MATH2923 Analysis

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Abstract

Hi there! This is the set of Analysis notes I wrote during my self study. Feel free to send me feedback via email.

I write about math and programming on my personal website, and share non-technical snippets and other opinions through Medium.

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1. Real Numbers

Introduction to Real Numbers

1.1 Introduction to Real Numbers

1.1.1 Axiomatic Description of Real Numbers

Natural numbers seem fairly trivial, so we start by defining the set of natural numbers using a set of axioms.

Definition 1.1 *(Peano Axioms for $\mathbb{N}$)*

The set of natural numbers is a set $\mathbb{N}$ with a distinguished element 0 and a map $S : \mathbb{N} \to \mathbb{N} \setminus 0$ such that

(i) $S : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ is injective.

(ii) If $N \subseteq \mathbb{N}, \{0\} \in \mathbb{N}$ and $S(n) \in \mathbb{N}\ \forall n \in \mathbb{N} \Rightarrow N = \mathbb{N}$

*S is the **successor function** - the map $n \mapsto n + 1$ once you define addition.*

*The second axiom is the **Principle of Mathematical Induction** which then leads to addition and multiplication properties as well as the concept of ordering.*

To define $\mathbb{R}$ uniquely up to *isomorphism*, we layer the three sets of axioms on top of the Peano Axioms. They are

1. *Field Axioms*
2. *Order Axioms*
3. *Completeness Axiom*

1.1.2 Field Axioms

The field axioms are as follows:

1. $x + y = y + x\ \forall x, y \in \mathbb{R}$: (commutativity)
2. $(x + y) + z = x + (y + z)\ \forall x, y, z \in \mathbb{R}$: (associative law of addition)
3. $\exists 0 \in \mathbb{R}$ with $x + 0 = x\ \ \forall x \in \mathbb{R}$: (natural element for addition)
4. for every $x \in \mathbb{R}\ \exists y \in \mathbb{R}$ s.t. $x + y = 0$: (existence of additive inverse)
5. $x \cdot y = y \cdot x\ \ \forall x, y \in \mathbb{R}$: (associative law of multiplication)
6. $x \cdot (y \cdot z) = (x \cdot y) \cdot z\ \ \forall x, y, z \in \mathbb{R}$: (associative law of multiplication)

7. for every $x \in \mathbb{R}$ $\exists 1 \in \mathbb{R}$ s.t. $1 \cdot x = x$: (neutral element of multiplication)

8. for every $x \in \mathbb{R} \setminus \{0\}$ $\exists y \in \mathbb{R}$ s.t. $x \cdot y = 1$: (existence of multiplicative inverse)

9. $x \cdot (y + z) = x \cdot y + x \cdot z$: (distributive law)

These are sufficient in building up the framework with which you use to do elementary operations.

1.1.3 Order Axioms

1. $x < y$ IFF $0 < y - x$

2. If $0 < x, y$ then $0 < x + y$

3. If $0 < x, y$ then $0 < x \cdot y$

4. For every $x \in \mathbb{R}$ precisely one of the following is true:

$$0 < x, \ x = 0 \text{ or } 0 < -x$$

1.1.4 Completeness Axiom

In order to properly explain the *Completeness Axiom*, we will need to introduce the idea of upper and lower bounds, supremum and infimum.

Definition 1.2 *(upper and lower bounds)*

Let $A \subseteq \mathbb{R}$ be a set. We call A a bounded from above (below) if $\exists m \in \mathbb{R}$ s.t. $x \leq m$ $(\geq m)$ $\forall x \in A$. Every such m is a upper (lower) bound.

If A is not bounded, then A is unbounded.

Definition 1.3 *(supremum and infimum)*

The supremum and infimum is the 'optimal' upper and lower bound of the set $A \subseteq \mathbb{R}$.

If a set in $\mathbb{R}$ has an upper bound in $\mathbb{R}$, then

*$M \in \mathbb{R}$ is the **supremum** (least upper bound) of A if*

(i) M is an upper bound of A

(ii) For every upper bound m of A, $M \leq m$

Define $M := \sup A$. If A is unbounded from above, $M := \infty$ and if $A = \varnothing$ then $M := -\infty$.

*Similarly for infimum, $M \in \mathbb{R}$ is the **infimum** (greatest lower bound) of A if*

(i) M is an lower bound of A

(ii) For every lower bound m of A, $M \geq m$

Define $M := \inf A$. If A is unbounded from below, $M := -\infty$ and if $A = \varnothing$ then $M := \infty$.

Be very careful here, as $\sup A$ can only exist if it is a least upper bound within the SAME FIELD as A. This ties in directly with the *Completeness Axiom*

Definition 1.4 *(maximum and minimum)*

Let $A \subseteq \mathbb{R}$ be a non-empty set. M is a maximum (or minimum) or A if

(i) $M \in A$

(ii) $x \leq M$ $(\geq M)$ $\forall x \in A$

If M is the maximum of A we define $M := \max A (\min A)$

Remark 1.5 *If the maximum (minimum) of $A \subseteq \mathbb{R}$ exists, it coincides with $\sup A$ $(\inf A)$.*

The idea that a supremum (and/or infimum) alway exists for a bounded set A is called the **Completeness Axiom**.

Definition 1.6 *(Completeness or Least Upper Bound Axiom)*

Every non-empty set $A \subseteq \mathbb{R}$ which is bounded from above admits a least upper bound (supremum).

Remark 1.7 *Only the real $\mathbb{R}$ field satisfies all of the three axioms and the Peano axioms.*

1.1.5 Supremum and Infimum

We discuss a little more about supremum and infimum.

Lemma 1.8 *Let $A \neq \varnothing$ be a subset of $\mathbb{R}$. Then the following assertions are true*

(i) For every $t < \sup A$ $\exists a \in A \to t < a$

(ii) For every $t > \inf A$ $\exists a \in A \to t > a$

The proof is fairly trivial. Since t is less than the supremum, t cannot be an upper bound. Then by the definition of the upper bound there must exist another value in A that is larger than t.

A similar proof can be constructed for the infimum.

We now show equivalence of a few propositions

Proposition 1.9 *The following assertions are equivalent:*

(i) $\sup A$ exists for every non-empty $A \subset \mathbb{R}$ bounded from above.

(ii) $\inf A$ exists for every non-empty $A \subset \mathbb{R}$ bounded from below.

(iii) If $A, B \subseteq \mathbb{R}$ are non-empty subsets such that $a \leq b$ $\forall a \in A, b \in B$, then $\exists c \in \mathbb{R}$ s.t. $a \leq c \leq b, \forall a \in A, b \in B$

1.1.6 Basic Properties of Real Numbers

Definition 1.10 *(Archimedean Property of $\mathbb{N}$)*

The set $\mathbb{N}$ is not bounded from above in $\mathbb{R}$. For every $x \in \mathbb{R}$ $\exists n \in \mathbb{N}$ s.t. $x < n$.

(Sketch of Proof) Assume $s := \sup \mathbb{N} < \infty$ exists. Then $\exists n \in \mathbb{N}$ s.t.

$$s - \frac{1}{2} < n \leq s \Rightarrow s + \frac{1}{2} < n \leq s$$

$n + 1 \in \mathbb{N}$ which is strictly greater than the defined $s = \sup A$.

Hence we have reached a contradiction.

Corollary 1.11 *For every $\epsilon > 0$, $\exists n \in \mathbb{N}$ s.t.*

$$0 < \frac{1}{n} < \epsilon$$

(Sketch of Proof) Let $\epsilon > 0 \in \mathbb{R}$. Then by the Archimedean property we have

$$\frac{1}{\epsilon} < n \Rightarrow \frac{1}{n} < \epsilon$$

Proposition 1.12 *(Denisty of rational and irrational numbers)*

Suppose $a, b \in \mathbb{R}$ with $a < b$. Then

(i) $\exists r \in \mathbb{Q}$ with $a < r < b$;

(ii) $\exists x \in \mathbb{R} \setminus \mathbb{Q}$ with $a < x < b$;

(Sketch of Proof) Fix a, b with $0 \leq a < b$. Then

$$\frac{1}{n} < b - a \Rightarrow an + 1 < bn$$

Choose a minimal $m \in \mathbb{N}$ s.t.

$$m - 1 \leq an < m \Rightarrow an < m \leq an + 1 < bn$$

and since m and n are both natural numbers we have $a < \frac{m}{n} < b$.

If $a < 0$, then we choose $p \in \mathbb{N}$ s.t. $a + p > 0$, possible by the Archimedean property of $\mathbb{N}$. Then by what we just proved we get our result.

Now for $x\mathbb{R}$. By (i) we have for $m \in \mathbb{Q}$

$$a - \sqrt{2} < m < b - \sqrt{2} \Rightarrow a < m + \sqrt{2} < b$$

and $m + \sqrt{2}$ is in $\mathbb{R}$.

1.2 Revision of Complex Numbers

1.2.1 Quick Overview of Basic Properties

Define $z = x + iy$, $z \in \mathbb{C}$ and $x, y \in \mathbb{R}$. Then we have

1. $Re(z) = x$: the real part
2. $Im(z) = y$: the imginary part
3. $\bar{z} = x - iy$: the conjugate
4. $|z| = |z\bar{z}| = \sqrt{x^2 + y^2}$: the modulus

Proposition 1.13 *Let $z, w \in \mathbb{C}$. Then*

1. $|zw| = |z||w|$
2. $|z + w| \leq |z| + |w|$
   (Sketch of Proof)

$$\begin{aligned} |zw| &= \sqrt{(z\bar{z})(w\bar{w})} \\ &= \sqrt{(|z|^2)(|w|^2)} \\ &= |z||w| \end{aligned}$$

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + 2Re(zw) + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

1.2.2 Higher Dimensional Spaces

Definition 1.14 *(Inner product)*

Given $\mathbf{x} = x_1, x_2, ..., x_n$ and $\mathbf{y} = y_1, y_2, ..., y_n$, $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$. We define their inner product as

$$x \cdot y := \sum_{i=1}^{n} x_i \bar{y}_i$$

Proposition 1.15 *(Properties of the Modulus)*

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{K}^n$ and $\alpha, \beta \in \mathbb{K}$. Then

(i) $\mathbf{x} \cdot \mathbf{x} \geq 0$ *with equality IFF $x = 0$.*

(ii) $\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}}$

(iii) $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$

Remark 1.16 *From the above properties, we also have*

(i) $(\alpha\mathbf{x})\cdot\mathbf{y} = \alpha(\mathbf{x}\cdot\mathbf{y})$

(ii) $(\mathbf{y}\cdot\alpha\mathbf{y}) = \bar{\alpha}(\mathbf{x}\cdot\mathbf{y})$

Definition 1.17 *(Euclidean Norm)*

Given $\mathbf{x} = x_1, x_2, ..., x_n,\ \mathbf{x}\in\mathbb{K}^n$, *we define the Euclidean norm by*

$$\|\mathbf{x}\| := \sqrt{\mathbf{x}\cdot\mathbf{x}} = \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}}$$

Definition 1.18 *(Cauchy-Schwarz Inequality)*

For all $\mathbf{x},\mathbf{y}\in\mathbf{K}^n$

$$\|\mathbf{x}\cdot\mathbf{y}\| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

with equality IFF x and y are ***linearly dependent****, that is, $\exists t\in\mathbb{R}$ with $x = ty$ or $y = tx$.*

(Sketch of proof)

$$\begin{aligned}
0 \leq \|\mathbf{x} - t(\mathbf{x}\cdot\mathbf{y})\mathbf{y}\|^2 &= (\mathbf{x} - t(\mathbf{x}\cdot\mathbf{y})\mathbf{y})\cdot(\mathbf{x} - t(\mathbf{x}\cdot\mathbf{y})\mathbf{y}) \\
&= \mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot(t(\mathbf{x}\cdot\mathbf{y})\mathbf{y}) - (t(\mathbf{x}\cdot\mathbf{y})\mathbf{y})\cdot\mathbf{x} - t(\mathbf{x}\cdot\mathbf{y})\mathbf{y}\cdot t(\mathbf{x}\cdot\mathbf{y})\mathbf{y} \\
&= \|x\|^2 - t(\overline{\mathbf{x}\cdot\mathbf{y}})(\mathbf{x}\cdot\mathbf{y}) - t(\mathbf{x}\cdot\mathbf{y})(\mathbf{y}\cdot\mathbf{x}) + t^2(\mathbf{x}\cdot\mathbf{y})^2\|y\|^2 \\
&= \|x\|^2 - 2t|\mathbf{x}\cdot\mathbf{y}| + t^2(\mathbf{x}\cdot\mathbf{y})^2\|y\|^2
\end{aligned}$$

and this is a quadratic, with real coefficients that are non-negative. This is only possible if

$$|\mathbf{x}-\mathbf{y}|^4 - |\mathbf{x}-\mathbf{y}|^2\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$$

Rearranging the inequality will lead to the result.

