FM2 Exam December 2019 Name: Jerry Jiang 98%

Excellent!

1. The relation R is defined on $\mathbb{Z}$ by $x\ R\ y$ if 5 divides $x + y$. Prove that R is not an equivalence relation.

for number 1 in $\mathbb{Z}$, $1+1=2$ and isn't a multiple of 5.

Hence, $1 \not R 1$, so it's not reflexive, thus not an equivalence relation.

2. Let S be the set of positive irrational numbers together with the number 1. Does $(S, \times)$ form a group?

$\sqrt{2}$ is a positive irrational number.

$\sqrt{2} \times \sqrt{2} = 2$ isn't a positive irrational number.

Without closure $(S, \times)$ isn't a group.

3. Use the Maclaurin series for e^{-x} and $\sin 2x$ to evaluate the limit $\lim\limits_{x\to 0} \dfrac{1-e^{-x}}{\sin 2x}$.

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$$\sin 2x = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \cdots$$

$$L = \lim_{x\to 0} \frac{1 - (1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots)}{2x - \frac{8x^3}{6} + \frac{32x^5}{120}}$$

$$= \lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \cdots}{2x - \frac{8x^3}{6} + \frac{32x^5}{120}}$$

$$= \lim_{x\to 0} \frac{1 - \frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{24} + \cdots}{2 - \frac{8x^2}{6} + \frac{32x^4}{120}}$$

$$= \frac{1}{2}$$

$$\frac{e^{-x}}{2\cos 2x} = \frac{1}{2}$$

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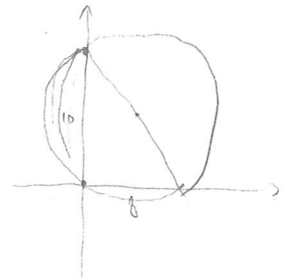
4. A circle intersects the axes at $(0, 10)$, $(0, 0)$ and $(8, 0)$. A line through $(2, -3)$ cuts the circle in half. Find the y-intercept of the line.

The centre of the circle is $(\frac{8}{2}, \frac{10}{2}) = (4, 5)$.

l pass through $(4, 5)$ and $(2, -3)$, so

l: $y = 4x - 11$.

The y-intercept of l is $(0, -11)$.



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$4x - 11$

5. State the mean value theorem. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

For function f if its differentiable in $]a, b[$ and continuous in $[a, b]$,

then there is a $c \in]a, b[$ that : $f'(c) = \frac{f(b) - f(a)}{b - a}$

$\frac{f(4) - f(1)}{4 - 1} = f'(x)$ for all $x \in [1, 4]$.

$\frac{f(4) - 10}{3} \geqslant 2$.

$f(4) \geqslant 16$, so the minimum value of $f(4) = 16$.

6. Let $y = f(x)$ be the particular solution to the differential equation $y' = x^2 + y^2$ for which $f(1) = 2$. Use Euler's method starting at $x = 1$ with a step size of 0.1 to approximate $f(1.2)$. Set out your work in a table.

n	x_n	y_n	h	$h \cdot (y_n)'$
1	1	2	0.1	0.5
2	1.1	2.5	0.1	0.746
3	1.2	3.246	0.1	—

$f(1.2) \approx 3.246$

1.21 6.25

7.46

$\frac{dy}{dx} = x^2 + y^2$

$\frac{dy}{dx} - y^2 = x^2$

$\int -1 = -x$

$e^{-x} \frac{dy}{dx} - e^{-x} y^2 = e^{-x} x^2$

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Simple but not connected.

7. Prove that a simple graph with more than one vertex contains two vertices of the same degree.

$1\cdot 2\cdot 3 = 6$

Since it's simple, the graph must be connected, so minimum degree is 1.

批：除了第一句没有问题，其余证明过程无问题，说明最小degree直接从1开始即可

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Suppose every vertex has a different degree, then starting from 1, the sequence of degree is : $1, 2, 3, \cdots, v$.

