4. Integrate $y = P\tan P + \log \cos P$
where $P = \frac{dy}{dx}$

→ Given that:

$$y = P\tan P + \log \cos P \longrightarrow (1)$$

∵ It is solvable for y

Diff eqn (1) both sides w.r.t x

$$\frac{d}{dx}(y) = \frac{d}{dx}(P\tan P) + \frac{d}{dx}(\log \cdot \cos P)$$

$$\frac{dy}{dx} = P\cot P + \tan P \frac{dP}{dx} + \frac{1}{\cos P} + \sin P =$$

$$P = \frac{Pd}{dx}(\tan P) + \tan P \frac{d}{dx}(P) + \frac{1}{\cos P} x - \sin P \frac{dP}{dx}$$

$$P = P\sec^2 P \frac{dP}{dx} + \cancel{\tan \frac{dP}{dx}} - \cancel{\tan \frac{dP}{dx}}$$

$$\cancel{P} = \cancel{P}\sec^2 P \frac{dP}{dx}$$

$$dx = \sec^2 P \, dP$$

by integrating both sides
$$\int dx = \int \sec^2 P \, dP$$

$$(y - 4c^2) = 4cx$$

$$\boxed{x = \tan P + c}$$

Proved

Theorem - 2

V.V.I

Q11. (2) Prove that the inverse of each element of a group is unique.

Proof: let $a \in G$ and e is identity of G. if possible let b & c are the inverse a.

Then,

b is inverse of a

$\therefore ab = e \rightarrow (I)$

again, c is inverse of a

$\therefore ac = e \rightarrow (II)$

From eq (I) & (II)

we, have

$ab = ac$

$b = c$ (by left cancellation law)

$\therefore$ The inverse of each element of group is unique

Theorem - 3

V.V.I

Q12. (3) If the inverse of a is a^{-1} then the inverse of a^{-1} is a i.e $(a^{-1})^{-1} = a$.

Proof: let e is the identity of G

then,

we, have

$$a^{-1} a = e \rightarrow (1)$$

multiply eqn (1) by $(a^{-1})^{-1}$ both side

$\Rightarrow (a^{-1})^{-1} [a^{-1} a] = (a^{-1})^{-1} e$

$\Rightarrow (a^{-1})^{-1} [a^{-1} a] = (a^{-1})^{-1}$

$[\underbrace{(a^{-1})^{-1}}_{\text{element}} \underbrace{a^{-1}}_{\text{inverse}}] a = (a^{-1})^{-1}$ identity

$\Rightarrow e a = (a^{-1})^{-1}$ (Inverse law)

$$\boxed{a = (a^{-1})^{-1}}$$

Proved

Theorem:

5. Prove that the identity of a subgroup is the same as that of the group.

Proof:

let H is subgroup of G
and e is identity of G
and e' is identity of H

Now,

$a \in H \Rightarrow e'a = a$ [e' is identity of H]

again

$a \in G = e a$ [e is identity of G]

$\Rightarrow e'a = ea$

$\Rightarrow \boxed{e' = e}$ [Right distibutive law]

Year 2021

GROUP - A

$1 \times 10 = 10$

(a) Define monotonic sequence.

⇒ Sequences which are either increasing or decreasing are called monotone.

(b) Define greatest lower bound of a sequence.

⇒ The unique element that is less than or equal to all elements in a subset.

(c) Define convergent sequence?

⇒ if there exist a real no. such that every neighbourhood of real no contains all but finite many terms of the sequence.

(d) Define D' Alembert's ratio test?

⇒ The ratio test is a test for the convergence of a series where each term is a real or complex no. and a_n is non zero when n is large.

(e) Define group.

⇒ A group is the combination of a set and binary operations.

(f) Define abelian group?

⇒ A abelian group is also called commutative group, is a group in which the result of applying the group operation of two group elements does not depend on the order in which they are written.

Q2. Prove that the identity element of a group is unique?

⇒

Let,

if possible e and e' be two identity elements of group $(G, *)$, ~~Here~~

Hence

$e, e' \in G$

Now,

e is an identity $e' * e = e' \rightarrow (1)$

e' is an identity $e * e' = e \rightarrow (11)$

From eqn (1) and (11)

LHS = LHS ⇒ RHS = ~~B~~LHS

⇒ $e = e'$

Thus the identity element is unique in group $(G, *)$.

