CASACT Exam 7 Notes

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1 Benktander Method

1.1 General Relationship Between Reserve & Ultimate Loss Estimates

Suppose that C_k is the actual claims amount paid after k years of development. Given a **reserve estimate** $\widehat{R}$ and **ultimate loss estimate** $\widehat{U}$, we have the following general relationship:

$$\widehat{U} = C_k + \widehat{R} \tag{1}$$

This relationship always holds. Note that C_k is the cumulative paid amount.

1.2 Bornhuetter/Ferguson (BF) Method

The **Bornhuetter/Ferguson (BF) Method** estimates reserves based on an **a priori** expectation of ultimate losses. Mathematically:

$$R_{BF} = q_k * U_0 \tag{2}$$

where:

- R_{BF} is the **reserve estimate**
- $q_k = 1 - \frac{1}{CDF}$ is the proportion of the ultimate claims amount which is expected to remain unpaid after k years of development
- U_0 is the a priori expectation of ultimate losses

Since R_{BF} uses U_0, it assumes that the current claims amount C_k is **NOT predictive of future claims**.

Using the general relationship described above, we obtain the **BF ultimate loss**:

$$U_{BF} = C_k + R_{BF} \tag{3}$$

1.3 Chain Ladder Method

The **Chain Ladder Method** estimates ultimate losses and reserves based on claims to date. In other words, it assumes that the current claims amount C_k is **fully predictive of future claims**. Mathematically:

$$U_{CL} = \frac{C_k}{p_k} \tag{4}$$

$$R_{CL} = q_k * U_{CL} \tag{5}$$

1.4 Benktander Method

Since the CL and BF methods represent extreme positions, where the CL method fully believes C_k and the BF method does not rely on C_k at all, Gunnar Benktander replaced U_0 with a credibility mixture:

$$U_c = c * U_{CL} + (1-c) * U_0 \tag{6}$$

where c is the credibility weight.

As the claims C_k develop, credibility should increase, Benktander proposed the following:

- Set $c = p_k$
- Set $R_{GB} = R_{BF} * \frac{U_{pk}}{U_0}$

$$\begin{aligned} R_{GB} &= R_{BF} && * \frac{U_{pk}}{U_0} \\ &= (q_k * \cancel{U_o}) && * \frac{U_{pk}}{\cancel{U_o}} \\ &= q_k * U_{pk} \end{aligned} \tag{7}$$

1.4.1 BF Method as a Credibility-Weighted Average

Using our credibility mixture with $c = p_k$, we can show the following:

$$\begin{aligned} U_{pk} &= p_k * U_{CL} + (1 - p_k) * U_0 \\ &= p_k * U_{CL} + q_k * U_0 \\ &= C_k + R_{BF} \\ &= U_{BF} \end{aligned} \tag{8}$$

then

$$R_{GB} = q_k * U_{BF} \tag{9}$$

Hence, the **BF method** is a credibility-weighted average of the CL method and the a priori expectation.

1.4.2 Benktander Method as a Credibility-Weighted Average

The **Benktander method** is a credibility-weighted average of the CL and BF methods:

$$\begin{aligned} U_{GB} &= C_k && + R_{GB} \\ &= C_k && + q_k * U_{BF} \\ &= (1 - q_k) * U_{CL} && + q_k * U_{BF} \end{aligned} \tag{10}$$

The **Benktander reserve** is also a credibility-weighted average of the CL and BF methods. This was proposed by Esa Hovinen (1981):

$$
\begin{aligned}
R_{EH} &= (1 - q_k) * R_{CL} + q_k * R_{BF} \\
&= p_k * q_k * U_{CL} + q_k * q_k * U_0 \\
&= p_k * q_k * U_{CL} + (1 + p_k) * q_k * U_0 \\
&= q_k * [p_k * U_{CL} + (1 - p_k) * U_0] \\
&= q_k * U_{pk} \\
&= R_{GB}
\end{aligned}
\tag{11}
$$

