Robust Auction Design with Support Information

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Abstract

A seller wants to sell an item to n buyers. Buyer valuations are drawn i.i.d. from a distribution unknown to the seller; the seller only knows that the support is included in $[a, b]$. To be robust, the seller chooses a DSIC mechanism that optimizes the *worst-case* performance relative to the ideal expected revenue the seller could have collected with knowledge of buyers' valuations. Our analysis unifies the *regret* and the *ratio* objectives.

For these objectives, we derive an optimal mechanism and the corresponding performance in quasi-closed form, as a function of the support information $[a, b]$ and the number of buyers n. Our analysis reveals three regimes of support information and a new class of robust mechanisms. i.) When a/b is below a threshold, the optimal mechanism is a second-price auction (SPA) with random reserve, a focal class in earlier literature. ii.) When a/b is above another threshold, SPAs are strictly suboptimal, and an optimal mechanism belongs to a class of mechanisms we introduce, which we call *pooling auctions* (POOL); whenever the highest value is above a threshold, the mechanism still allocates to the highest bidder, but otherwise the mechanism allocates to a *uniformly random buyer*, i.e., pools low types. iii.) When a/b is between two thresholds, a randomization between SPA and POOL is optimal.

We also characterize optimal mechanisms within nested central subclasses of mechanisms: standard mechanisms that only allocate to the highest bidder, SPA with random reserve, and SPA with no reserve. We show strict separations in terms of performance across classes, implying that deviating from standard mechanisms is necessary for robustness.

Lastly, we show that the same results hold under other distribution classes that capture "positive dependence," namely: i.i.d., mixture of i.i.d., and exchangeable and affiliated distributions, as well as i.i.d. regular distributions.

Keywords: robust mechanism design, minimax regret, maximin ratio, support information, prior-independent, standard mechanisms, second-price auctions, pooling.

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1 Introduction

The question of how to optimally sell an item underlies much of modern marketplaces, from online advertising and e-commerce to art auctions. Selling mechanisms are widely used in practice, and in turn they are studied in economics, computer science, and operations research under *optimal mechanism design*, starting from the pioneering work of [Myerson, 1981]. The literature often assumes that the seller knows the environment perfectly, but (i) this knowledge is often either not available or reliable, and (ii) the optimal mechanism prescribed by the theory is often too complicated or fine-tuned to the details of the environment, to be used in practice. There is therefore a need to develop mechanisms that depend less on market details, and this need is often referred to as the "Wilson doctrine" [Wilson, 1987].

The emerging literature on *robust mechanism design*, in turn, aims to design mechanisms that perform "well" in the worst case against "any" environment. This line of work often leads to interesting insights but, taken literally, they can lead to mechanisms that are too conservative. In practice, while we do not have complete knowledge about the environment, we often do have *partial* knowledge, and how to incorporate additional side information into the robust framework is essential to bring the robust theory closer to practice. In this paper, we make progress in this direction by analyzing the role of *support information* of bidder valuations, as captured by *lower bounds* and *upper bounds* on bidder valuations.

The motivation of the knowledge of such bounds is that we operate in a world with minimal or no data. The bounds need not be learned from data but rather are derived from asking experts, using domain knowledge, or common sense. Examples include launching a new product, or auctioning rarely traded goods such as fine art, collectibles, and jewelry. In these contexts, the support information is a natural form of partial knowledge because it is easier and more intuitive to come up with a reasonable range of values than to guess something like the shape of the valuation distribution (either parametric or nonparametric like regularity or monotone hazard rate) or distributional parameters like the mean or the optimal monopoly price.

More formally, consider a seller who wants to sell an item to n bidders. The bidders' valuations are unknown to the seller and are assumed to be drawn from a joint distribution $\boldsymbol{F}$. The seller does not know $\boldsymbol{F}$, and knows only a lower bound a and an upper bound b on the support of $\boldsymbol{F}$,

and that the valuations belong to the class $\mathcal{F}$ of i.i.d. distributions. Similarly, the bidders also do not know $\boldsymbol{F}$. Therefore, we focus on mechanisms that are *dominant strategy incentive compatible* (DSIC). Under such a mechanism, every bidder optimally reports her true value regardless of other bidders' valuations and strategies.

We will quantify the performance of mechanisms by the *gap* between the benchmark oracle revenue, the ideal expected revenue the seller could have collected with knowledge of the buyers' valuations, and the expected revenue garnered by the actual mechanism. Our framework will be general and apply to two classical notions of gaps considered in the literature: (i) the *regret* (absolute gap) is the difference between these two revenues, and (ii) the *approximation ratio* (relative gap) is the ratio of these two revenues. The seller selects a mechanism that performs well (minimizes regret or maximizes approximation ratio) in the *worst case* against all admissible distributions.[1]

The interval $[a, b]$ associated with the admissible distribution class captures the amount of uncertainty of the decision maker. We will parameterize this uncertainty through a/b, which we call the *relative support information*, and which is a unitless quantity ranging from 0 to 1. When $a/b \sim 0$ (either because $a \sim 0$ or $b \gg a$), we have minimal relative support information while when $a/b \sim 1$ we have maximal support information as the endpoints are close.

The understanding of the interplay of support information and robust auctions is very limited outside of very particular special cases ($n = 1$ [Bergemann and Schlag, 2008, Eren and Maglaras, 2010] and $a = 0$ [Anunrojwong et al., 2022]), leading to the following question: how does support information affect the structure of optimal robust auctions and achievable performance? We study optimal performance and associated mechanisms across the relative support information spectrum and establish richness in the structure of the resulting robust mechanisms with three distinct information regimes corresponding to three mechanism types. In particular, our work subsumes and unifies the three studies mentioned above, characterizing an optimal mechanism and the associated performance for an arbitrary number of bidders n and any support information $[a, b]$, for both the regret and ratio objectives. See Table 1 for a high level summary of known results and the results we develop in this paper.

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[1]Our framework also applies to the traditional maxmin (worst-case) revenue, but the worst-case revenue is trivial in the present case of support information.

Problem	Information	Objective	
Type	level	Regret	Ratio
pricing ($n = 1$)	all a/b	Bergemann and Schlag [2008]	Eren and Maglaras [2010]
auctions ($n \geq 1$)	$a/b = 0$	Anunrojwong et al. [2022]	0
auctions ($n \geq 1$)	all a/b	—**This work**—	

Table 1: Comparison with the closest previous studies along the dimension of the number of buyers (pricing ($n = 1$) vs. auctions (arbitrary $n \geq 1$) and the level of relative support information a/b.

1.1 Summary of Main Contributions

We develop a unified framework for regret and approximation ratio through a single quantity, the minimax λ-regret, where the λ-regret is the difference between λ times the benchmark revenue and the mechanism revenue, and $\lambda \in (0, 1]$ is a constant. It is clear that when $\lambda = 1$, the λ-regret reduces to the regret. The fact that the λ-regret can be used to characterize the maximin ratio relies on an epigraph reformulation of the latter problem, which is fairly standard in the context of optimization with fractional objectives. Our main contribution, however, is the full characterization of a minimax optimal mechanism and its associated performance for λ-regret for *any* value of $\lambda \in (0, 1]$, *any* number of buyers n and *any* support information $[a, b]$. Since we are primarily interested in the effect of the support $[a, b]$, we initially assume that the valuations are n i.i.d. random variables given the canonical nature of this setting. Our family of optimality results across this spectrum brings to the foreground a very rich structure of optimal mechanisms, and establishes how relative support information critically impacts the structure of optimal mechanisms.

Novel mechanism class.   A natural candidate for an optimal mechanism is a second-price auction with appropriate random reserve. Previous work [Anunrojwong et al., 2022] shows that this is optimal with zero relative support information, i.e., for $a = 0$. Suppose for a moment that relative support information is high (i.e., $a \sim b$) and we are restricted to the class of second-price auctions (SPAs). Setting any nontrivial reserve is *risky* because when the highest buyer's value is below the reserve the seller does not allocate and gets zero revenue. At the same time, the benefits of a reserve price are limited since the highest and lowest values are close. Indeed, the seller can guarantee a revenue of a with no reserve, which is close to the maximal revenue achievable of b. Hence, it should be intuitive that when relative support information is high, a SPA with no reserve is optimal among

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the class of SPAs. (A formal result is presented in Section 4.2.) A natural question is then whether there are mechanisms that can outperform a SPA with no reserve from a robust perspective, and what structure they take.

We define a new mechanism class, with the aim of softening the trade-offs associated with reserve pricing in second-price auctions. These mechanisms, that we dub "pooling auctions" (POOL), have an associated threshold. When the highest bid is above the threshold, the mechanism allocates to the highest bidder, as in a SPA when the highest bid is above the reserve price. However, when the highest bid is below the threshold, rather than not allocating as a SPA would do, the seller allocates uniformly at random to any of the bidders. In other words, this auction pools the low types and the lowest bidder may get the item. By increasing the allocation at low values, the mechanism increases the revenue derived from lower-valued bidders, but in doing so, the mechanism can extract less revenue from higher-valued bidders due to incentive compability. In this sense, the pooling auction makes allocation and payment more "uniform" across values and softens the tradeoffs from reserve pricing.

Characterization of an optimal mechanism. Our main result, Theorem 1, establishes that there always exists an optimal mechanism that is a randomization over second-price auctions (SPA) with different reserves and pooling auctions (POOL) with different thresholds. Therefore, an optimal mechanism can be implemented in terms of a random instance of one of these "base" mechanisms. Furthermore, three fundamental relative support information regimes emerge. There are thresholds k_l and k_h such that: if $a/b \leq k_l$ (low information regime), SPA with random reserves is optimal; if $a/b \geq k_h$ (high information regime), POOL with random thresholds is optimal; if $k_l \leq a/b \leq k_h$ (moderate information regime), a randomization over SPA and POOL (i.e. interpolation between the two extremes) is optimal.

We note that SPA is a "standard" mechanism, meaning that it never allocates to non-highest bidders, but POOL is not. Therefore, the optimal mechanism we have identified is standard if and only if $a/b \leq k_l$. Secondly, POOL always allocates, meaning it allocates with probability one, whereas SPA does not (because it does not allocate below the reserve). Therefore, the optimal mechanism always allocates if and only if $a/b \geq k_h$.

While the result above applies for any λ, we note that for the maximin ratio problem (the problem of maximizing the worst-case ratio of revenue to the benchmark), the value of λ is endogenous,

and it is not clear a priori in which information regime one falls. Quite interestingly, we can prove that the optimal maximin ratio mechanism is never in the SPA regime and thus some amount of pooling is always necessary in this case (see Section 3.2 and Proposition 4).

Methodology and closed-form characterization. We characterize the optimal mechanism and worst-case distribution in closed form via a saddle-point argument. In particular, if we assume that a saddle point exists and the optimal mechanism has the form outlined in the previous paragraph, we derive necessary conditions for Nature's worst-case distribution (cf. Section 3.1) as well as the distributions of random reserve r and threshold τ, under a few fairly mild technical conditions. We then prove that the resulting mechanism is optimal without any additional assumptions. Our methodology provides a unified treatment across all support information levels, and objectives (regret and approximation ratio) in one framework. We also characterize Nature's worst-case distribution as part of our analysis, which takes the following form: for $a/b \leq k_h$, the worst-case distribution is an isorevenue distribution (i.e., zero virtual value), whereas for $a/b > k_h$, the worst-case distribution has a constant positive virtual value in the interior of the support.[2]

Quantifying the value of scale information and competition. Using the machinery we develop, we can exactly compute the minimax regret and maximin ratio for any support information $[a, b]$ and number of buyers n (cf. Figure 4). We show that even a small amount of knowledge can lead to nontrivial guarantees on revenue. For example, even when we only know that values can vary over a full order of magnitude ($a/b = 0.10$), we can guarantee 40.38% of the ideal benchmark with only 2 buyers. When the knowledge of the scale is more precise, say, if we know the value up to a factor of two ($a/b = 0.50$), we get a guarantee of 74.63% with 2 buyers. With more agents, the guarantees improve (around 5% and 3% more, respectively, for an additional buyer).

Quantifying the power of mechanism features. We have identified an optimal mechanism that is a randomization over base mechanisms in the SPA and POOL classes. A distinguishing feature of the latter mechanism is that it is *non-standard*, i.e., it allocates to non-highest bidders. We show that this feature is *necessary for optimality* by characterizing the minimax optimal mechanism and performance within the class of all standard mechanisms and showing that the op-

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[2]For a distribution with CDF F and density f, the virtual value at v is defined by $v - (1 - F(v))/f(v)$.

timal mechanism strictly improves over optimal standard mechanisms. More broadly, in Section 4, we quantify the value of different features in the mechanism class by computing the worst-case λ-regret (and thus, regret and ratio) for different nested mechanism subclasses of all DSIC mechanisms: all DSIC mechanisms ($\mathcal{M}_{\text{all}}$), all standard mechanisms ($\mathcal{M}_{\text{std}}$), SPA with random reserve ($\mathcal{M}_{\text{SPA-rand}}$), SPA with deterministic reserve ($\mathcal{M}_{\text{SPA-det}}$), and SPA with no reserve ($\mathcal{M}_{\text{SPA-a}}$). These results are also of independent interest, as they characterize the worst-case performance of commonly used mechanisms. In terms of maximin ratio, we find strict separation for all subclasses except $\mathcal{M}_{\text{SPA-det}}$ versus $\mathcal{M}_{\text{SPA-a}}$ (cf. Figure 7). These results show that introducing some features (such as non-standardness) can lead to significant performance improvements.

1.2   Related Work

Auction Design and Mechanism Design   Vickrey [1961], Myerson [1981] and Riley and Samuelson [1981] pioneered a long line of work on the design of auctions and other economic mechanisms with strategic agents. In particular, Myerson [1981] shows that if agent valuation distributions are known, i.i.d. and *regular*, then the optimal (expected-revenue-maximizing) mechanism is a *second-price auction with reserve*. This is the classical paradigm of Bayesian mechanism design. However, once we go beyond the simplest settings, this paradigm quickly leads to very complicated "optimal" mechanisms that are too detail-dependent and potentially fragile. In response, part of the algorithmic game theory literature instead focuses on proving approximation guarantees for specific "simple" mechanisms [Roughgarden and Talgam-Cohen, 2019]. This line of work still assumes that the value distribution is known to both the designer and all players, and the players play a Bayes-Nash equilibrium. In our setting, however, the value distribution is *not* known, the performance is evaluated in the *worst case* rather than the "Bayesian" average case, and we require a dominant strategy equilibrium.

Robust Mechanism Design   The closest line of work to ours is how to robustly sell an item with non-Bayesian uncertainty on valuation distributions. This question has been studied in the "prior-independent approximation" literature in algorithmic game theory [Hartline, 2020, Chapter 5], often assuming the shape of the distribution (such as regular or monotone hazard rate) is known but not the "scale" of the distribution. In the present paper, we assume we know the scale of the distribution (as captured by the bounds $[a, b]$) but not the shape and we derive both regret and

ratio guarantees. The one-agent case reduces to a pricing problem; Bergemann and Schlag [2008] and Eren and Maglaras [2010] provide exact characterization for minimax regret and maximin ratio pricing, respectively. Koçyiğit et al. [2020], Koçyiğit et al. [2022] analyze minimax regret against any number n of agents whose valuation distributions are arbitrarily correlated with a known upper bound on the support. They show that their problem "reduces" to the one-agent case because Nature can choose the worst-case distribution to only have one effective bidder.

Technically and conceptually, Anunrojwong et al. [2022] is the closest to our work. They show that the second-price auction is robustly optimal for any number n of agents when only the upper bound of the valuations is known, whereas our work assumes that both the lower bound a and the upper bound b are known. This allows us to capture the entire spectrum of support information. Whereas both their work and ours share the guess-and-verify saddle point framework, the main difficulty of this framework is to identify the *form* of the optimal mechanism in the first place. This makes our departure from the case $a = 0$ challenging: it requires us to explore the space of DSIC mechanisms beyond second price auctions. As soon as one departs from SPAs, the space of mechanisms is much larger and it is not clear what should be a good candidate class a priori. We identify new focal mechanisms (namely, pooling auctions) and show that qualitatively different forms of optimal mechanisms emerge, depending on the amount of information a/b. Lastly, they only focus on regret, whereas we unify both regret and ratio objectives in a single framework.

Previous works that study robust mechanism design tend to identify second-price auctions (SPA) as optimal [Anunrojwong et al., 2022, Bachrach and Talgam-Cohen, 2022, Koçyiğit et al., 2024, Zhang, 2022a,b, Che, 2022, Allouah and Besbes, 2020]. One of the main contributions of this paper is to show why SPA fails to be optimal when we have sufficient relative support information and propose a new *building block* for robust mechanism design, the *pooling auction* mechanism. Other than the fact that the optimal mechanism in our setting is composed of these new mechanisms, this new class may also be of independent interest in other robust mechanism design problems.

Our work is also related to a broader literature on robustness in mechanism design and contracting [Carroll, 2019]. In particular, while we highlight the work on robustness to distributions here because they are most related to our work, there are other forms of robustness as well, e.g., robustness to higher-order beliefs [Bergemann and Morris, 2005, 2013], robustness to collusion and renegotiation [Che and Kim, 2006, 2009, Carroll and Segal, 2018], and robustness to strategic be-

havior that is weaker than dominant strategy [Chung and Ely, 2007, Babaioff et al., 2009, Arya et al., 2009]. Robust mechanism design also has conceptual links to robust and distributionally robust optimization; see Bertsimas et al. [2011] and Rahimian and Mehrotra [2022] for overviews.

Optimal Mechanisms with Partial Information   Our work is also related to the design of robustly optimal pricing and mechanisms with partial information about the distribution. Some works assume access to samples drawn from the i.i.d. distribution [Cole and Roughgarden, 2014, Dhangwatnotai et al., 2015, Allouah et al., 2022, Feng et al., 2021, Fu et al., 2021] while others assume that summary statistics of distributions are known [Azar et al., 2013, Suzdaltsev, 2020, 2022, Bachrach and Talgam-Cohen, 2022, Allouah et al., 2023].

Pooling in Auctions   While the specific form of the pooling auction POOL that we propose is new, the more general notion of pooling in auctions has appeared in the literature, starting from the pioneering work on revenue-maximizing auctions of Myerson [1981]. When F is not regular, it is shown the distribution must be "ironed" such that the all bidders in the same ironing interval have the same allocation, that is, their types are pooled. The main difference is that in Myerson [1981], there is a known focal distribution F to iron, whereas there is no single distribution in our problem, and the pooling emerges naturally from the worst-case analysis over all feasible distributions. In fact, we can see from the proof of the main theorem that the worst case distributions are all regular. Therefore, *even if we only consider the worst case over all regular distributions, the robustly optimal mechanism will still pool*, whereas if we know the true distribution to be any specific regular distribution, the Bayesian optimal mechanism will not pool by Myerson. The robust auction framework therefore gives qualitatively different prescriptions. Beyond Myerson [1981], pooling in auctions has been shown to be optimal in a variety of settings [Bergemann et al., 2022, Feldman et al., 2022, Laffont and Robert, 1996, Pai and Vohra, 2014] but to the best of our knowledge, these work all operate in the Bayesian setting and as such the driver of pooling appears different.

2   Problem Formulation

The seller wants to sell an indivisible object to one of n buyers. The n buyers have valuations drawn from a joint cumulative distribution $\boldsymbol{F}$. The seller does not know $\boldsymbol{F}$, and only knows a lower bound a and an upper bound b of the valuation of each buyer. That is, the seller only knows that the support of the buyers' valuations belongs to $[a,b]^n$.

Seller's Problem.   We model our problem as a game between the seller and Nature, in which the seller first selects a selling mechanism from a given class $\mathcal{M}$ and then Nature may counter such a mechanism with any distribution from a given class $\mathcal{F}$. Buyers' valuations are then drawn from the distribution chosen by Nature and they participate in the seller's mechanism.

We will now consider the choice of the mechanism class $\mathcal{M}$. A selling mechanism $m = (\boldsymbol{x}, \boldsymbol{p})$ is characterized by an allocation rule $\boldsymbol{x}$ and a payment rule $\boldsymbol{p}$, where $\boldsymbol{x} : [a,b]^n \to [0,1]^n$ and $\boldsymbol{p} : [a,b]^n \to \mathbb{R}$. Given buyers' valuations $\boldsymbol{v} \in [a,b]^n$, $x_i(\boldsymbol{v})$ gives the probability that the item is allocated to buyer i, and $p_i(\boldsymbol{v})$ his expected payment to the seller. In our main result, we will consider the class $\mathcal{M}_{\mathrm{all}}$ of all dominant strategy incentive compatible (DSIC) direct mechanisms. A mechanism is DSIC if and only if it is optimal for every buyer to report her true valuation (IR) and participate in the mechanism (IC), regardless of the realization of valuations of the other buyers, and the seller can allocate at most one item (AC). More formally, we require that the mechanism $m = (\boldsymbol{x}, \boldsymbol{p})$ satisfies the following constraints:

$$v_i x_i(v_i, \boldsymbol{v}_{-i}) - p_i(v_i, \boldsymbol{v}_{-i}) \geq 0, \quad \forall i, v_i, \boldsymbol{v}_{-i} \tag{IR}$$

$$v_i x_i(v_i, \boldsymbol{v}_{-i}) - p_i(v_i, \boldsymbol{v}_{-i}) \geq v_i x_i(\hat{v}_i, \boldsymbol{v}_{-i}) - p_i(\hat{v}_i, \boldsymbol{v}_{-i}) \quad \forall i, v_i, \boldsymbol{v}_{-i}, \hat{v}_i \tag{IC}$$

$$\sum_{i=1}^{n} x_i(v_i, \boldsymbol{v}_{-i}) \leq 1 \quad \forall \boldsymbol{v}. \tag{AC}$$

Note that we allow the seller's mechanism to be randomized. We can now define the class of all DSIC mechanisms

$$\mathcal{M}_{\mathrm{all}} = \{(\boldsymbol{x}, \boldsymbol{p}) : (\mathrm{IR}), (\mathrm{IC}), (\mathrm{AC})\}. \tag{1}$$

Seller's Objective. Informally, the seller seeks to minimize the "gap" between the expected revenue $\mathbb{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{v}) \right]$ relative to the benchmark associated with the revenues that could be collected when the valuations of the buyers are known $\mathbb{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\max(\boldsymbol{v}) \right]$.[3] We consider two notions of gaps. First is the *absolute gap*, or *regret*, defined by

$$\text{Regret}(m, \boldsymbol{F}) = \mathbb{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\max(\boldsymbol{v}) - \sum_{i=1}^{n} p_i(\boldsymbol{v}) \right]. \tag{2}$$

Second is the *relative gap* or *approximation ratio*, defined by

$$\text{Ratio}(m, \boldsymbol{F}) = \frac{\mathbb{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{v}) \right]}{\mathbb{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\max(\boldsymbol{v}) \right]}. \tag{3}$$

After the seller chooses a mechanism m, Nature then chooses a distribution $\boldsymbol{F}$ from a given class of distributions $\mathcal{F}$ such that the valuation of the n agents $\boldsymbol{v} \in \mathbb{R}_+^n$ are drawn from $\boldsymbol{F}$. The seller aims to select the mechanism m to either minimize the worst-case regret or maximize the worst-case approximation ratio. Our goal, therefore, is to characterize the minimax regret and maximin ratio for different classes of mechanisms $\mathcal{M}$ and classes of distributions $\mathcal{F}$:

$$\text{MinimaxRegret}(\mathcal{M}, \mathcal{F}) := \inf_{m \in \mathcal{M}} \sup_{\boldsymbol{F} \in \mathcal{F}} \text{Regret}(m, \boldsymbol{F}), \tag{4}$$

$$\text{MaximinRatio}(\mathcal{M}, \mathcal{F}) := \sup_{m \in \mathcal{M}} \inf_{\boldsymbol{F} \in \mathcal{F}} \text{Ratio}(m, \boldsymbol{F}). \tag{5}$$

For $\lambda \in (0, 1]$, define the λ-regret

$$R_\lambda(m, \boldsymbol{F}) = \mathbb{E}_{\boldsymbol{v} \sim \boldsymbol{F}} \left[\lambda \max(\boldsymbol{v}) - \sum_{i=1}^{n} p_i(\boldsymbol{v}) \right].$$

To unify the minimax regret and maximin ratio objectives, we will focus on the minimax λ-regret

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[3]The benchmark we use, maximum revenue with known *valuations*, is called the first-best benchmark. Another plausible benchmark we can use is the second-best benchmark, maximum revenue with known *distributions*. Both benchmarks are extensively used in the economics, operations, and computer science literatures; examples of papers using the first-best benchmark include [Bergemann and Schlag, 2008, Caldentey et al., 2017, Guo and Shmaya, 2023, Kleinberg and Yuan, 2013]. The first-best benchmark is also reminiscent of the *offline optimum* benchmark, which is extensively used in the analysis of algorithms [Borodin and El-Yaniv, 2005].

defined by

$$R_\lambda(\mathcal{M}, \mathcal{F}) := \inf_{(x,p) \in \mathcal{M}} \sup_{\boldsymbol{F} \in \mathcal{F}} R_\lambda(m, \boldsymbol{F}). \tag{6}$$

The following proposition, whose proof is given in Appendix A, formalizes that the values of problems (4) and (5) can be obtained from a characterization of the problem in (6).

Proposition 1. MinimaxRegret$(\mathcal{M}, \mathcal{F}) = R_1(\mathcal{M}, \mathcal{F})$ *and* MaximinRatio$(\mathcal{M}, \mathcal{F})$ *is the largest constant* $\lambda \geq 0$ *such that* $R_\lambda(\mathcal{M}, \mathcal{F}) \leq 0$.

We note that the traditional notion of worst-case performance can also be obtained from MaximinRevenue$(\mathcal{M}, \mathcal{F}) = -R_0(\mathcal{M}, \mathcal{F})$. However, with only support information $[a, b]$, the problem is trivial: without the F-dependent benchmark counteracting, Nature will simply put all the weight of the worst-case distribution F at a.

Admissible distributions. Lastly, we consider the choice of the class of admissible distributions $\mathcal{F}$. This class can be seen as capturing the "power" of Nature: the larger the class, the more powerful/adversarial Nature becomes. To simplify exposition, we will assume for most of the paper that $\mathcal{F}$ is a class of independently and identically distributed (i.i.d.) distributions $\mathcal{F}_{\text{iid}}$, defined formally as follows.

