TCS+	**Speaker: Shayan Oveis Gharan**
Effective-Resistance-Reducing Flows,	
Spectrally Thin Trees, and Asymmetric TSP	
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This talk [AG15] is about the new integrality gap result for the *Asymmetric Traveling Salesman Problem (ATSP)*. Shayan Oveis Gharanand Nima Anari improved the original $O(\log n/\log\log n)$ gap to poly-$\log\log n$. Since it takes a long way to the result, I will summarize the whole structure of the talk in Section 1. In Section 2, I will introduce a key ingredient: *thin spanning tree* in depth. Finally, in Section 3 I will wrap up the whole talk.

1 Overview

The traveling salesman problem is defined as below.

Definition 1 (traveling salesman problem (TSP)) *Given a graph $G = (V, E)$ with cost function $c : V \times V \to \mathbb{R}^{>0}$. The goal is to minimize the total cost of a path P that travels every vertex. Specifically, the c should obey the triangle inequality. Also, when c is asymmetric, i.e., $c(u, v) \neq c(v, u)$ for some $u, v \in V$, the problem is called the asymmetric traveling salesman problem.*

There's a natural LP relaxation for ATSP by Held and Karp [HK70] as follows.

$$\begin{aligned}
\underset{x_{u,v}}{\text{minimize}} \quad & \sum_{u,v\in V} c(u,v)x_{u,v} & \\
\text{subject to} \quad & \sum_{u\in S, v\notin S} x_{u,v} \geq 1 & \forall S \subseteq V, \\
& \sum_{v\in V} x_{u,v} = \sum_{v\in V} x_{v,u} & \forall u \in V, \\
& x_{u,v} \geq 0 & \forall u, v \in V.
\end{aligned}$$

With the LP relaxation in (1), denote OPT^{LP} as the optimal value of LP and OPT^{ATSP} as the optimal value of ATSP, define the integrality gap as follows.

$$\max_{c(\cdot,\cdot)} \frac{OPT^{ATSP}}{OPT^{LP}}. \tag{1}$$

The integrality gap have been conjectured to be **constant** while before this work, there's a 2 lower bound [CGK06] and $O(\log n/\log\log n)$ upper bound [AGM+10]. In this work, the Shayan Oveis Gharanand Nima Anari improved the upper bound to poly-$\log\log n$ with constant roughly around 10-12.

Note that this is only for the integrality gap for the LP relaxation of ATSP. As to the approximation algorithms for ATSP, there have been also no known constant factor results. The state-of-the-art approximation algorithm has $O(\log n/\log\log n)$ factor [AGM+10] and there's also constant factor algorithms for planar or bounded genus graph [GS11, ES14].

1.1 ATSP, thin spanning tree, and effective resistance

The high-level idea of the proof is based on a reduction from ATSP to the *thin spanning tree* [AGM+10]. Intuitively, finding a good approximation from the LP relaxation in (1) can be reduced to upper bounding the **combinatorial thinness** of the graph.

Next, upper bounding the combinatorial thinness can be reduced to upper bounding the **spectral thinness** of the graph. Finally, the spectral thinness can be reduced to upper bounding the effective resistance of the graph. The contribution of this work is finding a good way to modify the structure of the graph so that the combinatorial structure is preserved while the spectral property becoming better.

2 Thin Spanning Tree

First, let's see some definitions.

Definition 2 (k-edge connected) *A graph G is k-edge connected if any cut of G consists at least k edges.*

Remark 3 There are some equivalent definition for k-edge connectivity.

- $\forall$ pair of vertices, $\exists$ k edge-disjoint paths.
- G is the union of k edge-disjoint spanning trees.

Definition 4 (α-thin spanning tree) *Given a k-edge connected graph $G = (V, E)$. A spanning tree T of G is α-thin if for all cut $S \subseteq V$,*

$$|E_T(S, \bar{S}) \leq \alpha \cdot E_G(S, \bar{S}).$$

Remark 5

- Ideally, we want $\alpha = O(\frac{1}{k})$, but even $\alpha < 0.99$ is interesting.
- An α-thin spanning tree can be regarded as an one-sided unweighted sparsifier cut.

Remark 6 Some exercises for α-thin three.

- The Hamiltonian path in complete graph K_n is a $\frac{2}{k}$-thin tree.
- Show that the k-dimensional hypercube has $O(\frac{1}{k})$-thin tree. Note that, the Hamiltonian path will no longer be a thin spanning tree.

3 Summary

References

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