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Non-manifold hyperbolic groups of high cohomological dimension

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Introduction

The asymptotic properties of the fundamental group of a compact, negatively curved Riemannian manifold or orbifold were captured by Gromov's concept of a *hyperbolic group* [Gro87]. Examples of hyperbolic groups of arbitrary cohomological dimension were already known, from Borel's construction of cocompact arithmetic lattices acting on any of the negatively curved symmetric spaces: real, complex, and quaternionic hyperbolic spaces and the Cayley hyperbolic plane [Bor63]. Starting from these examples, one can produce other hyperbolic groups using the combination theorem of Bestvina and Feighn [BF92], which says that under suitable conditions, amalgams of hyperbolic groups are hyperbolic. As the cohomological dimension of an amalgam is bounded below by the maximal cohomological dimension of its factors, we can thus produce many more examples of hyperbolic groups of arbitrarily high cohomological dimension. Note, however, that each of these examples contains a subgroup of the same dimension which is isomorphic to a cocompact lattice acting on a negatively curved symmetric space. A conjecture attributed to Gromov, found in Bestvina's problem list [Bes], asserts that for large enough dimension this is always the case. Specifically, the conjecture says that there exists an $N > 0$, such that if $n > N$, then any hyperbolic group of rational cohomological dimension n contains as a subgroup a cocompact lattice of the same rational cohomological dimension. The conjecture asserts furthermore that the lattice subgroup must be arithmetic.

The aim of this paper is to provide examples of hyperbolic groups in every dimension ≥ 3 which serve as counterexamples to this conjecture. These examples are not only hyperbolic: each of them is the fundamental group of a compact, geodesic metric space X which is negatively curved, meaning that each sufficiently small geodesic triangle in X is "no thicker" than the corresponding triangle in the hyperbolic plane.

The examples are as follows. Start with a complete, noncompact, finite volume hyperbolic n-manifold, of arbitrary dimension. Cut off a horoball neighborhood of each cusp, leaving a manifold whose boundary consists of concave horospherical Euclidean manifolds. Using residual finiteness, pass to a finite cover N so that each boundary component has sufficiently large injectivity radius. Now collapse each boundary component to a point, to obtain X.

These examples were first described by Gromov in [Gro93], section 7.A.VI, where it is claimed that if the injectivity radius of ∂N is sufficiently large then X is negatively curved (see our remarks after the *Negative Curvature Theorem* below). We justify this claim by proving that if the injectivity radius of each component of ∂N is at least π, then X has a negatively curved metric. Our proof combines elements of the Gromov-Thurston 2π-Theorem [BH96] with Berestovskii's Theorem on "hyperbolic cosine law metrics" [Ber83].

This construction produces a compact negatively curved metric space which is a pseudomanifold of dimension n, for any $n \geq 3$. It is a manifold off of a finite, nonempty set of singular points, the links of which are tori of dimension $n - 1$. It is then not hard to see that the complex has cohomological dimension n. Moreover, an argument similar to one used for fundamental groups of aspherical manifolds tells us that any subgroup of cohomological dimension n must have finite index. Finally, an examination of the boundary of our group tells us that it differs from the boundary of any negatively curved manifold group (in fact, it differs from the boundary of any hyperbolic Poincaré duality group), ruling out the existence of finite index subgroups which are fundamental groups of negatively curved manifolds.

Preliminaries

Hyperbolic spaces and negatively curved spaces   References for the material on hyperbolic metric spaces and word hyperbolic groups include [Gro87], [GdlH90], and [CDP90]. For negatively curved spaces see [Bal91]

and [BH95].

A metric space X is said to be a *path space* if, for any $x, y \in X$, the distance $d(x, y)$ is equal to the infimum of the lengths of paths in X connecting x to y. Every Riemannian manifold has an induced path metric, where the distance $d(x, y)$ is defined to be the infimum of the Riemannian length of piecewise smooth paths connecting x to y. A path space X is said to be a *geodesic space* if the infimum is realized: $d(x, y)$ is equal to the length of some path connecting x to y. Such a path is called a *geodesic* from x to y, and it is denoted $\overline{xy}$ (the notation is not meant to suggest that geodesics are unique). A metric space X is *proper* if closed balls are compact. Every proper path space is a geodesic space.

We shall need the following fact whose proof is left to the reader:

Completion Lemma. *The metric completion of a totally bounded path space is a proper geodesic space.* ◇

Next we recall the definition of a hyperbolic metric space, and a word hyperbolic group. A proper, geodesic metric space X is said to be *hyperbolic* (in the sense of Gromov) if there exists $\delta \geq 0$ such that for all geodesic triangles $T = \overline{xy} \cup \overline{yz} \cup \overline{zx}$ and for all $p \in \overline{xy}$ we have $d(p, \overline{yz} \cup \overline{zx}) \leq \delta$. A finitely generated group G is said to be *word hyperbolic* if some Cayley graph for G is hyperbolic. Using the fact that hyperbolicity is a quasi-isometry invariant of proper, geodesic metric spaces, it follows that word hyperbolicity of G is equivalent to each of the following:

- There exists a hyperbolic metric space X on which G acts properly discontinuously and cocompactly.

- For each proper geodesic metric space X on which G acts properly discontinuously and cocompactly, the space X is hyperbolic.

