Tiling Problems

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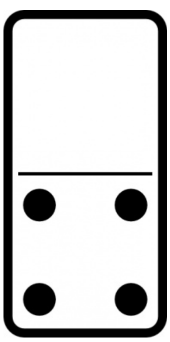
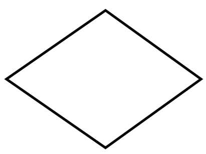
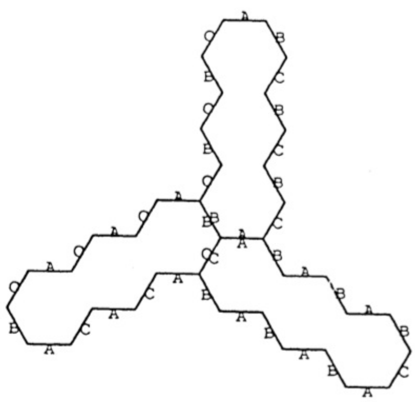
Introduction

Tiles:





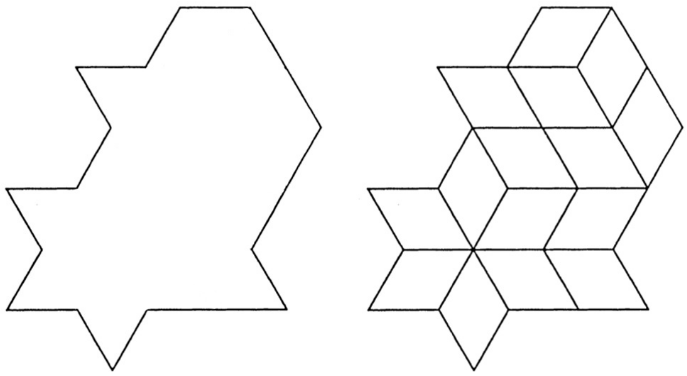


Definition (Tile)

A **tile** *is a (closed) plane polygon.*

Introduction (cont.)

Tiling:



Definition (Region)

A **region** *is a (closed) plane polygon.*

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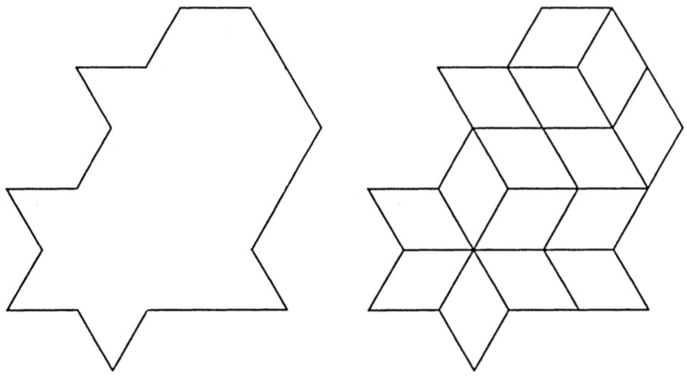
Introduction (cont.)

Tiling:



Definition (Tiling)

A tiling of a region R is a decomposition of R into tiles, $R = \bigcup_i T_i$, such that if x is a point in the interior of a tile T_i, then it is not contained in any T_j for $j \neq i$.

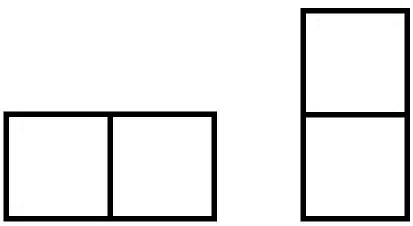
Introduction (cont.)

Tiling Problems:

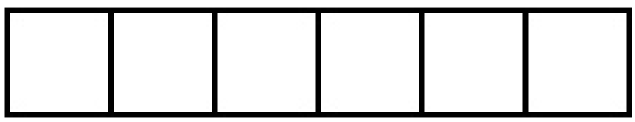
- Counting Problems: How many ways are there to tile a region with a fixed set of tiles?
- Feasibility Problems: Can a region be tiled with a fixed set of tiles?
- Tiling problems are hard:
  - Counting is $\#P$-complete.
  - Feasibility of tiling bounded regions is NP-complete.
  - Given a set of tiles, can one tile the plane with them? This is undecidable.

