Lecture notes in Ph.D.

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Contents

Conventions

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{N}$ denotes the set $\{1, 2, 3, ...\}$ of natural numbers (excluding 0).

Chapter 1

AMP

References:

- [8] https://arxiv.org/pdf/2105.02180

Chapter 2

High-dim statistics

Chapter 3

Statistical inference

References:

- [1] https://arxiv.org/pdf/2010.14863
- reference file [18] https://web.stanford.edu/~montanar/OTHER/TALKS/oops_refs.pdf

Chapter 4

Statistical learning

Chapter 5

Optimal transport and its application in ML

Topics and Reference:

- Wasserstein barycenter, Fréchet means, empirical and population, Fast computation of Fréchet means. [21] [1]
- Statistical estimation of Monge map, consistency and asymptotic. [11][2]
- Knothe-Rosenblatt transport and trangular maps, Estimation and smoothness class. [24][3],[16]
- Kullback-Leilbler estimatioin and normalizing flows, Asymptotic consistency. [12], [13]
- Sinkhorn divergence, Sinkhorn algorithm, asymptotic convergence. [5], [14]
- Deep learning as measure optimization, Global convergence of measure optimization. [4]
- Asymptotic convergence of gradient flows, Particle limits and time limits. [10]

5.1 Optimal transport

5.1.1 The existence of solution of (KP)

Monge problem(MP). Given two probability measure $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \to [0, \infty]$, solve

$$\inf \left\{ M(T) := \int c(x, T(x)) d\mu(x) : T_{\#}\mu = \nu \right\} \tag{5.1}$$

where $T_{\#}\mu(A) := \mu(T^{-1}(A))$.

Kantorovich problem(KP). Given $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$, and $c : X \times Y \to \mathbb{R}_+$, we consider the problem

$$\inf \left\{ K(\gamma) := \int_{X \times Y} c d\gamma : \gamma \in \Gamma(\mu, \nu) \right\}, \tag{5.2}$$

where $\Gamma(\mu, \nu)$ is the set of the so-called transport plans, i.e.,

$$\Gamma(\mu, \nu) = \{\gamma \in \mathcal{P}(X \times Y) : (\pi_x)_{\#}\gamma = \mu, (\pi_y)_{\#}\gamma = \nu\},$$

where π_x and π_y are the two projections of $X \times Y$ onto X and Y, and $(\pi_x)_{\#}\gamma(A) := \gamma(\pi_x^{-1}(A))$ is called the pushforward of γ through π_x.

Remark. Should γ be of the form $(id, T)_{\#}\mu$ for a measurable map $T : X \to Y$, the map T would be called

[1] https://arxiv.org/pdf/1806.05500
[2] https://arxiv.org/pdf/1905.05828
[3] https://www.jmlr.org/papers/volume19/17-747/17-747.pdf

the optimal transport map from μ to ν. It is clear that $(id, T)_{\#}\mu \in \Gamma(\mu, \nu)$ iff T pushes μ onto ν.

$$\int_X c(x, T(x))d\mu(x) = \int_{X\times Y} c(x, y)d\gamma(x, y).$$

We will use the "continuity-compactness argument" in [23] to show that a minimum does exist. We use the Weierstrass criterion for the existence of minimizers.

Definition 5.1.1. (lower semi-continuous) On a metric space X, a function $f : X \to \mathbb{R} \cup \{\infty\}$ is said to be lower semi-continuous if for every sequence $x_n \to x$ we have

$$f(x) \leqslant \liminf_n f(x_n).$$

We often use the converging in norm for a sequence $\{x_n\}$ in a normed vector spaces. However, in some cases it is useful to work in topologies on vector spaces that are weaker than a norm topology. One reason for this is that many important modes of convergence are not captured by a norm topology. Another reason (of particular importance in PDE) is that the norm topology on infinite-dimensional spaces is so strong that very few sets are compact or pre-compact in these topologies, making it difficult to apply compactness methods in these topologies. Instead, one often first works in a weaker topology, in which compactness is easier to establish, and then somehow upgrades any weakly convergent sequences obtained via compactness to stronger modes of convergence.

Two basic weak topologies for this purpose are the weak topology on a normed vector space X, and the weak* topology on a dual vector space X'. Compactness in the latter topology is usually obtained from the **Banach-Alaoglu theorem** (and its sequential counterpart), which will be a quick consequence of the Tychonoff theorem (and its sequential counterpart) from the previous lecture.

Remark. For more definition of the strong and weak topologies, see 245B, Notes 11: The strong and weak topologies.

Theorem 5.1.2. (Weierstrass) If $f : X \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous and X is compact, then there exists $\overline{x} \in X$ such that $f(\overline{x}) = \min\{f(x) : x \in X\}$.

Theorem 5.1.3. (Banach-Alaoglu theorem) The closed unit ball of the dual space of a Banach space is compact in the weak-* topology.

Proof.

□

Theorem 5.1.4. (Sequential Banach-Alaoglu theorem) If $\mathcal{X}$ is separable and ξ_n is a bounded sequence in $\mathcal{X}'$, then there exists a subsequence ξ_{n_k} weakly converging to some $\xi \in \mathcal{X}'$.

We donate by $\mathcal{M}(X)$ the set of finite signed measures on X.

Theorem 5.1.5. Suppose that X is a separable and locally compact metric sapce. Let $\mathcal{X} = C_0(X)$ be the space of continuous function on X vanishing at infinity. Then every element of $\mathcal{X}'$ is represented in a unique way as an element of $\mathcal{M}(X)$: for all $\xi \in \mathcal{X}'$, there exists a unique $\lambda \in \mathcal{M}(X)$ such that

$$< \xi, \phi >= \int \phi d\lambda$$

for all $\phi \in \mathcal{X}$.

We will call it weak convergence and denote it through $\mu_n \rightharpoonup \mu$ iff $\phi \in C_b(X)$ we have

$$\int \phi d\mu_n \to \int \phi d\mu$$

where $C_b(X)$ is the space of bounded continuous functions on X. Note that $C_b(X) = C_0(X) = C(X)$ if X is compact, so the weak-* convergence is same as weak convergence.

Theorem 5.1.6. Let X and Y be compact metric spaces, $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$, and $c : X \times Y \to \mathbb{R}$ a continuous function. Then (KP) admits a solution.

Proof. We just need to show that the set $\Gamma(\mu, \nu)$ is compact and $K(\gamma)$ is continuous and apply Weierstrass's theorem. Since $c \in C(X \times Y)$, this gives weak-* continuity of functional $\gamma \mapsto \int c d\gamma$ on $\mathcal{M}(X \times Y)$. $\Gamma(\mu, \nu)$ is non-empty since $\mu \times \nu \in \Gamma(\mu, \nu)$. By the Banach-Alaoglu theorem, we need to show that $\Gamma(\mu, \nu)$ is weak-* closed subset of $B = \{\gamma : \|\gamma\| \leqslant 1\}$ which is weak-* compact. Note that a sequence of probability measures $\gamma_n \in \Gamma(\mu, \nu)$ are mass 1, so they are bounded in the dual of $C(X \times Y)$. Hence, this guarantees the existence of a subsequence $\gamma_{\gamma_k} \to \gamma$ converging to a probability γ, We need to check $\gamma \in \Gamma(\mu, \nu)$. Indeed, fix $\phi \in C(X)$,

$$\int_X \phi(x) d\mu(x) = \int_{X \times Y} \phi(x) d\gamma_{n_k}(x, y) \to \int_{X \times Y} \phi(x) d\gamma(x, y)$$

This shows that $(\pi_x)_{\#}\gamma = \mu$. The same may be done for π_y. □

Theorem 5.1.7 (Minimax Theory)**.** We have

$$\inf_{\gamma \in \mathcal{M}_+(X \times Y)} \sup F(\gamma, (\phi, \psi)) = \sup \inf_{\gamma \in \mathcal{M}_+(X \times Y)} F(\gamma, (\phi, \psi))$$

Remark. See KAKUTANI'S fixed point theorem and the minimax theorem in game theory.

5.1.2 Probabilistic interpretation

See here. Convex function.

5.1.3 Duality

The problem (KP) is a linear optimization under convex constraints, given by linear qualities or inequalities. Hence, an important tool will be duality theory, which is typically used for convex problems. The Kantorovich motivation for (KP):
If μ is a distribution of mines producing iron. Let $f(x)$ be production capacity of mine at x which $d\mu(x) = f(x)$. And ν is a distribution of factories consuming iron that $d\nu(y) = g(y)$ and $g(y)$ is the consumption requirements of factory at location y. Let $c(x, y)$ be the transport cost per mass. Then (KP) asks which mine should supply which factory to minize overall transport cost. Now, an independent company might offer to buy iron from the mines and sell iron to the factories say they offer to pay the mine at x, $\phi(x)$ dollars per mass and sell iron to the factory at y for $\psi(y)$ dollars per mass. Their profits are

$$\int_Y \psi(y) d\nu(y) + \int_X \phi(x) d\mu(x).$$

But if for (x, y) we have $\phi(x) + \psi(y) > c(x, y)$, the mine at x could sell and deliver directly to the factory at y for price $\psi(y) - \varepsilon$. The mine's profit would be

$$\psi(y) - \varepsilon - c(x, y) > -\phi(x)$$

for all $\varepsilon > 0$ samll enough.

So, to entice all mines and factories to take the deal, the company must ensure

$$\phi(x) + \psi(y) \leqslant c(x, y) \, everywhere$$

We write the dual optimization problem:

Dual Problem(DP). Given $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and the cost function $c : X \times Y \to [0, \infty)$. We consider the problem

$$\max \left\{ \int_Y \psi(y) d\nu(y) + \int_X \phi(x) d\mu(x) : \phi \in C_b(X), \psi \in C_b(Y) : \phi(x) + \psi(y) \leqslant c(x, y) \right\} \tag{5.3}$$

Theorem 5.1.8. $\sup(DP) = \min(KP)$

5.1.4 The existence of solution to (DP)

Definition 5.1.9 (c-transform, c-concave)**.** Given a function $\phi \in C(X)$, define $\phi^c : Y \to \mathbb{R}$ by

$$\phi^c(y) := \min_{x \in X}(c(x, y) - \phi(x)).$$

Similary, for $\psi \in C(Y)$, define $\psi^c : X \to \mathbb{R}$ by

$$\psi^c(x) := \min_{y \in Y}(c(x, y) - \psi(y)).$$

We say ϕ is **c-concave** if $\phi = \psi^c$ for some ψ.

Lemma 5.1.10. If $(\phi, \psi) \in C(X) \times C(Y)$ and $\phi(x) + \psi(y) \leqslant c(x, y)$. Then (ϕ^{cc}, ϕ^c) satisfies

$$\phi^c(y) \geqslant \psi(y), \phi^{cc}(x) \geqslant \phi^c(x)$$

for all (x, y). Therefore,

$$\int_X \phi d\mu + \int_Y \psi d\nu \leqslant \int_X \phi^{cc} d\mu + \int_Y \phi^c d\nu.$$

Definition 5.1.11 (k-Lip)**.** We say the cost function c is $k-$Lipschitz in X, if

$$c(x_1, y) - c(x_0, y) \leqslant k|x_1 - x_0|$$

for all $x0, x_1, y$.

Theorem 5.1.12. Assume X, Y are compact and c is $k-$Lip. Then there exists a maxmizer (ϕ, ψ) in (DP). It has the form that ϕ and ψ are $c-$concave. In particular

$$\max(DP) = \max_{\phi \in c-concave(X)} \int_X \phi d\mu + \int_Y \phi^c d\nu.$$

Proof.

□

5.1.5 c-cyclical monotonicity

Definition 5.1.13 ($c-$monotone of order 2)**.** We say that a set $S \subseteq X \times Y$ is c-monotone of order 2, for a given cost function $c : X \times Y \to \mathbb{R}$, if for all $(x_0, y_0), (x_1, y_1) \in S$, we have

$$c(x_0, y_0) + c(x_1, y_1) \leqslant c(x_0, y_1) + c(x_1, y_0)$$

Definition 5.1.14 (c-cyclical monotone)**.** A subset $S \subseteq X \times Y$ is **c-cyclical monotone**(c-CM) if for all integers k, all permutation σ, and finite family of points $(x_1, y_1), \dots, (x_N, y_N) \in S$, we have

$$\sum_{i=1}^{N} c(x_i, y_i) \leqslant \sum_{i=1}^{N-1} c(x_i, y_{i+1}) + c(x_N, y_1). \tag{5.4}$$

Definition 5.1.15. On a separable metric space X, the support of a measure γ is defined as the smallest closed set on which γ is concentrated, i.e.,

$$spt(\gamma) := \bigcap \{A : A \text{ is closed and } \gamma(X \setminus A) = 0\}$$

Theorem 5.1.16. Assume that X, Y are compact and c is $k-$Lip. If $\gamma \in \Gamma(\mu, \nu)$ is optimal in (KP). Then $spt(\gamma)$ is $c-$cyclically monotone.

Proof. Let (ϕ, ψ) be a continuous solution to dual problem (DP). We have
$$\phi(x) + \psi(y) \leqslant c(x, y)$$
and $\phi(x) + \psi(y) = c(x, y)$, $\gamma-$a.e. The set satisfying "=" is closed. So it contains the support of γ. Let $(x_1, y_1), \dots, (x_N, y_N) \in spt(\gamma)$, we have
$$\begin{aligned}\sum_{i=1}^{N} c(x_i, y_i) &= \sum_{i=1}^{N} (\phi(x_i) + \psi(y_i)) \\ &= \sum_{i=1}^{N-1} (\phi(x_i) + \psi(y_{i+1})) + \phi(x_N) + \psi(y_1) \\ &\leqslant \sum_{i=1}^{N-1} c(x_i, y_{i+1}) + c(x_N, y_1).\end{aligned}$$
□

Remark. The converse is also true, see [23, Theorem 1.49, p39]

If we admit the duality result $\min(KP) = \max(DP)$ as Theorem 5.1.12(the following theorem), then we have
$$\min(KP) = \max_{\phi \in c-concave(X)} \int_X \phi d\mu + \int_Y \phi^c d\nu.$$

Theorem 5.1.17. Suppose that X and Y are Polish spaces[a] and that $c : X \times Y \to \mathbb{R}$ is uniformly continuous and bounded. Then the problem (DP) admits a solution (ϕ, ϕ^C) and we have $\min(KP) = \max(DP)$

[a] A Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

Proof. See [23, Theorem 1.39]. □

5.1.6 1-dimensional solutions

Definition 5.1.18. For $X \subseteq \mathbb{R}^d$, we say $c : X \times X \to \mathbb{R}$ satisfies the twist condition if c is differentiable w.r.t. x at every point and the map $y \mapsto \nabla_x c(x_0, y)$ is injective for every x_0

This condition is also known in economics as Spence-Mirrlees condition.
Examples of costs satisfying the twist condition:

Example. (i) If n=1,

Theorem 5.1.19. Assume that $X, Y \subseteq \mathbb{R}$ are compact intervals and cost function c satisfies $\frac{\partial^2 c}{\partial x \partial y} < 0$ throughout $X \times Y$. Then $spt(\gamma)$ is monotone increasing for any optimal γ (i.e. for any $(x_0, y_0), (x_1, y_1) \in spt(\gamma)$, $(x_1 - x_0)(y_1 - y_0) \geqslant 0$.

Proof.
□

$\frac{\partial^2 c}{\partial x \partial y} < 0$ is often called submodularity.

Remark. (1) In 1 dimension, the sets which are $c-$monotone of order 2 are $c-$cyclically monotone. But this is not true in $d \geqslant 2$ dimension. Here is an example:

5.1.7 Uniqueness of Monge solutions

Theorem 5.1.20. Assume that X, Y are compact. If $c \in C(X \times Y)$ satisfies the twist condition and μ is absolutely continuous w.r.t. Lebesgue measure. Then any minimizer γ in (KP) is of Monge form.

Proof.

□

Corollary 5.1.21. Under the assumptions of the Theorem 5.1.20, the solutions γ and T to (KP) and (MP) are unique.

5.1.8 Wasserstein's Distance

Given two probability measures $\mu, \nu \in \mathcal{P}(X)$ and $X \subseteq \mathbb{R}^n$ be compact, define

$$W_2(\mu, \nu) := \left(\min_{\gamma \in \Gamma} \int_{X \times X} |x - y|^2 d\gamma(x, y) \right)^{1/2} \tag{5.5}$$

Theorem 5.1.22. W_2 defined in 5.5 is a metric on $\mathcal{P}(X)$.

See arXiv:1412.7726

Chapter 6

Statistical mechanics of mean-field disordered systems: a PDE approach

These are notes from the CRM-PIMS Summer School in Probability 2021. Webpage for CRM-PIMS Summer School: https://secure.math.ubc.ca/Links/ssprob21/. The course is taught by Jean-Christophe Mourrat: A PDE approach to mean-field disordered systems and course webpage is http://perso.ens-lyon.fr/jean-christophe.mourrat/Montreal.html.

6.1 Large Deviations and Convex Analysis

Scribe: Tim Banova

We give a primer on some fundamental results from the theory of *large deviations* and *convex analysis*

6.1.1 Motivating example: LDP for IID Bernoulli random variables

Let $(X_n)_{n\in\mathbb{N}}$ denote a collection of independent Bernoulli random variables with parameter $p \in [0,1]$; that is,

$$\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0), \qquad n \in \mathbb{N}. \tag{6.1}$$

Define the (rescaled) partial sum via

$$S_N := \frac{1}{N}\sum_{n=1}^{N} X_n, \qquad N \in \mathbb{N}. \tag{6.2}$$

We know by the large large number, the sum S_N converges to its mean p as N goes to infinity. Now, one may ask what is the speed that the probability of S_N converging to a number x other than p tends to zero? This questions is not answered by the central limit theorem because it only tells us typical deviations from p, which are of order $\sqrt{N}$, while the questions is about large deviations, which are of order N.

For this example, we can calculate the probability exactly by the following formula

$$\mathbb{P}(S_N = k/N) = \binom{N}{k} p^k (1-p)^{N-k}, \quad k \in \{0,\dots,N\},\ N \in \mathbb{N}. \tag{6.3}$$

Letting $x := k/N \in (0,1)$, we have that

$$\begin{aligned}\binom{N}{k} &= \exp\Big(N\log(N/e) - Nx\log(Nx/e) + O(\log N) - N(1-x)\log(N(1-x)/e)\Big)\\ &= \exp\Big(-Nx\log x - N(1-x)\log(1-x) + O(\log N)\Big).\end{aligned} \tag{6.4}$$

Above, we have used *Stirling's formula*, which is

$$n! = \exp\Big(n\log(n/e) + O(\log n)\Big), \quad n \to \infty. \tag{6.5}$$

Thus, letting

$$I(x) := x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}, \quad x \in (0,1), \tag{6.6}$$

we have that for every $\delta > 0$ and $x \in [\delta, 1-\delta]$ of the form $x = k/N$ for some $k \in \mathbb{Z}$,

$$\mathbb{P}(S_N = x) = \exp\Big(-NI(x) + O(\log N)\Big), \quad N \to \infty. \tag{6.7}$$

Moreover, note that for any $x \in (p,1)$, as $N \to \infty$

$$\begin{aligned}\mathbb{P}(S_N \geqslant x) \geqslant \mathbb{P}\Big(S_N = \frac{\lceil Nx \rceil}{N}\Big) &\geqslant \exp\Big(-NI(\lceil Nx\rceil/N) + O(\log N)\Big)\\ &\exp\Big(-NI(x) + O(\log N)\Big)\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}\mathbb{P}(S_N \geqslant x) \leqslant \sum_{k=\lfloor Nx \rfloor}^{N} \mathbb{P}(S_n = k/N) &\leqslant (N+1)\mathbb{P}(S_n = \lfloor Nx \rfloor/N)\\ &\leqslant \exp\Big(-NI(x) + O(\log N)\Big)\end{aligned} \tag{6.9}$$

so that

$$\mathbb{P}(S_N \geqslant x) = \exp\Big(-NI(x) + O(\log N)\Big), \quad N \to \infty \tag{6.10}$$

Analogously, for every $x \in (0,p)$,

$$\mathbb{P}(S_N \leqslant x) = \exp\Big(-NI(x) + O(\log N)\Big), \quad N \to \infty \tag{6.11}$$

This leads to an important definition.