Definition 1. *The class* $\mathcal{F}_{\text{iid}}$ *consists of all distributions such that there exists a distribution F with support on $[a, b]$, referred to as the marginal, such that $\boldsymbol{F}(\boldsymbol{v}) = \prod_{i=1}^{n} F(v_i)$ for every $\boldsymbol{v} \in [a, b]^n$.*

3   Optimal mechanisms over the class of all DSIC mechanisms

In this section, we characterize an optimal mechanism over the class of all DSIC mechanisms $\mathcal{M}_{\text{all}}$ for the minimax λ-regret problem against i.i.d. distributions $\mathcal{F}_{\text{iid}}$ for any $\lambda \in (0, 1]$, support information $[a, b]$, and number of bidders n. Our main theorem presents an optimal mechanism for each uncertainty regime. We will then use the main theorem to gain insights into the structure and performance of the optimal mechanism.

Let $v^{(1)}$ and $v^{(2)}$ be the highest and second-highest entry of $\boldsymbol{v}$ and let k denote the number of buyers with value equal to $v^{(1)}$. We define the SPA and POOL mechanism classes as follows.

Definition 2 (second-price and pooling auctions). *A second-price auction with reserve r, denoted* $\mathrm{SPA}(r)$, *is defined by the allocation rule* $x : [a,b]^n \to [0,1]^n$ *and the payment rule* $p : [a,b]^n \to [0,1]^n$ *given by, for each $i \in [n]$,*

$$x_i(\boldsymbol{v}) = \begin{cases} \frac{\mathbf{1}(v_i = v^{(1)})}{k} & \text{if } v^{(1)} \geq r\,, \\ 0 & \text{if } v^{(1)} < r\,, \end{cases} \quad \text{and} \quad p_i(\boldsymbol{v}) = \begin{cases} v^{(2)} \frac{\mathbf{1}(v_i = v^{(1)})}{k} & \text{if } v^{(2)} \geq r\,, \\ r \frac{\mathbf{1}(v_i = v^{(1)})}{k} & \text{if } v^{(1)} \geq r > v^{(2)}\,, \\ 0 & \text{if } v^{(1)} < r\,, \end{cases}$$

Given a distribution of reserve prices Φ, we denote by $\mathrm{SPA}(\Phi)$ a second-price auction with random reserve prices drawn from Φ.

A pooling auction *with threshold τ, denoted* $\mathrm{POOL}(\tau)$, *is defined by the allocation rule* $x : [a,b]^n \to [0,1]^n$ *and the payment rule* $p : [a,b]^n \to [0,1]^n$ *given by, for each $i \in [n]$,*

$$x_i(\boldsymbol{v}) = \begin{cases} \frac{\mathbf{1}(v_i = v^{(1)})}{k} & \text{if } v^{(1)} \geq r\,, \\ \frac{1}{n} & \text{if } v^{(1)} < r\,, \end{cases} \quad \text{and} \quad p_i(\boldsymbol{v}) = \begin{cases} v^{(2)} \frac{\mathbf{1}(v_i = v^{(1)})}{k} & \text{if } v^{(2)} \geq r\,, \\ \frac{(n-1)\tau + a}{n} \frac{\mathbf{1}(v_i = v^{(1)})}{k} & \text{if } v^{(1)} \geq r > v^{(2)}\,, \\ \frac{a}{n} & \text{if } v^{(1)} < r \end{cases}$$

Given a distribution of thresholds Ψ, we denote by $\mathrm{POOL}(\Psi)$ a pooling auction with random thresholds drawn from Ψ.

Note that in any DSIC mechanism (including SPA and POOL defined above), the payment rule is uniquely determined from the allocation rule via Myerson's envelope formula. In a pooling auction, we can still use the threshold to differentiate between bidders with different values and potentially extract more revenue, without risking the zero payoff that comes from not allocating the item. Of course, this has implications for payments. We illustrate this interplay in Figure 1, where we depict, for the case of two agents, the allocation rule $x(\boldsymbol{v})$ and revenue $p_1(\boldsymbol{v}) + p_2(\boldsymbol{v})$ at each valuation vector $\boldsymbol{v} = (v_1, v_2)$ for three mechanisms: SPA (no reserve), $\mathrm{SPA}(r)$, and $\mathrm{POOL}(r)$.

We can see intuitively that pooling low types indeed softens the tradeoff associated with reserve pricing. By increasing the allocation for the low types, we increase the payment accrued from lower-value bidders but, at the same time, we decrease the payment accrued from higher-value bidders to guarantee incentive compatibility (so higher-value bidders do not pretend to be lower-value ones).

When the relative support information is high (i.e., $a \sim b$), the lower-value and higher-value bidders are not too different, and this softer tradeoff has the potential to lead to more robust mechanisms.
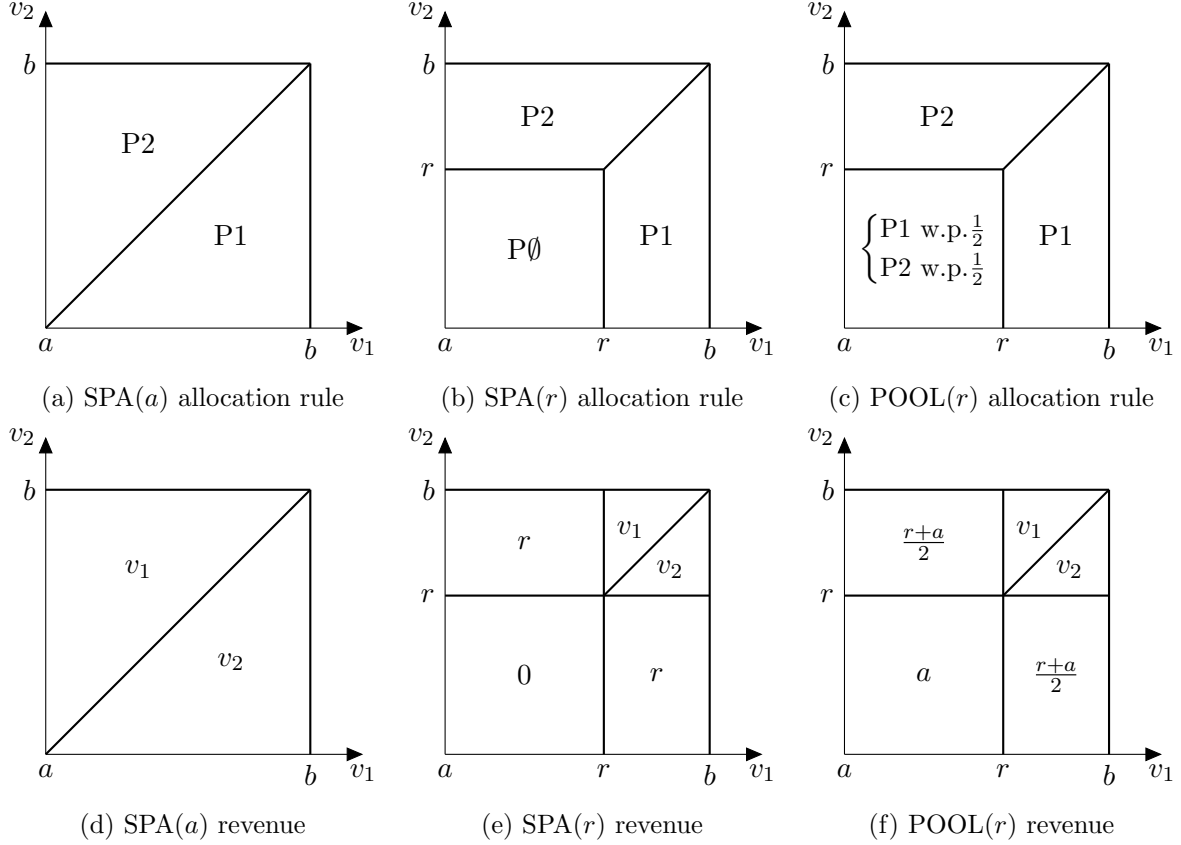

(a) SPA(a) allocation rule   (b) SPA(r) allocation rule   (c) POOL(r) allocation rule

(d) SPA(a) revenue   (e) SPA(r) revenue   (f) POOL(r) revenue

Figure 1: Allocation rules and revenue of SPA without reserve, SPA(r), and POOL(r). In the allocation rule, P1 stands for allocating to player 1, P2 for allocating to player 2, and P$\emptyset$ for not allocating.

Before we state our main theorem, we first define two thresholds that will demarcate three information regimes arising in our analysis.

Definition 3. *Fix n and $\lambda \in (0, 1]$. Define $k_l \in (0, 1)$ as a unique solution to*

$$\lambda \int_{t=k_l}^{t=1} \frac{(t - k_l)^{n-1}}{t^n} dt = (1 - k_l)^{n-1}.$$

Define k_h to be 1 for $n = 1$, and if $n \geq 2$, define $k_h \in (0, 1)$ to be the unique solution to

$$\int_{t=k_h}^{t=1} \left[\frac{(t - k_h)^{n-1}}{t^n} - (1 - \lambda)\frac{(t - k_h)^n}{t^{n+1}} \right] dt = (1 - k_h)^n .$$

We are now ready to state our main theorem.

Theorem 1 (Main Theorem). *Depending on the value of a/b, the minimax λ-regret problem admits the following mechanism as robustly optimal.*

- *Suppose $a/b \leq k_l$, and let $r^* = k_l b$. An optimal mechanism is SPA(Φ^*) with*

$$\Phi^*(v) = \lambda \frac{v^{n-1}}{(v - r^*)^{n-1}} \int_{t=r^*}^{t=v} \frac{(t - r^*)^{n-1}}{t^n} dt .$$

- *Suppose $a/b \geq k_h$, and let $\phi_0 = (a - k_h b)/(1 - k_h)$. An optimal mechanism is POOL(Ψ^*) with*

$$\Psi^*(v) = \frac{n\lambda}{n - 1} \left(\frac{v - \phi_0}{v - a} \right)^n \int_{t=a}^{t=v} \frac{(t - a)^n}{(t - \phi_0)^{n+1}} dt .$$

- *Suppose $k_l \leq a/b \leq k_h$. Let (v^*, α) be the unique solution to*

$$\frac{(v^* - a)^{n-1}}{(v^*)^{n-1}}(1 - n\alpha) = \lambda \int_{t=a}^{t=v^*} \frac{(t - a)^{n-1}}{t^n} dt ,$$

$$\frac{(b - a)^n}{b^n} - \frac{(v^* - a)^n}{(v^*)^n}(1 - (n - 1)\alpha) = \int_{t=v^*}^{t=b} \left[\frac{(t - a)^{n-1}}{t^n} - (1 - \lambda)\frac{(t - a)^n}{t^{n+1}} \right] dt .$$

An optimal mechanism is based on a unified threshold distribution $\mathcal{D}$ such that if we draw a sample $r \sim \mathcal{D}$, if $r \leq v^*$ the mechanism is SPA(r), otherwise the mechanism is POOL(r), where the CDF of $\mathcal{D}$ is given by

$$\mathcal{D}(v) = \begin{cases} \lambda \left(\frac{v}{v-a} \right)^{n-1} \int_{t=a}^{t=v} \frac{(t-a)^{n-1}}{t^n} dt & \text{for } v \in [a, v^*] , \\ -\frac{1}{n-1} + \frac{n}{n-1} \left(\frac{v}{v-a} \right)^n \left[\left(\frac{b-a}{b} \right)^n - \int_{t=v}^{t=b} \left[\frac{(t-a)^{n-1}}{t^n} - (1 - \lambda)\frac{(t-a)^n}{t^{n+1}} \right] dt \right] & \text{for } v \in [v^*, b] . \end{cases}$$

We present the proof of Theorem 1 in §3.1, with some computational details deferred to Appendix B. We then use our closed form characterization to gain insights into the structure and

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performance of the optimal mechanism in §3.2 and show how our results reduce to the known pricing case with $n = 1$ in §3.3.

3.1 Proof of the Main Theorem

The key idea for the proof of our main theorem (Theorem 1) is to explicitly exhibit a saddle point of the zero-sum game between seller and Nature, defined as follows.

Definition 4. $(m^*, \boldsymbol{F}^*)$ *is a* saddle point *of* $R_\lambda(m, \boldsymbol{F})$ *defined in (6) if and only if*

$$R_\lambda(m^*, \boldsymbol{F}) \leq R_\lambda(m^*, \boldsymbol{F}^*) \leq R_\lambda(m, \boldsymbol{F}^*) \quad \text{for all } m \in \mathcal{M}, \boldsymbol{F} \in \mathcal{F}.$$

As discussed in the related work section, the saddle point guess-and-verify technique has been used before in robust auction design problems [Anunrojwong et al., 2022, Bachrach and Talgam-Cohen, 2022], but these all identify second-price auctions as robustly optimal. Because the space of DSIC mechanisms is so big and unstructured, it is unclear a priori what the candidate mechanism family should be. Part of our technical contribution is identifying the right mechanism class depending on the support information regime. Importantly, this mechanism class is parameterized by a "one-dimensional distribution" (e.g. the Φ in SPA(Φ) or Ψ in POOL(Ψ) or a "unified threshold distribution" combining both SPA and POOL), which enables us to apply first-order condition techniques to establish the saddle point conditions.

We proceed in three steps. Firstly, we introduce a class of (g_u, g_d) mechanisms, which unifies the optimal mechanisms in all three regimes (second-price auctions with random reserves, pooling auctions with random thresholds, and randomization between SPA and POOL). Secondly, we derive sufficient conditions for a particular (g_u, g_d) mechanism and a particular distribution F^* to be a saddle point. The proof of the main theorem is then reduced to checking these sufficient conditions, which we defer to the Appendix. Thirdly, we illustrate how we can use our framework to guess and verify an optimal mechanism in the high information regime. While this step is not strictly necessary for verifying the optimality of a candidate saddle point, it offers valuable insights into the construction of our mechanism. Such intuition may prove beneficial for future researchers exploring related robust mechanism design problems.
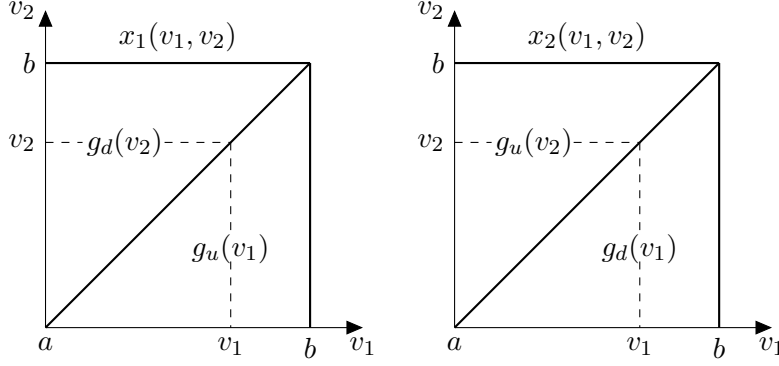
Figure 2: Allocation rules of a (g_u, g_d) mechanism for $n = 2$. For the first buyer, we have $x_1(v_1, v_2) = g_u(v_1)$ if $v_1 > v_2$, and $x_1(v_1, v_2) = g_d(v_2)$ if $v_1 < v_2$. When $v_1 = v_2$ the allocation is not shown but equal to $x_1(v_1, v_2) = (g_u(v_1) + g_d(v_1))/2$.

3.1.1   A Unifying Class of Mechanisms

We now introduce the class of (g_u, g_d) mechanisms, parameterized by two functions g_u, g_d.

Definition 5 ((g_u, g_d) mechanisms). *Let $g_u, g_d : [a, b] \to [0, 1]$ be given functions. A mechanism (g_u, g_d) is defined by the allocation rule $x : [a, b]^n \to [0, 1]^n$ given by, for each $i \in [n]$,*

$$x_i(\boldsymbol{v}) = \begin{cases} \frac{1}{k}g_u(v_{\max}) + \frac{k-1}{k}g_d(v_{\max}) & \text{if } v_i = \max(\boldsymbol{v}) := v_{\max} \text{ and there are } k \text{ entries in } \boldsymbol{v} \text{ equal to } v_{\max} \\ g_d(v_{\max}) & \text{if } v_i < \max(\boldsymbol{v}) := v_{\max} \end{cases}$$

and the payment rule $p : [a, b]^n \to \mathbb{R}_{++}^n$ is determined uniquely from Myerson's formula such that the resulting mechanism (x, p) is dominant strategy incentive compatible.

In other words, the mechanism allocates $g_u(v_{\max})$ to the highest bidder(s) and $g_d(v_{\max})$ to the non-highest bidder(s). If there are k highest bidders, the mechanism breaks ties symmetrically by selecting one of them to be the "winner" with g_u allocation uniformly at random. The (g_u, g_d) mechanism class is a powerful abstraction in our settings for two reasons we delineate below. Figure 2 illustrates the allocation under a (g_u, g_d) mechanism for the case of two buyers.

First, all mechanisms in the three different support information regimes (SPA with random reserves, POOL with random thresholds, and a randomization between SPA and POOL) can be represented in this form. Note that both SPA(r) and POOL(τ) are (g_u, g_d) with the following

specification:

$$\text{SPA}(r): \qquad g_u(v) = \mathbf{1}(v \geq r) \qquad\qquad\qquad g_d(v) = 0$$

$$\text{POOL}(\tau): \qquad g_u(v) = \frac{1}{n} + \frac{n-1}{n}\mathbf{1}(v \geq \tau) \qquad\qquad g_d(v) = \frac{1}{n} - \frac{1}{n}\mathbf{1}(v \geq \tau)$$

Furthermore, this class of mechanisms is closed under randomization. A randomization over a family of (g_u, g_d) mechanisms is still a (g_u, g_d) mechanism with the resulting mechanism having g_u and g_d that are "convex combinations" over the base g_u and g_d functions. Therefore, all mechanisms having the form in Theorem 1 are (g_u, g_d) mechanisms. The formal statement and the proof are deferred to Proposition B-1 in the Appendix.

Second, the (g_u, g_d) mechanism, while general, still captures the sense in which our mechanisms are analytically tractable for saddle point calculations. More precisely, we have the following expression for the expected regret of a (g_u, g_d) mechanism under an arbitrary distribution F.

Proposition 2 (Expected Regret of a (g_u, g_d) mechanism). *Let $R_\lambda(\boldsymbol{g}, F) := R_\lambda((g_u, g_d), F)$ be the expected λ-regret of a (g_u, g_d) mechanism under i.i.d. distribution F.*

If we assume that g_u and g_d are continuous everywhere and differentiable everywhere except at a finite number of points, then

$$R_\lambda(\boldsymbol{g}, F) = a(\lambda - g_u(a) - (n-1)g_d(a)) + \int_{v=a}^{v=b} (\lambda - g_u(v) + g_d(v) - vg_u'(v) + (v - na)g_d'(v))$$

$$+ \int_{v=a}^{v=b} \left(-\lambda - (n-1)(g_u(v) - g_d(v)) + v(g_u'(v) + (n-1)g_d'(v))\right) F(v)^n dv$$

$$+ \int_{v=a}^{v=b} n(g_u(v) - g_d(v) - (v - a)g_d'(v))F(v)^{n-1} dv. \qquad\qquad \text{(Regret-}F\text{)}$$

If, instead, we let g_u and g_d be arbitrary but we assume that F has a density $f = F'$ on $[a, b]$ (so it potentially has point masses only at a and b of size $F(a)$ and $F(\{b\}) := f_b$ respectively), then

*we have the (Regret-**g**) expression*

$$R_\lambda(\boldsymbol{g}, F) = \lambda b - a\left(g_u(a) + (n-1)g_d(a)\right)F(a)^n - (bg_u(b) + (n-1)ag_d(b))\left(1 - (1 - f_b)^n\right)$$

$$+ (b-a)g_d(b)\left(1 - (1 - f_b)^{n-1}(1 + (n-1)f_b)\right)$$

$$+ \int_{v=a}^{v=b} -\lambda F(v)^n + g_u(v)nF(v)^{n-1}(1 - F(v) - vF'(v))dv$$

$$+ \int_{v=a}^{v=b} g_d(v)n(n-1)F(v)^{n-2}F'(v)\left\{(v-a)(1-F(v)) - aF(v)\right\}dv. \qquad \text{(Regret-}\boldsymbol{g}\text{)}$$

(Regret-**g**) *is valid for $n \geq 1$ if we take the expression $n(n-1)F(v)^{n-2}$ to be zero for $n = 1$.*

3.1.2 Verification of a Saddle Point

Importantly, the (Regret-F) shows that the expected regret can be written as an explicit polynomial function of the marginal CDF $F(v)$ and is "separable" as a function of $F(v)$. This allows us to maximize regret as a function of F by maximizing each individual $F(v)$ expression "pointwise" independently for each v, subject only to the constraint that $F(v)$ is weakly increasing in v (which is automatically satisfied for our specific F). By deriving first and second conditions using (Regret-F), we derive the following sufficient conditions for Nature's saddle.

Proposition 3. *Suppose that $\boldsymbol{g}^*$ is a (g_u^*, g_d^*) mechanism and $F^*(v)$ is an increasing function that satisfy the following conditions:*

$$\left(-\lambda - (n-1)(g_u^*(v) - g_d^*(v)) + v(g_u^{\prime}(v) + (n-1)g_d^{\prime *}(v))\right)F^*(v)$$

$$+(n-1)(g_u^*(v) - g_d^*(v) - (v-a)g_d^{\prime *}(v)) = 0 \qquad \text{(FOC)}$$

$$g_u^*(v) - g_d^*(v) - (v-a)g_d^{\prime *}(v) > 0. \qquad \text{(SOC)}$$

Then $R(\boldsymbol{g}^*, F^*) \leq R(\boldsymbol{g}^*, F)$ for any F.

Note that first-order and second-order conditions together do *not* imply global optimality in general. It is only true in this case due to the special structure of the integrand, which has the form $\alpha F(v)^{n-1} - \beta F(v)^n$ for each $F(v)$, that we analyze directly.

To verify Nature's saddle it is sufficient to check (FOC) and (SOC) for the specific g_u^* and g_d^* and F^* for each of the three regimes. We defer these calculations to Appendix B.1.4.

The seller's saddle is $R(m, F^*) \leq R(m^*, F^*)$. Optimizing over m given i.i.d. F^* is a standard Bayesian mechanism design problem, and optimality of m^* in each respective regime follows from applying the classical theory from Myerson [1981].

3.1.3 Derivation of the Saddle Point for the High Information Regime

To give the reader a sense of how these calculations work, we will work out the guess-and-verify procedure in the high support information regime below. Throughout the rest of this subsection, we assume $a/b \geq k_h$. We first use the formula (Regret-g) and the sellers' saddle to pin down the worst-case distribution F^*. In turn, plugging F^* in the first-order condition of Nature's saddle in Proposition 3 gives us a differential equation involving g_u and g_d that can be used to solve for the mechanism. We then conclude by formally verifying that the candidate saddle point is optimal using the approach delineated in the previous section.

In the high information regime, the pooling auction is conjectured to be optimal and, thus, we should always allocate the item. So, we must have $g_u(v) + (n-1)g_d(v) = 1$ for every v. Using (Regret-g), we can write the expected regret purely in terms of g_u as

$$
\begin{aligned}
& \lambda b - aF(a)^n - b + b(1 - f_b)^n \\
& + \int_{v=a}^{v=b} \Big[-\lambda F(v)^n + nF(v)^{n-2}F'(v)\{(v-a)(1 - F(v)) - aF(v)\} \\
& \qquad\qquad + nF(v)^{n-2}\{F(v) - F(v)^2 - (v-a)F'(v)\} g_u(v) \Big] dv \,. \qquad \text{(Regret-}g\text{)}
\end{aligned}
$$

Note that (Regret-g) depends on g_u only through $g_u(v)$ and is linear in g_u. This is useful for the seller's saddle $\inf_m R_\lambda(m, F^*)$. If the seller maximizes over the POOL mechanism parameterized by g_u, then by the first-order conditions, under the worst-case distribution F^*, the coefficient of each $g_u(v)$ should be zero. Otherwise, the seller could decrease her regret by changing the distribution of reserves. Therefore,

$$
F^*(v) - F^*(v)^2 - (v-a)(F^*)'(v) = 0 \Rightarrow \frac{d}{dv}\left(v - \frac{v-a}{F^*(v)}\right) = 0 \Rightarrow v - \frac{v-a}{F^*(v)} = \phi_0 \,.
$$

This pins down Nature's candidate distribution as $F^*(v) = (v-a)/(v-\phi_0)$, a distribution with constant virtual value ϕ_0. We note this part of the argument (guessing F^*) is technically not needed

in the formal proof, because it is sufficient to simply verify the saddle point to confirm that our mechanism is robustly optimal. Therefore, we can make certain smoothness assumptions to derive this F^* which are *not needed* for the formal saddle verification proof. Nevertheless, we think it is useful for the readers to understand where our candidate saddle point comes from.

We now derive the mechanism g_u^*. Proposition 3 gives sufficient conditions to imply Nature's saddle. The (FOC) also gives an Ordinary Differential Equation (ODE) on g_u^* that uniquely determines it as follows. Substituting $g_d^*(v) = (1 - g_u^*(v))/(n-1)$ and $F^*(v) = (v - a)/(v - \phi_0)$ in (FOC), we get

$$\frac{d}{dv}\left[\frac{(v-a)^n}{(v-\phi_0)^n}g_u^*(v)\right] = \frac{(v-a)^{n-1}}{(v-\phi_0)^n} - (1-\lambda)\frac{(v-a)^n}{(v-\phi_0)^{n+1}}.$$

We therefore get

$$g_u^*(v) = \frac{(v-\phi_0)^n}{(v-a)^n}\int_{t=a}^{t=v}\left[\frac{(t-a)^{n-1}}{(t-\phi_0)^n} - (1-\lambda)\frac{(t-a)^n}{(t-\phi_0)^{n+1}}\right]dt = \frac{1}{n} + \lambda\sum_{k=n+1}^{\infty}\frac{(v-a)^{k-n}}{k(v-\phi_0)^{k-n}}.$$

The expression makes it clear that $g_u^*(a) = 1/n$ and $g_u^*(v)$ is increasing in v, the latter of which is necessary for the function to correspond to a true feasible mechanism. We also impose the condition that $g_u^*(b) = 1$, which gives an equation that ϕ_0 must satisfy. By definition of k_h, we see by inspection that the resulting equation has an explicit solution

$$\phi_0 = \frac{a - k_h b}{1 - k_h}.$$

We need $\phi_0 \geq 0$ for Nature's saddle to hold: this is why this mechanism and the corresponding saddle is valid only in the $a/b \geq k_h$ regime.