Note: $\mathbf{x}\cdot(t(\mathbf{x}\cdot\mathbf{y})\mathbf{y})$ here is a dot product where $t(\mathbf{x}\cdot\mathbf{y})$ is the scalar!

Note Two: $|\mathbf{z}|^2 = \mathbf{z}\bar{\mathbf{z}}$

Note Three: If all three coefficients $a, b, c\in\mathbb{R}$ for a quadratic equation $ax^2 + bx + c$, then for 2 complex conjugate roots $\Delta < 0$.

Proposition 1.19 *(Properties of the Norm)*

Let $\mathbf{x},\mathbf{y},\in\mathbb{K}^n$ *and* $\alpha\in\mathbb{K}$. *Then*

(i) $\mathbf{x}\geq 0$ *with equality IFF* $\mathbf{x} = 0$.

(ii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$

(iii) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$*: (triangle inequality)*

Note: The triangle inequality is the most commonly used tool to prove theorems and illustrate ideas in this course. If you're stuck, try the triangle inequality =).

Corollary 1.20 *(Reversed Triangle Inequality)*

If $\mathbf{x}, \mathbf{y} \in \mathbf{K}^n$, *then*

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$$

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2. Sequences and Limits

Sequences and Convergence

2.3 Sequences and Convergence

Definition 2.21 *(Sequence)*

A sequence in a set X is a function $x : \mathbb{N} \to X$.

We usually denote this as $x_n := x(n) \longrightarrow x_1, x_2, x_3 \ldots$.

Most of the time we shorten this to $\{x_n\}$ or (x_n)

Definition 2.22 *(Bounded sequences)*

A sequence in $\{x_n\}$ in $\mathbb{R}$ is called bounded from above (below) if $\exists m \in \mathbb{R}$ s.t. $x_n \leq m$ $(\geq m)$ $\forall n \in \mathbb{N}$.

A sequence is bounded if it has both a supremum and infimum.

A sequence $\{x_n\}$ in $\mathbb{K}^n$ is bounded if it's norm is bounded in $\mathbb{R}$.

Definition 2.23 *(Convergence and limit)*

Suppose $\{x_n\}$ is a sequence in $\mathbb{K}^n$. Then convergence of the sequence $\{x_n\} \to x$ as $n \to \infty$ exists if $\forall \epsilon > 0$ $\exists n_\epsilon \in \mathbb{N}$ s.t.

$$\|x_n - x\| < \epsilon$$

$\forall n \geq n_\epsilon$. We can also denote this as

$$x = \lim_{n\to\infty} x_n$$

x is called the limit of x_n.

If a sequence has no limit the (i.e. does not converge) we say it diverges.

Proposition 2.24 *Suppose that $\{x_n\}$ is a convergent sequence in $\mathbb{K}^n$ with the limit x. Then $\{x_n\}$ is bounded and $\|x_n\| \to \|x\|$ in $\mathbb{R}$.*

Theorem 2.25 *(Limit Laws)*

Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{K}^n$ with limits $x_n \to x$ and $y_n \to y$. We also have $\alpha_n \in \mathbb{K}$ a scalar sequence with $\alpha_n \to \alpha$.

Then the following are the basic limit laws

(i) $x_n + y_n \to x + y$

(ii) $\alpha_n x_n \to \alpha x$

(iii) $x_n \cdot y_n \to x \cdot y$

(iv) if $\alpha \neq 0$*, then* $\exists m \in \mathbb{N}$ *s.t.* $|\alpha_n| > \left|\frac{\alpha}{2}\right|$, $\forall n > m$ *and* $\frac{1}{\alpha_n} \to \frac{1}{\alpha}$.

(Sketch of proof for (iv)) We rewrite the fraction as

$$\left|\frac{1}{\alpha_n} - \frac{1}{\alpha}\right| = \frac{|\alpha_n - \alpha|}{|\alpha_n| \cdot |\alpha|}$$

by the definition of convergence, $\exists m \in \mathbb{N}$ *s.t.*

$$|\alpha_n| \in \left(\frac{|\alpha|}{2}, \frac{3|\alpha|}{2}\right)$$

Now we utilise the result

$$|\alpha_n| > \frac{|\alpha|}{2} \Rightarrow \frac{1}{|\alpha_n|} < \frac{2}{|\alpha|}$$

to obtain

$$\begin{aligned}\left|\frac{1}{\alpha_n} - \frac{1}{\alpha}\right| &= \frac{|\alpha_n - \alpha|}{|\alpha_n| \cdot |\alpha|} \\ &< \frac{2}{|\alpha|^2}|\alpha_n - \alpha| \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

as required.

Remark 2.26 *The sequence* $\{z_n\} \in \mathbb{C}$ *converges IFF the real and imaginary parts converge.*

You can think of a sequence in $\mathbb{C}^n$ *to have the same convergence properties the equivalent sequence in* $\mathbb{R}^{2n}$ *where a sequence* $z_n = x_n + iy_n \in \mathbb{C}^n$ *can be written as*

$$w := (x_1, y_1, x_2, y_2, ..., x_n, y_n) \in \mathbb{R}^{2n}$$

which gives identical norms:

$$\begin{aligned}\|z\| &= \left(\sum_{i=1}^{n} |z_i|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{n} x_i^2 + y_i^2\right)^{\frac{1}{2}} \\ &= \|w\|\end{aligned}$$

Definition 2.27 *(Squeeze Law)*

Suppose that a_n, b_n *and* x_n *are sequences in* $\mathbb{R}$ *s.t.* $\exists m \in \mathbb{N}$ *s.t.* $a_n \leq x_n \leq b_n,\ \forall n \geq m$. *If* $a_n \to x$ *and* $b_n \to x$ *then* $x_n \to x$.

Proposition 2.28 *(Preservation of Inequalities)*

Let $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* $\mathbb{R}$ *with* $x_n \to x$ *and* $y_n \to y$.

If $\exists m \in \mathbb{N}$ *s.t.* $x_n \leq y_n,\ \forall n \geq m$ *then* $x \leq y$.

Remark 2.29 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* $\mathbb{R}$ *with* $x_n \to x$ *and* $y_n \to y$.

$x_n < y_n,\ \forall n \in \mathbb{N}$ *DOES NOT imply that* $x < y$*!*

Strict inequality in general is not preserved. Think $x_n = -\frac{1}{n}$ *and* $y_n = \frac{1}{n}$.

2.3.1 Monotone Sequences and Basic Limits

Definition 2.30 *(Monotone Sequences)*

A sequence $\{x_n\} \in \mathbb{R}$ *is called monotone increasing if* $x_{n+1} \geq x_n,\ \forall n \in \mathbb{N}$.

If $x_{n+1} > x_n,\ \forall n \in \mathbb{N}$ *then the sequence* ***strictly*** *monotone increasing.*

Theorem 2.31 *Every bounded and montone sequence in* $\mathbb{R}$ *converges to it's supremum (or infimum).*

(i) If $\{x_n\}$ *is bounded and increasing, then* $x_n \to x := \sup_{n \in N} x_n$

(i) If $\{x_n\}$ *is bounded and decreasing, then* $x_n \to x := \inf_{n \in N} x_n$

Sometimes might have an arbitrary sequence with starting values that are not necessarily monotone - $x_1 = 10, x_2 = 3, x_3 = 11, x_4 = \frac{1}{n}$. The above theorem still holds if x_n is bounded as we are mostly concerned with what happens in the limit.

So as long as $x_{n+1} \leq x_n,\ \forall n > m$ for some $m \in \mathbb{N}$ large enough, we say that x_n is eventually monotone and thus will converge for $\sup_{n \geq m} x_n$.

Hence we can relax the theorem and say it only needs to be bounded and monotone eventually.

Definition 2.32 *(Divergence)*

Let $\{x_n\}$ *be a sequence in* $\mathbb{R}$. *Then*

$$\lim_{n \to \infty} x_n = \infty$$

if $\forall M \in \mathbb{R},\ \exists n_M$ *s.t.* $x_n > M,\ \forall n \geq n_M$. *and likewise for a divergence to* $-\infty$.

We note that the construction of the definition of divergence is the opposite of convergence. Of course this was a deliberate choice by our predecessors.

Proposition 2.33 *Let* $\{a_n\}$ *be a sequence in* $\mathbb{R}$.

(i) If a_n is unbounded and monotone increasing, then $x_n \to \infty$

(ii) If a_n is unbounded and monotone decreasing, then $x_n \to -\infty$

The definitions of monotonicity and convergence is fairly straightforward, but proving them without much thought requires familarity with the building blocks.

One more 'computational' types of questions in this course will involve evaluating whether sequences converge. The below is the first example of such a method.

Proposition 2.34 *(Ratio Test for Sequences)*

Suppose that $\{a_n\}$ is a sequence in $\mathbb{R}$, and $a_n > 0$, $\forall n \in \mathbb{N}$.

(i) If $\frac{a_{n+1}}{a_n} \to L < 1$ then $a_n \to 0$

(ii) If $\frac{a_{n+1}}{a_n} \to L > 1$ then $a_n \to \infty$

(iii) If $\frac{a_{n+1}}{a_n} \to L = 1$, then it is the convergence is unclear from this test.

Some more limit results below;

Proposition 2.35 *Fix $k \in \mathbb{N}$ be fixed and $a \geq 0$. Then*

$$\lim_{n\to\infty} n^k a^n = \begin{cases} 0 & \text{if } a \in [0, 1) \\ \infty & \text{if } a > 1 \end{cases}$$

The exponential decay is stronger than polynomial growth!

Proposition 2.36 *Let $a \in \mathbb{C}$. Then*

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

We can show this result using the ratio test.

2.3.2 Arithmetic-geometric Mean Inequality

First we mention the **Bernoulli's inequality** which is used in the proof for the arthimetic-geometric mean inequality.

Lemma 2.37 *(Bernoulli's Inequality)*

For $x \geq -1$ and $n \geq 1$, $n \in \mathbb{N}$, we have

$$(1 + x)^n \geq 1 + nx$$

(Sketch of proof)

Use induction.

Theorem 2.38 *(AM-GM Inequality)*

Suppose that $n \in \mathbb{N}$ and that $x_1, x_2, ..., x_n \geq 0$. Then

$$x_1 x_2 ... x_n \leq \left(\frac{x_1 + x_2 + ... + x_n}{n} \right)^n$$

(Sketch of proof)

We define

$$a_k := \frac{x_1, x_2, ..., x_k}{k}$$

and use the fact that $a_{n+1}, a_n > 0$ and the Bernoulli inequality to write

$$\begin{aligned} \left(\frac{a_{n+1}}{a_n} \right)^{n+1} &= \left(1 + (\frac{a_{n+1}}{a_n} - 1) \right)^{n+1} \\ &\geq 1 + (n+1) \left(\frac{a_{n+1}}{a_n} \right) \\ &\dots \\ &= \frac{x_{n+1}}{a_n} \\ \Rightarrow a_{n+1}^{n+1} \geq a_n^n x_{n+1} \end{aligned}$$

Applying this idea to prove by induction yields

$$\begin{aligned} x_1 x_2 ... x_{n+1} &= (x_1 x_2 ... x_n) x_{n+1} \\ &= a_n^n x_{n+1} \\ &\leq a_{n+1}^{n+1} \\ &= \left(\frac{x_1 + x_2 + ... + x_{n+1}}{n+1} \right)^{n+1} \end{aligned}$$

2.3.3 Existence of nth roots

Theorem 2.39 *For every $a > 0$ and $n \in \mathbb{N}$, $\exists_{=1} x > 0$ (read exactly one x greater than zero) s.t. $x^n = a$. We denote that number by $a^{\frac{1}{n}}$.*

(Sketch of proof) The solution to the nth root must be unique since if $x^n = a = y^n \Rightarrow x = y$.

To show existence, we use Newton's method of finding the nth root

$$x_0 = a \qquad\qquad x_{k+1} := \frac{1}{n} \left((n-1)x_k + \frac{a}{x_k^{n-1}} \right)$$

By assuming $x_k \to x$ we find that $x^n = a$.

It remains to show that x_{k+1} is monotonically decreasing and thus converges.

We use the AM-GM inequality to obtain

$$x_{k+1}^n \geq a \Rightarrow \frac{a}{x_k^n} \leq 1$$

and substitute this result into Newton's method to obtain our result.