Next we recall the definition of a CAT(κ) space, where κ is any real number. Let M_κ denote the complete, simply connected 2-dimensional Riemannian manifold of constant sectional curvature κ. To be explicit: if $\kappa = 1$ then M_κ is the unit 2-sphere in Euclidean 3-space; if $\kappa = 0$ then M_κ is the Euclidean plane; and if $\kappa = -1$ then M_κ is the hyperbolic plane. Otherwise, M_κ is obtained by multiplying the Riemannian metric on $M_{\pm 1}$ by the constant factor $1/\sqrt{|\kappa|}$. For example, if $\kappa > 0$ then a great circle in the sphere M_κ has length $2\pi/\sqrt{\kappa}$.

Let X be a geodesic space, not necessarily proper. Let $T = \overline{xy} \cup \overline{yz} \cup \overline{zx}$ be a geodesic triangle, with perimeter $p(T) = d(x, y) + d(y, z) + d(z, x)$. Consider $\kappa \in \mathbf{R}$, and if $\kappa > 0$ we assume that $p(T) \leq 2\pi/\sqrt{\kappa}$. Then the space M_κ

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contains a triangle $T' = \overline{x'y'} \cup \overline{y'z'} \cup \overline{z'x'}$ unique up to isometry such that $d(x, y) = d(x', y')$, $d(y, z) = d(y', z')$, and $d(z, x) = d(z', x')$. The triangle T' is called a *comparison triangle* for T in M_κ, and there is a unique bijection $T \leftrightarrow T'$ denoted $p \to p'$ taking $x \to x'$, $y \to y'$, $z \to z'$, and preserving path length. We say that T is *no thicker* than T' if for all $p, q \in T$ we have

$$d(p, q) \le d(p', q')$$

We say that X is a CAT(κ) *space* if for every geodesic triangle T, either $\kappa > 0$ and $p(T) > 2\pi/\sqrt{\kappa}$, or T is no thicker than its comparison triangle in M_κ. A basic result of Hadamard says that a simply connected complete Riemannian manifold of sectional curvature $\le \kappa$ is CAT(κ). The same conclusion holds if "complete" is replaced by "geodesic".

A geodesic space X is *locally* CAT(κ) if X is a union of convex open subsets each of which is CAT(κ). It follows that a geodesic Riemannian manifold of sectional curvature $\le \kappa$ is locally CAT(κ). Finally, a proper geodesic space X is *negatively curved* if it is locally CAT(κ) for some $\kappa < 0$; by rescaling the metric we often assume that $\kappa = -1$.

Some basic facts are that any any simply connected, locally CAT(-1) space is CAT(-1), and any CAT(-1) space is hyperbolic. It follows that if G is the fundamental group of a compact, negatively curved metric space then G is word hyperbolic.

If G is a word hyperbolic group then there is a canonical boundary ∂G which can be viewed in several ways. For us the following statement will be sufficient: if G acts properly discontinuously and cocompactly on a CAT(-1) metric space X, then $\partial G = \partial X$ is the space of all geodesic rays based at a fixed point $x_0 \in X$, with the compact open topology.

Cohomological dimension The primary reference for this material is [Bro82].

Given a finitely generated group G, the *cohomological dimension* of G, denoted $\mathrm{cd}(G)$, is defined to be the smallest integer $d \ge 0$ such that for all $\mathbf{Z}G$-modules M and all $n > d$ we have $H^n(G; M) \approx 0$. We shall be working almost exclusively with torsion free groups of finite cohomological dimension. In this context it follows that $\mathrm{cd}(G)$ is equal to the largest integer n such that $H^n(G; \mathbf{Z}G) \not\approx 0$. It also follows that if H has finite index in G then $\mathrm{cd}(H) = \mathrm{cd}(G)$.

If G is the fundamental group of an aspherical simplicial complex of dimension n then $\mathrm{cd}(G) \le n$. In order to then show that $\mathrm{cd}(G) = n$, it suffices to find some abelian group A such that $H^n(G; A)$ or $H_n(G; A)$ is

nontrivial. For example, the fundamental group of a compact, aspherical n-manifold has cohomological dimension n, because $H_n(G; \mathbf{Z}/2)$ is nontrivial. In general, however, it may be necessary to use cohomology with some kind of twisted coefficients, that is, coefficients in a $\mathbf{Z}G$-module M where the G action is not trivial. If the "canonical cohomology" $H^n(G; \mathbf{Z}G)$ is easily computed then, as mentioned above, that gives a guaranteed computation of $\mathrm{cd}(G)$. One way to compute $H^n(G; \mathbf{Z}G)$ is to use the following:

Proposition. *Let G be the fundamental group of a compact, aspherical complex X with universal cover $\tilde{X}$. Then $H^n(G; \mathbf{Z}G)$ is isomorphic to $H^n_c(X; \mathbf{Z})$, where H^*_c denotes cohomology with compact supports.*

When G is word hyperbolic then we can go further with the results of Bestvina and Mess:

Theorem ([BM91]). *If G is word hyperbolic and if G acts properly discontinuously and cocompactly on a geodesic metric space Y (such as the universal cover of a compact, aspherical complex with fundamental group G), then*

$$H^n_c(Y; \mathbf{Z}) \approx \check{H}^{n-1}(\partial G; \mathbf{Z})$$

When G is torsion free it follows that $\mathrm{cd}(G)$ equals one more than the top dimension of nontrivial Čech cohomology $\check{H}^i(\partial G; \mathbf{Z})$.