However in a simple graph, the maximum degree you can possibly get is $(v-1)$ by connecting a vertex to all $(v-1)$ others, so the case above doesn't work.

By contradiction, we know that when $v>1$ in a simple graph, there're always two vertices with the same degree.

8. Find a basis for the null space of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. let $x_3 = s$, $x_4 = t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, the basis for null space is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

9. The circle $\mathscr{C}$ has centre O, the point X lies outside $\mathscr{C}$, the point Z lies on $\mathscr{C}$ and the secant $[XZ]$ cuts $\mathscr{C}$ at Y. If $XY = 6$, $YZ = 5$ and $XO = 9$, find the area of the circle.

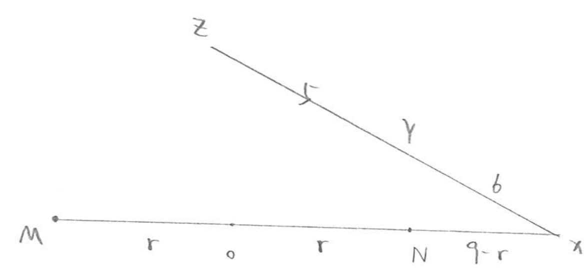
According to the tangent-secant theorem,

$XY \cdot XZ = XM \cdot XN$.

$6 \cdot 11 = (9-r)(9+r)$

$81 - r^2 = 66$

$r^2 = 15$



$r^2 = 15$, since $r>0$,

$r = \sqrt{15}$.

$A = \pi r^2 = 15\pi$.

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10. The function $f : \mathbb{Z}_{91} \to \mathbb{Z}_7 \times \mathbb{Z}_{13}$ with rule $f(x) = (x \bmod 7, x \bmod 13)$ is a bijection. Find $f^{-1}(1, 4)$.

$x = 7a+1 = 13b+4$.

only when $a=6$, $b=3$, $x=43$, satisfy the equation.

Hence, $f^{-1}(1,4) = 43$.

11. Prove that the order of a non-Abelian group cannot be prime.

Since the order of a element in a group divides the order of a group p, a prime number, elements in the group either have order 1, which is the identity element; or have order p, which is the generator of the group.

Since all cyclic groups are abelian, we then know a group with prime order must be abelian.

Thus, the order of a non-abelian group can't be prime.

12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\ker T$ is a subspace of $\mathbb{R}^n$.

- $T(\vec{0}) = \vec{0}$. $\vec{0} \in \ker T$.
- Suppose $\vec{a}, \vec{b} \in \ker T$, $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{0}+\vec{0} = \vec{0}$, $\vec{a} + \vec{b} \in \ker T$.
- Suppose $\vec{a} \in \ker T$, $T(k\vec{a}) = k\,T(\vec{a}) = k \cdot \vec{0} = \vec{0}$. $k\vec{a} \in \ker T$.

According to the three-step subspace test, $\ker T$ is a subspace of R^n.

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13. Let $f : G \to G'$ be a homomorphism of groups whose respective identity elements are e and e'. Prove

(a) $f(e) = e'$;

$f(e * e) = f(e) \circ f(e)$. $f(e * e) = f(e) = f(e) \circ e'$.

Hence, $f(e) \circ f(e) = f(e) \circ e'$. $[f(e)]^{-1} \circ f(e) \circ f(e) = [f(e)]^{-1} \circ f(e) \circ e'$.

Therefore, $f(e) = e'$.

(b) $f(a^{-1}) = [f(a)]^{-1}$.

$f(a * a^{-1}) = f(a) \circ f(a^{-1})$

$f(a * a^{-1}) = f(e) = e'$.

Hence, $e' = f(a) \circ f(a^{-1})$, $[f(a)]^{-1} \circ e' = [f(a)]^{-1} \circ f(a) \circ f(a^{-1})$.

Therefore, $[f(a)]^{-1} = f(a^{-1})$.

