Shaded Triangle Problem: Fractal Solution

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Problem Statement

Find the proportion of area that is shaded in figure 1.

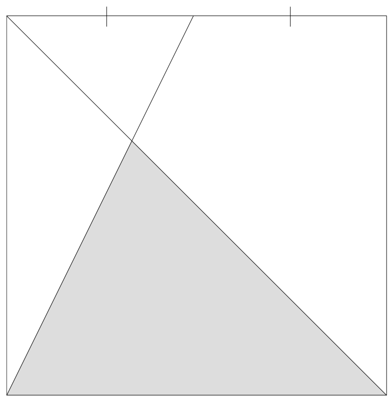

Figure 1: The enclosing shape is a perfect square and the line coming from the lower left corner bisects the top line of the square evenly.

Solution

Let x be the length of one side of the square and h be the height of the leftmost triangle; i.e., the length of a line extending from the left side of the square (and orthogonal to it) and reaching the intersection of the two diagonal lines within the square. See figure 2.

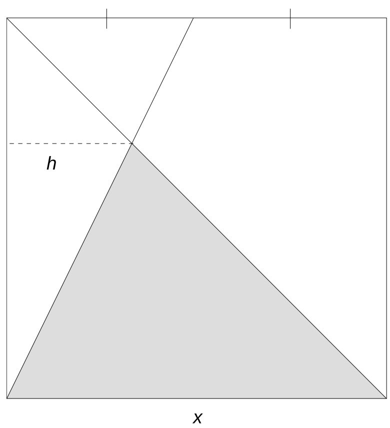

Figure 2: Variables

For the moment, assume that h can be expressed as a linear function of x:

$$h(x) = \alpha x \tag{1}$$

where α is some constant. In that case, the proportion of area that is shaded, A, can be expressed as

$$\begin{aligned} A &= \frac{\frac{1}{2}x^2 - \frac{1}{2}xh}{x^2} \\ &= \frac{\frac{1}{2}x^2 - \frac{1}{2}\alpha x^2}{x^2} \\ &= \frac{1-\alpha}{2} \end{aligned} \tag{2}$$

Thus, we need only to show that h is in fact a linear function of x and find the value of α.

Finding h

Divide the square in figure 1 evenly into four quadrants. See figure 3.

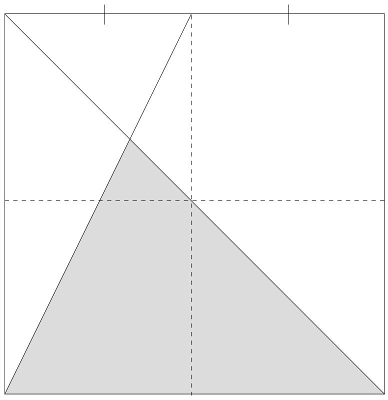

Figure 3: Evenly divided into four quadrants

The square in the top left resembles the statement of the original problem just on a smaller scale. In fact, the top left square is an inverted version of the original problem scaled by one fourth. Divide this (top-left) square evenly into four quadrants. See figure 4.

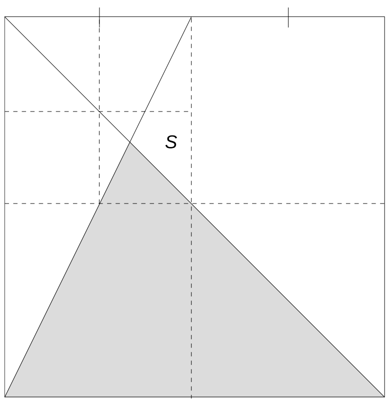

Figure 4: Top-left evenly divided into four quadrants

The square labeled s in the resulting figure (figure 4) is exactly the original problem scaled by one sixteenth. Thus, the problem is fractal (i.e. exhibits self-similarity). We will use this fact to find h.

We can express h as

$$h = \frac{1}{4}x + h', \tag{3}$$

where h' is the length in s that corresponds to h in the original problem; i.e. it is the scaled version of h. See figure 5.

Now, since s is exactly the same problem, only scaled, we can express h' as

$$h' = \frac{1}{4}x' + h'' \tag{4}$$

where x' corresponds to the scaled version of x, and h'' corresponds to h' in an even smaller version of the original problem, call it s' (not shown in the figures). From the figure it is easy to see that $x' = \frac{1}{4}x$, so we can write (4) as

$$h' = \left(\frac{1}{4}\right)^2 x + h''.$$

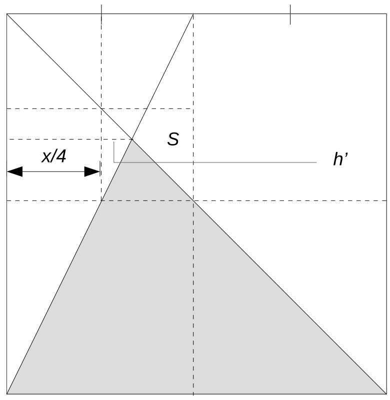

Figure 5: Top-left evenly divided into four quadrants

Using this equation, we can write (3) as

$$h = \frac{1}{4}x + \left(\frac{1}{4}\right)^2 x + h''.$$

Now, as mentioned above, h'' corresponds to another fractal version of the original problem, so we can apply the same reasoning to get

$$h = \frac{1}{4}x + \left(\frac{1}{4}\right)^2 x + \left(\frac{1}{4}\right)^3 x + h'''.$$

We can continue this process indefinitely (since the problem is fractal), and so by mathematical induction we have that

$$h = x \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i.$$

The sum above is just a geometric series and so converges:

$$\begin{aligned} h &= x \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i \\ &= x \left(\frac{1}{1 - \frac{1}{4}} - 1\right) \\ &= \frac{1}{3}x. \end{aligned}$$

Thus, we have shown that h is a linear function of x with $\alpha = \frac{1}{3}$. Using (2) from the first part of the solution, we have that the proportion of area that is shaded is

$$A = \frac{1 - \alpha}{2} = \frac{1 - \frac{1}{3}}{2} = \frac{1}{3}.$$