Counting Problems (warmup)

Tiles: 2×1 rectangles (dominos):

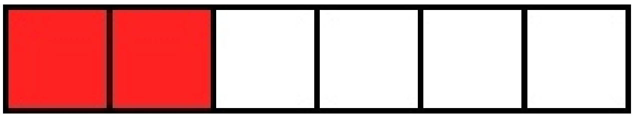


Region: $1 \times n$ grid.



Counting Problems (warmup)

Let T_n be the number of tilings of a $1 \times n$ grid by dominos.



$$T_n = T_{n-2}$$

$$T_1 = 0, T_2 = 1$$

$$T_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

We'll return to this example.

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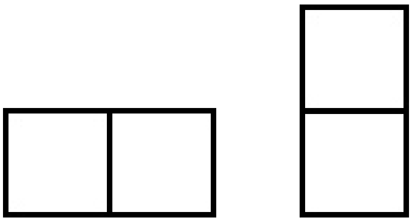
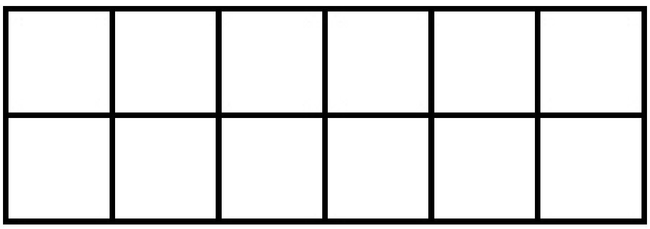
Counting Problems (trickier example)

Tiles: 2×1 rectangles (dominos):



Region: $2 \times n$ grid.



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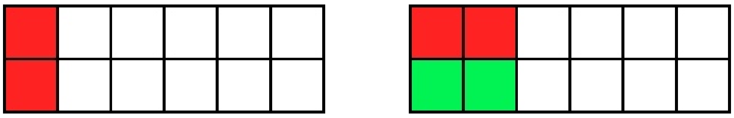
Counting Problems (tricker example)

Let T_n be the number of tilings of a $2 \times n$ grid by dominos. There are two ways to cover the leftmost column:



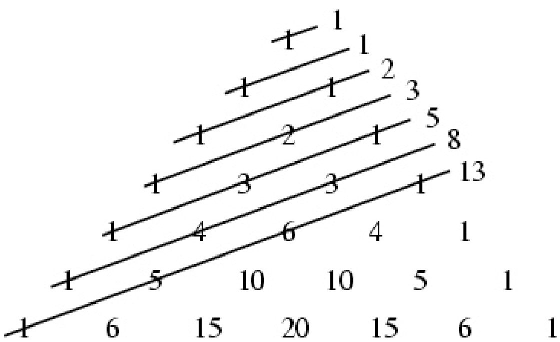
$$T_n = T_{n-1} + T_{n-2}$$

$$T_1 = 1, T_2 = 2$$

Fibonacci numbers!

Applications (Fibonacci identities)

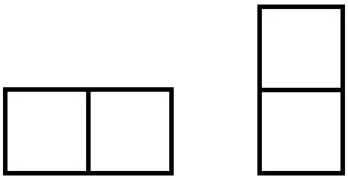
- $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots = f_n$:



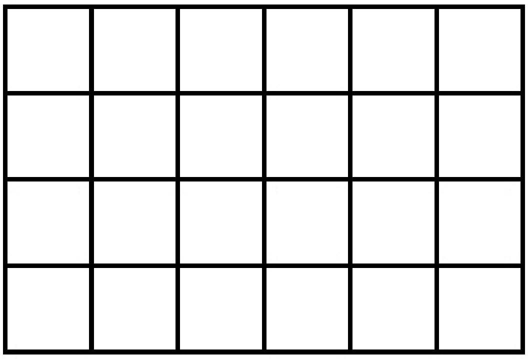
- $\sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} = f_{2n+1}$
- For $m \geq 1, n \geq 0$, if $m|n$ then $f_{m-1}|f_{n-1}$.
- $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$

Counting Problems (dominos, $m \times n$ case)

Tiles: 2×1 rectangles (dominos):



Region: $m \times n$ grid.



Counting Problems (dominos, $m \times n$ case)

Let $T_{m,n}$ be the number of tilings of a $m \times n$ grid by dominos. As you might expect:

$$T_{m,n} = 2^{mn/2} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{\pi j}{m+1} + \cos^2 \frac{\pi k}{n+1} \right)^{1/4}$$

Kasteleyn (1961)
Q: How does one prove this?
A: Pfaffians!

