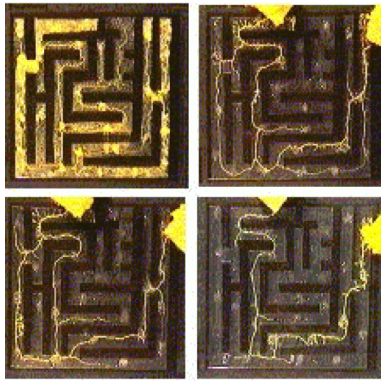
Computational Aspects of Spiking Neural Networks

Chi-Ning Chou

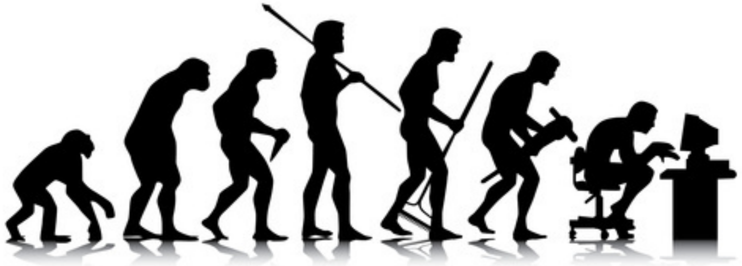
Advisor: Dr. Kai-Min Chung and Dr. Chi-Jen Lu

Natural Algorithm



slime system
[NYT00, TKN07, BMV12]

bird flocking
[Cha12, Cha09]



Evolution
[LP16, LPR+14]

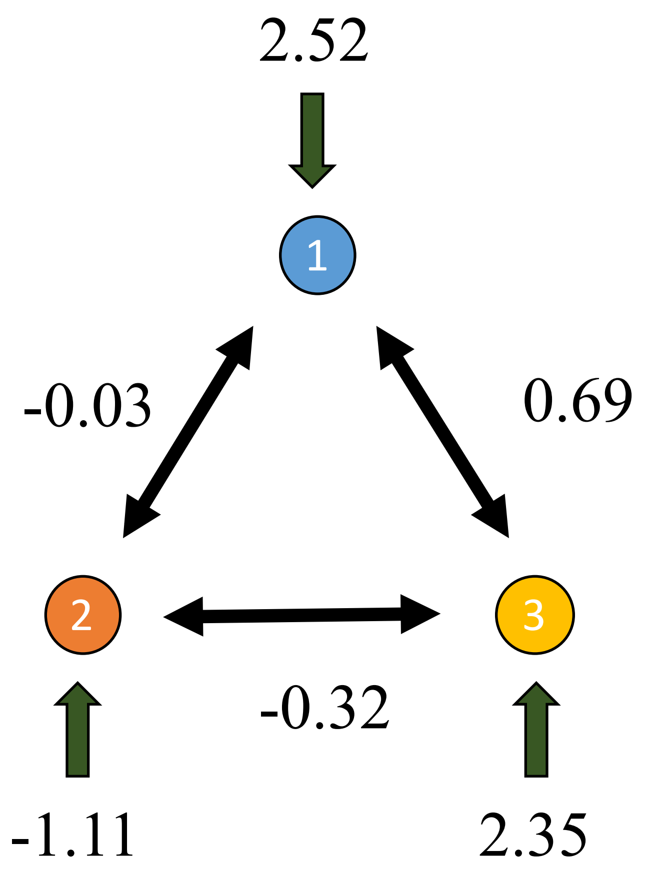
sandpile model
[CV12]

``understanding and explaining algorithms observed from natural systems and dynamics''

Spiking Neural Networks (SNN) Dynamics

- Neuron i has a time-varying **potential** $\mathbf{u}_i(t)$.
- **External charging rate** $\mathbf{I} = (\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n)$.
- **Firing threshold** $\eta > 0$:

  neuron i *fires* a spike if $\mathbf{u}_i(t) > \eta$ or $\mathbf{u}_i(t) < \eta$.
- **Spike** causes inhibition or exhibition:
  When neuron i fires a spike, neighbors' $\mathbf{u}_j(t)$ decrease by C_{ij}.



