Demostración teorema pag 17, i) $\Rightarrow$

· $W \underset{SE}{\subseteq} V$, $v \in V$, $\bar{w}$ es mejor aproximación de v por vectores de $W \Longleftrightarrow v - \bar{w}$ es ortogonal a W

$$\cancel{v} - u = (v - \bar{w}) + (\bar{w} - u)$$

$$\|v - u\|^2 = \langle (v - \bar{w}) + (\bar{w} - u), (v - \bar{w}) + (\bar{w} - u) \rangle = \|v - \bar{w}\|^2 + \|\bar{w} - u\|^2 + 2 R_e \langle v - \bar{w}, \bar{w} - u \rangle$$

Si $\bar{w}$ es una m.a. a v en W $\quad \|v - \bar{w}\|^2 \leq \|v - u\|^2 \; \forall u \in W$

$$0 \leq \|v - u\|^2 - \|v - \bar{w}\|^2$$

$$0 \leq \|\bar{w} - u\|^2 + 2 R_e \langle v - \cancel{w}, \bar{w} - u \rangle \; \forall u \in W$$

$\bar{w} \in W$ fijo $u \in W$ es cualquiera $\bar{w} - u = z$

$$0 \leq 2 R_e \langle v - \bar{w}, z \rangle + \|z\|^2 \quad \forall z \in W$$

Queremos probar que $\langle v - \bar{w}, z \rangle = 0 \quad \forall z \in W$

$$\langle v - \bar{w}, \bar{w} - u \rangle = 0 \quad \forall u \in W$$

$u \neq \bar{w}$

$z = \alpha(\bar{w} - u)$

$$z = \underbrace{- \frac{\langle v - \bar{w}, \bar{w} - u \rangle}{\|\bar{w} - u\|^2}}_{\alpha \in K} (\bar{w} - u) \qquad u \in W$$

$$0 \le 2\,\mathrm{Re}\langle v-\bar{w}, z\rangle + \|z\|^2 \qquad \forall z \in W$$

en particular vale para $u \in W$, $u \neq \bar{w}$

$$z = -\frac{\langle v-\bar{w}, \bar{w}-u\rangle}{\|\bar{w}-u\|^2}(\bar{w}-u)$$

$$0 \le 2\,\mathrm{Re}\left\langle v-\bar{w}, -\frac{\langle v-\bar{w}, \bar{w}-u\rangle}{\|\bar{w}-u\|^2}(\bar{w}-u)\right\rangle + \left\|-\frac{\langle v-\bar{w}, \bar{w}-u\rangle}{\|\bar{w}-u\|^2}(\bar{w}-u)\right\|^2$$

$$0 \le 2\,\mathrm{Re} \qquad + \frac{|\langle v-\bar{w}, \bar{w}-u\rangle|^2 \, \cancel{\|\bar{w}-u\|^2}}{\|\bar{w}-u\|^{\cancel{4}\,2}}$$

$$2\,\mathrm{Re}\left(-\overline{\frac{\langle v-\bar{w}, \bar{w}-u\rangle}{\|\bar{w}-u\|^2}}\,\langle v-\bar{w}, \bar{w}-u\rangle\right)$$

$$0 \le -2a + a$$
$$a > 0$$

$$2\,\mathrm{Re}\left(-\frac{1}{\|\bar{w}-u\|^2}\,\overline{\langle v-\bar{w}, \bar{w}-u\rangle}\,\langle v-\bar{w}, \bar{w}, u\rangle\right)$$

$$2\,\mathrm{Re}\,\frac{-1}{\|\bar{w}-u\|^2}\,|\langle v-\bar{w}, \bar{w}-u\rangle|^2$$

$$0 \le -2\,\frac{|\langle v-\bar{w}, \bar{w}-u\rangle|^2}{\|\bar{w}-u\|^2} + \frac{|\langle v-\bar{w}, \bar{w}-u\rangle|^2}{\|\bar{w}-u\|^2} \qquad \cancel{u \neq \bar{w}}$$

$$\Rightarrow \langle v-\bar{w}, \bar{w}-u\rangle = 0 \qquad \forall u \in W$$

$\tau = \alpha(\dot{w} - u)$; buscamos el α tal que

$\langle \tau - \dot{w}, \dot{w} - u \rangle = 0$

<u>Ejercicio</u>.

