Numerical Linear Algebra: Homework 2

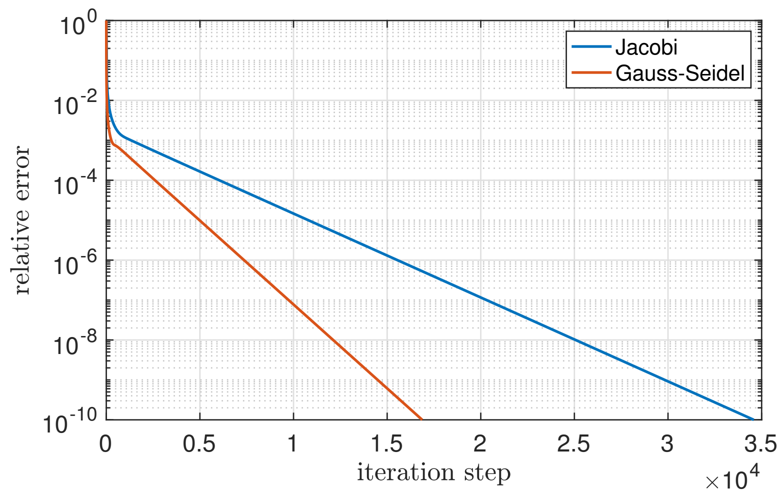
1. Solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{100\times 100} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{100\times 1},$$

by Jacobi and Gauss-Seidel iterative methods. Set the initial guess $\mathbf{x}^{(0)} = (1, 0, 0, \cdots, 0)^T$, the tolerance $TOL = 10^{-10}$ with the stopping strategy $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty / \|\mathbf{x}^{(k)}\|_\infty < TOL$, and the maximum number of iterations $N_0 = 10^5$. Note that, in Gauss-Seidel method, you can just use left division "\" to solve the linear system $-(D+L)\mathbf{z} = U\mathbf{x}^{(k-1)}$.

   (a) Output the number of iterations N and the error $\|\mathbf{x}^{(N)} - \mathbf{x}\|_\infty$ for both iterative methods. Compare the consuming time for these two methods. [Hint: You may use the MATLAB commands `tic` and `toc`.]

   (b) The following figure shows the number of iteration step k as a function of relative error $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty / \|\mathbf{x}^{(k)}\|_\infty$. This demonstration evidently shows that the Gauss-Seidel method attains a "better" convergence than Jacobi's method. Depict it. [Hint: You may use the MATLAB commands `semilogy`, `xlabel`, `ylabel`, `legend`, and `grid`.]



2. Let $(v_1, v_2, v_3, v_4, v_5, v_6) = (1, 1.2, 1.4, 1.6, 1.8, 2)$. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^5 \\ 1 & v_2 & v_2^2 & v_2^3 & v_2^4 & v_2^5 \\ 1 & v_3 & v_3^2 & v_3^3 & v_3^4 & v_3^5 \\ 1 & v_4 & v_4^2 & v_4^3 & v_4^4 & v_4^5 \\ 1 & v_5 & v_5^2 & v_5^3 & v_5^4 & v_5^5 \\ 1 & v_6 & v_6^2 & v_6^3 & v_6^4 & v_6^5 \end{bmatrix}_{6\times 6}$$

and $\mathbf{b} = A^{(1)}+A^{(2)}+A^{(3)}+A^{(4)}+A^{(5)}+A^{(6)}$ ($A^{(j)}$ being the jth column of matrix A). So the exact solution to the linear system is $\mathbf{x} = \mathbf{1}$. Solve the linear system using Jacobi and Gauss-Seidel iterative methods. Set the initial guess $\mathbf{x}^{(0)} = (1, 0, 0, 0, 0, 0)^T$, the tolerance $TOL = 10^{-10}$ with the stopping strategy $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty / \|\mathbf{x}^{(k)}\|_\infty < TOL$, and the maximum number of iterations $N_0 = 3 \times 10^5$. Output the number of iterations N and the error $\|\mathbf{x}^{(N)} - \mathbf{x}\|_\infty$ for both iterative methods. [Hint: Use `fliplr` and `vander` to construct the Vandermonde matrix A.]