Graph Theory **December 10, 2016**

Topological Method: Kneser Conjecture

Notes: Chi-Ning Chou

In which we introduce the basic idea of topological method in graph theory.

Topology studies the abstract structures in both geometry and set theory. It concerns the continuity, compactness, and open set in a general setting. Surprisingly, topology connects combinatorics with geometry and provides beautiful techniques for graph theorist to prove the problem that are difficult to be dealt with with pure combinatorics method.

In these line of survey, I want to study how people apply topological method in graph theory and other combinatorics studies. Let's start with a simple example to motivate this mysterious techniques.

1 Kneser's conjecture

In 1955, Kneser posed a problem about the mapping over some sets, which had been called *the Kneser's conjecture*. Although the problem is purely combinatorics, surprisingly, Lovász proved the Kneser's conjecture with a theorem in topology, the Borsuk-Ulam theorem. Before we state the Borsuk-Ulam theorem and the proof for Kneser's conjecture, let's first see the statement of the conjecture.

Definition 1 (Kneser's graph) *Let n, k be two positive integers where $n > k$. Define the Kneser's graph $KG_{n,k} = (V, E)$ as*

- $V = \binom{[n]}{k}$, *which is the set containing all size k subsets in $[n]$.*
- $E = \{(S_1, S_2) | S_1, S_2 \in \binom{[n]}{k}, S_1 \cap S_2 = \emptyset\}$.

The Kneser's conjecture is about the *chromatic number*, a.k.a. the minimum coloring, of Kneser's graph, i.e. $\chi(KG_{n,k})$. Since Lovász proved the correctness of this conjecture, we state it as a theorem in the following.

Theorem 2 (Lovász-Kneser) *For all $k > 0$ and $n \geq 2k - 1$, we have $\chi(KG_{n,k}) = n - 2k + 2$.*

The achievability part, i.e. the upper bound, is relatively easy. The difficulty lies in the lower bound part, which was proved with Borsuk-Ulam theorem.

Theorem 3 (Borsuk-Ulam) *The following are equivalent and true.*

- *(BU1a)* $\forall$ *continuous mapping* $f : S^n \to \mathbb{R}^n$, $\exists x \in S^n$ *such that* $f(x) = f(-x)$.

- *(BU1b) Let $g : S^n \to \mathbb{R}^n$ be continuous and $\forall x \in S^n$, $g(x) = -g(-x)$. Then, $\exists x \in S^n$ such that $g(x) = 0$.*
- *(LSc) S^n is covered by closed sets $F_1, \ldots, F_{n+1}$, then $\exists F_i$ and $x \in S^n$ such that $x, -x \in F_i$.*
- *(LSd) S^n is covered by open sets $U_1, \ldots, U_{n+1}$, then $\exists U_i$ and $x \in S^n$ such that $x, -x \in U_i$.*

2 Proof of Kneser's conjecture

PROOF:

- (Achievability) Construct a mapping $f : \binom{[n]}{k} \to [n-2k+2]$ for $KG_{n,k}$ by

$$f(S) = \min\{\min_{s \in S} s, n-2k+2\} \tag{1}$$

  To check the validity of this coloring, first assume two sets S_1, S_2 have the same color i. If $i < n-2k+1$, it means that $i \in S_1$ and $i \in S_2$. That is, there's no edge among S_1, S_2. If $i \geq n-2k+1$, it means that both S_1 and S_2 lie in $\{n-2k+1, \ldots, n\}$, which only has $2k-1$ elements. That is, S_1, S_2 must share at least one element. As a result, we can see that once two vertices in $KG_{n,k}$ have the same color, there's no edge among them.

- (Lower bound) The idea is to take n points on a S^d where $d = n-2k+1$. Then, assume the existence of coloring with d colors and use it to construct a covering. Finally, by applying Borsuk-Ulam theorem, one can find a contradiction. Thus, the coloring does not exist.

  1. Take X to be n points on S^d such that there's no more than d points lying on the same hyperplane through the center. Assume there's a d-coloring for $KG_{n,k}$, we color the n points with this d-coloring.
  2. Define the covering $\{A_1, \ldots, A_{d+1}\}$ by $\forall x \in S^d$, $x \in A_i$ for $i \in [d]$ if there are at least one tuple of $S \in \binom{X}{k}$ of color i that lie on the hemisphere centered by x, i.e. $H(x) = \{y \in S^d | \langle x, y \rangle > 0\}$. As to A_{d+1}, take it by $S^d - \cup_{i \in [d]} A_i$.
  3. By Borsuk-Ulam's theorem, $\exists j \in [d+1]$ and $y \in S^d$ such that both y and $-y$ lie in A_j.
     - If $1 \leq j \leq d$, $\exists S_1, S_2 \in \binom{X}{k}$ of color i such that $\langle x, y \rangle > 0$ for all $x \in S_1$ and $\langle x, -y \rangle > 0$ for all $x \in S_2$. Namely, $S_1 \cap S_2 = \emptyset$. Thus, $(S_1, S_2) \in E(KG_{n,k})$, which is a contradiction.
     - If $j = d+1$, then exists $d+1$ points lie on an equator.

□

In this application of topological method on Kneser's conjecture, we map the coloring over sets to a covering of a high-dimensional sphere. Then, apply Borsuk-Ulam theorem, which concerned the covering over sphere, and yield the impossibility. Intuitively, there's a deep connection between sets and high-dimensional structure. In this post, we get a taste of it and in our later survey, we will see more interplay among geometry and combinatorics.