1 Introduction

This project belongs to the broad research of *natural algorithms*, which aim for understanding and explaining algorithms observed from natural systems and dynamics. In recent years, there have been several interesting research on natural algorithms in wide range of contexts, such as bird flocking [Cha12, Cha09], evolution [LP16, LPR$^+$14], slime system [NYT00, TKN07, BMV12], and sandpile model [CV12]. As a successful example, in a seminal work of Nakagaki Yamada, and Tóth [NYT00] published in Nature, they reported an experiment demonstrating that *Physarum Polycephalum*, a simple amoeboid organism, can find shortest path collectively in a maze. This phenomena was investigated by a sequence of works [TKN07, MO08, BMV12] mathematically, where Bonifaci et al. [BMV12] showed that the phenomena can be explained as a distributed algorithm among the slimes based on connection to electrical network.

In this proposal, we will focus on investigating certain classes of *spiking neural networks (SNN)* as natural algorithms for solving convex optimization problems. In general, a SNN refers to a set of neurons that can signal to each other by spikes through their synaptic connections (i.e., edges between two neurons). There has been rich literature in neural science searching for mathematical models to explain and predict various (statistical) behaviors of different types of SNN [I$^+$03, Adr26, Lap07]. In particular, Barrett et al. [BDM13] was interested in the firing rate (i.e., the number of spikes of a neuron in a fixed time interval) of a certain class of SNN, and observed that it can be characterized as the optimal solution of certain quadratic programming problems defined by the parameters of the SNN (in an so-called integrate-and-fire model).

On the other hands, Shapero et al. [SRH13, SZHR14] viewed SNN dynamics as a computational framework, and proposed a class of SNN to solve the *least absolute shrinkage and selection operator (Lasso)* optimization problem, an important problem widely used in the machine learning context. Furthermore, from personal communications with an Intel researcher Dr. Tsung-Han Lin [], there are experimental evidence that such SNN dynamics have comparable efficiency to other Lasso solvers with an advantage of fast identification of sparse variables in Lasso solutions. Indeed, Tsung-Han mentioned potential plan of hardware implementation of SNN dynamics at Intel as alternative computation architecture. However, to the best of our knowledge, no explicit complexity bounds on the runtime of the SNN dynamics (defined appropriately) for solving optimization problems have been proved.

Viewing SNNs as natural computation model, it has several peculiar features. In particular, the computation result is encoded as *firing rate* of the system. It is also a highly parallel and distributed computation with extremely simple communication (i.e., via spike signals). We are intrigued by the power of such simple distributed model for solving complex optimization problems with potentially competitive practical performance.

The major goal of this proposal is to theoretically better understand the computational power of the SNN dynamics. In particular, we identified a simple SNN dynamics which can potentially solve the ℓ_1 minimization problem (a.k.a. the basis pursuit problem in the context of compressed sensing) in a seemingly different way from existing algorithms. As our first step, we will focus on analyzing the complexity of this SNN dynamics and understanding the algorithmic ideas behind it. Next, we hope that the findings there can help us investigating the computational power of the SNN in general as well as designing new SNN inspired algorithms for other optimization problems. As a long term goal, we will also compare SNN with other related natural algorithm models, such as the sandpile model [CV12] to systematically understand the power of natural algorithms.

2 Spiking Neuron Networks

In this section, we introduce the spiking neuron network (SNN) model we will investigate in this proposal. We will focus on the simple SNN dynamics mentioned above, and briefly discuss other variants in literature. We then discuss our observation about the simple SNN dynamics as an algorithm for the ℓ_1 minimization problem, and interesting research questions we will investigate in this proposal.

The simple SNN model. A SNN model is a dynamic system consists of n neurons, where each neuron i is associated with a time-varying *potential* denoted by $\mathbf{u}_i(t) \in \mathbb{R}$ at time $t \geq 0$ with $\mathbf{u}_i(0) = 0$. Each neuron i receives a steady rate of external charging specified by a value $\mathbf{I}_i \in \mathbb{R}$; namely, the potential of neuron i is added by $\mathbf{I}_i$ in one unit of time (note that $\mathbf{I}_i \in \mathbb{R}$ may be negative). There is also a *threshold* parameter $\eta > 0$ such that a neuron i will *fire* a positive (resp., negative) spike when its potential $\mathbf{u}_i(t) > \eta$ (resp., $\mathbf{u}_i(t) < -\eta$), which will cause inhibition or exhibition to its neighboring neurons.[1] Specifically, this is described by a *connectivity matrix* C, where when neuron i fires a positive (resp., negative) spike, the potential of neuron j will decrease (resp., increase) by C_{ij}. In particular, the potential of neuron i itself is decreased (resp., increase) by C_{ii}. An example with $n = 3$ is illustrated in Figure 1.