Proved

Q4. Prove that every convergent sequence is bounded?

⇒ Proof:

Let $\{a_n\}$ be a convergent sequence and converge to limit l.

let $\varepsilon > 0$ be given, then by defination $\exists$ a positive no. m.

Such that:

$$|a_n - l| < \varepsilon \quad \forall n \geq m$$

$$\varepsilon - l < a_n < \varepsilon + l, \quad \forall n \geq m$$

let $G = \max \{\varepsilon + l, a_1, a_2, a_3 \cdots a_{n+1}\}$

And

$g = \min \{\varepsilon - l, a_1, a_2, a_3 \cdots a_{m-1}\}$

Hence,

$$\boxed{g \leq a_n \leq G \; \forall n \geq m}$$

Therefore, the sequence $\{a_n\}$ is bounded.

Q5

(6) State and prove cauchy's genral principal principle of convergence.

⇒ The neccessary and subbicient condition for the convergence of a sequence $\langle a_n \rangle$ is that,

for every given $\varepsilon > 0$, $\exists$ n_0, $P \in N$ such that:

$$|a_{n+p} - a_n| < \varepsilon \quad \forall \; n > n_0$$

or

A sequence is convergent $\Leftrightarrow$ it is cauchy sequence.

Proof: (i) ⇒ given: $\langle a_n \rangle$ is convergent sequence.

$\because$ every convergent sequence has limit.

let limit of $\langle a_n \rangle = l$

$\therefore$ $|a_n - l| < \frac{\varepsilon}{2}$ $\longrightarrow$ (1)

$\therefore$ $|a_{n+p} - l| < \frac{\varepsilon}{2}$ $\longrightarrow$ (2)

$\Rightarrow |a_{n+p} - a_n| = |a_{n+p} - a_n + l - l|$

$\Rightarrow |a_{n+p} - a_n| = |(a_{n+p} - l) - (a_n - l)| < |a_{n+p} - l| + |a_n - l|$

$\Rightarrow |a_{n+p} - a_n| < |a_{n+p} - l| + |a_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ [from eq (1) & (2)]

$\Rightarrow \boxed{|a_{n+p} - a_n| < \varepsilon \quad \forall \; n > n_0}$

$\therefore$ sequence $\langle a_n \rangle$ is a cauchy sequence.

Proved.

Q. State the Prove Pringsheim's Theorem.

⇒ Statement: ΣU_n is a convergent series of +ve terms and sequence $\{U_n\}$ is monotonic decreasing then prove that $nU_n \to 0$ as $n \to \infty$.

Proof: As given infinite series ΣU_n is convergent series of positive terms.

Also it is given that $\{U_n\}$ is monotonically decre...

$\cancel{U_{n+1} \geq U_{n+n} = U_{2n}}$

$U_{n+1} \geq U_{2n}$

$U_{n+2} \geq U_{2n}$

$U_{n+3} \geq U_{2n}$

$\cdots$

$\cdots$

$\cancel{U_{2n} \geq U_{2n}}$

adding above we get.

$\cancel{U_{n+1} + U_{n+2} + U_{n+3} + \cdots U_{2n} \geq U_{2n}}$

$U_{n+1} + U_{n+2} + U_{n+3} + \cdots U_{n+n}$ is divergent.

$\therefore U_{n+1} + U_{n+2} + \cdots U_{2n} \geq nU_{2n} \longrightarrow ①$

$\because \Sigma U_n$ is convergent series from Cauchy's genral principle of convergence,

$|S_{n+n} - S_n| \leq \varepsilon$

$\Rightarrow |U_{n+1} + U_{n+2} + \cdots U_{n+1}| < \varepsilon$

$\Rightarrow U_{n+1} + U_{n+2} + U_{n+3} + \cdots U_{n+n} \to 0$ as $n \to \infty$

$\Rightarrow U_{n+1} + U_{n+2} + U_{n+3} + \cdots U_{2n} \to 0$ as $n \to \infty$

Proved

So from ① we have

P.T.O.

Q6. Pg 2

$nU_{2n} \to 0$ as $n \to \infty$

$U_n \to \infty$ $nU_n = 0$

$\Rightarrow \lim_{n\to\infty} 2n\, U_{2n} = 0 \quad \to (1)$

So Pringshein's theorem is free for even int $2n$.