Motivation

- Mathematical models to **explain** and **predict** the behaviors of different neuron systems. (e.g., [I+03, Adr26, Lap07])

$$\frac{d\mathbf{u}(t)}{dt} = -C\mathbf{s}(t) + \mathbf{I}dt$$

- Barrett et al. [BDM13]: *spike firing rate* can be characterized as optimal solution of quadratic optimization problem (least square problem).

$$\min_{\mathbf{x}} \mathbf{x}^{\mathrm{T}} C\mathbf{x} - 2\mathbf{I}^{\mathrm{T}}\mathbf{x}$$

- Shapero et al. [SRH13,SZHR14]: SNN dynamics as algorithm for solving sparse optimization problems, e.g., Lasso.

First Convergence Result

Theorem *(discrete SNN to linear system)*

For any $\epsilon > 0$ *, take* $\eta = \lambda_{\max}$*, the step size* $\Delta t < \frac{\sqrt{\lambda_{\min}}}{12\cdot\sqrt{n}\|\mathbf{x}^*\|_{A^{\mathrm{T}}A}}$*, and the discrete time* $T \geq \frac{2\kappa(A^{\mathrm{T}}A)\cdot n}{\epsilon}$*, we have* $\|\hat{\mathbf{x}}(T) - \mathbf{x}^*\|_{A^{\mathrm{T}}A} \leq \epsilon \cdot \|\mathbf{x}^*\|_{A^{\mathrm{T}}A}$.

Key Observation

- Conservation argument:

$$\frac{\mathbf{u}(T) - \mathbf{u}(0)}{T \cdot \Delta \mathrm{t}} = -A^{\mathrm{T}} A \hat{\mathbf{x}}(T) + A^{\mathrm{T}} \mathbf{b}$$

- Bounded potential

Lemma *(bounded potential)*

For any $t > 0$, take $\eta = \lambda_{\max}$ and step size $\Delta t < \frac{\sqrt{\lambda_{\min}}}{12 \cdot \sqrt{n} \|\mathbf{x}^*\|_{A^{\mathrm{T}} A}}$, we have

$\|\mathbf{u}(t)\|_{(A^{\mathrm{T}} A)^+} \leq 2\sqrt{\kappa(A^{\mathrm{T}} A) \cdot \lambda_{\max} \cdot n}.$

Q: What Solution does SNN converges to?

- Observed that a simple SNN dynamics may solve the basis pursuit problem (a.k.a. ℓ_1 minimization problem).

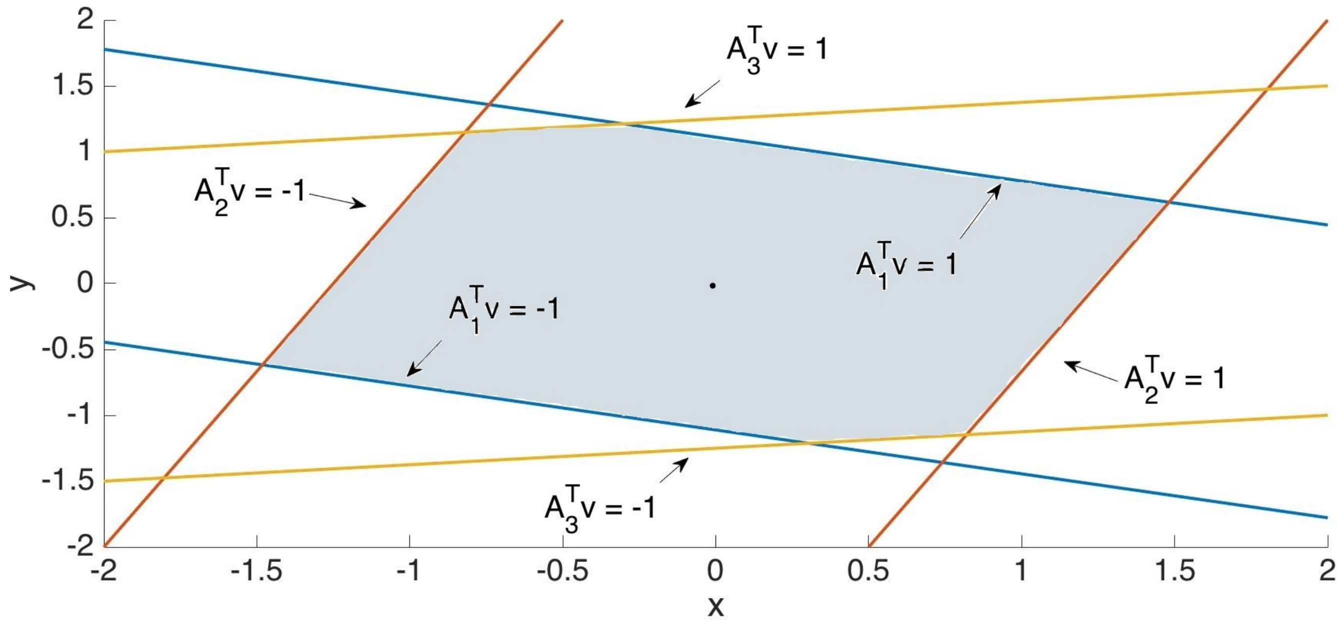
$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \|\mathbf{x}\|_1 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $C = A^{\mathrm{T}}A$ and $\mathbf{I} = A^{\mathrm{T}}\mathbf{b}$.

