Convex Optimization

Chapter 2: Convex Sets

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1 Convex Sets

1.1 Definition and convexity

1.2 Examples

2.14 (*expanded and restricted sets*)
Concept:Scaling a set (either extending or shrinking) preserves convexity.
Proof Idea: Drawing a trapezoid.

1.3 Operations that preserve convexity

2.16 (*partial sum*)
Concept:As sum preserves convexity, partial sum is just summing over a subspace and clearly preserves convexity.

2.18 (*Invertible linear fractional function*)
Concept:Consider the *Projective interpolation* in page 41. **Details**: We associate a point in $x \in \mathbb{R}^n$ with a ray $\mathcal{P}(x) = \{t(x, 1) | t \geq 0\}$. Then we have

- $f(x) = \mathcal{P}(Q(\mathcal{P}^{-1}(x)))$
- $f^{-1}(y) = \mathcal{P}^{-1}(Q^{-1}(\mathcal{P}))$

1.4 Separation theorem and supporting hyperplanes

In this subsection we investigate the powerful *separating hyperplane theorem*. The theorem states that

If C and D are **disjoint** convex sets $\Rightarrow \exists$ separating hyperplane

This can be easily proved by finding one. But this is not the end of the story, we also care about the *converse* of the theorem. That is,

C and D are convex and there's a separting affine function $\Rightarrow$ C and D are disjoint

Another important aspect is to find **alternative** conditions for inequalities.
For example: (*Alternatives for strict inequalities*)

$$Ax \prec b \Leftrightarrow \exists \lambda \text{ s.t. } \lambda \neq 0,\ \lambda \succeq 0,\ A^T\lambda = 0,\ \lambda^T b \leq 0$$

2.20 (*strictly positive solution of linear equations*)
Goal: $\exists x \succ 0,\ Ax = b \Leftrightarrow$ there's **no** λ such that $A^T\lambda \succeq 0, A^T\lambda \neq 0, b^T\lambda \leq 0$
Details: link

Intuition (strictly positive solution of linear equations)

"there's **no** λ such that $A^T\lambda \succeq 0, A^T\lambda \neq 0, b^T\lambda \leq 0$" is a sufficient condition for the existence of strictly positive solution.

2.22 (*general-case separating hyperplane theorem*)
Goal: If convex sets C and D are disjoint $\Rightarrow \exists$ separating hyperplane.

2.24 (*examples of supporting hyperplane*)

(a) A convex set $C = \bigcap$ halfspaces constructed by the supporting hyperplanes that contain C

(b) How to explicitly write down the supporting hyperplane of infinite norm.

2.26 (*support function*)

Definition 1 (support function) *The support function of $C \subseteq \mathbb{R}^n$ is*

$$S_C(y) = \sup\{y^T x | x \in C\}$$

Goal: For two sets C and D, $C = D \Leftrightarrow S_C = S_D$.
Proof Idea: If $\exists x \in D,\ x \notin C$, then by *separating hyperplane theorem*, there is a hyper-plane $a^T x + b$ that separates x and C. Then either $a^T x > S_C(a)$ or $-a^T x > S_C(-a)$.

Intuition (supporting function)

The maximizer of the supporting function is the point that being supported by the hyperplane with normal vector y.

1.5 Convex cones and generalized inequalities

Here, we want to use a *proper cone* to define a generalized inequality and explore some useful results. First, we define the proper cone

Definition 2 (proper cone) *We say a cone $K \subseteq \mathbb{R}^n$ is proper if*

- *K is convex. (contains every line segments)*
- *K is closed. (consists every limit points)*
- *K is solid. (has interior point)*
- *K is pointed. (contains no line or contain* $\mathbf{0}$*)*

Then, we define a partial ordering on $\mathbb{R}^n$ with K as follow

$$x \preceq_K y \Leftrightarrow y - x \in K$$

, which is called *generalized inequality*.