Definition 6.1.1 (Large deviations principle for sequences of random variables)**.** For $N \in \mathbb{N}$, let $S_N : \Omega_N \to \mathbb{R}$ be a random variable defined on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$. We say that $(S_N)_{N\in\mathbb{N}}$ *satisfies a large deviations principle (LDP)* with *speed* N and *rate function* I if for all Borel-measurable $A \subseteq \mathbb{R}$,

$$\limsup_{N\to\infty} \frac{1}{N}\log\mathbb{P}_N(S_N \in A) \leqslant -\inf_{x\in\overline{A}} I(x), \quad \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N(S_N \in A) \geqslant -\inf_{x\in A^\circ} I(x), \tag{6.12}$$

where $\overline{A}$ and A° denote the closure and interior of A.
Informally, we write this as

$$\mathbb{P}(S_N \simeq x) \simeq \exp(-NI(x)) \tag{6.13}$$

Exercise. Apply (6.10) and (6.11) to show that $(S_N)_{N\in\mathbb{N}}$ defined earlier satisfies a LDP with speed N and rate function I given in (6.6).

6.1.2 Cramér's Theorem: a LDP for i.i.d random variables

In the previous section, we derived (from first principles) a LDP for i.i.d Bernoulli random variables with parameter $p \in [0,1]$. Naturally, this leads us to the following question: when do random variables with an *arbitrary* distribution satisfy a LDP?

Suppose that $(X_n)_{n\in\mathbb{N}}$ are a family of i.i.d random variables with $\mathbb{E}[X_1] = 0$ and $x, \lambda \geqslant 0$.

Then, by Markov's inequality,

$$\mathbb{P}(S_N \geqslant x) = \mathbb{P}(e^{\lambda N S_n} \geqslant e^{\lambda N x}) \leqslant e^{-\lambda N x}\mathbb{E}[e^{\lambda N S_N}], \tag{6.14}$$

where by independence of $(X_n)_{n\in\mathbb{N}}$

$$\mathbb{E}[e^{\lambda N S_N}] = \mathbb{E}[e^{\lambda X_1}]^N. \tag{6.15}$$

Thus, letting

$$\psi(\lambda) := \log\mathbb{E}[\exp(\lambda X_1)] \tag{6.16}$$

denote the *cumulant generating function* of X_1, we have

$$\mathbb{P}(S_N \geqslant x) \leqslant \exp\Big(-N\sup_{\lambda\in\mathbb{R}_+}\big(\lambda x - \psi(\lambda)\big)\Big) \tag{6.17}$$

Define the *Fenchel-Legendre transform* of ψ, $\psi^* : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ via

$$\psi^*(x) := \sup_{\lambda \in \mathbb{R}}(\lambda x - \psi(\lambda)) \tag{6.18}$$

Theorem 6.1.2 (Cramér's Theorem : LDP for IID)**.** Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d random vairables such that for all $\lambda \in \mathbb{R}$, $\mathbb{E}[e^{\lambda X_1}] < \infty$, with cumulant generating function ψ and Legendre transform ψ^*.
Then, $(S_N)_{N\in\mathbb{N}}$, where $S_N := \frac{1}{N}\sum_{k=1}^N X_k$ satisfies a large deviations principle with speed N and rate function ψ^*.

Proof. Apply Theorem 6.1.3 below or see Section 2.2.2 of [6]. □

Exercise. Show that if X is a Bernoulli random variable with parameter $p \in [0, 1]$, then $\psi^* = I$, where I is given in (6.6).

In fact, we can derive LDPs for arbitrary sequences of random variables (under some fairly strong conditions).

Theorem 6.1.3 (General LDP on $\mathbb{R}$)**.** For $N \in \mathbb{N}$, let $S_N : \Omega_N \to \mathbb{R}$ be a random variable defined on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ and define

$$\psi_N(\lambda) := \frac{1}{N}\log \mathbb{E}_N[\exp(\lambda N S_N)]. \tag{6.19}$$

Suppose that there exists a $\psi \in C^1(\mathbb{R})$ such that $\psi_N \to \psi$ pointwise. Define

$$\psi^*(x) := \sup_{\lambda\in\mathbb{R}}(\lambda x - \psi(\lambda)). \tag{6.20}$$

Then, S_N satisfies a large deviation principle with speed N and rate function ψ^*.

Remark. This is a special case of the Gärtner–Ellis theorem [6, Theorem 2.3.6].

Proof. We claim that for every $x \in \mathbb{R}$,

$$\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}(S_N \geqslant x) \leqslant -\inf_{z\geqslant x} \psi^*(x), \tag{6.21}$$

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty} \frac{1}{N}\mathbb{P}(S_N \in [x-\varepsilon, x+\varepsilon]) \geqslant -\psi^*(x). \tag{6.22}$$

Denote $m = \psi'(0)$ (Recall that ψ is of C^1). By convexity, $\psi(\lambda) \geqslant m\lambda$. Thus, for all $x \geqslant m$,

$$\sup_{\lambda\leqslant 0}(\lambda x - \psi(\lambda))$$

and ψ^* is increasing on $[0, +\infty)$. Moreover, $\psi^*(x) = 0$. Thus, it suffices to prove (6.21) via showing that for all $x \geqslant m$,

$$\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}(S_N \geqslant x) \leqslant -\inf_{z\geqslant x} \psi^*(x) = \sup_{\lambda\leqslant 0}(\lambda x - \psi(\lambda)). \tag{6.23}$$

We have

$$\mathbb{P}(S_N \geqslant x) \leqslant \exp(-\lambda N x) \cdot \mathbb{E}[\exp(\lambda N S_N)] \leqslant \exp(\lambda x - \psi_N(\lambda)). \tag{6.24}$$

Also,

$$\frac{1}{N}\log\exp(\lambda x - \psi_N(\lambda)) = \frac{\lambda x}{N} - \frac{1}{N}\psi_N^*(\lambda) \to -\psi^*(x) \tag{6.25}$$

as $N \to \infty$, so (6.24) follows from (6.24) and (6.25).
It remains to prove (6.21). TODO: to be continued... □

Example (Necessity of $\psi \in C^1(\mathbb{R})$)**.** Consider the sequence of random variables $(S_N)_{N\geqslant 1}$ where

$$\mathbb{P}(S_N = 1) = 1/2 = \mathbb{P}(S_N = -1).$$

We have

$$\psi_N(\lambda) = \frac{1}{N}\log\left(\frac{\exp(\lambda N)}{2} + \frac{\exp(-\lambda N)}{2}\right).$$

Since

$$\psi_N(\lambda) = \begin{cases} \lambda + \frac{1}{N}(\log\big(1+e^{-2\lambda N}\big) - \log 2) \to \lambda & \text{when } \lambda > 0 \\ \\ -\lambda + \frac{1}{N}(\log\big(1+e^{2\lambda N}\big) - \log 2) \to -\lambda & \text{when } \lambda \leqslant 0, \end{cases}$$

we derive that

$$\psi(\lambda) = \lim_{N\to\infty} \psi_N(\lambda) = |\lambda|.$$

Note that

$$\psi^*(x) = \sup_{\lambda\in\mathbb{R}}(\lambda x - |\lambda|) = \begin{cases} 0 & x \in [-1,1] \\ \infty & \text{otherwise} \end{cases}$$

TODO: to be continued...

6.1.3 Some convex analysis

We now develop some of the essentials of the analysis of *convex* functions $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ on $\mathbb{R}^d$, for $d \geqslant 1$.

Definition 6.1.4 (Convex functions)**.** A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is *convex* if for all $x, y \in \mathbb{R}^d$ and $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha)f(y). \tag{6.26}$$

Exercise. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}^d$ an $\mathbb{R}^d$-random variable (ie. with respect to the Borel σ-algebra on $\mathbb{R}^d$). Show that the map

$$\mathbb{R}^d \to \mathbb{R} \cup \{\infty\}, \quad \lambda \mapsto \log \mathbb{E}[e^{\lambda \cdot X}] \tag{6.27}$$

is convex.

Solution to Exercise 6.1.3. Let $x, y \in \mathbb{R}^d$ and $\alpha \in [0,1]$. Note that by Hölder's inequality, with conjugate exponents $p = \alpha^{-1}$ and $q = (1-\alpha)^{-1}$, we have

$$\mathbb{E}[e^{\alpha x\cdot X} e^{(1-\alpha)y\cdot X}] \leqslant \mathbb{E}[e^{x\cdot X}]^\alpha \mathbb{E}[e^{y\cdot X}]^{1-\alpha}. \tag{6.28}$$

Taking logarithms gives the desired inequality.

Definition 6.1.5 (Lower semi-continuous functions)**.** A function $f : \mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ is *lower semi-continuous (l.s.c.)* if for every sequence $x_n \to x \in \mathbb{R}^d$,

$$f(x) \leqslant \liminf_{n\to\infty} f(x_n). \tag{6.29}$$

Remark. For $\alpha \in I$, let $f_\alpha : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. If for all $\alpha \in I$, f_α is convex (resp., l.s.c.) then $\sup_{\alpha\in I} f_\alpha$ is convex (resp., l.s.c).

Definition 6.1.6 (Convex duals)**.** Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. The *convex dual (or convex conjugate) of* f is the function $f^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined via

$$f^*(\lambda) := \sup_{x\in\mathbb{R}^d} (\lambda x - f(x)), \qquad \lambda \in \mathbb{R} \tag{6.30}$$

Moreover, define the *bidual (or biconjugate)* $f^{**} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ via

$$f^{**}(x) := \sup_{\lambda\in\mathbb{R}}(\lambda x - f^*(\lambda)) \tag{6.31}$$

Note that for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$f^*(\lambda) \geqslant \lambda x - f(x), \quad f(x) \geqslant \lambda x - f^*(\lambda) \tag{6.32}$$

so that $f(x) \geqslant f^{**}(x)$. Moreover, the case that $f(x) \leqslant f^{**}(x)$ for all $x \in \mathbb{R}^d$ characterises functions which are convex and l.s.c., by the following theorem.

Theorem 6.1.7 (Fenchel-Moreau)**.** A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is convex and l.s.c if and only if $f^{**} = f$.

6.2 Curie-Weiss Models

Scribe: Tim Banova

6.2.1 Gibbs measures

We give a probabilisitic description of an *isolated system* in which the *energy is fixed.*

In particular, suppose for some $N, K \in \mathbb{N}$ we are given a system of N units, which can each be in one of K states $[K] = \{1, 2, \ldots, K\}$.

For each $k \in [K]$, let N_k denote the number of units in state k. To each each state $k \in [K]$, we associate an *energy* $e_k \in \mathbb{R}$, where $e : [K] \to \mathbb{R}$ is the *Hamiltonian* of the system.

Throughout, we assume that there exists $\bar{e} \in \mathbb{R}$ such that

$$\sum_{k \in [K]} N_k e_k = N\bar{e}. \tag{6.33}$$

According to Boltzmann's hypothesis, we postulate that the system is distributed according to the uniform measure.

A natural question arises; for each $k \in [K]$, what is the likelihood of finding a given unit N_k in state k? In other words, what does N_k/N look like?

By a clear counting argument, the number of configurations consistent with $(N_1, \ldots, N_k)$ is

$$\binom{N}{N_1}\binom{N - N_1}{N_2} \cdots \binom{N - \sum_{k \in [K-1]} N_k}{N_K} = \frac{N!}{N_1! \cdots N_K!} \tag{6.34}$$

Supposing that $N_k/N \simeq p_k$ where $\sum_{k \in [K]} p_k = 1$, Stirling's formula implies that

$$\log\left(\frac{N!}{N_1! \cdots N_K!}\right) = NS(p) + O(\log N), \tag{6.35}$$

where

$$S(p) := -\sum_{k \in [K]} p_k \log p_k = \sum_{k \in [K]} p_k \log \frac{1}{p_k} \tag{6.36}$$

is the *(Shannon) entropy* of p.

Thus, the system will tend to concentrate on p which maximise the Shannon entropy S, subject to the constraints

$$\sum_{k \in [K]} p_k e_k = \bar{e}, \qquad \sum_{k \in [K]} p_k = 1. \tag{6.37}$$

The maximizer p is such that there exists α, β satisfying

$$\partial_{p_k} S(p) = \alpha + \beta e_k, \quad k \in [K], \tag{6.38}$$

or

$$p_k = \frac{1}{Z} \exp(-\beta e_k) \tag{6.39}$$

where

$$Z := \sum_{k \in [K]} \exp(-\beta e_k) \tag{6.40}$$

is called the *partition function*, and β is the *inverse temperature*. Such a $(p_k)_{k \in [K]}$ is called the *(canonical) Gibbs measure* associated with $(e_k)_{k \in [K]}$.

For motivation on the physical terminology, as well as an account of the *macroscopic* thermodynamical theory which this *microscopic* approach describes, refer to [9, Chapter 1].

6.2.2 A brief look at the Ising model

Before introducing the main models of interest, we introducing

6.2.3 The Curie-Weiss model

6.3 Curie-Weiss Model and Generalized Curie-Weiss Model

Scribe: Haotian Gu

6.3.1 Curie-Weiss Model

Let us first recall the definition of the Curie-Weiss model on $\Omega = \{1, \cdots, N\}$, which is obtained by trivializing the geometry of the Ising model. The energy associated with each configuration $\sigma \in \{\pm 1\}^N$, at inverse temperature t and with an external magnetic field h, is

$$H_N(t,h,\sigma) := \frac{t}{N}\sum_{i,j=1}^{N}\sigma_i\sigma_j + h\sum_{i=1}^{N}\sigma_i.$$

Note that here the energy is the opposite of that in [9], which means with more spins align the energy is larger.

The free energy of the system is then defined as

$$F_N(t,h) = \frac{1}{N}\log 2^{-N}\sum_{\sigma\in\{\pm1\}^N}\exp(H_N(t,h,\sigma)).$$

The expectation for function $f(\sigma)$ with respect to the Gibbs measure is denoted as

$$\langle f\rangle := Z_N^{-1}(t,h)\sum_{\sigma\in\{\pm1\}^N} f(\sigma)\exp(H_N(t,h,\sigma)),$$

where $Z_N(t,h)$ is the partition function of the Gibbs measure:

$$Z_N(t,h) = \sum_{\sigma\in\{\pm1\}^N}\exp(H_N(t,h,\sigma)).$$

What we want to study here is the mean magnetization $m_N := \frac{1}{N}\sum\limits_{i=1}^{N}\sigma_i$. However, it will be easier to do it in another way: via studying the free energy, from which we can reproduce the information about magnetization.

By treating $2^{-N}\sum_\sigma \mathbf{1}(\frac{1}{N}\sum\limits_{i=1}^{N}\sigma_i \approx m)$ as the probability of some event under product Bernoulli measures, we know from large deviation result that

$$2^{-N}\sum_\sigma \mathbf{1}(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\approx m) \approx \exp(-N\psi^*(m)),$$

where $\psi(\lambda) = \log\left(\frac{1}{2}\sum\limits_{\{\pm1\}}\exp(\lambda\sigma_1)\right) = \log\cosh(\lambda)$, and $\psi^*(m) = \sup\limits_\lambda(\lambda m - \psi(\lambda)) = \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m)$. Therefore,

$$\begin{aligned}
2^{-N}\sum_\sigma\exp(H_N(t,h,\sigma)) &= 2^{-N}\sum_\sigma\exp\Big(tN(\frac{1}{N}\sum\sigma_i)^2 + hN(\frac{1}{N}\sum\sigma_i)\Big)\\
&\approx \sum_m\left(2^{-N}\sum_\sigma\mathbf{1}(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\approx m)\exp\Big(tN(\frac{1}{N}\sum\sigma_i)^2 + hN(\frac{1}{N}\sum\sigma_i)\Big)\right)\\
&\approx \sum_m\exp\big(tNm^2 + hNm\big)\exp(-N\psi^*(m))\\
&= \sum_m\exp\big(N(tm^2 + hm - \psi^*(m))\big).
\end{aligned}$$

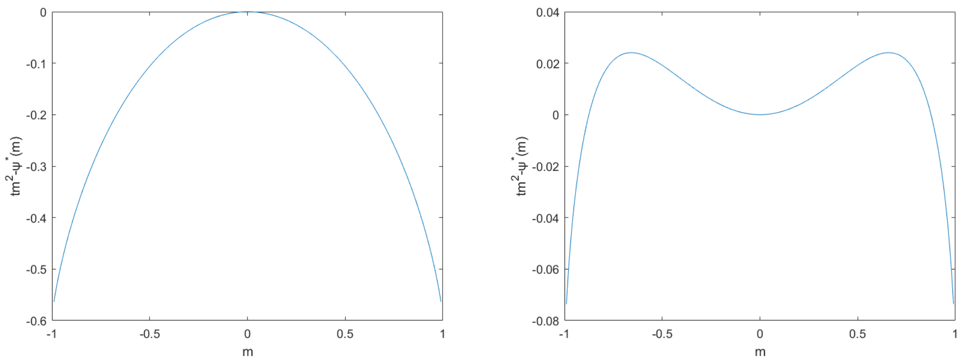

Figure 6.1: Graphs of $tm^2 - \psi^*(m)$ with $t = 0.1$(left) and $t = 0.6$(right).

When $N \to \infty$, the sum of m becomes integration over $[-1, 1]$, and the maximum of the exponentials dominates in the limit:

Proposition 6.3.1. $\forall t \geqslant 0, h \in \mathbb{R}$,

$$\lim_{N\to\infty} F_N(t,h) = \sup_{m\in[-1,1]} (tm^2 + hm - \psi^*(m)).$$

Let $f(t, h)$ be the limit function. Then if it is differentiable in h at some point, then the limit of partial derivatives of F_N w.r.t. h converges to the partial derivative of f as well.

Proposition 6.3.2. If f is differentiable in h at (t, h), then

$$\lim_{N\to\infty} \partial_h F_N(t,h) = \partial_h f(t,h).$$

Proof. Observe that F_N and f are both convex in h(the first obtained using Hölder's inequality, and the second is the supremum of affine functions). Therefore for any $h' \in \mathbb{R}$,

$$F_N(t, h + h') \geqslant F_N(t, h) + \partial_h F_N(t, h)h'.$$

Note that $\partial_h F_N(t,h) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \sigma_i \right\rangle$, which is bounded for all N, therefore has a convergent subsequence. Along this subsequence(or w.l.o.g. assume the original sequence converges) we have $\lim\limits_{N\to\infty} \partial_h F_N(t,h) = p$ for some p. Hence

$$f(t, h + h') \geqslant f(t, h) + ph'.$$

Meanwhile from the Taylor expansion of f we know

$$f(t, h + h') = f(t, h) + \partial_h f(t, h)h' + o(h'),$$

so $p = \partial_h f(t, h)$ since the inequality holds for both positive and negative h'. □

Note that actually if a sequence of differentiable convex functions f_n converges pointwise to f(thus also convex) and f is differentiable at x, then $f_n'(x) \to f'(x)$.

Now we take a look at the case $h = 0$, which means there is no external magnetic field. Then

$$f(t) = \sup_{m\in[-1,1]} (tm^2 - \psi^*(m)),$$

where $\psi^*(m)$ is a function behaves like a parabola. See Figure 6.1 for the graphs of $tm^2 - \psi^*(m)$ for different t. So this competition between tm^2 and $\psi^*(m)$ leads to the phase transition in the limit behavior of the mean magnetization we are interested in.