$\because$ Series $\{U_n\}$ is divergent

So,

$U_{2n+1} \le U_{2n}$

$\therefore (2n+1)\, U_{2n+1} \le (2n+1)\, U_{2n}$

$(2n+1)\, U_{2n+1} \le \frac{2n+1}{2n} \; 2n\, U_{2n}$

$\lim_{n\to\infty} (2n+1)\, U_{2n+1} \le \lim_{n\to\infty} \left(\frac{2n}{2n} + \frac{1}{2n}\right) 2n\, U_{2n}$

$\le \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right) 2\, nU_{2n}$

$\le \left(1 + \frac{1}{\infty}\right) 0$ from (1)

$\le (1+0)\, 0$

≤ 0

$\lim_{n\to\infty} (2n+1)\, U_{2n+1} = 0$

So, Pringshein's theorem is free for odd integer $(2n+1)$.

$\therefore$ This theorem is free for both even and odd integer.

$\therefore \lim_{n\to\infty} nU_n = 0$

Proved

Q7. Prove that the infinite series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots$ to ∞ is convergent if $p>1$ and divergent $p\le 1$.

⇒ Proof: We consider the following cases:

Case 1: when $p>1$,

$$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$

$$\Rightarrow \Sigma \frac{1}{n^p} = \frac{1}{1} + \left(\frac{1}{2^p}+\frac{1}{3^p}\right) + \left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right) + \dots$$

$$\Sigma \frac{1}{n^p} \le 1 + \left(\frac{1}{2^p}+\frac{1}{2^p}\right) + \left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right) + \dots$$

$$\Rightarrow \Sigma \frac{1}{n^p} < 1 + 2\times\frac{1}{2^p} + 4\times\frac{1}{4^p} + \dots$$

$$\Rightarrow \Sigma \frac{1}{n^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \dots$$

$$\Rightarrow \Sigma \frac{1}{n^p} < \left[\frac{1}{1-\left(\frac{1}{2^{p-1}}\right)}\right]$$

∵ $p>1$

∴ $p-1>0$

⇒ $2^{p-1} > 2^0$

⇒ $2^4 > 1$

⇒ $\frac{1}{2^{1-1}} < \frac{1}{1}$

$r<1$.

∴ The sum of this geometric series.

⇒ $\frac{1}{1-\left(\frac{1}{2}\right)^{p-1}}$ is finite quantity

⇒ Hence the series is convergence,

therefore the given series is convergent.

For $p<1$.

P.T.O.

Case 2: $\Rightarrow$ when $P=1$

$$\Sigma \frac{1}{n^P} = \Sigma \frac{1}{n^1} = \Sigma \frac{1}{n}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$$

$$\Rightarrow \frac{1}{n^P} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$\Rightarrow \Sigma \frac{1}{n^P} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$

$$\Sigma \frac{1}{n^P} > 1 + \frac{1}{2} + \cancel{2} \times \frac{1}{\cancel{4}\,2} + \cancel{4} \times \frac{1}{\cancel{8}\,2} + \cdots$$

$$\Rightarrow \Sigma \frac{1}{n^P} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$\because \lim_{n \to \infty} U_n = \lim_{n \to \infty} \left(\frac{1}{2}\right) = \frac{1}{2} \neq 0$$

$\therefore$ Series on R.H.S is not convergence.

Since, all terms are positive,

$\therefore$ it is not oscillatory.

Hence, the series on R.H.S is divergent.

So, the given series as divergent series. For $P=1$

Case III: when $P<1$

$$n^P < n^1$$

$$\Rightarrow \frac{1}{n^P} > \frac{1}{n} \quad \text{Putting } n = 1, 2, 3 \cdots$$

$$\Rightarrow \frac{1}{1^P} > \frac{1}{1}, \quad \frac{1}{2^P} > \frac{1}{2}, \quad \frac{1}{3^P} > \frac{1}{3} \cdots\cdots$$

adding coloumn wise.

$$\left(\frac{1}{{}^P} + \frac{1}{2^P} + \frac{1}{3^P} + \cdots\right) > \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \cdots\right)$$

$$\Sigma \frac{1}{n^P} > 1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots$$

The Series on R.H.S is divergent by case II;

Hence, the given series is divergent $P<1$.