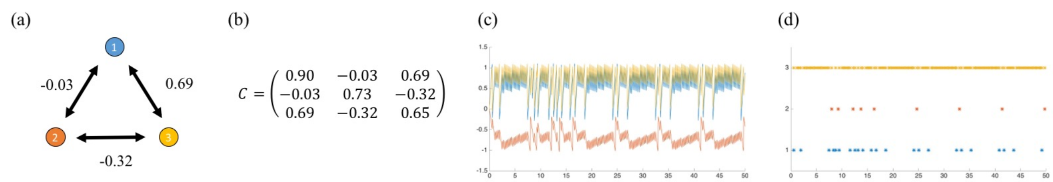

Figure 1: An example when $n = 3$. There are three neurons: blue, orange, and yellow. The connectivities among them are represented geometrically in (a) and written down in matrix form in (b). The external charging rate $\mathbf{I} = (2.175, -0.95, 2.025)$. The neuron potentials in the first 50 seconds are plotted in (c). In (d), the razer plot, recorded the timing when neurons fired.

The dynamics of a SNN can be described by a differential equation as follows.

$$\frac{d\mathbf{u}(t)}{dt} = -C\mathbf{s}(t) + \mathbf{I},$$

where $\mathbf{s}(t)$ are variables that record the spike trains of the system. Namely, for each $i \in [n]$, $\mathbf{s}_i(t) = \sum_k \delta(t - t^+_{i,k}) - \sum_k \delta(t - t^-_{i,k})$, where neuron i fires positive (resp., negative) spikes at time $\{t^+_{i,k}\}$ (resp., $\{t^-_{i,k}\}$). For any time $T > 0$, let $\mathbf{S}(T) := \int_0^T \mathbf{s}(t)dt$ be the vector recording total number spikes from time 0 to T. The *firing rate* at time T is defined to be $\hat{\mathbf{x}}(T) := \mathbf{S}(T)/T$. Figure 1 is an example when $n = 3$.

The SNN dynamics defined above can be viewed as a simplified variant of existing models in literature [BDM13, TUKM15, SRH13, SZHR14] which have certain extra terms in the differential equation. In the variant of [BDM13], they observed that when $C = A^\intercal A$ and $\mathbf{I} = A^\intercal \mathbf{b}$

[1] We remark that the SNN models in neural science typically consider only positive spike. Here we consider two-sided (both positive and negative) spikes for convenience and note that there is a simple reduction to one-sided model.

for some matrix A and vector b, the firing rate is characterized by the optimal solution of the quadratic programming problem $\min_{\mathbf{x}} \mathbf{x}^\intercal A^\intercal A\mathbf{x} - 2\mathbf{x}^\intercal \mathbf{b}$. In the variant of [SRH13, SZHR14] (where certain "drift" term λ is added), the firing rate is characterized by the following Lasso instance $\min_{\mathbf{x}}(1/2)\|A\mathbf{x} - \mathbf{b}\|_2^2 + \lambda \cdot \|\mathbf{x}\|_1$.

The simple SNN dynamics as an ℓ_1 minimization algorithm. We now discuss our preliminary findings about the simple SNN dynamics, where we observed that it is a potential new algorithm to solve the ℓ_1 minimization problem. Specifically, consider

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \|\mathbf{x}\|_1 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}. \end{aligned} \tag{1}$$

We conjecture that the firing rate $\hat{\mathbf{x}}(t)$ of the simple SNN dynamics with $C = A^\intercal A$ and $\mathbf{I} = A^\intercal \mathbf{b}$ converges to the optimal solution of the above problem, denoted by $\mathbf{x}^*$.