Finally, by the so-called "envelope theorem", we know that if f has a unique optimizer $m_0(t, h)$(true for $h = 0$ and small t), the derivative can be taken into the supremum and we obtain $\partial_h f(t, h) = m_0(t, h)$. For more information on the envelope theorem, please refer to SPS3 Problem 1, where JCM has provided a thorough proof to the statement.

6.3.2 Generalized Curie-Weiss Model

The Curie-Weiss model can be generalized in two directions:

1. We use $\xi(\frac{1}{N}\sum_i \sigma_i)$ instead of $(\frac{1}{N}\sum_i \sigma_i)^2$ with given smooth ξ to be the interaction energy, and
2. we use generic measure P_N on $\mathbb{R}^N$ instead of product Bernoulli measure on the space of configurations. In this case, we assume $|\sigma| \leqslant \sqrt{N}$, $P_N(\sigma)$-a.s.

Under these generalizations, the free energy is now

$$F_N(t,h) := \frac{1}{N}\log\int \exp\left(tN\xi(\frac{1}{N}\sum_i \sigma_i) + h\sum_i \sigma_i\right) dP_N(\sigma).$$

We now have the following result.

Theorem 6.3.3. Assume $\forall h \in \mathbb{R}$, $\lim_{N\to\infty} F_N(0,h) = \psi(h)$ and $\psi \in C^1$. Then

$$\lim_{N\to\infty} F_N(t,h) = f(t,h) := \sup_{m\in\mathbb{R}} (t\xi(m) + hm - \psi^*(m)),$$

where $\psi^*(m) = \sup_{h\in\mathbb{R}}(hm - \psi(h))$.

Proof. Assume $|\frac{1}{N}\sum_i \sigma_i| \leqslant (\frac{1}{N}\sum_i \sigma_i^2)^{1/2} \leqslant 1$, P_N-a.s. Then $|\partial_h F_N| = |\left\langle \frac{1}{N}\sum_i \sigma_i\right\rangle| \leqslant 1$, which implies F_N, and therefore ψ, are all 1-Lip. Then we can apply the LDP result(with condition assumed in the context, i.e. $t = 0$ case) to get a lower bound as follows.
Let m_0 be one of the optimization point. Denote U_ϵ to be $(m_0 - \epsilon, m_0 + \epsilon)$. Then

$$F_N(t,h) \geqslant \frac{1}{N}\log\int \exp\left(tN\xi + h\sum_i \sigma_i\right)\mathbf{1}(\frac{1}{N}\sum_i \sigma_i \in U_\epsilon) dP_N(\sigma),$$

which implies (with standard technique)

$$\liminf_{N\to\infty} F_N \geqslant \inf_{m\in U_\epsilon}(t\xi(m) + hm) - \psi^*(m_0).$$

By sending ϵ to 0 we have

$$\liminf_{N\to\infty} F_N(t,h) \geqslant \sup_m (t\xi(m) + hm - \psi^*(m)).$$

On the other hand,

$$F_N(t,h) \leqslant \frac{1}{N}\log\sum_{k=-K}^{K}\int \exp\left(tN\xi + h\sum_i \sigma_i\right)\mathbf{1}(\frac{1}{N}\sum_i \sigma_i \in [\frac{k-1}{K}, \frac{k}{K}]) dP_N(\sigma),$$

which by same method gives

$$\limsup_{N\to\infty} F_N(t,h) \leqslant \max_{-K\leqslant k\leqslant K}\Big(\sup_{m\in[\frac{k-1}{K},\frac{k}{K}]}(t\xi(m) + hm) - \inf_{m\in[\frac{k-1}{K},\frac{k}{K}]}\psi^*(m)\Big).$$

Sending K to infinity completes the proof. □

6.4 Hamilton–Jacobi Equations

Scribe: Fu-Hsuan

Recall that in Chapter 6.3, we studied the limiting free energy via Large deviation methods. This approach, however, seems to be inapplicable for the models coming from statistical inference or the spin glass models, which we are interested in. In this chapter, we study the Curie–Weiss model by a Hamilton–Jacobi equation arised from differentiating the finite free energies and then taking limits. This equation doesn't have a unique solution. In fact, we can construct infinitely many solutions that satisfy this solutions. Fortunately, by restricting ourselves to the class of viscosity solutions, the solution is unique.

6.4.1 Revisit of the Curie–Weiss Model

Recall that in Section 6.3.1, the free energy of the centered Curie–Weiss model was defined as

$$F_N(t,h) = \frac{1}{N}\log 2^{-N}\sum_{\sigma\in\{\pm1\}^N}\exp\left(\frac{t}{N}\sum_{i,j=1}^N \sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right). \tag{6.41}$$

For any function $f : \{\pm1\}^N \to \mathbb{R}$, define

$$\langle f(\sigma)\rangle = \frac{\sum_{\sigma\in\{\pm1\}^N} f(\sigma)\exp\left(\frac{t}{N}\sum_{i,j=1}^N \sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right)}{\sum_{\sigma\in\{\pm1\}^N}\exp\left(\frac{t}{N}\sum_{i,j=1}^N \sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right)}$$

Note that

$$\partial_t F_N = \frac{1}{N}\left\langle \frac{1}{N}\sum_{i,j=1}^N \sigma_i\sigma_j\right\rangle = \left\langle\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right)^2\right\rangle \tag{6.42}$$

$$\partial_h F_N = \left\langle \frac{1}{N}\sum_{i=1}^N \sigma_i\right\rangle. \tag{6.43}$$

Thus,

$$\begin{aligned}\partial_t F_N - (\partial_h F_N)^2 &= \left\langle\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right)^2\right\rangle - \left\langle\frac{1}{N}\sum_{i=1}^N\sigma_i\right\rangle^2 \\ &= \operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right).\end{aligned} \tag{6.44}$$

On the other hand, we have

$$\partial_h^2 F_N = \left\langle \frac{1}{N}\left(\sum_{i=1}^N\sigma_i\right)^2\right\rangle - \frac{1}{N}\left\langle\sum_{i=1}^N\sigma_i\right\rangle^2 = N\cdot\operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right). \tag{6.45}$$

Thus, we conclude from (6.44) and (6.45) that

$$\partial_t F_N - (\partial_h F_N)^2 = \frac{1}{N}\cdot\partial_h^2 F_N. \tag{6.46}$$

Recall that the limit is

$$f(t,h) = \sup_{m\in[-1,1]}(tm^2 + hm - \psi^*(m)). \tag{6.47}$$

Let $m_0(t,h)$ be the minimizer of (6.47). By the envelope theorem (see SPS3.1), at any differentiable point (t,h) of f, we have

$$\partial_t f(t,h) = m_0(t,h)^2 \quad\text{and}\quad \partial_h f(t,h) = m_0(t,h).$$

Thus, at any differentiable point (t,h)

$$\partial_t f(t,h) - (\partial_h f(t,h))^2 = 0. \tag{6.48}$$

The initial condition can be written as

$$F_N(0,h) = \frac{1}{N}\log 2^{-N}\sum_{\sigma\in\{\pm1\}^N}\exp\left(h\sum_{i=1}^N\sigma_i\right) = \log\cosh(h) = \psi(h). \tag{6.49}$$

By Rademacher's theorem and the fact f is Lipschitz, f is differentiable almost everywhere. Thus, we conclude from (6.48) and (6.49) that

$$\begin{cases}\partial_t f(t,h) - (\partial_h f(t,h))^2 = 0. \\ f(t=0,\cdot) = \psi,\end{cases} \tag{6.50}$$

almost surely. One may ask if (6.50) characterizes the function f. The answer is no. In fact, the following construction will give us infinitely many solutions.

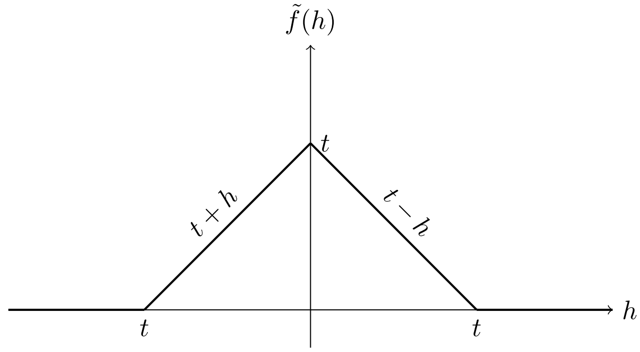

Figure 6.2: Graph of the function $\widetilde{f}$.

Example. Note that $(t,h) \mapsto 0$, $(t,h) \mapsto t+h$ and $(t,h) \mapsto t-h$ are all solutions of (6.50). From these solutions, we can construct the following function (See Figure 6.2)

$$\widetilde{f}(t,h) = \begin{cases} t+h & h \in [-t,0] \\ t-h & h \in [0,t] \\ 0 & \text{otherwise.} \end{cases} \tag{6.51}$$

We can put the corner at anywhere or anytime we want, so there are uncountably many solutions of 6.50 which are Lipschitz.

One may wonder if stronger regularity assumptions than Lipschitz continuity will solve the uniqueness problem. However, we are not allowed to do so since the desired solution in our mind can have corner singularities. Thus, what we really need is to restrict ourselves to a smaller class of solutions which preserves some properties that we required to be true even after passing to limit.

To be precise, if there exists two solutions f and g of the following PDEs

$$\partial_t f - (\partial_h f)^2 = \varepsilon \cdot \Delta f$$
$$\partial_t g - (\partial_h g)^2 = \varepsilon \cdot \Delta g$$

with intial conditions satisfying

$$f(t=0,\cdot) \leqslant g(t=0,\cdot),$$

then the inequality holds for all time t. This is called the maximum principle. We want the maximum principle still be valid when we take $\varepsilon \to 0$. In the next section, we will see that this true if we consider the viscosity solutions.

6.4.2 Viscosity solutions

Let $\mathsf{H} \in C^1(\mathbb{R}^d, \mathbb{R})$. The goal of this section is to develop a well-posed theory of the equation

$$\partial_t f - \mathsf{H}(\nabla f) = 0, \tag{HJ}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. Consider the following PDE with the same initial condition of (HJ).

$$\partial_t f_\varepsilon - \mathsf{H}(\nabla f_\varepsilon) = \varepsilon \Delta f_\varepsilon. \tag{HJ$_\varepsilon$}$$

We want to define the solution of (HJ) as the limit of (HJ$_\varepsilon$) when $\varepsilon \to 0$. Suppose that f_ε indeeds converges to a limit f as $\varepsilon \to 0$. We will see that f does retains some property in the same spirit of the maximum principle.

Let $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ and $(t,x) \in (0,\infty) \times \mathbb{R}^d$ be such that $f - \phi$ has a strict local maximum at (t,x).

For each $\varepsilon > 0$, there should be a point $(t_\varepsilon, x_\varepsilon)$ such that $f_\varepsilon - \phi$ has a local maximum at $(t_\varepsilon, x_\varepsilon)$, and $(t_\varepsilon, x_\varepsilon)$ coverges to (t,x) as $\varepsilon \to 0$. Since f_ε is smooth, at $(t_\varepsilon, x_\varepsilon)$ we have

$$\partial_t (f_\varepsilon - \phi) = 0, \quad \nabla (f_\varepsilon - \phi) = 0, \quad \text{and} \quad \Delta(f_\varepsilon - \phi) \leqslant 0. \tag{6.52}$$

Thus, we have

$$(\partial_t \phi - \mathsf{H}(\nabla \phi))(t_\varepsilon, x_\varepsilon) = (\partial_t f_\varepsilon - \mathsf{H}(\nabla f_\varepsilon))(t_\varepsilon, x_\varepsilon) = \varepsilon \cdot \Delta f_\varepsilon(t_\varepsilon, x_\varepsilon) \leqslant \varepsilon \cdot \Delta \phi(t_\varepsilon, x_\varepsilon). \tag{6.53}$$

Since ϕ is smooth, when $\varepsilon \to 0$, we obtain that

$$(\partial_t \phi - \mathsf{H}(\nabla \phi))(t, x) \leqslant 0. \tag{6.54}$$

Therefore, we conclude that if ϕ is smooth and touches f from above, then (6.54) holds at the contact point. Similarly, if ϕ touches f from below, then the converse inequality of (6.54) holds at the contact point. This motivates the following definition.

Definition 6.4.1. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be continuous. We say that f is a *viscosity subsolution* to (HJ) if for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ such that $f - \phi$ has a local maximum at (t, x), we have

$$(\partial_t \phi - \mathsf{H}(\nabla \phi))(t, x) \leqslant 0.$$

We say that f is a *viscosity supersolution* to (HJ) if for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ such that $f - \phi$ has a local maximum at (t, x), we have

$$(\partial_t \phi - \mathsf{H}(\nabla \phi))(t, x) \geqslant 0.$$

Finally, the function f is called a *viscosity solution* to (HJ) if it is both a subsolution and a supersolution.

Remark. Replacing the locally maximal condition by strictly locally maximal condition yields an equivalent definition. We can also replace $\phi \in C^\infty$ by $\phi \in C^2$. (See SPS.4).

If f is a C^2 fucntion and it solves (HJ) everywhere, then f is a viscosity solution.

Example. Now, we verify that the function $\widetilde{f}$ constructed in Example 6.4.1 is not a viscosity solution. Define $\phi(t, x) = t$. Then,

$$\widetilde{f} - \varphi \leqslant 0 \quad \text{and} \quad \widetilde{f}(t, 0) - \varphi(t, 0) = 0,$$

so $f - \phi$ has a local maximum at $(t, 0)$. If f was a viscosity solution, the we should have

$$(\partial_t \phi - (\partial_x \phi)^2)(t, 0) \leqslant 0.$$

However,

$$(\partial_t \phi - (\partial_x \phi)^2) = 1,$$

which leads to a contradiction.

Back to the Curie–Weiss model, recall that $F_N(0, \cdot)$ does not depend on N, and that $|\partial_h F_N|, |\partial_t F_N| \leqslant 1$. Also, the sequence F_N are all one Lipschitz both in t and x. Therefore, by the Arzelà–Ascoli theorem, the sequence F_N is precompact with respect to the topology induced by local uniform convergence.

Proposition 6.4.2. Recall that F_N is defined as in (6.41). Let f be any subsequential limit of F_N. Then f is a viscosity solution to the PDE with initial condition

$$\begin{cases} \partial_t f - (\partial_h f)^2 = 0 \\ f(0, \cdot) = F_N(0, \cdot) = \psi. \end{cases} \tag{6.55}$$

Proof. Recall that

$$\partial_t F_N - (\partial_h F_N)^2 = \frac{1}{N} \cdot \partial_h^2 F_N.$$

Let $(t, h) \in (0, \infty) \times \mathbb{R}$ and $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ be such that $f - \phi$ has a strict local maximum at (t, h). Since F_N converges to f, there exists $(t_N, h_N) \in (0, \infty) \times \mathbb{R}$ such that for every N sufficiently large, $F_N - \phi$ has a local maximum at (t_N, h_N) and $(t_N, h_N) \to (t, h)$ as $N \to \infty$ (See exercise in PS4.3 for an argument in solution).

Since $t_N > 0$ as N sufficiently large, we have

$$\begin{aligned}
(\partial_t \phi - (\partial_h \phi)^2)(t_N, h_N) &= (\partial_t F_N - (\partial_h F_N)^2)(t_N, h_N) \\
&= \frac{1}{N} \partial_h^2 F_N(t_N, h_N) \\
&\leqslant \frac{1}{N} \partial_h^2 \phi(t_N, h_N).
\end{aligned}$$

Since ϕ is smooth, passing to the limit as $N \to \infty$ we obtain

$$(\partial_t \phi - (\partial_h \phi)^2)(t,h) \leqslant 0.$$

This shows that f is a subsolution. The argument for supersolution is similar. We complete the proof.

□

6.4.3 Comparison principle

Notice that Propostion 6.4.2 already gives us the existence of a viscosity solution. It remains to argue that for a given initial conditon, there exists at most one viscosity solution. We will show the more general result called the comparison principle.

Theorem 6.4.3 (comparison principle)**.** Let u be a subsolution and v be a supersolution to (HJ) such that both u and v are uniformly Lipschitz in the x variable. We have

$$\sup_{\mathbb{R}_+ \times \mathbb{R}^d} (u - v) = \sup_{\{0\} \times \mathbb{R}^d} (u - v)$$

Remark. In particular, if u and v are solutions to (HJ) and $u(0,\cdot) = v(0,\cdot)$, then $u = v$. Indeed, from the Theorem 6.4.3 we have $u \leqslant v$ and by symmetry we get the result.

Sketch of proof of Theroem 6.4.3. Here we prove the statement in a special case for torus but not $\mathbb{R}^d$. We argue by contradiction. Suppose that for some $T < \infty$ we have

$$\sup_{[0,T] \times \mathbb{R}^d} (u - v) > \sup_{\{0\} \times \mathbb{R}^d} (u - v)$$

For $\varepsilon > 0$, denote $\chi(t) := \frac{\varepsilon}{T-t}$. For $\varepsilon > 0$ sufficiently small, we get

$$\sup_{[0,T] \times \mathbb{R}^d} (u - v - \chi) > \sup_{\{0\} \times \mathbb{R}^d} (u - v - \chi)$$

The main problem is that u and v are not differentiable. The idea is to double the variable as follows. For every $\alpha \geqslant 1$, let

$$\psi_\alpha(t,x,t',x') := u(t,x) - v(t',x') - \frac{\alpha}{2}((t'-t)^2 + |x'-x|^2) - \chi(t).$$

Suppose that the supremum of ψ_α is achieved at some points $(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha)$ and $(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \to (t_0, x_0, t'_0, x'_0)$ as $\alpha \to \infty$. We first argue that we must have $t_0 = t'_0$ and $x_0 = x'_0$. Indeed, since u and v are continous on the torus in the space variable, they remain bounded over any bounded set. Then for some constant $C > 0$ we have

$$\frac{\alpha}{2}((t'_\alpha - t_\alpha)^2 + |x'_\alpha - x_\alpha|^2) \leqslant C$$

Therefore, $t_0 = t'_0$ and $x_0 = x'_0$. (otherwise, $\psi_\alpha(t,x,t',x') \to -\infty$ as $\alpha \to \infty$)
Note that $0 \leqslant \chi(t_\alpha) \leqslant C$ for some constant $C > 0$, so we have $t_0 < T$. Next, we claim that we must have $t_0 > 0$. Indeed, observe that

$$\psi_\alpha(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \geqslant \sup_{[0,T) \times \mathbb{R}^d} (u - v - \chi) \tag{6.56}$$

and

$$\psi_\alpha(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \leqslant u(t_\alpha, x_\alpha) - v(t'_\alpha, x'_\alpha) - \chi(t_\alpha). \tag{6.57}$$

By continuity, combing (6.56) and (6.57) we obtain

$$\sup_{[0,T) \times \mathbb{R}^d} (u - v - \chi) \leqslant (u - v - \chi)(t_0, x_0). \tag{6.58}$$

Sine the LHS of (6.58) is strictly greater than $\sup_{\{0\} \times \mathbb{R}^d} (u - v - \chi)$, we must have $t_0 > 0$.
As a consequence, for α suffifiently large we have

$$0 < t_\alpha, t'_\alpha < T.$$

Notice that the mapping

$$(t,x) \mapsto u(t,x) - v(t'_\alpha, x'_\alpha) - \frac{\alpha}{2}((t - t'_\alpha)^2 + |x'_\alpha - x|^2) - \chi(t)$$

is maximal at (t_α, x_α). Let $\phi(t,x) := \frac{\alpha}{2}((t'_\alpha - t)^2 + |x'_\alpha - x|^2)$. Since u is a viscosity subsolution to (HJ), we infer that

$$(\partial_t(\phi + \chi) - H(\nabla(\phi+\chi))(t_\alpha, x_\alpha) \leqslant 0. \tag{6.59}$$

From (6.59), we have

$$(\partial_t \phi - H(\nabla \phi))(t_\alpha, x_\alpha) \leqslant -\frac{\varepsilon}{(T-t_\alpha)^2}. \tag{6.60}$$

Similarly, if the mapping

$$(t', x') \mapsto v(t', x') - u(t_\alpha, x_\alpha) + \frac{\alpha}{2}((t'-t_\alpha)^2 + |x' - x_\alpha|^2) + \chi(t_\alpha)$$

is minimal at (t'_α, x'_α). Let $-\widetilde{\phi}(t', x') := \frac{\alpha}{2}((t'-t_\alpha)^2 + |x'-x_\alpha|^2)$. Since v is a viscosity supersolution to (HJ), we get

$$(\partial_t \widetilde{\phi} - H(\nabla \widetilde{\phi}))(t'_\alpha, x'_\alpha) \geqslant 0. \tag{6.61}$$

Notice that $\partial_t \phi(t_\alpha, x_\alpha) = \partial_t \widetilde{\phi}(t'_\alpha, x'_\alpha)$ and $\nabla \phi(t_\alpha, x_\alpha) = \nabla \widetilde{\phi}(t'_\alpha, x'_\alpha)$. Combining (6.60) and (6.60) gives us a contradiction.

