FM2 Test #4

Name: Jerry Jiang

38

1. Use the Maclaurin series for $\cos x$ to evaluate the limit $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

$$\lim_{x \to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}{x^2} = \lim_{x \to 0} \frac{\frac{x^2}{2!} - O(x^4)}{x^2} = \lim_{x \to 0} \frac{1}{2} - O(x^2) = \frac{1}{2}$$

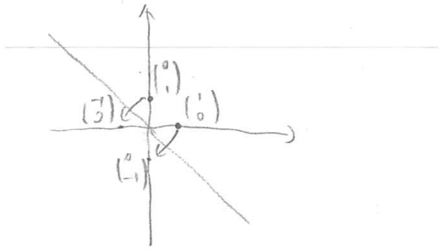
Excellent!

usually written as $+O(x^4)$

2. Find the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under reflection in the line $y = -x$. Hence write down the matrix for the reflection.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

$$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$



3. Find the values of a and b that make the given function f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x < 2 \\ ax^2 - bx + 3 & 2 \le x < 3 \\ 2x - a + b & x \ge 3 \end{cases}$$

- $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} x + 2 = 4.$

$ax^2 - bx + 3 = 4$ when $x = 2$, $\quad 4a - 2b + 3 = 4.$

- $\lim_{x \to 3} ax^2 - bx + 3 = \lim_{x \to 3} 2x - a + b, \quad 9a - 3b + 3 = 6 - a + b.$

- $\begin{cases} a = \frac{1}{2} \\ b = \frac{1}{2}. \end{cases}$

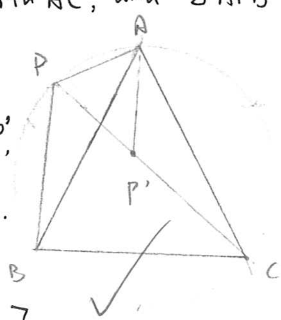
4. Let G be a group containing the element a. We say a has a cube root in G if there is an $x \in G$ such that $x^3 = a$. Prove that if $a^2 = e$ then a has a cube root in G.

$a^3 = a \cdot a^2 = a \cdot e = a.$

Therefore, a is the cube root of itself.

5. Point P is on arc AB of the circumcircle of equilateral triangle ABC, $AP = 3$, and $BP = 4$. Find CP.

Rotate $\triangle APB$ 60° anticlockwise so that AB overlaps with AC, and $\triangle APB$ becomes $\triangle AP'C$.



As cyclic quadrilateral, $\angle APB + \angle ACB = 180^\circ$. $\angle ACB = 60^\circ$,

So $\angle APB = \angle AP'C = 180^\circ - 60^\circ = 120^\circ$. Since $AP = AP'$, $\angle PAP' = 60^\circ$,

so $\triangle APP'$ is an equilateral triangle and $\angle AP'P = 60^\circ$.

$\angle AP'P + \angle AP'C = 180^\circ$, so P, P' and C are collinear.

We know $PP' = AP = 3$, $P'C = PB = 4$, so $CP = PP' + P'C = 7$.

A

Ptolemy's Theorem is an easier approach.

托勒密做会更简单.

6. Let $T\colon R^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\operatorname{ran} T$ is a subspace of $\mathbb{R}^m$.

- $T(\vec{0}) = \vec{0}$, so $\vec{0} \in \operatorname{ran} T$.
- Suppose $T(\vec{u}) = \vec{u}'$, $T(\vec{v}) = \vec{v}'$, then $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{u}' + \vec{v}'$.

  so $\vec{u}' + \vec{v}' \in \operatorname{ran} T$.
- Suppose $T(\vec{v}) = \vec{v}'$, $k \in R$, then $T(k\vec{v}) = kT(\vec{v}) = k\vec{v}'$.

  so $k\vec{v}' \in \operatorname{ran} T$.

Therefore, according to the 3-step subspace test, $\operatorname{ran} T$ is a subspace of R^m.

7. Find the radius of convergence and interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{3x^n}{2n}$.

Let $u_n = \frac{3x^n}{2n}$.

Ratio Test : $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{3x^{n+1}}{2(n+1)}}{\frac{3x^n}{2n}}\right| = |x| \cdot \left|\frac{n}{n+1}\right| = |x|$ as $n \to \infty$.

so $|x| < 1$, $\underline{\underline{R = 1}}$. ✓

- $x = 1$, $\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}$, diverges.
- $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n 3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$, converges.

Hence, the interval of convergence is $[-1, 1[$. ✓

8. Diagonalize the matrix of the ellipse $7x^2 - 8xy + 13y^2 = 150$. Hence determine the ellipse's eccentricity.

$(x \ \ y) \begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 150$. $\lambda^2 - 20\lambda + 75 = 0$, $\lambda_1 = 5$, $\vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\lambda_2 = 15$, $\vec{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$\left[(x \ \ y) \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\right] \begin{pmatrix} 15 & 0 \\ 0 & 5 \end{pmatrix} \left[\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right] = 150.$$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $(x' \ \ y') \begin{pmatrix} 15 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 150$.