14. Find the general solution of the differential equation $\frac{dy}{dx} + y \cot x = x$, $0 < x < \pi$. Give your answer in the form $y = f(x)$.

Multiply both sides by $\sin x$,

$$\sin x \frac{dy}{dx} + y \cdot \frac{\cos x}{\cancel{\sin x}} \cdot \cancel{\sin x} = x \sin x.$$

$$(\sin x \cdot y)' = x \sin x.$$

$$\sin x \cdot y = \int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \sin x + C$$

Therefore, $y = \dfrac{-x \cos x + \sin x + C}{\sin x}$.

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15. Prove that the intersection of two subgroups of a group is also a subgroup of that group.

Let the group be G and the two subgroups be H and I.

$H \cap I = J$.

- $e \in H$, $e \in I$, so $e \in J$. <u>identity</u>.
- Suppose $a, b \in J$. since $a, b \in H$, $ab \in H$; $a, b \in I$, $ab \in I$. So $ab \in J$.

  <u>closure</u>
- Suppose $a \in J$, since $a \in H$, $a^{-1} \in H$; since $a \in I$, $a^{-1} \in I$. so $a^{-1} \in J$.

  <u>inverse</u>.

According to the 3-step subgroup test, $J \leq G$.

16. Determine the interval of convergence for the power series $1 + \frac{x+2}{3 \times 1} + \frac{(x+2)^2}{3^2 \times 2} + \frac{(x+2)^3}{3^3 \times 3} + \ldots$.

The power series is $1 + \sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n \times n}$.

Ratio test : $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{\frac{(x+2)^{n+1}}{3^{n+1} \times (n+1)}}{\frac{(x+2)^n}{3^n \times n}}\right| = \left|\frac{(x+2)}{3}\right| \cdot \frac{n}{n+1} = \left|\frac{x+2}{3}\right|$ as $n \to \infty$.

$\left|\frac{x+2}{3}\right| < 1$, $-3 < x+2 < 3$, $-5 < x < 1$. $R = 3$.

- at $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as it is the harmonic series.
- at $x = -5$, $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges as it is the alternating harmonic series.

Hence, the interval of convergence is $[-5, 1[$.

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17. Find the equation of the line containing the major axis of the ellipse $2x^2 - 4xy + 5y^2 = 6$.

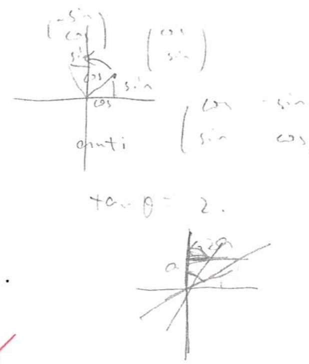
$$⑥\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \quad ①\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$(x \quad y)\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6.$$

$$\lambda^2 - 7\lambda + 6 = 0. \quad (\lambda-6)(\lambda-1) = 0. \quad \lambda_1 = 6, \ \vec{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}; \ \lambda_2 = 1, \ \vec{v_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

$$\left[(x \quad y)\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\right]\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] = 6$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \quad 6x'^2 + y'^2 = 6.$$



$\tan\theta = 2.$

So by rotating anti-clockwise $\arctan(2)$, the major axis is on y.

Therefore, before the rotation, the line was $y = \frac{1}{2}x$.

18. Solve the differential equation $x^2\frac{dy}{dx} = y^2 + xy + 4x^2$ given $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 4. \quad \text{let } y = vx, \quad v = \frac{y}{x}. \quad \frac{dy}{dx} = v + \frac{dv}{dx}x.$$

$$v^2 + \cancel{v} + 4 = \cancel{v} + \frac{dv}{dx}x. \qquad \frac{1}{v^2+4}\,dv = \frac{1}{x}\,dx.$$

$$\int \frac{1}{v^2+4}\,dv = \int \frac{1}{x}\,dx = \ln|x| + C.$$

$$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = \ln|x| + C.$$

$$\arctan\left(\frac{y}{2x}\right) = 2\ln|x| + C. \qquad \tan C = 1$$

$$\frac{y}{2x} = \tan(2\ln|x| + c). \qquad \frac{2}{2} = \tan(2\ln 1 + c).$$

when $y = 2$, $x = 1$, so $C = \frac{\pi}{4}$.