Proved

Q8. Prove that $(ab)^{-1} = b^{-1}a^{-1}$, a, b be any elements of a Group.

⇒ Proof: Suppose a & b are any elements of G. If a^{-1} and b^{-1} are the inverses of a & b respectively,

Given:

G be a group

$a, b \in G \quad \exists\, a^{-1}, b^{-1} \in G$

$\therefore\ a a^{-1} = a^{-1} a = e$

And

$b b^{-1} = b^{-1} b = e \longrightarrow (1)$ where e is the identity element of G.

Now,

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$$
$$= a e a^{-1}$$
$$= a a^{-1}$$
$$= e.$$

$\therefore\ (ab)\ (b^{-1}a^{-1})$

$\therefore\ (ab)^{-1} = b^{-1}a^{-1}$

Proved

Q9. Prove the set of real numbers forms an abelian Group under addition.

$\Rightarrow$ Let $I = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5 \ldots\}$ be the set of inte.

To prove: $(I, +)$ is an abelian group.

(1) Closure Property:

$\forall\ a, b \in I$ we have $a + b \in I$

(2) Associative Property:

$\forall\ a, b, c \in I$ such that $(a+b)+c = a+(b+c)$.

(3) Existance of identity:

$\forall\ a \in I\ \exists\ 0 \in I$ such that $a + \overset{\text{zero}}{(0)} = \overset{\text{zero}}{(0)} + a = 0$

Thus 0 is an identity element

(4) Existance of inverse.

$\forall\ a \in I\ \exists\ -a \in I$. Such that $a + (-a) = (-a) + a = 0$

$\boxed{e = 0}$ identity element

Thus $(-a)$ is the additive inverse of a.

(5) Commutative / abelian

$\forall\ a, b \in I$ we have $a + b = b + a$

Thus, $(I, +)$ is an abelian group.

Proved

Q 10. State and prove Lagrange's theorem ?

⇒ Statement: " The order of a each subgroup of a finite group is a divisor of the order of the group.

Proof: let, $O(G) = n$ [order of a group] and
$O(H) = m$ [order of subgroup].

We need to show only -
$\lambda = n/m$
$\lambda = O(G) / O(H)$

Let $H = \{h_1, h_2, h_3 \ldots\ldots h_m\}$

Be m distinct elements of H.

If $a \in G$
Then a left cosets.
$aH = \{ah_1, ah_2, ah_3 \ldots\ldots ah_m\}$

These are m different elements.
Since,
if $ah_i = ah_j$
$\boxed{h_i = h_j}$ [By L.C.L]

∴ every left coset of H in G will have m distinct elements

We know that a finite group G must be decompose in a finite number of distinct left coset say λ.

P.T.O.

i.e a_1H, a_2H, a_3H .... $a\lambda H$.
are the λ disjoint left cosets of H in G.

$G = a_1 H \cup a_2 H \cup a_3 H \cup \cdots \cup a_\lambda H$

Since every left coset contains m elements and these are λ cosets. So total number of elements of all the coset is λm. which must to be equal to be equal to be total number of elements of G.

$$n = \lambda m$$

$$\Rightarrow \lambda = \frac{n}{m}$$

$$\lambda = \frac{O(G)}{O(H)}$$

$\Rightarrow$ $\boxed{O(H) \text{ is a divisor of } O(G)}$

Hence, the order of each subgroup of a finite group is a divisor of a group.

Proved

Q11. Solve: $3x^2\,dx + 3y^2\,dy - (x^3 + y^3 + e^{2z})\,dz = 0$.

$\Rightarrow$ let us take z to be a constant, so that $dz = 0$.

$3x^2\,dx + 3y^2\,dy - (x^3 + y^3 + e^{2z})\,dz = 0$

$\Rightarrow 3x^2\,dx + 3y^2\,dy = 0$

integrating, we get $x^3 + y^3 = \text{const} = \phi \rightarrow (1)$

where ϕ may be regarded as function of z.

