{1}_{(i,j)} = 1$ if $(i,j) \in \mathcal{S}$, otherwise 0)

$$M = \begin{cases} \dfrac{2}{z}\dfrac{1}{\sqrt{(\mathbf{1}_{(1,0)} - 4\mathbf{1}_{(1,1)}\mathbf{1}_{(1,-1)})(x_3x_4 - x_2x_3 - x_1x_4 + x_1x_2)}} & \text{if } x_4 \neq \infty, \\ \dfrac{2}{\sqrt{(2z\mathbf{1}_{(1,0)} + 4z^2[\mathbf{1}_{(1,1)}\mathbf{1}_{(0,-1)} + \mathbf{1}_{(1,-1)}\mathbf{1}_{(0,1)}])(x_3 - x_1)}} & \text{if } x_4 = \infty. \end{cases} \tag{7.25}$$

The expressions of ω_2 and ω_3 written in (7.22), (7.23), (7.24) and (7.25) are obtained from (3.1) and (3.2) by making simple changes of variables.

We are now in position to analyze the behavior of ω_2/ω_3 (or equivalently, thanks to (7.22), that of Ω_2/Ω_3) in the neighborhood of $z=0$. First, with (2.2) and [23, Proposition 6.1.8], we obtain that as $z\to 0$, $x_1,x_2\to 0$ and $x_3,x_4\to\infty$. For this reason,

$$k=\sqrt{\frac{(x_4-x_1)(x_3-x_2)}{(x_4-x_2)(x_3-x_1)}}\to 1. \tag{7.26}$$

The behavior of $X(y_1)$ as $z\to 0$ is not so simple as that of the branch points x_ℓ (indeed, as $z\to 0$, $X(y_1)$ can converge to 0, to ∞ or to some non-zero constant), but we can show that for all 51 models,

$$w=\sqrt{\frac{(x_4-x_2)(x_1-X(y_1))}{(x_4-x_1)(x_2-X(y_1))}}\to 1. \tag{7.27}$$

Due to (7.26) and (7.27), in order to determine the behavior of Ω_2/Ω_3 near $z=0$ it suffices to know

(i) the expansion of $K(k)$ as $k\to 1$;
(ii) the expansion of $F(w,k)$ as $k\to 1$ and $w\to 1$.

Point (i) is classical, and is known as Abel's identity (it can be found, e.g., in [11]): there exist two functions A and B, holomorphic at $z=0$, such that $K(k)=A(k)+\ln(1-k)B(k)$. Both A and B can be computed in an explicit way, see [11], and from all this we can deduce an expansion of $K(k)$ as $k\to 1$ up to any level of precision. For our purpose, it will be enough to use the following:

$$\begin{aligned}A(k)&=(3/2)\ln(2)+((k-1)/4)(1-3\ln(2))+O(k-1)^2,\\ B(k)&=-1/2+(k-1)/4+O(k-1)^2.\end{aligned}$$

As for Point (ii), we proceed as follows. We have $F(w,k)=K(k)-\widetilde{F}(w,k)$, with

$$\widetilde{F}(w,k)=\int_w^1\frac{\mathrm{d}t}{[1-t^2]^{1/2}[1-k^2t^2]^{1/2}}.$$

Then, introduce the expansion $1/([1+t]^{1/2}[1+kt]^{1/2})=\sum_{\ell=0}^\infty\mu_\ell(k)(1-t)^\ell$, so that

$$\widetilde{F}(w,k)=\sum_{\ell=0}^\infty\mu_\ell(k)\int_w^1\frac{\mathrm{d}t}{[1-t]^{1/2-\ell}[1-kt]^{1/2}}. \tag{7.28}$$

In Equation (7.28), all $\mu_\ell(k)$ as well as all integrals can be computed. As an example (that we shall use), we have

$$\begin{aligned}\mu_0(k)\int_w^1\frac{\mathrm{d}t}{[1-t]^{1/2}[1-kt]^{1/2}}&=\frac{1}{[2k(1+k)]^{1/2}}\times\\ &\times[\ln\{(1-k)/k^{1/2}\}-\ln\{-[1-(1+k)w+kw^2]^{1/2}+[(k+1)/2-kw]/k^{1/2}\}].\end{aligned}$$

Moreover, it should be noticed that as $k\to 1$ and $w\to 1$, the speed of convergence to zero of the integrals in (7.28) increases with ℓ. This way, we can write an expansion of $\widetilde{F}(w,k)$—and thus of $F(w,k)$—up to any level of precision.

Unfortunately, the end of the proof cannot be done simultaneously for all 51 models, but should be done model by model. For the sake of shortness, we choose to present the details only for one model, namely for the model with $\mathcal{S} = \{(-1,0),(-1,1),(0,1),(1,-1)\}$ (which belongs to Subcase II.D of Figure 17). For this model, we easily obtain from (2.2) that

$$\begin{aligned} x_1 &= z - 2z^2 + 3z^3 + O(z^4),\\ x_2 &= z + 2z^2 + 5z^3 + O(z^4),\\ x_3 &= 1/(4z^2) - 1 - 2z - 8z^3 + O(z^4),\\ x_4 &= \infty,\\ X(y_1) &= 0. \end{aligned}$$