Proposition 2.40 *We have* $\sqrt[n]{n} \to 1$ *as* $n \to \infty$.

(Sketch of proof)

Since $n^k a^n = 0$ if $|a| < 1$, we can write

$$\lim_{n\to\infty} n \left(\frac{1}{1+\epsilon}\right)^n = 0$$

Hence by the definition of the limit, $\exists n_0$ s.t.

$$0 < n \left(\frac{1}{1+\epsilon}\right)^n < 1$$

$\forall n > n_0$. We know that $\sqrt[n]{n} \geq 1 \; \forall n \in \mathbb{N}$ and thus we can the inequalities as follows

$$\begin{aligned} 1 - \epsilon < 1 \leq \sqrt[n]{n} < 1 + \epsilon \\ \Rightarrow \left| \sqrt[n]{n} - 1 \right| < \epsilon \end{aligned}$$

$\forall n > n_0$.

Corollary 2.41 $\sqrt[n]{a} \to 1$ *as* $n \to \infty$ *for every* $a > 0$

(Sketch of proof)

Use the squeeze law. Consider what happens if $a < 1$ as a separate case to be more explicit.

2.3.4 Euler Number

Theorem 2.42 *(Euler number)*

$e_n := \left(1 + \frac{1}{n}\right)^n$ *which converges to* e *as* $n \to \infty$.

The limit $e := \lim_{n\to\infty} e_n$ *exists and lies between* $2 < e \leq 4$.

(Sketch of proof)

Apply the AM-GM inequality to show that e_n is monotonically increasing. Then use the AM-GM inequality again with

$$e_n = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (1 + 1_n)^n$$

to show that it has an upper bound of 4.

Proposition 2.43 $e_n = \left(1 + \frac{1}{n}\right)^n$ *converges very slowly and thus we introduce another series that converges faster.*

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

2.3.5 Limit Superior and Limit Inferior

Not all sequences converge, but can we always construct at least 2 arbitrary monotone sequences in $\mathbb{R}$ that converge.

Note that you can converge to $\pm\infty$. If a sequence converges to infinity, it just means that the limit for the sequence does not exist!

Definition 2.44 *(Limit Superior and Inferior)*

Let $\{x_n\}$ be an arbitary bounded sequence in $\mathbb{R}$. For $n \in \mathbb{N}$, consider a subset

$$\mathbf{X}_n := \{x_k : k \geq n\} = \{x_1, x_2, ...x_n\}$$

Now define the two sequences

$$a_n = \inf \mathbf{X}_n \leq \inf \mathbf{X}_{n+1} = a_{n+1}$$
$$b_n = \sup \mathbf{X}_n \geq \sup \mathbf{X}_{n+1} = b_{n+1}$$

both of which are clearly monotone. Hence a_n will be well defined if x_n is bounded from below and b_n well defined if x_n is bounded from above.

Then we have

$$\lim_{n\to\infty} inf(x_n) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{k\geq n}(x_k))$$

if x_n is bounded from below, otherwise $\lim_{n\to\infty} inf(x_n = -\infty$.

Conversely, we have

$$\lim_{n\to\infty} sup(x_n) := \lim_{n\to\infty} b_n = \lim_{n\to\infty} (\sup_{k\geq n}(x_k))$$

if x_n is bounded from above, otherwise $\lim_{n\to\infty} sup(x_n) = \infty$.

I remember thinking that this idea was straightforward, but it took me a few examples to confirm my understanding.

The important thing to remember is that we're still talking about the limit here, even if it's the inf if the function is $f(x) = x$ then we're still going all the way to infinity.

Theorem 2.45 *For a sequence in $\{x_n\} \in \mathbb{R}$, the following statements hold:*

(i) $\lim_{n\to\infty} inf(x_n) \leq \lim_{n\to\infty} sup(x_n)$

(ii) $\{x_n\}$ is a convergent sequence IFF $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} inf(x_n) \geq \lim_{n\to\infty} sup(x_n)$$

(iii) If $\{x_n\}$ converges, then

$$\lim_{n\to\infty} inf(x_n) = \lim_{n\to\infty} sup(x_n) = \lim_{n\to\infty}(x_n)$$

So what happens if you try and sum the limit superiors of 2 sequences?

Theorem 2.46 *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathbb{R}$. If the RHS of the equation is not of the form $\infty - \infty$ or $-\infty + \infty$, then*

$$\underset{n\to\infty}{lim\,sup}(x_n + y_n) \leq \underset{n\to\infty}{lim\,sup}(x_n) + \underset{n\to\infty}{lim\,sup}(y_n)$$

and for products, if RHS of the equation is not of the form $0 \cdot \infty$ or $\infty \cdot 0$, then

$$\underset{n\to\infty}{lim\,sup}(x_n y_n) \leq (\underset{n\to\infty}{lim\,sup}(x_n))(\underset{n\to\infty}{lim\,sup}(y_n))$$

Corollary 2.47 *Suppose $\beta_n \to \beta > 0$ and that $\{x_n\}$ is a sequence with $x_n \geq 0$ $\forall n \in \mathbb{N}$. Then*

$$\underset{n\to\infty}{lim\,sup}(\beta_n x_n) = \beta \underset{n\to\infty}{lim\,sup}(x_n)$$

2.3.6 Subsequences and Bolzano-Weierstrass Theorem

Definition 2.48 *(Subsequence)*

Let $\{x_n\}$ be a sequence in a set $\mathbf{X}$, and n_k be a strictly increasing sequence in $\mathbb{N}$. Then $\{x_{n_k}\}_{k\in\mathbb{N}}$ or $x_{n_1}, x_{n_2}, ...$ is called a ***subsequence*** *of x_n.*

Definition 2.49 *(Accumulation Point)*

If a subsequence x_{n_k} is convergent, then we call it a convergent subsequence and it's limit is an ***accumulation point*** *of x_n.*

Proposition 2.50 *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If they exist in $\mathbb{R}$, $\underset{n\to\infty}{lim\,inf}(x_n)$ and $\underset{n\to\infty}{lim\,sup}(x_n)$ are the smallest and largest accumulation points respectively.*

Theorem 2.51 *(Bolzano-Weierstrauss Theorem)*

Every bounded sequence in $\mathbb{K}^n$ has convergence subsequence.

(Sketch of Proof)

Use induction. You will need to consider subsequences all the way down to $z_{n_{k_j}}$.

2.3.7 Cauchy Sequences and Completeness

Definition 2.52 *(Cauchy Sequence)*

A sequence $\{x_n\} \in \mathbb{K}^n$ is called a ***Cauchy Sequence*** *if for every $\epsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t.*

$$|x_n - x_m| < \epsilon$$

$\forall n, m > n_0$.

Proposition 2.53 *Every convergence sequence in $\mathbb{K}^n$ is a Cauchy sequence.*

Proposition 2.54 *Every Cauchy sequence in $\mathbb{K}^n$ is bounded.*

Proposition 2.55 *Every Cauchy sequence in $\mathbb{K}^n$ having a convergent subsequence converges.*

If we combine the above propositions, we reach what is called the **Cauchy criterion for sequences**.

Theorem 2.56 *(Completeness of $\mathbb{K}^n$)*

A sequence in $\mathbb{K}^n$ is convergent IFF it is a Cauchy sequence.

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3. Series

Discussion of Different Types of Series

3.4 Definition and Basic Properties of Series

Definition 3.57 *(Series)*

Suppose that $\{a_k\}$ *is a sequence in* $\mathbb{K}^n$ *and set*

$$s_n := \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + ... + a_n$$

We call s_n *the nth partial sum of* $\sum_{k=0}^{n} a_k$ *and* $\{s_n\}$ *the sequence of partial sums. If* $\{s_n\}$ *converges we set*

$$\sum_{k=0}^{n} a_k := \lim_{n\to\infty} s_n$$

Definition 3.58 *(Geometric series) For* $a \in \mathbb{C}$*, the partial sum of a geometric series is*

$$s_n := \sum_{k=0}^{\infty} a^k$$

and for $a \neq 1$ *the summation is*

$$s_n = \frac{1 - a^{n+1}}{1 - a}$$

which can be proven with induction. If $|a| < 1$

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - a} \\ &= \frac{1}{1 - a} \end{aligned}$$

Lemma 3.59 *If* $\sum_{k=0}^{n} a^k$ *converges then* $a_k \to 0$ *in* $\mathbb{K}^n$*.*

Be careful as this is not a IFF relationship, the harmonic series (see below) has $\frac{1}{n}$ converging to zero but it diverges!

Definition 3.60 *(Harmonic Series)*

$$s_n = \sum_{k=1}^{n} \frac{1}{k}$$

diverges, which you can show if you show that the partial sum is not a Cauchy sequence.

Theorem 3.61 *(Cauchy Criterion for Series)*

Let $\{a_k\}$ be a sequence in $\mathbb{K}^n$. Then $\sum_{k=0}^{n} a^k$ converges in $\mathbb{K}^n$ IFF $\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}$ s.t.

$$\left\| \sum_{k=n+1}^{m} a_k \right\| < \epsilon$$

$\forall m > n \geq n_0$

3.4.1 Non-negative Series

Definition 3.62 *(Non-negative series) If $a_k \geq 0\ \forall k \in \mathbb{N}$. Then*

$$s_n := \sum_{k=0}^{n} a_k$$

converges if the sequence of partial sums is bounded. We can write this as

$$s_n \leq \sum_{k=0}^{\infty} a_k$$

We can apply this idea to formulate the **Comparison test**

Theorem 3.63 *(Comparison Test)*

Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences in $\mathbb{R}$ with $a_k, b_k \geq 0\ \forall k \in \mathbb{N}$. Further suppose $K > 0$ and $m \in \mathbb{N}$ s.t.

$$a_k \leq K b_n$$

$\forall k \geq m$. Then

(i) If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.

(ii) If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.

Corollary 3.64 *(Limit Comparison Test)*

Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences in $\mathbb{R}$ with $a_k, b_k \geq 0\ \forall k \in \mathbb{N}$. Further suppose that

$$\underset{n\to\infty}{lim\,sup} \frac{a_n}{b_n} < \infty$$

Then

(i) If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.

(ii) If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.

Proposition 3.65 *(Cauchy Condensation Test)*

Suppose that $\{a_k\}$ *is a decreasing sequence with* $a_k \geq 0\ \forall k \in \mathbb{N}$. *Then* $\sum_{k=0}^{\infty} a_k$ *converges IFF* $\sum_{k=0}^{\infty} 2^n a_{2^n}$ *converges.*

Definition 3.66 *(p-series)*

The p-series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and converges IFF $p > 1$.

(Sketch of proof) You can prove this using the Cauchy condensation test.

3.4.2 Alternating Series and Conditional Convergence

Definition 3.67 *(Alternating Series)*

$$s_n = \sum_{k=0}^{n} (-1)^k a_k$$

To test for converge for alternating series we use the Leibniz test.

Theorem 3.68 *(Leibniz Test)*

Let $\{a_k\}$ *be a* ***non-negative decreasing*** *sequence with* $a_k \to 0$. *Then the alternating series*

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges. Moreover

$$a_0 - a_1 \leq \sum_{k=0}^{\infty} (-1)^k a_k \leq a_0$$

3.4.3 Absolutely Convergent Series

Now we discuss the importance of differentiating between **conditionally convergent** and **absolute convergence**.

First we make an important remark.

Remark 3.69 *(Commutative Law for Infinite Sums)*

the commutative law is not valid for infinite sums!

Definition 3.70 *(Conditional Convergence)*

A series $\sum_{k=0}^{\infty} a_k$ *is conditionally convergent if it's limit exists and is a finite number but*

$$\sum_{k=0}^{\infty} |a_k| = 0$$

diverges.

Definition 3.71 *(Absolute Convergence)*

A series $\sum_{k=0}^{\infty} a_k$ *in* $\mathbb{K}^n$ *is called absolutely convergent if* $\sum_{k=0}^{\infty} \|a_k\|$ *is a convergent series in* $\mathbb{R}$*.*

Theorem 3.72 *(Infinite Triangle Inequality)*

$$\left\| \sum_{k=0}^{\infty} a_k \right\| \leq \sum_{k=0}^{\infty} \|a_k\|$$

To test for absolute convergence we have another ratio and root test.