Demostración corolario página 18

V e.v.p.i. $/\mathbb{K}$, $W \underset{sc}{\subseteq} V$, $\dim W$ finita $P_W : V \longrightarrow W$

$v \longmapsto P_W v = \bar{w} = \sum_{k=1}^{n} \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k$

$\{w_1, \dots, w_k\}$ base ortogonal de W

Entonces la aplicación

$$V \longrightarrow W^\perp$$
$$v \longmapsto v - P_W v = P_{W^\perp} v$$

$$\boxed{P_{W^\perp} = I - P_W}$$

es la proyección ortogonal de V sobre $W^\perp$

<u>Demos</u> $v \in V$, $v - \underbrace{P_W v}_{\bar{w}} \in W^\perp$ para cualquier $u \in W^\perp$

se tiene que $v - u = P_W v + (v - P_W v - u)$

como $P_W v \in W$ y $v - P_W v - u \in W^\perp$

$$\|v - u\|^2 = \|P_W v + (v - P_W v - u)\|^2 = \langle P_W v + (v - P_W v - u), P_W v + (v - P_W v - u)\rangle$$
$$= \|P_W v\|^2 + \|v - P_W v - u\|^2 \geq \|P_W v\|^2 = \|v - \underbrace{(v - P_W v)}\|^2$$

$\forall u \in W^\perp$

Q.E.D.

$$P_W : V \to W$$
$$v \mapsto P_W v = \sum_{k=1}^{n} \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k \quad , \{w_1, \dots, w_n\} \text{ base ortogonal de } W$$

P_W es una transformación lineal $\quad P_W \in \mathcal{L}(V)$

$$P_W(v+u) = \sum_{k=1}^{n} \frac{\langle v+u, w_k \rangle}{\|w_k\|^2} w_k = \sum_{k=1}^{n} \left(\frac{\langle v, w_k \rangle + \langle u, w_k \rangle}{\|w_k\|^2} \right) \cdot w_k$$

$$= \sum_{k=1}^{n} \left(\frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k + \frac{\langle u, w_k \rangle}{\|w_k\|^2} w_k \right)$$

$$= \sum_{k=1}^{n} \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k + \sum_{k=1}^{n} \frac{\langle u, w_k \rangle}{\|w_k\|^2} w_k = P_W v + P_W u$$

$$P_W(\alpha v) = \sum_{k=1}^{n} \frac{\langle \alpha v, w_k \rangle}{\|w_k\|^2} w_k = \alpha \sum_{k=1}^{n} \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k = \alpha P_W v$$

$$\therefore \quad P_W \in \mathcal{L}(V)$$

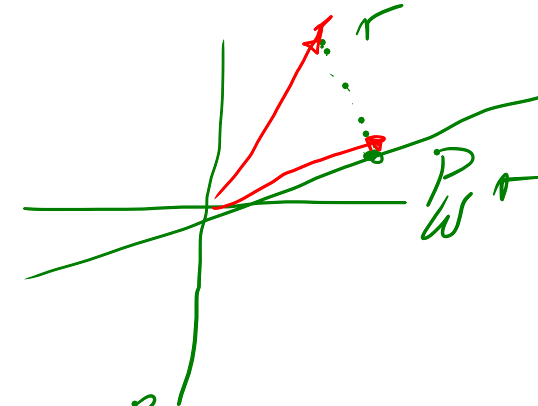
Demostración Teorema página 20

V e.v. s/ $\mathbb{K}$, $W \underset{se}{\subseteq} V$ dim W finita P_W es una transf. lineal idempotente

