FM2 Assignment #27 Name: Jerry Jiang

1. A radioactive source emits 482 alpha particles in two hours. What is the probability that the source will emit more than three alpha particles in the next minute?

$\mu = \frac{482}{120} = 4.02$ (3s.f.)

$\frac{10}{10}$ Excellent!

$X \sim Po(4.02)$.

$P(X>3) = 1 - \text{poissoncdf}(4.02, 3) = 0.570$ (3s.f.)

2

2. Prove that a non-Abelian group must contain a proper subgroup.

Let take identity e and another element a in a non-Abelian group.

The proper subgroup contains $e, a, a^2, a^3, \cdots a^n, \cdots$.

This is a proper subgroup because the non-Abelian group can't be cyclic, so there must be some other element b not generated by a^n.

Also, the element a other than e must exist as the group with order less than 6 are all Abelian. Thus, the proposed group has to be a proper subgroup.

2

3. The angle between the asymptotes of the hyperbola $x^2 - k^2y^2 = k^2$ is $\pi/3$. Find the equations of the directrices.

$\frac{x^2}{k^2} - y^2 = 1$.

asymptotes: $y = \pm\frac{1}{k}x$

$k = \frac{\sqrt{3}}{3}$ or $\sqrt{3}$.

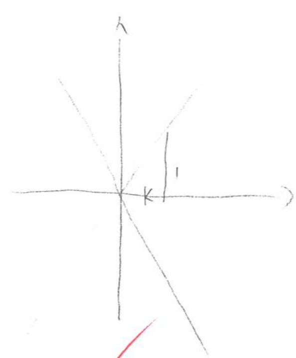
$1^2 = k^2(e^2-1)$

$k^2 \cdot e^2 - k^2 = 1$

$e^2 = \frac{1}{k^2} + 1 = 4$ or $\frac{4}{3}$

$e = \pm 2$ or $\pm\frac{2}{3}\sqrt{3}$.

$\frac{k}{2}$ $\frac{a}{e}$ $\frac{\sqrt{3}}{6}$ $\frac{3}{2}$



6

The equations of the directrices: $x = \pm\frac{\sqrt{3}}{6}$ or $x = \pm\frac{3}{2}$.

4. Suppose $X \sim \mathrm{Po}(\ln 2)$. Find the probability that X is even.

$$P = e^{-\ln 2}\left(\frac{(\ln 2)^0}{0!} + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^4}{4!} + \cdots\right)$$

$$= e^{-\ln 2}\left(1 + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^4}{4!} + \cdots\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right)$$

$$= \frac{1}{4}\left(2 + \frac{1}{2}\right)$$

$$= \frac{5}{8}$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1}{2}(e^{x} + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

5. A cylindrical water tank has its axis vertical. The area of its base is $2\,\mathrm{m}^2$. Initially the tank is empty. Starting at time $t = 0$, water is poured into the tank at a constant rate of $0.2\,\mathrm{m}^3\mathrm{s}^{-1}$, and leaks out a small hole in the base at a rate of $0.1x\,\mathrm{m}^3\mathrm{s}^{-1}$, where x m is depth of the water in the tank at time t. Show that

$$20\frac{dx}{dt} = 2 - x,$$

and solve the differential equation to obtain x as a function of t. Deduce that the depth of the water in the tank never exceeds 2 m, and find in the form $a \ln b$ for integers a and b, how long the water takes to reach a depth of 1 m.

water height increase : 0.1 m/s. decrease : $0.05x$ m/s.

$$\frac{dx}{dt} = 0.1 - 0.05x \iff 20\frac{dx}{dt} = 2 - x.$$

$$\int \frac{1}{2-x}\,dx = \int \frac{1}{20}\,dt$$

$$-\ln(2-x) = \frac{1}{20}t + C$$

$$\frac{1}{2-x} = Ce^{\frac{1}{20}t}$$

$$2 - x = Ce^{-\frac{1}{20}t}$$

$$x = 2 - Ce^{-\frac{1}{20}t}$$

when $t=0$, $x=0$.

$$0 = 2 - C\cdot 1$$

$$C = 2$$

$$\underline{x = 2 - 2e^{-\frac{1}{20}t}}$$

Since $2e^{-\frac{1}{20}t} > 0$, x is always smaller than 2.

$$2 - 2e^{-\frac{1}{20}t} = 1.$$

$$2e^{-\frac{1}{20}t} = 1.$$

$$e^{-\frac{1}{20}t} = \frac{1}{2}$$

$$-\frac{1}{20}t = \ln\frac{1}{2}$$

$$t = -20\ln\frac{1}{2} = 20\ln 2$$

It takes $20\ln 2$ seconds to reach 1m.

4