Theorem 3.73 *(Ratio Test for Absolute Convergence)*

Consider the series $\sum_{k=0}^{\infty} a_k \in \mathbb{K}^n$*.*

(i) The series converges absolutely if

$$R := \limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$$

(ii) The series diverges if

$$R := \limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} > 1$$

(iii) The test is inconclusive if $R \geq 1$ *or* $r \leq 1$*.*

Theorem 3.74 *(Root Test for Absolute Convergence)*

Consider the series $\sum_{k=0}^{\infty} a_k \in \mathbb{K}^n$ *with*

$$r := \limsup_{n \to \infty} \sqrt[n]{\|a_n\|}$$

(i) If $r < 1$ *the series converges absolutely*

(ii) If $r > 1$ *the series diverges*

(iii) If $r = 1$ *the test is inconclusive.*

3.5 Power Series and their Convergence Properties

Power series have nice convergence properties, and are important when we discuss analytic functions and complex analysis later on in the course.

Definition 3.75 *(Power Series)*

Let $\{a_k\}$ be a sequence in $\mathbb{K}^n$ and $x \in \mathbb{K}$. Then the series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is called a power series in $\mathbb{K}^n$ centred around x_0.

Theorem 3.76 *(Cauchy-Hadamard Theorem)*

Every power series $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ in $\mathbb{K}^n$ either converges $\forall x \in \mathbb{C}$ or there exists $\rho \in [0, \infty)$ s.t. it

(i) Converges absolutely if $|x - x_0| < \rho$

(ii) Diverges if $|x - x_0| > \rho$

(iii) For $|x - x_0| = \rho$, the series could diverge, converge, or even converge absolutely.

If it converges for every x we set $\rho := \infty$.

Definition 3.77 *(Radius of Convergences)*

The radius of convergence, denoted ρ, as specified in the Cauchy-Hadamard Theorem is defined as

$$\rho := \frac{1}{\mathop{lim\,sup}\limits_{n \to \infty} \sqrt[n]{\|a_n\|}}$$

By convention we set $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

Remark 3.78 *We note that from the Cauchy-Hadamard theorem imples that*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad\qquad \sum_{k=0}^{\infty} \|a_k\| (x - x_0)^k$$

have the same radius of convergence.

Proposition 3.79 *If the limit exists, we can also define the radius of convergence as*

$$\rho := \frac{1}{R}$$

where

$$R := \mathop{lim}\limits_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|}$$

Let's consider an example where the limit doesn't exist. Suppose we have a series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + ...$$

Then

$$\mathop{lim\,inf}\limits_{k \to \infty} \frac{a_{k+1}}{a_k} = \frac{1}{2}$$

and

$$\lim_{k\to\infty} sup \frac{a_{k+1}}{a_k} = 2$$

Hence using the limit of the ratio we can only obtain that $\frac{1}{2} \leq \rho \leq 2$.

However, the root test will tell us that $\rho = 1$, as $\sqrt[n]{\frac{1}{2}} = \sqrt[n]{2} = 1$.

3.6 Double Series and Cauchy Products

Consider a double series

$$\sum_{j,k}^{\infty} x_{j,k}$$

We can choose to sum this double series in many ways.

1) By column

$$\sum_{k}^{\infty}\sum_{j}^{\infty} x_{j,k}$$

2) By row

$$\sum_{j}^{\infty}\sum_{k}^{\infty} x_{j,k}$$

3) Specify a bijection $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

$$\sum_{k=0}^{\infty} x_{\sigma(k)}$$

For the summation of a double series to be equal for all possible bijections of the summation, the series must converge absolutely.

Theorem 3.80 *(Absolute convergence of double series)*

Define $x_{j,k} \in \mathbb{K}^n$ *s.t.*

$$M := \sup_{m,n\in N} \sum_{j=0}^{m}\left(\sum_{k=0}^{n} \|x_{j,k}\|\right) < \infty$$

Then

$$\sum_{j}^{\infty}\sum_{k}^{\infty} x_{j,k} = \sum_{j}^{\infty}\sum_{k}^{\infty} x_{j,k} = \sum_{k=0}^{\infty} x_{\sigma(k)}$$

Corollary 3.81 *Suppose that* $\sum_{k=0}^{\infty} a_k$ *is an absolutely convergent series in* $\mathbb{K}^n$ *and* $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ *is a bijection. Then* $\sum_{\sigma(k)}^{\infty} a_k$

$$\sum_{j=0}^{\infty} a_j = \sum_{\sigma_(k)}^{\infty} a_{\sigma(k)}$$

Now let's look at the product of 2 convergent series. Suppose we have $a = \sum_{j=0}^{\infty} a_j$ and $b = \sum_{k=0}^{\infty} b_k$.

Then

$$\begin{aligned} ab &= \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_j b_k\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} a_j b_k\right) \end{aligned}$$

However, we have another way of summing their products.

Theorem 3.82 *(Cauchy Products)*

Suppose $a = \sum_{j=0}^{\infty} a_j$ *and* $b = \sum_{k=0}^{\infty} b_k$ *are absolutely convergent in* $\mathbb{K}$. *Then their Cauchy product converges absolutely and*

$$\left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$$

The RHS is called the Cauchy product.

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4. Limits and Continuity

4.7 Limits and Continuity

4.7.1 Introduction to Topology

Definition 4.83 *(Open Ball)*

Let $x, y \in \mathbb{K}^n$ and $r > 0$. Define

$$B(x, r) := \{y \in \mathbb{K}^n \mid \|y - x\| < r\} \tag{4.1}$$

Using the definition of the open ball we define an open set. Note that $r > 0$ allows no points on the "boundary" of the set.

Definition 4.84 *(Open Set)*

Let $U \subseteq \mathbb{K}^n$ be a subset. U is open if $U = \varnothing$ or if $\forall x \in U \ \ \exists r_x > 0$ s.t.

$$B(x, r_x) \subseteq U \tag{4.2}$$

Definition 4.85 *(Closed Set)*

A set $U \subseteq \mathbb{K}^n$ is closed if it's complement U^c is open.

The way we defined closed sets is interesting, and at first sight a set that appears closed may not be once you consider it's complement.

We can use the definition of open sets to show that open balls are open sets using the triangle inequality, where it can be shown that any element that forms an arbitrary open ball $B_z(z, r_z)$ is also inside the 'bigger' ball.

Proposition 4.86 *The following are important properties of open and closed sets.*

1. *$\varnothing$ and $\mathbb{K}^n$ are both open and closed sets simultaneously.*
2. *The union of infinite open sets is open.*
3. *The intersection of finite open sets is open.*
4. *The intersection of infinite closed sets is closed.*
5. *The union of finite closed sets is closed.*

For item 2, think of an intersection of open sets that has a component that converges to 0 in the limit.

Similarly for item 4, think of an infinite union of a closed set and consider the fact that it's an intersection.

So... is $\cup_{i=1}^{\infty}[i, i+1]$ closed or open?

Definition 4.87 *(Interior)*

The interior of a set A is the largest open set of $A \subset K^n$. We write this as

$$int(A) := \{x \in A \mid B(x, r_x) \subseteq A, r_x > 0\}$$

Definition 4.88 *(Closure)*

The closure of a set A is the smallest closed set that contains $A \subset K^n$. We write this as

$$\bar{A} := \{x \in A \mid B(x, r_x) \cap A \neq \varnothing, \forall r_x > 0\}$$

Definition 4.89 *(Boundary)*

We define the boundary as

$$\delta x := \bar{A} \setminus int(A)$$

Before we move back to our discussion of limits, sequences and functions, we try to tie all of this with an idea.

Proposition 4.90 *(Sequential Characterisation of Closure)*

If a set $A \subseteq \mathbb{K}^n$ has a sequence $\{x_n\}$ with a limit x. Then we can say that

1. *if $x_n \to x$, then $x \in \bar{A}$*
2. *if $x \in \bar{A}$ then $\exists \{x_n\} \in A$ s.t. $x_n \to x$*

4.7.2 Limits for Functions

Let $f : D \to \mathbb{K}^n$, where D is the domain, a subset of $\mathbb{K}^d$.

We first define what a limit point in the domain D is.

Definition 4.91 *(Limit Point in the Domain)*

If there exists a $x_0 \in \bar{D}$ s.t. $\forall \epsilon > 0,\ B(x_0, \epsilon) \cap (D \setminus \{x_0\}) \neq \varnothing$, then we say that x_0 is a limit point.

This means that $\exists \{x_n\} \in D \setminus \{x_0\}$ s.t. $\underset{n\to\infty}{lim} x_n = x_0$.

Essentially, we are saying that we are only worried about what happens as x_n approaches the limit and hence the $\setminus\{x_0\}$. The $\setminus\{x_0\}$ and ball can also be re-written as $0 < \|x - x_0\| < \delta_\epsilon$.

Note that the intersection between the ball and domain must be non-empty for ALL ϵ values. This ensures that 'gaps' do not happen.

Definition 4.92 *(Limit for f(x))*

Now for $f(x)$. Suppose $f(x)$ converges to $k \in \mathbb{K}^n$ as $x \to x_0$, where x_0 is a limit point in D. Then $\forall \epsilon > 0,\ \exists \delta = \delta_\epsilon > 0$ s.t.

$$\|f(x) - k\| < \epsilon$$

$\forall x \in D \setminus \{x_0\}$ with $\|x - x_0\| < \delta$.

Definition 4.93 *(Limit for f(x) as $x \to \infty$)*

Suppose that the domain $D \subseteq \mathbb{R}$ is unbounded above from $\mathbb{K}^n$. Then $\exists \{x_n\} \in D$ s.t. $x_n \to \infty$ as $n \to \infty$.

If $\forall \epsilon > 0,\ \exists \delta = \delta_\epsilon > 0$ s.t.

$$\|f(x) - k\| < \epsilon$$

$\forall x \in D$ with $x > \delta$, then we can say that $\lim_{x \to \infty} f(x) = k$.

As a result, the following propositions are true.

Proposition 4.94 *The following are equivalent*

1. $\lim_{x \to x_0} f(x) = k$
2. $\forall \epsilon > 0,\ \exists \delta_\epsilon > 0$ *s.t.* $f(x) \in B(k, \epsilon),\ \forall x \in (D \setminus \{x_0\} \cap B(x_0, \delta_\epsilon)$.
3. $\lim_{n \to \infty} f(x_n) = k$ *for every* $\{x_n\} \in$ *with* $x_n \to x_0,\ x_n \neq x_0$.

(i) and (iii) allow us to apply the results on limits for sequences to limits for functions.

4.7.3 Continuity

Let $f : D \to \mathbb{K}^n$, where D is the domain, a subset of $\mathbb{K}^d$.

Now we want to define continuity for $f(x)$ at a particular point in the domain $x_c \in D$. Notice that this time, x_c must be in the domain itself, and not necessarily the closure.

Definition 4.95 *(Continuity for functions at a point in the domain)*

A function f is continuous at $x_c \in D$ if $\forall \epsilon > 0,\ \exists \delta = \delta_{\epsilon, x_c} > 0$ s.t.

$$\|f(x) - f(x_c)\| < \epsilon$$

$\forall x \in D$ with $\|x - x_0\| < \delta_{\epsilon, x_c}$.

This time, instead of having some k as the limit, we are explicitly stating that the codomain at the domain point of interest x_c exists by denoting it $f(x_c)$.

By this definition, we are also implying that x_c is a limit point of the domain D as per the definition of a limit point.

Definition 4.96 *(Continuous functions)*

If f is continuous at every $x \in D$, then f is said to be a continuous function. We denote this as

$$C(D, \mathbb{K}^n) := \{f : D \to \mathbb{K}^n \mid f \text{ is continuous}\}$$

An equivalent definition is given as

$\forall x \in D$ and $\forall \epsilon > 0,\ \exists \delta_{\epsilon, x} > 0$ s.t.

$$\|f(y) - f(x)\| < \epsilon$$

$\forall y \in D$ with $|y - x| < \delta_{\epsilon, x}$

Remark 4.97 *(isolated points)*

If $x_i \in D$ is isolated, then $\exists r_i > 0$ s.t. $B(x_i, r_i) \cap D = \{x_0\}$

Remark 4.98 *It turns out that $(C(D, \mathbb{K}^n), \times, \cdot)$ is a vector space with scalar multiplication and vector addition.*

The behaviour is as you would expect:

- $(f + g)(x) = f(x) + g(x)$
- $(\alpha f)(x) = \alpha f(x)$

$\forall a \in \mathbb{K}^n,\ x \in D$

Definition 4.99 *(Relatively open and closed sets)*

Suppose $S \subset \mathbb{K}^n$. Then

1. *$U \subseteq S$ is relatively open in S if $U = \varnothing$ or $\forall x \in U\ \exists r_x > 0$ s.t. $B(x, r_x) \cap S \subseteq U$*
2. *$U^c \subseteq D$ is relatively closed if $U \cap D$ is relatively open in D.*

Relatively open sets are much like open sets, but only considering it's 'openness' within a specific domain. That is what the ball intersects S allows us to take care of.

