Minimal Steiner tree

- Steiner tree introduction
- Approximation algorithm

Minimal Steiner tree

- Given: connected graph $G = (V, E)$, set of vertices N: terminals
- Problem: find a subtree T of G covering N with minimal total length
    - Steiner tree spanning N
    - N: required vertices
    - $V - N$: Steiner vertices

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- Trivial cases
    - $N = V$: minimal spanner tree
    - $|N| = 2$: shortest path problem

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- Applications
  - Wire routing of VLSI
  - Customer's bill for renting communication networks
  - Facility location

Minimal Steiner tree Problem is NP-hard

- Prove the following decision version is NP-complete
- Instance:
  - an undirected graph $G = (V, E)$
  - a subset of the vertices $N \subseteq V$, called terminal nodes
  - a natural number k
- Question: is there a tree T with $\leq k$ edges covering N?

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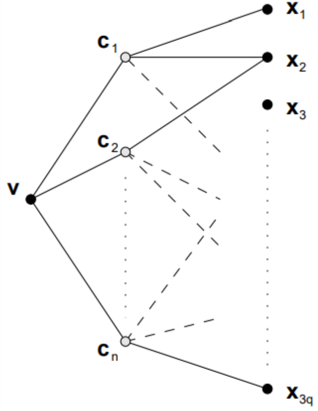
- How to show a problem P is NP-complete
  - show P is in NP
  - select a known NP-complete problem P'
  - construct a transformation f from P' to P
  - prove f is a polynomial transformation.

Minimal Steiner tree Problem is NP-hard

- Steiner Tree is in NP: easy
- Select a known NP-complete problem P'
- Exact Cover by 3-Sets (X3C)
- Instance:
    - a finite set X with $|X| = 3q$
    - a collection C of 3-element subsets of X, $C = \{C_1, \cdots, C_n\}$, $C_i \subseteq X, |C_i| = 3$
- Question: does C contain an exact cover for X, i.e., a subset $C' \subseteq C$ s.t. every element of X occurs in exactly one member of C'?
    - members of the solution C' form a partition of the set X
    - $|C'| = q$

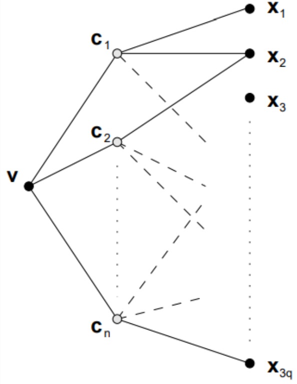
Transform X3C to ST

- Given an instance of X3C, build the Steiner Tree instance
  - create a new node v, a node for each member of C and each element of X: $V = \{v\} \cup \{c_1, \cdots, c_m\} \cup \{x_1, \cdots, x_{3q}\}$
  - $E = \{vc_1, \cdots, vc_n\} \cup \left(\bigcup_{x_i \in C_j} \{x_i c_j\}\right)$
  - $N = \{v, x_1, \cdots, x_{3q}\}$
  - $k = 4q$



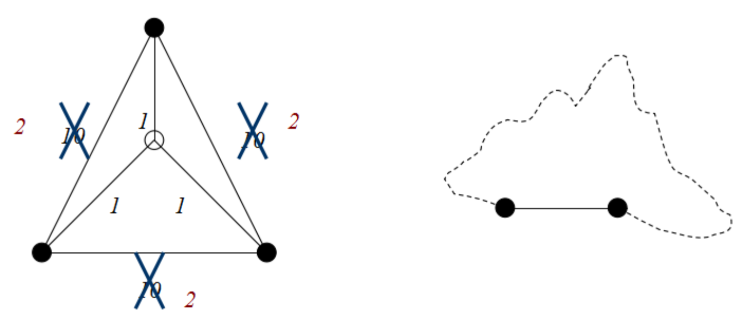
Transform X3C to ST

- If there is an exact 3-cover C'
- Suppose $C' = \{C_1, \cdots, C_q\}$
- the following tree is a solution of the Steiner tree problem
    - $vc_1, \cdots, vc_q$
    - $c_i x_j$, if $x_j \in C_i, 1 \le i \le q$



Metric Closure

- Replace the cost of uv by the cost of a shortest path between them
- Resulting graph: metric closure
- metric closure is a complete graph



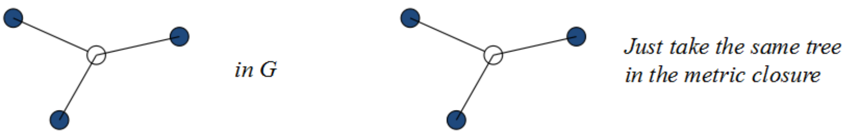
Lemma

In metric closure, edge costs satisfy triangle inequalities

Metric Closure

Lemma

The cost of a minimum Steiner tree in a graph is at least the cost of a minimum Steiner tree in its metric closure



Proof

For each edge, its cost in metric closure is at most its cost in G, so T in metric closure is no more expensive

Metric Closure

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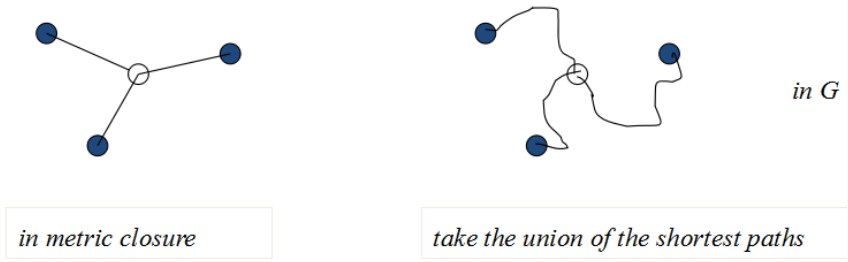
The cost of a minimum Steiner tree in a graph is at most the cost of a minimum Steiner tree in its metric closure



Proof

Let T' be a MST in metric closure. Consider the union U of the shortest paths in G. U has cost at most the cost of T'. U connects all the required vertices. So, U contains a ST with most the cost of T'.