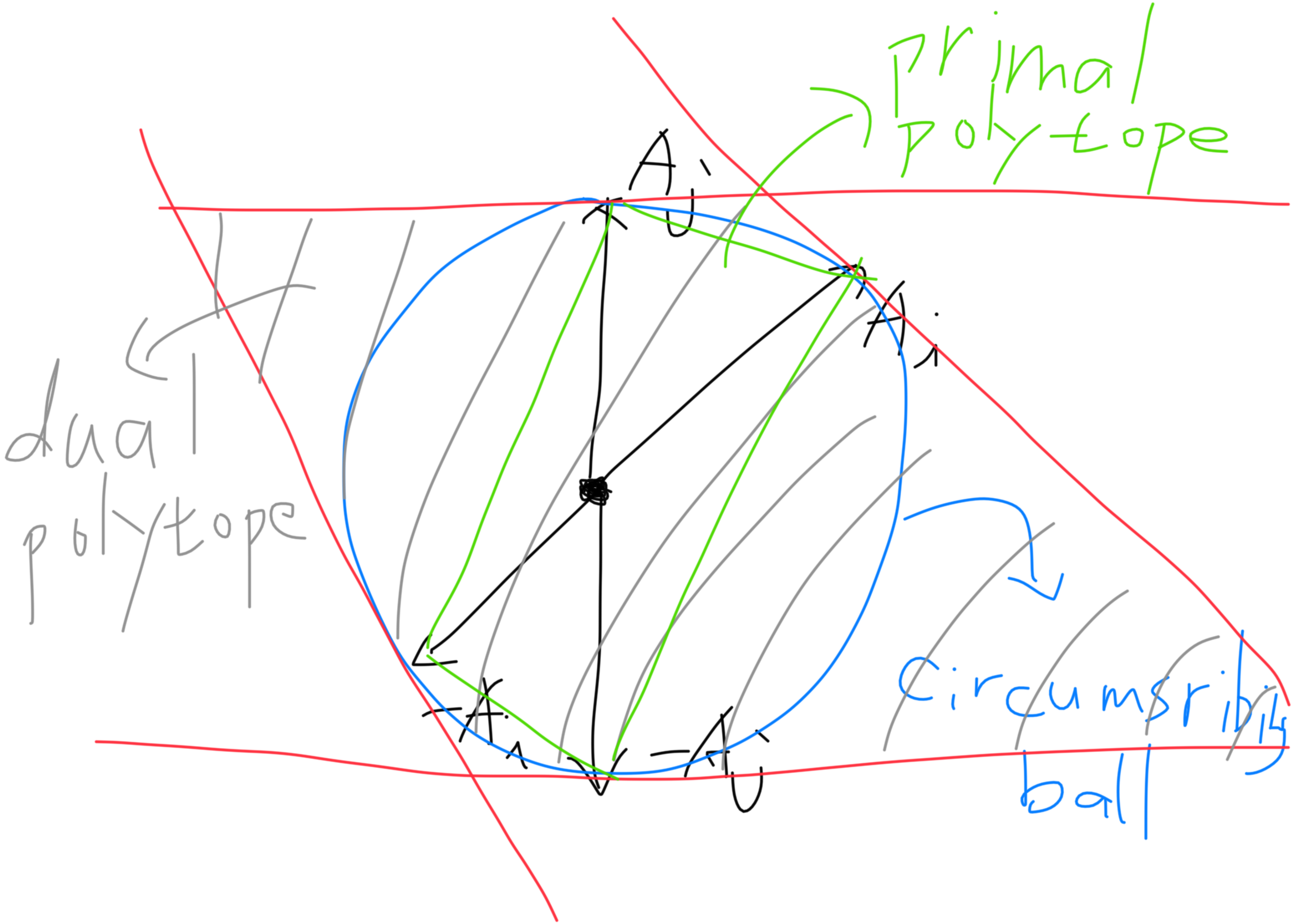
- Interpret SNN as a natural algorithm for solving its dual linear programming.
- Seems a new algorithm for the ℓ_1 minimization problem.