The above ODE manipulation not only determines g_u^*, but also makes sure that the resulting g_u^* satisfies (FOC). The only thing that remains for Nature's saddle is to verify (SOC). By substituting g_d^* with g_u^* and write $(g_u^*)'(v)$ in terms of $g_u^*(v)$ using the ODE from (FOC), (SOC) reduces to $ng_u^*(v) - 1 + \lambda > 0$ which is true because $g_u^*(v) \geq g_u^*(a) = 1/n$. Lastly, verifying Seller's saddle is a standard Bayesian mechanism design problem. Fixing F^*, the optimality of a POOL mechanism follows because under a constant positive virtual value F^*, every mechanism that always allocates is optimal.

3.2 Structure of Optimal DSIC Mechanisms

We next discuss the structure of optimal mechanisms under the minimax regret and maximin ratio objectives.

3.2.1 Minimax Regret Objective

The case of minimax regret is obtained by setting $\lambda = 1$ in Theorem 1. In Figure 3, we fix $b = 1$, and depict optimal mechanisms for $a = 0$ (low information), $a = 0.25$ (moderate information), and $a = 0.5$ (high information). We show the allocation rule $x_1(v_1, v_2)$ to buyer 1, and the total allocation $x_1(v_1, v_2) + x_2(v_1, v_2)$ to both buyers. Note that the mechanism is symmetric, so $x_2(v_1, v_2) = x_1(v_2, v_1)$ and is therefore not explicitly shown. The corresponding g_u and g_d functions are shown in the last row of Figure 3.

The top heatmaps of Figure 3 for $x_1(v_1, v_2)$ show that in the low information regime ($a = 0$), the upper triangle is all zero because in that regime the optimal mechanism randomizes among SPAs, which never allocate to the non-highest buyer. The upper triangle is not zero for moderate (mixture of SPA and POOL) and high information regimes (POOL). We can also see this from the bottom row of Figure 3 because g_d is zero for $a = 0$ but strictly positive for $a \in \{0.25, 0.50\}$.

The middle heatmaps of Figure 3 show the total allocation $x_1(\boldsymbol{v}) + x_2(\boldsymbol{v}) = g_u(v_{\max}) + g_d(v_{\max})$. In the low information regime, it is always less than 1 because SPA discards the item below the reserve. In the high information regime, it is always 1 because POOL always allocates. In the moderate information regime, it is 1 in the POOL region when $v_{\max} \geq v^* \approx 0.8$ and is less than 1 in the SPA region when $v_{\max} < v^*$ in the lower left corner. We can also see this from the value of $g_u(v) + g_d(v)$ in the last row of Figure 3.

In all heatmaps, we see that in the lower triangle ($v_1 \geq v_2$), the values are constant on each horizontal line because the allocations are $g_u(v_1)$ for bidder 1 and $g_d(v_1)$ for bidder 2 which depend on v_1, while in the upper triangle ($v_1 \leq v_2$), the values are constant on each vertical line because the allocations are $g_d(v_2)$ for bidder 1 and $g_u(v_2)$ for bidder 2 which depend only v_2, just as Figure 2 suggests.
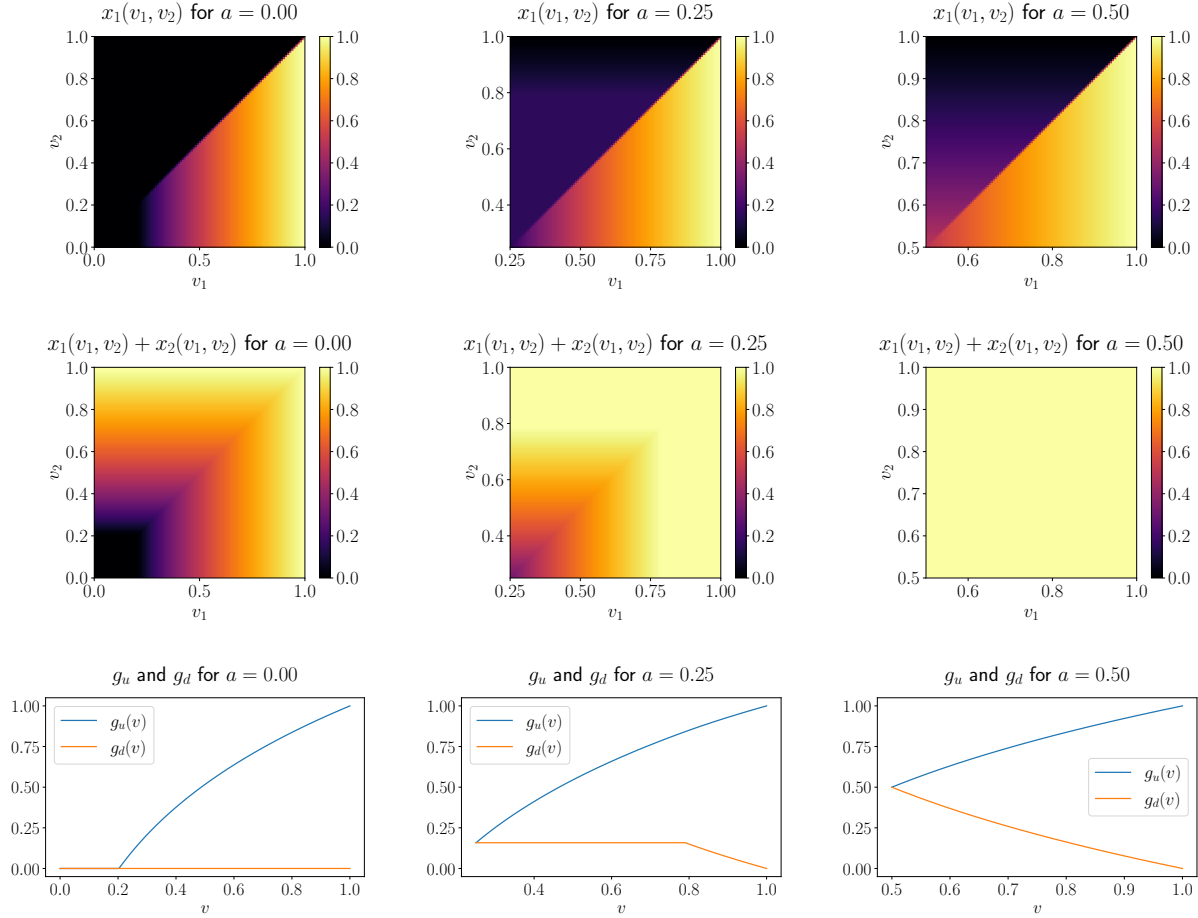
Figure 3: Optimal mechanisms for minimax regret. The top row shows the allocation rule $x_1(v_1, v_2)$, the second row shows the total allocation $x_1(v_1, v_2) + x_2(v_1, v_2)$, and the last row shows the g_u and g_d functions. The upper end of the support is $b = 1$ and each columns shows a different value of the lower end $a \in \{0, 0.25, 0.50\}$ (low, moderate, and high support information regimes).

3.2.2   Maximin Ratio Objective

Figure 4 depicts the maximin ratio value as a function of relative support information a/b for different numbers of buyers n. The ratio quantity gives us the fraction of ideal revenue that the mechanism can achieve. This figure provides quantitative evidence that even a small amount of knowledge can lead to nontrivial guarantees on revenue.

Unlike the minimax regret case where $\lambda = 1$ is set exogenously, here λ is obtained from bisection search to find the value of λ such that the minimax λ-regret is zero (cf. Proposition 1) and $\lambda = \lambda^*(k, n)$ is a function of $k \equiv a/b$, i.e., the maximin ratio given k that we computed earlier.
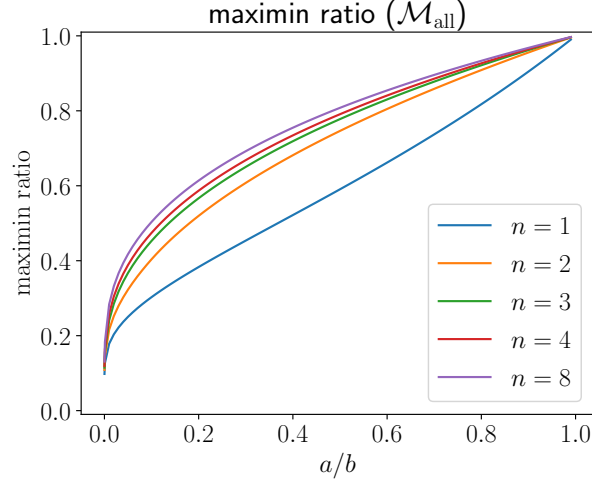
Figure 4: Maximin ratio as a function of relative support information a/b and the number of buyers n.

As a result, the regime is determined endogenously. For each n, we then compare $k = a/b$ with $k_l = k_l(\lambda^*(k,n), n)$ and $k_h = k_h(\lambda^*(k,n), n)$ to determine the regime. The next result shows that the maximin ratio mechanism is either a pure POOL, or a mixture of SPA and POOL.

Proposition 4. *The optimal mechanism identified in Theorem 1, when specialized to the maximin ratio objective, is never in the pure SPA regime.*

The proof is given in Appendix B.2. Numerically, we find that the mechanism is in a pure pooling auction regime for "reasonable" values of a/b, namely, $a/b \geq 0.0978$ for $n = 2$, $a/b \geq 0.0155$ for $n = 3$, and $a/b \geq 0.0035$ for $n = 4$. We can visualize the pool threshold distribution Ψ of POOL(Ψ) as follows. Because Ψ is supported on $[a,b]$, with varying parameters a and b we normalize the threshold τ by $\tilde{\tau} = (\tau - a)/(b - a)$ so the normalized thresholds are on the same scale $[0,1]$. For $n \in \{2, 4\}$ and $a/b \in \{0.10, 0.25, 0.50, 0.75, 0.99\}$, we plot the normalized POOL-threshold CDFs in Figure 5. We see that for low a/b, the distribution puts more weight on lower thresholds, and vice versa. However, the distributions are quite close for a wide range of a/b and for reasonably high values of a/b, the normalized distribution is close to uniform.

3.3 Remark on the case $n = 1$

An important corollary of Theorem 1 is the pricing case (one-bidder / no-competition). Applying the result with $n = 1$ directly recovers the minimax regret result of Bergemann and Schlag [2008]
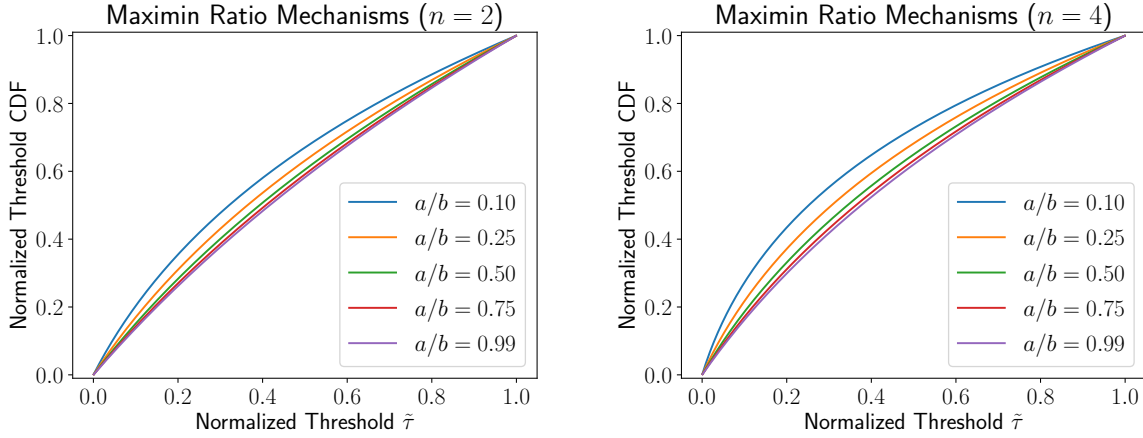
Figure 5: Normalized threshold distributions for $n \in \{2, 4\}$ and $a/b \in \{0.10, 0.25, 0.50, 0.75, 0.99\}$

and the maximin ratio result of Eren and Maglaras [2010] as special cases.[4] We give the proof of this corollary in Appendix B.3.

Corollary 1 (Pricing). *Suppose $n = 1$ and fix λ in $[0, 1]$. For $a/b \leq e^{-1/\lambda}$, the minimax λ-regret is $\lambda e^{-1/\lambda} b$, achieved with the price distribution CDF $\Phi(v) = 1 + \lambda \log(v/b)$ for $v \geq e^{-1/\lambda} b$ and 0 otherwise. For $a/b \geq e^{-1/\lambda}$, the minimax λ-regret is $-a + \lambda a + \lambda \log(b/a)$ achieved with the price distribution CDF $\Phi(v) = 1 + \lambda \log(v/b)$ for $v \in [a, b]$.*

In particular, the minimax regret is b/e if $a/b \leq 1/e$ and $a \log(b/a)$ if $a/b \geq 1/e$. For $a > 0$, the maximin ratio is $1/(1 + \log(b/a))$, achieved by the price distribution $\Phi(v) = 1 + \log(v/b)/(1 + \log(b/a))$ for $v \in [a, b]$.

We remark that in the one-bidder case, $k_l = e^{-1/\lambda}$ and $k_h = 1$, so there are only two regimes (low and moderate support information), and this is reflected in the corollary statement. Moreover, with only one bidder, POOL becomes a degenerate mechanism that always allocates. This is why in the $a/b > e^{-1/\lambda}$ regime (moderate information), the optimal mechanism, which is a mixture of SPA and POOL, always allocates with positive probability. This can be seen in the pricing CDF $\Phi(v) = 1 + \lambda \log(v/b)$, which has a point mass of positive size $1 + \lambda \log(a/b) > 0$ at $v = a$.

---

[4]More precisely, Eren and Maglaras [2010] derives the maximin ratio against the second-best benchmark in the discrete price setting, whereas our result is against the first-best benchmark in the continuous setting. However, their numerical value for the ratio approaches ours as the grid resolution becomes finer. We can also show that for the $n = 1$ case, the maximin ratio for two benchmarks are the same. Eren and Maglaras [2010] also does not explicitly derive the optimal mechanism, whereas we do.

4 Minimax λ-Regret across Mechanism Classes

Our main theorem (Theorem 1) gives a complete characterization of the optimal robust performance when Nature's distribution is i.i.d. ($\boldsymbol{F} \in \mathcal{F}_{\text{iid}}$) and the seller can choose any DSIC mechanism ($m \in \mathcal{M}_{\text{all}}$). It turns out that the optimal mechanism is generally a randomization over SPA and POOL mechanisms. This optimal mechanism has interesting features, and we would like to quantify how much each feature contributes to the performance. That is, without that feature, how much (robust) performance, if any, we will lose. Equivalently, our results quantify the "cost of simplicity" or the performance loss if the seller is restricted to simpler classes of mechanisms. We formalize this problem by solving minimax λ-regret problems, $\lambda \in (0, 1]$, when the mechanism classes $\mathcal{M}$ are successively smaller, omitting one feature at a time. The subclasses under consideration are shown in Figure 6.
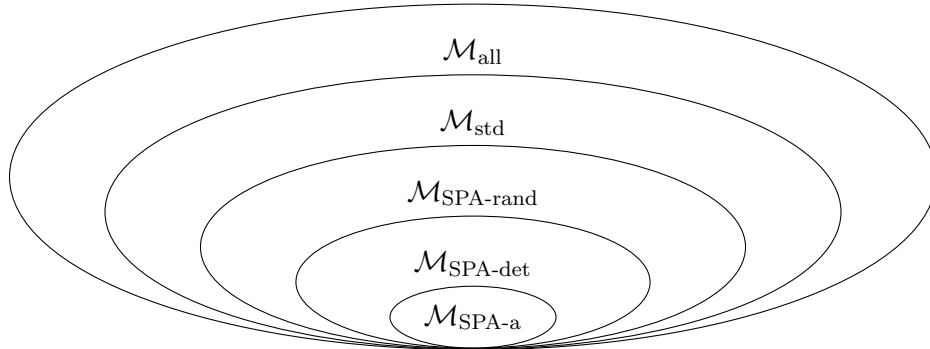

Figure 6: Nested mechanism subclasses we consider, from biggest to smallest: DSIC mechanisms ($\mathcal{M}_{\text{all}}$), standard mechanisms ($\mathcal{M}_{\text{std}}$), SPA with random reserve ($\mathcal{M}_{\text{SPA-rand}}$), SPA with deterministic reserve ($\mathcal{M}_{\text{SPA-det}}$), SPA with no reserve ($\mathcal{M}_{\text{SPA-a}}$)

First, our optimal mechanism is not standard because POOL might allocate to a bidder who is not the highest. To isolate the role of the pooling feature, we study the class of standard mechanisms that only allocate to the maximum bidder. Second, we study the need to deviate from SPAs in standard mechanisms, and hence study SPAs with randomized reserves. Lastly, we quantify the power of randomness and the power of using a reserve by computing minimax regret under the class $\mathcal{M}_{\text{SPA-det}}$ of SPA with a deterministic reserve and the class $\mathcal{M}_{\text{SPA-a}}$ of SPA with no reserve. Interestingly, we show that there are strict separations in terms of maximin ratio between $\mathcal{M}_{\text{all}}$, $\mathcal{M}_{\text{std}}$, $\mathcal{M}_{\text{SPA-rand}}$, and $\mathcal{M}_{\text{SPA-a}}$ (but not between $\mathcal{M}_{\text{SPA-det}}$ and $\mathcal{M}_{\text{SPA-a}}$). In other words, pooling and deviations from SPAs are critical for robust performance, and so is the randomization of reserve

prices.

4.1 Minimax λ-Regret Over Standard Mechanisms

A mechanism is said to be *standard* if it never allocates to an agent that does not have the highest value. Formally, it satisfies the following constraint:

$$x_i(v_i, \boldsymbol{v}_{-i}) = 0 \quad \forall i, v_i, \boldsymbol{v}_{-i} \text{ such that } v_i < \max(\boldsymbol{v}) \, . \tag{STD}$$

We can now define the class of all standard mechanisms.

Definition 6. *The class of all standard mechanisms is given by*

$$\mathcal{M}_{\mathrm{std}} = \{(\boldsymbol{x}, \boldsymbol{p}) : (\mathrm{IR}), (\mathrm{IC}), (\mathrm{AC}), (\mathrm{STD})\} \, . \tag{7}$$

It is clear that any second-price auction (SPA) with random reserve is standard, and intuitively, SPAs seem like "natural" and "typical" elements of this class, but as it turns out, other standard mechanisms lead to higher performance than SPAs when relative support information is high. Let $v^{(1)}$ and $v^{(2)}$ be the highest and second highest values in the vector $\boldsymbol{v}$. We now introduce the following mechanism class.

Definition 7 (Generous SPA). *A generous SPA with reserve distribution Φ, denoted* $\mathrm{GenSPA}(\Phi)$*, is defined by the allocation rule x given by, for each $i \in [n]$,*

$$x_i(\boldsymbol{v}) = \begin{cases} \Phi(v^{(1)}) & \textit{if } v_i \textit{ is the highest and } v^{(2)} > a \, , \\ 1 & \textit{if } v_i \textit{ is the highest and } v^{(2)} = a \, , \end{cases}$$

and zero otherwise, breaking ties uniformly at random. The payment rule $p : [a, b]^n \to \mathbb{R}_+^n$ is determined uniquely from Myerson's formula such that the resulting mechanism (x, p) is dominant strategy incentive compatible.

We call this mechanism *generous* SPA because it behaves like SPA, except in the case when all other non-highest agents have the lowest possible value a, then it always allocates ("generously"). We now state the main theorem of this section.

Theorem 2 (Optimal Standard Mechanism). *Fix n and $\lambda \in (0,1]$, and let $\tilde{a} = a/b \in [0,1)$. Define k_l as in Theorem 1. Then, the problem admits an optimal minimax λ-regret standard mechanism m^*, depending on a/b as follows.*

- *(Low Relative Support Information) For $a/b \leq k_l$, $m^* = \mathrm{SPA}(\Phi)$ is the same as in Theorem 1.*

- *(High Relative Support Information) For $a/b \geq k_l$, there is a probability distribution Φ such that $m^* = \mathrm{GenSPA}(\Phi)$.*

Note that by Theorem 1, if $a/b \leq k_l$, then SPA with random reserve is optimal in $\mathcal{M}_{\mathrm{all}}$, and it is also standard, so it is immediate that it is also optimal in the class $\mathcal{M}_{\mathrm{std}}$. Similar to Theorem 1, Theorem 2 highlights the structural features of our optimal mechanism and is a corollary of Theorem 5 in the Appendix which fully characterizes the saddle point in closed form.

The proof of Theorem 2 follows a similar outline to that of Theorem 1, although the calculations are nontrivial. In particular, we need to derive the expressions of *conditional* distributions of order statistics for arbitrary F, taking into account potential ties, which complicate the calculations.[5] In contrast, the regret of any (g_u, g_d) mechanism (whose class contains all other mechanisms in this paper) depends only on the marginal distributions of the first- and second-order statistics, which are simpler (cf. Proposition 2). However, the hardest part is *coming up with the right structural class* GenSPA that contains the optimal mechanism (within the subclass of standard mechanisms) and is tractable, because our techniques based on solving differential equations can pin down the candidate mechanism only once we fix the mechanism up to a one-dimensional functional parameter. We discuss key technical challenges and give the full proof in Appendix C.1.

4.2 Minimax λ-Regret over SPA with random and deterministic reserve

We can characterize the minimax λ-regret mechanism and its corresponding worst-case distribution and performance in the following theorem.

Theorem 3 (Optimal SPA with Random Reserve). *Fix n and $\lambda \in (0,1]$. Define k_l as in Theorem 1 and $k_h' = \lambda n/((1+\lambda)n-1)$. Then, the problem admits a minimax λ-regret $m^* = \mathrm{SPA}(\Phi^*)$, depending on a/b, as follows.*

---

[5]The existing results on conditional distributions of order statistics assume that F has a density, see e.g. David and Nagaraja [2003]. These results do not apply because we do not make any assumptions on F. In fact, the worst case F has point masses.

- *(Low Relative Support Information) For $a/b \leq k_l$, $m^* = \mathrm{SPA}(\Phi^*)$ is the same as in Theorem 1.*

- *(High Relative Support Information) For $a/b \geq k'_h$, Φ^* is a point mass only at a, i.e., $m^* = \mathrm{SPA}(a)$ is a SPA with no reserve.*

- *(Moderate Relative Support Information) For $k_l \leq a/b \leq k'_h$, there is $r^* \in [a, b]$ such that Φ^* has a point mass at a and a density on $[r^*, b]$.*

The second bullet point of Theorem 3 formalizes the intuition highlighted in the introduction that in the high scale information regime (a/b is close enough to 1), the optimal SPA with random reserve sets no reserve at all. Similar to Theorem 1, Theorem 3 highlights the structural features of our optimal mechanism and is a corollary of Theorem 6 in Appendix C.2 which fully characterizes the saddle point in closed form. The proof of the moderate information regime of Theorem 3 is the most challenging. It is different from previous saddle problems because in this case, the increasing condition on the reserve price distribution Φ is *binding*; if we optimize pointwise, the resulting distribution is not increasing, which is infeasible. We characterize an optimal distribution of reserves using a Lagrangian approach that involves introducing a Lagrange multiplier for the monotonicity constraint and then designing a primal-dual pair that satisfies complementary slackness and Lagrangian optimality. We discuss key technical challenges and give the full proof in Appendix C.2.

Lastly, we characterize the optimal SPA with deterministic reserve $\mathcal{M}_{\mathrm{SPA\text{-}det}}$ and SPA with no reserve $\mathcal{M}_{\mathrm{SPA\text{-}a}}$. Proposition C-5 in the Appendix gives the the minimax λ-regret for $\mathrm{SPA}(r)$ with a fixed deterministic reserve r. In particular, it subsumes the problem of choosing the regret-minimizing reserve r as well as computing worst-case regret of SPA without reserve ($r = a$).

4.3 Performance Separation Between Mechanism Classes

Figure 7 shows the maximin ratio as a function of a/b of all mechanism classes for $n \in \{2, 4\}$. This metric captures the performance of the optimal mechanism. We can see that while $\mathcal{M}_{\mathrm{SPA\text{-}det}}$ and $\mathcal{M}_{\mathrm{SPA\text{-}a}}$ have the same maximin ratios (so a fixed reserve does not improve over no reserve), there are *strict separations* between $\mathcal{M}_{\mathrm{all}}$, $\mathcal{M}_{\mathrm{std}}$, $\mathcal{M}_{\mathrm{SPA\text{-}rand}}$, and $\mathcal{M}_{\mathrm{SPA\text{-}a}}$.

The gap between $\mathcal{M}_{\mathrm{std}}$ and $\mathcal{M}_{\mathrm{SPA\text{-}rand}}$ shows that no SPA is optimal within the class of standard
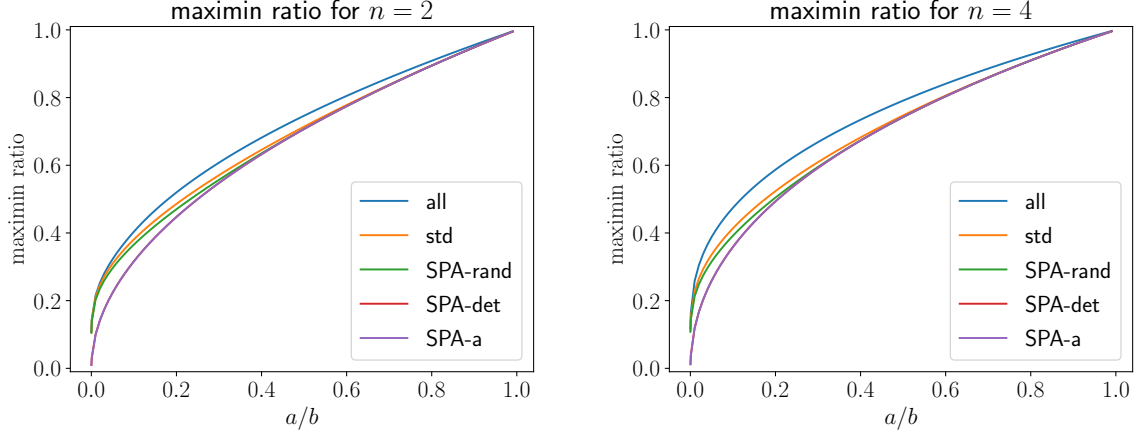
Figure 7: Maximin ratio as a function of a/b for $n \in \{2, 4\}$.

mechanisms, even though the gap is quantitatively small. In contrast, the gap between $\mathcal{M}_{\text{all}}$ and $\mathcal{M}_{\text{std}}$ is significant. This means that in robust settings, *it is important to sometimes allocate to non-highest bidders*. We can see from the plots with $n = 2$ and $n = 4$ that the non-standard gap becomes bigger and dominates all other gaps as n gets large, so this becomes more important with more bidders.