Metric Closure

Theorem

The cost of a minimum Steiner tree in a graph equals to the cost of a minimum Steiner tree in its metric closure

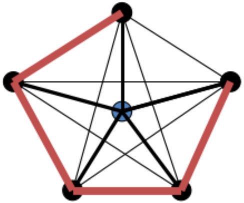
Metric Steiner Tree Problem

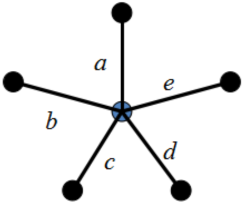
- Given a complete graph with edge costs satisfying triangle inequalities
- Find a minimum Steiner tree

Solve Metric Steiner Tree Problem

- A Steiner tree connects all the required vertices
- Can we just find a minimum spanning tree connecting all the required vertices?
  - In other words, forget about all Steiner vertices!
  - How bad can it be?

Solve Metric Steiner Tree Problem: idea





Cost of a minimum Steiner tree

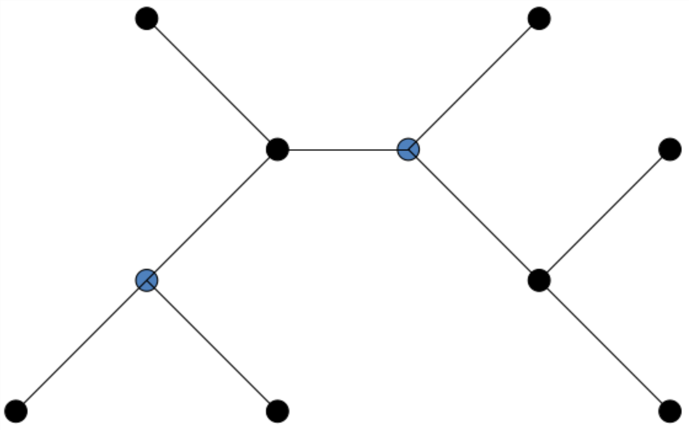
$$\mathrm{MST} \leq \underline{a + b} + \underline{b + c} + \underline{c + d} + \underline{d + e}$$

$$\leq 2a + 2b + 2c + 2d + 2e = 2(a + b + c + d + e)$$

$$\mathrm{MST} \leq 2\mathrm{OPT}$$

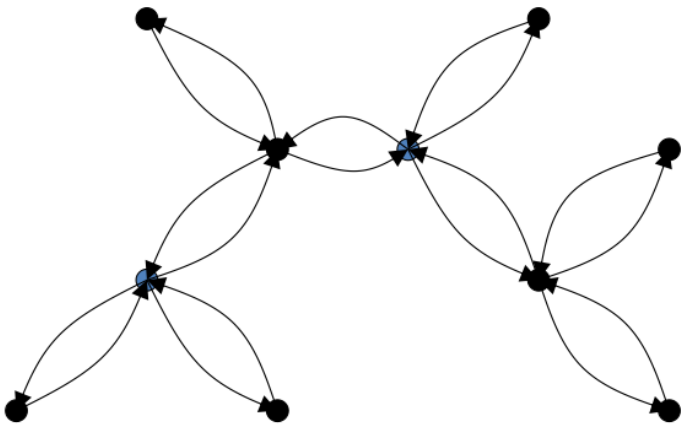
How to formalize the idea?

Let the cost of minimal Steiner tree be OPT



How to formalize the idea?

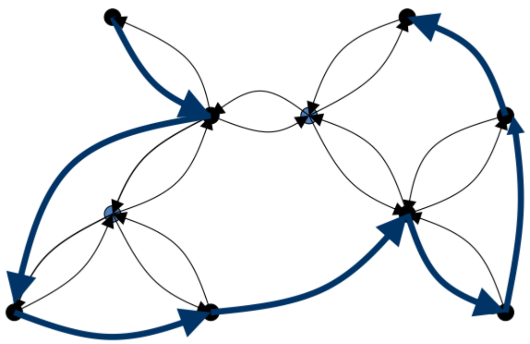
Double all the edges and find a deep first search traversal: cost=2OPT



How to formalize the idea?

Shortcut the traversal

- By triangle inequalites, the shortcut is not longer than the traversal
- The cost of MST $\leq$ the cost of traversal $\leq$ 2OPT
    - More precisely $(2 - 1/|T|)$OPT



Algorithm

- Compute metric closure G^* of G
- Find a minimum spanning tree T^* of G^*
- Construct the union U of shortest paths in G corresponding to edges in T^*
- Output the Steiner tree T contained in U

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- Find a minimum spanning tree T^* of G^*
- Construct the union U of shortest paths in G corresponding to edges in T^*
- Output the Steiner tree T contained in U
- T is contained in U: $cost(T) \leq cost(U)$
- G^* is metric closure of G: $cost(U) \leq cost(T^*)$
- Short-cutting: $cost(T^*) \leq 2OPT$
- Putting together: $cost(T) \leq 2OPT$

A worst-case scenario