$W^\perp = \text{nul}\, P_W \qquad V = W \oplus W^\perp$

<u>Demos</u>: $v \in V$, $P_W v$ es la mejor aproximación de v por un vector de W $\quad z = P_W v \in W$

$$P_W(\underbrace{P_W v}_{\in W}) = P_W v$$



$$P_W^2 v = P_W(P_W v) = P_W(z) = z = P_W v \quad \forall v \in V$$

$\therefore \quad P_W^2 = P_W$ } P_W es idempotente

Sean $u, v \in V$, $\alpha \in K \qquad u - P_W u \in W^\perp \qquad v - P_W v \in W^\perp$

$$\alpha(v - P_W v) + (u - P_W u) = (\alpha v + u) - (\alpha P_W v + P_W u) \in W^\perp$$

$$\alpha P_W v + P_W u = P_W(\alpha v + u)$$

ii) $v \in V$ $\quad P_W v$ es el único vector de W tal que

$$v - P_W v \in W^\perp$$

Luego $P_W v = 0 \iff v \in W^\perp$

$$v \in \operatorname{nul} P_W \iff P_W v = 0 \iff v \in W^\perp$$

$$\therefore \quad \operatorname{nul} P_W = W^\perp$$

iii) $v \in V$ $\quad v = \underbrace{P_W v}_{\in W} + \underbrace{v - P_W v}_{\in W^\perp}$ $\qquad$ i) $V = W + W^\perp$

Pero ii) $W \cap W^\perp = \{0\}$ por i) y ii) $\therefore$ $V = W \oplus W^\perp$

QED

Demostración Corolario Desigualdad de Bessel, página 21

V e.p.i. $/\mathbb{K}$, $\{v_1, \ldots, v_n\}$ conjunto ortogonal de vectores no nulos

Sea $u \in V$ $\quad \underbrace{\|P_W u\|^2}_{} \overset{(*)}{=} \sum_{k=1}^{n} \frac{|\langle u, v_k\rangle|^2}{\|v_k\|^2} \leq \|u\|^2$

y vale el $=$ si y solo si $u = \underbrace{\sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k}_{P_{S.E(\{v_1, \ldots, v_k\})} u} = P_W u$

Demos: Sea $W = S.E(\{v_1, \ldots, v_n\})$

$$P_W u = \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k \qquad \underbrace{P_W u}_{\in W} + \underbrace{(u - P_W u)}_{\in W^\perp}$$

Por el teorema $V = W \oplus W^\perp$, luego $u = \underbrace{w}_{\in W} + \underbrace{z}_{\in W^\perp}$

$$\|u\|^2 = \langle u, u\rangle = \langle w+z, w+z\rangle = \|w\|^2 + \|z\|^2 \geq \|w\|^2$$
$$= \|P_W u\|^2 + \underbrace{\|z\|^2}_{=0} \geq \|P_W u\|^2$$

$$\circledast \quad \boxed{\|P_W u\|^2} = \left\| \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} v^k \right\|^2 = \left\langle \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k , \sum_{j=1}^{n} \frac{\langle u, v_j\rangle}{\|v_j\|^2} v_j \right\rangle$$

$\{v_1 \ldots v_n\}$ es ortogonal

$$= \sum_{k=1}^{n} \left\langle \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k , \sum_{j=1}^{n} \frac{\langle u, v_j\rangle}{\|v_j\|^2} v_j \right\rangle$$

$$= \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} \left(\sum_{j=1}^{n} \left\langle v_k , \frac{\langle u, v_j\rangle}{\|v_j\|^2} v_j \right\rangle \right)$$

$$= \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} \left(\sum_{j=1}^{n} \frac{\overline{\langle u, v_j\rangle}}{\|v_j\|^2} \langle v_k , v_j\rangle \right) =$$

$$= \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} \frac{\overline{\langle u, v_k\rangle}}{\cancel{\|v_k\|^2}} \cancel{\|v_k\|^2} = \boxed{\sum_{k=1}^{n} \frac{|\langle u, v_k\rangle|^2}{\|v_k\|^2}}$$