We can also express the Benktander method as a credibility-weighted average of the CL method and the a priori expectation:

$$
\begin{aligned}
U_{GB} &= C_k + R_{GB} \\
&= U_{CL} - R_{CL} + q_k * U_{pk} \\
&= U_{CL} - q_k * U_{CL} + q_k * ([1 - q_k] * U_{CL} + q_k * U_0) \\
&= U_{CL} - \cancel{q_k * U_{CL}} + \cancel{q_k * U_{CL}} - q_k^2 * U_{CL} + q_k^2 * U_0 \\
&= U_{CL} - q_k^2 * U_{CL} + q_k^2 * U_0 \\
&= (1 - q_k^2) * U_{CL} + q_k^2 * U_0 \\
&= U_{1-q_k^2}
\end{aligned}
\tag{12}
$$

1.4.3 Iterative Relationships

As shown above, the Benktander reserve is obtained by applying the BF method twice, first to U_0 (which produces R_{BF}) and then to U_{BF} (which produces R_{GB}). Hence, the Benktander method is also called the **iterated BF method**.

Using a starting point of $U^{(0)} = U_0$ and iteration rules $R^{(m)} = q_k * U^{(m)}$ and $U^{(m+1)} = C_k + R^{(m)}$, we obtain the following iterative relationships:

$$
\begin{aligned}
U^{(m)} &= (1 - q_k) * U_{CL} + q_k^m * U_0 \\
R^{(m)} &= (1 - q_k) * R_{CL} + q_k^m * R_{BF}
\end{aligned}
\tag{13}
$$

These relationships lead to the following:

- $U^{(0)} = (1 - q_k^0) * U_{CL} + q_k^0 * U_0 = U_0$
- $R^{(0)} = (1 - q_k^0) * R_{CL} + q_k^0 * R_{BF} = R_{BF}$
- $U^{(1)} = (1 - q_k) * U_{CL} + q_k * U_0 = C_k + R_{BF} = U_{BF}$
- $R^{(1)} = (1 - q_k) * R_{CL} + q_k * R_{BF} = R_{GB}$
- $U^{(2)} = (1 - q_k^2) * U_{CL} + q_k^2 * U_0 = U_{GB}$
- ...
- $U^{(\infty)} = (1 - q_k^\infty) * U_{CL} + q_k^\infty * U_0 = U_{CL}$
- $R^{(\infty)} = (1 - q_k^\infty) * R_{CL} + q_k^\infty * R_{BF} = R_{GB}$

1.5 Benktander vs. BF vs. CL

The Benktander method is superior to the BF and CL methods for the following reasons:

- The mean squared error (MSE) is almost always smaller than the BF or CL methods.
- Better approximation of the exact Bayesian procedure.

- Superior to the CL method since it give more weight to the a priori expectation of ultimate losses.
- Superior to the BF method since it gives more weight to actual loss experience.

1.5.1 Mean Squared Error (MSE)

The Benktander reserve R_{GB} has a smaller mean squared error (MSE) than R_{BF} whenever $c^* > p_k/2$ holds.

c^* is the optimal credibility reserve credibility factor.

2 Hurlimann

	development period				
period (i)	1	2	3	4	5
1	S_{1,1}	S_{1,2}	S_{1,3}	S_{1,4}	S_{1,5}
2	S_{2,1}	S_{2,2}	S_{2,3}	S_{2,4}	
3	S_{3,1}	S_{3,2}	S_{3,3}		
4	S_{4,1}	S_{4,2}			
5	S_{5,1}				

Note that each of the highlighted diagonal rows represent the same calendar development period.

3 Brosius

3.1 Least Squares Method

- x = losses to date
- y = losses at a future evaluation

3.1.1 Link Ratio Method

where c is the selected link ratio. For the purposes of this paper, we will assume that c **is the volume-weighted average LDF.**

3.2 Testing the least squares method using loss & loss reporting distributions

3.2.1 Simple Model

- The number of claims incurred each year is a random variable Y which is either 0 or 1 with equal probability.
- The number of claims reported by year-end is a random variable X. If there is a claim, there is a 50 chance it will be reported by year-end. Hence, X is a binomial random variable with $n = y$ and $p = 0.5$.

Let $Q(x)$ represent the expected total number of claims, and $R(x)$ represent the expected number of claims outstanding, both given that $X = x$,. Mathematically:

$Q(0) = 1/3$ $Q(1) = 1$

3.3 The linear approximation (Bayesian Credibility)

Let L be the best linear approximation to Q. Mathematically, L is the linear function that minimizes $E_X[(Q(x) - L(x))^2]$.

Assuming $L(x) = a + bx$, we must minimize $E_X[(Q(x) - a - bX)]^2$.

3.3.1 Development Formula 1