These structural results shows that there are interesting mechanism classes in DSIC mechanisms beyond SPA in the sense that they are robustly optimal in natural settings. In particular, SPA is not optimal even within the class of standard mechanisms; GenSPA is. It is an open question whether GenSPA will also be useful in other settings as well.

5 Extensions and Conclusion

In this paper, we give an explicit characterization of a *robustly* optimal mechanism to sell an item to n buyers knowing only a lower bound and an upper bound of the support of values, where the seller's performance is evaluated in the worst case. Our general framework is broadly applicable to an arbitrary number n of buyers and several mechanism classes $\mathcal{M}$ and captures both regret and ratio objectives.

Furthermore, we note that it is possible to extend the framework to other classes of distributions. It is possible to show that the minimax λ-regret we have obtained for the case of i.i.d. distributions (and the corresponding optimal mechanism) does not change if Nature optimizes over broader

classes of distributions capturing positive dependence: exchangeable and affiliated values, a common class considered with knowledge of the distributions [Milgrom and Weber, 1982]; and mixtures of i.i.d. distributions, another common class. The results also do not change if Nature optimizes over the smaller class of i.i.d. regular distributions.

There are many avenues for future work. This present paper is a step in the more general agenda of *robust mechanism design with partial information*, and it would be interesting to investigate how other forms of side information (such as moments, samples, and shapes of distributions) impact the structure and performance of optimal or near-optimal mechanisms, and the value of such information. Another direction is to consider other benchmarks, especially the second-best benchmark rather than the first-best benchmark considered in this paper.

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Electronic Companion:

Robust Auction Design with Support Information

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Appendix

Table of Contents

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A   Proofs for Section 2

Proof of Proposition 1. The definition of MaximinRatio$(\mathcal{M}, \mathcal{F})$ says that it is a solution to

$$\sup_{(x,p)\in\mathcal{M}} \lambda \text{ s.t. } \frac{\mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\sum_{i=1}^n p_i(\boldsymbol{v})\right]}{\mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\max(\boldsymbol{v})\right]} \leq \lambda \quad \forall \boldsymbol{F}\in\mathcal{F},$$

or

$$\sup_{(x,p)\in\mathcal{M}} \lambda \text{ s.t. } \mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\lambda\max(\boldsymbol{v}) - \sum_{i=1}^n p_i(\boldsymbol{v})\right] \leq 0 \quad \forall \boldsymbol{F}\in\mathcal{F},$$

or

$$\sup_{(x,p)\in\mathcal{M}} \lambda \text{ s.t. } \sup_{\boldsymbol{F}\in\mathcal{F}} \mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\lambda\max(\boldsymbol{v}) - \sum_{i=1}^n p_i(\boldsymbol{v})\right] \leq 0.$$

That is, the maximin ratio is the highest value of λ such that *there exists* $(x,p)\in\mathcal{M}$ such that $\sup_{\boldsymbol{F}\in\mathcal{F}}\mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\lambda\max(\boldsymbol{v}) - \sum_{i=1}^n p_i(\boldsymbol{v})\right] \leq 0$. Equivalently, it is the highest value of λ such that

$$R_\lambda(\mathcal{M}, \mathcal{F}) = \inf_{(x,p)\in\mathcal{M}} \sup_{\boldsymbol{F}\in\mathcal{F}} \mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\lambda\max(\boldsymbol{v}) - \sum_{i=1}^n p_i(\boldsymbol{v})\right] \leq 0. \quad \square$$

B   Proofs for Section 3

B.1   Proofs for Section 3.1

This subsection contains the technical details deferred from §3.1, the proof of the main theorem.

We first state and derive technical lemmas in §B.1.1. We then give a reformulation of our main theorem in terms of the (g_u, g_d) mechanism and the distribution F^* in a saddle point in §B.1.2. We prove key results supporting the saddle calculation in §B.1.3 and verify the saddle in §B.1.4, thus proving the main theorem.

B.1.1   Technical Lemmas

Lemma B-1. *Let ϕ_0 be a constant, then for any positive integer n and $v \geq a$ we have the identity*

$$\int_{t=a}^{t=v} \frac{(t-a)^{n-1}}{(t-\phi_0)^n}dt = \log\left(\frac{v-\phi_0}{a-\phi_0}\right) - \sum_{k=1}^{n-1}\frac{(v-a)^k}{k(v-\phi_0)^k} = \sum_{k=n}^{\infty}\frac{(v-a)^k}{k(v-\phi_0)^k}.$$

Proof of Lemma B-1. We first check the equality of the first and the second expressions. Note that both expressions are zero when $v = a$. It is then sufficient to check that the derivatives of the two

expressions agree. The derivative of the second expression is

$$\frac{1}{v - \phi_0} - \sum_{k=1}^{n-1} \frac{1}{k} k \left(\frac{v - a}{v - \phi_0} \right)^{k-1} \frac{(a - \phi_0)}{(v - \phi_0)^2}$$

$$= \frac{1}{v - \phi_0} - \frac{(a - \phi_0)}{(v - \phi_0)^2} \frac{1 - \left(\frac{v-a}{v-\phi_0} \right)^{n-1}}{1 - \frac{v-a}{v-\phi_0}} = \frac{1}{v - \phi_0} \left(\frac{v - a}{v - \phi_0} \right)^{n-1} = \frac{(v - a)^{n-1}}{(v - \phi_0)^n},$$

which is the derivative of the first expression.

Now we check the third expression. We have the Taylor Series

$$-\log(1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{k}.$$

Substituting $x = \frac{v-a}{v-\phi_0}$ gives

$$\log \left(\frac{v - \phi_0}{a - \phi_0} \right) = -\log \left(1 - \frac{v - a}{v - \phi_0} \right) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{v - a}{v - \phi_0} \right)^k.$$

We see that the first $n - 1$ terms of k cancel out, and we get the third expression. $\square$

Lemma B-2. *The following holds*

$$\int_{t=a}^{t=v} \left[\frac{(t - a)^{n-1}}{(t - \phi_0)^n} - (1 - \lambda) \frac{(t - a)^n}{(t - \phi_0)^{n+1}} \right] dt = \frac{(v - a)^n}{n(v - \phi_0)^n} + \lambda \sum_{k=n+1}^{\infty} \frac{(v - a)^k}{k(v - \phi_0)^k}$$

$$= \lambda \log \left(\frac{v - \phi_0}{a - \phi_0} \right) - \lambda \sum_{k=1}^{n-1} \frac{(v - a)^k}{k(v - \phi_0)^k} + (1 - \lambda) \frac{(v - a)^n}{n(v - \phi_0)^n}.$$

Proof of Lemma B-2. We apply Lemma B-1 to obtain

$$\int_{t=a}^{t=v} \frac{(t - a)^{n-1}}{(t - \phi_0)^n} dt = \log \left(\frac{v - \phi_0}{a - \phi_0} \right) - \sum_{k=1}^{n-1} \frac{(v - a)^k}{k(v - \phi_0)^k} = \sum_{k=n}^{\infty} \frac{(v - a)^k}{k(v - \phi_0)^k}$$

$$\int_{t=a}^{t=v} \frac{(t - a)^n}{(t - \phi_0)^{n+1}} dt = \log \left(\frac{v - \phi_0}{a - \phi_0} \right) - \sum_{k=1}^{n} \frac{(v - a)^k}{k(v - \phi_0)^k} = \sum_{k=n+1}^{\infty} \frac{(v - a)^k}{k(v - \phi_0)^k}.$$

Therefore,

$$\int_{t=a}^{t=v} \left[\frac{(t-a)^{n-1}}{(t-\phi_0)^n} - (1-\lambda)\frac{(t-a)^n}{(t-\phi_0)^{n+1}} \right] dt$$

$$= \lambda \log\left(\frac{v-\phi_0}{a-\phi_0}\right) - \lambda \sum_{k=1}^{n-1} \frac{(v-a)^k}{k(v-\phi_0)^k} + (1-\lambda)\frac{(v-a)^n}{n(v-\phi_0)^n}$$

$$= \frac{(v-a)^n}{n(v-\phi_0)^n} + \lambda \sum_{k=n+1}^{\infty} \frac{(v-a)^k}{k(v-\phi_0)^k},$$

where the last equation follows from the Taylor series for the logarithm. □

Lemma B-3. *If h is a differentiable function, then*

$$\int_{w\in[a,b]} h(w)dG(w) = h(a) + \int_{w\in[a,b]} h'(w)(1-G(w))dw.$$

Proof of Lemma B-3.

$$\int_{w\in[a,b]} h(w)dG(w) = \int_{w\in[a,b]} \left(h(a) + \int_{\tilde{w}=a}^{\tilde{w}=w} h'(\tilde{w})d\tilde{w} \right) dG(w)$$

$$= h(a) + \int_{\tilde{w}=a}^{\tilde{w}=b} h'(\tilde{w}) \int_{w\in(\tilde{w},b]} dG(w)dw$$

$$= h(a) + \int_{\tilde{w}=a}^{\tilde{w}=b} h'(\tilde{w})(G(b) - G(\tilde{w}))d\tilde{w}$$

$$= h(a) + \int_{\tilde{w}=a}^{\tilde{w}=b} h'(\tilde{w})(1-G(\tilde{w}))d\tilde{w}. \quad \square$$

Now, we give a formal proposition that (g_u, g_d) mechanisms and convex combinations of $\{\mathrm{SPA}(r),$ $\mathrm{POOL}(\tau)\}$ are *almost* equivalent representations of the same mechanism class in the sense that one can be converted to another.

Proposition B-1. *We have the following correspondence between the (g_u, g_d) mechanisms arising in our main theorem (Theorem 4) and convex combinations of SPAs and POOLs.*

(1) *A mechanism is a (g_u, g_d) mechanism with $g_u(v) \in [0,1]$ increasing in v, $g_u(a) = 0$, and $g_d(v) = 0$ for all v if and only if it is $\mathrm{SPA}(\Phi)$, Φ has measure 1, and $\Phi(v) = g_u(v)$.*

(2) *A mechanism is a (g_u, g_d) mechanism with $g_u(v) \in [0,1]$ increasing in v, $g_u(a) = g_d(a) = 1/n$, and $g_u(v) + (n-1)g_d(v) = 1$ for all v if and only if is $\mathrm{POOL}(\Psi)$, Ψ has measure 1, and $\Psi(v) = 1 - ng_d(v)$.*

(3) *A mechanism is a (g_u, g_d) mechanism with $g_u(v) \in [0,1]$ increasing in v for $v \in [a,b]$, $g_u(a) = g_d(a) := \alpha$, $g_d(v) = \alpha$ for $v \in [a, v^*]$ for some constant $v^* \in [a,b]$, $g_u(v) + (n-1)g_d(v) = 1$ for $v \in [v^*, b]$ if and only if it is a randomization over $\mathrm{SPA}(\Phi)$, with Φ supported on $[a, v^*]$ and $\mathrm{POOL}(\Psi)$ with Ψ supported on $[v^*, b]$. Furthermore, their cumulative probabilities are given by $\Phi(v) = g_u(v) - \alpha$ for $v \in [a, v^*]$ and $\Psi(v) = n(\alpha - g_d(v))$ for $v \in [v^*, b]$.*

Proof of Proposition B-1. Note that the (g_u, g_d) representation of SPA(r) is $g_u(v) = \mathbf{1}(v \geq r)$ and $g_d(v) = 0$, and the (g_u, g_d) representation of POOL(r) is $g_u(v) = \frac{1}{n} + \frac{n-1}{n}\mathbf{1}(v \geq r)$ and $g_d(v) = \frac{1}{n} - \frac{1}{n}\mathbf{1}(v \geq r)$. All three cases follow from computing the convex combination of these.

(1) is straightforward. For (2), the (g_u, g_d) representation of POOL(Ψ) is

$$g_u(v) = \int \left(\frac{1}{n} + \frac{n-1}{n}\mathbf{1}(v \geq r) \right) d\Psi(r) = \frac{1}{n} + \frac{n-1}{n}\Psi(v),$$

$$g_d(v) = \int \left(\frac{1}{n} - \frac{1}{n}\mathbf{1}(v \geq r) \right) d\Psi(v) = \frac{1}{n} - \frac{1}{n}\Psi(v).$$

We can then see that $g_u(a) = g_d(a) = 1/n$, $\Psi(v) = 1 - ng_d(v)$ and $g_u(v) + (n-1)g_d(v) = 1$. Conversely, given this (g_u, g_d), we can let $\Psi(v) = 1 - ng_d(v)$ giving a valid POOL(Ψ).

For (3), the (g_u, g_d) mechanism is

$$g_u(v) = \int \mathbf{1}(v \geq r)d\Phi(r) + \int \left(\frac{1}{n} + \frac{n-1}{n}\mathbf{1}(v \geq \tau) \right) d\Psi(r) \quad = \Phi(v) + \frac{1}{n}|\Psi| + \frac{n-1}{n}\Psi(v),$$

$$g_d(v) = \int 0 d\Phi(r) + \int \left(\frac{1}{n} - \frac{1}{n}\mathbf{1}(v \geq \tau) \right) d\Psi(v) \quad = \frac{1}{n}|\Psi| - \frac{1}{n}\Psi(v).$$

We therefore have a formula that transforms (Φ, Ψ) to (g_u, g_d). From these formula, we immediately see that $g_u(a) = g_d(a)$; we let this be α. We also see that $g_u(v) + (n-1)g_d(v) = \Phi(v) + |\Psi|$ is increasing in v, while $g_d(v) = \frac{1}{n}(|\Psi| - \Psi(v))$ is decreasing in v, because Φ and Ψ are increasing functions.

Conversely, assume that (g_u, g_d) has these properties. We will show that we can invert these formulas and find the corresponding (Φ, Ψ). From $g_u(v) = \Phi(v) + \frac{1}{n}|\Psi| + \frac{n-1}{n}\Psi(v)$, setting $v = a$ gives $\alpha = g_u(a) = \frac{1}{n}|\Psi|$, so $|\Psi| = n\alpha$, and $|\Phi| = 1 - |\Psi| = 1 - n\alpha$. From $g_d(v) = \frac{1}{n}(|\Psi| - \Psi(v))$, we get $\Psi(v) = |\Psi| - ng_d(v) = n(\alpha - g_d(v))$, and from $g_u(v) = \Phi(v) + \frac{1}{n}|\Psi| + \frac{n-1}{n}\Psi(v) = \Phi(v) + \frac{1}{n}(n\alpha) + \frac{n-1}{n} \cdot n(\alpha - g_d(v)) = \Phi(v) + n\alpha - (n-1)g_d(v)$, we get $\Phi(v) = g_u(v) + (n-1)g_d(v) - n\alpha$. $\square$

B.1.2 Reformulation of the Main Theorem

Armed with Proposition B-1, we can reformulate our main theorem (Theorem 1) as follows.

Theorem 4 (Main Theorem in (g_u, g_d)). *Fix n and $\lambda \in (0, 1]$. Define $k_l \in (0, 1)$ as a unique solution to*

$$\lambda \int_{t=k_l}^{t=1} \frac{(t - k_l)^{n-1}}{t^n} dt = (1 - k_l)^{n-1},$$

and if $n = 1$, define $k_h = 1$ and if $n \geq 2$, define $k_h \in (0, 1)$ to be a unique solution to

$$\int_{t=k_h}^{t=1} \left[\frac{(t - k_h)^{n-1}}{t^n} - (1 - \lambda)\frac{(t - k_h)^n}{t^{n+1}} \right] dt = (1 - k_h)^n.$$

Then we have $R_\lambda(m, \mathbf{F}^*) \leq R_\lambda(m^*, \mathbf{F}^*) \leq R_\lambda(m^*, \mathbf{F})$ for any $m \in \mathcal{M}_{\text{all}}$ and $\mathbf{F} \in \mathcal{F}_{\text{iid}}$, where m^* and $\mathbf{F}^*$ (which is n i.i.d. with marginal F^*) is defined depending on the value of a/b as follows.

- *Suppose $a/b \leq k_l$, and let $r^* = k_l b$. We define m^* as a (g_u^*, g_d^*) mechanism with $g_u^*(v) =$*

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$\Phi^*(v), g_d^*(v) = 0$, *where*

$$\Phi^*(v) = g_u^*(v) = \lambda \frac{v^{n-1}}{(v - r^*)^{n-1}} \int_{t=r^*}^{t=v} \frac{(t - r^*)^{n-1}}{t^n} dt \,,$$

and

$$F^*(v) = \begin{cases} 0 & \text{if } v \in [a, r^*] \\ 1 - \frac{r^*}{v} & \text{if } v \in [r^*, b) \\ 1 & \text{if } v = b \,. \end{cases}$$

- *Suppose* $a/b \geq k_h$, *and let* $\phi_0 = (a - k_h b)/(1 - k_h) \in [0, a]$. *We define* m^* *as a* (g_u^*, g_d^*) *mechanism with*

$$g_u^*(v) = \frac{1}{n} + \lambda \left(\frac{v - \phi_0}{v - a} \right)^n \int_{t=a}^{t=v} \frac{(t - a)^n}{(t - \phi_0)^{n+1}} dt$$

$$g_d^*(v) = \frac{1 - g_u^*(v)}{n - 1} \,,$$

and

$$F^*(v) = \begin{cases} 1 - \frac{a - \phi_0}{v - \phi_0} & \text{if } v \in [a, b) \\ 1 & \text{if } v = b \,. \end{cases}$$

- *For* $k_l \leq a/b \leq k_h$, *we define* m^* *as a* (g_u^*, g_d^*) *mechanism with*

$$g_u^*(v) = \begin{cases} \alpha + \lambda \left(\frac{v}{v-a} \right)^{n-1} \int_{t=a}^{t=v} \frac{(t-a)^{n-1}}{t^n} dt & \text{for } v \in [a, v^*] \\ \frac{v^n}{(v-a)^n} \left[\frac{(b-a)^n}{b^n} - \int_{t=v}^{t=b} \left[\frac{(t-a)^{n-1}}{t^n} - (1 - \lambda) \frac{(t-a)^n}{t^{n+1}} \right] dt \right] & \text{for } v \in [v^*, b] \,, \end{cases}$$

and

$$g_d^*(v) = \begin{cases} \alpha & \text{if } v \in [a, r^*] \\ \frac{1 - g_u^*(v)}{n-1} & \text{if } v \in [r^*, b] \,, \end{cases}$$

and

$$F^*(v) = \begin{cases} 1 - \frac{a}{v} & \text{if } v \in [a, b) \\ 1 & \text{if } v = b \,, \end{cases}$$

where (v^*, α) *is the unique solution to*

$$\frac{(r^* - a)^{n-1}}{(r^*)^{n-1}} (1 - n\alpha) = \lambda \int_{t=a}^{t=r^*} \frac{(t - a)^{n-1}}{t^n} dt$$

$$\frac{(b - a)^n}{b^n} - \frac{(r^* - a)^n}{(r^*)^n} (1 - (n-1)\alpha) = \int_{t=r^*}^{t=b} \left[\frac{(t - a)^{n-1}}{t^n} - (1 - \lambda) \frac{(t - a)^n}{t^{n+1}} \right] dt.$$

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B.1.3 Key Results Supporting the Saddle Calculation

Proof of Proposition 2. We first derive the (Regret-$\boldsymbol{F}$) expression, i.e. the expected regret of a (g_u, g_d) mechanism under an arbitrary joint distribution $\boldsymbol{F}$, assuming that g_u and g_d are continuous everywhere and differentiable everywhere except a finite number of points. Let $\boldsymbol{F}_n^{(1)}$ and $\boldsymbol{F}_n^{(2)}$ be the distributions of $v^{(1)}$ and $v^{(2)}$, the highest and second-highest entry of $\boldsymbol{v}$, respectively.

From Myerson's lemma,

$$p_i(\boldsymbol{v}) = v_i x_i(\boldsymbol{v}) - \int_{\tilde{v}_i=a}^{\tilde{v}_i=v_i} x_i(\tilde{v}_i, \boldsymbol{v}_{-i}) d\tilde{v}_i \,.$$

the allocation rule (g_u, g_d) gives

$$p_i(\boldsymbol{v}) = \begin{cases} v_i g_u(v_i) - (v^{(2)} - a)g_d(v^{(2)}) - \int_{t=v^{(2)}}^{t=v_i} g_u(t) dt & \text{if } v_i \text{ is the highest and} \\ & v^{(2)} \text{ is the second-highest}, \\ a g_d(v^{(1)}) & \text{if } v_i \text{ is not the highest}, \end{cases}$$

so the pointwise regret is

$$v^{(1)}(\lambda - g_u(v^{(1)})) - (n-1)a g_d(v^{(1)}) + (v^{(2)} - a)g_d(v^{(2)}) + \int_{t=v^{(2)}}^{t=v^{(1)}} g_u(t) dt \,.$$

Therefore, by Lemma B-3,

$$\mathbb{E}[v^{(1)}(\lambda - g_u(v^{(1)})) - (n-1)a g_d(v^{(1)})]$$
$$= a(\lambda - g_u(a) - (n-1)g_d(a)) + \int_{v \in [a,b]} (\lambda - g_u(v) - v g_u'(v) - (n-1)a g_d'(v))(1 - \boldsymbol{F}_n^{(1)}(v)) dv \,.$$

Now we compute the second term.

$$\mathbb{E}[(v^{(2)} - a)g_d(v^{(2)})] = (a-a)g_d(a) + \int_{v \in [a,b]} (g_d(v) + (v-a)g_d'(v))(1 - \boldsymbol{F}_n^{(2)}(v)) dv$$
$$= \int_{v \in [a,b]} (g_d(v) + (v-a)g_d'(v))(1 - \boldsymbol{F}_n^{(2)}(v)) dv \,.$$

Lastly, we compute the third term

$$\mathbb{E}\left[\int_{t=v^{(2)}}^{t=v^{(1)}} g_u(t) dt\right] = \mathbb{E}\left[\int_{v \in [a,b]} g_u(v) \mathbf{1}(v^{(2)} < v \le v^{(1)}) dv\right]$$
$$= \int_{v \in [a,b]} g_u(v) \Pr(v^{(2)} < v \le v^{(1)}) dv$$
$$= \int_{v \in [a,b]} g_u(v)(\boldsymbol{F}_n^{(2)}(v) - \boldsymbol{F}_n^{(1)}(v)) dv \,.$$

Therefore, the regret is

$$a(\lambda - g_u(a) - (n-1)g_d(a)) + \int_{v \in [a,b]} (\lambda - g_u(v) - vg'_u(v) - (n-1)ag'_d(v))(1 - \boldsymbol{F}_n^{(1)}(v))dv$$

$$+ \int_{v \in [a,b]} (g_d(v) + (v-a)g'_d(v))(1 - \boldsymbol{F}_n^{(2)}(v))dv + \int_{v \in [a,b]} g_u(v)(\boldsymbol{F}_n^{(2)}(v) - \boldsymbol{F}_n^{(1)}(v))dv \,.$$

Rearranging this gives the (Regret-$\boldsymbol{F}$) expression.

$$R(g, \boldsymbol{F}) = a(\lambda - g_u(a) - (n-1)g_d(a)) + \int_{v \in [a,b]} (\lambda - g_u(v) + g_d(v) - vg'_u(v) + (v-na)g'_d(v))$$

$$+ \int_{v \in [a,b]} \left(-\lambda - (n-1)(g_u(v) - g_d(v)) + v(g'_u(v) + (n-1)g'_d(v)) \right) \boldsymbol{F}_n^{(1)}(v)dv$$

$$+ \int_{v \in [a,b]} n(g_u(v) - g_d(v) - (v-a)g'_d(v))\boldsymbol{F}_{n-1}^{(1)}(v)dv \,.$$

To derive (Regret-F), the expected regret expression with i.i.d. F, substitute $\boldsymbol{F}_n^{(1)}(v) = F(v)^n$ and $\boldsymbol{F}_n^{(2)}(v) = nF(v)^{n-1} - (n-1)F(v)^n$. $\qquad\square$

Proof of Proposition 3. We use the following (Regret-F) expression for λ-regret

$$R(g, F) = a(\lambda - g_u(a) - (n-1)g_d(a)) + \int_{v \in [a,b]} (\lambda - g_u(v) + g_d(v) - vg'_u(v) + (v-na)g'_d(v))$$

$$+ \int_{v \in [a,b]} \left(-\lambda - (n-1)(g_u(v) - g_d(v)) + v(g'_u(v) + (n-1)g'_d(v)) \right) F(v)^n dv$$

$$+ \int_{v \in [a,b]} n(g_u(v) - g_d(v) - (v-a)g'_d(v))F(v)^{n-1} dv$$

In Nature's saddle, we fix the mechanism (g_u, g_d) and optimize over F. The integral expression is separable over $F(v)$ for $v \in (a, b)$. Here we will assume that the optimization is done pointwise.

The first order condition on F on the regret pointwise is

$$\left(-\lambda - (n-1)(g_u^*(v) - g_d^*(v)) + v(g_u'^*(v) + (n-1)g_d'^*(v)) \right) \cdot nF(v)^{n-1}$$

$$+ n(g_u(v) - g_d(v) - (v-a)g'_d(v)) \cdot (n-1)F(v)^{n-2} = 0$$

Nature's saddle states that over all F, F^* maximizes the λ-regret. If pointwise optimization is valid, then F^* must satisfy the above FOC equation. Dividing both sides by $nF^*(v)^{n-2}$ gives (FOC) as required.