Diff (1) w r to x, y and z we get:

$3x^2\,dx + 3y^2\,dy = d\phi$

comparing with the given eqn we, get

$d\phi = (x^3 + y^3 + e^{2z})\,dz$

$\Rightarrow \dfrac{d\phi}{dz} = x^3 + y^3 + e^{2z} = \phi + e^{2z}$; $\because x^3 + y^3 = \phi$

$\Rightarrow \dfrac{d\phi}{dz} - \phi = e^{2z}$

which is linear and its I.F $= e^{\int -1\,dz} = e^{-z}$

Hence the solution is:

$\phi \cdot e^{-z} = \int e^{2z} \cdot e^{-z}\,dz = \int e^{z}\,dz = e^{z} + c$

$\Rightarrow \phi = e^{2z} + ce^{z}$

$\Rightarrow \boxed{x^3 + y^3 = e^{2z} + ce^{z}}$

Proved

Q19. Solve the differential equation $(y+1)p - xp^2 + 2 = 0$:

$\Rightarrow$ Given:

$$(y+1)\, p - xp^2 + 2 = 0$$

$$\Rightarrow (y+1)\, p = xp^2 - 2$$

$$\Rightarrow y+1 = xp - \frac{2}{p}$$

$$\Rightarrow y = px - 1 - \frac{2}{p}$$

which is in Clairaut's form

Hence, its general solution is.

$$y = cx - 1 - \frac{2}{c} \longrightarrow (1)$$

inorder to obtain its singular solution.

Putting the value of (1) in (c) Thus

$$yc = c^2 x - c - 2$$

i.e. $c^2 x - c\,(1+y) - 2 = 0 \longrightarrow (2)$

Now, the two roots of [2] will be equal if

$$(1+y)^2 = 4x(-2)$$

which $\Rightarrow (y+1)^2 + 8x = 0$

Proved

This is the required singular solution.

Q 13. Prove that the system of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

$\Rightarrow$ Given: $y^2 = 4a(x+a)$

$$y^2 = 4ax + a^2 \longrightarrow (1)$$

Diff w r to y

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow y\frac{dy}{dx} = 2a \longrightarrow (2)$$

Hence eliminating (a) from eqn (1) & (11)

we get,

$$y^2 = 4x\frac{1}{2}y\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$= 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

Proved

which is the required differential equation.

Q14. If a finite group of order n contains an element of order n, then prove that the group must be cyclic

⇒ Proof: suppose G is a finite group of order n.

Let $a \in G$ and let n be the order of a i.e. n is the least positive integer such that G is cyclic.

if H is the cyclic subgroup of G generated by a i.e.

if $H = \{a^r : r \in A\}$, then the order of H is n,

because the order of the generator a of H is n.

Thus, H is a cyclic subgroup of G and the order of H is equal to the order of G. Hence $H = G$ and therefore G itself is a cyclic group and a is the generator of G.

Proved

Q15. Prove that any two right (left) cosets of a subgroup are either disjoint or identical.

Proof:
⇒ let aH and bH be two left cosets of H in G. Suppose aH and bH are not disjoint. Then there exists at least one element say c such that $c \in aH$ and $c \in bH$.

Let $c = ah_1$ and $c = bh_2$ where $h_1, h_2 \in H$.
Then $ah_1 = bh_2$ (each $= c$)

$\Rightarrow ah_1h_1^{-1} = bh_2h_1^{-1} \Rightarrow ae = (bh_2 \cdot h_1^{-1})$
$\Rightarrow a = b(h_2h_1^{-1})$.

Since H is a subgroup, therefore $h_2h_1^{-1} \in H$.

Let $h_2h_1^{-1} = h_3$. Then $a = bh_3$.

Now, $aH = bh_3H = b(h_3H) = bH$,
Since $h_3 \in H$ and
$\therefore h_3H = H$

Thus, we find that the two left cosets are identical if they are not disjoint.

Thus, either $aH \cap bH = \phi$ (null) or $aH = bH$.

Proved

Q16. If H_1 and H_2 are any two subgroups of a group G, then prove that $H_1 \cap H_2$ is also a subgroup of G.

⇒ Proof:

Let H_1 and H_2 be any subgroup of G then ~~H_2~~ $H_1 \cap H_2 \neq \phi$ since atleast the identity element e is common to both H_1 and H_2.

