CASACT Exam 7 Notes

Jerrison Li

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1 Benktander Method

1.1 General Relationship Between Reserve & Ultimate Loss Estimates

Suppose that C_k is the actual claims amount paid after k years of development. Given a **reserve estimate** $\widehat{R}$ and **ultimate loss estimate** $\widehat{U}$, we have the following general relationship:

$$\widehat{U} = C_k + \widehat{R} \tag{1}$$

This relationship always holds. Note that C_k is the cumulative paid amount.

1.2 Bornhuetter/Ferguson (BF) Method

The **Bornhuetter/Ferguson (BF) Method** estimates reserves based on an **a priori** expectation of ultimate losses. Mathematically:

$$R_{BF} = q_k * U_0 \tag{2}$$

where:

- R_{BF} is the **reserve estimate**
- $q_k = 1 - \frac{1}{CDF}$ is the proportion of the ultimate claims amount which is expected to remain unpaid after k years of development
- U_0 is the a priori expectation of ultimate losses

Since R_{BF} uses U_0, it assumes that the current claims amount C_k is **NOT predictive of future claims**.

Using the general relationship described above, we obtain the **BF ultimate loss**:

$$U_{BF} = C_k + R_{BF} \tag{3}$$

1.3 Chain Ladder Method

The **Chain Ladder Method** estimates ultimate losses and reserves based on claims to date. In other words, it assumes that the current claims amount C_k is **fully predictive of future claims**. Mathematically:

$$U_{CL} = \frac{C_k}{p_k} \tag{4}$$

$$R_{CL} = q_k * U_{CL} \tag{5}$$

1.4 Benktander Method

Since the CL and BF methods represent extreme positions, where the CL method fully believes C_k and the BF method does not rely on C_k at all, Gunnar Benktander replaced U_0 with a credibility mixture:

$$U_c = c * U_{CL} + (1 - c) * U_0 \tag{6}$$

where c is the credibility weight.

As the claims C_k develop, credibility should increase, Benktander proposed the following:

- Set $c = p_k$
- Set $R_{GB} = R_{BF} * \frac{U_{pk}}{U_0}$

$$\begin{aligned} R_{GB} &= R_{BF} & &* \frac{U_{pk}}{U_0} \\ &= (q_k * \cancel{U_o}) & &* \frac{U_{pk}}{\cancel{U_o}} \\ &= q_k * U_{pk} \end{aligned} \tag{7}$$

1.4.1 BF Method as a Credibility-Weighted Average

Using our credibility mixture with $c = p_k$, we can show the following:

$$\begin{aligned} U_{pk} &= p_k * U_{CL} + (1 - p_k) * U_0 \\ &= p_k * U_{CL} + q_k * U_0 \\ &= C_k + R_{BF} \\ &= U_{BF} \end{aligned} \tag{8}$$

then

$$R_{GB} = q_k * U_{BF} \tag{9}$$

Hence, the **BF method** is a credibility-weighted average of the CL method and the a priori expectation.

1.4.2 Benktander Method as a Credibility-Weighted Average

The **Benktander method** is a credibility-weighted average of the CL and BF methods:

$$\begin{aligned} U_{GB} &= C_k & &+ R_{GB} \\ &= C_k & &+ q_k * U_{BF} \\ &= (1 - q_k) * U_{CL} & &+ q_k * U_{BF} \end{aligned} \tag{10}$$

The **Benktander reserve** is also a credibility-weighted average of the CL and BF methods. This was proposed by Esa Hovinen (1981):

$$
\begin{aligned}
R_{EH} &= (1 - q_k) * R_{CL} + q_k * R_{BF} \\
&= p_k * q_k * U_{CL} + q_k * q_k * U_0 \\
&= p_k * q_k * U_{CL} + (1 + p_k) * q_k * U_0 \\
&= q_k * [p_k * U_{CL} + (1 - p_k) * U_0] \\
&= q_k * U_{pk} \\
&= R_{GB}
\end{aligned} \tag{11}
$$

We can also express the Benktander method as a credibility-weighted average of the CL method and the a priori expectation:

$$
\begin{aligned}
U_{GB} &= C_k + R_{GB} \\
&= U_{CL} - R_{CL} + q_k * U_{pk} \\
&= U_{CL} - q_k * U_{CL} + q_k * ([1 - q_k] * U_{CL} + q_k * U_0) \\
&= U_{CL} - \cancel{q_k * U_{CL}} + \cancel{q_k * U_{CL}} - q_k^2 * U_{CL} + q_k^2 * U_0 \\
&= U_{CL} - q_k^2 * U_{CL} + q_k^2 * U_0 \\
&= (1 - q_k^2) * U_{CL} + q_k^2 * U_0 \\
&= U_{1-q_k^2}
\end{aligned} \tag{12}
$$

1.4.3 Iterative Relationships

As shown above, the Benktander reserve is obtained by applying the BF method twice, first to U_0 (which produces R_{BF}) and then to U_{BF} (which produces R_{GB}). Hence, the Benktander method is also called the **iterated BF method**.

