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Topological Method - Simplicial Complex

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In which we introduce simplicial complex, which is an important element in topology.

Simplicial complex is an important bridge that connects topology with combinatorics. In the world of geometry, we care about *continuity, close, and neighborhood.* When we define a topological space, the three notions will become well-defined. That is to say, in a topological space, the *nearness* and *continuity* will be captured. However, when we turn to combinatorics, things become nontrivial. It's difficult to imagine what is neighbor, what is continuous. Surprisingly, there's a connection between topology and combinatorics through simplicial complex. With this connection, we can rephrase the theorems in topology into combinatorics and derive nice results.

1 Basic Topology

1.1 Topological Space

Topology, unlike geometry, captures only the *nearness* and *continuity* in a space instead of following concrete distance measure. Formally, we define a topological space as follow:

Definition 1 (topological space) *A topological space is a pair* $(X, \mathcal{O})$ *where*

- X *is the ground set.*
- $\mathcal{O} \subseteq 2^X$ *is a set system whose members are called the open sets such that their intersection/union are also open.*

For two topological spaces, we can define mapping among their ground sets to understand their relationship. However, we are only interested in the mappings that are *continuous.*

Definition 2 (continuous mapping) *Let* $(X_1, \mathcal{O}_1)$ *and* $(X_2, \mathcal{O}_2)$ *be two topological spaces. A mapping* $f : X_1 \to X_2$ *is continuous if* $\forall V \in \mathcal{O}_2$, $f^{-1}(V) \in \mathcal{O}_1$.

Furthermore, when the inverse of a function is also continuous, we call it *homeomorphism.*

Definition 3 (homeomorphism) $\phi : X_1 \to X_2$ *is a homeomorphism among topological spaces* $(X_1, \mathcal{O}_1)$ *and* $(X_2, \mathcal{O}_2)$ *if* $\forall U \subseteq X_1$, $\phi(U) \in \mathcal{O}_2 \Leftrightarrow U \in \mathcal{O}_1$. *Equivalently,* ϕ *and* ϕ^{-1} *are continuous.*

In topology, when there exists a homeomorphism between two topological spaces, we consider the two as *the same.* Note that as two spaces might be very different geometrically under certain metric, they might be topologically the same as long as there's a homeomorphism among them.

1.2 Homotopy Equivalence and Homotopy

Previously, we say that when there's a homeomorphism between two topological spaces, they are considered topologically the same. However, in the view of *algebraic topology*, this definition is not useful enough. Namely, we need to define a **coarser** equivalence relation for topological spaces.

First, we introduce the notion of *deformation retract*.

Definition 4 (deformation retraction) *A deformation retraction from X onto Y is a family $\{f_t\}_{t\in[0,1]}$ of continuous maps $f_t : X \to X$ such that*

1. $f_0(x) = x$, $\forall x \in X$.
2. $f_t(y) = y$, $\forall y \in Y$ *and* $t \in [0, 1]$.
3. $f_1(X) = Y$.
4. *Define* $F : X \times [0, 1] \to X$ *by* $F(x, t) = f_t(x)$, F *is continuous.*

Intuition (deformation retraction)

Intuitively, if there exists a deformation retraction from X onto Y, it means that one can *continuously shrink* X to Y while all the points of Y are fixed. With this new notion mapping, one can define an equivalence relation called homotopy equivalence.

Definition 5 (homotopy equivalence) *If Y is a deformation refract of X, then we say X and Y are homotopy equivalence.*

There's another definition of homotopy equivalence based on homotopic mapping.

Definition 6 (homotopic mapping) *Two continuous mapping $f, g : X \to Y$ are homotopic if there exists a continuous interpolation $\{f_t\}_{t\in[0,1]}$ which is continuous in t and $f_0 = f$, $f_1 = g$.*

That is, with some nontrivial argument, one can show that homotopy equivalence is equivalent to the following functional definition.