To see how the simple SNN dynamics solves the ℓ_1 minimization problem, our key observation is to consider the SNN dynamics in a dual "$\mathbf{v}$-space" that is informally connected by $\mathbf{u} = A^\intercal \mathbf{v}$. Concretely, let $\mathbf{v}(0) = \mathbf{0}$. The external charging rate in the $\mathbf{v}$-space is $\mathbf{b}$, and the firing rule in the $\mathbf{v}$-space becomes as follows. Neuron i fires a positive (resp., negative) spike when $A_i^\intercal \mathbf{v}(t) > \eta$ (resp., $A_i^\intercal \mathbf{v}(t) < -\eta$), where A_i is the i-th column of A, and $\mathbf{v}$ is updated by adding $-A_i$ (resp., A_i). Namely, the dynamics in the $\mathbf{v}$-space can be described by the following differential equation.

$$\frac{d\mathbf{v}(t)}{dt} = -A\mathbf{s}(t) + \mathbf{b}.$$

It is easy to see by construction that $\mathbf{u}(t) = A^\intercal \mathbf{v}(t)$ for all $t \geq 0$, i.e., there is a bijection between the primal $\mathbf{u}$-space and the dual $\mathbf{v}$-space. Now the dynamics in the $\mathbf{v}$-space has the following clean geometric interpretation: Define a "feasible" polytope $\mathcal{P} = \{\mathbf{v} : |A_i^\intercal \mathbf{v}| \leq \eta \ \forall i \in [n]\}$. One can view $\mathbf{v}(t)$ as walking along the $\mathbf{b}$ direction, and when it hits the boundary of $\mathcal{P}$, say, the "wall" defined by $A_i^\intercal \mathbf{v} = \eta$, it is bounced back by the normal vector of the wall, A_i. See Figure 2 for an illustration.

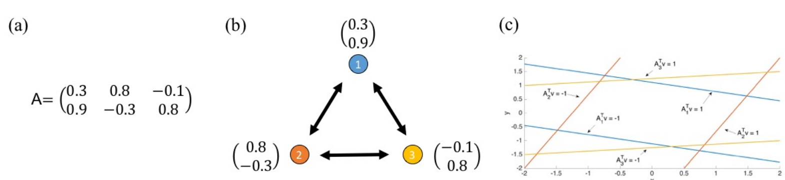

Figure 2: (a) The corresponding SNN dynamics as Figure 1 with $C = A^\intercal A$. (b) Each neuron i is associated with A_i. (c) The feasible polytope $\mathcal{P}$ of dual $\mathbf{v}$-space defined by the "walls" A_i's.

Intuitively, the dynamics takes $\mathbf{v}(t)$ towards the corner of $\mathcal{P}$ defined by the b direction, which is the optimal solution of the following optimization problem.

$$\begin{aligned} \underset{\mathbf{v}\in\mathbb{R}^m}{\text{maximize}} \quad & \mathbf{b}^\intercal \mathbf{v} \\ \text{subject to} \quad & \|A^\intercal \mathbf{v}\|_\infty \leq \eta. \end{aligned} \tag{2}$$

This is exactly the dual linear program of the above ℓ_1 minimization problem. To summarize, we believe that the simple SNN dynamics solves the ℓ_1 minimization problem by solving the above dual linear program implicitly in the dual $\mathbf{v}$-space.

Research problems about the simple SNN dynamics. A major concrete goal of this proposal is to prove that the simple SNN dynamics indeed solves the ℓ_1 minimization problem. While we do not know how to prove it yet, we have partial evidence that the simple SNN dynamics may be a new algorithm for the ℓ_1 minimization problem with competitive parallel time complexity. In particular, we know how to analyze an "idealized" simple SNN dynamics where the spike strength is sufficiently small, i.e., the effect of a spike is changed to $\alpha \cdot C_{ij}$ for sufficiently small $\alpha > 0$. In the dual $\mathbf{v}$-space, this corresponds to sufficiently small bounced back force αA_i when $\mathbf{v}$ hits the wall defined by A_i, and intuitively, $\mathbf{v}$ can move "smoothly" along the walls towards the optimal corner defined by the $\mathbf{b}$ direction. In this setting, we are able to prove a converge rate of the firing rate to the optimal solution via a perturbation theorem [**?**] for linear programming. The way that the problem is solved in the dual $\mathbf{v}$-space is similar to a gradient projection algorithm, but the simple SNN dynamics do not need to compute expensive projections. We formulate the following optimistic conjecture, which if true, can solve the ℓ_1 minimization problem in $O(n^2)$ parallel time and in $O(n^3)$ sequential setting for instances with well-conditioned matrix A.