For F^* to be maximizing, we also need the second-order conditions to hold, namely, that the second derivative with respect to $F(v)$ evaluated at $F^*(v)$ is negative[4]:

$$\left(-\lambda - (n-1)(g_u^*(v) - g_d^*(v)) + v(g_u'^*(v) + (n-1)g_d'^*(v)) \right) \cdot n(n-1)F^*(v)^{n-2}$$

$$+ n(g_u(v) - g_d(v) - (v-a)g'_d(v)) \cdot (n-1)(n-2)F^*(v)^{n-3} < 0$$

---

[4] Note that if $n = 2$ the last term disappear, so we can write $F^*(v)^{n-3}$ there with the understanding that the entire term becomes zero for $n = 2$.

or

$$\left(-\lambda - (n-1)(g_u^*(v) - g_d^*(v)) + v(g_u'^*(v) + (n-1)g_d'^*(v))\right) F^*(v)$$
$$+ (n-2)(g_u(v) - g_d(v) - (v-a)g_d'(v)) < 0$$

but from the (FOC) equality that we have just derived,

$$\left(-\lambda - (n-1)(g_u^*(v) - g_d^*(v)) + v(g_u'^*(v) + (n-1)g_d'^*(v))\right) F^*(v)$$
$$+ (n-2)(g_u(v) - g_d(v) - (v-a)g_d'(v))$$
$$= \left(-\lambda - (n-1)(g_u^*(v) - g_d^*(v)) + v(g_u'^*(v) + (n-1)g_d'^*(v))\right) F^*(v)$$
$$+ (n-1)(g_u^*(v) - g_d^*(v) - (v-a)g_d'^*(v)) - (g_u^*(v) - g_d^*(v) - (v-a)g_d'^*(v))$$
$$= -(g_u^*(v) - g_d^*(v) - (v-a)g_d'^*(v)) .$$

Therefore, our condition reduces to (SOC), as required. $\qquad\square$

B.1.4 Verification of the Saddle

The last step of the proof is to verify the (FOC) and (SOC) conditions for the $((g_u^*, g_d^*), F^*)$ pair given in Theorem 4. To do this, we first show that the (g_u^*, g_d^*) satisfies a certain ordinary differential equation (ODE).

Proposition B-2. *Define $(v^*, \alpha) = (b, 0)$ in the $a/b \leq k_l$ regime, $(v^*, \alpha) = (a, 1/n)$ in the $a/b \geq k_h$ regime, and (v^*, α) be defined as stated in Theorem 4 in the the $k_l \leq a/b \leq k_h$ regime. Also define ϕ_0 as in Theorem 4 in the $a/b \geq k_h$ regime, and $\phi_0 = 0$ in other regimes. Let (g_u^*, g_d^*) be given as in Theorem 4. Then (g_u^*, g_d^*) satisfies $g_u^*(v) = g^*(v)$, $g_d^*(v) = \alpha$ for $v \in [a, v^*]$ and $g_d^*(v) = (1 - g^*(v))/(n-1)$ for $v \in [v^*, b]$, and g^* is an increasing continuous function that satisfies the ODE*

$$(g^*)'(v) + \frac{(n-1)r^*}{v(v - r^*)}(g^*(v) - \alpha) = \frac{\lambda}{v} \qquad\qquad \text{for } v \in (r^*, v^*) \qquad\qquad \text{(ODE-}g\text{-1)}$$

$$(g^*)'(v) + \frac{n(a - \phi_0)}{(v - \phi_0)(v - a)}g^*(v) = \frac{1}{v - a} - \frac{1 - \lambda}{v - \phi_0} \qquad \text{for } v \in (v^*, b) . \qquad \text{(ODE-}g\text{-2)}$$

Furthermore, in the $k_l \leq a/b \leq k_h$ regime, the system of equations defining (v^*, α) actually has a unique solution, and if we view r^* and α as a function of a/b, then we have $r^* \uparrow b$ and $\alpha \downarrow 0$ as $a/b \downarrow k_l$, while $r^* \downarrow a$ and $\alpha \uparrow 1/n$ as $a/b \uparrow k_h$.

Proof of Proposition B-2. We first consider the regime $v \leq v^*$ where the ODE is (ODE-g-1). By multiplying both sides of (ODE-g-1) by $(v - r^*)^{n-1}/v^{n-1}$, we observe that (ODE-g-1) is equivalent to

$$\frac{d}{dv}\left[\frac{(v - r^*)^{n-1}}{v^{n-1}}(g^*(v) - \alpha)\right] = \lambda \frac{(v - r^*)^{n-1}}{v^n} .$$

For the case $a/b \leq k_l$, we set $\alpha = 0, g^*(r^*) = 0$ and integrate the above equation from $v = r^*$ to arbitrary v to get the $g_u^* \equiv g^*$ as stated in the theorem statement.

For the case $k_l \leq a/b \leq k_h$, we set $g^*(a) = \alpha$ and integrate the above equation from $v = a$ to arbitrary v to get the $g_u^* \equiv g^*$ as stated in the theorem statement.

In both cases, we can use Lemma B-1 to write $g^*(v)$ in the valid region as

$$g^*(v) = \alpha + \lambda \sum_{k=n}^{\infty} \frac{1}{k} \left(\frac{v - r^*}{v} \right)^{k-(n-1)} ,$$

which immediately implies that g^* is an increasing continuous function in v.

In the $a/b \leq k_l$ case, the valid region starts at $v = r^*$ and the expression immediately implies that $g^*(r^*) = 0 = \alpha$, and

$$g^*(b) = \lambda \sum_{k=n}^{\infty} \frac{1}{k} \left(\frac{b - r^*}{b} \right)^{k-(n-1)} ,$$

which equals 1 because r^*/b satisfies the same defining equation as k_l, so we can set $r^* = k_l b$, and the expression is decreasing in r^*, so the equation setting the above to 1 has a unique solution in r^* (equivalently, in k_l) if and only if as $r^* \downarrow a$, the expression is ≥ 1, which is equivalent to $a/b \leq k_l$ that we had just assumed.

In the $k_l \leq a/b \leq k_h$ case, the valid region is from $v = a$ to $v = v^*$. The expression implies $g^*(a) = \alpha$. The defining equation for (v^*, α) in this regime implies that

$$\frac{(v^* - a)^{n-1}}{(v^*)^{n-1}} (g^*(v^*) - \alpha) = \lambda \int_{t=a}^{t=v^*} \frac{(t - a)^{n-1}}{t^n} dt = \frac{(v^* - a)^{n-1}}{(v^*)^{n-1}} (1 - n\alpha) ,$$

so $g^*(v^*) = 1 - (n-1)\alpha$. (We still need to prove that the two equations defining (v^*, α) has a unique solution; we will defer this to the end of the proof.)

Now we consider the regime $v \geq v^*$ where the ODE is (ODE-g-2). By multiplying both sides by $(v - a)^n/(v - \phi_0)^n$, we observe that (ODE-g-2) is equivalent to

$$\frac{d}{dv} \left[\frac{(v - a)^n}{(v - \phi_0)^n} g^*(v) \right] = \frac{(v - a)^{n-1}}{(v - \phi_0)^n} - (1 - \lambda) \frac{(v - a)^n}{(v - \phi_0)^{n+1}} .$$

In the case $a/b \geq k_h$, this equation applies for all $v \in [a, b]$, so we integrate this equation from $v = a$ to arbitrary v and note that $\frac{(v-a)^n}{(v-\phi_0)^n} g^*(v)$ is 0 when $v = a$ (because of the $(v - a)$ factor), so we get

$$\frac{(v - a)^n}{(v - \phi_0)^n} g^*(v) - 0 = \int_{t=a}^{t=v} \left[\frac{(t - a)^{n-1}}{(t - \phi_0)^n} - (1 - \lambda) \frac{(t - a)^n}{(t - \phi_0)^{n+1}} \right] dt ,$$

which is equivalent to the g^* as stated in the theorem statement. By Lemma B-2, we can write $g^*(v)$ as

$$g^*(v) = \frac{1}{n} + \lambda \sum_{k=n+1}^{\infty} \frac{(v - a)^{k-n}}{k(v - \phi_0)^{k-n}} .$$

The expression immediately implies that $g^*(v)$ is an increasing continuous function of v and $g^*(a) =$

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$1/n$. We also have

$$g^*(b) = \frac{1}{n} + \lambda \sum_{k=n+1}^{\infty} \frac{(b-a)^{k-n}}{k(b-\phi_0)^{k-n}} = 1 \,,$$

by the defining equation of ϕ_0, and by inspecting the defining equations for ϕ_0 and k_h we see that $\phi_0 = (a - k_h b)/(1 - k_h)$ as claimed. The defining equation of k_h is

$$\frac{1}{n} + \lambda \sum_{k=n+1}^{\infty} \frac{(1-k_h)^{k-n}}{k} = 1 \,.$$

The expression is decreasing in k_h and it is $1/n < 1$ as $k_h \uparrow 1$, and $\frac{1}{n} + \lambda \sum_{k=n+1}^{\infty} \frac{1}{k} = \infty$ as $k_h \downarrow 0$ because the harmonic series is divergent, so the equation has a unique solution k_h.

In the case $k_l \le a/b \le k_h$, this equation applies for $v \in [v^*, b]$. By requiring that $g^*(b) = 1$, integrating the equation from arbitrary v to $v = b$ gives

$$\frac{(b-a)^n}{b^n} - \frac{(v-a)^n}{v^n} g^*(v) = \int_{t=v}^{t=b} \left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}} \right] dt \,.$$

Just as before, Lemma B-2 implies that the right hand side is decreasing and continuous in v, so $g^*(v)$ is increasing and continuous in v. We also have

$$\frac{(b-a)^n}{b^n} - \frac{(v-a)^n}{v^n} g^*(v^*) = \int_{t=v^*}^{t=b} \left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}} \right] dt$$

$$= \frac{(b-a)^n}{b^n} - \frac{(v-a)^n}{v^n}(1 - (n-1)\alpha) \,,$$

by the defining equations for (v^*, α), so $g^*(v^*) = 1 - (n-1)\alpha$. We therefore see that the value of g^* at v^* from both the $v \le v^*$ and the $v \ge v^*$ regions are equal, so g^* is continuous at v^* as well.

Finally, we will prove that the equations defining (v^*, α) in the $k_l \le a/b \le k_h$ regime have a unique solution.

Eliminating α from the two equations gives

$$n - \frac{n(v^*)^n}{(v^* - a)^n} \left(\frac{(b-a)^n}{b^n} - \int_{t=v^*}^{t=b} \left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}} \right] dt \right)$$

$$= (n-1) - \frac{(n-1)(v^*)^{n-1}}{(v^* - a)^{n-1}} \lambda \int_{t=a}^{t=v^*} \frac{(t-a)^{n-1}}{t^n} dt \,,$$

or

$$\frac{n(b-a)^n}{b^n} = \frac{(r^* - a)^n}{(r^*)^n} + \frac{(n-1)(r^* - a)}{(r^*)} \lambda \int_{t=a}^{t=r^*} \frac{(t-a)^{n-1}}{t^n} dt$$

$$+ n \int_{t=r^*}^{t=b} \left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}} \right] dt \,.$$

Let $\mathrm{fn}(r^*)$ denote the right hand side viewed as a function of r^*, namely,

$$\mathrm{fn}(v) := \frac{(v-a)^n}{v^n} + \frac{(n-1)(v-a)}{v}\lambda\int_{t=a}^{t=v}\frac{(t-a)^{n-1}}{t^n}dt + n\int_{t=v}^{t=b}\left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}}\right]dt \,.$$

We claim that $\mathrm{fn}(v)$ is a decreasing function. That is, we want to show that $d\mathrm{fn}(v)/dv = \mathrm{fn}'(v) \leq 0$. We compute

$$\mathrm{fn}'(v) = n\left(\frac{v-a}{v}\right)^{n-1}\frac{a}{v^2} + (n-1)\lambda\frac{d}{dv}\left[\frac{(v-a)}{v}\int_{t=a}^{t=v}\frac{(t-a)^{n-1}}{t^n}dt\right]$$
$$- n\left(\frac{(v-a)^{n-1}}{v^n} - (1-\lambda)\frac{(v-a)^n}{v^{n+1}}\right)\,.$$

Note that

$$n\left(\frac{v-a}{r}\right)^{n-1}\frac{a}{v^2} = \frac{na(v-a)^{n-1}}{v^{n+1}} = \frac{n(v-(v-a))(v-a)^{n-1}}{v^{n+1}} = \frac{n(v-a)^{n-1}}{v^n} - \frac{n(v-a)^n}{v^{n+1}}\,.$$

We then have

$$\mathrm{fn}'(v) = -n\lambda\frac{(v-a)^n}{v^{n+1}} + (n-1)\lambda\frac{d}{dv}\left[\frac{(v-a)}{v}\int_{t=a}^{t=v}\frac{(t-a)^{n-1}}{t^n}dt\right]$$
$$= -n\lambda\frac{(v-a)^n}{v^{n+1}} + (n-1)\lambda\left[\frac{(v-a)}{v}\frac{(v-a)^{n-1}}{v^n} + \frac{a}{v^2}\int_{t=a}^{t=v}\frac{(t-a)^{n-1}}{t^n}dt\right]$$
$$= \lambda\left[-\frac{(v-a)^n}{v^{n+1}} + \frac{(n-1)a}{v^2}\int_{t=a}^{t=v}\frac{(t-a)^{n-1}}{t^n}dt\right]\,.$$

Therefore, we have $\mathrm{fn}'(v) \leq 0$ if and only if

$$\int_a^v\frac{(t-a)^{n-1}}{t^n}dt \leq \frac{1}{(n-1)a}\frac{(v-a)^n}{v^{n-1}}\,.$$

We can prove this inequality as follows. Both sides are zero for $v = a$, so it is sufficient to show that the derivative of the LHS is $\leq$ the derivative of the RHS. This is true because the derivative of the LHS is $(v-a)^{n-1}/v^n$ and the derivative of the RHS is

$$\frac{1}{(n-1)a}\frac{v^{n-1}n(v-a)^{n-1} - (v-a)^n(n-1)v^{n-2}}{v^{2n-2}} = \frac{1}{(n-1)a}\frac{(v-a)^{n-1}}{v^n}(nv - (n-1)(v-a))$$
$$= \frac{(v-a)^{n-1}}{v^n} + \frac{(v-a)^{n-1}}{(n-1)av^{n-1}} \geq \frac{(v-a)^{n-1}}{v^n}\,.$$

Therefore, we have proved that $\mathrm{fn}(v)$ is decreasing in v.

To show that the equation $\mathrm{fn}(v^*) = n\left(\frac{b-a}{b}\right)^n$ has a unique solution $v^* \in [a, b]$, it is sufficient to

show that $\text{fn}(a) \geq n \left(\frac{b-a}{b}\right)^n \geq \text{fn}(b)$, or

$$n \int_a^b \left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}}\right] dt \geq n \left(\frac{b-a}{b}\right)^n$$

$$\geq \left(\frac{b-a}{b}\right)^n + \frac{(n-1)(b-a)}{b}\lambda \int_a^b \frac{(t-a)^{n-1}}{t^n} dt.$$

The first inequality

$$\int_a^b \left[\frac{(t-a)^{n-1}}{t^n} - (1-\lambda)\frac{(t-a)^n}{t^{n+1}}\right] dt \geq \left(\frac{b-a}{b}\right)^n$$

is true by the definition of k_h and $a/b \leq k_h$. The second inequality is equivalent to

$$\lambda \int_{t=a}^{t=b} \frac{(t-a)^{n-1}}{t^n} dt \leq \frac{(b-a)^{n-1}}{b^{n-1}},$$

which is true by the definition of k_l and $a/b \geq k_l$.

We also conclude from the above that as $a/b \uparrow k_h$ we have $r^* \downarrow a$, while as $a/b \downarrow k_l$, we have $r^* \uparrow b$.

Now we will show that $\alpha \in [0, 1/n]$ and as $a/b \uparrow k_h$ we have $\alpha \uparrow 1/n$, while as $a/b \downarrow k_l$, we have $\alpha \downarrow 0$.

We will first show that for any $v \in [a, b]$, we have $\int_a^v \frac{(t-a)^{n-1}}{t^n} dt \leq \frac{1}{\lambda}\frac{(v-a)^{n-1}}{v^{n-1}}$. Let $\text{fn}(v) := \int_a^v \frac{(t-a)^{n-1}}{t^n} dt - \frac{1}{\lambda}\frac{(v-a)^{n-1}}{v^{n-1}}$. (We overload the fn notation here — it has nothing to do with the earlier fn; it is just a shorthand that we discard after we finish proving the technical statement.) Note that $\text{fn}(a) = 0$ and $\text{fn}(b) \leq 0$ by the definition of $a/b \geq k_l$. We have

$$\text{fn}'(v) = \frac{(v-a)^{n-1}}{v^n} - \frac{1}{\lambda}(n-1)\frac{(v-a)^{n-2}}{v^{n-2}}\frac{a}{v^2} = \frac{(v-a)^{n-2}}{v^n}\left(v - \left(1 + \frac{n-1}{\lambda}\right)a\right).$$

So $\text{fn}(v)$ is decreasing for $v \leq \left(1 + \frac{n-1}{\lambda}\right)a$ and increasing for $v \geq \left(1 + \frac{n-1}{\lambda}\right)a$. Regardless of whether $b \leq \left(1 + \frac{n-1}{\lambda}\right)a$ or not, we have $\text{fn}(v) \leq \max(\text{fn}(a), \text{fn}(b)) \leq 0$, and we are done. Note also that as $a/b \downarrow k_l$, we have $r^* \uparrow b$ so every inequality here approaches equality, and $\alpha \downarrow 0$.

From the defining equation we have

$$1 - n\alpha = \lambda \frac{(r^*)^{n-1}}{(r^* - a)^{n-1}} \int_a^{r^*} \frac{(t-a)^{n-1}}{t^n} dt,$$

but by the lemma we have just proved,

$$\int_a^{r^*} \frac{(t-a)^{n-1}}{t^n} dt \leq \frac{1}{\lambda}\frac{(r^* - a)^{n-1}}{(r^*)^{n-1}},$$

so $1 - n\alpha \leq 1$, which implies $\alpha \geq 0$.

From the same equation, it is clear that $\frac{(r^*)^{n-1}}{(r^* - a)^{n-1}} \int_a^{r^*} \frac{(t-a)^{n-1}}{t^n} dt \geq 0$, so $1 - n\alpha \geq 0$, so $\alpha \leq 1/n$.

Furthermore, as $a/b \uparrow k_h$, we have $r^* \downarrow a$ so

$$1 - n\alpha = \lambda \frac{(r^*)^{n-1}}{(r^*-a)^{n-1}} \int_a^{r^*} \frac{(t-a)^{n-1}}{t^n} dt \leq \lambda \frac{(r^*)^{n-1}}{(r^*-a)^{n-1}} \int_a^{r^*} \frac{(t-a)^{n-1}}{a^n} dt = \lambda \frac{(r^*)^{n-1}(r^*-a)}{na^n} \downarrow 0 \,,$$

so $\alpha \uparrow 1/n$.

We conclude that for $k_l \leq a/b \leq k_h$, there is a unique valid solution (r^*, α). Furthermore, as $a/b \uparrow k_h$ we have $r^* \downarrow a$ and $\alpha \uparrow 1/n$, while as $a/b \downarrow k_l$, we have $r^* \uparrow b$ and $\alpha \downarrow 0$, as desired. $\qquad\square$

Proposition B-2 gives a unifying description of the mechanism across three support information regimes and makes it clear that the intermediate regime interpolates between the SPA regime and the POOL regime.

We will now use (ODE-g-1) and (ODE-g-2) given in Proposition B-2 to check that (FOC) and (SOC) hold.

Checking (FOC) We first consider the case $v \leq v^*$, where (ODE-g-1) applies. We only need to check this case in the low information $(a/b \leq k_l)$ and moderate information $(k_l \leq a/b \leq k_h)$ regimes, because this case becomes vacuous $(v^* = a)$ in the high information $(a/b \geq k_h)$ regime. In both of these regimes, $F^*(v) = 1 - r^*/v$. Therefore, the (FOC) equation is

$$\left[-\lambda - (n-1)(g^*(v) - \alpha) + v(g^*)'(v) \right] \left(1 - \frac{r^*}{v} \right) + (n-1)(g^*(v) - \alpha)$$

$$= \left(1 - \frac{r^*}{v} \right) \left[-\lambda + \frac{(n-1)r^*}{v(v-r^*)}(g^*(v) - \alpha) + v(g^*)'(v) \right] = 0 \,,$$

where the last equality holds by (ODE-g-1).

Now we consider the case $v \geq v^*$ where (ODE-g-1) applies. We only need to check this case in the high information $(a/b \geq k_h)$ and moderate information $(k_l \leq a/b \leq k_h)$ regimes, because this case becomes vacuous $(v^* = b)$ in the low information $(a/b \geq k_h)$ regime. In both regimes, $F^*(v) = 1 - (a - \phi_0)/(v - \phi_0) = (v-a)/(v-\phi_0)$. ($\phi_0 = 0$ for the moderate information regime, and $\phi_0 > 0$ for the high information regime.) Therefore, the (FOC) is

$$\left[-\lambda - (n-1)\left(g^*(v) - \frac{1 - g^*(v)}{n-1} \right) + v \cdot 0 \right] \left(\frac{v-a}{v-\phi_0} \right)$$

$$+ (n-1)\left[g^*(v) - \frac{1 - g^*(v)}{n-1} - (v-a)\left(\frac{-(g^*)'(v)}{n-1} \right) \right]$$

$$= (-\lambda - (n-1)g^*(v) - 1 + g^*(v)) \left(\frac{v-a}{v-\phi_0} \right) + (n-1)g^*(v) - 1 + g^*(v) + (v-a)(g^*)'(v)$$

$$= (v-a)(g^*)'(v) - \left(\frac{v-a}{v-\phi_0} \right) \lambda + (ng^*(v) - 1)\left(\frac{a-\phi_0}{v-\phi_0} \right)$$

$$= (v-a)\left(\frac{1}{v-a} - \frac{1-\lambda}{v-\phi_0} - \frac{n(a-\phi_0)}{(v-\phi_0)(v-a)}g^*(v) \right) - \left(\frac{v-a}{v-\phi_0} \right) \lambda + (ng^*(v) - 1)\left(\frac{a-\phi_0}{v-\phi_0} \right) = 0 \,,$$

where the second-to-last equality holds by (ODE-g-1).

Checking (SOC). The following applies whether we are in the regime $a/b \leq k_l$, $a/b \geq k_h$, or $k_l \leq a/b \leq k_h$.

In the case $v \leq v^*$, we have $g_u^*(v) = g^*(v)$ and $g_d^*(v) = \alpha$, so (SOC) reduces to $g^*(v) - \alpha > 0$, which is true by definition of g^* in the interior.

In the case $v \geq v^*$, we have $g_u^*(v) = g^*(v)$ and $g_d^*(v) = (1 - g^*(v))/(n-1)$, so (SOC) reduces to $ng^*(v) - 1 + (v-a)(g^*)'(v) > 0$.

Here, g^* satisfies (ODE-g-2) with $\phi_0 = 0$ in the case $k_l \leq a/b \leq k_h$ and $\phi_0 \geq 0$ in the case $a/b \geq k_h$. Substituting $(g^*)'(v) = \frac{1}{v-a} - \frac{1-\lambda}{v-\phi_0} - \frac{n(a-\phi_0)}{(v-\phi_0)(v-a)}g^*(v)$, we get that (SOC) reduces to $ng^*(v) - 1 + (v-a)(g^*)'(v) = \frac{(v-a)}{(v-\phi_0)}(ng^*(v) - 1 + \lambda) > 0$, so we have to prove that $ng^*(v) - 1 + \lambda > 0$. From $v \geq v^*$ we have $g^*(v) \geq 1 - (n-1)\alpha$, we have $ng^*(v) - 1 + \lambda \geq n(1 - (n-1)\alpha) - 1 + \lambda = (n-1)(1-\alpha) + \lambda \geq 0$, because $\alpha \leq 1, \lambda \geq 0$, with strict inequality everywhere but the boundary, as desired.

