$9\frac{1}{2}/10$ Very good!

FM2 Assignment #24 Name: Jerry Jiang

1. Prove that the set $S = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid ax + by + cz = 0\}$ is a subspace of $\mathbb{R}^3$.

- $a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$. $\vec{0} \in S$.
- $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$. $(u_1+v_1)a + (u_2+v_2)b + (u_3+v_3)c =$ $(u_1 a + u_2 b + u_3 c) + (v_1 a + v_2 b + v_3 c) = 0 + 0 = 0$. $\vec{u} + \vec{v} \in S$.
- $\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$, $n \in R$. $n\vec{k} = \begin{pmatrix} nk_1 \\ nk_2 \\ nk_3 \end{pmatrix}$ $ank_1 + bnk_2 + cnk_3 =$ $(ak_1 + bk_2 + ck_3) = n \cdot 0 = 0$. ✓

  $n\vec{k} \in S$.

Therefore, according to the 3-step subspace test, S is a subspace of R^3.

2. Let $T \colon R^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\operatorname{ran} T$ is a subspace of $\mathbb{R}^m$.

- $T(\vec{0}) = \vec{0}$, $\vec{0} \in \operatorname{ran} T$. ✓
- Suppose $T(\vec{u}) = \vec{u}'$, $T(\vec{v}) = \vec{v}'$. $T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$.

  so $\vec{u}' + \vec{v}' \in \operatorname{ran} T$.
- Suppose $T(\vec{u}) = \vec{u}'$, $k \in R$, then $T(k\vec{u}) = kT(\vec{u}) = k\vec{u}'$. ✓ so $k\vec{u}' \in \operatorname{ran} T$.

Therefore, according to the 3-step subspace test, $\operatorname{ran} T$ is a subspace of R^m.

3. The system below has a particular solution $x = -1.5, y = 1.5, z = 0$. Find the general solution.

$$\begin{aligned} x + y - 2z &= 0 \\ x - y &= -3 \\ 3x - y - 2z &= -6 \\ 2y - 2z &= 3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & -3 \\ 3 & -1 & -2 & -6 \\ 0 & 2 & -2 & 3 \end{array}\right) = A.$$

$$\operatorname{rref}(A) = \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1.5 \\ 0 & 1 & -1 & 1.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$\operatorname{rank}(A) = 2$, nullity $(A) = 1$.

$\ker T$ = null space = $\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$. $\operatorname{ran} T = \langle \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix} \rangle$.

$S = \{ \begin{pmatrix} -1.5 \\ 1.5 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t \mid t \in R \}$. ✓

/6

4. Let T be the linear transformation with matrix $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 3 \\ 0 & -1 & 5 \end{pmatrix}$. Find Cartesian equations for $\ker T$ and $\operatorname{ran} T$.

$$\text{rref}(M) = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{rank}(M) = 2, \quad \text{nullity}(M) = 1.$$

$$\ker T = \left\langle \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \right\rangle. \quad \operatorname{ran} T = \left\langle \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right\rangle.$$

$$\ker T: \ \frac{x}{3} = \frac{y}{5} = z. \qquad ✓$$

$$\operatorname{ran} T: \ x = 2a - b, \quad y = -a, \quad z = -b. \qquad x + 2y - z = 0. ✓$$

5. Find a formula for M^n where $M = \begin{pmatrix} 2b-a & a-b \\ 2b-2a & 2a-b \end{pmatrix}$. Hence calculate M^{10} when $a = 1$ and $b = 2$.

$$\lambda^2 - (a+b)\lambda + ab = 0.$$

$$\lambda_1 = a, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ \boxed{-2} \end{pmatrix} +2 \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 2b-2a & a-b \\ 2b-2a & a-b \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\lambda_2 = b, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ \underset{2}{\boxed{-2}} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \boxed{\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}} \to \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \qquad 1\tfrac{1}{2} ✓$$

$$M^n = \begin{pmatrix} 1 & 1 \\ \underset{2}{\boxed{-2}} & 1 \end{pmatrix} \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix} \boxed{\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}}$$

$$M^{10} = \begin{pmatrix} 1 & 1 \\ \underset{2}{\boxed{-2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix} \boxed{\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}} = \begin{pmatrix} 683 & 341 \\ 682 & 342 \end{pmatrix}. \qquad 3\tfrac{1}{2} ✓$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2047 & -1023 \\ 2046 & -1022 \end{pmatrix}.$$