In order to prove that $H_1 \cap H_2$ is a subgroup then it is sufficient to prove that:

$A \in H_1 \cap H_2$, $b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

$(a \in H,\ b \in H \Rightarrow ab^{-1} \in H)$

$A \in H_1 \cap H_2 \Rightarrow A \in H_1$ and $a \in H_2$

$B \in H_1 \Rightarrow B \in H_1$ and $B \in H_2$

Now,

Given: H_1 is subgroup

$a \in H_1,\ b \in H_1 \Rightarrow ab^{-1} \in H_1 \rightarrow (1)$

Again

H_2 is subgroup

$a \in H_1$, ~~and~~ $b \in H_2 \Rightarrow ab^{-1} \in H_2 \rightarrow (11)$

From (1) and (11)

$ab^{-1} \in H_1$ and $ab^{-1} \in H_2$

$\Rightarrow ab^{-1}\ H_1 \cap H_2$

$\therefore \boxed{H_1 \cap H_2}$ is subgroup.

Proved

Page No.

Q.17. Show that the set $G = \{a + b\sqrt{2} : a, b \in Q\}$ is a group with respect to addition.

$\Rightarrow$ $G = \{a + b\sqrt{2} : ab \; 2Q\}$

Given: $G = \{a + b\sqrt{2} : a, b \in Q\}$

(I) closure property:

let $x, y \in G$ and $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$

$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2})$

$= (a+c) + (b + d\sqrt{2}) \in G \; \forall a, b, c, d \in Q$

It holds the clouser property.

(II) Associative

let $x, y, z \in G$

$x = a + b\sqrt{2}$

$y = c + d\sqrt{2}$

$z = e + f\sqrt{2}$

$\forall a, b, c, d, e, f \in Q$

$(x+y)+z = x + (y+z)$

L.H.S $(x+y)z$

$\Rightarrow [(a + b\sqrt{2}) (b+d)\sqrt{2}) + e + f\sqrt{2})$

$(a + c + e) + (b + d + f)\sqrt{2} \longrightarrow (I)$

R.H.S.

$x + (y+z)$

$= (a + b\sqrt{2}) + [(c + d + \sqrt{2}) + (e + f\sqrt{2})]$

$= (a + c + e) + (b + d + f))\sqrt{2} \longrightarrow (II)$

From eqn (I) & (II)

L.H.S = R.H.S.

$(a + c + e) + (b + d + f)\sqrt{2} = (a + c + e) + (b + d + f)\sqrt{2}$

It holds associative property.

P.T.O.

(iii) Existance of identity

$0 \in Q$

$(0 + 0\sqrt{2}) \in G$

$(0 + 0\sqrt{2}) + (a + b\sqrt{2})$

$\Rightarrow (0 + a) + (0 + b)\sqrt{2}$

$\Rightarrow a + b\sqrt{2}$

Hence $(0 + 0\sqrt{2})$ is the identity of G.

(iv) Existance of Inverse

$-a, -b \in Q$

$(-a) + (-b)\sqrt{2} \in G$

$(a + b\sqrt{2}) + (-a - b\sqrt{2})$

$\Rightarrow (a - a) + (b - b)\sqrt{2}$

$\Rightarrow 0 + 0\sqrt{2}$

Hence $(-a - b\sqrt{2})$ is the inverse of G.

The set G holds the all four property
Therefore the set G is a group.

Proved

Q18. Test the convergence of the series whose general term is $\left(1-\frac{1}{n}\right)^{n^2}$.

$\Rightarrow$ given $\left(1-\frac{1}{n}\right)^{n^2}$

$$U_n = \left(1-\frac{1}{n}\right)^{n^2}$$

$$\lim_{n\to\infty} U_n^{1/2} = \lim \left\{\left(1-\frac{1}{n}\right)^{n^2}\right\}^{1/n}$$

$$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n}$$

$$= \lim_{n\to\infty} \left[\left\{1+\left(-\frac{1}{n}\right)\right\}\right]^{-n \times -1}$$

$$\Sigma U_n = u_1 + U_2 \cdots\cdots u_n$$

$$U_n = \left(1-\frac{1}{n}\right) n^2$$

$$\lim_{n\to\infty} (1+n)^{1/x} = e$$

$$\lim_{n\to\infty} \left(1+\frac{1}{}\right)^n = e$$

$$l = \lim_{n-\infty} \left[\left\{1+\left(-\frac{1}{n}\right)\right\}^n\right]^{-1}$$

$\Rightarrow$ $l = e^{-1}$

$\Rightarrow$ $l = \frac{1}{e} < 1$ $\{\because 2<e<3$

$\Rightarrow$ $l < 1$

Σu_n is convergent.