Step 1: The examples

For the rest of the paper, fix $n \geq 3$.

Start with an n-dimensional complete, noncompact hyperbolic orbifold M of finite volume, as constructed for example by [Bor63]. The group $\pi_1(M)$ is residually finite and virtually torsion free, by a result of Selberg [Sel60] (see also [Alp87]). Therefore, by passing to a finite, regular covering space of M we may assume that M is an orientable manifold and each cusp of M is a torus cusp. Remove open horoball neighborhoods of the cusps, leaving a compact manifold N whose boundary is a finite union of $(n-1)$-dimensional Euclidean toruses. Applying residual finiteness once again, by passing to a finite regular covering space we may assume that each closed geodesic on the Euclidean manifold ∂N has length $> 2\pi$.

Now collapse each component of ∂N to a point (different points for different boundary components), and let G be the fundamental group of the resulting space X. Note that G has a presentation obtained from any presentation of $\pi_1(N)$ by adding a relator for each generator of the fundamental group of each boundary component.

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Negative Curvature Theorem. *X admits a negatively curved metric.*

Proof. The proof uses the method of the Gromov-Thurston 2π theorem [BH96] by interpolating the hyperbolic metric near a component T of ∂N with the "hyperbolic cosine law metric" described by Berestovskii's Theorem [Ber83] (see also [BH95]). (We first learned about Berestovskii's Theorem from the recent paper of Ancel and Guilbeault [AG97]).

We'll start by describing X in a slightly different manner, which is more amenable to imposing a negatively curved metric.

For each component T of ∂N, choose $r_T < 0$ in a manner to be described later, and let $\mathcal{C}(T)$ be obtained from $T \times [r_T, 0]$ by collapsing $T \times \{r_T\}$ to a point p_T. Glue $\mathcal{C}(T)$ to N by identifying $x \times 0 \in T \times 0$ with the point $x \in T \subset \partial N$. Do this for each component T of ∂N, and let X be the resulting topological space.

To put a metric on X, we impose a Riemannian metric ds on the manifold $X -$(cone points), and then we show that the completion of the induced path metric is X. On the subset $N \subset X$ the metric ds is the given hyperbolic metric on N. Now fix a component T of ∂N, and we explain how to extend the metric to $\mathcal{C}(T)$.

Let ds_T be the restriction of ds to T, a Euclidean metric on T. A collar neighborhood $T \times [0, \epsilon) \subset N$ of T may be chosen so that the metric on N restricted to this neighborhood has the form

$$ds^2 = dr^2 + e^{2r}ds_T^2, \quad r \in [0, \epsilon)$$

Using the same formula, extend this metric over $T \times (-\epsilon, \epsilon) \subset X$.

The shortest closed geodesic on T has a length $L > 2\pi$. Choose $r_T \in (-L/2\pi, -1)$. For sufficiently small ϵ, the metric on $T \times (r_T, r_T + \epsilon)$ is defined by the formula

$$ds^2 = dr^2 + \left(\frac{2\pi}{L} \sinh(r - r_T) \right)^2 ds_T^2$$

Having defined the metric on $T \times \left[(r_T, r_T + \epsilon) \cup (-\epsilon, +\epsilon) \right]$ we now interpolate, using the method described in [BH96]. The metric on $T \times (r_T, \epsilon)$ will have the form

$$ds^2 = dr^2 + f(r)^2 ds_T^2$$

where

$$f(r) = \begin{cases} \frac{2\pi}{L} \sinh(r - r_T), & r \in (r_T, r_T + \epsilon) \\ e^r, & r \in (-\epsilon, +\epsilon) \end{cases}$$

The function f can be extended over the entire interval $(r_T, +\epsilon)$ to be smooth, positive, increasing, and concave upward, because at the point
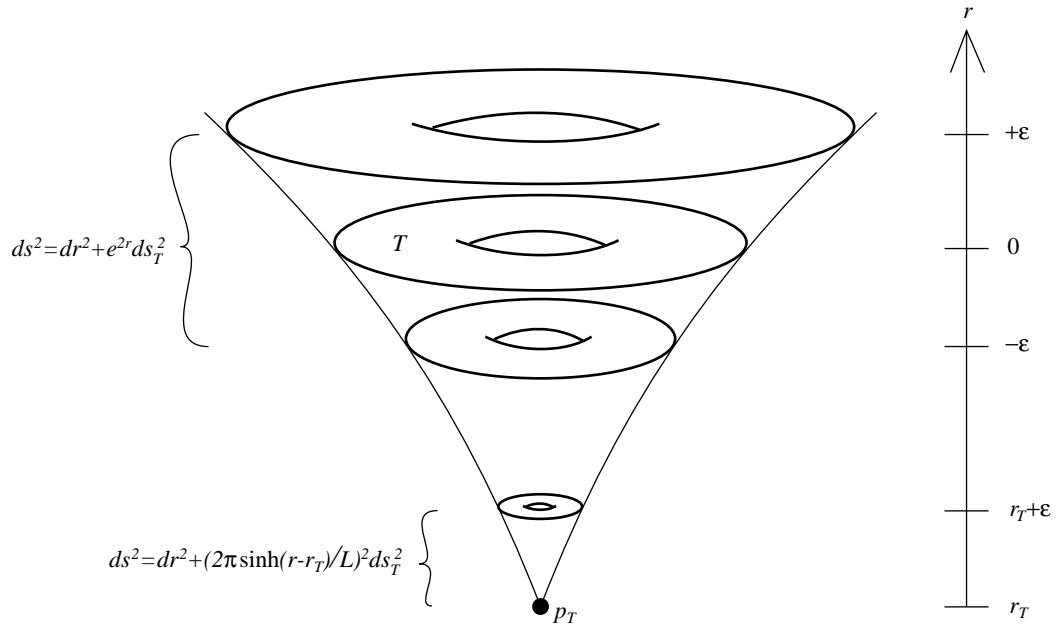
Figure 1: Interpolating the hyperbolic metric on $T \times (-\epsilon, \epsilon)$ and the coned-off metric on $T \times (r_T, r_T + \epsilon)$.