Matrices and Counting

Let $A = a_{ij}$ be a $2n \times 2n$ skew-symmetric matrix, i.e. $a_{ij} = -a_{ji}$.

Definition (Pfaffian)

Let Π be the set of partitions of $\{1, 2, ..., 2n\}$ into pairs

$$\alpha = \{(i_1, j_1), (i_2, j_2), ..., (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < i_3 < \cdots < i_n$. The **Pfaffian** *of A is defined to be*

$$\mathrm{pf}(A) = \sum_{\alpha \in \Pi} \mathrm{sign}(\alpha) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}.$$

Theorem

$$\mathrm{pf}(A) = \pm\sqrt{\det(A)}$$

Matrices and Counting

$$\mathrm{pf}(A) = \sum_{\alpha \in \Pi} \mathrm{sign}(\alpha) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}.$$

Label the squares of a grid from 1 to mn. If square i is next to square j, let $a_{ij} = \pm 1$, with $a_{ji} = -a_{ij}$. Let $a_{ij} = 0$ otherwise.

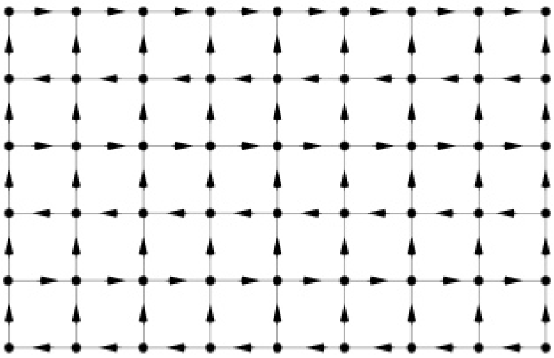
$$|a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}| = 1 \iff (i_1, j_1), \cdots, (i_n, j_n) \text{ is a tiling!}$$

Have to pick signs right.

Remark

If A is the adjacency matrix of an oriented graph, $\mathrm{pf}(A)$ *counts oriented perfect matchings.*

Counting Problems (dominos, $m \times n$ case)



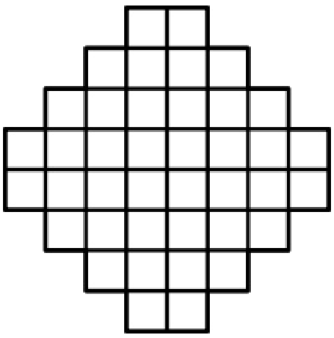
Let $a_{ij} = 1$ if there is an edge $i \to j$, with $a_{ij} = -a_{ji}$. Let $a_{ij} = 0$ otherwise. Then $\text{sign}(\alpha) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}$ is always positive!
So $T_{m,n} = \text{sqrt}(\det(A))$. Compute by diagonalizing A.

Counting Problems (dominos)

Remark

In general, any planar graph has a "Pfaffian Orientation," which makes the above argument work.

But this isn't the end of the story. Consider the aztec diamond:



Number of tilings is $2^{(n+1)n/2}$. But add one more row in the middle, and the number of tilings only grows exponentially.

Feasibility Problems

- Special case of counting problems (Is the number of tilings equal to zero?)
- But still hard, even if we can count: Given a sequence $(x_n$ defined via an integer linear recurrence, is the truth of the statement "$x_n \neq 0$ for all n" decidable in finite time? This is an **open problem**.
- Given a set of tiles, can they tile the plane? This is undecidable.

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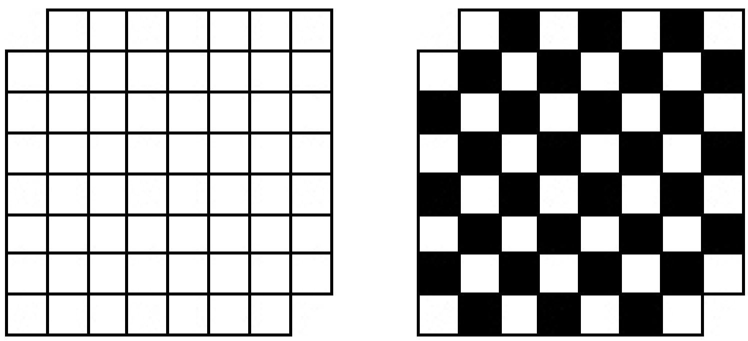
A Classical Example

Can this region be tiled by dominos?



