$\frac{12}{12}$ Excellent!

FM2 Assignment #26 Name: Jerry Jiang

1. The solution to the differential equation $\dfrac{dy}{dx} + 5y = 0$ for which $y = 2$ when $x = 0$ can be written in the form $y = Ae^{kx}$. Find the values of A and k.

$\frac{dy}{dx} = -5y.$

$\int \frac{1}{y}\, dy = \int -5\, dx$

$\ln y = -5x + c$

$y = e^{-5x} \cdot e^{c} = ce^{-5x}.$

$y = 2,\ x = 0.$

$2 = ce^{0}.$

$c = 2.$

Therefore, $A = 2,\ k = -5.$ ✓

2. A fair coin is tossed until five heads occur. Find the probability that the fifth head occurs on the tenth toss.

$X \sim NB(5, \frac{1}{2})$

$P(X = 10) = \binom{9}{4} (\frac{1}{2})^5 (\frac{1}{2})^5 = 0.123$ (3 s.f.) ✓

3. An isomorphism from a group to itself is called an *automorphism*. Show that the function $f : \mathbb{C}^* \to \mathbb{C}^*$ with rule $f(z) = z^*$ is an automorphism of the group $(C^*, \times)$.

- let $z = a + bi,\ w = c + di.$

$f(z \cdot w) = f(ac + adi + bci - bd) = ac - bd - (ad + bc)i$

$f(z) \cdot f(w) = (a - bi)(c - di) = ac - bd - (ad + bc)i.$ ✓

so $f(z \cdot w) = f(z) \cdot f(w).$

- for any $z = a + bi$ in C^*, there will be a $w = a - bi$ that satisfies $f(z) = w$, which shows surjection. ✓
- Suppose there is $f(z_1) = f(z_2) = w = c + di.$ then $z_1 = c - di,\ z_2 = c - di,$ z_1 has to be equal to z_2. injection. ✓

Hence, f is a isomorphism so it's automorphism. 6

4. The random variable X has probability generating function $G(t) = \dfrac{t}{3-2t}$, mean μ and variance σ^2. Find $P(|X-\mu| < \sigma)$.

$G'(t) = \dfrac{3}{(3-2t)^2}$. $G'(1) = 3 = \mu$.

$G''(t) = \dfrac{12}{(3-2x)^3}$. $G''(1) = 12$.

$Var(X) = \sigma^2 = 12 + 3 - 3^2 = 6$.

$P(|X-3| < \sqrt{6})$. $3-\sqrt{6} < X < \sqrt{6}+3$, $x = 1, 2, 3, 4, 5$.

$G(t) = \dfrac{\frac{1}{3}t}{1-\frac{2}{3}t} = \frac{1}{3}t + \frac{2}{9}t^2 + \frac{4}{27}t^3 + \frac{8}{81}t^4 + \frac{16}{243}t^5 + \cdots$

$P = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \frac{16}{243} = 0.868$ (3 s.f.).

5. Newton's law of cooling states that a body cools at a rate proportional to the difference between the temperature of the body and its surroundings. Sherlock Holmes finds that the core temperature of a corpse is 17 °C at 6.30 am and three hours later that this temperature has fallen to 11 °C. If the temperature of the surroundings had been approximately 5 °C throughout the night and the normal living body temperature is 37 °C, what did Holmes estimate as the time of death?

$\dfrac{dT}{dt} = -k(T-5)$

$\int \frac{1}{T-5}\, dT = \int -k\, dt$.

$\ln(T-5) = -kt + c$

$T - 5 = ce^{-kt}$

$T = ce^{-kt} + 5$

at $t=0$, $T = 37$.

$37 = ce^0 + 5$.

$c = 32$.

$T = 32e^{-kt} + 5$.

$17 = 32e^{-kt_1} + 5$

$11 = 32e^{-k(t_1+3)} + 5 = 32e^{-kt_1} \cdot e^{-3k} + 5$

$32e^{-kt_1} = 12$, $32e^{-kt_1} \cdot e^{-3k} = 6$.

$e^{3k} = 2$.

$3k = \ln 2$

$k = \dfrac{\ln 2}{3}$

So $T = 32e^{-\frac{t\ln 2}{3}} + 5$

when $T = 17$, $t = 4.2451125$ h = 4h 15 min.

Therefore the time of death is 2:15 am.

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