E.g. $D = [0, 2)$ and $U = [0, 1)$. Then U not open, and instead relatively open in D since even if $u \in U$ is at zero the subset is strictly within U since we are taking the intersection.

However, for $U' = [\frac{1}{2}, 1)$ is no longer the case, as the intersection at $\frac{1}{2}$ any arbitrarily small radius will cause $B(x, r_x)$ to contain a point outside of U'.

Now we consider some equivalent definitions of continuous functions. Have fun =)

Theorem 4.100 *(Equivalent definitions of continuous functions)*

For a function $f : D \to \mathbb{K}^n$, the following is equivalent.

1. *$f \in C(D, \mathbb{K}^n)$, i.e. $f : D \to \mathbb{K}^n$ is continuous.*
2. *$\forall x \in D$ and $\forall \epsilon > 0$, $\exists \delta_{\epsilon,x} > 0$ s.t. $\|f(y) - f(x)\| < \epsilon$, $\forall y \in D$ with $|y - x| < \delta_{\epsilon,x}$*
3. *$\forall x \in D$ and $\forall \epsilon > 0$, $\exists \delta_{\epsilon,x} > 0$ s.t. $f(y) \in B(f(x), \epsilon)$, $\forall y \in B(x, \delta_{\epsilon,x}) \cap D$*
4. *$\forall x \in D$ and every sequence $\{x_n\}$ in D with $x_n \to x$ as $n \to \infty$, we have $f(x_n) \to f(x)$*
5. *$f^{-1}[U] := \{x \in D \mid f(x) \in U\}$ is relatively open in D for every open set $U \in \mathbb{K}^n$*
6. *$f^{-1}[U]$ is relatively closed in D for every closed set $U \subseteq \mathbb{K}^n$*

Definition 4.101 *(Intermediate Value Theorem)*

Let $f \in C([a, b], \mathbb{R})$ with $f(a) < f(b)$. Then for every $c \in (f(a), f(b))\ \exists x \in (a, b)$ s.t. $f(x) = c$.

4.7.4 Properties of Continuous Functions

In this section we discuss functions that are closed and bounded in $\mathbb{K}^n$.

Definition 4.102 *(bounded set)*

A set $A \subseteq \mathbb{K}^n$ is bounded if $\exists R > 0$ s.t. $\|x\| < R \ \ \forall x \in A$.

Proposition 4.103 *($A \subseteq \mathbb{K}^n$ is bounded $\iff$ convergent subsequence)*

This can be proved using the Bolzano-Weiestrauss Theorem.

You can extend this idea to bounded AND closed sets.

Corollary 4.104 *The following are equivalents*

1. *A is closed and bounded*
2. *Every sequence $\{x_n\}$in A has a convergent subsequence with it's limit in A.*

Note: (2) is also known as the '(sequential) **compactness**' property of A. The lecturer refers to (1) as the compactness of A, but the lecture notes refer to (2). Either way, since these are equivalent, then we can say that a set A with the above properties is a sequentially compact set.

Theorem 4.105 *($f \in C(D, \mathbb{K}^n)$ maps compact sets (D) to compact sets ($\mathbb{K}^n$))*

If $A \subseteq D$ is closed and bounded, then

$$f(A) := \{x \in A \mid \exists x \in A \text{ s.t. } y = f(x)\}$$

is closed and bounded.

Proposition 4.106 *(Continuity of inverse functions)*

Let $D \subseteq \mathbb{K}^d$ be closed and bounded and $f \in C(D, \mathbb{K}^n)$ be injective. Then the inverse function $f^{-1} : f(D) \to \mathbb{K}^d$ is continuous.

Theorem 4.107 *(Extreme value theorem)*

Let $f \in C(D, \mathbb{R})$, $D \subseteq \mathbb{K}^n$ is closed and bounded. Then f has a minimum and maximum, i.e.

$$m := f(a) \leq f(x) \leq f(b) =: M$$

where $m := \min$ and $M := \max$, and $\exists a, b \in D$.

We have reached an important theorem - uniform continuity. Note that before we present the definition, note that this is just a subtle change on the original continuity. We change our dependence of δ_{ϵ, x_c} to just δ_ϵ.

Theorem 4.108 *(Uniform continuity)*

Let $D \subseteq \mathbb{K}^d$, $f : D \to \mathbb{K}^n$. f is uniformly continuous on D if $\forall \epsilon > 0$, $\exists \delta = \delta_\epsilon > 0$ s.t.

$$\|f(x) - f(y)\| < \epsilon$$

$\forall x, y \in D$ with $\|x - y\| < \delta_\epsilon$.

we first define uniform continuity separately, unlike the lecture notes which take the approach of defining the above theorem directly with compact sets. Now we directly state that a continuous function mapping a compact set to the image is uniformly continuous. Note that the image is also compact as continuous functions map compact sets to compact sets.

Theorem 4.109 *(Any $f \in C(D, \mathbb{K}^n)$ that has compact set $D \subseteq \mathbb{K}^d \mapsto \mathbb{K}^n$ is uniformly continuous)*

So, why is $f(x) = x^2$ not uniformly continuous on $\mathbb{R}$?

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5. Uniform Convergence

5.8 Uniform Convergence of Functions

5.8.1 Pointwise and Uniform Convergence

Now we move on from sequences, functions, functions of sequences in the domain, to dealing with sequences of functions.

Define a sequence of functions $f_n : D \to \mathbb{K}^n$, where $D \subseteq \mathbb{K}^d, n \in \mathbb{N}$.

This time, we fix x. Now we consider pointwise and uniform convergence.

Definition 5.110 *(Pointwise convergence)*

$\{f_n\}$ convergences pointwise to f on D (for a fixed x) if $\forall x \in D$, $\forall \epsilon > 0$, $\exists n_{\epsilon,x} \geq 1$ s.t.

$$\|f_n(x) - f(x)\| < \epsilon$$

$\forall n \geq n_{\epsilon,x}$.

More concisely, $f(x) = \lim_{n\to\infty} f_n(x)$.

Note that a sequence of function convergencing pointwise must be for ALL $x \in D$.

As with uniform continuity, uniform convergence removes the dependence of n_ϵ on x.

Definition 5.111 *(Uniform convergence)*

$\{f_n\}$ convergences uniformly to f on D (for a fixed x) if $\forall \epsilon > 0$, $\exists n_\epsilon \geq 1$ s.t.

$$\|f_n(x) - f(x)\| < \epsilon$$

$\forall n \geq n_\epsilon$.

More concisely, $f_n \xrightarrow{unif} f$.

5.8.2 Supremum Norm

The supremum norm allows us to relate concepts involving norms of sequences we saw earlier in the notes.

Definition 5.112 *(Supremum norm)*

If $f : D \to \mathbb{K}^n$ is a function, the supremum norm is

$$\|f\|_{\infty,D} = \sup_{x\in D} \|f(x)\| = \sup\{\|f(x)\| \mid x \in D\}$$

Unsurprisingly, all the properties of a norm apply to the Supremum norm. For completeness, we include them here:

Proposition 5.113 *(Properties of the supremum norm, $\|\cdot\|_\infty$)*

Let $f, g : D \to \mathbb{K}^n$ be functions. Then the following properties hold:

1. $\|f\|_\infty \geq 0$, *with equality IFF* $f(x) = 0 \;\; \forall x \in D$
2. $\alpha\|f\|_\infty = |\alpha|\|f\|_\infty$, $\forall \alpha \in \mathbb{K}$
3. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Now we can state the idea of uniform convergence using the supremum norm

Proposition 5.114 *$f_n \to f$ uniformly $\iff \|f_n - f\|_{\infty,D} \to 0$ on the domain D.*

Definition 5.115 *(Uniform Cauchy sequences)*

$\{f_n\}$ is uniformly Cauchy in D if $\forall \epsilon > 0 \; \exists n_\epsilon \geq 1$ s.t.

$$\|f_n - f_m\|_{\infty,D} < \epsilon$$

$\forall m > n \geq n_\epsilon$

and similar to dealing with sequences, we have that

Theorem 5.116 *$\{f_n\}$ is uniformly convergence in $D \iff$ it is a uniform Cauchy sequence in D.*

Let's quickly define pointwise and uniform convergence for series of functions. Given the function $g_k : D \to \mathbb{K}^n$ we have the following 2 definitions:

Definition 5.117 *(uniform convergence for series of functions)*

The series of functions $f_n(x) := \sum_{k=0}^{n} g_k(x)$ converge uniformly on D if the sequence $\{f_n\}$ of partial sums is uniformly convergent on D.

Definition 5.118 *(absolute convergence for series of functions)*

The series of functions $\sum_{k=0}^{\infty} g_k$ is absolutely convergent on D if $\forall x \in D$, $\sum_{k=0}^{\infty} g_k(x)$ converges absolutely.

Recall that the definition of absolute convergence is $\sum_{k=0}^{\infty}\|g_k(x)\|$ converges in $\mathbb{R}$.

Note: A diagram will make it very intuitive, but pointwise convergence is not sufficient for f to be continuous in D. If $\{f_n\}$ is uniformly convergent however, f is continuous on D.

Be careful here, since f being continous does not imply $\{f_n\}$ is uniformly convergent.

Theorem 5.119 *Suppose $f_n : D \to \mathbb{K}^n$ is continuous on D, $\forall n \geq 1$.*

If $f_n \to f$ uniform on D, then f is continuous on D.

Theorem 5.120 *(Weierstrauss M-test)*

Let $g_k : D \to \mathbb{K}^n,\ k \in \mathbb{N}$ be functions.

If $\sum_{k=0}^{\infty} \|g_k\|_{\infty,D}$ converges, $\implies \sum_{k=0}^{\infty} g_k$ converges absolutely and uniformly on D.

The intuition here is that we are using the supremum norm for the Comparison test to show absolute convergence.

For uniform convergence, we apply the uniform Cauchy sequence and apply the infinite triangle inequality since our assumption is that the sequence of norms converge..

Note: This test is a sufficient but not necessary condition for the uniform convergence of a series.

5.8.3 Uniform Convergence and Continuity

Theorem 5.121 *(Uniform convergence preserves continuity on D)*

Let $D \subseteq \mathbb{K}^d$. Then $f_n : D \to \mathbb{K}^n$ is continuous on D $\forall \geq 1$.

If $f_n \to f$ is uniform on D, then f is continuous on D.

We then have the contrapositive.

Corollary 5.122 *Assume that $f_n \in C(D, \mathbb{K}^n)$, $D \subseteq \mathbb{K}^d$ and $f_n \to f$ pointwise on D.*

If f is not continuous on D, then f_n cannot converge uniformly on D.

Corollary 5.123 *(Continuity of power series)*

Given a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathbb{K}^n$ with a radius of convergence $\mathcal{Q} > 0$.

Then $f(z)$ converges absolutely and uniformly on $B(0, r)$ $\forall r \in (0, \mathcal{Q})$.

Additionally, $f : B(0, \mathcal{Q}) \to \mathbb{K}^n$ is continuous.

5.9 Differentiation and Integration

5.9.1 Differentiation and Integration Take 2

Definition 5.124 *(Derivative)*

Let $D \subseteq \mathbb{K}^n$ be open and $f : D \to \mathbb{K}^n$ and $x_0 \in D$. We say f is differentiable at x_0 if

$$\lim_{x \to x_0} f(x) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

f is differentiable if $f(x_0)$ exists $\forall x_0 \in D$.

f is continuously differentiable is f' is continuous. We use the notation

$$C^k(D, \mathbb{K}^n) := \left\{ f : D \to \mathbb{K}^n \mid f^j \text{ is continous for } j = 0, 1, ..., k \right\}$$

to denote that the set of functions f is k times continuously differentiable.

Proposition 5.125 *(Differentiability of an Inverse Function)*

Let $D \subseteq \mathbb{K}^n$ be open and $f : D \to \mathbb{K}^n$ and $x_0 \in D$ is injective and differentiable at $f^{-1}(y_0)$, for $y_0 \in f(D)$. If $f^{-1}(f(y_0)) \neq 0$, then f^{-1} is differentiable at y_0.

