FM2 Assignment #21 Name: Jerry Jiang $\frac{10}{10}$ Excellent!

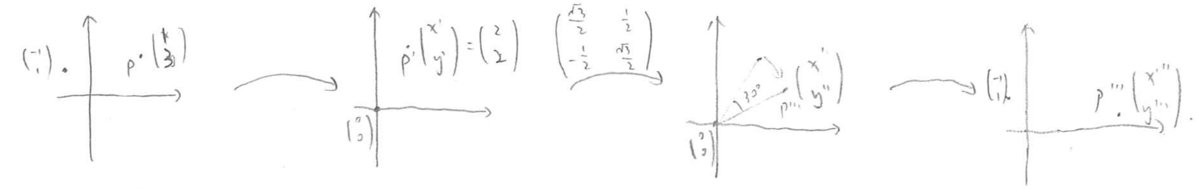
1. Must a matrix with an eigenvalue of 0 be a singular matrix? Make sure to justify your answer.

The characteristic equation of matrix A is $\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$.

when the eigenvalue is 0, we have $0^2 - \mathrm{tr}A \cdot 0 + \det A = 0$.

Therefore $\det A = 0$, so A must be singular. ✓

2. Find the image of the point $(1, 3)$ under a rotation of 30° clockwise about $(-1, 1)$ by first translating the centre of rotation to the origin, then rotating about the origin using a rotation matrix and lastly translating back.



$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+1 \\ \sqrt{3}-1 \end{pmatrix}.$$

✓

$$\therefore \begin{pmatrix} \sqrt{3}+1-1 \\ \sqrt{3}-1+1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ \sqrt{3} \end{pmatrix}.$$

$\therefore$ $p'''(\sqrt{3}, \sqrt{3})$.

3. Find the characteristic equation, eigenvalues and eigenvectors of the matrix $M = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$.

$\lambda^2 - 2\lambda + (-3) = 0$.

$(\lambda - 3)(\lambda + 1) = 0$

$\therefore \lambda = 3$ or -1. ✓

- $\lambda = 3$, $\begin{pmatrix} 1 & -5 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$.

The corresponding eigenvector is $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$. ✓

- $\lambda = -1$ $\begin{pmatrix} 5 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$.

The corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. ✓

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4. By integrating the binomial series for $\frac{1}{\sqrt{1-x^2}}$ find the seventh derivative of arcsin x at $x = 0$.

$$(1-x^2)^{-\frac{1}{2}} = 1+(-\tfrac{1}{2})(-x^2) + \frac{-\frac{1}{2}(-\frac{3}{2})}{2}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{6}(-x^2)^3 + \cdots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

$$\int (1-x^2)^{-\frac{1}{2}} = \arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$

$$C_7 = \frac{f^{(7)}(0)}{7!} = \frac{5}{112}$$

✓

$$\therefore f^{(7)}(0) = 225.$$

5. Write the matrix $M = \begin{pmatrix} -11 & 4 \\ -6 & 2 \end{pmatrix}$ as the product of elementary matrices and hence describe the transformation represented by M as a sequence of basic transformations.

$$\begin{pmatrix} -11 & 4 \\ -6 & 2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} A} \begin{pmatrix} 1 & 0 \\ -6 & 2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} B} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ $C \times B \times A \times M = I.$

$\therefore$ $M = A^{-1} \times B^{-1} \cdot C^{-1}$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Therefore, it's a stretch in y-axis direction with factor of 2, followed by a shear downwards with facter of 6 ✓ followed by a shear to the right by a factor of 2.

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