B.2 Proofs for Section 3.2

Proof of Proposition 4. Suppose for the sake of contradiction that the SPA regime is possible. By Theorem 1, the λ-regret

$$-1 + (1 - k_l)^n + \lambda\left(1 - \int_{t=k_l}^{t=1}\left(1 - \frac{k_l}{t}\right)^n dt\right)$$

is zero, while the corresponding λ satisfies

$$\lambda = \frac{(1 - k_l)^{n-1}}{\int_{t=k_l}^{t=1} \frac{(t-k)^{n-1}}{t^n}dt}.$$

Substituting the expression of λ gives

$$-1 + (1 - k_l)^n + (1 - k_l)^{n-1}\frac{1 - \int_{t=k_l}^{t=1}\left(1 - \frac{k_l}{t}\right)^n dt}{\int_{t=k_l}^{t=1} \frac{(t-k_l)^{n-1}}{t^n}dt} = 0. \tag{B-1}$$

Let $f(k_l)$ be the left hand side of (B-1). We will derive a contradiction by showing that $f(k_l) > 0$ for all $0 < k_l \leq 1$. We will prove this by viewing $k_l \in (0, 1]$ as a free variable. Let

$$I(k_l) = \int_{t=k_l}^{t=1} \frac{(t - k_l)^{n-1}}{t^n}dt = \sum_{i=n}^{\infty} \frac{1}{i}(1 - k_l)^i.$$

Note that

$$I'(k_l) = \sum_{i=n}^{\infty}(1 - k_l)^{i-1}(-1) = -\frac{(1 - k_l)^{n-1}}{1 - (1 - k_l)} = -\frac{(1 - k_l)^{n-1}}{k_l}.$$

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Also, by integration by parts,

$$\int_{t=k_l}^{t=1} \left(1 - \frac{k_l}{t}\right)^n dt = \left[\left(1 - \frac{k_l}{t}\right)^n t\right]_{t=k_l}^{t=1} - \int_{t=k_l}^{t=1} tn \left(1 - \frac{k_l}{t}\right)^{n-1} \frac{k}{t^2} dt = (1 - k_l)^n - nkI(k_l) .$$

Therefore, both integrals in $f(k_l)$ can be written in terms of $I(k_l)$. We want to show that

$$-1 + (1 - k_l)^n + (1 - k_l)^{n-1} \frac{1 - (1 - k_l)^n + nkI(k_l)}{I(k_l)} > 0 .$$

This is equivalent to

$$I(k_l) < \frac{1 - (1 - k_l)^n}{(1 - k_l)^{-(n-1)} - (1 + (n - 1)k_l)} .$$

Note that $(1 - k_l)^{-(n-1)} > (1 + (n - 1)k_l)$, so the above manipulation is valid, and both sides of the inequality are positive. Let

$$g(k_l) = I(k_l) - \frac{1 - (1 - k_l)^n}{(1 - k_l)^{-(n-1)} - (1 + (n - 1)k_l)} .$$

It is clear from the integral definition of $I(k_l)$ that $\lim_{k_l \uparrow 1} I(k_l) = 1$. We now compute, by L'Hopital's rule,

$$\lim_{k_l \uparrow 1} \frac{1 - (1 - k_l)^n}{(1 - k_l)^{-(n-1)} - (1 + (n - 1)k_l)} = \lim_{k_l \uparrow 1} \frac{-n(1 - k_l)^{n-1}(-1)}{(-n + 1)(1 - k_l)^{-n}(-1) - (n - 1)} = 0 .$$

Therefore, $\lim_{k_l \uparrow 1} g(k_l) = 0$ To prove that $g(k_l) < 0$ it is sufficient to prove that $g(k_l)$ is strictly increasing in k_l, i.e., $g'(k_l) > 0$. This is very convenient because $I'(k_l)$ does not involve an integral. We compute

$$g'(k_l) = I'(k_l) - \frac{d}{dk_l} \left[\frac{1 - (1 - k_l)^n}{(1 - k_l)^{-(n-1)} - (1 + (n - 1)k_l)}\right]$$

$$= -\frac{(1 - k_l)^{n-1}}{k_l} - \frac{1}{[(1 - k_l)^{-n+1} - (1 + (n - 1)k_l)]^2} \times \Bigg\{$$

$$\left[(1 - k_l)^{-n+1} - (1 + (n - 1)k_l)\right] \left[-n(1 - k_l)^{n-1}(-1)\right]$$

$$- [1 - (1 - k_l)^n] \left[(-n + 1)(1 - k_l)^{-n}(-1) - (n - 1)\right] \Bigg\}$$

$$= -\frac{(1 - k_l)^{n-1}}{k_l} + \frac{\frac{(n-1)(1-(1-k_l)^n)^2}{(1-k_l)^n} - n \left[1 - (1 + (n - 1)k_l)(1 - k_l)^{n-1}\right]}{\frac{1}{(1-k_l)^{2n-2}} \left[1 - (1 + (n - 1)k_l)(1 - k_l)^{n-1}\right]^2} .$$

Then $g'(k_l) > 0$ is equivalent to

$$(n - 1)k_l(1 - (1 - k_l)^n)^2 - n(1 - k_l)^n k_l \left[1 - (1 + (n - 1)k_l)(1 - k_l)^{n-1}\right]$$

$$-(1 - k_l) \left[1 - (1 + (n - 1)k_l)(1 - k_l)^{n-1}\right]^2 > 0 .$$

App-16

Let $x = 1 - k_l \in [0, 1)$, algebraic simplification gives that the left hand side is

$$(n-1)(1-x)(1-x^n)^2 - nx^n(1-x)\left[1 - (1 + (n-1)(1-x))x^{n-1}\right]$$
$$- \left[1 - (1 + (n-1)(1-x))x^{n-1}\right]^2$$
$$= (1 - x^n)(n - 1 - nx + x^n).$$

It is clear that $1 - x^n > 0$. We also have

$$n - 1 - nx + x^n = n(1-x) - (1 - x^n) = (1 - x)\left(n - \sum_{i=0}^{n-1} x^i\right) > 0.$$

We conclude that the optimal maximin ratio mechanism is never in the SPA regime. □

B.3   Proofs for Section 3.3

Proof of Corollary 1. We will use the equivalent formulation of Theorem 4. k_l is a solution to $\lambda \log(1/k_l) = 1$, so $k_l = \exp(-1/\lambda)$. k_h is a solution to $(1 - \lambda)(1 - k_h) + \lambda \log(1/k_h) = (1 - k_h)$ or $\log(1/k_h) = (1 - k_h)$, so $k_h = 1$.

Therefore, we only need to consider the regime $a/b \leq k_l$ and $k_l \leq a/b \leq k_h$.

For $a/b \leq k_l$, the regret is b/e. For $a/b \geq k_l$, we know that (v^*, α) is a solution to the following system of equations

$$1 - \alpha = \lambda \log\left(\frac{v^*}{a}\right)$$
$$\frac{b-a}{b} - \frac{v^* - a}{v^*} = \lambda \log\left(\frac{b}{v^*}\right) - \frac{(1-\lambda)a}{b} + \frac{(1-\lambda)a}{v^*}.$$

The solution to this is $v^* = b$ and $\alpha = 1 - \lambda \log(b/a)$.

We now substitute $n = 1$ to the minimax λ-regret and $g_u^*(v)$ expressions. (We have $\Phi(v) = g_u^*(v)$, the price distribution CDF.) For $a/b \leq k_l$, the minimax λ-regret is

$$-(1-\lambda)b + \left[(1 - k_l) - \lambda \int_{\tilde{r}=k_l}^{\tilde{r}=1} \left(1 - \frac{k_l}{\tilde{r}}\right) d\tilde{r}\right]b$$
$$= -(1-\lambda)b + \left[(1 - k_l) - \lambda(1 - k_l + k_l \log(k_l))\right]b$$
$$= (-1 + \lambda + 1 - e^{-1/\lambda} - \lambda + \lambda e^{-1/\lambda} + e^{-1/\lambda})b$$
$$= \lambda e^{-1/\lambda}b,$$

and the price CDF is

$$g_u^*(v) = \lambda \frac{v^{n-1}}{(v - r^*)^{n-1}} \int_{t=r^*}^{t=v} \frac{(t - r^*)^{n-1}}{t^n} dt = \lambda \int_{t=r^*}^{t=v} \frac{1}{t} dt = \lambda \log\left(\frac{v}{r^*}\right) = \lambda \log\left(\frac{v}{e^{-1/\lambda}b}\right)$$
$$= 1 + \lambda \log\left(\frac{v}{b}\right) \quad \text{for } v \in [r^*, b].$$

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For $a/b \geq k_l$, the minimax λ-regret is

$$-(1-\lambda)b + b\left(1 - \frac{a}{b}\right) + (b-a)\alpha(1-1) - \lambda \int_{v=a}^{v=b}\left(1 - \frac{a}{v}\right)dv$$

$$= -(1-\lambda)b + (b-a) - \lambda(b - a - a\log(b/a)) = -a + \lambda a + \lambda\log(b/a),$$

and the price CDF is

$$g_u^*(v) = \alpha + \lambda\left(\frac{v}{v-a}\right)^{n-1}\int_{t=a}^{t=v}\frac{(t-a)^{n-1}}{t^n}dt = 1 - \lambda\log\left(\frac{b}{a}\right) + \lambda\int_{t=a}^{t=v}\frac{1}{t}dt$$

$$= 1 - \lambda\log\left(\frac{b}{a}\right) + \lambda\log\left(\frac{v}{a}\right) = 1 + \lambda\log\left(\frac{v}{b}\right) \text{ for } v \in [a,b].$$

We get the minimax regret by substituting $\lambda = 1$, which is b/e for $a/b \leq 1/e$ and $a\log(b/a)$ for $a/b \geq 1/e$. The maximin ratio is λ such that the minimax λ-regret is zero. Because $\lambda e^{-1/\lambda}b > 0$ always, minimax λ-regret cannot be zero in this regime. In the other regime, $k_l \leq a/b \leq k_h$, The λ such that $-a + \lambda a + \lambda\log(b/a) = 0$ is $\lambda^* = 1/(1 + \log(b/a))$, so this is the maximin ratio value, and the corresponding price CDF is

$$\Phi^*(v) = 1 + \lambda^*\log\left(\frac{v}{b}\right) = 1 + \frac{\log(v/b)}{1 + \log(b/a)} \text{ for } v \in [a,b].\ \square$$

C   Proofs and Discussions from Section 4

C.1   Proofs and Discussions from Section 4.1

Before we prove the main theorem characterizing the minimax λ-regret standard mechanism (Theorem 2), we first derive the regret expression of generous SPA GenSPA(Φ).

Proposition C-3. *The regret of the mechanism* GenSPA(Φ) *under distribution $\boldsymbol{F}$ that is n i.i.d. with marginal F is*

$$a(\lambda - \Phi(a)) + \int_{v=a}^{v=b}(\lambda - \Phi(v) - v\Phi'(v))(1 - F(v)^n) + \Phi(v)(nF(v)^{n-1} - nF(v)^n)dv$$

$$+ F(a)^{n-1}(b-a)(n - (n-1)F(a))$$

$$- F(a)^{n-1}\int_{v=a}^{v=b}\left((n - (n-1)F(a))\Phi(v) + v(nF(v) - (n-1)F(a))\Phi'(v)\right)dv.$$

Proof of Proposition C-3. The λ-regret expression, pointwise at $\boldsymbol{v}$, is

$$\lambda v^{(1)} - \left(v^{(1)}\Phi(v^{(1)}) - \int_{t=v^{(2)}}^{t=v^{(1)}}\Phi(t)dt\right)\mathbf{1}(v^{(2)} > a) - \left(v^{(1)}\cdot 1 - \int_{t=v^{(2)}}^{t=v^{(1)}}1dt\right)\mathbf{1}(v^{(2)} = a)$$

$$= \lambda v^{(1)} - \left(v^{(1)}\Phi(v^{(1)}) - \int_{t=v^{(2)}}^{t=v^{(1)}}\Phi(t)dt\right)\mathbf{1}(v^{(2)} > a) - \left(v^{(2)}\right)\mathbf{1}(v^{(2)} = a).$$

Writing $\mathbf{1}(v^{(2)} > a) = 1 - \mathbf{1}(v^{(2)} = a)$, the regret expression becomes

$$\left(\lambda v^{(1)} - v^{(1)}\Phi(v^{(1)}) + \int_{t=v^{(2)}}^{t=v^{(1)}} \Phi(t)dt\right) + \left(v^{(1)}\Phi(v^{(1)}) - \int_{t=v^{(2)}}^{t=v^{(1)}} \Phi(t)dt - v^{(2)}\right)\mathbf{1}(v^{(2)} = a)$$

$$= \left(\lambda v^{(1)} - v^{(1)}\Phi(v^{(1)}) + \int_{t=v^{(2)}}^{t=v^{(1)}} \Phi(t)dt\right) + \left(v^{(1)}\Phi(v^{(1)}) - \int_{t=a}^{t=v^{(1)}} \Phi(t)dt - a\right)\mathbf{1}(v^{(2)} = a).$$

We will now calculate the distribution of $v^{(1)}|v^{(2)} = a$.

We note that, for any $x \in [a, b]$,

$$\Pr(v^{(1)} > x, v^{(2)} = a) = \Pr(\ n - 1\ v\text{'s are } a,\ \text{one is} > x) = nF(a)^{n-1}F((x, b])) = nF(a)^{n-1}(1 - F(x)),$$

and

$$\Pr(v^{(2)} = a) = \Pr(\ \text{exactly } n - 1 \text{ are } a\ ) + \Pr(\ \text{exactly } n \text{ are } a\ )$$
$$= nF(a)^{n-1}(1 - F(a)) + F(a)^n = F(a)^{n-1}(n - (n-1)F(a)).$$

If $\Pr(v^{(2)} = a) > 0$ (that is, if F has an atom at a), then dividing the two equations gives

$$\Pr(v^{(1)} > x|v^{(2)} = a) = \frac{n(1 - F(x))}{n - (n-1)F(a)} \quad \text{for } x \in [a, b].$$

This gives us the CDF of $v^{(1)}|v^{(2)} = a$ as

$$\tilde{F}(x) := \Pr(v^{(1)} \le x|v^{(2)} = a) = 1 - \frac{n(1 - F(x))}{n - (n-1)F(a)} = \frac{nF(x) - (n-1)F(a)}{n - (n-1)F(a)} \quad \text{for } x \in [a, b].$$

(If $\Pr(v^{(2)} = a) = 0$, then the expression we want to evaluate is zero and we won't need this conditional distribution anyway.)

We can now calculate the expected regret. The first term is exactly the expected regret expression from SPA:

$$\mathbb{E}_{\boldsymbol{v}\sim F}\left[\lambda v^{(1)} - \left(v^{(1)}\Phi(v^{(1)}) - \int_{t=v^{(2)}}^{t=v^{(1)}} \Phi(t)dt\right)\right]$$

$$= a(\lambda - \Phi(a)) + \int_{v=a}^{v=b} (\lambda - \Phi(v) - v\Phi'(v))(1 - F(v)^n) + \Phi(v)(nF(v)^{n-1} - nF(v)^n)dv.$$

The second term can be written as

$$\mathbb{E}_{\boldsymbol{v}\sim F}\left[\left(v^{(1)}\Phi(v^{(1)}) - \int_{t=v^{(2)}}^{t=v^{(1)}}\Phi(t)dt - a\right)\mathbf{1}(v^{(2)} = a)\right]$$

$$= \Pr(v^{(2)} = a)\mathbb{E}\left[\left(v^{(1)}\Phi(v^{(1)}) - \int_{t=a}^{t=v^{(1)}}\Phi(t)dt - a\right)\bigg|v^{(2)} = a\right]$$

$$= F(a)^{n-1}(n - (n-1)F(a))\left[-a + \int_{v\in[a,b]}\left(v\Phi(v) - \int_{t=a}^{t=v}\Phi(t)dt\right)d\tilde{F}(v)\right].$$

We now use the following integration-by-part-like statements.

$$\int_{w\in[a,b]}h(w)dG(w) = h(b) - \int_{v=a}^{v=b}h'(v)G(v)dv.$$

Therefore,

$$\int_{v\in[a,b]}v\Phi(v)d\tilde{F}(v)$$

$$= b\Phi(b) - \int_{v=a}^{v=b}(\Phi(v) + v\Phi'(v))\tilde{F}(v)dv$$

$$= b - \int_{v=a}^{v=b}(\Phi(v) + v\Phi'(v))\frac{nF(v) - (n-1)F(a)}{n - (n-1)F(a)}dv.$$

We also have

$$\int_{v\in[a,b]}\int_{t=a}^{t=v}\Phi(t)dtd\tilde{F}(v)$$

$$= \int_{t=a}^{t=b}\int_{v\in(t,b]}\Phi(t)d\tilde{F}(v)dt$$

$$= \int_{t=a}^{t=b}\Phi(t)(1 - \tilde{F}(t))dt$$

$$= \int_{v=a}^{v=b}\Phi(v)\frac{n(1 - F(v))}{n - (n-1)F(a)}dv.$$

Therefore, the second term is

$$F(a)^{n-1}(n - (n-1)F(a))\left[-a + b - \int_{v=a}^{v=b}(\Phi(v) + v\Phi'(v))\frac{nF(v) - (n-1)F(a)}{n - (n-1)F(a)} - \int_{v=a}^{v=b}\Phi(v)\frac{n(1 - F(v))}{n - (n-1)F(a)}dv\right]$$

$$= F(a)^{n-1}(n - (n-1)F(a))\left[b - a - \int_{v=a}^{v=b}\left(\Phi(v) + v\Phi'(v)\frac{nF(v) - (n-1)F(a)}{n - (n-1)F(a)}\right)dv\right]$$

$$= F(a)^{n-1}\left[(b-a)(n - (n-1)F(a)) - \int_{v=a}^{v=b}\left((n - (n-1)F(a))\Phi(v) + v(nF(v) - (n-1)F(a))\Phi'(v)\right)dv\right].$$

We therefore have the regret expression

$$= a(\lambda - \Phi(a)) + \int_{v=a}^{v=b} (\lambda - \Phi(v) - v\Phi'(v))(1 - F(v)^n) + \Phi(v)(nF(v)^{n-1} - nF(v)^n)dv$$

$$+ F(a)^{n-1}(b - a)(n - (n-1)F(a))$$

$$- F(a)^{n-1} \int_{v=a}^{v=b} \left((n - (n-1)F(a))\Phi(v) + v(nF(v) - (n-1)F(a))\Phi'(v)\right)dv.$$

Note that the extra term (in the second line) is linear in F. □

We are now ready to state and prove the main theorem.

Theorem 5. *Fix n and $\lambda \in (0,1]$ and let $k = a/b \in [0,1)$. Define k_l as in Theorem 2. The minimax λ-regret problem $R_\lambda(\mathcal{M}_{\mathrm{std}}, \mathcal{F}_{\mathrm{iid}})$ admits the following saddle point (m^*, F^*), depending on a/b, as follows.*

- *For $a/b \le k_l$, the optimal mechanism m^* and worst-case distribution F^* are the same as those of Theorem 1.*

- *For $a/b \ge k_l$, the optimal mechanism m^* is GenSPA(Φ^*) with*

$$\Phi^*(v) = \frac{\int_{t=a}^{t=v} \frac{\lambda(1-c/t)^{n-1}}{t}dt}{(1 - c/v)^{n-1} - (1 - c/a)^{n-1}} \quad \text{for } v \in [a, b],$$

  where $c \in [0, a]$ is a unique constant such that $\Phi^*(c) = 1$. The worst case distribution F^* is an isorevenue distribution defined by $F^*(v) = 1 - c/v$ for $v \in [a, b)$ and $F^*(b) = 1$.

Remark. Note that the worst case distribution has two point masses at $v = a$ and $v = b$ (of size $1 - c/a$ and c/b respectively), whereas the worst case distributions of $\mathcal{M}_{\mathrm{all}}$ in Theorem 1 each only has one point mass at $v = b$. Also, the distributions Φ and Ψ of the reserve of SPA and threshold of POOL in Theorem 1 do not have any point masses, whereas the distribution Φ of GenSPA in Theorem 5 has a point mass at $v = a$ of size $\Phi^*(a) = \lim_{v\downarrow a} \Phi^*(v) = \lambda(a/c - 1)/(n-1)$.

Proof of Theorem 5. Seller's saddle is that, fixing F^*, the given mechanism m^* gives the lowest regret over all standard mechanisms. Because F^* is fixed, this is equivalent to that m^* maximizes expected revenue under F^*. This is a standard Bayesian mechanism design problem, and we check with Monteiro and Svaiter [2010] that m^* is indeed optimal, even over all DSIC mechanisms. Henceforth, we will focus on Nature's Saddle.

Note that if $a/b \le k_l$, then Theorem 1 immediately tells us that the same SPA(Φ^*) is optimal over $\mathcal{M}_{\mathrm{all}}$, and thus over $\mathcal{M}_{\mathrm{std}}$ also. Henceforth we assume $a/b \ge k_l$.

We have the regret expression from Proposition C-3:

$$a(\lambda - \Phi(a)) + \int_{v=a}^{v=b} (\lambda - \Phi(v) - v\Phi'(v))(1 - F(v)^n) + \Phi(v)(nF(v)^{n-1} - nF(v)^n)dv$$

$$+ F(a)^{n-1}(b-a)(n - (n-1)F(a))$$

$$- F(a)^{n-1}\int_{v=a}^{v=b} \left((n - (n-1)F(a))\Phi(v) + v(nF(v) - (n-1)F(a))\Phi'(v)\right) dv.$$

The additional term is linear in F, so if pointwise optimization gives a global maximum then, it will also give a global maximum now. The first-order condition gives

$$(\lambda - \Phi(v) - v\Phi'(v))(-nF(v)^{n-1}) + \Phi(v)(n(n-1)F(v)^{n-2} - n^2F(v)^{n-1}) - nvF(a)^{n-1}\Phi'(v) = 0,$$

or

$$v(F(v)^{n-1} - F(a)^{n-1})\Phi'(v) + (n-1)F(v)^{n-2}(1 - F(v))\Phi(v) = \lambda F(v)^{n-1}.$$

We want Φ such that $F(v) = 1 - c/v$ is a solution to that equation. With $F(v) = 1 - c/v$ we have $F'(v) = \frac{c}{v^2} = \frac{1-F(v)}{v}$. So

$$(F(v)^{n-1} - F(a)^{n-1})\Phi'(v) + (n-1)F(v)^{n-2}F'(v)\Phi(v) = \frac{\lambda F(v)^{n-1}}{v},$$

or

$$\frac{d}{dv}\left[(F(v)^{n-1} - F(a)^{n-1})\Phi(v)\right] = \frac{\lambda F(v)^{n-1}}{v}.$$

Because $(F(v)^{n-1} - F(a)^{n-1})\Phi(v)$ is zero when $v = a$, we have

$$\Phi(v) = \frac{\int_{t=a}^{t=v} \frac{\lambda F(t)^{n-1}}{t}dt}{F(v)^{n-1} - F(a)^{n-1}} = \frac{\int_{t=a}^{t=v} \frac{\lambda\left(1 - \frac{c}{t}\right)^{n-1}}{t}dt}{\left(1 - \frac{c}{v}\right)^{n-1} - \left(1 - \frac{c}{a}\right)^{n-1}}.$$

We will now show that, for any $c \in (0, a]$, $\Phi(v)$ is increasing in v.

Let $\tilde{v} = 1 - c/v$, $\tilde{a} = 1 - c/a$, and use the substitution $u = 1 - c/t$, $t = \frac{c}{1-u}$, $dt = \frac{c}{(1-u)^2}du$ to get

$$\frac{1}{\lambda}\Phi(v) = \frac{\int_{u=\tilde{a}}^{u=\tilde{v}} \frac{u^{n-1}}{1-u}du}{\tilde{v}^{n-1} - \tilde{a}^{n-1}}.$$

Showing that $\Phi(v)$ is increasing in v is equivalent to showing that the right hand side is increasing in $\tilde{v} = 1 - c/v$. From $v \geq a$ we have $\tilde{v} \geq \tilde{a}$. The derivative with respect to $\tilde{v}$ of the right hand side is

$$\frac{(\tilde{v}^{n-1} - \tilde{a}^{n-1})\frac{\tilde{v}^{n-1}}{1-\tilde{v}} - \left(\int_{u=\tilde{a}}^{u=\tilde{v}} \frac{u^{n-1}}{1-u}du\right)(n-1)\tilde{v}^{n-2}}{(\tilde{v}^{n-1} - \tilde{a}^{n-1})^2}.$$

This expression is nonnegative if and only if

$$\int_{u=\tilde{a}}^{u=\tilde{v}} \frac{u^{n-1}}{1-u} du \le \frac{\tilde{v}^n - \tilde{a}^{n-1}\tilde{v}}{(n-1)(1-\tilde{v})}.$$

It is clear that this is true if it holds for $\tilde{v} = a$ and the derivative of LHS is $\le$ the derivative of RHS. For $\tilde{v} = a$, both sides are zero, so the inequality holds. The condition that the derivative of LHS is $\le$ the derivative of RHS is

$$\frac{\tilde{v}^{n-1}}{1-\tilde{v}} \le \frac{1}{n-1} \frac{(1-\tilde{v})(n\tilde{v}^{n-1} - \tilde{a}^{n-1}) - (\tilde{v}^n - \tilde{a}^{n-1}\tilde{v})(-1)}{(1-\tilde{v})^2}.$$

This is equivalent to

$$(n-1)\tilde{v}^{n-1} - (n-1)\tilde{v}^n \le n\tilde{v}^{n-1} - \tilde{a}^{n-1} - n\tilde{v}^n + \tilde{a}^{n-1}\tilde{v} + \tilde{v}^n - \tilde{a}^{n-1}\tilde{v},$$

or

$$\tilde{a}^{n-1} \le \tilde{v}^{n-1},$$

which is true because $\tilde{v} \ge \tilde{a}$.

Lastly, we will show that if $a/b \ge k_l$, then there is a $c \in (0, a]$ such that $\Phi(b) = 1$, making this a valid solution. From

$$\Phi(b) = \frac{\int_{t=a}^{t=b} \frac{\lambda\left(1-\frac{c}{t}\right)^{n-1}}{t} dt}{\left(1-\frac{c}{b}\right)^{n-1} - \left(1-\frac{c}{a}\right)^{n-1}},$$

consider the right hand side as a function of c. The numerator is clearly a decreasing function of c, because $\frac{\lambda\left(1-\frac{c}{t}\right)^{n-1}}{t}$, for each fixed t, is a decreasing function of c. The denominator is an increasing function of c because

$$\left(1-\frac{c}{b}\right)^{n-1} - \left(1-\frac{c}{a}\right)^{n-1} = c\left(\frac{1}{a} - \frac{1}{b}\right)\left(\sum_{k=0}^{n-1}\left(1-\frac{c}{b}\right)^k\left(1-\frac{c}{a}\right)^{n-1-k}\right),$$

and each of c, $1-\frac{c}{b}$, $1-\frac{c}{a}$ are positive and increasing in c, and $\frac{1}{a} - \frac{1}{b} > 0$.

Therefore, the right hand side is decreasing in c. As $c \downarrow 0$, the numerator converges to $\int_{t=a}^{t=b} \frac{\lambda}{t} dt = \lambda \log(b/a) > 0$, and the denominator converges to 0, so the expression converges to $+\infty$. At $c = a$, the expression is

$$\frac{\int_{t=a}^{t=b} \frac{\lambda\left(1-\frac{a}{t}\right)^{n-1}}{t} dt}{\left(1-\frac{a}{b}\right)^{n-1}} \le 1,$$

where the ≤ 1 holds because $a/b \ge k_l$. Therefore, there is a $c \in (0, a]$ such that $\Phi(b) = 1$.

Lastly, we note that

$$\Phi(a) = \lim_{v \downarrow a} \Phi(v) = \lambda \lim_{v \downarrow a} \frac{\frac{\left(1 - \frac{c}{v}\right)^{n-1}}{v}}{(n-1)\left(1 - \frac{c}{v}\right)^{n-2}\frac{c}{v^2}} = \lambda \frac{(a-c)}{(n-1)c},$$

where the second equality holds by L'Hopital's rule. $\qquad\square$

C.2 Proofs and Discussions from Section 4.2

The main theorem in the main text (Theorem 3) is an immediate corollary of the main theorem in this Appendix (Theorem 6). We first outline key challenges of the proof before diving into the full proof of the main theorem.

Key Challenges of the Proof of The Main Theorem. The proof of the moderate information regime is the hardest, and we outline key technical ideas here. Our candidate Φ^* has a point mass Φ_0 at a and a density on $[r^*, b]$, while the candidate worst-case distribution F^* has a point mass F_0, an isorevenue density on $[r^*, b)$, and a point mass at b. The seller's saddle is to find Φ that minimizes the regret $R(\Phi) := R(\Phi, F^*)$ such that $\Phi(v) \in [0, 1]$ is an increasing function. This is different from previous saddle problems because in this case, the increasing condition is *binding*; if we optimize pointwise, the result is nonincreasing, which is infeasible. We write the Lagrangian $\mathcal{L}(\Phi, \mu) = R(\Phi) - \int_a^b \mu(v)d\Phi(v)$. Here, $\mu : [a, b] \to \mathbb{R}_+$ is the dual variable associated with the increasing constraint. Complementary slackness requires that μ^* is zero wherever Φ^* is strictly increasing; this immediately suggests the correct form of Φ^*, and that $\mu^*(v) = 0$ on $[r^*, b]$. Lagrangian optimality requires that Φ^* also minimizes $\mathcal{L}(\Phi, \mu^*)$, which is linear in Φ (as can be made explicit by integration by parts on the $d\Phi(v)$ term). We satisfy this by requiring that the coefficients of every $\Phi(v)$ term to be zero. These, together with complementary slackness, pin down μ^*. Nature's saddle is significantly simpler because the condition that F is increasing does not bind here, so pointwise optimization (as before) works. The full proof is given in Appendix C.2. $\qquad\square$

Before we proceed to the proof, we derive the regret expression for SPA(Φ), which is just (g_u, g_d) with $g_u = \Phi$ and $g_d \equiv 0$.