Definition 5.126 *(Fundamental Theorem of Calculus)*

(i) If $f \in C^1([a,b], \mathbb{K}^n)$ then

$$f(b) - f(a) = \int_a^b f'(t)\ \delta t$$

(ii) If $f \in C([a,b], \mathbb{K}^n)$ then for every $c \in [a,b]$

$$\frac{\delta}{\delta t} \int_c^t f(s)\ \delta s = f(t)$$

with the Fundamental Theorem of Calculus we can define

Definition 5.127 *(Mean Value Theorem)*

Suppose that $D \subset \mathbb{K}^n$ is open and $f \in C^1(D, \mathbb{K}^n)$. If $a, b \in D$ are such that $a + t(b-a) \in D$ for all $t \in [0,1]$ then

$$f(b) - f(a) = (b-a) \int_0^1 f(a + t(b-a)\ \delta t$$

Definition 5.128 *(Caratheodory's Theorem)*

Let $f : D \to \mathbb{K}^n$ be a function with $D \subseteq \mathbb{K}$ open. Then the two statements are equivalent

(i) f is differentiable at $x_0 \in D$.

(ii) There exists a function $\psi : D \to \mathbb{K}^n$ continuous at x_0 such that

$$f(x) - f(x_0) = \psi(x)(x - x_0)$$

and if f is differentiable at x_0, then $f'(x_0) = \psi(x_0)$.

Lemma 5.129 *If $f \in C([a,b], \mathbb{K}^n$, then*

$$\left\| \int_a^b f(t)\ \delta t \right\| \leq \int_a^b \|f(t)\|\ \delta t$$

Proposition 5.130 *(Parameter Integrals)*

Let $D \subseteq \mathbb{K}^n$ be open and $f \in C([a,b] \times D, \mathbb{K}^n$. Define, $\forall x \in D$

$$g(x) := \int_a^b f(t,x)\ \delta t$$

Then the following hold

(i) $g \in C(D, \mathbb{K}^n)$.

(ii) If

$$\frac{\delta f}{\delta x} \in C([a,b] \times D, \mathbb{K}^n)$$

then $g \in C^1(D, \mathbb{K}^n)$ *and*

$$g'(x) = \int_a^b \frac{\delta}{\delta x}(f(t,x))\ \delta t$$

Essentially what we are saying is that, if f is continous on both variables t and x, then g on just one variable is also continuous.

Furthermore, if f is continuously differentiable with respect to x, then g is also continuously differentiable with respect to x!

The parameter integral theorem is intuitive, but like the Caratheodory theorem it might take some time to get used to. You'll see it again in complex analysis when it is used to prove results of certain path integrals.

5.9.2 Uniform Convergence, Integration and Differentiation

Theorem 5.131 *(Uniform Convergence allows Interchange of Limit and Integral)*

Let $f_n \in C([a,b], \mathbb{K}^n$ *and assume that* $f_n \to f$ *uniformly on* $[a,b]$. *Then* $f \in C([a,b], \mathbb{K}^n$ *and*

$$\begin{aligned} \lim_{n\to\infty} \int_a^b f_n(x)\ \delta x &= \int_a^b \lim_{n\to\infty} f_n(x)\ \delta x \\ &= \int_a^b f(x)\ \delta x \end{aligned}$$

(Sketch of proof)

Note that the limit function f *is continuous and hence integrable.*

We also know the the supremum norm exists if $f_n \to f$ *uniformly, and we also know that the triangle inequality for integrals can only be used if* $f \in C([a,b], \mathbb{K}^n$.

Combining both leads to an inequality that upper bounds LHS to an expression that converges to zero as $n \to \infty$.

We saw earlier that uniform convergence preserves continuity of the function in the limit.

However, we can actually relax this requirement. It turns out that only locally uniform convergence is required to preserve continuity of the limit function.

Definition 5.132 *(Locally Uniformly Convergent)*

Define $f_n, f : D \to \mathbb{K}^n$, *where* $D \subseteq \mathbb{K}^n$ *open. We say that* $f_n \to f$ *locally uniformly on* D *if* $\forall x \in D\ \exists r > 0$ *s.t.* $f_n \to f$ *uniformly on* $B(x,r) \cap D$.

Lemma 5.133 *If $f_n \to f$ locally uniformly on D and $f_n \in C(D, \mathbb{K}^n)$ $\forall n \geq 1$ then $f \in C(D, \mathbb{K}^n)$.*

It turns out that we can apply this to show that pointwise convergence and locally uniform convergence is sufficient for the derivative to be continuously differentiable.

Theorem 5.134 *Define $f_n \in C(D, \mathbb{K}^n)$, $D \subseteq \mathbb{K}$ open, $\forall n \in \mathbb{N}$.*

If $f_n \to f$ pointwise and $f_n' \to g$ locally uniformly, then $f \in C^1(C, \mathbb{K}^n)$ and $f' = g$.

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6. Analytic Functions

6.10 A precursor to Complex Analysis

6.10.1 Power Series Take 2

Theorem 6.135 *Define the power series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

where $z \in B(0, \rho)$, ρ being the radius of convergence. Then we have the derivative

$$g(z) := \sum_{k=1}^{\infty} k a_k z^{k-1}$$

and primitive

$$F(z) := \sum_{k=1}^{\infty} \frac{1}{k+1} a_k z^{k+1}$$

F and f are differentiable, with $F' = f$ and $f' = g$.

Lemma 6.136 *Let $f(z)$ be the power series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with radius of convergence ρ. Then $\forall z \in B(0, \rho)$, f has derivatives of all orders and

$$f^{(k)}(z) = k! \sum_{k=0}^{\infty} \binom{n}{k} a_n z^{n-k}$$

moreover

$$a_k = \frac{1}{k!} f^{(k)}(0)$$

This shows that the power series coincides with the Taylor series if we expand around the centre $z = 0$.

We can generalise this to expansions around every $z_0 \in B(0, \rho)$.

Theorem 6.137 $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is a power series with radius of convergence $\rho > 0$. If $z_0 \in B(0, \rho)$ then*

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_0)(z - z_0)^k$$

Theorem 6.138 *(Uniqueness Theorem for Power Series)*

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) = \sum_{k=0}^{\infty} g_k z^k$. *Both are power series in* $\mathbb{K}^n$ *converging with* $|z| < r$.

Suppose $z_n \neq 0\ \forall n \in \mathbb{N}$ *and* $z_n \to 0$.

If $f(z_n) = g(z_n)\ \forall n \in \mathbb{N}$, *then* $a_n = b_n\ \forall n \in \mathbb{N}$.

6.10.2 Analytic Functions

Definition 6.139 *(Analytic Functions)*

Let $D \subseteq \mathbb{K}^n$ *be open and* $f : D \to \mathbb{K}^n$. *We say that* f *is analytic if for every* $x_0 \in D\ \exists r > 0$ *and a sequence* $a_k \in \mathbb{K}^n$*s.t.*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

$\forall x \in B(x_0, r)$. *If the domain is real, then the function is real analytic.*

Definition 6.140 *(Path)*

A path in a set D *is a continuous function* $\gamma : [a, b] \to D$. $\gamma \in C([a, b], D)$.

Definition 6.141 *(Connected Set)*

An open set $D \subseteq \mathbb{K}$ *is called path connected if for every* $x, y \in D$ *there exists a path* $\gamma \in C([0, 1], D)$ *with* $\gamma(0) = x$ *and* $\gamma(1) = y$

Theorem 6.142 *(Uniqueness Theorem for Analytic Functions)*

Suppose $D \subseteq \mathbb{K}$ *is a connected open set, and* $f, g : D \to \mathbb{K}^n$ *are both analytic functions on* D.

Suppose that $\{x_n\}$ *is a sequence with* $x_n \to x_0$, $x_0 \in D$ *and* $x_n \neq x_0\ \forall n \in \mathbb{N}$.

If $f(x_n) = (x_n)\ \forall n \in \mathbb{N}$, *then* $f(x) = g(x)\ \forall x \in D$.

6.10.3 Exponential, Logarithms and Powers

Some functions defined in the real field are analytic and we are able to extend them into the complex plane using power series.

The uniqueness theorem then guarantees that the extension is unique.

Let's start with the exponential function, look at the region of analyticity of the logarithmic function and then conclude with the idea of principal powers.

Theorem 6.143 *(exponential function)*

The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ *is analytic. We have that* *(i)* $\exp(0) = 1$ *and* $\exp(1) = e$

(ii) $\exp(z + w) = \exp(z)\exp(w) \;\; \forall z, w \in \mathbb{C}$

(iii) $\exp(z) \neq 0$ *and* $\exp(-z) = \frac{1}{\exp(z)} \;\; \forall z \in \mathbb{C}$

(iv) $\overline{\exp(z)} = \exp(\bar{z}) \;\; \forall z \in \mathbb{C}$.

(v) $|\exp(it)| = 1 \;\; \forall t \in \mathbb{R}$

(vi) $\frac{\delta}{\delta z}\exp(z) = \exp(z) \;\; \forall z \in \mathbb{C}$

(vii) $\exp : \mathbb{R} \to (0, \infty)$ *is strictly increasing and bijective.*

We can define the logarithm to be the inverse of the exponential function.

Theorem 6.144 *(Real logarithm)*

The function $\log : \mathbb{R} \to (0, \infty)$ *is analytic. We have that*

(i) $\log(1) = 0$ *and* $\log(e) = 1$

(ii) $\log(xy) = \log(x) + \log(y) \;\; \forall x, y > 0$

(iii) $\frac{\delta}{\delta x}\log(x) = \frac{1}{x} \;\; \forall x > 0$

(iv) $\log : (0, \infty) \to \mathbb{R}$ *is strictly increasing and bijective.*

(v) The Taylor series expansion of $\log$ *about* $x_0 > 0$ *is given by*

$$\log(x) = \log(x_0) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k z_0^k}(z - z_0)^k$$

for $x \in (0, 2x_0)$

It turns out that the complex Logarithm is only analytic on the domain $\mathbb{C} \setminus (-\infty, 0]$.

Theorem 6.145 *(Complex logarithm)*

There exists a unique function $Log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *with the properties*

(i) $Log(1) = 0$

(ii) $\frac{\delta}{\delta z} Log(z) = \frac{1}{z} \; \forall z \in \mathbb{C}$

(iii) $\exp(Log(z)) = z \; \forall z \in \mathbb{C}$.

Moreover, $Log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ *is analytic.*

Definition 6.146 *(Principal logarithm)*

The function $Log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ *is called the* ***principal logarithm****.*

Definition 6.147 *(Principal argument)*

The unique number $\theta \in (-\pi, \pi]$ *satisfying* $z = |z|e^{it}$ *is called the principal argument of* z *and is denoted by* $Arg(z)$

We can represent the Principal logarithm as a combination of the logarithm and its argument.

Remark 6.148 *Suppose*

$$w = Log(z)$$

Since $w = u + iv$ $u, v \in \mathbb{R}$ *we have*

$$z = e^w$$

Taking the absolute value we have

$$\begin{aligned} |z| &= e^u \left|e^{iv}\right| \\ &= e^u \\ \Rightarrow u &= \log(|z|), \\ v &= Arg(z)\ \forall z \in \mathbb{C} \setminus \{0\} \end{aligned}$$

which then gives

$$Log(z) = \log(|z|) + iArg(z)$$

with $Arg(z) \in (-\pi, \pi]$.

Definition 6.149 *(Prinicipal powers)*

For $w \in \mathbb{C}$ *and* $z \in \mathbb{C} \setminus \{0\}$ *we define*

$$z^w := \exp(wLog(z))$$

and this is a principal power.

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7. Complex Analysis

7.11 Introduction to Complex Analysis

7.11.1 Cauchy-Riemann Equations

Define an open set $D \subseteq \mathbb{C}$ and let $f : D \to \mathbb{C}$. Let $z = x + iy \in \mathbb{C}$ be a complex number with $x, y \in \mathbb{R}$. We can define $f(z)$ as

$$f(z) = u(x, y) + v(x, y)i$$

Suppose f is differentiable at a point z_0 in the open set D. Then if $u(x, y)$ and $v(x, y)$ are real differentiable, i.e. their partial derivatives exist, then f is complex differentiable IFF the partial derivatives are u and v satisfy the Cauchy-Riemann equations

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$$
$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

or alternatively

$$\left(\frac{\delta}{\delta x} + i\frac{\delta}{\delta y}\right) f = 0$$

on D

Corollary 7.150 *Let f be an analytic function on a connected open set $D \subseteq \mathbb{C}$. If f is real-valued, then f is constant on D. In other words, a non-constant real-valued function f cannot be analytic if defined in a subset of the complex domain - you can check this using the Cauchy-Riemann equations.*

7.11.2 Path Integrals

When dealing with functions in the complex domain, we have a few common assumptions.

i) The domain of the function is usually open: $D \subseteq \mathbb{C}$ open

ii) The function f is continuous: $f \in C(D, \mathbb{C}^n)$

The reason for this becomes clear as we walk through the ensuing theorems.