□

Remark. The reasons for using function χ are to assume that $\partial_t u - H(\nabla u) \leqslant -\varepsilon$ and detach these optimizatin points from the endpoint.

Theorem 6.4.4 (stronger version of comparison principle)**.** Let $T \in (0, \infty)$ and let u and v be respectively subsolution and supersulotion to (HJ) on $[0,T) \times \mathbb{R}^d$ that are both uniformly L-Lipschitz in the x variable. Define

$$V := \sup \left\{ \frac{|H(p') - H(p)|}{|p'-p|} : |p|, |p'| \leqslant L \right\}.$$

For every $R, M \in \mathbb{R}$ such that $M > 2L$, the mapping

$$(t,x) \mapsto u(t,x) - v(t,x) - M(|x| + Vt - R)_+ \tag{6.62}$$

achieves its supremum at a point in $\{0\} \times \mathbb{R}^d$.

Proof.
Without loss of generality, we assume that u and v are continuous on $[0,T]$. We argue by contradiction. Assume that the supremum of the mapping (6.62) is not achieved on $\{0\} \times \mathbb{R}^d$. We start by replacing

$$(t,x) \mapsto M(|x| + Vt - R)_+$$

by a smooth function.
Let $\theta \in C^\infty(\mathbb{R})$ be such that for every $\omega \in \mathbb{R}$

$$(\omega - \varepsilon_0)_+ \leqslant \theta(\omega) \leqslant \omega + 1.$$

Consider

$$\Phi(t,x) := M\theta((\varepsilon_0 + \sum_{k=1}^d |x_k|^2)^{1/2} + Vt - R).$$

By choosing $\varepsilon_0 \in (0,1]$ sufficiently small, we can make such that

$$\sup_{[0,T)\times\mathbb{R}^d} (u - v - \Phi) > \sup_{\{0\}\times\mathbb{R}^d} (u - v - \Phi).$$

Observe that

$$\partial_t \Phi(t,x) = MV\theta'((\varepsilon_0 + \sum_{k=1}^d |x_k|^2)^{1/2} + Vt - R)$$

and

$$\partial_{x_k} \Phi(t,x) = \frac{Mx_k}{(\varepsilon_0 + \sum_{k=1}^d |x_k|^2)^{1/2}} \theta'((\varepsilon_0 + \sum_{k=1}^d |x_k|^2)^{1/2} + Vt - R).$$

Hence,

$$\partial_t \Phi \geqslant V|\nabla \Phi| \tag{6.63}$$

Notice that

$$\Phi(t,x) \geqslant M(|x| + Vt - R - 1)_+$$

For some $\varepsilon > 0$ to be determined, we set

$$\chi(t,x) := \Phi(t,x) + \frac{\varepsilon}{T-t}.$$

We can choose $\varepsilon > 0$ small enough so that

$$\sup_{[0,T)\times\mathbb{R}^d}(u - v - \chi) > \sup_{\{0\}\times\mathbb{R}^d}(u - v - \chi).$$

For every $\alpha \geqslant 1$, let

$$\Psi_\alpha(t,x,t',x') := u(t,x) - v(t',x') - \frac{\alpha}{2}((t'-t)^2 + |x'-x|^2) - \chi(t,x).$$

Since $M > 2L$, one can check that the supremum of Ψ_α is achieved at a point $(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha)$. Moreover, this point remains in a bounded region, so up to the extaction of a subsequence, we have $(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \to (t_0, x_0, t'_0, x'_0)$ as $\alpha \to \infty$. Since for some constants $C > 0$, we have

$$\frac{\alpha}{2}((t'_\alpha - t_\alpha)^2 + |x'_\alpha - x_\alpha|^2) \leqslant C$$

Thus, we must have $t_0 = t'_0$ and $x_0 = x'_0$. Thanks to the term $\frac{\varepsilon}{T-t}$, we have $t_0 < T$. Arguing as in Theorem 6.4.3, we also have

$$\Psi_\alpha(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \geqslant \sup_{[0,T)\times\mathbb{R}^d}(u - v - \chi) \geqslant (u - v - \chi)(t_0, x_0) \tag{6.64}$$

while

$$\Psi_\alpha(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \leqslant u(t_\alpha, x_\alpha) - v(t'_\alpha, x'_\alpha) - \chi(t_\alpha, x_\alpha). \tag{6.65}$$

Thus, by (6.64) and (6.65)

$$(u - v - \chi)(t_0, x_0) \geqslant \sup_{[0,T)\times\mathbb{R}^d}(u - v - \chi). \tag{6.66}$$

Hence, $t_0 > 0$ and thus $t_\alpha > 0$ for every α large enough.
Notice that the mapping

$$(t,x) \mapsto u(t,x) - v(t'_\alpha, x'_\alpha) - \frac{\alpha}{2}((t - t'_\alpha)^2 + |x'_\alpha - x|^2) - \chi(t,x)$$

achieves its maximum at (t_α, x_α). Let $\phi(t,x) := \frac{\alpha}{2}((t - t'_\alpha)^2 + |x'_\alpha - x|^2)$. Hence,

$$\partial_t(\phi + \chi) - H(\nabla(\phi + \chi)) \leqslant 0 \tag{6.67}$$

at some points (t_α, x_α). Hence,

$$\partial_t\phi(t_\alpha, x_\alpha) + \partial_t\Phi(t_\alpha, x_\alpha) - H(\nabla\phi(t_\alpha, x_\alpha) + \nabla\Phi(t_\alpha, x_\alpha)) \leqslant -\frac{\varepsilon}{(T - t_\alpha)^2} \tag{6.68}$$

By the Lipschitz property of H,

$$|H(\nabla\phi(t_\alpha, x_\alpha) + \nabla\Phi(t_\alpha, x_\alpha)) - H(\nabla\phi(t_\alpha, x_\alpha))| \leqslant V|\nabla\Phi(t_\alpha, x_\alpha)|. \tag{6.69}$$

By (6.63) and (6.69), we get

$$(\partial_t\phi - H(\nabla\phi))(t_\alpha, x_\alpha) \leqslant -\frac{\varepsilon}{(T - t_\alpha)^2}. \tag{6.70}$$

Similarly, the mapping

$$(t',x') \mapsto v(t',x') - u(t_\alpha, x_\alpha) + \frac{\alpha}{2}((t' - t_\alpha)^2 + |x' - x_\alpha|^2) + \chi(t_\alpha, x_\alpha)$$

ahcieves its minimum at (t'_α, x'_α). Let $-\widetilde{\phi}(t',x') := \frac{\alpha}{2}((t' - t_\alpha)^2 + |x' - x_\alpha|^2)$. Thus, we have

$$\partial_t\widetilde{\phi} - H(\nabla\widetilde{\phi}) \geqslant 0 \tag{6.71}$$

at some points (t'_α, x'_α).
Since $\partial_t\phi(t_\alpha, x_\alpha) = \partial_t\widetilde{\phi}(t'_\alpha, x'_\alpha)$ and $\nabla\phi(t_\alpha, x_\alpha) = \nabla\widetilde{\phi}(t'_\alpha, x'_\alpha)$, (6.71) contradicts with (6.70). We complete the proof.

□

Theorem 6.4.4 is indeed stronger than Theorem 6.4.3. Indeed, we can prove Theorem 6.4.3 using Theorem 6.4.4 in the following way.

Another proof of Theorem 6.4.3. We argue by contradiction. Suppose that

$$\sup_{\mathbb{R}_+\times\mathbb{R}^d}(u - v) > \sup_{\{0\}\times\mathbb{R}^d}(u - v)$$

There exist t_0 and x_0 such that
$$(u-v)(t_0,x_0) > \sup_{\{0\}\times\mathbb{R}^d}(u-v).$$
Fix $M = 2L+1$ and $R = |x_0| + Vt_0$ so that
$$\begin{aligned} u(t_0,x_0) - v(t_0,x_0) - M(|x_0| + Vt_0 - R)_+ &= (u-v)(t_0,x_0) \\ &> \sup_{\{0\}\times\mathbb{R}^d}(u-v) \\ &> \sup_{\{0\}\times\mathbb{R}^d}(u - v - M(|x_0| + Vt_0 - R))_+, \end{aligned}$$
which is a contradiction by Theorem 6.4.4. □

To sum up, we have seen that

- The sequence F_N is precompact by the Arzelà–Ascoli theorem.
- Any limit point of the sequence F_N must be a viscosity solution to (6.55).
- There is a unique solution to (6.55).

We conclude that the sequence F_N converges to the unique viscosity solution to (6.55).
The initial condition is
$$0 \leqslant \psi(h) = F_N(0,h) = \log\cosh(h) \sim \frac{h^2}{2}, \quad h \to 0.$$
Then there exists $C < \infty$ such that
$$0 \leqslant \psi(h) \leqslant Ch^2.$$
By Theorem 6.4.3,
$$0 \leqslant f(t,h) \leqslant \frac{Ch^2}{1-4Ct}, \quad t < \frac{1}{4C}.$$
In particular, $\partial_h f(t,0) = 0$ for all $t < \frac{1}{4C}$.

6.4.4 Variational formulas for HJ equations

Let f be the viscosity solution to
$$\begin{cases} \partial_t f - \mathsf{H}(\nabla\phi) = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ f(t=0,\cdot) = \psi \end{cases}.$$
We assume throughout that ψ is Lipschitz. Recall that we write
$$\mathsf{H}^*(q) := \sup_{p\in\mathbb{R}}(p\cdot q - \mathsf{H}(p)) \tag{6.72}$$

Theorem 6.4.5 (Hopf-Lax formula)**.** If H is convex then
$$f(t,x) = \sup_{y\in\mathbb{R}^d}\left(\psi(y) - t\mathsf{H}^*\left(\frac{y-x}{t}\right)\right). \tag{6.73}$$

For a proof see []

Theorem 6.4.6 (Hopf formula)**.** If ψ is convex, then
$$f(t,x) = \sup_{p\in\mathbb{R}^d}\inf_{y\in\mathbb{R}^d}\big(\psi(y) + p\cdot(x-y) + t\mathsf{H}(p)\big). \tag{6.74}$$

Observe that the Hopf-Lax formula can be rewritten as
$$f(t,x) = \sup_{y\in\mathbb{R}^d}\inf_{p\in\mathbb{R}^d}\big(\psi(y) + p\cdot(x-y) + t\mathsf{H}(p)\big). \tag{6.75}$$
I don't know how the following comment fits in the continuation of the text and I think it is ok just ignore it: The optimization over y of $g(y) := \psi(y) + p\cdot(x-y) + t\mathsf{H}(p)$ imposes that $\nabla\psi(y) = p$ at the optimal y.
When both H and ψ are convex, can we verify that the two formulas (6.75) and (6.74) coincide?

Proposition 6.4.7. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be two convex functions. We have

$$\sup_{x\in\mathbb{R}^d} \inf_{y\in\mathbb{R}^d} \big(f(x) + g(y) - x\cdot y\big) = \sup_{y\in\mathbb{R}^d} \inf_{x\in\mathbb{R}^d} \big(f(x) + g(y) - x\cdot y\big) \tag{6.76}$$

Proof. Recall the Fenchel-Moreau Theorem: $f(x) = \sup_y \big(x\cdot y - f^*(y)\big)$. We write,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \inf_{y\in\mathbb{R}^d} \big(f(x) + g(y) - x\cdot y\big) &= \sup_{x\in\mathbb{R}^d} \big(f(x) - g^*(x)\big) \\
&= \sup_{x\in\mathbb{R}^d} \sup_{y\in\mathbb{R}^d} \big(x\cdot y - f^*(y) - g^*(x)\big) \\
&= \sup_{y\in\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \big(x\cdot y - f^*(y) - g^*(x)\big) \\
&= \sup_{y\in\mathbb{R}^d} \big(g(y) - f^*(y)\big) \\
&= \sup_{y\in\mathbb{R}^d} \inf_{x\in\mathbb{R}^d} \big(g(y) + f(x) - x\cdot y\big).
\end{aligned} \tag{6.77}$$

□

6.4.5 Convex Selection Principle

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We will rely on a new selection principle that leverages the convexity of F_N (in some loose sense, counterexamples must locally look like the hat function 6.2, so must be neither convex nor concave).

Theorem 6.4.8. (Convex Selection Principle): Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a convex function such that the equation

$$\partial_t f - \mathsf{H}(\nabla f) = 0$$

is satisfied on a dense subset of $\mathbb{R}_+ \times \mathbb{R}^d$, where f is assumed to be uniformly Lipschitz and H is assumed to be locally Lipschitz. Assume also that $f(0,\cdot)$ is of class C^1. Then f is a viscosity solution to the equation. [Clarification: The assumption is that the set $\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d : f \text{ is differentiable at } (t,x) \text{ and } (\partial_t f - \mathsf{H}(\nabla f))(t,x) = 0\}$ is dense in $\mathbb{R}_+ \times \mathbb{R}^d$.]

We will first prove the following Corollary (assuming the statement of the Theorem).

Corollary 6.4.9. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a convex function such that $f(0,\cdot) =: \psi$ is of class C^1. Suppose that for every $(t,x) \in (0,\infty) \times \mathbb{R}^d$ and $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $f - \phi$ has a [strict] local maximum at (t,x), we have $(\partial_t \phi - \mathsf{H}(\nabla\phi))(t,x) = 0$. Then f is the unique viscosity solution to

$$\begin{cases} \partial_t f - \mathsf{H}(\nabla\phi) = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ f(t=0,\cdot) = \psi \end{cases}.$$

Proof. Let (t,x), ϕ be as in the statement. We have
$f(t+s,x+y)-f(t,x)\leqslant \phi(t+s,x+y)-\phi(t,x)$
$=\nabla\phi(t,x)\cdot y+\partial_t\phi(t,x)s+\mathcal{O}(|y|^2+s^2)$.
Since f is convex, by the hyperplane separation theorem, there exists $a\in\mathbb{R}, p\in\mathbb{R}^d$ such that
$f(t+s,x+y)-f(t,x)\geqslant as+p\cdot y$.
And, hence, we have
$as+p\cdot y\leqslant f(t+s,x+y)-f(t,x)\leqslant \partial_t\phi(t,x)s+y\cdot\nabla\phi(t,x)+\mathcal{O}(|y|^2+s^2)$.
It follows that $a=\partial_t\phi(t,x)$, $p=\nabla\phi(t,x)$, and f is differentiable at (t,x) and its derivatives coincide with those of ϕ. [Recalling that $f-\phi$ is maximal at (t,x), we infer that $\partial_t f(t,x)=\partial_t\phi(t,x)$ and $(\nabla f-\nabla\phi)(t,x)=0$. So $(\partial_t f-\mathsf{H}(\nabla f))(t,x)=0$.]
In order to show the corollary, it thus suffices to show that the set $\{(t,x)\in(0,\infty)\times\mathbb{R}^d : \exists\,\phi\in C^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ such that $f-\phi$ has a local maximum at $(t,x)\}$ is dense in $\mathbb{R}_+\times\mathbb{R}^d$. For a fixed $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^d$, consider

$$(t,x)\mapsto f(t,x)-\frac{\alpha}{2}((t-t_0)^2-|x-x_0|^2). \tag{6.78}$$

Let V be a fixed compact neighborhood of (t_0,x_0), and (t_α,x_α) be the maximum of (6.78) over V. It is easy to see that $(t_\alpha,x_\alpha)\to(t_0,x_0)$ as $\alpha\to\infty$, so for α sufficiently large, (t_α,x_α) is a local maximum of (6.78). Since $(t_\alpha,x_\alpha)\to(t_0,x_0)$, we obtain the result. □

6.4.6 Identification of the limit free energy of the Generalized Curie-Weiss model

By definition, we have $F_N(0,\cdot)\to\psi$ as $N\to\infty$, and we assume that ψ is C^1. Let f be a subsequential limit of F_N. [Recall that F_N is precompact by Arzelà-Ascoli Theorem]. Let $(t_0,h_0)\in(0,\infty)\times\mathbb{R}$, and $\phi\in C^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ be such that $f-\phi$ has a strict local maximum at (t_0,h_0). We wish to verify that $(\partial_t\phi-\xi(\partial_h\phi))(t,h)=0$. There exists $(t_N,h_N)\to(t_0,h_0)$ such that for every N sufficiently large, $F_N-\phi$ has a local maximum at (t_N,h_N). [Recall that $|\partial_t F_N-\xi(\partial_h F_N)|\leqslant\frac{C}{N}(\partial_h^2 F_N)$]. At (t_N,h_N), we have

$$\partial_t(F_N-\phi)=0, \partial_h(F_N-\phi)=0, \text{ and } \partial_h^2(F_N-\phi)\leqslant 0.$$

So, $|\partial_t\phi-\xi(\partial_h\phi)|(t_N,h_N)\leqslant\frac{C}{N}(\partial_h^2\phi(t_N,h_N))$; Letting $N\to\infty$, we get $(\partial_t\phi-\xi(\partial_h\phi))(t,h)=0$, as desired. Hence, by the corollary, any subsequential limit f of F_N is the (unique for a fixed initial condition, ψ) viscosity solution to

$$\begin{cases}\partial_t f-\xi(\partial_h f)=0\\ f(t=0,\cdot)=\psi\end{cases}.$$

Note that by the Hopf formula, the solution to

$$\begin{cases}\partial_t f-\xi(\partial_h f)=0\\ f(t=0,\cdot)=\psi\end{cases}$$

is $f(t,h)=\sup\limits_{m\in\mathbb{R}}(t\xi(m)+hm-\psi^*(m))$, recovering our formula from large deviations. There remains to show that the convex selection is valid. We first state a couple of properties of convex functions.

Definition 6.4.10. For every convex function $f:U\to\mathbb{R}, U\subseteq\mathbb{R}^d$, we define the subdifferential at $x\in U$ by

$$\partial f(x):=\left\{p\in\mathbb{R}^d:\ \text{for all } y\in U, f(y)\geqslant f(x)+p\cdot(y-x)\right\}$$

i.e., the set of slopes that give us a supporting hyperplane at x.

Note that this set is not empty.

Proposition 6.4.11. If $x_m\to x$ in the interior of U and $p_m\in\partial f(x_m)$, $p_m\to p$, then $p\in\partial f(x)$.

