Initially choose door one as having the car. Game show host reveals that door 3 doesn't have the car. What is the probability that door one has the car given that the host reveals that door 3 doesn't have the car?

$$A \equiv \text{car is behind door one}$$
$$B \equiv \text{host reveals door 3 as not having the car}$$

Want to know $P(A|B)$.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(B|A)$ is the probability that the host reveals door 3 as not having the car given that the car is behind door one. Since the car is behind door one, the host could have revealed door 2 or door 3 as not having the car; therefore,

$$P(B|A) = \frac{1}{2}.$$

$P(A)$ is the probability that the car is behind door one; that is just

$$P(A) = \frac{1}{3}.$$

$P(B)$ is the probability that the host reveals door 3 as not having the car. There are 4 possibilities: the car is behind door 1 and the host reveals door 2 as not having the car, the car is behind door 1 and the host reveals door 3 as not having the car, the car is behind door 2 and the host reveals door 3 as not having the car, and, finally, the car is behind door 3 and the host reveals door 2 as not having the car. Two of the four cases involve the host revealing door 3 as not having the car, therefore

$$P(B) = \frac{2}{4} = \frac{1}{2}.$$

Thus,

$$P(A|B) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}.$$

And so we have that $P(\neg A|B) = 2/3$ which is the same as the probability that door 2 has the car after the host revealed door 3 as not having the car. Hence, the optimal move is to choose door 2. By symmetry of the argument, the optimal move is to always switch doors from the original guess.

Why do smart people get this wrong?

Change B to

$$B' \equiv \text{door 3 does not have car}$$

This induces the following changes:

$$P(B'|A) = 1$$
$$P(A) = \frac{1}{3}$$
$$P(B') = \frac{2}{3},$$

and so

$$P(A|B') = \frac{(1)\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{2}.$$

Thus, with the change to proposition B', the optimal strategy is indifferent between switching or staying with the original guess; that is, given information B', the probability that the car is behind door 1 is equal to the probability that the car is behind door 2.

Conclusion

My conjecture as to why people tend to choose the incorrect answer to this problem is that they are reasoning with information B' rather than B. Since the difference between B and B' is essentially the phrase "host reveals", this amounts to concluding that people tend to not account for the hosts explicit role in revealing information. And so, how the information is revealed itself contains pertinent information.