Paths are continuous by definition, but could be piecewise continuous. We distinguish between the two.

Definition 7.151 *(Line integral/Path)*

$\gamma \in C([a,b],D)$ is a C^1 path defined by the integral

$$\int_\gamma f(z)\ \delta z := \int_b^a f(\gamma(t))\gamma'(t)\ \delta t$$

Note that C^1 indicates that it is a single piece path. This notation becomes clear when you consider the piecewise path.

The function $\gamma \in C([a,b],D)$ is a piecewise path C^k if it can be k-partitioned into $a = t_0 < t_1 < ... < t_k = b$ where $\gamma \in C([t_i, t_{i+1}], D)$, $i = 0, ..., k-1$.

$$\int_\gamma f(z)\ \delta z := \sum_{i=1}^{k} \int_{t_{i-1}}^{i} f(\gamma(t))\gamma'(t)\ \delta t$$

It was not explicitly mentioned, but you can assume that the paths don't cross.

Proposition 7.152 *(Supremum of the line integral)*

Given that the curve is of finite length, it is unsurprising that a supremum exists for the norm of the line integral.

$$\begin{aligned}\int_\gamma f(z)\ \delta z &= \int_a^b f(\gamma(t))\gamma'(t)\ \delta t\\ &\le \max_{t\in[a,b]} \|f(\gamma(t))\| \cdot \int_a^b |\gamma'(t)|\ \delta t\end{aligned}$$

To show this, use the triangle inequality. For a C^k path, you can sum the individual partitions.

For path integrals, the existence of primitives is a desirable property.

This is because if a primitive exists, we can integral f in the 'usual' way, and then we don't need to know what happens inbetween a and b - because you'll just have $F(\gamma(a)) - F(\gamma(b))$ as the result.

This is what is referred to as **path independent**.

Furthermore, suppose you have a path is that closed - $\gamma \in C([a,b],D)$ where $a = b$, then you can try evaluating the integral and see what you get =).

Definition 7.153 *(Primitive)*

Let $D \in \mathbb{C}$ open, $f : D \to \mathbb{C}^n$. If $F' = f$ then we say that $F : D \to \mathbb{C}^n$ exists and is a primitive of f.

Then it follows that

$$\int_\gamma f(z)\ \delta z = F(\gamma(b)) - F(\gamma(a))$$

This can be proved with the fundamental theorem of calculus.

Definition 7.154 *(Closed paths)*

Let $D \subseteq \mathbb{C}$ open. If $f \in C(D, \mathbb{C}^n)$ has a primitive then for every closed piecewise C^1 path in D

$$\int_\gamma f(z)\ \delta z = 0$$

Note that we need a C^1 path.

The existence of the primitive is very important here for the above definition to hold.

Now we consider a special $f(z) = (z - z_0)^n$ for which we investigate the line integrals.

Proposition 7.155 *Let $f(z) = (z - z_0)^n$, γ be a C^1 **closed path** in $\mathbb{C}$ with $z, z_0 \in \mathbb{C}$.*

Let's consider the case where $n > 0$.

Then for $z \in \mathbb{C}$

$$\int_\gamma (z - z_0)^n\ \delta z = 0$$

For $n = 0$, it is pretty clear that

$$\int_\gamma (z - z_0)^n\ \delta z = 0$$

Now let's consider what happens when $n < 1$.

$$\int_\gamma (z - z_0)^n\ \delta z = 0$$

ONLY if $z \in \mathbb{C} \setminus \{z_0\}$ because the denominator cannot be zero.

In all of the above, if the domain constraints are taken care of, the existence of the primitive comes naturally;

$$\int (z - z_0)^n\ \delta z = \frac{(z - z_0)^{n+1}}{n+1}$$

Now we come to a special case: $n = -1$. The domain is $z \in \mathbb{C} \setminus \{z_0\}$.

If you're dealing with $x \in \mathbb{R}$ then you can integrate to get a logarithm, but here we have to be more careful.

We now have to distinguish an extra 2 cases of the special case! Here we just state the result.

$$\int \frac{1}{z - z_0}\ \delta z = \begin{cases} 0 & \text{if } |z_0| > r \\ 2\pi i & \text{if } |z_0| < r \end{cases}$$

So now we have some good properties. However, we have yet to discuss the conditions for which a function f has a primitive. This requires some restrictions on the domain and the function.

7.11.3 The Existence of Primitives

We mentioned that the existence of primitives is desirable. Now we consider what is required for $f \in C(D, \mathbb{C}^n)$ to have a primitive.

Definition 7.156 *(Star-shaped domain)*

An open set $D \subseteq \mathbb{C}$ is called star-shaped w.r.t to $z_0 \in D$ if the curve joining z_0 and any $z \in D$ is a straight line s.t. $(1-t)z_0 + (t)z \in D \;\; \forall t \in [0,1]$.

If the domain is star-shaped w.r.t $z_0 \;\; \forall z_0 \in D$, then the domain is convex.

Definition 7.157 *(Existence of Primitives)*

Let $D \subseteq \mathbb{C}$ be open and star-shaped w.r.t to $z_0 \in D$.

If $f \in C(D, \mathbb{C}^n) \cup C^1(D \setminus \{z_0\}, \mathbb{C}^n)$, then f admits a primitive on D defined as

$$F(z) = (z - z_0) \int_0^1 f((1-t)z_0 + (t)z) \; \delta t$$

$\forall z \in D$.

Once we prove this, we can immediately argue that for the exact same properties of f and D, the closed path has line integral equal to zero.

7.11.4 The Cauchy Integral Theorem and Formula

Definition 7.158 *(Cauchy Integral Theorem)*

Let $D \subseteq \mathbb{C}$ be open and star-shaped w.r.t to $z_0 \in D$.

*If $f \in C(D, \mathbb{C}^n) \cup C^1(D \setminus \{z_0\}, \mathbb{C}^n)$, then for every **closed piecewise** C_1 path γ in D, we have*

$$\int_\gamma f(z) \; \delta z = 0$$

It turns out that we don't really need a star-shaped domain, just a simply connected domain. For simplicity, just think of it as a domain with no 'holes'.

The proof of this theorem comes from the existence of a primitive given the domain of f, and the fact we defined earlier where a closed piecewise path that has a primitive results in the line integral being zero.

With all of these ideas, can we now prove the **Cauchy Integral Formula** and subsequently show that f is analytic on D using all of the ideas we discussed above.

Definition 7.159 *(Cauchy Integral Formula)*

Let $D \subseteq \mathbb{C}$ *be open and* $f \in C^1(D, \mathbb{C}^n))$. *Fix an arbitrary* $z_0 \in D$ *and let* $r > 0$ *s.t.* $\overline{B(z_0, r)} \subseteq D$. *Define the boundary of the ball as* $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. *Then*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{f(w)}{w - z} \, \delta w$$

$\forall z \in B(z_0, r)$.

Note that the closed ball is possible because D is an open set. The prove this, we write

$$\int_\gamma \frac{f(w)}{w - z} \, \delta w = \int_\gamma \frac{f(w) - f(z)}{w - z} \, \delta w - f(z) \int_\gamma \frac{1}{w - z} \, \delta w$$

From previous theorems $f(z) \int_\gamma \frac{1}{w-z} \, \delta w = 2\pi i f(z)$ so we just have to show that the first part is equal to zero.

To do this, we define $g(z) := \frac{f(w) - f(z)}{w - z}$ $\forall w \neq z$, show that $g \in C^1(D \setminus \{z\}, \mathbb{C}^n) \cap C(D, \mathbb{C}^n)$. We know g is differentiable since f is differentiable (except at the point z), but we need to show that $g(z)$ exists.

Now, we don't know z is star-shaped with respect to the entire domain D, so we need to pick a open set carefully to ensure this.

To do this we create a slightly larger ball with $R > r$ - $B(z_0, R)$. This is possible because $B(z_0, r)$ is closed and D is open. Now we have a star-shaped domain, and since we have already established $g \in C^1(D \setminus \{z\}, \mathbb{C}^n) \cap C(D, \mathbb{C}^n)$, we also have $g \in C^1(B(z_0, R) \setminus \{z\}, \mathbb{C}^n) \cap C(B(z_0, R), \mathbb{C}^n)$

Now we can apply the Cauchy Integral Theorem and obtain what we are looking for.

We need the closed ball because we use the boundary to create the closed path - and that closed path won't be part of the domain if the ball is open. Also note that for g has a limit for $w = z$.

7.12 Investigating Analytic Functions in $\mathbb{C}$

7.12.1 Analyticity of Differentiable Functions

One of the most important ideas in complex analysis is that a differentiable complex function (a holomorphic function) on a domain D is analytic on the same domain. This ties in to what we have been doing in the previous section - which is finding the existence of a primitive.

First we state the theorem for analyticity of complex functions and then backtrack to components required for its proof.

Definition 7.160 *(Analyticity of* C^1 *functions on* $\mathbb{C}$*))*

Let $D \subseteq \mathbb{C}$ *be open and* $f \in C^1(D, \mathbb{C}^n)$. *Then* f *is analytic on* D.

This does not necessarily hold for real functions. Consider $f(x) = |x|x$ ...

Remark 7.161 *Let $D \subseteq \mathbb{C}$ be open and $f \in C^1(D, \mathbb{C}^n)$.*

We know from the above that f is analytic on D, so if we fix an arbitrary $z_0 \in D$, and ρ is the radius of convergence of the Taylor series of f about $z = z_0$ for all $z_0 \in \mathbb{D}$, then

$$\rho \geq \sup \left\{ r > 0 \mid \overline{B(z_0, r)} \subseteq D \right\}$$

Recall that for Taylor series in a domain D, the radius of convergence must be at least r, and hence we take the supremum over all possible balls with radius r.

The requirements for analyticity seems deceptively simple.

In fact, we also have

Corollary 7.162 *Let $D \subseteq \mathbb{C}$ be open. Then*

(i) If $f \in C(D, \mathbb{C}^n)$ admits a primitive F on D, then f is analytic on D.

(ii) If $z_0 \in D$ and $f \in C\left(D, \mathbb{C}^n\right) \cap C^1\left(D \setminus \{z_0\}, \mathbb{C}^n\right)$, then f is analytic on D.

For (i), the primitive F exists on D and thus $f = F'$ exists on domain D and is continuous as assumed. Hence $F \in C^1(D, \mathbb{C}^n)$ is analytic $\Rightarrow$ f is analytic.

This is because once some function is analytic, it is infinitely differentiable.

For (ii), we know that $D \setminus \{z_0\}$ is open in $\mathbb{C}$ since D is open in $\mathbb{C}$. Hence f is analytic at $D \setminus \{z_0\}$.

Now show that f is analytic on the remaining point.

Since D is open, we con construct $B(z_0, r) \subsetneq D$, $r > 0$, s.t. $f \in C(B(z_0, r), \mathbb{C}^n) \cap C^1(B(z_0, r) \setminus \{z_0\}, \mathbb{C}^n)$. Since $B(z_0, r)$ is open and start-shaped with respect to z_0, then by the existence of the primitive F, f is analytic on $B(z_0, r)$ using (i).

We will need to following proposition to prove for analyticity of C^1 functions.

Proposition 7.163 *Let $z_0 \in \mathbb{C}$, $r > 0$. Set $S := \{z \in \mathbb{C} \mid |z - z_0| = r\}$, the path of the circle. and suppose that $f \in C(S, \mathbb{C}^n)$.*

Then depending on whether z is within the circle or outside of the circle, we have

$$\begin{aligned}
\int_S \frac{f(w)}{w - z}\ \delta w &= \sum_{k=0}^{\infty} a_k (z - z_0)^k \\
&= \sum_{k=0}^{\infty} \int_S \frac{f(w)}{(w - z_0)^{k+1}}\ \delta w\ (z - z_0)^k
\end{aligned}$$

if $|z - z_0| < r$ and

$$\begin{aligned}
\int_S \frac{f(w)}{w - z}\ \delta w &= -\sum_{k=0}^{\infty} a_{-k} (z - z_0)^k \\
&= -\sum_{k=0}^{\infty} \int_S f(w)(w - z_0)^{k-1}\ \delta w\ \frac{1}{(z - z_0)^k}
\end{aligned}$$

if $|z - z_0| > r$

where we can define $a_k = \int_S \frac{f(w)}{(w-z_0)^{k+1}}\ \delta w$ with the appropriate substitution.