Proposition 6.4.12. If f is differentiable at x, and x is in the interior of U, then $\partial f(x)=\{\nabla f(x)\}$.

Consequence: If a Lipschitz, convex function $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ satisfies $\partial_t f-\mathsf{H}(\nabla f)=0$ on a dense set, then it satisfies the equation at every point of differentiability in $(0, \infty) \times \mathbb{R}^d$. Moreover, for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists $(a, p) \in \partial f(t, x)$ such that $a - \mathsf{H}(p) = 0$. [Note that on this dense set of points, $\partial f(t, x)$ is a singleton, $\partial f(t, x) = \{(\partial_t f(t, x), \nabla f(t, x))\}$]. We will now prove the Convex Selection Principle.

Proof.

(Step 1): We show that f is a subsolution. Let $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and $\phi \in C^\infty$ be such that $f - \phi$ has a local maximum at (t, x). Then
$f(t', x') - f(t, x) \leqslant \phi(t', x') - \phi(t, x)$
$= (t' - t)\partial_t \phi(t, x) + (x' - x) \cdot \nabla\phi(t, x) + \mathcal{O}((t' - t)^2 + |x' - x|^2)$.
Since f is convex, we can argue as in the proof of Corollary 6.4.9 and deduce that f is differentiable at (t, x), with $\partial f(t, x) = \{(\partial_t \phi(t, x), \nabla\phi(t, x))\}$ is a singleton. Thus $(\partial_t \phi - \mathsf{H}(\nabla\phi))(t, x) = 0$. Indeed, f is a subsolution.

(Step 2): We give a partial argument showing that f is a supersolution; to be completed in the last step. If ϕ touches f from below at (t, x), then $(\partial_t \phi(t, x), \nabla\phi(t, x)) \in \partial f(t, x)$. It thus suffices to show that, for every $(a, p) \in \partial f(t, x)$, we have $a - \mathsf{H}(p) \geqslant 0$. By definition, we have
$f(t', x') \geqslant f(t, x) + (t' - t)a + (x' - x) \cdot p$.
So the mapping $y \mapsto f(0, y) - y \cdot p$ is bounded from below. In this step, we assume that

$$\inf_{y \in \mathbb{R}^d} (f(0, y) - y \cdot p)$$

is acheived and show how to conclude. [Note that the point where the infimum is achieved will be a point with slope p].
Call y the optimizer. By the "consequence" preceding this proof, there exists $(b, p') \in \partial f(0, y)$ such that $b - \mathsf{H}(p') = 0$. Since $f(0, \cdot)$ in C^1, we must have $p' = \nabla f(0, y) = p$, and thus $(b, p) \in \partial f(0, y)$ and $b - \mathsf{H}(p) = 0$. Notice that the mapping

$$g : \begin{cases} [0, 1] \to \mathbb{R} \\ \lambda \mapsto f(\lambda(t, x) + (1 - \lambda)(0, y)) \end{cases}$$

is convex. Since $(b, p) \in \partial f(0, y)$, the right derivative at 0 satisfies

$$g'_+(0) \geqslant bt + p \cdot (x - y).$$

Since $(a, p) \in \partial f(t, x)$, the left derivative at 1 satisfies

$$g'_-(1) \leqslant at + p \cdot (x - y).$$

Since g is convex, we must have $g'_+(0) \leqslant g'_-(1)$, so $a \geqslant b$. Recalling that $b - \mathsf{H}(p) = 0$, we obtain $a - \mathsf{H}(p) \geqslant 0$, as desired.

(Step 3): We now address the possibility that the infimum is not attained. For every $\varepsilon > 0$, we consider

$$\inf_{y \in \mathbb{R}^d} (f(0, y) + \varepsilon|y| - y \cdot p)$$

This infimum is acheived at some $y_\varepsilon \in \mathbb{R}^d$, and $|\nabla f(0, y_\varepsilon) - p| \leqslant \varepsilon$. Moreover,

$$\lim_{\varepsilon \to 0} \inf_{y \in \mathbb{R}^d} (f(0, y) + \varepsilon|y| - y \cdot p) = \inf_{y \in \mathbb{R}^d} (f(0, y) - y \cdot p), \text{ and}$$

$$f(0, y_\varepsilon) - y_\varepsilon \cdot p \geqslant \inf_{y \in \mathbb{R}^d} (f(0, y) - y \cdot p), \text{ so that}$$

$$\lim_{\varepsilon \to 0} \varepsilon|y_\varepsilon| = 0.$$

As in the previous step, there exists $b_\varepsilon \in \mathbb{R}$ such that $(b_\varepsilon, \nabla f(0, y_\varepsilon)) \in \partial f(0, y_\varepsilon)$ and $b_\varepsilon - \mathsf{H}(\nabla f(0, y_\varepsilon)) = 0$. Continuing as before, we find that

$$b_\varepsilon t + \nabla f(0, y_\varepsilon) \cdot (x - y_\varepsilon) \leqslant at + p \cdot (x - y_\varepsilon).$$

Using $|\nabla f(0, y_\varepsilon) - p| \leqslant \varepsilon$ and $\lim_{\varepsilon \to 0} \varepsilon|y_\varepsilon| = 0$, we find that $b_\varepsilon \leqslant a + o(1)$ as $\varepsilon \to 0$. Since $b_\varepsilon - \mathsf{H}(\nabla f(0, y_\varepsilon)) = 0$, and $\nabla f(0, y_\varepsilon) \to p$, we conclude that $a - \mathsf{H}(p) \geqslant 0$.

□

6.5 Statistical inference of low-rank matrices

Scribe: Yang Chu

6.5.1 Brief discussion of Convex Selection Principle

Recall from last lecture, the condition in Corollary 6.4.9 requires $f(0,\cdot) =: \psi$ to be C^1. Similar with the Large Deviation Principle, we show this assumption is necessary by giving a counter example.

Let $P_N = \frac{1}{2}\delta_{(1,\dots,1)} + \frac{1}{2}\delta_{(-1,\dots,-1)}$, $\xi(p) = p^2$.

In this case, the free energy is given by

$$F_N(t,h) = \frac{1}{N}\log\left(\frac{1}{2}\exp(-tN + hN) + \frac{1}{2}\exp(-tN - hN)\right) \to |h| - t$$

as $N \to \infty$ The relevant Hamilton-Jacobi equation is

$$\partial_t f + (\partial_h f)^2 = 0$$
$$f(0,h) = |h|$$

By the comparison principle, we have $f \geqslant 0$. Then $f(t,h) \neq |h| - t$. Note that $g(t,h) := |h| - t$ is convex, and solves the equation almost everywhere.

Remark. One can construct P_N such that the initial condition $f(0,h) = |h|$, but the free energy does converge to the solution of Hamilton-Jacobi equations.

6.5.2 Introduction

For a vector $\overline{x} \in \mathbb{R}^N$, we oberve a noisy version of $\overline{x}\overline{x}^*$, more precisely, $\sqrt{\frac{2t}{N}}\overline{x}\overline{x}^* + W$, where W is a matrix with independent standard Gaussian entries. We want to estimate $\overline{x}$.

Motivation of this model comes from community detection and recommendation systems.

Suppose we have notes $\{1, 2, \dots, N\}$ and two communities $+1$ and -1, indicated by a vector $\overline{x} = (\overline{x}_1, \dots, \overline{x}_N)$ of i.i.d. random variables with values 1 and -1 with equal probability $\frac{1}{2}$. We draw an edge between i and j with probability $\frac{dp}{N}$ if $\overline{x}_i = \overline{x}_j$, or $\frac{dq}{N}$ if $\overline{x}_i \neq \overline{x}_j$. with $p + q = 2$. (so that average degree of a node is d).

Question: if we only oberse the graph of connections, can we revover information about the groups?

For each pair (i,j), we observe $Ber(\frac{dp}{N})$ if $\overline{x}_i = \overline{x}_j$ and $Ber(\frac{dq}{N})$ if $\overline{x}_i \neq \overline{x}_j$, i.e. $Ber(d\frac{p+q}{2N} + d\frac{p-q}{2N}\overline{x_i x_j})$.

The mean of this random variable is $\frac{d}{N} + d\frac{p-q}{2N}\overline{x_i x_j}$. The variance is $\frac{d}{N} +$ some small constant.

For the Gaussian case instead of Bernoulli, it's similar to oberserve, where W_{ij} are i.i.d. standard Gaussians,

$$d\frac{p-q}{2N}\overline{x_i x_j} + \sqrt{\frac{d}{N}}W_{ij}.$$

Divide both sides by $\sqrt{\frac{d}{N}}$, we have the setting as given in the beginning of this section. In the limit $N \to \infty$, and then $d \to \infty$, fixing $\sqrt{d}(p-q)$ remains $O(1)$.

Remark. More complex models can be imagined, e.g. more communities. This could be encoded by $\overline{X} \in \mathbb{R}^{N \times K}$ for N people and K communities. In this case, we are going to observe a noisy version of

$$\overline{X}^{\otimes 2}A, \text{where } \overline{X}^{\otimes 2} \in \mathbb{R}^{N^2 \times K^2} \text{ is the tensor product, and } K \text{ is a fixed } K \times K \text{ matrix.}$$

More generally, $\overline{X}^{\otimes p}A$ (see papers by Reeves).

6.5.3 Approach by viscosity solutions

We will focus on the $\overline{xx^*}$ model (but the methods will be appliable to the general model mentioned above):

$$Y := \sqrt{\frac{2t}{N}}\overline{x}\overline{x}^* + W, \text{ where } \overline{x} \text{ is a vector of iid bounded random variables with law } P_N$$

W is a matrix of independent standard Gaussians. $t \geqslant 0$ denote the signal-to-noise ratio.

Our goal is to understand the "information-theoretic" limit to the possibility of recovering infromation of $\overline{x}$.

Reminder: if $f \in L^2(\mathbb{P})$ and $\mathcal{F}$ is a σ-algebra, then $\inf_{g \text{ is } \mathcal{F} \text{ measurable}} \mathbb{E}[(f-g)^2]$ is achieved for $g = \mathbb{E}[f \mid \mathcal{F}]$.

We are interested in :

$$\inf_{Z:Y\text{-measurable}} \mathbb{E}[|\overline{x} - Z|^2] = \mathbb{E}[|\overline{x} - \mathbb{E}[\overline{x} \mid Y]|^2]$$

Call $\mathbb{E}[|\overline{x} - \mathbb{E}[\overline{x} \mid Y]|^2]$ minimum mean-square error $=: mmse_N(t)$. May also consider $\mathbb{E}[\overline{xx}^* - \mathbb{E}[|\overline{x}^* \mid Y]|^2]$, here we write $a \cdot b$ for the entrywise produce, and $|a|^2 = a \cdot a$.

To write down the conditional law of $\overline{x}$ given Y, by Bayes' rule , we informally haveL

$$\begin{aligned}\mathbb{P}[\overline{x} = x \mid Y = y] &= \frac{\mathbb{P}[\overline{x} = x, Y = y]}{\mathbb{P}[Y = y]} \\ &= \frac{dP_n(x)\exp\left(-\frac{1}{2}|y - \sqrt{\frac{2t}{N}}xx^*|^2\right)}{\int dP_N(x')\exp\left(-\frac{1}{2}|y - \sqrt{\frac{2t}{N}}xx^*|^2\right)}\end{aligned}$$

Define $H_N^0 := \sqrt{\frac{2t}{N}}Y \cdot (xx^*) - \frac{t}{N}|xx^*|^2$, so that for any bounded measurable f:

$$\mathbb{E}[f(\overline{x}) \mid Y] = \frac{\int f(x)\exp\left(H_N^0(t,x)\right)dP_N(x)}{\int \exp(H_N^0(t,x'))dP_N(x')} \tag{6.79}$$

This is a Gibbs measure! One difference with Curie-Weiss is that now $H_N^0(t,x)$ is random.

We will use the notation

$$\langle f(x)\rangle := \frac{\int f(x)\exp\left(H_N^0(t,x)\right)dP_N(x)}{\int \exp(H_N^0(t,x'))dP_N(x')} = \mathbb{E}[f(\overline{x}) \mid Y] = \int f(x)dP_{\overline{x}|Y}(x). \tag{6.80}$$

References: [3], [20], [19].

6.5.4 Background

Now we consider rank-one matrix inference. The question is statistical: we only observe a noisy version of a rank-one matrix. Can we recover information about it? This has already been solved by other people. The first that gave a complete solution of this problem is [15] and then [2] gave another proof.

We consider the problem in [20] using the approach in [19].

Problem: Let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_N)$ be a vector of bounded independent random variables with law $P_N = P^{\otimes N}$. For some $t > 0$, we observe

$$Y = \sqrt{\frac{2t}{N}}\overline{x}\,\overline{x}^\top + W \tag{6.81}$$

where $W = (W_{ij})$ are independent standard Gaussians. (We don't assume symmetry.)

Our main goal is to understand the large N behavior of minimum mean square error

$$\mathrm{mmse}_N(t) = \mathbb{E}\left[|\overline{x} - \mathbb{E}[\overline{x}|Y]|^2\right]. \tag{6.82}$$

Informally,

$$\mathbb{P}(\overline{x} = x \text{ and } Y = y) = dP_N(x)\,dy\,\exp\left(-\frac{1}{2}\left|Y - \sqrt{\frac{2t}{N}}xx^\top\right|^2\right).$$

Now compute the conditional probability,

$$\mathbb{P}(\overline{x} = x|Y) = \frac{\exp\left(-\frac{1}{2}\left|Y - \sqrt{\frac{2t}{N}}xx^\top\right|^2\right)dP_N(x)}{\int \exp\left(-\frac{1}{2}\left|Y - \sqrt{\frac{2t}{N}}x'x'^\top\right|^2\right)dP_N(x')}$$

Expand the exponent and remove $-\frac{1}{2}|Y|^2$ term, which doesn't depend on t, x to get the following.

Define

$$H_N^\circ(t,x) = \sqrt{\frac{2t}{N}} Y \cdot xx^\top - \frac{t}{N}|xx^\top|^2. \tag{6.83}$$

Substituting (6.81) and expanding (6.83) gives

$$H_N^\circ(t,x) = \sqrt{\frac{2t}{N}} x \cdot Wx + \frac{2t}{N}(x\cdot\overline{x})^2 - \frac{t}{N}|x|^4. \tag{6.84}$$

The first term is the important term, which is like the spin glass model. In fact, the conditional law of $\overline{x}|Y$ has the form of a Gibbs measure, with the most important part looking like the spin glass model.

We have

$$\mathbb{E}[f(\overline{x})|Y] = \frac{\int f(x)\exp(H_N^\circ(t,x))\,dP_N(x)}{\int \exp(H_N^\circ(t,x))\,dP_N(x)}. \tag{6.85}$$

The important difference with the Curie-Weiss model is that $H_N^\circ(t,x)$ is still random.

Denote

$$\langle f(x)\rangle := \int f(x)\,dP_{\overline{x}|Y}(x) = \frac{\int f(x)\exp(H_N^\circ(t,x))\,dP_N(x)}{\int \exp(H_N^\circ(t,x))\,dP_N(x)}. \tag{6.86}$$

So we have $\langle f(x)\rangle = \mathbb{E}[f(\overline{x})|Y]$.

In general, we have

$$\mathbb{E}\langle f(x)\rangle = \mathbb{E}[f(\overline{x})], \tag{6.87}$$

and

$$\begin{aligned} \mathbb{E}\langle f(x)g(x')\rangle &= \mathbb{E}[\langle f(x)\rangle\langle g(x)\rangle] && (6.88)\\ &= \mathbb{E}[\langle f(x)\rangle\,\mathbb{E}[g(\overline{x})|Y]] && (6.89)\\ &= \mathbb{E}[\mathbb{E}[\langle f(x)g(\overline{x})\rangle\,|Y]] && (6.90)\\ &= \mathbb{E}\langle f(x)g(\overline{x})\rangle && (6.91) \end{aligned}$$

where x' is an independent copy of x under $\langle\cdot\rangle$.

Proposition 6.5.1 (Nishimori indentity)**.** For any bounded measurable function f, we have

$$\mathbb{E}\langle f(x,x')\rangle = \mathbb{E}\langle f(x,\overline{x})\rangle$$
$$\mathbb{E}\langle f(x,x',x'')\rangle = \mathbb{E}\langle f(x,x',\overline{x})\rangle.$$

Similar identities hold with more replicas.

Proof. By a monotone class argument, it suffices to verify the claim for f of the form

$$f(x,x') = g_1(x)g_2(x')$$

for measurable functions g_1 and g_2.

Note that by (6.91) we have

$$\mathbb{E}\langle g_1(x)g_2(x')\rangle = \mathbb{E}\langle g_1(x)g_2(\overline{x})\rangle.$$

Other indentities derived in the same way. We complete the proof. □

Recall that we are interested in the mmse defined as in (6.82). Note that

$$mmse_N(t) = \mathbb{E}[|x-\langle x\rangle|^2] = \mathbb{E}[|\overline{x}|^2 - 2\mathbb{E}\langle\overline{x}\cdot x\rangle + \mathbb{E}\langle x\cdot x'\rangle] = \mathbb{E}\left[|\overline{x}|^2\right] - \mathbb{E}\langle x\cdot\overline{x}\rangle$$

where the last indetity from Proposition 6.5.1. Also, we have

$$\mathbb{E}\left[|\overline{x}|^2\right] = \sum_{i=1}^N \mathbb{E}\,\overline{x}_i^2 = N\mathbb{E}\overline{x}_1^2,$$

So we want to understand the asymptotic of $\mathbb{E}\langle x\cdot\overline{x}\rangle$. In analogy with the Curie-Weiss model, we are interested in the free energy

$$F_N^\circ(t) = \frac{1}{N}\log\int \exp(H_N^\circ(t,x))\,dP_N(x), \tag{6.92}$$

and

$$\overline{F}_N^\circ(t) = \mathbb{E}F_N^\circ(t). \tag{6.93}$$

As will be seen shortly, we have

$$\partial_t \overline{F}^\circ_N(t) = \frac{1}{N^2}\mathbb{E}\left\langle (x\cdot\overline{x})^2\right\rangle \tag{6.94}$$

and $\overline{F}^\circ_N$ should be easier to study than its derivative.

In the context of inference models, a last requirement is that we do not want to destroy the fact that the Gibbs measure is a conditional expectation, since the Nishimori indentity will be essential. So we propose the following: given a parameter $h \geqslant 0$, we observe that

$$\widetilde{Y} = \sqrt{2h}\overline{x} + z$$

where $z = (z_1, z_2, \ldots, z_N)$ is a vector of independent standard Gaussians.

So in total ,we observe $\mathcal{Y} = (Y, \widetilde{Y})$. A similar computtaion as before

$$\mathbb{E}[f(\overline{x})|\mathcal{Y}] = \frac{\int f(x)\exp(H_N(t,h,x))\,dP_N(x)}{\int \exp(H_N(t,h,x))\,dP_N(x)} \tag{6.95}$$

where

$$H_N(t,h,x) = H^\circ_N(t,x) + \sqrt{2h}\widetilde{Y}\cdot x - h|x|^2. \tag{6.96}$$

Substituting (6.84) and expand (6.96) gives

$$\begin{aligned} H_N(t,h,x) &= \sqrt{\frac{2t}{N}}x\cdot Wx + \frac{2t}{N}(x\cdot\overline{x})^2 - \frac{t}{N}|x|^4 + \sqrt{2h}\widetilde{Y}\cdot x - h|x|^2 \\ &= \underbrace{\sqrt{\frac{2t}{N}}x\cdot Wx}_{\text{spin-glass type}} + \frac{2t}{N}(x\cdot\overline{x})^2 - \frac{t}{N}|x|^4 + \underbrace{\sqrt{2h}x\cdot z}_{\text{spin-glass type}} + 2hx\cdot\overline{x} - h|x|^2. \end{aligned}$$

From now on, $\langle\cdot\rangle$ is the Gibbs measure w.r.t. $H_N(t,h,\cdot)$. Now, one may ask why we use the normalizations $\sqrt{2t}$ and $\sqrt{2h}$ in place of t and h. The reason is that $\sqrt{t}\times$ 'standard Gaussian' G is natural (think of Brownian motion B_t where $\sqrt{2t}G$ is like B_{2t}).

