FM2 Assignment #23 Name: Jerry Jiang

$\frac{9\frac{1}{2}}{10}$ Very good

1. Calculate the discriminant of the conic $4x^2 + 10xy + y^2 = 1$. Hence determine if the conic is an ellipse, hyperbola or parabola. Confirm your result using desmos.

$\Delta = 10^2 - 4\cdot 4\cdot 1 = 84 > 0.$

hyperbola.

✓

2. The ellipse $x^2 + xy + y^2 = 1$ is rotated 45° anticlockwise about the origin. Find the equation of the rotated ellipse.

$(x \quad y)\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1.$ $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}.$

$(x' \quad y')\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$

✓

$(x' \quad y')\begin{pmatrix} 0.5 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$

$\frac{1}{2}x^2 + \frac{3}{2}y^2 = 1$

3. Diagonalize the matrix of the hyperbola $x^2 + 2\sqrt{3}xy - y^2 = 2$. Hence determine the hyperbola's eccentricity.

$(x \quad y)\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2$

$\lambda^2 + (-4) = 0.$ $\vec{v}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \lambda_1 = 2.$ $\vec{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \lambda_2 = -2.$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$ $(x' \quad y')\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 2.$

✓

$\therefore -2x^2 + 2y^2 = 2,$ $-x^2 + y^2 = 1.$

$\therefore 1 - e^2 = -1$

$\therefore e = \sqrt{2}.$

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4. Diagonalize the matrix of the ellipse $5x^2 + 8xy + 11y^2 = 42$. Through what acute angle must the ellipse be rotated to align its major axis with the x-axis?

$$(x \quad y)\begin{pmatrix} 5 & 4 \\ 4 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 42.$$

$$\lambda^2 - 16\lambda + 39 = 0. \quad \lambda_1 = 3,\ \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \quad \lambda_2 = 13,\ \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$\left[(x\ y)\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\right]\begin{pmatrix} 13 & 0 \\ 0 & 3 \end{pmatrix}\left[\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] = 0.$$

$1\frac{1}{2}$

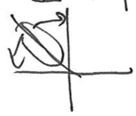
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$13x'^2 + 3y'^2 = 42.$ → This way major axis is on y axis.

aligns major axis with y-axis.

$\therefore$ clockwise $\arctan(2)$.

解决方法： 逆时针旋转 $\frac{\pi}{4} - \arctan(2)$



5. Prove that a graph with no odd cycles is bipartite.

Suppose G has no odd cycles.

Choose $v \in G$ and all vertices can be partition into two sets:

A: vertices such that the shortest path to v is of odd length.

B: vertices such that - - - - - - - even.

Suppose a_1 and $a_2 \in A$ are adjacent.

Then $(v, \dots, a_1, a_2, \dots v)$ is a closed walk of odd length.

This indicates that it has odd cycle, which is a contradiction.

This means that no vertices in A can be adjacent.

Similarly, no vertices in B can be adjacent.

Therefore, G is bipartite.

$3\frac{1}{2}$