FM2 Assignment #22 Name: Jerry Jiang $\frac{9}{12}$ v. good

1. Let A be an invertible matrix with eigenvalue λ and corresponding eigenvector $\vec{v}$. Prove that A^{-1} must have the eigenvalue $1/\lambda$ together with the same corresponding eigenvector $\vec{v}$.

$A\vec{v} = \lambda\vec{v}$.

$A^{-1}A\vec{v} = \lambda A^{-1}\vec{v}$

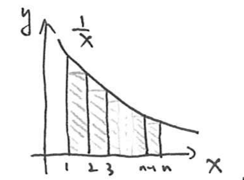
$\therefore \vec{v} = \lambda A^{-1}\vec{v}$

$\therefore \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$ ✓

Therefore A^{-1} have eigenvalue $\frac{1}{\lambda}$ with vector $\vec{v}$.

2. Let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. Show that $H_n \le 1 + \ln n$. Hence show that $H_{1\,000\,000\,000} < 22$.

- L_n for $\int_1^n \frac{1}{x}\,dx$ is $\frac{n-1}{n-1}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.



- $L_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le [\ln x]_1^n = \ln n - \ln 1 = \ln n$.
- $H_n = 1 + L_n \le 1 + \ln n$.

$\because H_{10^9} \le 1 + \ln 10^9 = 1 + 9\ln 10 \approx 21.7$ (3 s.f.).

$\therefore H_{1\,000\,000\,000} < 22$. ✓

3. Prove that similarity is an equivalence relation on the set of $n \times n$ matrices.

- $A = IAI^{-1}$, A is similar to itself. reflexive ✓
- Suppose $A = PBP^{-1}$, P is non-singular. Then $P^{-1}AP = B$. ✓

  Let $P^{-1} = Q$, so $B = QAQ^{-1}$. B and A are similar. symmetric
- Suppose A and B are similar, B and C are similar. Then there are non-singular matrices P_1, P_2 that: $A = P_1BP_1^{-1}$, $B = P_2CP_2^{-1}$.

  So $A = P_1(P_2CP_2^{-1})P_1^{-1} = (P_1P_2)C(P_2^{-1}P_1^{-1}) = (P_1P_2)C(P_1P_2)^{-1}$ ✓.

  A and C are similar. transitive

Therefore, similarity is an equivalence relation.

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4. Factorize the matrix $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$ in the form PDP^{-1} where D is a diagonal matrix. Hence find A^8.

$\lambda^2 - 3\lambda + 2 = 0, \quad \lambda = 1, \ \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \ \lambda = 2, \ \vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

$\therefore A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}.$

$\therefore A^8 = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^8 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 766 & -765 \\ 510 & -509 \end{pmatrix}.$ ✓

5. Use matrix methods to solve the recurrence relation $u_n = 5u_{n-1} - 6u_{n-2}$ given $u_1 = 1$ and $u_2 = 2$.

$\lambda^2 - 5\lambda - 11 = 0$

$R = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

$\begin{pmatrix} u_3 \\ u_2 \end{pmatrix} = R \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\begin{pmatrix} u_4 \\ u_3 \end{pmatrix} = R^2 \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}.$

$\underline{\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}.$

↑ True, but not considered a full solution to the problem as u_n is still difficult to calculate. We find that $u_n = 2^{n-1}$ is the explicit solution.

$\frac{1}{3}$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda_1 = 3, \ \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = 2, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{n-1} & 0 \\ 0 & 2^{n-1} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^n \\ 2^{n-1} \end{pmatrix}.$

$\therefore u_{n+1} = 2^n. \quad \underline{\underline{u_n = 2^{n-1}}}.$