Each domino covers exactly one black square and one white square; but there are more white squares than black squares.

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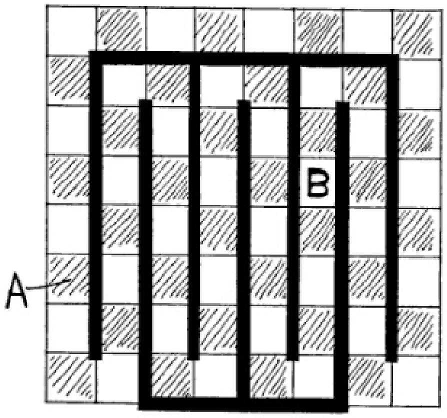
A Classical Example

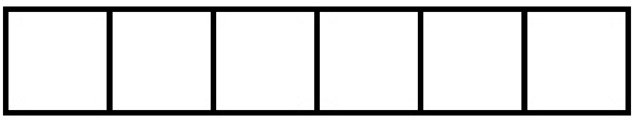
But is this the only obstruction? What if we remove two squares of different colors?



(Gomory)
Of course, if we remove more than 2 squares, a lot can go wrong.

Rectangle Tilings (Toy Example)

Let's return to tiling a $1 \times n$ rectangle R_n by dominos.



$1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5$

$p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$
$d(x) = 1 + x$
Note that $d(-1) = 0$.

$$p_{2n}(-1) = 0.$$

$$p_{2n}(x) = (1 + x^2 + x^4 + \cdots + x^{2n-2})(1 + x)$$

But $p_{2n+1}(x)$ is not a multiple of $d(x)$:

$$p_{2n+1}(-1) = 1$$

Rectangle Tilings

Label the upper-right quadrant of the plane as follows:

$$
\begin{array}{|c|c|c|c|c}
\vdots & \vdots & \vdots & \vdots & \\
\hline
y^2 & xy^2 & x^2y^2 & x^3y^2 & \cdots \\
\hline
y & xy & x^2y & x^3y & \cdots \\
\hline
1 & x & x^2 & x^3 & \cdots \\
\hline
\end{array}
$$

If R is a region consisting of unit squares (α, β) with non-negative integer coordinates, let

$$p_R(x, y) = \sum_{(\alpha,\beta)\in R} x^\alpha y^\beta.$$

If T_i are tiles made from unit squares, translate them so one square is at the origin, and let

$$p_{T_i}(x, y) = \sum_{(\alpha,\beta)\in T_i} x^\alpha y^\beta.$$

Rectangle Tilings

If R may be tiled by the T_i then there exist polynomials $a_i(x, y)$ with integer coefficients such that

$$p_R(x, y) = \sum_i a_i(x, y) p_{T_i}(x, y).$$

Definition

If there exist polynomials $a_i(x, y)$ with coefficients in a ring k, such that

$$p_R(x, y) = \sum_i a_i(x, y) p_{T_i}(x, y),$$

we say that the T_i can tile R over k.

Rectangle Tilings

Theorem

Let T_i be a (possibly infinite) set of tiles. Then there exists a finite subset T_{i_j} such that a region R may be tiled by the T_i over the integers if and only if it may be tiled by the T_{i_j}.

Proof.

Hilbert Basis Theorem. □

Rectangle Tilings

Let $k = \mathbb{C}$, the complex numbers. Let $V \subset \mathbb{C}^2$ be the set

$$V = \{(x, y) \mid T_i(x, y) = 0 \text{ for all } i\}.$$

If

$$p_R(x, y) = \sum_i a_i(x, y) p_{T_i}(x, y), \qquad (*)$$

then $p_R(x, y) = 0$ if $(x, y) \in V$.

Theorem

*Let I_T be the set of all polynomials that can be written as in $(*)$. If I_T is **radical**, and $p_R(x, y) = 0$ for all $(x, y) \in V$, then the T_i may tile R over $\mathbb{C}$.*

Rectangle Tilings

Theorem

*Let I_T be the set of all polynomials that can be written as in $(*)$. If I_T is **radical**, and $p_R(x,y)=0$ for all $(x,y)\in V$, then the T_i tile R over $\mathbb{C}$.*

Proof.