The main ingredient is **Itô calculus without Itô**. Let G be a standard gaussian, then from Itô calculus we have

$$\mathbb{E}[\exp\Big(\sqrt{2t}G - t\Big)] = 1.$$

Differentiating with respect to t we should get 0, that is,

$$\mathbb{E}\left[\left(\frac{1}{\sqrt{2t}}G - 1\right)\exp\Big(\sqrt{2t}G - t\Big)\right] = 0.$$

How to see this without Itô calculus? Indeed, apply integration by parts,

$$\int x\exp\Big(\sqrt{2t}x - t\Big)e^{-x^2/2}\,dx = \int \sqrt{2t}\exp\Big(\sqrt{2t}x - t\Big)e^{-x^2/2}\,dx.$$

So for a standard Gaussian G and any $f \in C_c^\infty$,

$$\mathbb{E}[Gf(G)] = \mathbb{E}[f'(G)]. \tag{6.97}$$

Lemma 6.5.2 (Gaussian integration by parts)**.** Let F be a bounded measurable function. Recall that z is the standard Gaussian noise in $\widetilde{Y}$. We have

$$\mathbb{E}\langle z\cdot F(x,\overline{x})\rangle = \sqrt{2h}\mathbb{E}\langle (x - x')\cdot F(x,\overline{x})\rangle \tag{6.98}$$

$$\mathbb{E}\langle z\cdot F(x,x',\overline{x})\rangle = \sqrt{2h}\mathbb{E}\langle (x + x' - 2x'')\cdot F(x,x',\overline{x})\rangle \tag{6.99}$$

$$\mathbb{E}\left\langle (z\cdot F(x,\overline{x}))^2\right\rangle = \sqrt{2h}\mathbb{E}\langle ((x-x')\cdot F(x,\overline{x}))(z\cdot F(x,\overline{x}))\rangle + \mathbb{E}\left\langle |F(x,\overline{x})|^2\right\rangle \tag{6.100}$$

Proof. For each $i \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}
\mathbb{E}\left\langle z_i \cdot F_i(x,\overline{x})\right\rangle &= \mathbb{E}\left[\frac{z_i \int F_i(x,\overline{x}) \exp(H_N(t,h,x))\, dP_N(x)}{\int \exp(H_N(t,h,x))\, dP_N(x)}\right] \\
&= \mathbb{E}\left[\partial_{z_i}\left(\frac{\int F_i(x,\overline{x}) \exp(H_N(t,h,x))\, dP_N(x)}{\int \exp(H_N(t,h,x))\, dP_N(x)}\right)\right] \\
&= \mathbb{E}\left\langle \sqrt{2h} x_i F_i(x,\overline{x})\right\rangle - \mathbb{E}\left[\left\langle F_i(x,\overline{x})\right\rangle \left\langle \sqrt{2h} x_i\right\rangle\right] \\
&= \mathbb{E}\left\langle \sqrt{2h} x_i F_i(x,\overline{x})\right\rangle - \mathbb{E}\left\langle \sqrt{2h} x_i' F_i(x,\overline{x})\right\rangle \\
&= \sqrt{2h}\mathbb{E}\left\langle (x_i - x_i') F_i(x,\overline{x})\right\rangle
\end{aligned}$$

where the second equality from (6.97) and the forth equality from (6.91). Summing over i, we obtain the result.

The second statement is very similar as the first one.

Now for the last statement, for $i \neq j$

$$\begin{aligned}
\mathbb{E}[z_i z_j \left\langle F_i(x,\overline{x}) F_j(x,\overline{x})\right\rangle] &= \mathbb{E}[z_i \partial_{z_j} \left\langle F_i(x,\overline{x}) F_j(x,\overline{x})\right\rangle] \\
&= \sqrt{2h}\mathbb{E}[z_i (x_i - x_i') \left\langle F_i(x,\overline{x}) F_j(x,\overline{x})\right\rangle]
\end{aligned}$$

by the first statement.

For $i = j$, we get

$$\mathbb{E}[z_i z_j \left\langle F_i(x,\overline{x}) F_j(x,\overline{x})]\right\rangle = \mathbb{E}\left\langle F_i^2(x,\overline{x})\right\rangle.$$

To sum up, this gives us the desired result. $\square$

We are now ready to compute the derivatives of the free energy

$$F_N(t,j) := \frac{1}{N} \log \int \exp(H_N(t,h,x)) dP_N(x),$$

and the derivatives of their expectations

$$\overline{F}_N(t,h) := \mathbb{E}\left[F_N(t,h)\right], \quad t \geqslant 0 \text{ and } h \geqslant 0.$$

We have

$$\partial_h F_N = \frac{1}{N}\left\langle \frac{1}{\sqrt{2h}} x \cdot z + 2x \cdot \overline{x} - |x|^2 \right\rangle. \tag{6.101}$$

Taking expectation, we obtain

$$\begin{aligned}
\partial_h \overline{F}_N &= \frac{1}{N}\mathbb{E}\left[\left\langle -x \cdot x' + 2x \cdot \overline{x} - |x|^2 \right\rangle\right] && \text{(Gaussian integartion by parts)} \\
&= \frac{1}{N}\mathbb{E}\left\langle x \cdot \overline{x}\right\rangle && \text{(Nishimori identity).} \qquad (6.102)
\end{aligned}$$

Next, we have

$$\partial_t F_N = \frac{1}{N}\left\langle \frac{1}{\sqrt{2tN}} W \cdot xx^* + \frac{2}{N} xx^* \cdot \overline{x}\overline{x}^* - \frac{1}{N}|xx^*|^2 \right\rangle \tag{6.103}$$

Then by a similar computation as in (6.102), we have

$$\partial_t \overline{F}_N = \frac{1}{N^2}\mathbb{E}\left[\left\langle -xx^* \cdot x'x'^* + 2xx^* \cdot \overline{x}\overline{x}^* \right\rangle\right] = \frac{1}{N^2}\mathbb{E}\left[\left\langle (x \cdot \overline{x})^2 \right\rangle\right]. \tag{6.104}$$

Thus, we derive that

$$\partial_t \overline{F}_N - (\partial_h \overline{F}_N)^2 = \mathrm{Var}\left(\frac{x \cdot \overline{x}}{N}\right) \tag{6.105}$$

which resembles (6.44) appeared in the context of Curie–Weiss model. If we take $P_N = P_1^{\otimes N}$ as we did for the Curie–Weiss model, then

$$F_N(0,h) = \frac{1}{N}\mathbb{E}\left[\log \int \exp\Big(\sqrt{2h}x\cdot z - 2hx\cdot\overline{x} + h|x|^2\Big)\right] dP_N(x) \tag{6.106}$$

$$= \mathbb{E}\left[\log \int \exp\Big(\sqrt{2h}x_1z_1 - 2hx_1\overline{x}_1 + hx_1^2\Big)\right] dP_1(x) \tag{6.107}$$

$$= F_1(0,h). \tag{6.108}$$

Remark. One technical difference from the Curie–Weiss model is that we restrict ourselves to $h \geqslant 0$ here.

In order to use the convex selection principle, we have to check that $\overline{F}_N$ is convex in (t,h). Recall that by (6.102), we have

$$N\partial_h\overline{F}_N = \mathbb{E}\langle x\cdot\overline{x}\rangle = \mathbb{E}\left[\frac{\int x\cdot\overline{x}\exp(H_N(t,h,x))dP_N(x)}{\int \exp(H_N(t,h,x))dP_N(x))}\right].$$

Next,

$$N\partial_h^2\overline{F}_N = \mathbb{E}\left[\left\langle (x\cdot\overline{x})\left(\frac{1}{\sqrt{2h}}x\cdot z + 2x\cdot\overline{x} - |x|^2\right)\right\rangle - \langle x\cdot\overline{x}\rangle\left\langle \frac{1}{\sqrt{2h}}x\cdot z + 2x\cdot\overline{x} - |x|^2\right\rangle\right]$$

$$= \mathbb{E}\left[\langle (x\cdot\overline{x})(x\cdot(x-x') + 2x\cdot\overline{x} - |x|^2)\rangle\right] - \mathbb{E}\left[(x\cdot\overline{x})\left(\frac{1}{\sqrt{2h}}x'\cdot z + 2x'\cdot\overline{x} - |x'|^2\right)\right], \tag{6.109}$$

where (6.109) is by applying the Gaussian integration formula. Continuing (6.109), we have

$$= \mathbb{E}\left[(x\cdot\overline{x})(2x\cdot\overline{x} - x\cdot x')\right] - \mathbb{E}\left[\langle (x\cdot\overline{x})(x'\cdot(x+x'-2x'') + 2x'\cdot\overline{x} - |x'|^2)\rangle\right]$$

$$= 2\mathbb{E}\langle (x\cdot\overline{x})^2\rangle - 4\mathbb{E}\langle (x\cdot\overline{x})(x\cdot x')\rangle + 2\mathbb{E}\langle (x\cdot\overline{x})(x'\cdot x'')\rangle$$

$$= 2\mathbb{E}\left[|\langle xx^*\rangle - \langle x\rangle\langle x^*\rangle|^2\right] \geqslant 0.$$

The joint convexity of F_N can be obtained in the same way, and we leave it as an exercise (see Exercise 3 in SPS8).

Theorem 6.5.3 (approximate Hamilton–Jacobi equation)**.** We have

$$\partial_h\overline{F}_N \geqslant 0 \tag{6.110}$$

and

$$0 \leqslant \partial_t\overline{F}_N - (\partial_h\overline{F}_N)^2 \leqslant \frac{1}{N}\partial_h^2\overline{F}_N + \mathbb{E}\left[(\partial_h F_N - \partial_h\overline{F}_N)^2\right]. \tag{6.111}$$

Proof. We first show (6.110). By the Nishimori identity, we have

$$\partial_h\overline{F}_N = \frac{1}{N}\mathbb{E}\langle x\cdot\overline{x}\rangle = \frac{1}{N}\mathbb{E}\langle x\cdot x'\rangle = \mathbb{E}\left[\langle x\rangle\cdot\langle x\rangle\right] = \mathbb{E}\left[|\langle x\rangle|^2\right] \geqslant 0.$$

It remains to prove (6.111). By (6.105), we have

$$\partial_t\overline{F}_N - (\partial_h\overline{F}_N)^2 = \mathrm{Var}\left(\frac{x\cdot\overline{x}}{N}\right) = \frac{1}{N^2}\mathbb{E}\langle (x\cdot\overline{x} - \mathbb{E}\langle x\cdot\overline{x}\rangle)^2\rangle \geqslant 0. \tag{6.112}$$

We define

$$H_N'(h,x) := \frac{1}{\sqrt{2h}}x\cdot z + 2x\cdot\overline{x} - |x|^2. \tag{6.113}$$

Note that we have

$$\mathbb{E}\langle H_N'(h,x)\rangle = \mathbb{E}\langle x\cdot\overline{x}\rangle \quad \text{and} \quad \partial_h F_N = \frac{\langle H_N'\rangle}{N}. \tag{6.114}$$

We claim that

$$\mathbb{E}\langle (x\cdot\overline{x} - \mathbb{E}\langle x\cdot\overline{x}\rangle)^2\rangle \leqslant \mathbb{E}\langle (H_N' - \mathbb{E}\langle H_N'\rangle)^2\rangle - \frac{1}{2h}\mathbb{E}\left\langle |x|^2\right\rangle, \tag{6.115}$$

$$\mathbb{E}\langle (H_N' - \mathbb{E}\langle H_N'\rangle)^2\rangle - \frac{1}{2h}\mathbb{E}\left\langle |x|^2\right\rangle \leqslant N\partial_h^2\overline{F}_N + N^2\mathbb{E}\left[(\partial_h F_N - \partial_h\overline{F}_N)^2\right]. \tag{6.116}$$

We start with the proof of (6.115)

□

Comments on Theorem 6.5.3

We are in same position as for our analysis of the Curie–Weiss model, except for the extra term

$$\mathbb{E}\left[(\partial_h F_N - \partial_h \overline{F}_N)^2\right],$$

and for the fact that our domain is now restricted to $\{h \geqslant 0\}$. In SPS9, one can show that for every $M < \infty$, there exists a constant $C < \infty$ such that

$$\mathbb{E}\left[\sup_{[0,M]^2} (F_N - \overline{F}_N)^2\right] \leqslant CN^{-1/3}, \tag{6.117}$$

where the argument is based on classical concentration inequalities. In fact, one can improve (6.117) to the following upper bound. For every $p < \infty$, ε, and $M < \infty$, there exists a constant $C_{p,\varepsilon,M} < \infty$ such that

$$\mathbb{E}\left[\sup_{[0,M]^2} (F_N - \overline{F}_N)^p\right]^{1/p} \leqslant C_{p,\varepsilon,M} N^{-\frac{1}{2}+\varepsilon}. \tag{6.118}$$

However, note that what we need is a concentration inequality for $\partial_h F_N$, not the function itself. In fact, one can construct Curie–Weiss type examples for which the variance of $\partial_h F_N$

$$\mathbb{E}\left[(\partial_h F_N - \partial_h \overline{F}_N)^2\right]$$

is large at some special points, for example, the points where the limit has corners. Thus, similar to the term

$$\frac{1}{N}\partial_h^2 \overline{F}_N,$$

we can only hope to assert that the variance of $\partial_h F_N$ is small at most points.

In view of our Convex Selection Principle, we are interested in showing the following:

Proposition 6.5.4. Let f be any subsequential limit of $\overline{F}_N$, $t, h > 0$, and $\phi \in C_0^\infty$ be such that $f - \phi$ has a strict local maximum at (t, h). [Recall that F_N is precompact by Arzelà-Ascoli Theorem]. We then have

$$\left(\partial_t \phi - (\partial_h \phi)^2\right)(t, h) = 0.$$

Proof. (We omit to denote the subsequence for convenience). Let $t, h > 0$ and $\phi \in C^\infty$ be as in the statement. There exist $(t_N, h_N) \to (t, h)$ such that for N sufficiently large, the function $\overline{F}_N - \phi$ has a local maximum at (t_N, h_N). Then $\partial_h^2\left(\overline{F}_N - \phi\right)(t_N, h_N) \leqslant 0$. We first show that for every $|h'| \leqslant C^{-1}$,

$$0 \leqslant \overline{F}_N(t_N, h_N + h') - \overline{F}_N(t_N, h_N) - h'\partial_h \overline{F}_N(t_N, h_N) \leqslant C|h'|^2. \tag{6.119}$$

Note that the lower bound is by convexity of $\overline{F}_N$. By Taylor's Formula,

$$\overline{F}_N(t_N, h_N + h') - \overline{F}_N(t_N, h_N) = h'\partial_h \overline{F}_N(t_N, h_N) + \int_0^{h'} (h' - u)\partial_h^2 \overline{F}_N(t_N, h_N + u)\, \mathrm{d}u \tag{6.120}$$

and we also have

$$\phi(t_N, h_N + h') - \phi(t_N, h_N) \geqslant \overline{F}_N(t_N, h_N + h') - \overline{F}_N(t_N, h_N).$$

For N sufficiently large, we have $t_N > 0$, $h_N > 0$, so

$$\partial_t(\overline{F} - \phi)(t_N, h_N) = 0 = \partial_h(\overline{F}_N - \phi)(t_N, h_N).$$

[Notice that we can replace $\overline{F}_N$ by ϕ in (6.120)]. Combining all this, we get

$$\int_0^{h'} (h' - u)\partial_h^2 \overline{F}_N(t_N, h_N + u)\, \mathrm{d}u \leqslant \int_0^{h'} (h' - u)\partial_h^2 \phi(t_N, h_N + u)\, \mathrm{d}u \leqslant C|h'|^2.$$

Using (6.120) once more, we get (6.119). In particular, $\partial_h^2 \overline{F}_N(t_N, h_N) \leqslant C$. [This is obvious since $\partial_h^2(\overline{F}_N - \phi)(t_N, h_N) \leqslant 0$].

We now aim to control $\mathbb{E}\left[\left(\partial_h F_N - \partial_h \overline{F}_N\right)^2\right]$. We want to leverage on the semiconvexity of F_N. For every $|h'| \leqslant C^{-1}$,

$$\partial_h^2 F_N(t_N, h_N + h') \geqslant -C\frac{|z|}{\sqrt{N}}, \text{ so}$$

$$F_N(t_N, h_N + h') \geqslant F_N(t_N, h_N) + h'\partial_h F_N(t_N, h_N) - C|h'|^2 \frac{|z|}{\sqrt{N}}.$$

Combining with (6.119), we get, for every $|h'| \leqslant C^{-1}$,

$$h'(\partial_h F_N - \partial_h \overline{F}_N)(t_N, h_N) \leqslant 2 \sup_V \left|F_N - \overline{F}_N\right| + C|h'|^2 \left(1 + \frac{|z|}{\sqrt{N}}\right),$$

where V is some neighborhood of (t, h). For some deterministic $\lambda \in [0, C^{-1}]$ to be chosen later, we fix $h' := \lambda \frac{\partial_h F_N - \partial_h \overline{F}_N}{|\partial_h F_N - \partial_h \overline{F}_N|}(t_N, h_N)$, so that

$\lambda \left|\partial_h F_N - \partial_h \overline{F}_N\right|(t_N, h_N) \leqslant 2 \sup_V \left|F_N - \overline{F}_N\right| + C\lambda^2 \left(1 + \frac{|z|}{\sqrt{N}}\right).$

Squaring and taking the expectation,

$\lambda^2 \mathbb{E}\left[\left(\partial_h F_N - \partial_h \overline{F}_N\right)^2 (t_N, h_N)\right] \leqslant 8\mathbb{E}\left[\sup_V \left(F_N - \overline{F}_N\right)^2\right] + C\lambda^4.$

But recall that $\mathbb{E}\left[\sup_V \left(F_N - \overline{F}_N\right)^2\right] \leqslant CN^{-\frac{1}{3}}$. Then fixing $\lambda^4 = N^{-\frac{1}{3}}$, we get

$\mathbb{E}\left[\left(\partial_h F_N - \partial_h \overline{F}_N\right)^2 (t_N, h_N)\right] \leqslant CN^{-\frac{1}{6}}.$

Together with the approximate Hamilton-Jacobi equation, this yields

$$0 \leqslant \left(\partial_t \overline{F}_N - (\partial_h \overline{F}_N)^2\right)(t_N, h_N) = \left(\partial_t \phi - (\partial_h \phi)^2\right)(t_N, h_N) \leqslant \frac{C}{N} + \frac{C}{N^{\frac{1}{6}}}.$$

Since ϕ is smooth, we conclude that $\left(\partial_t \phi - (\partial_h \phi)^2\right)(t, h) = 0$, as desired. □

The only thing left to do is address the fact that our system is only defined for $h \geqslant 0$, as opposed to every $h \in \mathbb{R}$ in the case of the Curie-Weiss model. Here we can solve this easily by using a symmetrization argument, i.e., extending the function by reflection, as follows. [Recall that $\partial_h \overline{F}_N \geqslant 0$].