Definition 7 ((functional version) homotopy equivalence) *X and Y are homotopy equivalence if $\exists f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.*

Intuition (homotopy equivalence)

Two topological spaces are homotopy equivalence iff one can twist, pull, or press one space to make it become the other.

2 Geometric Simplicial Complexes

In 2-dimensional space, every polyhedron can be constructed from triangles. As a result, it becomes a natural question to ask can we do so in high-dimensional space. The answer is yes, with little changes since we do not have triangle as the simplest building block. Instead, we use **simplex** to do so. As along as we can use simplexes to construct high-dimensional polyhedron, it becomes easier for us to analyze their topological properties. In this section, we will gp through the basic definition of simplexes and see how to mathematically construct high dimensional object with these small simplexes.

Definition 8 (affinely dependent) *Let $v_0,\dots,v_k \in \mathbb{R}^d$, we say they are affinely dependent if $\exists a_0,\dots,a_k$ not all zero such that $\sum_{i=0}^k a_i v_i = 0$ and $\sum_{i=0}^k a_i = 0$. Otherwise, we call them affinely independent.*

Lemma 9 (characterization of affinely independent) *The following are equivalent:*

- *$v_0,\dots,v_k$ are affinely independent.*
- *$v_1 - v_0,\dots,v_k - v_0$ are linearly independent.*
- *$(1,v_0),\dots,(1,v_k)$ are linearly independent.*

Intuition (affinely independent)

If $k+1$ vectors are affinely independent, it means that they do not simultaneously lie on a k-dimensional hyperplane.

Definition 10 (simplex) *A simplex σ is the convex hull of a finite affinely independent set A in $\mathbb{R}^d$.*

- *The points in A are called vertices.*
- *The dimension of σ is $|A|-1$.*
- *The convex hull of an arbitrary subset of vertices of σ is a face of σ.*
- *The relative interior of σ arises from σ by removing all faces of dimension less than the dimension of σ.*

Intuition (simplex)

A k-dimensional simplex is an unit piece in k-dimensional space which are homotopy equivalence to each other. Geometrically, one can also think of a k-dimensional simplex as a k-dimensional analog of triangle.

For example, the 1-dimensional simplex is a line segment, the 2-dimensional simplex is a triangle, the 3-dimensional simple is a triangular pyramid.

Definition 11 (simplicial complex) *A nonempty family Δ of simplexes is a simplicial complex if the following hold:*

1. *Each face of $\sigma \in \Delta$ is also a simplex of Δ.*
2. *$\forall \sigma_1, \sigma_2 \in \Delta$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.*

Moreover,

- *The union of simplexes in Δ is the* **polyhedron** *of Δ and is denoted as $||\Delta||$.*
- *The dimension of Δ is $dim(\Delta) := \max\{dim(\sigma)|\sigma \in \Delta\}$.*
- *The vertex set of Δ is $V(\Delta) := \cup_{\sigma\in\Delta}\{vertices\ in\ \sigma\}$.*
- *The relative interiors of all the simplexes in Δ form a partition of $||\Delta||$ in the sense that $\forall x \in ||\Delta||$, exists exactly one $\sigma \in \Delta$ containing x in its relative interior. Denote this simplex as $supp(x)$.*

For instance, the set of all faces of a simplex is a simplicial complex.

3 Triangularization

Finally, after defining simplicial complex, we can introduce the concept of *triangularization*. Intuitively, a simplex is a high-dimensional triangle and a simplicial complex is a family of closed simplexes. That is, one can think of a simplicial complex as being partitioned by lots of triangles in different dimensions. With this idea in mind, we can define the triangularization of a topological space by the homotopic equivalence to a simplicial complex.

Definition 12 (triangularization) *Let X be a topological space, and Δ be a simplicial complex. If $X \cong ||\Delta||$. Then, we call Δ a triangularization of X.*

As long as we know the triangularization of a topological space, we can work on the simplicial complex. Surprisingly, when it comes to simplicial complex, there are a bunch of nice connection with algebraic object. In the next post, we will focus on these geometric-algebraic connection.