Conjecture 1 *Let $\mathbf{x}^*$ be one of the solution of the ℓ_1 minimization problem defined in* (1) *for $m \times n$ matrix A and length m vector $\mathbf{b}$. Let $\mathbf{x}(T)$ be the firing rate of SNN at time T, then $|\mathbf{x}(T) - \mathbf{x}^*|/|\mathbf{x}^*| = O(n^2/T)$ and $\|A\mathbf{x}(T) - \mathbf{b}\|_2 = O(n^2/T)$.*

We hope that resolving this conjecture can reveal deeper algorithmic insights on the simple SNN dynamics, which on one hand, can potentially lead to new algorithms for solving other optimization problems such as general linear programs, and on the other hand, can help design other spike inspired algorithms. It may also be interesting to explore the practical efficiency of the simple SNN dynamics.

3 Potential Impact of the Project

As hinted above, we learned the fascinating SNN dynamics from inspiring personal communication with Dr. Tsung-Han Lin at Intel, who also actively investigate variant SNN dynamics as natural algorithms for solving optimization problems. We have already collected several non-trivial findings beyond our prior communication. We believe that our project, if successful, can lead to novel results that interest research labs at Intel. We also had several discussions about the SNN dynamics with several theorists such as Man Cho Anthony So, Luca Trevisan, Aleksander Madry, Richard Peng, Zhenming Liu, and Shengyu Zhang. They all provided valuable inputs and expressed interests on the subject. We believe our potential results will be interesting to the general theoretical CS communication.

References

[Adr26] Edgar D Adrian. The impulses produced by sensory nerve endings. *The Journal of physiology*, 61(1):49–72, 1926.

[BBDM12] Ralph Bourdoukan, David Barrett, Sophie Deneve, and Christian K Machens. Learning optimal spike-based representations. In *Advances in neural information processing systems*, pages 2285–2293, 2012.

[BDM13] David G Barrett, Sophie Denève, and Christian K Machens. Firing rate predictions in optimal balanced networks. In *Advances in Neural Information Processing Systems*, pages 1538–1546, 2013.

[BMV12] Vincenzo Bonifaci, Kurt Mehlhorn, and Girish Varma. Physarum can compute shortest paths. *Journal of Theoretical Biology*, 309:121–133, 2012.

[BRR12] Aurèle Balavoine, Justin Romberg, and Christopher J Rozell. Convergence and rate analysis of neural networks for sparse approximation. *Neural Networks and Learning Systems, IEEE Transactions on*, 23(9):1377–1389, 2012.

[Cha09] Bernard Chazelle. Natural algorithms. In *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 422–431. Society for Industrial and Applied Mathematics, 2009.

[Cha12] Bernard Chazelle. Natural algorithms and influence systems. *Communications of the ACM*, 55(12):101–110, 2012.

[CV12] Ayush Choure and Sundar Vishwanathan. Random walks, electric networks and the transience class problem of sandpiles. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pages 1593–1611. SIAM, 2012.

[CW05] Patrick L Combettes and Valérie R Wajs. Signal recovery by proximal forward-backward splitting. *Multiscale Modeling & Simulation*, 4(4):1168–1200, 2005.

[DDDM04] Ingrid Daubechies, Michel Defrise, and Christine De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications on pure and applied mathematics*, 57(11):1413–1457, 2004.

[DT08] David L Donoho and Yaakov Tsaig. Fast solution of-norm minimization problems when the solution may be sparse. *IEEE Transactions on Information Theory*, 54(11):4789–4812, 2008.

[Fri55] KR Frisch. The logarithmic potential method of convex programming. *Memorandum, University Institute of Economics, Oslo*, 5(6), 1955.

[HYZ07] Elaine T Hale, Wotao Yin, and Yin Zhang. A fixed-point continuation method for l1-regularized minimization with applications to compressed sensing. *CAAM TR07-07, Rice University*, 43:44, 2007.

[I$^+$03] Eugene M Izhikevich et al. Simple model of spiking neurons. *IEEE Transactions on neural networks*, 14(6):1569–1572, 2003.

[Kar84] Narendra Karmarkar. A new polynomial-time algorithm for linear programming. In *Proceedings of the sixteenth annual ACM symposium on Theory of computing*, pages 302–311. ACM, 1984.