$(r_T, 0)$, which lies on the negative r-axis, the tangent line to the graph has slope $2\pi/L < 1$ forcing that line to intersect the y-axis below the point $(0, 1)$; and at the point $(0, 1)$ the tangent line to the graph has slope 1, making that line intersect the r-axis at $(-1, 0)$ which lies to the right of $(r_T, 0)$ (see figure 1 on p. 819 of [BH96], but change the coordinates).

In order to compute sectional curvatures of the metric, look at the lifted metric in the universal cover, which can be written in the form

$$dx_{n+1}^2 + f(x_{n+1})^2 (dx_1^2 + \cdots + dx_n^2)$$

on $\mathbf{R}^n \times \mathbf{R}_+$, where $f \colon \mathbf{R}_+ \to \mathbf{R}$ is positive, increasing, and concave upward.

When $n = 2$ we can appeal to [BH96] and conclude that each sectional curvature of this metric is a convex combination of sectional curvatures of the coordinate tangent planes: for the x_1, x_2 plane the sectional curvature is $-(f')^2/f^2$, and for the x_1, x_3 and x_2, x_3 planes the sectional curvature is $-f''/f$, and it follows that all sectional curvatures are negative.

When $n > 2$ we can reduce to the case $n = 2$ by a projection argument, as follows. If P is a tangent plane at a point $x \in \mathbf{R}^n \times \mathbf{R}_+$, project P to a tangent line or plane P' in $\mathbf{R}^n$, and let S be the affine subspace of $\mathbf{R}^n$ passing through x parallel to P'. The set $S \times \mathbf{R}_+$ is a totally geodesic subspace of $\mathbf{R}_n \times \mathbf{R}_+$, because orthogonal projection from $\mathbf{R}_n \times \mathbf{R}_+$ to $S \times \mathbf{R}_+$ does not increase the length of any tangent vector. It follows that the sectional curvature of P in $S \times \mathbf{R}_+$ equals its sectional curvature in $\mathbf{R}_n \times \mathbf{R}_+$. By rotational symmetry, the metric restricted to $S \times \mathbf{R}_+$ is the same as in the case $n = 2$, and so all sectional curvatures are negative.

Now we have to see what happens near the cone point p_T. We have a Riemannian metric on $X-$(cone points). Let ρ be the induced path metric on $X-$(cone points). Near p_T, the ρ-diameter of the torus $T \times r$ approaches zero as $r \to r_T$. The completion of ρ is therefore obtained by adding the points p_T, one for each T, and so the completion is a metric on X which we continue to call ρ. The space $X-$(cone points) is totally bounded, and so by the Completion Lemma the metric space X is a compact geodesic space.

We will show that the metric ρ is locally $\mathrm{CAT}(-1)$ near p_T, by comparing the sinh metric with the cosine law metric coming from Berestovkii's Theorem, which we now explain. Rewrite the metric on $T \times (r_T, r_T + \epsilon)$ as

$$dr^2 + \sinh^2(r - r_T)\left(\frac{2\pi}{L}ds_T\right)^2$$

For each $r \in (r_T, r_T + \epsilon)$ the torus $T \times r$ is clearly concave inward with respect to this metric, and so the subcone $\mathcal{C}_\epsilon(T) = T \times [r_T, r_T + \epsilon)/T \times r_T$

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is a geodesically convex subset of X relative to the metric ρ. It suffices to prove that $\mathcal{C}_\epsilon(T)$ is a CAT(-1) metric space. By a translation of the r coordinate we rewrite

$$\mathcal{C}_\epsilon(T) = T \times [0, \epsilon)/T \times 0$$

with metric

$$dr^2 + \sinh^2(r)\left(\frac{2\pi}{L}ds_T\right)^2$$

Consider the torus T with the rescaled metric $\frac{2\pi}{L}ds_T$, and let Θ be the induced path metric on T. This is a globally CAT(1) geodesic metric, for the following reasons. The shortest closed geodesic in T has length exactly 2π. Let τ be any geodesic triangle in T whose perimeter has length $\leq 2\pi$. If the perimeter has length $< 2\pi$ then τ lifts to a triangle $\tilde\tau$ in $\mathbf{E}^n$; if the perimeter equals 2π then either τ lifts as before or τ is a closed geodesic isometric to a great circle on S^2. If τ lifts to $\tilde\tau \subset \mathbf{E}^n$ then τ is no thicker than its comparison triangle $\tilde\tau$ in $\mathbf{E}^2$, from which it follows that τ is no thicker than a comparison triangle in S^2. If τ is a closed geodesic of length 2π then its comparison triangle τ' in S^2 is a great circle, and clearly τ is no thicker than τ'.