Intuition (generalized inequality)

To check whether $x \preceq_K y$, we can move the center of the cone K to x and see whether y is inside.

Next, we define the concept of dual cone

Definition 3 (dual cone) *Let K be a cone, then*

$$K^* = \{y : y^T x \geq 0,\ \forall x \in K\}$$

is the dual cone of K.

Intuition (dual cone)

Geometrically, $y \in K^* \Leftrightarrow -y$ is the normal of a hyperplane that supports K at the **origin**.

2.28 (*positive semidefinite cone*)
Goal: Algebraic intuitions for positive semidefinite cone of dimension $n = 1, 2, 3$.

2.30 (*properties of generalized inequality*) The following is the properties of generalized inequality $\preceq_K$:

- Preserved addition.
- Transitive.
- Preserved nonnegative scaling.
- Reflexive.
- Antisymmetric.
- Preserved under limits.

2.32 (*dual cone of the image of a linear transformation with nonnegative domain*)
Goal: $K = \{Ax : x \succeq 0\} \Rightarrow K^* = \{y : A^T y \succeq 0\}$

2.34 (*lexicographic cone and ordering*)

Definition 4 (lexicographic cone) *The lexicographic cone is*

$$K_{lexi} = \{0\} \cup \{x \in \mathbb{R}^n : x_1 = \cdots = x_k = 0,\ x_{k+1} > 0,\ \textit{for some } k\}$$

Namely, the first nonzero entry is positive.

Directly, we can define the lexicographic ordering as $x \leq_{lexi} y \Leftrightarrow y - x \in K_{lexi}$.
We have

- K_{lexi} is a cone but not proper.
- $\leq_{lexi}$ is a linear ordering.
- $K^*_{lexi} = \mathbb{R}_+ e_1 = \{(t, 0, ..., 0) : t \geq 0\}$

Intuition (lexicographic cone and ordering)

Lexicographic ordering is like a **dictionary**!

2.36 (*Euclidean distance matrices*)
Goal: The set of Euclidean distance matrices is a convex cone.

Definition 5 (Euclidean distance matrix) *We say $D \in \mathbf{S}^n$ is a Euclidean matrix if $\exists x_1, ..., x_n \in \mathbb{R}^n$ such that $D_{ij} = ||x_i - x_j||_2^2$.*

Moreover, there's a necessary and sufficient condition for Euclidean distance matrix:

$$D \in \mathbf{S}^n \text{ is a Euclidean distance matrix} \Leftrightarrow D_{ii} = 0,\ x^T D x \leq 0\ \forall x\ s.t.\ \mathbf{1}^T x = 0$$

2.38 (*barrier cone, recession cone, normal cone*)

A Details

2.20 link
($\Rightarrow$) is trivial. ($\Leftrightarrow$) is shown by separating hyperplane theorem as follow:

1. If there's no $x \succ 0,\ Ax = b$, it means that $\mathbb{R}^n_{++}$ and $\{x : Ax = b\}$ are disjoint.
2. Thus, we can apply *separating hyperplane theorem*: $\exists c \neq 0, d$ such that
   - $c^T x \geq d$ for all x in $\mathbb{R}^n_{++}$
   - $c^T x \leq d$ for all x in $\{x : Ax = b\}$
3. By 2., we have
   - $c \geq 0$ and $d \leq 0$.

- As $\{x : Ax = b\}$ is **affine**, a linear function either be a constant or takes all value. And here it can only be the constant case. That is, $\exists d' \leq d$ such that $c^T x = d'$, $\forall x \in \{x : Ax = b\}$.

4. By lemma

$$c^T x = d',\ \forall x \in \{x : Ax = b\} \Leftrightarrow \exists \lambda \text{ such that } c = A^T\lambda, d' = b^T\lambda$$

There is a λ such that $A^T\lambda = c \geq 0$, $A^T\lambda = c \neq 0$, and $c^T\lambda = d' \leq d \leq 0$

References