$\frac{x'^2}{10} + \frac{y'^2}{30} = 1$, $10 = 30(1 - e^2)$, $e = \frac{\sqrt{6}}{3}$.

Therefore, the eccentricity of the ellipse is $\frac{\sqrt{6}}{3}$. ✓

10

9. Show that the series $\sum(-1)^{n+1}a_n$, where $a_n = 1/n$ if n is odd and $a_n = 1/n^2$ if n is even, is divergent. Why does the alternating series test not apply?

The series = $\sum_{n=1}^{\infty} \frac{1}{(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$.

- $\sum_{n=1}^{\infty} \frac{1}{(2n-1)}$ is divergent according to the integeral test, as it's continuous, decreasing, and possitive, $\int \frac{1}{2n-1} = \frac{1}{2}\ln(2n-1) + C = \infty$ as $n \to \infty$.

5

- $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as it's a p-series with $p>1$.

So (Divergent series) − (Convergent series) = (Divergent series). The series diverges.

→ when n is odd, $|U_{n+1}| < |U_n|$.

→ when n is even, $|U_{n+1}| > |U_n|$. ← That is, the sequence $\{|u_n|\}$ is not decreasing.

This is why the alternating series test is inconclusive. The ~~series~~ series isn't → decreasing

10. Suppose the group G has subgroup H. Define the relation $\sim$ on G by $a \sim b$ if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation and describe the equivalence classes.

- $a \cdot a^{-1} = e \in H$. $a \sim a$, reflexive. ✓

- Suppose $a \sim b$, then $ab^{-1} \in H$. $(ba^{-1})(ab^{-1}) = b(a^{-1}a)b^{-1} = bb^{-1} = e$.

  So $(ba^{-1}) = (ab^{-1})^{-1}$, $ba^{-1} \in H$. $b \sim a$. symmetric ✓

  subgroups have to include the inverse of its elements

- Suppose $a \sim b$, $b \sim c$, then $ab^{-1} \in H$, $bc^{-1} \in H$. Due to closure,

  $ab^{-1} \cdot bc^{-1} = ac^{-1} \in H$. $a \sim c$. transitive. ✓

Therefore, $\sim$ is an equivalence relation, and the equivalence classes are H and its right cosets in G.

right cosets.

$k \in H$, $h \in H$,

~~a~~ $ka \in Ha$, $ha \in Ha$.

$ka(ha)^{-1} = kaa^{-1}h^{-1} = kh^{-1} \in H$.

so $ka \sim ha$.

4

9

Solutions to FM2 Test #4

1. $(1-\cos x)/x^2 = [1-(1-x^2/2+O(x^4))]/x^2 = 1/2+O(x^2)$. So the limit is $1/2$.

2. $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The required images are the first and second columns of the matrix.

3. We have $4a-2b+3=4$ and $9a-3b+3=6-a+b$. Solving gives $a=b=\frac{1}{2}$.

4. If $a^2=e$ then $a^3=a$. So a has a cube root in G, namely itself.

5. Let the side length of the equilateral triangle be x. Using Ptolemy's theorem, we find $3x+4x=xCP$. So $CP=7$.

6. See assignment #24.

7. The radius of convergence is $R=1$ and the interval of convergence is $[-1.1[$.

8. $\begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. So the canonical form of the ellipse is $x^2/30 + y^2/10 = 1$. Hence $e=\sqrt{2/3}$.

9. Suppose $\sum(-1)^{n+1}a_n$ is convergent. Now we know $\sum 1/(2n)^2$ is convergent by comparison with the known convergent series $\sum 1/n^2$. Hence $\sum[(-1)^{n+1}a_n + 1/(2n)^2] = \sum 1/(2n-1)]$ must also converge. But this is a contradiction as $\sum 1/(2n-1)$ diverges, by for example the integral test. Hence $\sum(-1)^{n+1}a_n$ must diverge.

   The alternating series test does not apply as the terms do not decrease in absolute value.

10. To show that $\sim$ is an equivalence relation, we must show that $\sim$ is reflexive, symmetric and transitive.

    i. Since $aa^{-1}=e$ and $e\in H$, it follows that $\sim$ is reflexive.

    ii. If ab^{-1} is in H then $(ab^{-1})^{-1}=ba^{-1}$ is also in H since subgroups contain their inverses. It follows that $\sim$ is symmetric.

    iii. Suppose ab^{-1} and bc^{-1} are in H. Then their product $ab^{-1}bc^{-1}=ac^{-1}$ is also in H since subgroups are closed under the group operation. It follows that $\sim$ is transitive.

    Now $[g]=\{x\in G \mid xg^{-1}\in H\}=\{x\in G \mid x\in Hg\}=Hg$. So the equivalence classes are the right cosets of H in G.