On the infinite cone $\mathcal{C}_\infty(T) = T \times [0, \infty)/T \times 0$, define the *cosine law metric* as follows. Given two points (x_1, r_1), (x_2, r_2) in $\mathcal{C}_\infty(T)$ the metric is:

$$\beta\big((x_1, r_1), (x_2, r_2)\big) =$$
$$\cosh^{-1}\bigg(\cosh(r_1)\cosh(r_2) - \sinh(r_1)\sinh(r_2)\cos\big(\mathrm{Min}\{\Theta(x_1, x_2), \pi\}\big)\bigg)$$

A theorem of Berestovskii tells us that the cosine law metric on the infinite cone of a CAT(1) space is a CAT(-1) metric, and so the metric β is CAT(-1) on $\mathcal{C}_\infty(T)$.

To complete the proof it suffices to show that $\mathcal{C}_\epsilon(T)$ is a geodesically convex subset of $\mathcal{C}_\infty(T)$ with respect to β, and that the metrics ρ and β are equal on $\mathcal{C}_\epsilon(T)$. To do this, consider two points $A_1, A_2 \in \mathcal{C}_\epsilon(T)$ with coordinates $A_1 = (x_1, r_1)$, $A_2 = (x_2, r_2)$, and let $\gamma \subset \mathcal{C}_\epsilon(T)$ be the ρ-geodesic connecting A_1 and A_2. It suffices to show that γ is also a β-geodesic, whose β-length is equal to $\rho(A_1, A_2)$. The following lemma gives the key special case of this:

Lemma. *If* $\Theta(x_1, x_2) < \pi$ *then* $p_T \notin \gamma$ *and* γ *is a β-geodesic of β-length equal to $\rho(A_1, A_2)$.*

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Proof. Let $\alpha \subset T$ be the Θ-geodesic from x_1 to x_2, a path of length $\Theta(x_1, x_2) < \pi$. By a projection argument it follows that the set $\mathcal{C}_\epsilon(\alpha) = \alpha \times [0, \epsilon]/\alpha \times 0$ is a geodesically convex subset of $\mathcal{C}_\epsilon(T)$ with respect to ρ, and so $\gamma \subset \mathcal{C}_\epsilon(\alpha)$.

We fix the Poincaré disc model of the hyperbolic plane $\mathbf{H}^2 = \{z \in \mathbf{C} \mid |z| < 1\}$ with circle at infinity $S_\infty^1 = \{z \in \mathbf{C} \mid |z| = 1\}$, and with origin $\mathcal{O}$. We use hyperbolic polar coordinates, where a point $z \in \mathbf{H}^2$ different from $\mathcal{O}$ is represented by polar coordinates (r, θ) where $r = d_{\mathbf{H}^2}(z, \mathcal{O})$ and $\theta = z/|z|$.

Pick two points x_1', x_2' in the circle at infinity S_∞^1 of $\mathbf{H}^2$ which are connected by an arc $\alpha' = \overline{x_1' x_2'} \subset S_\infty^1$ of length $\Theta(x_1, x_2) < \pi$. Let $\mathcal{C}(\alpha')$ be the set of points in $\mathbf{H}^2$ with polar coordinates by (r, θ) where $r \geq 0$, $\theta \in \alpha'$.

Consider the set $\mathcal{C}_\infty(\alpha) = \alpha \times [0, \infty)/\alpha \times 0$ as a subset of $\mathcal{C}_\infty(T)$. There is an isometry $f_\infty \colon \alpha' \to \alpha$ taking x_1' to x_1 and x_2' to x_2, and using hyperbolic polar coordinates this extends to a map $f \colon \mathcal{C}(\alpha') \to \mathcal{C}_\infty(\alpha)$. The map f is a ρ-isometric embedding, because the hyperbolic metric on the Poincaré disc in polar coordinates is none other than

$$dr^2 + \sinh^2(\theta)d\theta^2$$

and it follows that $f^{-1}(\gamma)$ is the hyperbolic geodesic from $f^{-1}(A)$ to $f^{-1}(B)$ in $\mathbf{H}^2$. Since the angle at $\mathcal{O}$ subtended by $f^{-1}(\gamma)$ is less than π it follows that $\mathcal{O} \notin f^{-1}(\gamma)$, and so $p_T \notin \gamma$.

By the hyperbolic law of cosines in $\mathbf{H}^2$, for each $\xi, \eta \in \gamma$ we clearly have $d_{\mathbf{H}^2}(f^{-1}\xi, f^{-1}\eta) = \beta(\xi, \eta)$, and it follows that γ is a β-geodesic of length $d_{\mathbf{H}^2}(f^{-1}A, f^{-1}B) = \rho(A, B)$. $\diamond$

Now we prove the general case. As a consequence of the lemma one of the following two statements is true in general:

1. $p_T \notin \gamma$.

