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Linear System - Matrix Tree Theorem

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In which we discuss the connection between graph Laplacian and spanning tree.

1 Introduction and Notation

Given a connected graph $G = (V, E)$, we fix the following notations:

- L is the graph Laplacian of G.
- $\tau(G)$ denote the number of spanning trees in graph G.
- Let $e \in E$, $G\backslash e$ is the graph *deleting* edge e.
- Let $e \in E$, G/e is the graph *contracting* edge e, *i.e.,* contracting the endpoints of e.
- Let M be a square matrix and S is a subset of its indexes, $M[S]$ is the submatrix of M deleting the rows and columns in S.
- Let M be a square matrix, denote $M(i, j)$ be the submatrix of M deleting the i-th row and the j-th column.
- The ij-cofactor of M is $(-1)^{i+j} \det M(i, j)$.
- The *adjugate* of M, denoted by adj M has ij-entry being the ji-cofactor of M.

2 Theorems and Lemmas

2.1 Number of spanning trees

Theorem 1 *For any $u \in V$, $\det L[u] = \tau(G)$.*

Proof: We prove this theorem with four steps:

1. For any edge $e \in E$, $\tau(G) = \tau(G\backslash e) + \tau(G/e)$.
   Observe that $\tau(G\backslash) =$ the number of spanning trees without e and $\tau(G/e) =$ the number of spanning trees containing e.
2. Let $e = (u, v) \in E$, $L[u] = L(G\backslash e)[u] + \mathbf{1}_{i=j=v}$.
   This is because taking away e decrease the degree of v by 1.

3. $\det L[u] = \det\Big(L(G\backslash e)[u]\Big) + \det\Big(L(G\backslash e)[u,v]\Big)$.

$$\begin{aligned}\det L[u] &= \det\Big(L(G\backslash e)[u] + \mathbf{1}_{i=j=v}\Big) \\ &= \sum_{w\in V}(-1)^{w+v}\Big(L(G\backslash e)[u] + \mathbf{1}_{i=j=v}\Big)_{v,w}\cdot\det\Big(L(G\backslash e)[u,v,w]\Big) \\ &= \sum_{w\in V}(-1)^{w+v}\Big(L(G\backslash e)[u]\Big)_{v,w}\cdot\det\Big(L(G\backslash e)[u,v,w]\Big) \\ &+ (-1)^{v+v}\Big(\mathbf{1}_{i=j=v}\Big)\cdot\det\Big(\det L(G\backslash e)[u,v]\Big) \\ &= \det\Big(L(G\backslash e)[u]\Big) + \det\Big(L(G\backslash e)[u,v]\Big)\end{aligned}$$

4. Observe that $L(G\backslash e)[u,v] = L(G/e)[u,v]$, then by induction, the theorem holds.

□

Lemma 2 *For any square matrix M, $M\,adj(M) = \det(M)I$.*

PROOF: Consider the ij-entry of the left hand side.

$$\Big(M\text{adj}(M)\Big) = \sum_k M_{i,k}\text{adj}(M)_{k,j} = \sum_k M_{i,k}\cdot(-1)^{j+k}\cdot\det(M(j,k)) \tag{1}$$

When $i=j$, this is exactly $\det(M)$ and when $i\neq j$, this will be the determinant of replacing the j-th row of M by the i-th row of M, which is 0. □

The next theorem connects the graph Laplacian to the number of spanning tree via the matrix adjugate.

Theorem 3 *We have $adj(L) = \tau(G)J$, where J is the all 1 matrix.*

PROOF: By Lemma 2, we have $L\text{adj}(L) = \det(L)I = 0$. That is, the columns of $\text{adj}(L)$ are in the null space of L. If G is disconnected, then the entries of $\text{adj}(L)$ are all zero and $\tau(G) = 0$, the theorem holds. If G is connected, then the null space of L is spanned by the uniform vector, *i.e.,* the columns of $\text{adj}(L)$ are all uniform vector. As $\text{adj}(L)$ is symmetric, we know that every entries of $\text{adj}(L)$ are the same. Finally, by Theorem 1, we know that the diagonal of $\text{adj}(L)$ is $\tau(G)$ and thus the theorem holds. □

Theorem 4 (matrix tree theorem) *Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the graph Laplacian L, then $\tau(G) = \frac{1}{n}\prod_{i=2}^n \lambda_i$.*

PROOF: If G is disconnected, then $\lambda_2 = 0$ and the theorem holds. If G is connected, let's consider the characteristic polynomial $\phi(t) = \det(tI - L) = \prod_{i=1}^n (t-\lambda_i)$ of L. The linear term of $\phi(t)$ is

$$(-1)^{n-1}\sum_{i=1}^n \prod_{j\in[n]\backslash\{i\}} \lambda_j = (-1)^{n-1}\prod_{i=2}^n \lambda_i \tag{2}$$

On the other hand, if we directly expand $\det(tI - L)$ using Laplace's expansion, we will have

$$\det(tI - L) = \sum_{i=0}^{n-1} (-1)^{n-i} t^i \sum_{S \subseteq [n], |S|=i} \det(L[S]) \tag{3}$$

Thus, the coefficient of the linear term in $\phi(t)$ is also $(-1)^{n-1} \sum_{u \in [n]} \det(L[u]) = n\tau(G)$ from Theorem 1. As a result, the theorem holds. $\square$

References