If $f : \mathbb{R}_+^2 \to \mathbb{R}$ is convex and increasing in h, then the mapping

$$\begin{cases} \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \\ (t, h) \mapsto f(t, |h|) \end{cases}$$

is convex. In order to see this, for every $\alpha \in [0, 1]$, $t, t' \geqslant 0$, and $h, h' \in \mathbb{R}$, we verify that

$$f(\alpha t + (1-\alpha)t', |\alpha h + (1-\alpha)h'|) \leqslant \alpha f(t, |h|) + (1-\alpha) f(t', |h'|).$$

Up to $(h, h') \leftarrow (-h, -h')$, nothing changes, so we can assume $\alpha h + (1-\alpha)h' \geqslant 0$. By symmetry, we can also assume that $h \leqslant h'$. If $0 \leqslant h \leqslant h'$, then we are okay, since f is convex. The remaining case is if $h \leqslant 0 \leqslant h'$. Then,

$$f(\alpha t + (1-\alpha)t', \alpha h + (1-\alpha)h') \leqslant \\ f(\alpha t + (1-\alpha)t', \alpha|h| + (1-\alpha)h') \leqslant \alpha f(t, |h|) + (1-\alpha) f(t', h'),$$

by monotonicity and convexity.

So after we have performed this extension of $\overline{F}_N$, we are in position to apply the Convex Selection Principle. We obtain the following:

Theorem 6.5.5. The function $\overline{F}_N$ converges to the unique viscosity solution to

$$\begin{cases} \partial_t f - (\partial_h f)^2 = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R} \\ f(0, h) = \overline{F}_1(0, |h|) =: \psi(h) & (h \in \mathbb{R}) \end{cases}$$

By the Hopf-Lax formula, for every $t, h \geqslant 0$,

$$\lim_{N \to \infty} \overline{F}_N(t, h) = \sup_{h' \geqslant 0} \left(\psi(h') - \frac{(h'-h)^2}{4t}\right) = f(t, h).$$

[When $h \geqslant 0$, it suffices to take the sup over $h' \geqslant 0$].

As in the Curie-Weiss model, we can verify that if $\mathbb{E}[\overline{x}_1] = 0$, then $\psi(h) \underset{h \to 0}{\sim} Ch^2$. Indeed, recall that $\partial_h \overline{F}_1(0, 0) = \mathbb{E}\left[\langle x_1 \rangle^2\right] = \mathbb{E}[\overline{x}_1]$, the last equality following from the fact that there is no randomness, we are simply sampling according to P_1. And from there, we find that, for some $t_C \in (0, \infty)$,

$$\begin{cases} \partial_h f(t,0) = 0 & \text{for } t < t_C \\ \partial_h^+ f(t,0) = \partial_h^- f(t,0) > 0 & \text{for } t > t_C \end{cases}.$$

Recall also that if f is differentiable in h at (t, h), then

$$\partial_h f(t,h) = \lim_{N\to\infty} \partial_h \overline{F}_N(t,h) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\langle x \cdot \overline{x} \rangle\right].$$

And similarly for ∂_t we have

$$\begin{cases} \partial_t f(t,0) = 0 & \text{if } t < t_C \\ \partial_t f(t,0) > 0 & \text{if } t > t_C \text{ and } f \text{ differentiable in } t \text{ at } (t,0) \end{cases}.$$

And if f is differentiable in t at $(t, 0)$, then

$$\partial_t f(t,0) = \lim_{N\to\infty} \frac{1}{N^2} \mathbb{E}\left\langle (x \cdot \overline{x})^2 \right\rangle = \lim_{N\to\infty} \frac{1}{N^2} \mathbb{E}\left\langle x\ x^* \cdot \overline{x}\ \overline{x}^* \right\rangle = \lim_{N\to\infty} \partial_t \overline{F}_N(t,0).$$

So phase transition for the possiblity to detect "a proportion of the signal" as t increases. This can be identified numerically rather straightforwardly from the Hopf or Hopf-Lax formulas.

To conclude, let us say a few words about more general models. If we think again about the community detection problem, we may want to represent the different community belongings of a given individual as a vector of fixed length K. Then $\overline{x} \in \mathbb{R}^{N\times K}$, and we may want to stick with independent, identically-distributed rows. We observe a noisy version of $\overline{x}^{\otimes 2} A$, for a given matrix $A \in \mathbb{R}^{K^2 \times L}$, where L is another fixed integer, maybe corresponding to the number of different networks we observe, and where $\overline{x}^{\otimes 2} \in \mathbb{R}^{N^2 \times K^2}$ is given by $\left(\overline{x}^{\otimes 2}\right)_{(i,j),(k,l)} = \overline{x}_{ik}\ \overline{x}_{jl}$.

Even more generally, we can take an arbitrary integer, $p \geqslant 1$ and assume that we observe $\overline{x}^{\otimes p} A$ for some $A \in \mathbb{R}^{K^p \times L}$. In this case, not much changes. The nonlinearity, say H, that appears in the equation is not necessarily convex, so our approximate Hamilton-Jacobi equation looks like

$$\left|\partial_t \overline{F}_N - H\left(\nabla \overline{F}_N\right)\right| \leqslant \frac{C}{N} \Delta \overline{F}_N + C\mathbb{E}\left[\left|\nabla F_N - \nabla \overline{F}_N\right|^2\right].$$

This is essentially like for the generalized Curie-Weiss model, so no problem. The only extra difficulty is that the extra variable h is now a positive, semi-definite matrix, so we cannot do the "cheap symmetrization trick" to revert to an equation posed on the full space. There are a few technicalities related to managing the boundary condition (of Neumann type), but overall not much changes, and in particular, we still get the Hopf formula. [See [3] for more on this].

6.6 Spin Glasses

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6.6.1 Overview

We start with a brief introduction to some common mean field spin glass models.

Sherrington-Kirkpatrick Model

This model is the most classical mean field spin glass model, it was defined on a complete graph with i.i.d disorder couplings. Fix integer N, for a configuration $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$, the Hamiltonian is given by

$$H_N(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{i,j} \sigma_i \sigma_j \tag{6.121}$$

where the disorder coupling $J_{i,j}$ are inpendent standard Gaussians. This model has been understood very well. The limiting free energy was computed in various ways, and the most famous approach is the Parisi formula, which can be used to computed limit of free energy at all temperatures. For the proof about Parisi formula, see [22], and more classical monographs [25, 26].

Bipartite Model

This model has a different structure with the classical SK model, where the spins are now coded into two layers, and the interactions only happen between layers. For simplicity, take a configuration $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathbb{R}^N \times \mathbb{R}^N$, the Hamiltonian is given by

$$H_N(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{i,j} \sigma_{1,i} \sigma_{2,j} \tag{6.122}$$

This model is only partially understood in the high temperature regime, see [7]. Note that in both examples, the Hamiltonian are just some Gaussian random fields indexed by appropriate configurations, so it's useful to look at the covariance structure of $H_N(\boldsymbol{\sigma})$. For SK model, with configuration $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{R}^N$,

$$\mathbb{E}[H_N(\boldsymbol{\sigma}) H_N(\boldsymbol{\tau})] = N R^2_{\boldsymbol{\sigma},\boldsymbol{\tau}} \tag{6.123}$$

where $R_{\boldsymbol{\sigma},\boldsymbol{\tau}} := \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tau_i$ is known as overlap in spin glass literature, and the above expectation $\mathbb{E}$ is with respect to the disorder randomness in J. Similarly, for bipartite model,

$$\mathbb{E}[H_N(\boldsymbol{\sigma}) H_N(\boldsymbol{\tau})] = N \frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{\tau}_1}{N} \frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{\tau}_2}{N} \tag{6.124}$$

Here we remark that the covariance structure uniquely characterize the spin glass models, for SK model, whose covariance is given by usual quadratic function. There are some other general spin glasses, such as mixed p-spin models, the covariance is given as the polynomials up to order p of overlaps. In general, the discussions in this chapter can be applied to the following more general setting, for $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_D) \in \mathcal{H}_N^D$, where D is an fixed integer, and $\mathcal{H}_N^D$ is a Hilbert space, and for some smooth funciton $\xi : \mathbb{R}^{D\times D} \to \mathbb{R}$, the covariance of Hamiltonian is

$$\mathbb{E}[H_N(\boldsymbol{\sigma}) H_N(\boldsymbol{\tau})] = N \xi \left(\left(\frac{\boldsymbol{\sigma}_d \cdot \boldsymbol{\tau}'_d}{N} \right)_{1 \leqslant d, d' \leqslant D} \right)$$

This is known as the vector spin glass model. The structure function ξ will appear later in the Hamilton-Jacobi setting.

Before we turn to next section, we will denote the reference measure on the configuration space as P_N. In SK model where the configuration space is a product space, the measure will be simply a product measure of 1 dimensional cases, that is, $P_N = P_1^{\otimes N}$. Similarly for bipartite model, the reference measure is $P_N = \pi_1^{\otimes N} \otimes \pi_2^{\otimes N}$, where π_1, π_2 supported on $\{-1, +1\}$.

Our goal is to study the free energy:

$$\frac{1}{N} \log \int \exp\Big(\sqrt{2t} H_N(\boldsymbol{\sigma}) - Nt\Big) dP_N(\boldsymbol{\sigma}) \tag{6.125}$$

Remark. Note that $\mathbb{E} \exp\big(\sqrt{2t} H_N(\boldsymbol{\sigma}) - Nt\big) = 1$, where $\mathbb{E}$ is with respect to the disorder randomness in Hamiltonian. It forms a mean 1 martingale. One can also regard the Hamiltonian as a Brownian motion and understand it from the stochastic analysis perspective.

6.6.2 Associated Hamilton-Jacobi equation

To find the associated HJ PDE, one has to enrich the corresponding free energy in a propery way. For SK model, we first introduce:

$$F_N(t,h) := -\frac{1}{N} \log \int \exp\Big(\sqrt{2t} H_N(\boldsymbol{\sigma}) - Nt + \sqrt{2h} \boldsymbol{z} \cdot \boldsymbol{\sigma} - Nh\Big) dP_N(\boldsymbol{\sigma}) \tag{6.126}$$

where $\boldsymbol{z}$ is an N-dimenional vector with independent standard Gaussian entries. Recall h plays the role of magentic field like in Curie-Weiss model. We write

$$\overline{F}_N := \mathbb{E}[F_N].$$

Lemma 6.6.1 (Gaussian Integration by Parts). Let Σ be a finite set, $(x(\boldsymbol{\sigma}), y(\boldsymbol{\sigma}))_{\boldsymbol{\sigma} \in \Sigma}$ be a centered Gaussian field, and for every $\boldsymbol{\sigma}, \boldsymbol{\sigma}' \in \Sigma$, define

$$C(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \mathbb{E}[x(\boldsymbol{\sigma}) y(\boldsymbol{\sigma}')],$$

Let P be a probablity measure on Σ, and define the Gibbs measure

$$\langle f(\boldsymbol{\sigma})\rangle := \frac{\int f(\boldsymbol{\sigma})\exp(y(\boldsymbol{\sigma}))dP(\boldsymbol{\sigma})}{\int \exp(y(\boldsymbol{\sigma}))dP(\boldsymbol{\sigma})}, \tag{6.127}$$

then we have

$$\mathbb{E}\left\langle x(\boldsymbol{\sigma})\right\rangle = \mathbb{E}\left\langle C(\boldsymbol{\sigma},\boldsymbol{\sigma}) - C(\boldsymbol{\sigma},\boldsymbol{\sigma}')\right\rangle. \tag{6.128}$$

Proof of Lemma 6.6.1 . The proof is similar as we did in the statistical inference case, and also one can try to find the solution in the problem set 11. □

Now let's compute the derivative of $\overline{F}_N$.

$$\begin{aligned}
\partial_h \overline{F}_N &= -\frac{1}{N}\mathbb{E}\left\langle \frac{1}{\sqrt{2h}}\boldsymbol{z}\cdot\boldsymbol{\sigma} - N\right\rangle \\
&= -\frac{1}{N}\mathbb{E}\left\langle |\boldsymbol{\sigma}|^2 - \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' - N\right\rangle \\
&= \mathbb{E}\left\langle \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\right\rangle
\end{aligned}$$

The 2nd step is due to the Gaussian integration by parts. Next let's compute

$$\begin{aligned}
\partial_t \overline{F}_N &= -\frac{1}{N}\mathbb{E}\left\langle \frac{1}{\sqrt{2t}}H_N(\boldsymbol{\sigma}) - N\right\rangle \\
&= \mathbb{E}\left\langle \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\right)^2\right\rangle
\end{aligned}$$

For the bipartite model, one can compute this similarly, but the free energy will be enriched in a bit different way, more specifically enrich it in each layer as in SK model. So in that case, there will be h_1, h_2 in the corresponding formula (6.126). From those derivatives, it seems that the t-derivative and h-derivative are related.

For SK model, it suggests to consider the following HJ PDE:

$$\partial_t \overline{F}_N - (\partial_h \overline{F}_N)^2 = \mathrm{Var}\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}\right)$$

While for bipartite model,

$$\partial_t \overline{F}_N - \partial_{h_1}\overline{F}_N\partial_{h_2}\overline{F}_N = \mathbb{E}\left\langle \left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1'}{N} - \mathbb{E}\left\langle\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1'}{N}\right\rangle\right)\left(\frac{\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_2'}{N} - \mathbb{E}\left\langle\frac{\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_2'}{N}\right\rangle\right)\right\rangle$$

$$\left|\partial_t \overline{F}_N - \partial_{h_1}\overline{F}_N\partial_{h_2}\overline{F}_N\right| \leqslant \frac{1}{2}\mathrm{Var}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1'}{N}\right) + \frac{1}{2}\mathrm{Var}\left(\frac{\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_2'}{N}\right)$$

In the more generic setting with the structure function ξ, we should have a similar form,

$$\left|\partial_t \overline{F}_N - \xi(\nabla \overline{F}_N)\right| \leqslant \mathrm{Var}(\cdots)$$

Note that the convexity of function ξ is important, since it allows us to appeal to the Hopf-Lax formula to represent the solution of the PDE variationally. For SK model, this is true, but for bipartite model, $\xi(x_1, x_2) = x_1x_2$, which is not convex. We further point out that there are 2 main differences w.r.t previous models:

- $\overline{F}_N$ is neither convex nor concave (see excerise 3 in problem set 11).
- The variances of $\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}{N}$ are not trivial as t becomes large. We need to refine the enrichment of the free energy.

The enrichment will be encoded by probability measures, for SK model, h will be repaced by a probability measures supported on the positive half line. Actually this is originated from some deep implications about how the Gibbs measure behaves when t is large. In next lecture, we will introduce the so-called Poisson-Dirichelet Cascades, which actully is the limit of the SK model in some sense. When t is small, the variance is trivial. This is because the limiting Gibbs measure is also trivial which concentrates around some constant involving h, but for large t, it has a nontrivial distribution. That's why we need to replace h by some probability measures.

6.6.3 Extreme values for i.i.d random variables

We start as some classical extreme value theory for i.i.d random random vraiables. Let $\zeta > 0$, and $(X_m)_{m\in\mathbb{N}}$ are i.i.d random variable such that $\mathbb{P}[X_n \geqslant y] \sim \frac{1}{y^\zeta}$ for $y \to \infty$, then

$$\mathbb{P}[\max_{1\leqslant i\leqslant n} X_i \leqslant n^{1/\zeta} y] \sim (1 - \frac{1}{ny^\zeta})^n \to \exp\left(-\frac{1}{y^\zeta}\right)$$

as $n \to \infty$. This basically tells us that if the random variables has the above tail decay, then we can normalize the maximum by n^ζ such that it converges in distribution. It's more interesting to understand the extremal process for this collections of random variables. Actually another classical result says $\forall 0 < a < b$, one would have

$$\#\left\{i \leqslant n : \frac{X_i}{n^{1/\zeta}} \in (a,b)\right\} \to \text{Poisson}\left(\int_a^b \frac{\zeta}{x^{1+\zeta}} dx\right)$$

In SK spin glass model, the Hamiltonians can be regrad as some Gaussian random variables with hypercube indices, but they are not independent. One simplification of the SK model is known as Random Energy Model. It is a collection of i.i.d standard Gaussian random variables $\{X_i\}_{1\leqslant i\leqslant 2^N}$. Intutitively the gap between the 1st maximum and 2nd maximum is roughyly $\frac{1}{\sqrt{N}}$. Then we transform X_i by the function

$$\{\exp\left(\beta\sqrt{N}X_i\right)\}_{1\leqslant i\leqslant 2^N}$$

This transform is basically doing some "stretching" on $\{X_i\}$, and now the extremal process looks like the original extremal process with $\zeta = \frac{\sqrt{s\log 2}}{\beta}$. One thing to remark is that that this random energy model behaves in a similar fashion as SK model in the high temperature regime. But in low temperature regime, it's more complicated, the Gibbs measure is hierarchically structured, and the limiting object is more complicated than the Poisson point process, it will become some probbaility cascades.

6.6.4 Poisson-Dirichlet Process

We first recall the definition of the Poisson point process(PPP),

Definition 6.6.2. Let μ be a measure on $\mathbb{R}^d$ without atoms. We say that a random discrete set $\Pi \subseteq \mathbb{R}^d$ is a Poisson point process with intensity measure μ if

- For any $A_1, \cdots, A_m \subseteq \mathbb{R}^d$ pairwise disjoint, measurable, the random variables $(|\Pi \cap A_i|)_{1\leqslant i\leqslant n}$ are independent
- For any $A \subseteq \mathbb{R}^d$ measurable, the law of $|\Pi \cap A|$ is Poisson$(\mu(A))$

Here we introduce a important theorem about PPP.

Theorem 6.6.3. If $\Pi \sim PPP(\mu)$, and $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ measurable function, then

$$f(\Pi) \sim PPP(f(\mu)) \tag{6.129}$$

provided $f(\mu)$ has no atoms.

We will mainly interested in the PPP with intensity measures like $\frac{\zeta}{x^{\zeta+1}} dx 1_{x>0}$, for $\zeta \in (0,1)$. Notice that for every $a > 0$ with $\int_a^\infty \frac{\zeta}{x^{\zeta+1}} dx < \infty$. By using different values of a, we can find the largest and 2nd largest values etc. Therefore, one can represent Π as a decreasing sequence. Denote $\Pi = \{u_n, n \geqslant 1\}$ with $u_1 \geqslant u_2 \geqslant \cdots u_n \geqslant \cdots$. Notice also that $\sum_{x\in\Pi} = \sum_{n=1}^\infty u_n < \infty a.s.$. Then

$$\mathbb{E}[\sum_{x\in\Pi} x 1_{x\leqslant 1}] = \int_0^1 x \frac{\zeta}{x^{1+\zeta}} dx < \infty$$

The above expectation is finite can be seen from the fact the there are only finitely many very large terms, and summing with many small terms is still finite.

Because of some universality attracted by the PPP, there should be some sort "stability" properties. We will introduce this in the following sense.

Theorem 6.6.4 (Stability Property). Let $(X_n, Y_n)_{n\geqslant 1}$ be i.i.d random vectors with values in $(0,\infty)\times\mathbb{R}^d$, and ν_ζ be the law of Y_1 size biased by X_1^ζ:

$$\nu_\zeta(A) := \frac{\mathbb{E}[X_1^\zeta 1_{Y_1\in A}]}{\mathbb{E}[X_1^\zeta]}.$$

The PPP $\{(u_n X_n, Y_n); n\geqslant 1\}$ has the same law as $\{(\mathbb{E}[X_1^\zeta]^{1/\zeta}u_n, Y_n'; n\geqslant 1\}$, where Y_n' are i.i.d, and independent of (u_n) with law ν_ζ.

