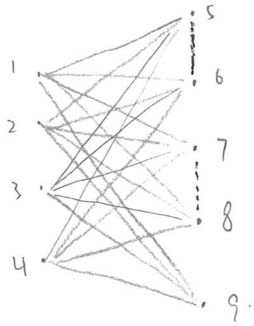
FM2 Assignment #32 Name: Jerry Jiang $\frac{12}{12}$ Excellent!

1. There are nine men at a party. By considering an appropriate graph, explain why it is impossible for each man to shake hands with exactly five other men.



The pencil ? indicates $K_{4,5}$. Vertices #1-4 have degrees 5 already. Suppose we connect 5 and 6, 7 and 8. This makes #5-8 have degrees 5 as well. There's no place #9 can shake with to achieve 5 shakes.

Better to quote sum of degrees of vertices is twice the number of edges (the hand shaking lemma)

2. For what values of x is the series $\sum_{k=1}^{\infty} e^{kx}$ convergent? For these values of x, find the sum as a simple function of x.

$$e^x + e^{2x} + e^{3x} + \cdots$$

$$= e^x \frac{1-(e^x)^n}{1-e^x}, \quad n \to \infty$$

For it to converge, $e^x < 1$,

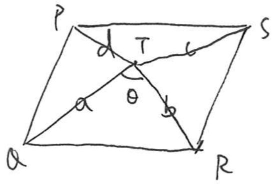
so $x < 0$.

When it converges,

$$S = e^x \cdot \frac{1}{1-e^x}$$

$$= \frac{e^x}{1-e^x}.$$

3. Point T lies inside parallelogram $PQRS$ so that $\angle PTQ + \angle RTS = 180°$. Show that $PT \times TR + ST \times TQ = PQ \times QR$.



$$RQ^2 = a^2+b^2-2ab\cos\theta$$

$$PS^2 = c^2+d^2+2ab\cos\theta.$$?

$$RQ^2 = \frac{cd(a^2+b^2)+ab(c^2+d^2)}{ab+cd} = \frac{(ac+bd)(ad+bc)}{ab+cd}$$

$$\text{Similarly, } PQ^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}$$

$$\text{So } RQ^2 \cdot PQ^2 = (ac+bd)^2$$

$$RQ \cdot PQ = QT \cdot TS + PT \cdot TR.$$

What about using Ptolemy's theorem?

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4. Is the series

$$\frac{1^1}{(101)!} + \frac{2^2}{(102)!} + \ldots + \frac{n^n}{(100+n)!} + \ldots$$

convergent or divergent? Justify your answer.

$\sum_{n=1}^{\infty} \frac{n^n}{(100+n)!}$, apply ratio test :

$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)^{n+1}}{(100+n+1)!} \cdot \frac{(100+n)!}{n^n}\right| = \left|\left(\frac{n+1}{n}\right)^n \cdot \frac{n+1}{100+n+1}\right| = e$ as $n \to \infty$. $e > 1$.

Hence, the series diverges.

5. Let $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q},\ a^2 + b^2 \neq 0\}$. Prove that $(S, \times)$ is a group. Is $(S, \times)$ a group if $a, b \in \mathbb{R}$, $a^2 + b^2 \neq 0$?

- $1 + 0 \cdot \sqrt{2} = 1$, so $1 \in \mathbb{Q}$. identity ✓
- Suppose $a + b\sqrt{2} \in S$, $\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}$
  - * if $a^2 = 2b^2$, then $a = \pm b \cdot \sqrt{2}$, but $a, b \in \mathbb{Q}$, so $a^2 \neq 2b^2$.

  $= \frac{a}{a^2-2b^2} + \frac{b}{2b^2-a^2}\sqrt{2}$.

  which is in S.

  inverse ✓
- multiplication is associative ✓.
- Suppose $a + b\sqrt{2}$, $c + d\sqrt{2} \in S$, $(a+b\sqrt{2}) \times (c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2} \in S$.

  closure ✓.

Therefore, $(S, \times)$ is a group.

→ if $a, b \in \mathbb{R}$, then a^2 is possible to equal to $2b^2$, so inverse might not exist. In this case, $(S, \times)$ is no longer a group.

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And what number is that?! Only one real number has no multiplicative inverse.