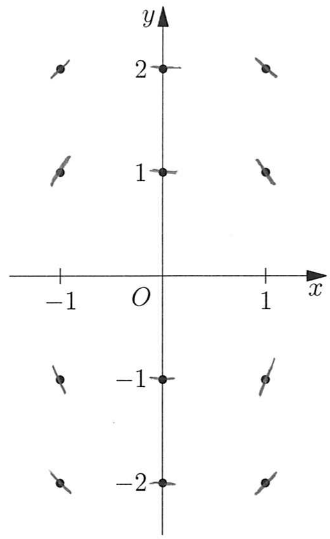
FM2 Differential Equations Name: Jerry Jiang

$\frac{31\frac{1}{2}}{32}$ Excellent!

1. Consider the differential equation $\dfrac{dy}{dx} = -\dfrac{2x}{y}$.

(a) Sketch the slope field for the differential equation at the twelve points indicated.



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

$y = 2x - 3$. $f(1.1) \approx -0.8$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

$\int -y \, dy = \int 2x \, dx$.

$-\frac{1}{2}y^2 = x^2 + C$

$x^2 + \frac{1}{2}y^2 = C$.

$1^2 + \frac{1}{2}(-1)^2 = C$. $C = \frac{3}{2}$.

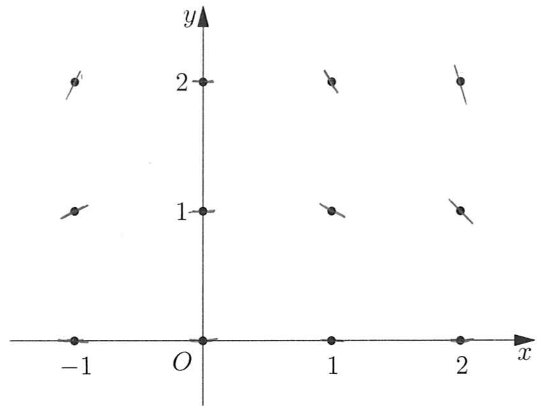
$x^2 + \frac{y^2}{2} = \frac{3}{2}$.

Hence, $y = \pm\sqrt{3 - 2x^2}$.

6

2. Consider the differential equation $\frac{dy}{dx} = -\frac{xy^2}{2}$.

(a) Sketch the slope field for the differential equation at the twelve points indicated.



(b) Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$. Write an equation for the line tangent to the graph of f at $x = -1$.

$y = 2x + 4.$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

$$\int \frac{1}{y^2}\,dy = \int -\frac{x}{2}\,dx.$$

$$-\frac{1}{y} = -\frac{1}{4}x^2 + c$$

$$\frac{1}{4}x^2 - \frac{1}{y} = C.$$

$$\frac{1}{4}(-1)^2 - \frac{1}{2} = C. \quad C = -\frac{1}{4}$$

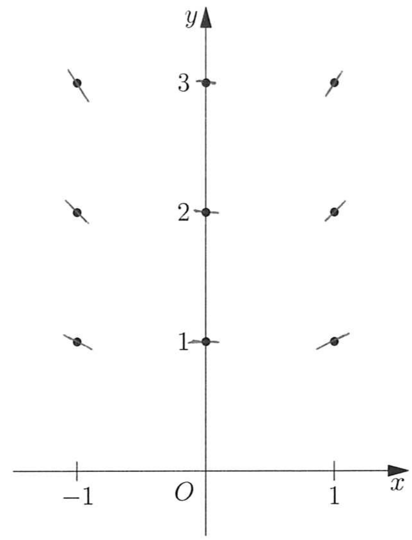
$$\frac{1}{4}x^2 - \frac{1}{y} = -\frac{1}{4}$$

Hence, $y = \frac{4}{x^2+1}$.

6

3. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{2}$.

(a) Sketch the slope field for the differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$ with a step size of 0.1 to approximate $f(0.2)$. Set out your work in a table.

n	x_n	y_n	h	$hf(x_n, y_n)$.
0	0	3	0.1	0
1	0.1	3	0.1	0.015
2	0.2	3.015	0.1	–

$f(0.2) \approx 3.015$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

$$\int \frac{1}{y}\,dy = \int \frac{x}{2}\,dx.$$

$$\ln|y| = \frac{1}{4}x^2 + C$$

$$|y| = Ce^{\frac{1}{4}x^2}$$

$$y = Ce^{\frac{1}{4}x^2}$$

$f(0) = 3$, $3 = C \cdot e^0$

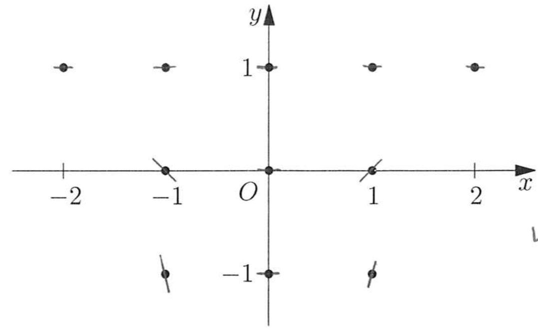
Hence, $y = 3e^{\frac{1}{4}x^2}$.