Proposition C-4. *Let the mechanism be a second-price auction with random reserve CDF Φ and distribution $\boldsymbol{F}$ with regret $R(\Phi, \boldsymbol{F})$.*

Suppose $\Phi : [a, b] \to [0, 1]$ is absolutely continuous, while $\boldsymbol{F}$ is arbitrary then we have the (Regret-$\boldsymbol{F}$) expression

$$R(\Phi, \boldsymbol{F}) = a(\lambda - \Phi(a)) + \int_{v \in [a,b]} (\lambda - \Phi(v) - v\Phi'(v))(1 - F^{(1)}(v))dv$$

$$+ \int_{v \in [a,b]} \Phi(v)(F^{(2)}(v) - F^{(1)}(v))dv \,.$$

If we further assume that $\boldsymbol{F}$ is n i.i.d. then

$$R(\Phi, \boldsymbol{F}) = a(\lambda - \Phi(a)) + \int_{v \in [a,b]} (\lambda - \Phi(v) - v\Phi'(v))(1 - F(v)^n) dv$$

$$+ \int_{v \in [a,b]} \Phi(v)(nF(v)^{n-1} - nF(v)^n) dv \,.$$

Suppose instead that $\boldsymbol{F}$ is i.i.d. with marginal F that has a density in (a, b), and we denote by $F(\{b\})$ and $\boldsymbol{F}^{(1)}(b)$ the mass at b. Then we have the (Regret-Φ) expression

$$R(\Phi, F) = \lambda b - a\Phi(a)F(a)^n - (1 - (1 - f_b)^n)b\Phi(b) +$$

$$\int_{v=a}^{v=b} -\lambda F(v)^n + \Phi(v)nF(v)^{n-1}(1 - F(v) - vF'(v)) dv \,.$$

Proof of Proposition C-4. These expressions follow immediately by taking $g_u(v) = \Phi(v)$ and $g_d(v) = 0$. $\qquad\square$

We are now ready to state and prove the main theorem.

Theorem 6 (Optimal SPA with Random Reserve). *Fix n and $\lambda \in (0, 1]$ and let $k = a/b \in [0, 1)$. Define k_l and k'_h as in Theorem 3. The minimax λ-regret problem $R_\lambda(\mathcal{M}_{\text{SPA-rand}}, \mathcal{F}_{\text{iid}})$ admits a saddle point $(m^* = \text{SPA}(\Phi^*), F^*)$ which is characterized as follows.*

- *For $a/b \le k_l$, an optimal mechanism m^* and worst-case distribution F^* are the same as those identified in Theorem 1 and its proof.*

- *For $a/b \ge k'_h$, an optimal mechanism is a SPA with no reserve and the worst-case distribution F^* is a two-point distribution with point masses at $v = a$ and $v = b$ with weights $f_a := \frac{n-1}{n-1+\lambda}$ and $f_b := \frac{\lambda}{n-1+\lambda}$.*

- *For $k_l \le a/b \le k'_h$, let $F_0 \in [0, 1)$ be a unique solution to*

$$\frac{\lambda F_0^n}{(n-1)(1-F_0)} - \left(1 - \frac{n - (n-1)F_0}{n}\tilde{a}\right)^{n-1}$$

$$= \lambda \left[\log\left(\frac{n-(n-1)F_0}{n(1-F_0)}\tilde{a}\right) - \sum_{k=1}^{n-1}\frac{1}{k}F_0^k + \sum_{k=1}^{n-1}\frac{1}{k}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^k\right],$$

and

$$r^* = \frac{n - (n-1)F_0}{n(1-F_0)}a,$$

$$\Phi_0 = \frac{\lambda F_0}{(n-1)(1-F_0)},$$

$$c = \frac{n - (n-1)F_0}{n}a,$$

$$d = \frac{F_0^n}{(n-1)(1-F_0)} + \log(1-F_0) + \sum_{k=1}^{n-1}\frac{1}{k}F_0^k \,.$$

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Then, an optimal mechanism (optimal reserve distribution) is

$$\Phi^*(v) = \begin{cases} \Phi_0 & \text{for } v \in [a, r^*] \\ \lambda \left(\frac{v}{v-c}\right)^{n-1} \left[d + \log\left(\frac{v}{c}\right) - \sum_{k=1}^{n-1} \frac{1}{k}\left(\frac{v-c}{v}\right)^k\right] & \text{for } v \in [r^*, b], \end{cases}$$

and a worst-case distribution is given by $F^*(v) = F_0$ for $v \in [a, r^*]$, $F^*(v) = 1 - c/v$ for $v \in [r^*, b)$ and $F^*(b) = 1$.

Proof of Theorem 6. We will prove the 3 cases separately.

Low Information $(a/b \le k_l)$ Regime. We know from Theorem 1 that $\text{SPA}(\Phi^*)$ is minimax optimal over $\mathcal{M}_{\text{all}}$ and so it is also minimax optimal over $\mathcal{M}_{\text{SPA-rand}}$.

High Information $(a/b \ge \frac{\lambda n}{(1+\lambda)n-1})$ Regime. We claim that the minimax optimal mechanism Φ^* in this regime is SPA without reserve and F^* is a two-point distribution with mass $f_a := \frac{n-1}{n-1+\lambda}$ at a and $f_b := \frac{\lambda}{n-1+\lambda}$ at b. Here, $R(\Phi^*, F^*) = -(1-\lambda)b + \left(\frac{n-1}{n-1+\lambda}\right)^{n-1}(b-a)$.

We want to show that $R(\Phi^*, F) \le R(\Phi^*, F^*) \le R(\Phi, F^*)$.

Part 1: $R(\Phi, F^*) \ge R(\Phi^*, F^*)$

We have the regret expression $R(\Phi, F)$ from Proposition C-4:

$$R(\Phi, F) = \lambda b - a\Phi(a)F(a)^n - (1 - (1 - f_b)^n)b\Phi(b)$$
$$+ \int_{v=a}^{v=b} -\lambda F(v)^n + \Phi(v)nF(v)^{n-1}(1 - F(v) - vF'(v))dv.$$

Under F^*, we have $F^*(v) = F(a) = f_a\frac{n-1}{n-1+\lambda}$ for all $v \in (a, b)$, which means every appearance of $F(v)$ becomes a constant:

$$R(\Phi, F^*) = \lambda b - af_a^n\Phi(a) - (1 - f_a^n)b\Phi(b) + \int_{v=a}^{v=b} -\lambda f_a^n + nf_a^{n-1}(1 - f_a)\Phi(v)dv.$$

We have $\Phi(b) \le 1$ and $-\lambda f_a^n + nf_a^{n-1}(1 - f_a)\Phi(v) \ge -\lambda f_a^n + nf_a^{n-1}(1 - f_a)\Phi(a)$, so

$$R(\Phi, F^*) \ge \lambda b - a\Phi(a)f_a^n - (1 - f_a^n)b + (b - a)(-\lambda f_a^n + nf_a^{n-1}(1 - f_a)\Phi(a))$$
$$= -(1-\lambda)b + (nf_a^{n-1} - (n - 1 + \lambda)f_a^n)(b - a) + (1 - \Phi(a))f_a^{n-1}(af_a - (b - a)(n - nf_a)).$$

The last term is ≥ 0 because we require that $af_a - (b - a)(n - nf_a) \ge 0 \Leftrightarrow \frac{a}{b} \ge \frac{n-nf_a}{n-(n-1)f_a} = \frac{\lambda n}{(1+\lambda)n-1}$. We now see that the choice $f_a = \frac{n-1}{n-1+\lambda}$ is chosen so the second term is maximized, and the bound on $\frac{a}{b}$ that is required to make the third term work follows accordingly. Therefore,

$$R(\Phi, F^*) \ge -(1-\lambda)b + (nf_a^{n-1} - (n - 1 + \lambda)f_a^n)(b - a) = -(1-\lambda)b + \left(\frac{n-1}{n-1+\lambda}\right)^{n-1}(b-a).$$

Part 2: $R(\Phi^*, F) \le R(\Phi^*, F^*)$

Because Φ^* is an SPA without reserve, we have $\Phi^*(v) = 1$ for every v. This gives

$$R(\Phi^*, F) = a(\lambda - 1) + \int_{v=a}^{v=b} (\lambda - 1)(1 - F(v)^n) + (nF(v)^{n-1} - nF(v)^n)dv$$

$$= -(1 - \lambda)b + \int_{v=a}^{v=b} nF(v)^{n-1} - (n - 1 + \lambda)F(v)^n dv$$

$$\leq -(1 - \lambda)b + (b - a) \sup_{z \in [0,1]} nz^{n-1} - (n - 1 + \lambda)z^n$$

$$= -(1 - \lambda)b + \left(\frac{n-1}{n-1+\lambda}\right)^{n-1} (b - a) = R(\Phi^*, F^*).$$

Moderate Information ($k_l \leq a/b \leq \frac{\lambda n}{(1+\lambda)n-1}$) Regime. We will exhibit a saddle point with the following structure: Φ^* has a point mass $\Phi^*(a)$ at a, then it is flat on $[a, r^*]$ (that is, $\Phi^*(v) = \Phi^*(a)$ on $[a, r^*]$), then it has a density on $[r^*, b]$, but no point mass at b (so $\Phi^*(b) = 1$). F^* has a point mass $F^*(a)$ at a, then it is flat on $[a, r^*]$, then it is $F^*(v) = 1 - c/v$ on $[r^*, b)$. That is, $F^*(v) = \max(1 - c/v, F^*(a))$ on $v \in [a, b)$. Importantly, we assume that F^* is continuous at r^*, so $1 - c/r^* = F^*(a)$, and this r^* is the same as r^* of Φ^*.

We use the regret expression (Regret-Φ) under F^*:

$$R_n(\boldsymbol{\Phi}, \boldsymbol{F}) = \lambda b - F(a)^n a\Phi(a) - (1 - (1 - f_b)^n)b\Phi(b)$$
$$+ \int_{v=a}^{v=b} -\lambda F(v)^n + \Phi(v)nF(v)^{n-1}(1 - F(v) - vF'(v))dv.$$

Note that $F^*(b^-) = 1 - f_b = 1 - c/b$. We will write $F_0 := F^*(a)$ for convenience. We can then write

$$R_n(\Phi, \boldsymbol{F}^*) = \lambda b - F_0^n a\Phi(a) - (1 - (1 - c/b)^n)b\Phi(b) + \int_{v=a}^{v=r^*} -\lambda F_0^n + \Phi(v)nF_0^{n-1}(1 - F_0)dv$$
$$+ \int_{v=r^*}^{v=b} -\lambda \left(1 - \frac{c}{v}\right)^n dv.$$

Consider the problem

$$R^* := \min_{\Phi} R_n(\Phi, \boldsymbol{F}^*) = \text{Regret}(\Phi) \text{ s.t. } \Phi(v) \in [0, 1] \text{ non-decreasing }.$$

We dualize the non-decreasing constraint.
Let

$$\mathcal{L}(\Phi, \mu) = \text{Regret}(\Phi) - \int_a^b \mu(v)d\Phi(v),$$

and

$$q(\mu) = \min_{\Phi(v) \in [0,1]} \mathcal{L}(\Phi, \mu) \text{ with } \mu(v) \geq 0.$$

Weak duality says that $R^* \geq q(\mu)$ for all $\mu : [a, b] \to \mathbb{R}_+$.

To get to strong duality, we want to choose a specific μ^* such that:

- Complementary Slackness (CS): $\int_a^b \mu^*(v)d\Phi^*(v) = 0$. That is, wherever Φ^* is strictly increasing, μ^* is zero.

- Lagrangian Optimality (LO): $\Phi^* \in \arg\min_{\Phi \in \text{CDF}} \mathcal{L}(\Phi, \mu^*)$.

Because the condition that $\Phi(v)$ is non-decreasing doesn't bind (it is strictly increasing) on $[r^*, b]$, by complementary slackness $\mu^*(v) = 0$ on $[r^*, b]$.

We can then use the integration by partss to get (using $\mu^*(r^*) = 0$)

$$\int_{v=a}^{v=b} \mu^*(v)d\Phi(v) = \int_{v=a}^{v=r^*} \mu^*(v)d\Phi(v) = \mu^*(r^*)\Phi(r^*) - \mu^*(a)\Phi(a) - \int_{v=a}^{v=r^*} (\mu^*)'(v)\Phi(v)dv$$

$$= -\mu^*(a)\Phi(a) - \int_{v=a}^{v=r^*} (\mu^*)'(v)\Phi(v)dv\,.$$

We substitute this into the expression for $\mathcal{L}(\Phi, \mu^*)$ to get

$$\mathcal{L}(\Phi, \mu^*) = \lambda b - \lambda(r^* - a)F_0^n + (\mu^*(a) - F_0^n a)\Phi(a) - \left(1 - \left(1 - \frac{c}{b}\right)^n\right)b\Phi(b) - \lambda\int_{v=r^*}^{v=b}\left(1 - \frac{c}{v}\right)^n dv$$

$$+ \int_{v=a}^{v=r^*}\left[(\mu^*)'(v) + nF_0^{n-1}(1 - F_0)\right]\Phi(v)dv\,.$$

We want to choose μ^* such that

$$\mu^*(a) - F_0^n a = 0$$
$$(\mu^*)'(v) + nF_0^{n-1}(1 - F_0) = 0\,,$$

so that the above expression for $\mathcal{L}(\Phi, \mu^*)$ becomes independent of Φ (zero out the coefficient of $\Phi(a)$ and $\Phi(v)$ between a and r^*; there still is $\Phi(b)$ but we will let this be 1). From

$$\mu^*(r^*) = \mu^*(a) + \int_{v=a}^{v=r^*} (\mu^*)'(v)dv$$
$$0 = aF_0^n - (r^* - a)nF_0^{n-1}(1 - F_0)$$
$$aF_0 = n(r^* - a)(1 - F_0)\,.$$

With that μ^*, we have

$$\mathcal{L}(\Phi, \mu^*) = \lambda b - \lambda(r^* - a)F_0^n - \left(1 - \left(1 - \frac{c}{b}\right)^n\right)b\Phi(b) - \lambda\int_{v=r^*}^{v=b}\left(1 - \frac{c}{v}\right)^n dv\,.$$

If we further assume that $\Phi(b) = 1$, then

$$\mathcal{L}(\Phi, \mu^*) = -(1 - \lambda)b - \lambda(r^* - a)F_0^n + b\left(1 - \frac{c}{b}\right)^n - \lambda\int_{v=r^*}^{v=b}\left(1 - \frac{c}{v}\right)^n dv\,.$$

Now we derive conditions from the fact that F maximizes regret given fixed Φ^*, that is, the saddle $R(\Phi^*, F) \leq R(\Phi^*, F^*)$. We use the Regret-$F$ equation

$$R(\Phi, F) = a(\lambda - \Phi(a)) + \int_{v=a}^{v=b} \left[(\lambda - \Phi(v) - v\Phi'(v))(1 - F(v)^n) + \Phi(v)nF(v)^{n-1}(1 - F(v)) \right] dv.$$

We do pointwise optimization for each v. For $v \in (a, r^*)$. $\Phi^*(v) = \Phi^*(a)$ is a constant, so $F(v)$ that maximizes that is a constant, the same for every v, given by

$$F^*(v) \in \arg\max_z (\lambda - \Phi^*(a))(1 - z^n) + \Phi^*(a)nz^{n-1}(1 - z).$$

Taking the derivative of z gives

$$-(\lambda - \Phi^*(a))nz^{n-1} + n\Phi^*(a)((n-1)z^{n-2} - nz^{n-1}) = 0$$
$$-(\lambda - \Phi^*(a))z + \Phi^*(a)(n - 1 - nz) = 0.$$

Write $\Phi^*(a) = \Phi_0$ for convenience. By the first-order condition, $z = F^*(a) = F_0$ satisfies this equation, so

$$-(\lambda - \Phi_0)F_0 + \Phi_0(n - 1 - nF_0) = 0.$$

For $v \in (r^*, b)$, $\Phi^*(v)$ is no longer a constant (but this is the regime that we have dealt with before). We have

$$F^*(v) \in \arg\max_z (\lambda - \Phi(v) - v\Phi'(v))(1 - z^n) + \Phi(v)nz^{n-1}(1 - z).$$

The first order condition gives

$$-(\lambda - \Phi(v) - v\Phi'(v))nz^{n-1} + \Phi(v)n((n-1)z^{n-2} - nz^{n-1}) = 0$$
$$-(\lambda - \Phi(v) - v\Phi'(v))z + \Phi(v)((n - 1) - nz) = 0.$$

By the first order condition, $z = F^*(v) = 1 - c/v$ satisfies this equation, so

$$-(\lambda - (\Phi^*)(v) - v(\Phi^*)'(v))\left(1 - \frac{c}{v}\right) + \Phi(v)\left((n-1) - n\left(1 - \frac{c}{v}\right)\right) = 0,$$

which simplifies to

$$(\Phi^*)'(v) = \frac{\lambda}{v} - \frac{(n-1)c}{v(v-c)}\Phi^*(v).$$

We have seen this ODE before. The solution is

$$\left(\frac{v-c}{v}\right)^{n-1} \Phi^*(v) = \lambda\left[d + \log\left(\frac{v}{c}\right) - \sum_{k=1}^{n-1} \frac{1}{k}\left(\frac{v-c}{v}\right)^k\right],$$

for some constant d.

$\Phi^*(r^*) = \Phi^*(a) = \Phi_0$ is the point mass of Φ^* at a, which is unknown. We get

$$\left(\frac{r^*-c}{r^*}\right)^{n-1}\Phi_0 = \lambda\left[d + \log\left(\frac{r^*}{c}\right) - \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{r^*-c}{r^*}\right)^k\right].$$

From $1 - c/r^* = F_0 \geq 0$ we have $c \leq r^*$ with strict inequality if $F_0 > 0$. So the summation makes sense (and also tells us it doesn't necessarily go away as zero like before). With $\Phi^*(b) = 1$ we get

$$\left(\frac{b-c}{b}\right)^{n-1} = \lambda\left[d + \log\left(\frac{b}{c}\right) - \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{b-c}{b}\right)^k\right], .$$

Therefore, we have 5 equations for 5 unknowns Φ_0, F_0, c, d, r^*:

$$aF_0 = n(r^* - a)(1 - F_0) \tag{C-2}$$

$$-(\lambda - \Phi_0)F_0 + \Phi_0(n - 1 - nF_0) = 0 \tag{C-3}$$

$$\left(\frac{r^*-c}{r^*}\right)^{n-1}\Phi_0 = \lambda\left[d + \log\left(\frac{r^*}{c}\right) - \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{r^*-c}{r^*}\right)^k\right] \tag{C-4}$$

$$\left(\frac{b-c}{b}\right)^{n-1} = \lambda\left[d + \log\left(\frac{b}{c}\right) - \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{b-c}{b}\right)^k\right] \tag{C-5}$$

$$1 - \frac{c}{r^*} = F_0 . \tag{C-6}$$

For notational convenience, we will write $F^*(a)$ as F_0.
We will write every variable in terms of F_0, so we have a single-variable equation we can solve. From (C-2), we get

$$r^* = a\left(\frac{F_0}{n(1-F_0)} + 1\right) = \frac{n - (n-1)F_0}{n(1-F_0)}a . \tag{C-7}$$

From (C-3) we get

$$\Phi_0 = \frac{\lambda F_0}{(n-1)(1-F_0)} . \tag{C-8}$$

From (C-6) we get

$$c = r^*(1 - F_0) = \frac{n - (n-1)F_0}{n}a . \tag{C-9}$$

Subtracting (C-4) and (C-5) gives

$$\left(\frac{r^*-c}{r^*}\right)^{n-1}\Phi_0 - \left(\frac{b-c}{b}\right)^{n-1} = \lambda\left[\log\left(\frac{r^*}{b}\right) - \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{r^*-c}{r^*}\right)^k + \sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{b-c}{b}\right)^k\right].$$

Substituting the expressions of r^*, Φ_0, c in terms of F_0 from (C-7), (C-8), (C-9) gives (writing

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$\tilde{a} = a/b$)

$$\frac{\lambda F_0^n}{(n-1)(1-F_0)} - \left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-1}$$

$$= \lambda\left[\log\left(\frac{n-(n-1)F_0}{n(1-F_0)}\tilde{a}\right) - \sum_{k=1}^{n-1}\frac{1}{k}F_0^k + \sum_{k=1}^{n-1}\frac{1}{k}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^k\right].$$

Let $\mathrm{fn}(F_0)$ be the left hand side minus the right hand side (taking $\tilde{a} = a/b$ as fixed):

$$\mathrm{fn}(F_0) = \frac{\lambda F_0^n}{(n-1)(1-F_0)} - \left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-1} - \lambda\log\left(\frac{n-(n-1)F_0}{n(1-F_0)}\tilde{a}\right)$$

$$+ \lambda\sum_{k=1}^{n-1}\frac{1}{k}F_0^k - \lambda\sum_{k=1}^{n-1}\frac{1}{k}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^k.$$

(When $\tilde{a}$ is not fixed, we will write the above expression instead as $L(F_0, \tilde{a})$, and we will use this notation later in the proof.)

We first note that, because $\tilde{a} \geq k_l$, by definition of k_l we have

$$\mathrm{fn}(0) = -(1-\tilde{a})^{n-1} - \lambda\log(\tilde{a}) - \lambda\sum_{k=1}^{n-1}\frac{1}{k}(1-\tilde{a})^k \leq 0,$$

with equality only when $\tilde{a} = k_l$.

We also note that as $F_0 \uparrow 1$, LHS grows as $\frac{1}{1-F_0}$ whereas RHS grows as $\log\left(\frac{1}{1-F_0}\right)$, so $\lim_{F_0 \uparrow 1} \mathrm{fn}(F_0) = +\infty$.

Lastly, we compute the derivative of fn as

$$\text{fn}'(F_0) = \frac{\lambda F_0^{n-1}(n-(n-1)F_0)}{(n-1)(1-F_0)^2} - (n-1)\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-2} \cdot \frac{(n-1)}{n}\tilde{a}$$

$$+ \lambda\left[\frac{(n-1)}{n-(n-1)F_0} - \frac{1}{1-F_0} + \sum_{k=1}^{n-1} F_0^{k-1} - \sum_{k=1}^{n-1}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{k-1} \cdot \frac{(n-1)}{n}\tilde{a}\right]$$

$$= \frac{\lambda F_0^{n-1}(n-(n-1)F_0)}{(n-1)(1-F_0)^2} - (n-1)\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-2} \cdot \frac{(n-1)}{n}\tilde{a}$$

$$+ \lambda\left[\frac{(n-1)}{n-(n-1)F_0} - \frac{1}{1-F_0} + \frac{1-F_0^{n-1}}{1-F_0} - \frac{(n-1)}{n-(n-1)F_0}\left[1 - \left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-1}\right]\right]$$

$$= \frac{\lambda F_0^{n-1}(n-(n-1)F_0)}{(n-1)(1-F_0)^2} - \frac{(n-1)^2}{n}\tilde{a}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-2}$$

$$+ \lambda\left[-\frac{F_0^{n-1}}{1-F_0} + \frac{(n-1)}{n-(n-1)F_0}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-1}\right]$$

$$= F_0^{n-1}\lambda\left(\frac{n-(n-1)F_0}{(n-1)(1-F_0)^2} - \frac{1}{1-F_0}\right)$$

$$+ \left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-2}\left[-\frac{(n-1)^2}{n}\tilde{a} + \frac{\lambda(n-1)}{n-(n-1)F_0}\left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)\right]$$

$$= \frac{\lambda F_0^{n-1}}{(n-1)(1-F_0)^2} + \left(1 - \frac{n-(n-1)F_0}{n}\tilde{a}\right)^{n-2}(n-1)\left(\frac{\lambda}{n-(n-1)F_0} - \frac{n-1+\lambda}{n}\tilde{a}\right).$$

We can see from the expression that $\text{fn}'(F_0)$ is increasing in F_0, so either fn is increasing for all F_0 in range (if $\text{fn}'(0) \geq 0$, or it is decreasing for $F_0 \leq F_0^*$ for some constant F_0^* and increasing for $F_0 \geq F_0^*$. Because, for $a/b > k_l$, $\text{fn}(0) < 0$ and $\lim_{F_0 \uparrow 1} \text{fn}(F_0) = +\infty$, we conclude that the equation $\text{fn}(F_0) = 0$ has a unique solution $F_0 \in (0,1)$.

We will also need to show that this unique solution F_0 leads to feasible values of other parameters as well. The ones that concern us are $\Phi^*(a)$ and r^*.

We must have

$$\Phi^*(a) = \frac{\lambda F_0}{(n-1)(1-F_0)} \leq 1 \Leftrightarrow F_0 \leq \frac{n-1}{n-1+\lambda},$$

and

$$r^* = \frac{n-(n-1)F_0}{n(1-F_0)}a \leq b \Leftrightarrow F_0 \leq \frac{n(1-\tilde{a})}{n-(n-1)\tilde{a}}.$$

We will show that $F_0 \leq \frac{n-1}{n}$.

We note that $F_0(\tilde{a} = \frac{\lambda n}{(1+\lambda)n-1}) = \frac{n-1}{n-1+\lambda}$ because when we plug in $\tilde{a} = \frac{\lambda n}{(1+\lambda)n-1}, F_0 = \frac{n-1}{n-1+\lambda}$ in the $L(F_0, \tilde{a})$ expression we get zero. We claim that $L(F_0, \tilde{a} = \frac{\lambda n}{(1+\lambda)n-1})$ is increasing in F_0 for $\frac{n-1}{n-1+\lambda} \leq F_0 \leq 1$. If this is true then we are done, because if for some $\tilde{a} < \frac{\lambda n}{(1+\lambda)n-1}$ we have $F_0(\tilde{a}) > \frac{n-1}{n-1+\lambda}$, then $0 = L(F_0(\tilde{a}), \tilde{a}) > L(F_0(\tilde{a}), \frac{\lambda n}{(1+\lambda)n-1}) > L(\frac{n-1}{n-1+\lambda}, \frac{\lambda n}{(1+\lambda)n-1}) = 0$, a contradiction.

