Algorithmic Spectral Graph Theory Boot Camp	**Speaker: David Stuerer**
Hierarchies of Relaxations and their Strengths and Limitations	
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Definition 1 (unique game) *For any $\epsilon > 0$, define the unique game $\mathcal{UG}(\epsilon)$ as follows.*

- *Given n variables $x_1, \dots, x_n$ over alphabets $[k]$, where most of the time set $k < \log n$.*
- *Consider a system of equations in the form $x_i - x_j = c_{ij}$ (mod k) for arbitrary $i, j \in [n]$.*
- *Promise there exists solution satisfying at least $1 - \epsilon$ fraction of constraints.*
- *The goal is to find a solution satisfying at least 0.9 fraction of constraints.*

Basically, one can view an unique game as a graph with n vertices where the constraints $x_i - x_j = c_{ij}$ are edges with weight c_{ij} between the ith and jth vertex. The following is the statement of the unique game conjecture by Khot [Kho02].

Conjecture 2 (The unique game conjecture) *For any $\epsilon > 0$, $\mathcal{UG}(\epsilon)$ is* **NP***-hard.*

Here, let's take a look at some examples to have more intuitions about the unique game.

- When $\epsilon = 0$, using Gaussian elimination based algorithm can sole the unique game in polynomial time.
- When $k = 2$, the unique game is equivalent to Max-Cut.

In this talk, the David Steurer is going to show a $O(2^{n^{\epsilon^{1/3}}})$ time algorithm based on SDP hierarchies to solve $\mathcal{UG}(\epsilon)$. The idea consists of two steps: first rewrite the unique game into boolean polynomial optimization problem, then construct pseudo-distribution for it.

Rewrite the unique game

Observe that, we can first formulate unique game as an optimization problem as follows.

- For any assignment $x \in [k]^n$, let value(x) be the fraction of satisfying constraints by x.
- Thus, the goal of unique game turns to solve the optimization problem $\max_{x \in [k]^n} \text{value}(x)$.

Nevertheless, we want value(x) to be a low-degree polynomial so that it would be easier to be optimized. Thus, define some new variables as follows.

- For any $i \in [n]$ and $a \in [k]$, define variable $[x_i = a]$ as an indicator function for $x_i = a$. Note that for any $i \in [n]$, exactly one $[x_i = a]$ is set to 1. Thus, we need to add additional constraints to our optimization problem.

- Take $\text{value}(x) = \mathbb{E}_{x_i - x_j = c_{ij}} \sum_{a \in [k]} [x_i = a] \cdot [x_i = a - c_{ij}]$. The expectation here is that giving uniform weight to each constraints and one can see that $\text{value}(x)$ is a degree 2 polynomial with these new variables.

Now, we successfully represent $\text{value}(x) : [k]^n \to \mathbb{R}$ as a low degree boolean polynomial. Later on, we'll see that this formulation will fit the framework of SDP hierarchies and yield efficient algorithm.

Pseudo-distribution for unique game

First, let's recall the definition of pseudo-distribution.

Definition 3 (pseudo-distribution) *For functions $D, f : [k]^n \to \mathbb{R}$, define the pseudo-expectation of D on f as*

$$\tilde{\mathbb{E}}_D f = \sum_{x \in [k]^n} D(x) \cdot f(x).$$

We say D is a degree-d pseudo-distribution if

- $\tilde{\mathbb{E}}_D \mathbf{1} = 1$.
- $\tilde{\mathbb{E}}_D p^2 \geq 0$ *for any polynomial p of degree at most $d/2$.*

Furthermore, for unique game, we D is degree-d pseudo-distribution for $\mathcal{UG}(\epsilon)$ if D is a degree-d pseudo-distribution and

$$\tilde{\mathbb{E}}_D p \cdot [\text{value}(x) - (1 - \epsilon)] = 0$$

for any polynomial p of degree at most $d - 2$.

Note that by [Sho88, Nes00, Par00, Las01], if exists, a degree-d pseudo-distribution can be found in time $n^{O(d)}$.

Next, we are going to see how to use pseudo-distribution of $\mathcal{UG}(\epsilon)$ to produce good approximation.

References

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