2. $p_T \in \gamma$ and $\Theta(x_1, x_2) \geq \pi$.

Note that *a posteriori* case 1 is equivalent to $\Theta(x_1, x_2) < \pi$, but we do not yet know this; *a priori* it could happen that $p_T \notin \gamma$ and $\Theta(x_1, x_2) \geq \pi$.

Now we break into cases, in each case proving that γ is a β-geodesic of length $\rho(A_1, A_2)$.

Case 1. Suppose $p_T \notin \gamma$. Since β is a CAT(-1) metric on $\mathcal{C}_\infty(T)$, it suffices to show that γ is a *local* β-geodesic of β-length $\rho(A_1, A_2)$. Since $p_T \notin \gamma$ it follows that the projection map $\gamma \mapsto T$, taking $(x, r) \mapsto x$, is a

well-defined continuous map on γ. It therefore suffices to show, for any two points $B_1 = (y_1, s_2)$, $B_2 = (y_2, s_2)$ in γ satisfying $\Theta(y_1, y_2) < \pi$, that the segment of γ connecting B_1 to B_2 is a β-geodesic of β-length $\rho(B_1, B_2)$. But this is an immediate consequence of the above lemma.

Case 2 Suppose $p_T \in \gamma$ and $\Theta(x_1, x_2) \geq \pi$. Write the path γ as a concatenation $\gamma_1^{-1} * \gamma_2$ where γ_i connects p_T to A_i. Note that any path from p_T to $A_i = (x_i, r_i)$ has ρ-length $\geq r_i$, with equality if and only if the path is perpendicular to each level surface $r =$(constant). Since γ is a ρ-geodesic it follows that each γ_i is the "radial path" $r \mapsto (x_i, r)$, $r \in [0, r_i]$. Given any two points $B_1 = (y_1, s_1)$, $B_2 = (y_2, s_2)$ in γ, if they are on the same side of p_T then $\rho(B_1, B_2) = |s_1 - s_2|$; but also $y_1 = y_2$ and a straightforward calculation shows $\beta(B_1, B_2) = |s_1 - s_2|$. On the other hand, if B_1, B_2 are on opposite sides of p_T then $\rho(B_1, B_2) = s_1 + s_2$; but also, up to an index transposition we have $y_1 = x_1$ and $y_2 = x_2$ and so $\Theta(y_1, y_2) \geq \pi$ and another calculation shows $\beta(B_1, B_2) = s_1 + s_2$.

This finishes the proof of the *Negative Curvature Theorem*. $\diamondsuit$

Remarks As remarked earlier, the *Negative Curvature Theorem* is claimed in [Gro93] section 7.A.VI, which refers back to [Gro87] for an explanation. The relevant portion of [Gro87] seems to be section 4.3. With hindsight we can now see that the discussion of [Gro87] section 4.3 cannot work in the context of the *Negative Curvature Theorem*, which makes all the more evident the subtlety of the 2π-Theorem and the discussion in [Gro93] 7.A.VI. Here is a brief discussion of [Gro87] section 4.3, in the simple context of cusped hyperbolic surfaces.

Start with a complete, finite area, oriented, hyperbolic surface with one cusp. Truncate a horodisc neighborhood of the cusp leaving a surface N with one circle boundary component. In section 4.3 one chooses a group Γ acting freely and isometrically on ∂N. For application to the *Negative Curvature Theorem*, choose Γ to be the group of rotational isometries of ∂N, which acts transitively on ∂N. Let X be obtained from N by collapsing ∂N to a point and taking the obvious geodesic metric on X, that is, $X = (N - \partial N) \cup \partial N / \Gamma$.

It is claimed in section 4.3, *without* any assumption on injectivity radius of ∂N, that X is CAT(-1). But this is not true: starting with a complete, finite area hyperbolic metric on the three-punctured sphere, cut off the cusps as above to obtain N, and collapse each cusp boundary to obtain X; evidently X is homeomorphic to the 2-sphere, and X has no CAT(-1) metric.

To find a small triangle in X which fails to be CAT(-1), consider the local picture near ∂N, namely a horodisc exterior in $\mathbf{H}^2$. Using the upper half plane model $\mathbf{H}^2 = \{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}_+\}$, consider the horodisc exterior $\tilde{N} = \{(x, y) \in \mathbf{H}^2 \mid y \geq 1\}$. The collapsed space in this situation is $\tilde{N}/\partial\tilde{N}$, and we show that this is not locally CAT(-1) near the collapsed image of $\partial\tilde{N}$.

Take $x \neq y \in \partial\tilde{N}$ such that $d(x, y)$ is very, very small. Take two geodesic segments l, m which end at right angles on ∂N at the points x, y. Suppose that Length(l) = Length(m) = $10d(x, y)$. Connect the opposite endpoints x', y' of l, m by a geodesic segment g. This produces a triangle $T = l \cup m \cup g$ in $\tilde{N}/\tilde{\partial}N$. The comparison triangle T' in $\mathbf{H}^2$ is obtained from T by rotating l, m about x', y' until they endpoints x, y meet, and evidently T is thicker than T'. In fact the triangle T is thicker than its comparison triangle in $\mathbf{E}^2$, and so $\tilde{N}/\partial\tilde{N}$ is not even locally CAT(0).