To show that $L(F_0, \frac{\lambda n}{(1+\lambda)n-1})$ is increasing in F_0 for $1 \geq F_0 \geq \frac{n-1}{n-1+\lambda}$, we calculate (taking the expression from the fn$'(F_0)$ earlier)

$$\frac{\partial L(F_0, \tilde{a} = \frac{\lambda n}{(1+\lambda)n-1})}{F_0} =$$

$$= \frac{\lambda F_0^{n-1}}{(n-1)(1-F_0)^2} + \left(1 - \frac{\lambda(n-(n-1)F_0)}{(1+\lambda)n-1}\right)^{n-2} \cdot (n-1)\lambda \left(\frac{1}{n-(n-1)F_0} - \frac{n-1+\lambda}{(1+\lambda)n-1}\right) \geq 0.$$

This is true because

$$\frac{1}{n-(n-1)F_0} \geq \frac{1}{n-(n-1)\frac{\lambda n}{(1+\lambda)n-1}} = \frac{n-1+\lambda}{(1+\lambda)n-1},$$

and

$$1 - \frac{\lambda(n-(n-1)F_0)}{(1+\lambda)n-1} > 0,$$

so the derivative is positive, as desired. (Note also that the expression $\frac{\partial L(F_0, \tilde{a} = \frac{\lambda n}{(1+\lambda)n-1})}{F_0}$ is increasing in F_0 so we can also just plug in $F_0 = \frac{n-1}{n-1+\lambda}$ in that expression and check that the resulting expression is > 0.

Lastly, we want to show that $F_0 \leq \frac{n(1-\tilde{a})}{n-(n-1)\tilde{a}}$. Note that $\tilde{a} < \frac{\lambda n}{(1+\lambda)n-1}$ implies that $\frac{n-1}{n-1+\lambda} < \frac{n(1-\tilde{a})}{n-(n-1)\tilde{a}}$, so this inequality is immediately implied by $F_0 \leq \frac{n-1}{n-1+\lambda}$ which we just proved.

We conclude that for $k_l \leq \tilde{a} \leq \frac{\lambda n}{(1+\lambda)n-1}$, these parameters give rise to a feasible mechanism that is minimax optimal, as desired. $\square$

Proposition C-5. *The worst-case λ-regret of* SPA(r), *the second-price auction with fixed reserve r, is*

$$R_\lambda(\text{SPA}(r), \mathcal{F}_{\text{iid}}) = \begin{cases} -(1-\lambda)b + (b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1} & \text{if } r = a \\ -(1-\lambda)b + \frac{(b-r)^n(n-1)^{n-1}}{((n-1+\lambda)(b-r)-r)^{n-1}} & \text{if } a < r \leq \frac{\lambda}{1+\lambda}b \\ \lambda r & \text{if } \frac{\lambda}{1+\lambda}b \leq r \leq b. \end{cases}$$

Therefore,

$$\inf_{r \in [a,b]} R_\lambda(\text{SPA}(r), \mathcal{F}_{\text{iid}}) = \begin{cases} -(1-\lambda)b + \left(\frac{n}{n+\lambda}\right)^n b & \text{if } \frac{a}{b} \leq 1 - \left(\frac{n}{n+\lambda}\right)^n \left(\frac{n-1+\lambda}{n-1}\right)^{n-1} \\ -(1-\lambda)b + \left(\frac{n-1}{n-1+\lambda}\right)^{n-1}(b-a) & \text{if } \frac{a}{b} \geq 1 - \left(\frac{n}{n+\lambda}\right)^n \left(\frac{n-1+\lambda}{n-1}\right)^{n-1}. \end{cases}$$

The optimal r in the first case ("low" a/b) is $r = \frac{\lambda}{n+\lambda}b$ and the optimal r in the second case ("high" a/b) is $r = a$.

This result is valid for any $n \geq 1$, if for $n = 1$ we interpret any term with the $n-1$ exponent as 1. In agreement with results from $\mathcal{M}_{\text{SPA-rand}}$ and our intuition, when scale information is important (a/b is high), the regret-minimizing reserve is no reserve $r = a$.

Before we prove the main result (Proposition C-5) characterizing the minimax λ-regret SPA with fixed reserve (including no reserve), we first derive the regret expression of $\mathrm{SPA}(r)$ for a fixed r.

Proposition C-6. *Fix $r \in [a,b]$. The regret of $\mathrm{SPA}(r)$ against a joint distribution $\boldsymbol{F}$ is*

$$-(1-\lambda)b + r\boldsymbol{F}_n^{(1)}(r^-) - \lambda \int_{v=a}^{v=r} \boldsymbol{F}_n^{(1)}(v)dv + \int_{v=r}^{v=b} (\boldsymbol{F}_n^{(2)}(v) - \lambda \boldsymbol{F}_n^{(1)}(v))dv \,,$$

where $\boldsymbol{F}_n^{(1)}(r^-) = \Pr(v^{(1)} < v)$. If we further assume that $\boldsymbol{F}$ is n i.i.d. with marginal F, then the regret is

$$-(1-\lambda)b + rF(r^-)^n - \lambda \int_{v=a}^{v=r} F(v)^n dv + \int_{v=r}^{v=b} (nF(v)^{n-1} - (n-1+\lambda)F(v)^n)dv \,.$$

Proof. The regret is

$$\lambda \mathbb{E}[v^{(1)}] - \mathbb{E}[\max(v^{(2)}, r)\mathbf{1}(v^{(1)} \geq r)]$$
$$= \lambda \mathbb{E}[v^{(1)}] - \mathbb{E}[v^{(2)}\mathbf{1}(v^{(2)} > r)] - \mathbb{E}[r\mathbf{1}(v^{(2)} \leq r \leq v^{(1)})] \,.$$

The first term is

$$\mathbb{E}[v^{(1)}] = \int_{v \geq 0} \Pr(v^{(1)} > v)dv = b - \int_{v=a}^{v=b} \boldsymbol{F}_n^{(1)}(v)dv \,.$$

The second term's calculation is analogous to that of Lemma B-3. We have

$$\mathbb{E}[v^{(2)}\mathbf{1}(v^{(2)} > r)] = \int_{v' \in (r,b]} v' d\boldsymbol{F}_n^{(2)}(v') = \int_{v' \in (r,b]} \left(a + \int_{v \in [a,v')} dv \right) d\boldsymbol{F}_n^{(2)}(v')$$

$$= a(\boldsymbol{F}_n^{(2)}(b) - \boldsymbol{F}_n^{(2)}(r)) + \int_{v \in [a,r]} \int_{v' \in (r,b]} d\boldsymbol{F}_n^{(2)}(v')dv + \int_{v \in (r,b]} \int_{v' \in (v,b]} d\boldsymbol{F}_n^{(2)}(v')dv$$

$$= a(1 - \boldsymbol{F}_n^{(2)}(r)) + \int_{v \in [a,r]} (1 - \boldsymbol{F}_n^{(2)}(r))dv + \int_{v \in (r,b]} (1 - \boldsymbol{F}_n^{(2)}(v))dv$$

$$= b - r\boldsymbol{F}_n^{(2)}(r) - \int_{v=r}^{v=b} \boldsymbol{F}_n^{(2)}(v)dv \,.$$

The third term is

$$\mathbb{E}[r\mathbf{1}(v^{(2)} \leq r \leq v^{(1)})] = r\Pr(v^{(2)} \leq r \leq v^{(1)}) = r(\Pr(v^{(2)} \leq r) - \Pr(v^{(1)} < r))$$
$$= r(\boldsymbol{F}_n^{(2)}(r) - \boldsymbol{F}_n^{(1)}(r^-)) \,.$$

Together, we have, for $r \in [a,b]$,

$$R_n(\mathrm{SPA}(r), \boldsymbol{F}_n) = -(1-\lambda)b + r\boldsymbol{F}_n^{(1)}(r^-) - \lambda \int_{v=a}^{v=r} \boldsymbol{F}_n^{(1)}(v)dv + \int_{v=r}^{v=b} (\boldsymbol{F}_n^{(2)}(v) - \lambda \boldsymbol{F}_n^{(1)}(v))dv \,,$$

as desired.

When $\boldsymbol{F}$ is n i.i.d. F, we have $\boldsymbol{F}_n^{(1)}(r^-) = \Pr(\max(\boldsymbol{v}) < r) = \prod_{i=1}^n \Pr(v_i < r) = F(r^-)^n$ where the second-to-last equality uses the fact that the v_i's are independent. We also have $\boldsymbol{F}_n^{(1)}(v) = F(v)^n$ and $\boldsymbol{F}_n^{(2)}(v) = nF(v)^{n-1} - (n-1)F(v)^n$. $\qquad\square$

Now we are ready to prove the main result.

Proof of Proposition C-5. From Proposition C-6, the λ-regret of SPA(r) is

$$R_n(\mathrm{SPA}(r), F) = -(1-\lambda)b + rF_-(r)^n - \lambda \int_{v=a}^{v=r} F(v)^n dv + \int_{v=r}^{v=b} nF(v)^{n-1} - (n-1+\lambda)F(v)^n dv.$$

We first assume that $n \geq 2$ and $r \in (a, b]$. Let $c = F_-(r)$. (Note here that we require $r > a$ in order for us to have the freedom to set the value of $c = \Pr(v < r)$, the mass *strictly* below r. If $r = a$, i.e. there is no reserve, then $c = 0$ by definition. This is why we consider the case $r = a$, i.e. no reserve, separately.) Note that the integrand $nF(v)^{n-1} - (n-1+\lambda)F(v)^n$ is increasing for $F(v) \leq \frac{n-1}{n-1+\lambda}$ and is decreasing for $F(v) \geq \frac{n-1}{n-1+\lambda}$. To minimize $\int_{v \in [a,r]^n} F(v)^n dv$ we must have $F(v) = 0$ for $v \in [a, r-\epsilon]$ for arbitrarily small $\epsilon > 0$, and to maximize $\int_{v \in (r,b]} nF(v)^{n-1} - nF(v)^n dv$, the only constraint we have is $F(v) \geq c$ so for $v \in (r, b]$ we set $F(v) = \frac{n-1}{n-1+\lambda}$ if $c \leq \frac{n-1}{n-1+\lambda}$ and $F(v) = c$ otherwise. Note that the sup over first case of $c \leq \frac{n-1}{n-1+\lambda}$ is simply the second case with $c = \frac{n-1}{n-1+\lambda}$. Because we take the sup over F, we can let $\epsilon \downarrow 0$ and get that the worst-case regret is

$$-(1-\lambda)b + \sup_{c \in [\frac{n-1}{n-1+\lambda}, 1]} rc^n + (b-r)(nc^{n-1} - (n-1+\lambda)c^n).$$

Now, the derivative of this expression of c is $nc^{n-2}\left[rc + (b-r)(n-1-(n-1+\lambda)c)\right]$. The expression in $[\cdots]$ is linear in c. At $c = \frac{n-1}{n-1+\lambda}$, the derivative expression is $nr\left(\frac{n-1}{n-1+\lambda}\right)^{n-1} \geq 0$. At $c = 1$, the expression is $n((1+\lambda)r - \lambda b)$. So if $r \geq \frac{\lambda}{1+\lambda}b$, the first derivative is always ≥ 0, so the maximum is achieved at $c = 1$ and the value is λr. If $r \leq \frac{\lambda}{1+\lambda}b$, the maximum is achieved at $c^* = \frac{(n-1)(b-r)}{(n-1+\lambda)(b-r)-r} \in \left[\frac{n-1}{n-1+\lambda}, 1\right]$ and the value is $-(1-\lambda)b + \frac{(b-r)^n(n-1)^{n-1}}{((n-1+\lambda)(b-r)-r)^{n-1}}$.

Now we consider the case $r = a$. In this case, by definition $c = 0$ and we have

$$R_n(\mathrm{SPA}(a), F) = -(1-\lambda)b + \int_{v \in (a,b]} nF(v)^{n-1} - (n-1+\lambda)F(v)^n dv,$$

so

$$R_n(\mathrm{SPA}(a), \mathcal{F}) = -(1-\lambda)b + (b-a) \sup_{z \in [0,1]} nz^{n-1} - (n-1+\lambda)z^n.$$

The maximum occurs at $z = \frac{n-1}{n-1+\lambda}$ which gives

$$R_n(\mathrm{SPA}(a), \mathcal{F}) = -(1-\lambda)b + (b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1}.$$

Now we deal with the case $n = 1$. The regret expression reduces to

$$R_1(\mathrm{SPA}(r), F) = -(1-\lambda)b + rF_-(r) - \lambda \int_{v\in[a,r]} F(v)dv + \int_{v\in(r,b]} (1-\lambda F(v))dv.$$

For $r > a$, we have

$$R_1(\mathrm{SPA}(r), \mathcal{F}) = -(1-\lambda)b + \sup_{c\in[0,1]} rc + (b-r)(1-\lambda c)$$

$$= \max(\lambda b - r, \lambda r) = \begin{cases} \lambda b - r & \text{if } a < r \le \frac{\lambda}{1+\lambda}b \\ \lambda r & \text{if } r \ge \frac{\lambda}{1+\lambda}b, \end{cases}$$

because the expression under sup is linear in c so it achieves the extrema at one of the end points, either at $c = 0$ or $c = 1$.

For $r = a$ we have

$$R_1(\mathrm{SPA}(a), F) = -(1-\lambda)b + \int_{v\in(a,b]} (1-\lambda F(v))dv.$$

This is maximized when $F(v) = 0$ for all $v \in (a, b]$ and we get

$$R_1(\mathrm{SPA}(a), \mathcal{F}) = -(1-\lambda)b + (b-a) = \lambda b - a.$$

We therefore have

$$R_1(\mathrm{SPA}(r), \mathcal{F}) = \begin{cases} \lambda b - a & \text{if } r = a \\ \lambda b - r & \text{if } a < r \le \frac{\lambda}{1+\lambda}b \\ \lambda r & \text{if } r \ge \frac{\lambda}{1+\lambda}b. \end{cases}$$

Note that the second regime and the first regime are continuous whenever the second regime is applicable, but we will keep them separate for clarity (because the first regime $r = a$ is always applicable, whereas the second regime $r \in (a, \frac{\lambda}{1+\lambda}b]$ is applicable only when $\frac{a}{b} < \frac{\lambda}{1+\lambda}$.

Now we want to choose the optimal r to minimize the worst-case regret. First consider the case $n \ge 2$. We note that

$$(b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1} \le \frac{(b-a)^n(n-1)^{n-1}}{((n-1+\lambda)(b-a)-a)^{n-1}},$$

with equality if and only if $a = 0$. Therefore, if $\frac{a}{b} < \frac{\lambda}{1+\lambda}$, that is, the regime $r \in (a, \frac{\lambda}{1+\lambda}b]$ is permissible, then the worst-case regret under $r = a$ is lower than under $r = a^+$, slightly above a. In contrast, the worst-case regret is continuous at $r = \frac{\lambda}{1+\lambda}b$. Given that the regret in the third regime λr is linear in r, the worst r (lowest regret) occurs at $r = \frac{\lambda}{1+\lambda}b$ with regret $\frac{\lambda^2}{1+\lambda}b$.

First consider the case $\frac{a}{b} < \frac{\lambda}{1+\lambda}$, so all 3 regimes of r are permissible.

In the $r \in [\frac{\lambda}{1+\lambda}b, b]$ regime, the regret is λr, so the lowest regret occurs at $r = \frac{\lambda}{1+\lambda}b$ and has value $\frac{\lambda^2}{1+\lambda}b$.

In the $r \in (a, \frac{\lambda}{1+\lambda}b]$ regime, the regret is

$$-(1 - \lambda)b + (n - 1)^{n-1} \exp\{n \log(b - r) - (n - 1) \log((n - 1 + \lambda)b - (n + \lambda)r)\}.$$

The derivative of the expression in $\{\cdots\}$ is

$$-\frac{n}{b - r} + \frac{(n - 1)(n + \lambda)}{(n - 1 + \lambda)b - (n + \lambda)r} = \frac{(n + \lambda)r - \lambda b}{(b - r)((n - 1 + \lambda)b - (n + \lambda)r)}.$$

Therefore, in this second regime, the worst-case regret is decreasing for $\frac{a}{b} \leq \frac{\lambda}{n+\lambda}$ and increasing for $\frac{a}{b} \geq \frac{\lambda}{n+\lambda}$. So if $\frac{a}{b} \leq \frac{\lambda}{n+\lambda}$, the r that minimizes worst-case regret is $r = \frac{\lambda}{n+\lambda}b$, which gives the regret

$$-(1 - \lambda)b + \left(\frac{n}{n + \lambda}\right)^n b.$$

Therefore, the overall worst-case regret, including $r = a$ also, has regret

$$\min\left(-(1 - \lambda)b + \left(\frac{n}{n + \lambda}\right)^n b, -(1 - \lambda)b + \left(\frac{n - 1}{n - 1 + \lambda}\right)^{n-1}(b - a)\right),$$

corresponding to $r = \frac{\lambda}{n+\lambda}b$ and $r = a$ respectively.

We can show that

$$\left(\frac{n - 1}{n - 1 + \lambda}\right)^{n-1} \geq \left(\frac{n}{n + \lambda}\right)^n.$$

So, for $\frac{a}{b} \leq 1 - \left(\frac{n}{n+\lambda}\right)^n \left(\frac{n-1+\lambda}{n-1}\right)^{n-1}$, $r = \frac{\lambda}{n+\lambda}b$ gives the lowest worst-case regret, and for $\frac{a}{b} \geq 1 - \left(\frac{n}{n+\lambda}\right)^n \left(\frac{n-1+\lambda}{n-1}\right)^{n-1}$, $r = a$ gives the lowest worst-case regret. We can show that

$$0 \leq 1 - \left(\frac{n}{n + \lambda}\right)^n \left(\frac{n - 1 + \lambda}{n - 1}\right)^{n-1} \leq \frac{\lambda}{n + \lambda},$$

so this threshold is always interior.

For $\frac{\lambda}{n+\lambda} \leq \frac{a}{b} \leq \frac{\lambda}{1+\lambda}$, the worst-case regret is increasing in r for the second regime, so the worst-case in the second regime is when $r = a^+$, but we already know that the worst-case regret is lower under $r = a$ than under $r = a^+$, so the best r is $r = a$ with regret

$$-(1 - \lambda)b + \left(\frac{n - 1}{n - 1 + \lambda}\right)^{n-1}(b - a).$$

For $\frac{a}{b} \geq \frac{\lambda}{1+\lambda}$, the second regime is not possible, and the third regime's worst case is again $r = a^+$ which has highest regret than $r = a$, so again the best r is $r = a$ with regret

$$-(1 - \lambda)b + \left(\frac{n - 1}{n - 1 + \lambda}\right)^{n-1}(b - a),$$

so the r that minimizes worst-case regret is $r = \frac{\lambda}{n+\lambda}b$, which gives the regret

$$-(1-\lambda)b + \left(\frac{n}{n+\lambda}\right)^n b.$$

If $\frac{a}{b} < \frac{\lambda}{n+\lambda}$, then the worst-case regret is minimized at $r = \frac{\lambda}{n+\lambda}b$ and the worst-case regret value is

$$-(1-\lambda)b + \frac{n^n}{(n+\lambda)^n}b.$$

If $\frac{a}{b} \geq \frac{\lambda}{n+\lambda}$, then we always have $r > a \geq \frac{\lambda}{n+\lambda}b$, so the worst-case regret is minimized at $r = a^+$, but we have already shown that $r = a^+$ always has higher regret (worse) than $r = a$.

Therefore, for $\frac{a}{b} < \frac{\lambda}{n+\lambda}$, the worst-case regret is

$$\min\left(-(1-\lambda)b + \frac{(b-a)(n-1)^{n-1}}{(n-1+\lambda)^{n-1}}, \left(\frac{n^n}{(n+\lambda)^n} - 1 + \lambda\right)b, \frac{\lambda^2}{1+\lambda}b\right).$$

The first, second, and third terms correspond to $r = a$, $r \in (a, \frac{\lambda}{1+\lambda}b]$ and $r \in [\frac{\lambda}{1+\lambda}b, b]$ respectively.

The third one is higher than the second one because the third one, as we have already shown, is the second one with $r = \frac{\lambda}{1+\lambda}b$ which by our proof has higher regret than that at $r = \frac{\lambda}{1+\lambda}b$. So the worst-case regret becomes

$$\min\left(\frac{(n-1)^{n-1}}{(n-1+\lambda)^{n-1}}(b-a) - (1-\lambda)b, \frac{n^n}{(n+\lambda)^n}b - (1-\lambda)b\right).$$

We note that

$$\left(\frac{n-1}{n-1+\lambda}\right)^{n-1} \geq \left(\frac{n}{n+\lambda}\right)^n.$$

This is true because $\left(\frac{x}{x+\lambda}\right)^x = \exp\left\{x\log(x) - x\log(x+\lambda)\right\}$ is a decreasing function of x: the derivative of the expression in the $\{\cdots\}$ is

$$\left(x\cdot\frac{1}{x} + \log(x)\right) - \left(x\cdot\frac{1}{x+\lambda} + \log(x+\lambda)\right) = 1 - \frac{x}{x+\lambda} + \log\left(\frac{x}{x+\lambda}\right) \leq 0,$$

because $1 + \log(u) \leq u$ for all u. (Let $u' = \log(u)$; this because the well-known $1 + u' \leq \exp(u')$.)

We however have

$$\left(\frac{n-1}{n-1+\lambda}\right)^{n-1} \leq \left(\frac{n}{n+\lambda}\right)^{n-1}.$$

Therefore, we have

$$\left(\frac{n-1}{n-1+\lambda}\right)^{n-1}(b-a) - (1-\lambda)b \geq \left(\frac{n}{\lambda+n}\right)^n b - (1-\lambda)b,$$

when $a = 0$ but

$$\left(\frac{n-1}{n-1+\lambda}\right)^{n-1}(b-a) - (1-\lambda)b \le \left(\frac{n}{\lambda+n}\right)^{n}b - (1-\lambda)b,$$

when $a = \frac{\lambda}{n+\lambda b}$. The threshold to define which one is better is therefore always in the middle, at $1 - \left(\frac{n}{n+\lambda}\right)^{n}\left(\frac{n-1+\lambda}{n-1}\right)^{n-1}$.

We now consider $\frac{\lambda}{n+\lambda} \le \frac{a}{b} \le \frac{\lambda}{1+\lambda}$. We have shown that the worst-case regret in the second regime occurs at $r = a^+$ which is always higher regret than $r = a$, so we only need to consider the first and third regime: the worst-case regret is

$$\min\left(-(1-\lambda)b + (b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1}, \frac{\lambda^2}{1+\lambda}b\right)$$

$$= \min\left((b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1}, \frac{1}{1+\lambda}b\right) - (1-\lambda)b.$$

We know that $\left(\frac{n-1}{n-1+\lambda}\right)^{n-1} \le \frac{1}{1+\lambda}$ because $n-1 \ge 1$ and $\left(\frac{x}{x+\lambda}\right)^{x}$ is a decreasing function of x. We also know that $b - a \le b$. Therefore, the first expression in the min (first regime) is always lower than the second expression (third regime). So the worst case regret in this case is just

$$-(1-\lambda)b + (b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1},$$

which is achieved at $r = a$.

Lastly, we consider the case $\frac{a}{b} \ge \frac{\lambda}{1+\lambda}$. Then the second regime is never applicable, and the worst-case regret is

$$\min\left(-(1-\lambda)b + (b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1}, \frac{\lambda^2}{1+\lambda}b\right) = -(1-\lambda)b + (b-a)\left(\frac{n-1}{n-1+\lambda}\right)^{n-1},$$

where we know the first term in the min is less than the second term by what we just proved. This is also achieved when $r = a$.

We therefore conclude that for $n \ge 2$ the worst-case regret (and the corresponding optimal reserve r^*) as a function of a and b is as follows.

$$\inf_{r \in [a,b]} R_n(\mathrm{SPA}(r), \mathcal{F}) = \begin{cases} \left(\frac{n-1}{n-1+\lambda}\right)^{n-1}(b-a) - (1-\lambda)b & \text{if } \frac{a}{b} \le 1 - \left(\frac{n}{n+\lambda}\right)^{n}\left(\frac{n-1+\lambda}{n-1}\right)^{n-1} \text{ or } \frac{a}{b} \ge \frac{\lambda}{n+\lambda} \\ \left(\frac{n}{n+\lambda}\right)^{n}b - (1-\lambda)b & \text{if } 1 - \left(\frac{n}{n+\lambda}\right)^{n}\left(\frac{n-1+\lambda}{n-1}\right)^{n-1} \le \frac{a}{b} < \frac{\lambda}{n+\lambda}. \end{cases}$$

In the first case, $r^* = a$. In the second case, $r^* = \frac{\lambda}{n+\lambda}b$.

Now we calculate the optimal r^* and the best worst-case regret for the case $n = 1$. In the case $\frac{a}{b} < \frac{\lambda}{1+\lambda}$, then all 3 regimes are possible. The lowest worst-case regret in the second regime is $\frac{\lambda^2}{1+\lambda}b$

when $r = \frac{\lambda}{1+\lambda}b$, which is the same as the lowest worst-case regret in the third regime. Therefore,

$$\inf_{r \in [a,b]} R_1(\mathrm{SPA}(r), \mathcal{F}) = \min\left(\lambda b - a, \frac{\lambda^2}{1+\lambda}b\right) = \frac{\lambda^2}{1+\lambda}b,$$

where the last part is true because $\frac{a}{b} < \frac{\lambda}{1+\lambda}$ implies $\lambda b - a > \frac{\lambda^2}{1+\lambda}b$. Here, $r^* = \frac{\lambda}{1+\lambda}b$.

Now consider the case $\frac{a}{b} \geq \frac{\lambda}{1+\lambda}$, then the second regime is inapplicable, and the third case holds for any $r \in (a, b]$, and the lowest worst-case regret in this regime is λa at $r = a^+$, so

$$\inf_{r \in [a,b]} R_1(\mathrm{SPA}(r), \mathcal{F}) = \min\left(\lambda b - a, \lambda a\right) = \lambda b - a,$$

where the last part is true because $\frac{a}{b} \geq \frac{\lambda}{1+\lambda}b$ implies $\lambda b - a \leq \lambda a$. Here, $r^* = a$. $\qquad\square$