Nullstellensatz. ☐

Theorem (Barnes)

Let T be a finite set of rectangular tiles, and R a rectangular region. There exists a constant K such that if the lengths of the sides of R are greater than K, then R is tileable by T if and only if it is tileable over $\mathbb{C}$.

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Rectangle Tilings

Theorem

*Let I_T be the set of all polynomials that can be written as in $(*)$. If I_T is **radical**, and $p_R(x,y)=0$ for all $(x,y)\in V$, then the T_i may tile R over $\mathbb{C}$.*

Example

If $T=(1+x,1+y)$, then I_T is radical. So one may detect domino tilings over $\mathbb{C}$ by evaluating $p_R(-1,-1)$.

$$\begin{array}{|c|c|c|c|c}
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \\
\hline
+1 & -1 & +1 & +1 & \cdots \\
\hline
-1 & +1 & -1 & +1 & \cdots \\
\hline
+1 & -1 & +1 & +1 & \cdots \\
\hline
\end{array}$$

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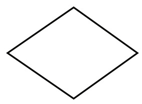
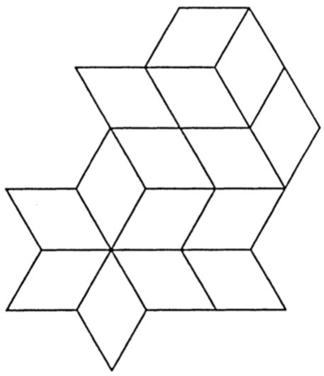
Lozenge Tilings

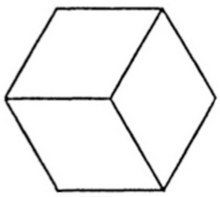
This is a lozenge:



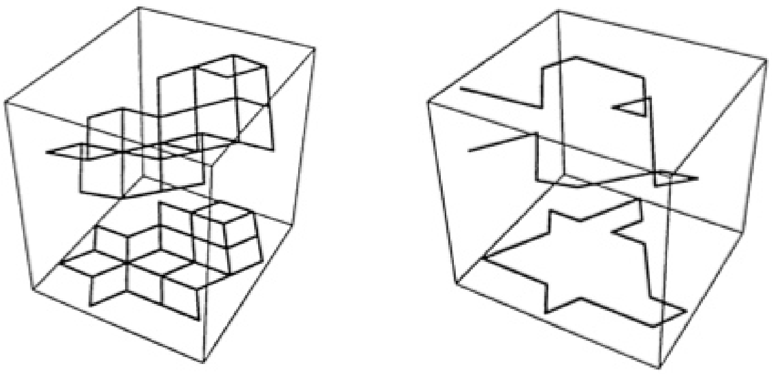
This is a lozenge tiling:

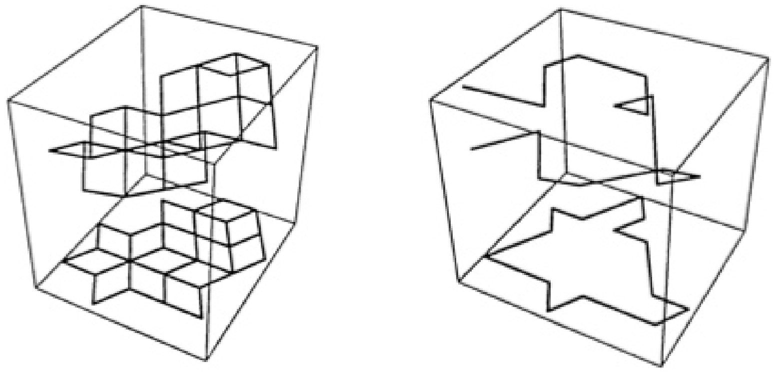


Squint a little:



Pick a direction for each edge. Does the outline of a region lift to a loop?





This condition is necessary but not sufficient; there is a sufficient geometric condition (Conway, Thurston).
This method generalizes, but is difficult to analyze except in special cases.

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Open Problems

- Can the plane be tiled by tiles with five-fold symmetry? (Kepler)
- Strengthen the algebra-geometric methods of Barnes.
- Find more sensitive obstructions to tiling.
- Characterize when coloring arguments forbid tilings.
- And more...

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