Dual Interpretation of SNN Dynamics

- Dual **potential**: $\mathbf{v}(t)$ where $\mathbf{u}(t) = A^{\mathrm{T}}\mathbf{v}(t)$.
- **External charging rate** $\mathbf{b}$ where $\mathbf{I} = A^{\mathrm{T}}\mathbf{b}$: walk along $\mathbf{b}$ direction.
- Spike when hit a ``wall'' $\{\mathbf{v}: A_i^{\mathrm{T}}\mathbf{v} = \eta\}$, and $\mathbf{v}(t)$ is bounced back by A_i.

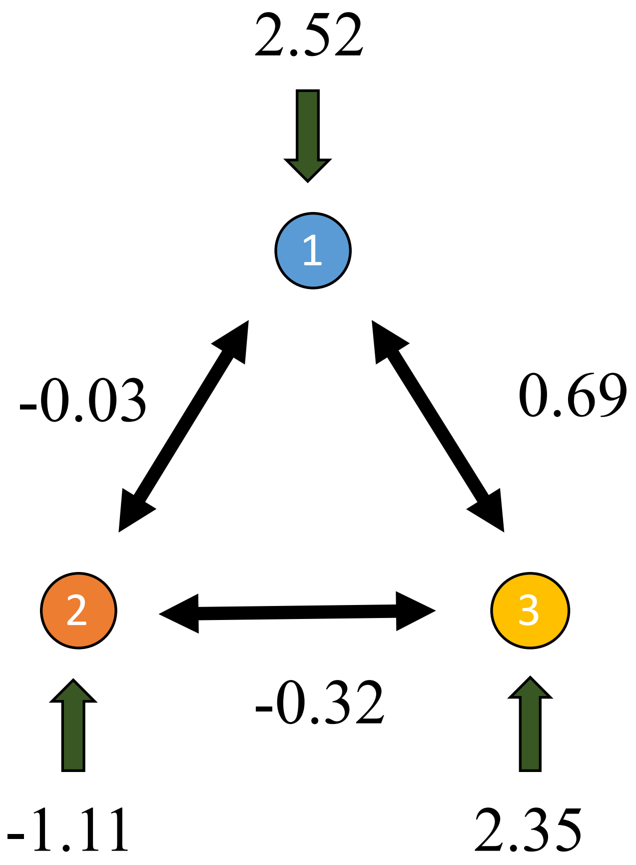


$$
\begin{aligned}
\underset{\mathbf{v}\in\mathbb{R}^m}{\text{maximize}} \quad & \mathbf{b}^{\mathsf{T}}\mathbf{v} \\
\text{subject to} \quad & \|A^{\mathsf{T}}\mathbf{v}\|_\infty \le \eta.
\end{aligned}
$$


primal polytope
A_j
A_i
dual polytope
$-A_i$
$-A_j$
circumsribing ball

Peculiar features of SNN algorithms

- Simple dynamics solves non-trivial optimization problems
- Distributed algorithm with extremely simple communication
- Solutions encoded as spike firing rate, not potential
- Potential practical efficiency for sparse optimization problem



Research questions

- **Complexity of SNN**:

  Prove convergence rate of firing rate to optimal solution.

- **Algorithms behind SNN**:

  Understand the underlying algorithmic ideas.

- **Power of SNN**:

  Understand the class of optimization problems solvable by SNN

- **Efficiency of SNN**:

  Understand practical efficiency of SNN as optimization solver.