[KKL$^+$07] Seung-Jean Kim, Kwangmoo Koh, Michael Lustig, Stephen Boyd, and Dimitry Gorinevsky. An interior-point method for large-scale-regularized least squares. *IEEE journal of selected topics in signal processing*, 1(4):606–617, 2007.

[KMM93] Masakazu Kojima, Nimrod Megiddo, and Shinji Mizuno. Theoretical convergence of large-step primaldual interior point algorithms for linear programming. *Mathematical Programming*, 59(1-3):1–21, 1993.

[Lap07] Louis Lapicque. Recherches quantitatives sur lexcitation électrique des nerfs traitée comme une polarisation. *J. Physiol. Pathol. Gen*, 9(1):620–635, 1907.

[LP16] Adi Livnat and Christos Papadimitriou. Sex as an algorithm: the theory of evolution under the lens of computation. *Communications of the ACM*, 59(11):84–93, 2016.

[LPR+14] Adi Livnat, Christos Papadimitriou, Aviad Rubinstein, Gregory Valiant, and Andrew Wan. Satisfiability and evolution. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*, pages 524–530. IEEE, 2014.

[MA89] Renato DC Monteiro and Ilan Adler. Interior path following primal-dual algorithms. part i: Linear programming. *Mathematical programming*, 44(1-3):27–41, 1989.

[Maa97] Wolfgang Maass. Networks of spiking neurons: the third generation of neural network models. *Neural networks*, 10(9):1659–1671, 1997.

[MCW05] Dmitry M Malioutov, Müjdat Cetin, and Alan S Willsky. Homotopy continuation for sparse signal representation. In *Proceedings.(ICASSP'05). IEEE International Conference on Acoustics, Speech, and Signal Processing, 2005.*, volume 5, pages v–733. IEEE, 2005.

[Meg89] Nimrod Megiddo. Pathways to the optimal set in linear programming. In *Progress in mathematical programming*, pages 131–158. Springer, 1989.

[MO08] Tomoyuki Miyaji and Isamu Ohnishi. Physarum can solve the shortest path problem on riemannian surface mathematically rigorously. *International Journal of Pure and Applied Mathematics*, 47(3):353–369, 2008.

[NW+07] Robert D Nowak, Stephen J Wright, et al. Gradient projection for sparse reconstruction: Application to compressed sensing and other inverse problems. *IEEE Journal of selected topics in signal processing*, 1(4):586–597, 2007.

[NYT00] Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth. Intelligence: Maze-solving by an amoeboid organism. *Nature*, 407(6803):470–470, 2000.

[OPT00] Michael R Osborne, Brett Presnell, and Berwin A Turlach. A new approach to variable selection in least squares problems. *IMA journal of numerical analysis*, 20(3):389–403, 2000.

[PMB12] Hélene Paugam-Moisy and Sander Bohte. Computing with spiking neuron networks. In *Handbook of natural computing*, pages 335–376. Springer, 2012.

[RJBO08] Christopher J Rozell, Don H Johnson, Richard G Baraniuk, and Bruno A Olshausen. Sparse coding via thresholding and local competition in neural circuits. *Neural computation*, 20(10):2526–2563, 2008.

[SRH13] Samuel Shapero, Christopher Rozell, and Paul Hasler. Configurable hardware integrate and fire neurons for sparse approximation. *Neural Networks*, 45:134–143, 2013.

[SZHR14] Samuel Shapero, Mengchen Zhu, Jennifer Hasler, and Christopher Rozell. Optimal sparse approximation with integrate and fire neurons. *International journal of neural systems*, 24(05):1440001, 2014.

[Tan16] Ping Tak Peter Tang. Convergence of lca flows to (c) lasso solutions. *arXiv preprint arXiv:1603.01644*, 2016.

[Tib96] Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 267–288, 1996.

[TKN07] Atsushi Tero, Ryo Kobayashi, and Toshiyuki Nakagaki. A mathematical model for adaptive transport network in path finding by true slime mold. *Journal of theoretical biology*, 244(4):553–564, 2007.

[TUKM15] Dominik Thalmeier, Marvin Uhlmann, Hilbert J Kappen, and Raoul-Martin Memmesheimer. Learning universal computations with spikes. *arXiv preprint arXiv:1505.07866*, 2015.

[WNF09] Stephen J Wright, Robert D Nowak, and Mário AT Figueiredo. Sparse reconstruction by separable approximation. *IEEE Transactions on Signal Processing*, 57(7):2479–2493, 2009.