$f(0.2) = 3e^{0.05} = \cancel{3.15}\ (3\text{s.f.})$ 3.03

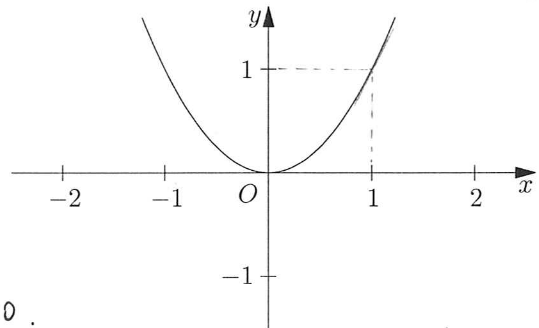
$3e^{0.05 \cdot 0.2^2} = 3.03$.

4. Consider the differential equation $\dfrac{dy}{dx} = x(y-1)^2$.

(a) Sketch the slope field for the differential equation at the eleven points indicated.



(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.



$\left.\dfrac{dy}{dx}\right|_{y=1} = 0.$

However in this graph, slope at $y=1$ isn't 0. Therefore, it can't be a solution.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

$$\int \frac{1}{(y-1)^2}\,dy = \int x\,dx.$$

$$-\frac{1}{y-1} = \frac{1}{2}x^2 + C.$$

$$-\frac{1}{(-1)-1} = \frac{1}{2}\cdot 0^2 + C. \quad C = \frac{1}{2}.$$

Hence, $y = \dfrac{x^2-1}{x^2+1}$.

$$+\frac{1}{y-1} = -\frac{1}{2}x^2 + 1 \qquad +\frac{1}{y-1} = -\frac{1}{2}x^2 - \frac{1}{2}$$

$$y-1 = \frac{1}{-\frac{1}{2}x^2+1} = \frac{2}{2-x^2} \qquad y-1 = \frac{-2}{x^2+1}$$

$$y = \quad + 1 = \frac{2+2-x^2}{2-x^2} = \frac{4-x^2}{2-x^2}$$

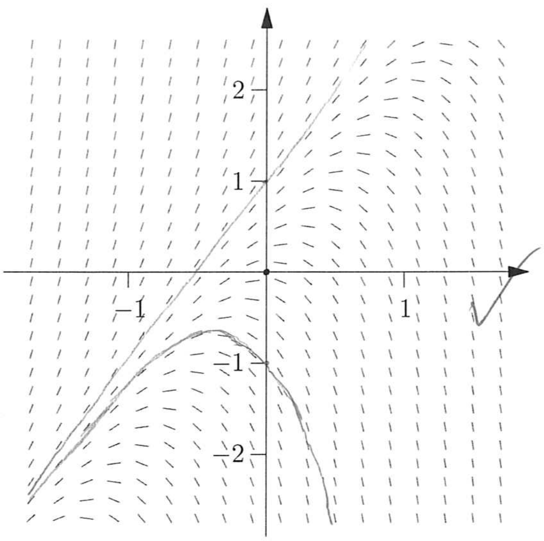
(d) Find the range of the solution found in part (c).

According to the GDC, range $= [-1, 1[$.

7

5. Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for this differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.



(b) Let f be the function that satisfies the differential equation with the initial condition $f(0) = 2$. Use Euler's method starting at $x = 0$ with a step size of 0.1 to approximate $f(0.2)$. Set out your work in a table.

n	x_n	y_n	h	$h \cdot f(x_n, y_n)$
0	0	2	0.1	0.4
1	0.1	2.4	0.1	0.44
2	0.2	2.84	0.1	—

$f(0.2) \approx 2.84$.

(c) Find the value of b for which $y = 2x + b$ is a solution to the differential equation. Justify your answer.

$\frac{dy}{dx} = 2 = 2(2x+b) - 4x$.

$4x + 2b - 4x = 2$.

$b = 1$.

(d) Let g be the function that satisfies the differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

$\left.\frac{dy}{dx}\right|_{x=0, y=0} = 2 \cdot 0 - 4 \cdot 0 = 0$, therefore it is a local extremum at $(0,0)$.

According to the slope field, $\frac{dy}{dx}$ when $x \to 0^-$ is positive while $x \to 0^+$ is negative.

Therefore, it's a local maximum.

7/7