Proved

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Q19. Prove that the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ is convergent. or $\sum \frac{1}{n^2}$

$\Rightarrow$

Soln: $\sum a_n = \sum \frac{1}{n^2}$

$n = 1, 2, 3, 4 \cdots n$

let $S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$

$n > m$

$S_m = \frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2}$

$n = m+1, m+2, m+3 \cdots n$

$S_{m+1} = \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \frac{1}{(m+3)^2} \cdots + \frac{1}{m^2}$

$\therefore |S_n - S_m| = \left| \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} \cdots \frac{1}{n^2} \right|$

$= \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots \frac{1}{n^2}$

$\Rightarrow \frac{\cancel{m+1}}{m(m+1)^{\cancel{2}}} + \frac{\cancel{(m+2)}}{(m+1)(m+2)^{\cancel{2}}} + \cdots + \frac{n}{(n-1)n}$

$\Rightarrow \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(n-1)n}$

$\Rightarrow \frac{(m+1)-m}{m(m+1)} + \frac{(m+2)}{(m+1)} - \frac{(m+1)}{(m+2)} \cdots \frac{n(n-1)}{\cancel{n(n-1)}}$

$\Rightarrow \frac{\cancel{m+1}}{m\cancel{(m+1)}} - \frac{\cancel{m}}{\cancel{m}(m+1)} + \frac{\cancel{m+2}}{(m+1)\cancel{(m+2)}} - \frac{m+1}{(m+1)(m+2)} \cdots$

$\cdots + \frac{n}{n(n-1)} - \frac{(n-1)}{n(n-1)}$

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$\Rightarrow \left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots \left(\frac{1}{(n-1)} - \frac{1}{n}\right)$

$\Rightarrow \frac{1}{m} - \cancel{\frac{1}{m+1}} + \cancel{\frac{1}{m+1}} - \frac{1}{m+2} + - \cdots \frac{1}{n-1} - \frac{1}{n}$

$= \frac{1}{m} - \frac{1}{n}$

$< \frac{1}{m}$

$\therefore |s_n - s_m| < \varepsilon$

$\frac{1}{m} < \varepsilon$

ie $m > \frac{1}{\varepsilon}$

P is a positive integer $\frac{1}{\varepsilon}$

$|s_n - s_m| < \varepsilon \ \forall \ n, m \geq P$ proved

It is convergent.

Q20- Determine the convergency of the series.

$$\frac{1}{4} + \frac{1\cdot3}{4\cdot7} + \frac{1\cdot3\cdot5}{4\cdot7\cdot10} + \frac{1\cdot3\cdot5\cdot7}{4\cdot7\cdot10\cdot13} + \cdots$$

$\Rightarrow$ Given: $\frac{1}{4} + \frac{1\cdot3}{4\cdot7} + \frac{1\cdot3\cdot5}{4\cdot7\cdot10} + \frac{1\cdot3\cdot5\cdot7}{4\cdot7\cdot10\cdot13} + \cdots$

let $U_n = \frac{1\cdot3\cdot5\cdot7}{4\cdot7\cdot10\cdot13} \cdots\cdots \frac{(2n-1)}{(3n+1)}$

$U_{n+1} = \frac{1\cdot3\cdot5\cdot7 \cdots\cdots (2n+1)(2n+1)}{4\cdot7\cdot10\cdot13 \quad (3n+1)(3n+4)}$

$\left\{ \begin{aligned} &1,3,5,7 \\ &t_n = a+(n-1)d \\ &= 1+(n-1)2 \\ &= 1+2n-2 \\ &= 2n-1 \end{aligned} \right.$

$\therefore \frac{U_n}{U_{n+1}} = \frac{3n+4}{2n+1}$

Now,

$$\lim_{n\to\infty}\left(\frac{U_n}{U_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{3n+4}{2n+1}\right)$$

$$= \lim_{n\to\infty}\left[\frac{n(3+4/n)}{n(2+1/n)}\right]$$

$$\boxed{= 3/2 > 1}$$

Proved