Cohomological dimension

Recall the notion of a pseudo-manifold of dimension n, defined in [Spa81]. This is an n-dimensional simplicial complex such that:

1. Each simplex of dimension $< n$ is a face of some simplex of dimension n.

2. Each simplex of dimension $n - 1$ is a face of exactly two simplices of dimension n.

3. For any two simplices σ, σ' of dimension n there exists a sequence of n-dimensional simplices $\sigma = \sigma_0, \ldots, \sigma_k = \sigma'$ such that σ_{i-1}, σ_i have a common $(n - 1)$-dimensional face τ_i, for $i = 1, \ldots, k$.

An *orientation* of a pseudo-manifold is an orientation on each n-simplex such that if σ, σ' are n-simplices sharing an $(n - 1)$ face τ, then σ, σ' induce opposite boundary orientations on τ. Note that a pseudo-manifold is orientable if and only if the complement of its codimension-2 skeleton is an orientable manifold.

Proposition. *X is an orientable pseudo-manifold of dimension n, and hence $G = \pi_1(X)$ has cohomological dimension n.*

Proof. Triangulate X so that each singularity is a vertex, using the Delauney construction for example. Since X is a manifold outside the vertices then properties (1) and (2) follow. Since the link of every vertex is either a sphere

or a torus, any path in X can be pushed off the vertices, and then can be perturbed into general position with respect to the positive dimensional simplices; property (3) follows, and so X is a pseudo-manifold of dimension n. Since $X-(\text{cone points})$ is an orientable manifold it follows that X is an orientable pseudo-manifold.

Since X is a simplicial complex of dimension n and $G = \pi_1(X)$ is torsion free it follows that $\text{cd}(G) \leq n$. Since X is a compact, orientable pseudo-manifold it follows that $H_n(X; \mathbf{Z}) \not\approx 0$, because the orientation cycle, which assigns coefficient 1 to each oriented n-simplex, is nontrivial in homology. Therefore, $\text{cd}(G) = n$.
$\diamondsuit$

Step 2: Subgroups of cohomological dimension n

Let Γ be a subgroup of G of cohomological dimension n for which there exists a finite $K(\Gamma, 1)$. In this step we will prove that Γ has finite index in G.

The cohomology $H^n(\Gamma; \mathbf{Z}\Gamma)$ is nontrivial, by [Bro82] proposition VIII.6.7. By [Bro82] exercise 4b page 198, there exists a $\mathbf{Z}\Gamma$-module Q such that $H_n(\Gamma; Q)$ is nontrivial.

Let Y be the covering space of X associated to the subgroup Γ of G. The space Y is a pseudo-manifold, by the same argument used to prove that X is a pseudo-manifold. Fix a pseudo-manifold triangulation of Y.

Since $\pi_1(Y) = \Gamma$, there is a local coefficient system $\mathcal{Q}$ on the triangulation of Y, where the local coefficients are isomorphic to Q, such that the associated action of Γ on Q determines the given $\mathbf{Z}\Gamma$-module structure on Q. Since Y is aspherical, it follows that $H_n(Y; \mathcal{Q}) \approx H_n(\Gamma; Q)$, and so $H_n(Y; \mathcal{Q})$ is nontrivial.

Let c be a nontrivial n-cycle on Y with coefficients in $\mathcal{Q}$, and so c is nonzero on some n-simplex of Y. It follows by connectivity of the dual 1-skeleton that c is nonzero on every n-simplex, because if σ, σ' are n-simplices with a common $(n-1)$-face F, then the inclusion maps $F \subset \sigma$, $F \subset \sigma'$ induce an isomorphism between the σ-coefficients and the σ'-coefficients, and c agrees with this isomorphism. But a cycle, by definition, is nonzero on only finitely many simplices, and so Y is compact. This shows that Γ has finite index in G.

Step 3: Boundaries

Now suppose that Γ is a finite index subgroup of G. It follows that Γ and G are quasi-isometric groups. In particular, Γ is word hyperbolic, and $\partial\Gamma \approx \partial G$.

According to [Bro82], if Γ is a Poincaré duality group of dimension n then $H^i(\Gamma; \mathbf{Z}\Gamma) = 0$ in all dimensions $i \neq n$. Next we show that Γ cannot be a Poincaré duality group of dimension $n \geq 3$, by showing that $H^2(\Gamma; \mathbf{Z}\Gamma)$ is nontrivial.

By [BM91], the groups $H^2(\Gamma; \mathbf{Z}\Gamma)$ and $H^2(G; \mathbf{Z}G)$ are isomorphic to the Čech cohomology group $\check{H}^1(\partial G; \mathbf{Z})$.

Look at the CAT(-1) space $\tilde{X}$. Fix a base point $\mathcal{O}$, to be one of the cone points. Let S_a be the sphere of radius a about $\mathcal{O}$. Radial projection defines a continuous map $p_{ab}\colon S_a \to S_b$ for all $0 < b < a$, and so we obtain an inverse system of topological spaces $\{S_a\}_{a \in (0,\infty)}$. The boundary ∂G is the inverse limit of this system.