Using a starting point of $U^{(0)} = U_0$ and iteration rules $R^{(m)} = q_k * U^{(m)}$ and $U^{(m+1)} = C_k + R^{(m)}$, we obtain the following iterative relationships:

$$
\begin{aligned}
U^{(m)} &= (1 - q_k) * U_{CL} + q_k^m * U_0 \\
R^{(m)} &= (1 - q_k) * R_{CL} + q_k^m * R_{BF}
\end{aligned} \tag{13}
$$

These relationships lead to the following:

- $U^{(0)} = (1 - q_k^0) * U_{CL} + q_k^0 * U_0 = U_0$
- $R^{(0)} = (1 - q_k^0) * R_{CL} + q_k^0 * R_{BF} = R_{BF}$
- $U^{(1)} = (1 - q_k) * U_{CL} + q_k * U_0 = C_k + R_{BF} = U_{BF}$
- $R^{(1)} = (1 - q_k) * R_{CL} + q_k * R_{BF} = R_{GB}$
- $U^{(2)} = (1 - q_k^2) * U_{CL} + q_k^2 * U_0 = U_{GB}$
- ...
- $U^{(\infty)} = (1 - q_k^\infty) * U_{CL} + q_k^\infty * U_0 = U_{CL}$
- $R^{(\infty)} = (1 - q_k^\infty) * R_{CL} + q_k^\infty * R_{BF} = R_{GB}$

1.5 Benktander vs. BF vs. CL

The Benktander method is superior to the BF and CL methods for the following reasons:

- The mean squared error (MSE) is almost always smaller than the BF or CL methods.
- Better approximation of the exact Bayesian procedure.

- Superior to the CL method since it give more weight to the a priori expectation of ultimate losses.
- Superior to the BF method since it gives more weight to actual loss experience.

1.5.1 Mean Squared Error (MSE)

The Benktander reserve R_{GB} has a smaller mean squared error (MSE) than R_{BF} whenever $c^* > p_k/2$ holds.

c^* is the optimal credibility reserve credibility factor.

2 Hurlimann

	development period				
period (i)	1	2	3	4	5
1	S_{1,1}	S_{1,2}	S_{1,3}	S_{1,4}	S_{1,5}
2	S_{2,1}	S_{2,2}	S_{2,3}	S_{2,4}	
3	S_{3,1}	S_{3,2}	S_{3,3}		
4	S_{4,1}	S_{4,2}			
5	S_{5,1}				

Note that each of the highlighted diagonal rows represent the same calendar development period.

3 Brosius

3.1 Least Squares Method

- x = losses to date
- y = losses at a future evaluation

3.1.1 Link Ratio Method

where c is the selected link ratio. For the purposes of this paper, we will assume that c **is the volume-weighted average LDF.**

3.2 Testing the least squares method using loss & loss reporting distributions

3.2.1 Simple Model

- The number of claims incurred each year is a random variable Y which is either 0 or 1 with equal probability.
- The number of claims reported by year-end is a random variable X. If there is a claim, there is a 50 chance it will be reported by year-end. Hence, X is a binomial random variable with $n = y$ and $p = 0.5$.

Let $Q(x)$ represent the expected total number of claims, and $R(x)$ represent the expected number of claims outstanding, both given that $X = x$,. Mathematically:

$Q(0) = 1/3$ $Q(1) = 1$

3.3 The linear approximation (Bayesian Credibility)

Let L be the best linear approximation to Q. Mathematically, L is the linear function that minimizes $E_X[(Q(x) - L(x))^2]$.

Assuming $L(x) = a + bx$, we must minimize $E_X[(Q(x) - a - bX)]^2$.

3.3.1 Development Formula 1

4 Clarks

4.1 Expected Loss Emergence

4.1.1 Cape Cod Method

In general, the Cape Cod method is preferred since data is summarized into a loss triangle with relatively few data points. Since the LDF method requires an estimation of a number of parameters (one for each AY ultimate loss, as well as θ and ω), it tends to be over-parameterized when few data points exist.

The difference between the Cape Cod method and the BF method is that the Cape Cod method uses the underlying data to estimate the ultimate loss while the BF method uses a priori information.

The ultimate is an expected ultimate, not the Cape Cod ultimate calculated using losses to date and adding the Cape Cod reserve.