Proof of Theorem 6.6.4.
Let $\widehat{\Pi}$ be the PPP with intensity measure $\frac{\zeta}{x^\zeta}dx \otimes P_{(X_1,Y_1)}$, then the first PPP in the statement can be realized as the image of $\widehat{\Pi}$ under the mapping $(u,(x,y))\mapsto(ux,y)$. By using the Theorem 6.6.3, this point process is indeed Poisson and we can compute the corresponding intensity measure. Similarly for the second point processs in the statement. For a more detailed proof, see the handwritten notes or Theorem 2.4 in [22]. $\square$

Corollary 6.6.5. With same notations as above, we have

$$\mathbb{E}\log\sum_{m=1}^{\infty}X_n u_n = \mathbb{E}\log\sum_{n=1}^{\infty}u_n + \frac{1}{\zeta}\log\mathbb{E}[X_1^\zeta] \tag{6.130}$$

Recall the free energy we discussed is very similar to the left handside.

Definition 6.6.6 (Poisson-Dirichlet Process). Let $\zeta\in(0,1)$, $\Pi\sim PPP(\frac{\zeta}{x^{1+\zeta}}dx 1_{x>0})$, and enumerate it in a decreasing way, $\Pi=\{u_n, n\leqslant 1\}, u_1\geqslant u_2\geqslant\cdots$, the random set $\{v_n, n\geqslant 1\}$ with

$$v_n := \frac{u_n}{\sum_{k=1}^\infty u_k}$$

is a Poisson-Dirichlet process with parameter ζ.

Note we already check the denominator is finite, so the above object is well-defined. We also point out the following fact, which will be used to represent the free energy of SK model.

$$\mathbb{E}\log\sum X_n v_n = \frac{1}{\zeta}\log\mathbb{E}[X_1^\zeta].$$

Proposition 6.6.7. Let $\mathbf{v} := (v_n)_{n\geqslant 1}$ be a P.D.(ζ), and conditionally on $\mathbf{v}$. Let α,α' be independent random variable with law $\sum_{n=1}^\infty v_n\delta_n$. We write $\langle\cdot\rangle$ for the law of α,α' given $\mathbf{v}$. Then

$$\mathbb{E}_{\mathbf{v}}\left\langle 1_{\{\alpha=\alpha'\}}\right\rangle = 1-\zeta \tag{6.131}$$

Proof. Let g_n be independent standard Gaussians, for $t\in\mathbb{R}$, consider

$$\left(u_n\exp\left(t(g_n-\frac{t\zeta}{2})\right), g_n - t\zeta\right)_{n\geqslant 1}$$

To use the stability property above, we know that $\exp\left(t(g_n-\frac{t\zeta}{2})\right)$ and $g_n-t\zeta$ play the role of X_n, Y_n respectively in the Theorem 6.6.4. By that theorem, it has the same law as

$$(u_n, g_n)_{n\geqslant 1}$$

Define $v_n^t := \frac{u_n\exp\left(t(g_n-\frac{t\zeta}{2})\right)}{\sum_{k=1}^\infty u_k\exp\left(t(g_k-\frac{t\zeta}{2})\right)}$, it has the same law as v_n up to reordering. Using the stability result, we know

$$v_n^t = \frac{u_n\exp(tg_n)}{\sum_{k=1}^\infty u_k\exp(tg_k)},$$

With this result, then $(v_n^t, g_n - t\zeta)_{n\geqslant 1}$ and $(v_n, g_n)_{n\geqslant 1}$ have the same law. Let $\langle\cdot\rangle_t$ be the Gibbs measure w.r.t v_n^t such that $\left\langle 1_{\{\alpha=n\}}\right\rangle_t = v_n^t$. We have $\mathbb{E}\langle g_\alpha\rangle_0 = 0$ Since $t=0$ implies that there is no correlation between the Gibbs weights and the Gaussians g_n. On the other hand

$$\begin{aligned}\mathbb{E}\langle g_\alpha\rangle_0 &= \mathbb{E}\langle g_\alpha - t\zeta\rangle_t\\ &= t\mathbb{E}\left\langle 1 - 1_{\{\alpha=\alpha'\}} - \zeta\right\rangle_t\\ &= t\mathbb{E}\left\langle 1 - \zeta - 1_{\{\alpha=\alpha'\}}\right\rangle_0 = 0\end{aligned}$$

where the first and third equality is due to the invariance principle we discussed. Second step is by Gaussin integration by parts. □

6.7 Poisson-Dirichlet Cascades

We now introduce Poisson-Dirichlet cascades. These are constructed iteratively using Poission-Dirichlet processes as building blocks. Roughly, we do the following:

For a fixed $K \geqslant 1$ and $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_K < \zeta_{K+1} = 1$,

$(u_n)_{n\geqslant 1}$ P.p.p. $\left(\dfrac{\zeta_1}{x^{\zeta_1+1}\mathbf{1}_{\{x>0\}}}\mathrm{d}x\right)$,

$(u_{1n})_{n\geqslant 1}, (u_{2n})_{n\geqslant 1}, \cdots$ independent P.p.p. $\left(\dfrac{\zeta_2}{x^{\zeta_2+1}\mathbf{1}_{\{x>0\}}}\mathrm{d}x\right)$,

$\cdots$ etc., independently on each branch, up to depth K.

We get weights which can be normalized to form a probability measure. The normalized weights are the Poission-Dirichlet cascade associated with the weights

$$0 = \zeta_0 < \zeta_1 < \cdots < \zeta_K < \zeta_{K+1} = 1.$$

We want to retain the whole hierarchical decomposition of the process, not just the law of the normalized weights. We need some notation. The tree is encoded by

$$\mathcal{A} := \mathbb{N}^0 \cup \mathbb{N}^1 \cup \cdots \cup \mathbb{N}^K,$$

where $\mathbb{N}^0 = \{\phi\}$ and ϕ denotes the root of $\mathcal{A}$, as in the Ulam-Harris tree.

For each $k \in \{0, \cdots, K-1\}$, and $\alpha \in \mathbb{N}^k$, we give ourselves an independent Poisson point process $(u_{\alpha n})_{n\geqslant 1}$ of intensity $\frac{\zeta_{k+1}}{x^{\zeta_{k+1}+1}}\mathbf{1}_{\{x>0\}}\mathrm{d}x$.

For $\ell \leqslant k \in \{0, \cdots, K\}$ and $\alpha = (n_1, \cdots, n_k) \in \mathbb{N}^k$, we write

$$\alpha|_\ell := (n_1, \cdots, n_\ell).$$

For every $\alpha \in \mathbb{N}^k$, we set

$$\omega_\alpha := \prod_{\ell=1}^{k} u_{\alpha|_\ell}.$$

The weights that we aim to normalize are the $(\omega_\alpha)_{\alpha\in\mathbb{N}^K}$.

Proposition 6.7.1. With probability 1, we have $\sum_{\alpha\in\mathbb{N}^K}\omega_\alpha < \infty$.

Proof. Clearly, we are ok if $K=1$. Otherwise, for every $\alpha \in \mathcal{A}\setminus\mathbb{N}^K$, we write $U_\alpha := \sum_{n\in\mathbb{N}} u_{\alpha n}$, so that $\sum_{\alpha\in\mathbb{N}^K}\omega_\alpha = \sum_{\alpha\in\mathbb{N}^{K-1}}\omega_\alpha U_\alpha = \sum_{\alpha\in\mathbb{N}^{K-2}}\omega_\alpha\sum_{n\in\mathbb{N}} u_{\alpha n}U_{\alpha n}$. For each $\alpha\in\mathbb{N}^{K-2}$, $\{u_{\alpha n}U_{\alpha n}, n\in\mathbb{N}\}$ has the same law as $\left\{\mathbb{E}\left[\left(U_{\alpha 1}^{\zeta_{K-1}}\right)^{\frac{1}{\zeta_{K-1}}}\right]u_{\alpha n}, n\geqslant 1\right\}$ and $U_{\alpha 1} = \sum_{n\in\mathbb{N}} u_{\alpha 1n}$, each of which is P.p.p. $\left(\frac{\zeta_K}{x^{\zeta_K+1}\mathbf{1}_{\{x>0\}}}\mathrm{d}x\right)$. So $\mathbb{E}\left[U_{\alpha 1}^{\zeta_{K-1}}\right] < \infty$. [See Exercise 2 in Problem Set 12]. Then we are left with investigating the finiteness of $\sum_{\substack{\alpha\in\mathbb{N}^{K-2}\\ n\in\mathbb{N}}}\omega_\alpha u_{\alpha n} = \sum_{\alpha\in\mathbb{N}^{K-1}}\omega_\alpha$. By induction, we are done. □

Definition 6.7.2. Recall that $K \geqslant 1$, $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_K < \zeta_{K+1} = 1$, and we constructed the random weights $(\omega_\alpha)_{\alpha \in \mathbb{N}^K}$. The Poisson-Dirichlet cascade associated with these parameters is the family of normalized weights $\left(\frac{\omega_\alpha}{\sum_{\beta \in \mathbb{N}^K} \omega_\beta} \right)_{\alpha \in \mathbb{N}^K} =: (v_\alpha)_{\alpha \in \mathbb{N}^K}$.

Remark. One can show that $\{v_\alpha, \alpha \in \mathbb{N}^K\}$ has the law of a Poisson-Dirichlet process with parameter ζ_K. The point of the construction is to provide a hierarchical decomposition of this process, which we can think of as some form of infinite divisibility.

6.8 Enriched Free Energy

We now proceed to define a suitable enriched free energy for the SK and bipartite models. We give ourselves $K \geqslant 1$, $(\zeta_k)_{1 \leqslant k \leqslant K}$ as above, $(v_\alpha)_{\alpha \in \mathbb{N}^K}$, the associated Poisson-Dirichlet cascade, and $0 = q_{-1} \leqslant q_0 < q_1 < \cdots < q_K < q_{K+1} = \infty$, and we denote

$$\mu := \sum_{k=0}^{K} (\zeta_{k+1} - \zeta_k)\, \delta_{q_k}.$$

We also give ourselves $(z_\alpha)_{\alpha \in \mathcal{A}}$, independent N-dimensional standard Gaussian random variables, independent of the rest, and set, for every $\alpha \in \mathbb{N}^K$,

$$Z_q(\alpha) := \sum_{k=0}^{K} (2q_k - 2q_{k-1})^{\frac{1}{2}}\, z_{\alpha|_k}.$$

[Note that our naive initial attempt in (6.126), with the random magnetic field $\sqrt{2h}z$, corresponds to the choice of $\mu = \delta_h$.]

This is our refined random field. It is such that, for every α, $\beta \in \mathbb{N}^K$,

$$\mathbb{E}\left[Z_q(\alpha) \cdot Z_q(\beta)\right] = 2Nq_{\alpha \wedge \beta},$$

where $\alpha \wedge \beta := \sup\{k : \alpha|_k = \beta|_k\}$. We define

$$F_N(t, \mu) := -\frac{1}{N} \log \int \sum_{\alpha \in \mathbb{N}^K} \exp\left(\sqrt{2t} H_N(\sigma) - Nt + Z_q(\alpha) \cdot \sigma - Nq_K\right) v_\alpha \, \mathrm{d}P_N(\sigma)$$

and $\overline{F}_N(t, \mu) := \mathbb{E}\left[F_N(t, \mu)\right]$. The associated Gibbs measure is $\langle \cdot \rangle$, with random variables (σ, α). Independent copies of (σ, α) under $\langle \cdot \rangle$ are denoted (σ', α'), (σ'', α''), etc. We still have $\partial_t \overline{F}_N = \mathbb{E}\left\langle \left(\frac{\sigma \cdot \sigma'}{N}\right)^2 \right\rangle$.

Now, for $k \in \{0, \cdots, K-1\}$,

$$\begin{aligned}
\partial_{q_k} \overline{F}_N &= -\frac{1}{N} \mathbb{E}\left\langle \left(2_{q_k} - 2_{q_{k-1}}\right)^{-\frac{1}{2}} z_{\alpha|_k} \cdot \sigma - \left(2_{q_{k+1}} - 2_{q_k}\right)^{-\frac{1}{2}} z_{\alpha|_{k+1}} \cdot \sigma \right\rangle \\
&= \frac{1}{N} \mathbb{E}\left\langle \left(\mathbf{1}_{\{\alpha|_k = \alpha'|_k\}} - \mathbf{1}_{\{\alpha|_{k+1} = \alpha'|_{k+1}\}}\right) \sigma \cdot \sigma' \right\rangle \\
&= \mathbb{E}\left\langle \frac{\sigma \cdot \sigma'}{N} \mathbf{1}_{\{\alpha \wedge \alpha' = k\}} \right\rangle.
\end{aligned}$$

Moreover, arguing as for Poisson-Dirichlet processes, one can show that

$\mathbb{E}\left\langle \mathbf{1}_{\{\alpha|_k = \alpha'|_k\}} \right\rangle = 1 - \zeta_k$, that is, $\mathbb{E}\left\langle \mathbf{1}_{\{\alpha \wedge \alpha' = k\}} \right\rangle = \zeta_{k+1} - \zeta_k$.

So $(\zeta_{k+1} - \zeta_k)^{-1} \partial_{q_k} \overline{F}_N = \mathbb{E}\left\langle \frac{\sigma \cdot \sigma'}{N} \middle| \alpha \wedge \alpha' = k \right\rangle$, where the conditional expectation is with respect to $\mathbb{E}\langle \cdot \rangle$.

Recall also that $(\zeta_{k+1} - \zeta_k)$ is the weight given to δ_{q_k} in the definition of μ. We will see shortly that it makes sense to think of $(\zeta_{k+1} - \zeta_k)^{-1} \partial_{q_k} \overline{F}_N(t, \mu)$ as a transport-type derivative, which we denote by $\partial_\mu \overline{F}_N(t, \mu, q_k)$. Instead of our error term being the variance of $\frac{\sigma \cdot \sigma'}{N}$, we can make it be the conditional variance given $\alpha \wedge \alpha'$:

$$\begin{aligned}
&\mathbb{E}\left\langle \left(\frac{\sigma \cdot \sigma'}{N} - \mathbb{E}\left\langle \frac{\sigma \cdot \sigma'}{N} \middle| \alpha \wedge \alpha' \right\rangle \right)^2 \right\rangle \\
&= \mathbb{E}\left\langle \left(\frac{\sigma \cdot \sigma'}{N} \right)^2 \right\rangle - \mathbb{E}\left\langle \left(\mathbb{E}\left\langle \frac{\sigma \cdot \sigma'}{N} \middle| \alpha \wedge \alpha' \right\rangle \right)^2 \right\rangle \\
&= \partial_t \overline{F}_N - \mathbb{E}\left\langle \sum_{k=1}^{K} \mathbf{1}_{\{\alpha \wedge \alpha' = k\}} \left(\mathbb{E}\left\langle \frac{\sigma \cdot \sigma'}{N} \middle| \alpha \wedge \alpha' = k \right\rangle \right)^2 \right\rangle \\
&= \partial_t \overline{F}_N - \sum_{k=1}^{K} (\zeta_{k+1} - \zeta_k) \left(\partial_\mu \overline{F}_N(t, \mu, q_k) \right)^2 .
\end{aligned}$$

We have shown that

$$\partial_t \overline{F}_N - \int \left(\partial_\mu \overline{F}_N \right)^2 \mathrm{d}\mu = \mathbb{E}\left\langle \left(\frac{\sigma \cdot \sigma'}{N} - \mathbb{E}\left\langle \frac{\sigma \cdot \sigma'}{N} \middle| \alpha \wedge \alpha' \right\rangle \right)^2 \right\rangle ,$$

where $\partial_t \overline{F}_N - \int \left(\partial_\mu \overline{F}_N \right)^2 \mathrm{d}\mu$ is computed implicitly at (t, μ). The integral can be written more explicitly as

$$\int \left(\partial_\mu \overline{F}_N \right)^2 \mathrm{d}\mu = \int \left(\partial_\mu \overline{F}_N(t, \mu, x) \right)^2 \mathrm{d}\mu(x).$$

We now motivate the notation

$$\partial \mu \overline{F}_N(t, \mu, q_k) = (\zeta_{k+1} - \zeta_k)^{-1} \partial_{q_k} \overline{F}_N(t, \mu).$$

This relates to optimal transport. For two probability measures μ, ν on $\mathbb{R}$, an optimal coupling between μ and ν can easily be realized as follows:

We take $U \sim \mathrm{Unif}([0,1])$ and then $X_\mu := F_\mu^{-1}(U)$, $X_\nu := F_\nu^{-1}(U)$, with $F_\mu^{-1}(u) = \inf\{s \geqslant 0 : \mu([0, s]) \geqslant u\}$, $u \in [0,1]$. For a "smooth" function g on the space of probability measures, we would then want that, as $\nu \to \mu$,

$$g(\nu) = g(\mu) + \mathbb{E}\left[\partial_\mu g(\mu, X_\mu)(X_\nu - X_\mu) \right] + o\left(\mathbb{E}\left[|X_\nu - X_\mu|^2 \right]^{\frac{1}{2}} \right).$$

If we specialize this to measures with fixed (ζ_k)'s and moving (q_k)'s, we find that $\partial_\mu g(q_k)$ should be as defined above. In practice, we do not need to give a more precise sense to this derivative, since we can approximate any measure by a sum of Dirac masses, and use the "explicit" definition of ∂_μ in this case. This is made possible by the Lipschitz continuity of $\overline{F}_N$.

Chapter 7

Interesting papers

- Replica method and random matrices (I)-(II)
- [17] [1]

[1] http://www.stat.ucla.edu/~ywu/research/documents/BOOKS/MontanariInformationPhysicsComputation.pdf

Chapter 8

Probability

8.1 Mixing and hitting times for Markov chains

These are notes from the Online Open Probability School (OOPS) 2020 [1]. The instructor of this course is Perla Sousi [2] (University of Cambridge).

Mixing times for Markov chains is an active area of research in modern probability and it lies at the interface of mathematics, statistical physics and theoretical computer science. The mixing time of a Markov chain is defined to be the time it takes to come close to equilibrium. There is a variety of techniques used to estimate mixing times, coming from probability, representation theory and spectral theory. In this mini course I will focus on probabilistic techniques and in particular, I will present some recent results (see references below) on connections between mixing times and hitting times of large sets.

This lecture note will cover results from 3 papers.

1. Equivalence (up to constants) between mixing times and hitting times of large sets
2. Hitting times: comparison for different sizes of sets
3. Refined mixing and hitting equivalence

8.1.1 Basic

Let X be an irreducible Markov chain in a finite state space S. (You can go from any state to any other state in a finite number of steps with positive probability.) Let P be the transition matrix of X. Let $P^t(i,j) = \mathbb{P}_i(X_t = j)$ for all $i, j \in S$ (starting at i, get to j in t steps.

There exists an invariant distribution π, $\pi = \pi P$. If X is also aperiodic, then $P^t(x,y) \to \pi(y)$ as $t \to \infty$, forall x, y.

We use the total variation distance. Let μ and ν be 2 probability distributions on S. Let

$$\|\mu - \nu\|_{TV} = \max_{A \subseteq S} |\mu(A) - \nu(A)|.$$

Let

$$d(t) = \max_x \left\|P^t(x,\cdot) - \pi\right\|_{TV}.$$

(Take over the worst starting state.). For all $\varepsilon \in (0,1)$,

$$t_{\text{mix}} = \min\{t \geqslant 0 : d(t) \leqslant \varepsilon\}.$$

Define $t_{\text{mix}}\left(\frac{1}{4}\right)$. X is called **reversible** if from the stationary distribution, running the Markov chain forwards

[1] Webpage for this course: https://secure.math.ubc.ca/Links/OOPS/abs_Sousi.php.

[2] http://www.statslab.cam.ac.uk/~ps422/

or backwards in time is indistinguishable: for all x, y,

$$\pi(x)P(x,y) = \pi(y)P(y,x).$$

I'll mostly talk about the reversible case.

We always consider the lazy verion of the chain. The lazy version of X: either stay with probability $\frac{1}{2}$, or choose a state with respect to the transition matrix. So $P_L = \frac{P+I}{2}$.

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