There are induced homomorphisms

$$p_{ab*}\colon H_*(S_a) \to H_*(S_b)$$
$$p_{ab}^*\colon H^*(S_b) \to H^*(S_a)$$

using $\mathbf{Z}$ coefficients. The Čech cohomology $\check{H}^1(\partial G; \mathbf{Z})$ is the direct limit of the system p_{ab}^*.

Choose a small $a_0 > 0$, so that $S_0 = S_{a_0}$ is a torus. Choose a sequence $a_n \to \infty$ so that $S_n = S_{a_n}$ does not contain a cone point. Let $p_{mn} = p_{a_m a_n}$. Since $H^1(S_0)$ is nontrivial, it suffices to show that p_{mn}^* is injective for $m > n$, and we do this by showing that p_{mn*} is surjective.

Consider a 1-dimensional homology class in S_n represented as a closed curve $\gamma\colon S^1 \to S_n$. There are only finitely many cone points whose distance from $\mathcal{O}$ lie in the interval $[a_n, a_m]$, and these points project radially to a finite subset F of S_n. Since S_n is a manifold of dimension ≥ 2, γ may be homotoped off of F. Any point not in F has a unique preimage in S_m, under the radial projection map p_{mn}, and moreover the induced map $S_n - F \to S_m$ is continuous. We may therefore project γ radially outward to a curve on S_m. This shows that p_{mn*} is surjective, completing the proof that G has no Poincaré duality subgroup of dimension n.

Final steps

Let Γ be a subgroup of cohomological dimension n in G. We show that Γ is not an arithmetic lattice of dimension n. Now Γ is a lattice in some semisimple Lie group G. Let X be the associated symmetric space, which is G/K where K is a maximal compact subgroup of G.

We know that Γ is torsion free, because G is torsion free, and so X/Γ is a manifold.

If the action of Γ on X is cocompact, then Γ is a Poincaré duality group of dimension n, contradicting step 3.

If the action of Γ on X is not cocompact then there are several cases, each leading to a contradiction. If $X = \mathbf{H}^2$ then Γ is a free group, contradicting that $\mathrm{cd}(\Gamma) = n \geq 3$. In all other cases, Γ has a solvable subgroup which is not virtually cyclic, which cannot happen inside the word hyperbolic group G (see [GdlH90] chapter 8 theorem 37). The existence of this solvable subgroup of Γ is a standard result about nonuniform lattices in semisimple Lie groups, although there does not seem to be a good reference. When $X = \mathbf{H}^m$ for $m \geq 3$ then Γ has a subgroup isomorphic to $\mathbf{Z}^{m-1}$. When X is complex or quaternionic hyperbolic space or the Cayley hyperbolic plane, with real dimension $m \geq 4$, then Γ has a 2-step nilpotent subgroup of rank $m - 1$. And in the remaining case, when the symmetric space X has rank ≥ 2, the group Γ has a solvable subgroup which is not virtually nilpotent.

Some speculations

In section 7.A.VI of [Gro93] Gromov claims that if M is any complete, finite volume, noncompact, locally symmetric space of negative sectional curvature, then by cutting off cusps, passing to a certain finite cover, and collapsing cusp boundaries, the resulting space has a negatively curved metric. We have proved this when M is locally modelled on real hyperbolic space, and our proof might extend to complex and quaternionic hyperbolic spaces and the Cayley hyperbolic plane, by using the right model of the metric near a cusp. These examples, and others described shortly thereafter in the same section of [Gro93], ought to provide additional counterexamples to the conjecture stated in the introduction.

There are several directions in which one can look for stronger counterexamples.

For example, does there exist a word hyperbolic group G which is a Poincaré duality group of dimension n, such that no subgroup of cohomo-

logical dimension n is isomorphic to a lattice? One possibility is the fundamental group of one of the Gromov-Thurston examples of negatively curved manifolds with arbitrarily large pinching constant [GT87]. These examples are discussed further in [Gro93] section 7.C_2. The main point of section 7.C is formulating asymptotic invariants based on pinching constants. Perhaps such invariants can be used to show that the Gromov-Thurston examples span infinitely many distinct quasi-isometry classes.

A different way to look for counterexamples was suggested to us by M. Bestvina. Given a large integer $k > 0$, does there exist a word hyperbolic group G, such that for any subgroup Γ with $cd(\Gamma) \geq cd(G) - k$, the group Γ is not isomorphic to a lattice in a semisimple Lie group?

Of course, this question would be settled if there were a word hyperbolic group G which has no subgroup isomorphic to a lattice in a semisimple Lie group (except for free groups which are noncocompact lattices in $PSL(2, \mathbf{R})$—these can be avoided by requiring $H^1(G; \mathbf{Z}) \approx 0$). But then one runs into the following question of Gromov on Bestvina's problem list: does every word hyperbolic group contain a subgroup isomorphic to the fundamental group of some compact hyperbolic surface?

Similar questions can be asked where "subgroup isomorphic to a lattice in a semisimple Lie group" is replaced by "Poincaré duality subgroup". Along these lines, the strongest counterexample one might ask for is a word hyperbolic group Γ which has no Poincaré duality subgroups of dimension ≥ 2—or, at least, no Poincaré duality subgroups of dimension $\geq cd(\Gamma) - k$, given an integer k.

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