Therefore, $y = 2x\tan\left(2\ln|x| + \frac{\pi}{4}\right)$.

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19. The random variable X has probability generating function $G(t)=\dfrac{t}{2-t}$, mean μ and variance σ^2. Find $P(|X-\mu|<\sigma)$.

更快的做法是写出$G(t)$的taylor series: $\frac{t}{2-t}=\frac{0.5t}{1-0.5t}$ 为首项$0.5t$，共比$0.5t$的等比数列之和。$=\frac{1}{2}t+\frac{1}{4}t^2+\frac{1}{8}t^3+\cdots$ 取前三项系数相加即可.

$E(X)=G'(1)$, $Var(X)=G''(1)+G'(1)-(G'(1))^2$.

$G'(t)=\frac{(2-t)-t(-1)}{(2-t)^2}=\frac{2-t+t}{(2-t)^2}=\frac{2}{(2-t)^2}=2(2-t)^{-2}$

$G''(t)=2\cdot(-2)(2-t)^{-3}(-1)=4(2-t)^{-3}$.

$G'''=-12(2-t)^{-4}\cdot(-1)$
$=12(2-t)^{-4}$.

$G'(1)=2$, $G''(1)=4$.

$G'''(0)=\frac{12}{16}=\frac{3}{4}=6P_3$.
$P_3=\frac{1}{8}$

$E(X)=2$, $Var(X)=4+2-2^2=2$. $\sigma^2=2$, $\sigma=\sqrt{2}$.

$|X-2|<\sqrt{2}$, $2-\sqrt{2}<X<2+\sqrt{2}$. $x=1,2,3$. $P(|X-\mu|<\sigma)=p_1+p_2+p_3$

$G'(0)=2\cdot 2^{-2}=2^{-1}=\frac{1}{2}=p_1$; $G''(0)=4(2)^{-3}=\frac{1}{2}=2!p_2$; $G'''(0)=\frac{1}{2}=3!p_3$.

$p_1=\frac{1}{2}$, $p_2=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$, $p_3=\frac{1}{2}\cdot\frac{1}{6}=\frac{1}{12}$

$P(|X-\mu|<\sigma)=\frac{1}{2}+\frac{1}{4}+\frac{1}{12}=\frac{10}{12}=\frac{5}{6}$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$.

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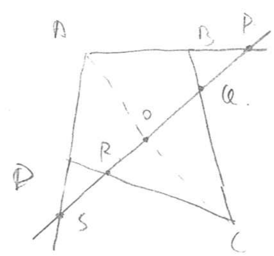
20. The sides $[AB]$, $[BC]$, $[CD]$, $[AD]$ of quadrilateral $ABCD$ (produced if necessary) are cut by a transversal in the points P, Q, R and S, respectively. Prove that

$$\frac{AP}{PB}\times\frac{BQ}{QC}\times\frac{CR}{RD}\times\frac{DS}{SA}=1.$$

Connect AC and intersect the line at O.



We have, according to Menelaus Theorem without direction:

$\frac{CO}{OA}\cdot\frac{AS}{SD}\cdot\frac{DR}{RC}=1$, $\frac{CO}{OA}\cdot\frac{AP}{PB}\cdot\frac{BQ}{QC}=1$.

Divide the second equation by the first, we have:

$$\frac{AP\cdot BQ\cdot SD\cdot RC}{PB\cdot QC\cdot AS\cdot DR}=1.$$

Hence, $\frac{AP}{PB}\times\frac{BQ}{QC}\times\frac{CR}{RD}\times\frac{DS}{SA}=1$.

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