Then, with (7.26) and (7.27), we reach the conclusion that

$$k = 1 - 8z^4 - 4z^5 + O(z^6), \tag{7.29}$$

$$w = 1 - 2z + z^2 - (7/4)z^3 - (65/8)z^4 + (613/64)z^5 + O(z^6). \tag{7.30}$$

Then, using Points (i) and (ii) above, we obtain

$$\begin{aligned} \Omega_2 &= -2\ln(z) - (1/4)z + (1/16)z^2 + O(z^3\ln(z)),\\ \Omega_3 &= -(1/2)\ln(z) - (1/2)\ln(2) + (1/4)z + (57/16)z^2 + O(z^3\ln(z)), \end{aligned}$$

so that

$$\Omega_2/\Omega_3 = 4 - 4\ln(2)/\ln(z) + O(1/(\ln(z))^2). \tag{7.31}$$

The latter proves (7.19), and thus Proposition 25, with $L = 4$ and $\widetilde{L} = -4\ln(2)$.

Making expansions of higher order of k and w in (7.29) and (7.30), we could obtain more terms in the expansion (7.31) of Ω_2/Ω_3. A contrario, we could also be interested in obtaining the first term only (the constant term L) in (7.31). (Indeed, we saw in the proof of Proposition 14 that it was sufficient for our purpose, i.e., for proving that in the infinite group case, the ratio ω_2/ω_3 is not constant in z.) To that aim, instead of (7.29) and (7.30), we just need two-terms expansions of k and w, say $k = 1 + \alpha z^p + o(z^p)$ and $w = 1 + \beta z^q + o(z^q)$, with $\alpha, \beta \neq 0$. Then with (i) and (ii) we deduce that $\Omega_2 = -(p/2)\ln(z) + o(\ln(z))$ and $\Omega_3 = -(q/2)\ln(z) + o(\ln(z))$, in such a way that $L = p/q$, which obviously is (non-zero and) rational.

□

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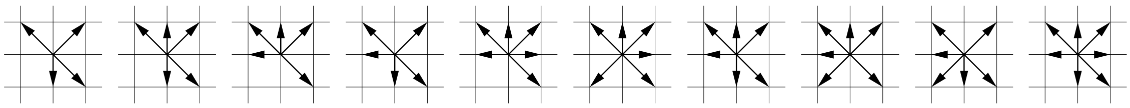
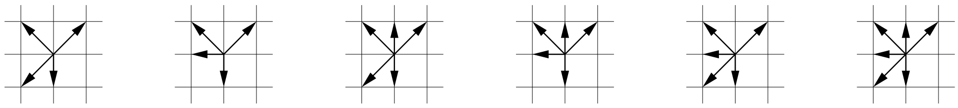
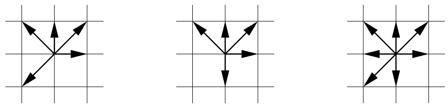
(Case I: $y_4 < 0$, Subcase I.A: $x_4 < 0$)



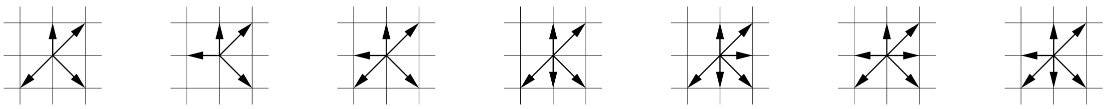
(Case I: $y_4 < 0$, Subcase I.B: $x_4 = \infty$)



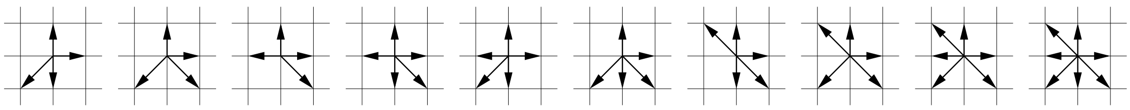
(Case I: $y_4 < 0$, Subcase I.C: $x_4 > 0$)



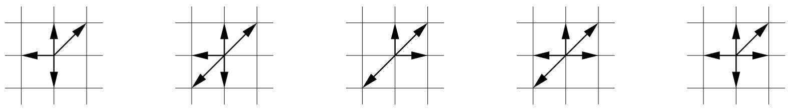
(Case II: $y_4 > 0$, Subcase II.A: $x_4 < 0$)



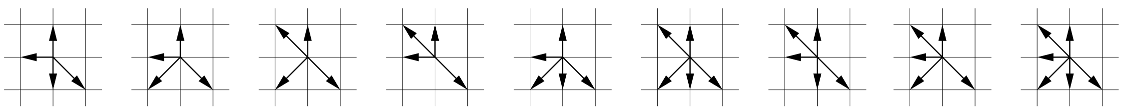
(Case II: $y_4 > 0$, Subcase II.B: $x_4 > 0$, one of $y^\circ, y^\bullet$ is ∞)



(Case II: $y_4 > 0$, Subcase II.C: $x_4 = \infty$ and $Y(x_4) \neq \infty$ as well as $x_4 > 0$, $y^\circ, y^\bullet \neq \infty$)



(Case II: $y_4 > 0$, Subcase II.D: $x_4 = \infty$ and $Y(x_4) = \infty$)



(Case III: $y_4 = \infty$)

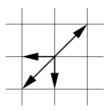


FIGURE 17. Different cases considered in the proof of Theorem 17—they correspond to the 51 non-singular walks with infinite group, see [5]

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