FM2 Assignment #29 Name: Jerry Jiang 10/10 Excellent!

1. Females enter Superstore at an average rate of two a minute and males enter it at an average rate of one a minute. Find the probability that three people enter Superstore in a given minute. Could you have got your answer in another way?

① $X \sim Po(2)$, $Y \sim Po(1)$. $X+Y \sim Po(1+2)$.

$$P(X+Y=3) = \frac{e^{-3} \cdot 3^3}{3!} = 0.224 \text{ (3s.f.)}$$

② $P(X=0) = 0.1353353\cdots$ $P(Y=0) = 0.3678794\cdots$ $P(X+Y=3) = \sum_{i=0}^{3} P(X=i)\cdot P(Y=3-i)$

$P(X=1) = 0.2706706\cdots$ $P(Y=1) = 0.3678794\cdots$ $= 0.224$ (3s.f.).

$P(X=2) = 0.2706706\cdots$ $P(Y=2) = 0.1839397\cdots$

$P(X=3) = 0.1804470\cdots$ $P(Y=3) = 0.1226265\cdots$

2. Consider the matrix $\begin{pmatrix} k & 1 & 1 \\ k & 2 & k-1 \\ k & 0 & k-2 \end{pmatrix}$. For which values of k is the matrix invertible?

$$\text{Determinant} = k(2k-4) - (k^2-2k-k^2+k) + (-2k)$$
$$= 2k^2 - 4k + k - 2k$$
$$= 2k^2 - 5k \neq 0$$

$k(2k-5) \neq 0$, $k \neq 0$ or $\frac{5}{2}$.

3. The joint probability distribution for X and Y is given by $P(X=x, Y=y) = \frac{xy}{18}$, $(x, y) \in \{1, 2\} \times \{1, 2, 3\}$.

(a) Tabulate the joint probability distribution for X and Y.

X \ Y	1	2	3	
1	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{3}{18}$	$\frac{6}{18}$
2	$\frac{2}{18}$	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{12}{18}$
	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{9}{18}$	

(b) Tabulate the probability distribution for $X+Y$.

$x+y$	2	3	4	5
$P(X+Y=x+y)$	$\frac{1}{18}$	$\frac{4}{18}$	$\frac{7}{18}$	$\frac{6}{18}$

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4. As we saw in class, the Poisson distribution can be derived as the limiting case of a binomial distribution with $p = \mu/n$ and $q = 1 - \mu/n$. Write down the pgf for the binomial distribution and show that the pgf for the Poisson distribution can be obtained from this by letting n go to infinity.

$X \sim Po(\mu)$, $X \sim B(n, \frac{\mu}{n})$ as $n \to \infty$.

For $B(n, \frac{\mu}{n})$,

$$G(t) = \sum_{x=0}^{n} \binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x} t^x \qquad = (q+pt)^n = \left(1 + \frac{\mu(t-1)}{n}\right)^n$$

$$= \sum_{x=0}^{n} \frac{n(n-1)\cdots(n-x+1) \cdot \mu^x}{x! \cdot n^x} \left(1 - \frac{\mu}{n}\right)^{n-x} t^x \qquad \to e^{\mu(t-1)} \text{ as } n \to \infty.$$

$$= \sum_{x=0}^{n} \frac{\mu^x}{x!} \cdot \left[\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n}\right] \left(1 + \frac{-\mu}{n}\right)^{n-x} t^x$$

works better.

when $n \to \infty$,

$$G(t) = \sum_{x=0}^{n} \frac{\mu^x}{x!} \cdot \left[\frac{n}{n} \cdot \frac{n}{n} \cdots \frac{n}{n}\right] \left(1 + \frac{-\mu}{n}\right)^n t^x$$

$$= \sum_{n=0}^{n} \frac{\mu^x}{x!} \cdot e^{-\mu} t^x, \text{ as expected.}$$

✓

5. Use the pigeon hole principle to show that some pair of any five points in a unit square will be at most $\frac{1}{\sqrt{2}}$ units apart.

Divide the square into four smaller squares by connecting mid-points of opposite sides.

As there're 5 points and 4 regions, there must be a region with 2 points.

Since the diagonal of that square region is $\frac{1}{\sqrt{2}}$ units long, that's as far as the two points in the region can possibly get.

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