Using this, we can equate

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma(t)} \frac{f(w)}{w-z}\ \delta w \\ &= \frac{1}{2\pi i}\sum_{k=0}^{\infty} a_k (z-z_0)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \end{aligned}$$

to obtain $\frac{a_k}{2\pi i} = \frac{f^{(k)}(z_0)}{k!} \Rightarrow \frac{a_0}{2\pi i} = f(z_0) = \int_{\gamma(t)} \frac{f(w)}{w-z_0}\delta w$.

The proof of the proposition is quite lengthy, but the idea here is that for each case we rewrite the function as a geometric series and then show that it converges absolutely and uniformly using the Weierstrauss M-test.

This will allow you to interchange the integration and summation and thus obtain what you are looking for.

7.12.2 Laurent Expansions

Laurent expansions come in handy when working with annuli.

Definition 7.164 *(annuli)*

Define an annulus $S := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$*, with* $0 \leq R_1 < R_2 \leq \infty$*. We consider* S *an "improper" annulus if* $R = 0$ *and/or* $R = \infty$*.*

Lemma 7.165 *Suppose* f *is analytic in the annulus* $R_1 < |z - z_0| < R_2$*, with values in* $\mathbb{C}^n$*, and* $\gamma_r(t) := z_0 + re^{it},\ t \in [0, 2\pi]$*. Then*

$$g(r) := \int_{\gamma_r} f(z)\ \delta z$$

is constant for $r \in (R_1, R_2)$*.*

We can prove the above lemma using the idea of **Homotopy invariance** or by utilising the parameter integral theorem to show that $g'(r) = 0$. Note that the Cauchy Integral Theorem does not apply since the domain is not simply connected.

Naturally we move onto the Cauchy Integral Formula is still valid for annuli, albeit being slightly different.

Definition 7.166 *(Cauchy Integral Formula for Annuli)*

Suppose f *is analytic in the annulus* $R_1 < |z - z_0| < R_2$*, with values in* $\mathbb{C}^n$*.*

Let $R_1 < r_1 < r_2 < R_2$ and let C_1 and C_2 be positively oriented circles centred at z_0 with radius r_1 and r_2 respectively. Then

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_2} \frac{f(w)}{w-z}\,\delta w - \int_{C_1} \frac{f(w)}{w-z}\,\delta w\right]$$

for $r_1 < |z - z_0| < r_2$.

The proof bears reminiscence to the proof for the original Cauchy Integral Formula, but is much more straight forward.

Now let's formally define a Laurent Series.

Definition 7.167 *(Laurent Series)*

Suppose $D \subseteq \mathbb{C}$ be open and $f : D \to \mathbb{C}^n$ is analytic. Fix a $z_0 \in \mathbb{C}$ such that

$$A := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\} \subseteq D$$

There exists a_k, $k \in \mathbb{Z}$ s.t.

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

$\forall z \in A$. The series converges absolutely and uniformly on closed and bounded subsets of A. For the Laurent series, we assume that f is analytic on the entire domain D!

Remark 7.168 *Using the proposition about equating the power series of $f(z)$ with the integrand of $f(z)$, we can derive*

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z_0)^{k+1}}\,\delta w \\ a_{-k} &= \frac{1}{2\pi i}\int_C f(w)(w - z_0)^{k-1}\,\delta w \end{aligned}$$

Note that the proposition used in this remark has nothing to do with analyticity or the Cauchy Integral formula. The only thing we need is the closed path of a circle.

7.12.3 Singularities and Poles

Definition 7.169 *(Isolated Singularity)*

Let $f : D \in \mathbb{C}^n$ be analytic on the open set $D \subseteq \mathbb{C}$. We call the point $z_0 \in \delta D$ an isolated singularity of f if $\exists r > 0$ s.t. $B(z_0, r) \setminus \{z_0\} \subseteq D$

Definition 7.170 *(Classifying Singularities)*

There are 3 types of singularities.

(i) ***Removable Singularity:*** $a_{-k} = 0 \;\; \forall k \geq 1$

$$\lim_{z \to z_0} f(z) = a_0$$

(ii) ***Pole of order m:*** $a_{-m} \neq 0, \; a_{-k} = 0, \; \forall k > m, \; k, m \in \mathbb{N}$

$$\lim_{z \to z_0} (z - z_0)^m f(z) = a_{-m} \neq 0$$

(iii) ***Essential Singularity:*** $a_{-k} \neq 0 \;\; k \in \mathbb{N}$

$$\lim_{z \to z_0} (z - z_0)^m f(z) \;\; \textit{does not exist} \; \forall m \in \mathbb{N}$$

Theorem 7.171 *Let $f : D \to \mathbb{C}^n$ be analytic and z_0 an isolated singularity of f.*

(i) If z_0 is a removable singularity of f, then $\lim_{z \to z_0} f(z)$ exists and f has an analytic extension $D \cup \{z_0\}$.

(ii) If f has a pole of order m at z_0 then there exists an analytic function $g : D \cup \{z_0\} \to \mathbb{C}^n$ with $g(z_0) \neq 0$ s.t. $f(z) = \frac{g(z)}{(z-z_0)^m}$.

7.12.4 Residues and the Residues Theorem

Definition 7.172 *(Residue)*

Let z_0 be an isolated singularity of the analytic function $f : D \to \mathbb{C}^n$. The coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion of f in $0 < |z - z_0| < r$ is called the residue of f at z_0. We write this as

$$Res[f, z_0] := a_{-1}$$

Proposition 7.173 *Suppose that $D \subseteq \mathbb{C}$ is open and z_0 is an isolated singularity of the analytic function $f : D \to \mathbb{C}^n$. Let $r > 0$ s.t. $B(z_0, r) \setminus \{z_0\} \subseteq D$ and set $\gamma(t) := z_0 + re^{it}, \; t \in [0, 2\pi]$. Then*

$$\int_\gamma f(z) \; \delta z = 2\pi i a_{-1}$$

Proposition 7.174 *Let $D \subseteq \mathbb{C}$ is open and z_0 is an isolated singularity of the analytic function $f : D \to \mathbb{C}^n$.*

(i) If z_0 is a removable singularity, then

$$Res[f, z_0] = 0$$

(ii) If z_0 is a simple pole of order $m = 1$, then

$$Res[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z) \neq 0$$

(iii) (ii) If z_0 is a simple pole of order $m > 1$, then

$$Res[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\delta^{m-1}}{\delta z^{m-1}} (z - z_0)^m f(z) \neq 0$$

Theorem 7.175 *(Residue Theorem)*

Suppose that $D \subseteq \mathbb{C}$ is open and $f : D \to \mathbb{C}^n$ an analytic function in D. Let γ be the positively oriented closed piecewise C^1 path in D s.t. f is analytic in the region enclosed by γ except for the isolated singularities $z_1, z_2, ..., z_n$. Then

$$\int_\gamma f(z)\ \delta z = 2\pi i \sum_{k=1}^{n} Res[f, z_k]$$

Proposition 7.176 *Let p and q be polynomials with $q(x) \neq 0$, $\forall x \in \mathbb{R}$ and $\deg(q) \geq \deg(p) + 2$. If $z_1, z_2, ..., z_n$ are the zeros of q with **positive imaginary parts**, then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\ \delta x = 2\pi i \sum_{k=1}^{n} Res[f, z_k]$$

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8. Bonus

Things to be wary of during exams

8.13 Miscellaneous Tips

8.13.1 Wk1-5

$$\frac{1}{k^2} = \frac{1}{k-1} - \frac{1}{k}$$

Just sayin'

Let $a, b \in \mathbb{R}$ s.t. $0 \leq a < b$.

$$\begin{aligned} x_n &= (a^n + b^n)^{\frac{1}{n}} \\ &= (b^n + b^n)^{\frac{1}{n}} \\ &= 2^{\frac{1}{n}} \cdot b \to b \end{aligned}$$

Don't forget to state the squeeze law where it's needed!!

And of course the zero infront, as in

$$0 \leq f_n(x) \leq f(x)$$

and variants.

Recall that for the reverse triangle inequality it has an absolute value sign on the LHS

$$|\|x\| - \|y\|| \leq \|x - y\|$$

because the proof involves both $\|x\| = \|x + y - y\|$ and $\|y\| = \|y + x - x\|$

In tutorial 3 we show that $\underset{n\to\infty}{lim\,sup}(x_n) = -\underset{n\to\infty}{lim\,inf}(-x_n)$.

This is necessary for us to show equality of

$$\lim_{n\to\infty} sup(x_n) + \lim_{n\to\infty} sup(y_n) = \lim_{n\to\infty} sup(x_n + y_n)$$

when $\lim_{n\to\infty}(x_n)$ (or $\lim_{n\to\infty} y_n$) exists because we are able to transform
$\lim_{n\to\infty} sup(x_n) = -\lim_{n\to\infty} inf(-x_n) = \lim_{n\to\infty}(x_n)$ back and forth once we move the inequality to the other side.

A finite geometric series goes from $k = 0$ to $n - 1$ not n!

$$\begin{aligned} s_n &= \sum_{k=0}^{n-1} x^n \\ &= \frac{1 - x^n}{1 - x} \end{aligned}$$

An easy way to remember this is that the finite geometric series for n terms start from zero. Hence $n - 1$ is the last term in the partial sum.

When dealing with a sequence $\{x_n\}$ that involves trig and complex powers, you can always try taking the norm and see what happens e.g.

$$\frac{\cos(n)}{(1+i)^n}$$

which in this case converges to zero in the limit.

A limit superior and limit inferior can exist in the form of ∞!

Hence this does not necessarily mean that the $\liminf$ and $\limsup$ coincide with the limit because the limit has to be well defined.

Hence, if $\lim_{n\to\infty} inf(x_n) = \lim_{n\to\infty} sup(x_n)$ it does not mean that a limit exists!

What is the supremum of

$$\sqrt{n}(\sqrt{n+1} - \sqrt{n})$$

Don't forget to show that this is monotone first! =)

What's happening here?

$$s_n = \left(1 + \frac{1}{n^2}\right)^n$$

Does s_n converge?

The limit below can be proven with l'hopital's rule.

$$\lim_{n\to\infty} \frac{\log(n)}{n^\alpha} = 0$$

$\forall \alpha > 0$

$\forall n \geq e^2$, we have $\ln(n) \geq 2$. Hence

$$\sum \frac{1}{\ln(n)^n} \leq \sum \frac{1}{2^n}$$

for n large enough.

8.13.2 Wk6-10

Recall that when answering questions about uniform convergence and using the supremum norm, don't forget to add the subscripts

$$\|f_n(x) - f(x)\|_{\infty,\mathcal{D}}$$

where $\mathcal{D}$ is the domain.

Suppose $z = a + bi$. Then $n^{ib} \neq 1$ - actually $e^{ib\ln(n)} = 1$.

Be very careful with this!

Suppose there is a open subset $A \subset \mathbb{R}$. An open interval where $\forall x \in \mathbb{R}\ \exists r_x$ such that $(x - r_x, x + r_x) \in A$

An open interval is for one dimension, don't write the wrong thing! And don't forget to mention that an open interval is an open set/ball!

If you have a domain $\left[0, t - \frac{1}{n}\right), \left[t - \frac{1}{n}, t + \frac{1}{n}\right], \left(t + \frac{1}{n}, 1\right]$ for some $f_n(t)$ then checking for pointwise limits requires you to check that the left and right limits coincide at the points $t - \frac{1}{n}$ and $t + \frac{1}{n}$.

Does

$$s_n = \sum_{n}^{\infty} \frac{(-1)^n}{n + |z|^3}$$

converge absolutely?

ALWAYS MENTION THE SET IN WHICH FUNCTION/SEQUENCE/SERIES CONVERGES ON!!!

E.g. hence $f_n(x)$ converges uniformly on D.

Adjusting the start index of series

$$\begin{aligned} g_n(x) &= \sum_{k=n}^{m} x^k \\ &= x^n \sum_{k=0}^{m-n} x^k \end{aligned}$$

to do partial sum of geometric series (Q4, T9) or (Q5, T8) is very useful.

$$\begin{aligned} \int_a^b 0 \ \delta x &= c - c \\ &= 0 \end{aligned}$$

Don't get tricked!

Proposition 8.177 *Let $s_n = \sum_{k=0}^{n} a_n(z-z_0)^k$ be a power series about a point z_0. Let $\mathcal{Q}$ be the radius of convergence. Let $r \in \mathbb{R}$ s.t. $0 \leq r < \mathcal{Q}$. Then*

s_n is uniformly convergent on $D := \{x \mid |z-z_0| \leq r\}$

See proof here.