FM2 Assignment #30 Name: Jerry Jiang 10/10 Excellent!

1. Use Rolle's theorem to show that the equation $x^3 + x + c = 0$ where c is a constant cannot have more than one real zero.

$f(x) = x^3 + x + c$ is continuous and differentiable for all real numbers.

According to Rolle's theorem, if $f(a) = f(b)$, then there's a $c \in]a, b[$ that $f'(c) = 0$.

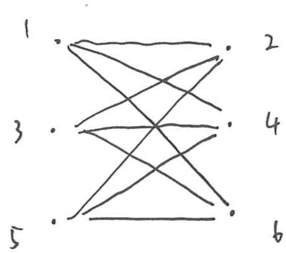
$f'(x) = 3x^2 + 1$ which is positive for all real x values, so $f'(x) \neq 0$. (contrapositive).

Hence, there's no pair of values $x_1, x_2 \in \mathbb{R}$ that satisfies $f(x_1) = f(x_2)$.

Therefore, there can't be more than one real zero.

2. Let $G = (V, E)$ be the graph with $V = \{1, 2, 3, 4, 5, 6\}$ and $uv \in E$ if $|u - v|$ is odd. To which well known graph is G isomorphic?

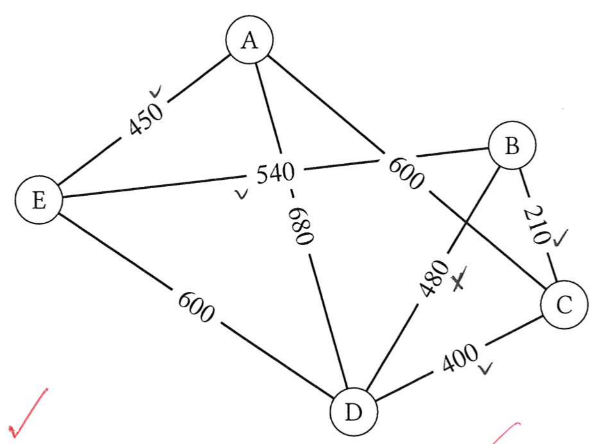
$E = \{12, 14, 16, 23, 25, 34, 36, 45, 56\}$.



Hence, $G \cong K_{3,3}$.

3. Use Kruskal's algorithm in table form to find a minimum spanning tree and its weight in the weighted graph below.

n	edge	weight
1	BC	210
2	CD	400
3	AE	450
4	BE	540



$W(MST) = 210 + 400 + 450 + 540 = 1600$.

4. Let X and Y be independent random variables with $X \sim \text{Po}(3)$ and $Y \sim \text{Po}(2)$.

(a) Find $\text{E}(2X + 3Y)$ and $\text{Var}(2X + 3Y)$.

$E(2X+3Y) = 2E(X) + 3E(Y) = 2\times 3 + 3\times 2 = 12.$

$Var(2X+3Y) = 4Var(X) + 9Var(Y) = 4\times 3 + 9\times 2 = 30.$

(b) Hence state with a reason whether or not $2X + 3Y$ has a Poisson distribution.

Poisson Distribution have characteristic of $E(X) = Var(X)$ but $E(2X+3Y) \neq Var(2X+3Y)$, so it isn't a Poisson Distribution.

5. Evaluate $\lim_{x\to 0}\left[\frac{\ln(1+x)}{x}\right]^{1/x}$.

Approach 1

$$L = \lim_{x\to 0} e^{\boxed{\frac{1}{x}\ln\left[\frac{\ln(1+x)}{x}\right]} \to M}$$

$$M = \lim_{x\to 0} \frac{\ln\left[\boxed{\frac{\ln(1+x)}{x}}\right] \to N}{x}$$

$$N = \lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

$$M \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{x}{\ln(1+x)} \cdot \frac{\frac{x}{1+x} - \ln(1+x)}{x^2}}{1} = \lim_{x\to 0} \frac{x-(1+x)\ln(1+x)}{(x^2+x)\ln(1+x)}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{1 - \frac{1+x}{1+x} - \ln(1+x)}{(2x+1)\ln(x+1) + \frac{x^2+x}{x+1}} = \lim_{x\to 0} \frac{-\ln(1+x)}{(2x+1)\ln(x+1)+x}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\frac{1}{1+x}}{\frac{2x+1}{x+1} + 2\ln(x+1)+1} = -\frac{1}{2}.$$

$$L = e^{-\frac{1}{2}}.$$

~~is there an easier way?~~

ok there's the second solution

Approach 2.

taylor series

$$L = \lim_{x\to 0}\left[\frac{\overbrace{x - \frac{x^2}{2} + O(x^3)}^{1}}{x}\right]^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(1 - \frac{x}{2} + O(x^2)\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(1-\frac{x}{2}\right)^{\frac{1}{x}} \cdot \left(1 + \frac{O(x^2)}{1-\frac{x}{2}}\right)^{\frac{1}{x}}$$

$$= \lim_{u\to\infty}\left(1 + \frac{-\frac{1}{2}}{u}\right)^u \cdot \lim_{x\to 0}\left(1 + O(x^2)\underbrace{\left(1-\frac{x}{2}\right)^{-1}}_{\text{binomial expansion}}\right)^{\frac{1}{x}}$$

$$= \lim_{u\to\infty}\left(1+\frac{-\frac{1}{2}}{u}\right)^u \cdot \lim_{x\to 0}\underbrace{\left(1 + O(x^2)O(1)\right)^{\frac{1}{x}}}_{\text{binomial expansion}}$$

$$= \lim_{u\to\infty}\left(1+\frac{-\frac{1}{2}}{u}\right)^u \cdot \lim_{x\to 0}\left(1 + O(x)\right)$$

$$= e^{-\frac{1}{2}}.$$

Very good.

法 三

$$L = \lim_{x\to 0}\left[\frac{x - \frac{x^2}{2} + O(x^3)}{x}\right]^{\frac{1}{x}} = \lim_{x\to 0}\left[1 - \frac{x}{2} + O(x^2)\right]^{\frac{1}{x}}$$

$\ln L = M.$

$$M = \frac{\ln\left[1 - \frac{x}{2} + O(x^2)\right]}{x} \overset{L'H}{=} \frac{\frac{1}{1-\frac{x}{2}+O(x^2)} \cdot [\cdots]'}{1}$$

$$= 1 \cdot \left[-\frac{1}{2} + O(x)\right]$$

$$= -\frac{1}{2}$$

$$L = e^{-\frac{1}{2}}.$$

$\frac{\ln(\quad)}{x} \to -\frac{1}{2}$ as $x\to 0$.