FURTHER MATHEMATICS
HIGHER LEVEL

84% V. good

Tuesday 28 January 2020

1 hour 10 minutes

Name in block letters

J	E	R	R	Y		J	I	A	N	G	

INSTRUCTIONS TO CANDIDATES

- Do not open this test until instructed to do so.
- Answer all 10 questions.
- A graphic display calculator is required for this test.
- A clean copy of the formula booklet is required for this test.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working or explanations. Where an answer is incorrect, some marks may be given for a correct method provided this is shown by written working. You are therefore advised to show all working. Working may be continued below the lines, if necessary.

1. Use L'Hôpital's rule to find $\lim_{x \to 0} (\csc x - \cot x)$.

$L = \lim_{x \to 0} \frac{1 - \cos x}{\sin x}$

$\cdot \lim_{x \to 0} 1 - \cos x = \lim_{x \to 0} \sin x = 0$

Therefore, apply L'Hôpital's rule.

$L = \lim_{x \to 0} \frac{\sin x}{\cos x} = 0.$

/5

2. The relation R is defined on $\mathbb{R}$ by $x\,R\,y$ if $|x| + |y| = |x + y|$.

   (a) Show that R is reflexive.

   (b) Show that R is symmetric.

   (c) Show by means of a counterexample that R is not transitive.

(a) $|x| + |x| = 2|x| = |2x| = |x + x|$ reflexive

(b) $|x| + |y| = |x + y|$.

Then $|y| + |x| = |x| + |y| = |x + y|$ symmetric

(c) Suppose xRy, yRz.

Then $|x| + |y| = |x+y|$, $|y| + |z| = |y+z|$.

Let $x = -1$, $y = 0$, $z = 1$

$|x| + |y| = |-1| + |0| = 1 = |x+y|$

$|y| + |z| = |0| + |1| = 1 = |y+z|$

$|x| + |z| = |-1| + |1| = 2 \neq |x+z|$

Therefore, R isn't transitive.

Turn over

3. Consider the differential equation $dy/dx = y^3 - x^3$ with $y = 1$ when $x = 0$. Use Euler's method in table form with a step length of 0.1 to approximate the value of y when $x = 0.4$.

n	x_n	y_n	h	$h \cdot f(x_n, y_n)$
0	0	1	0.1	0.1
1	0.1	1.1	0.1	0.133
2	0.2	1.233	0.1	0.187
3	0.3	1.42	0.1	0.284
4	0.4	1.70	0.1	—

when $x = 0.4$, $y \approx 1.70$ (3s.f.)

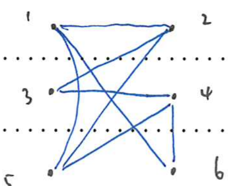
4. Consider the simple graph G with adjacency matrix
$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1&2&3&4&5&6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

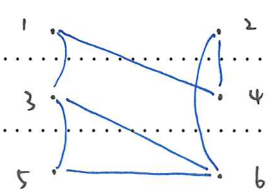
(a) Draw G and its complement G'.

(b) State whether or not G is planar giving a reason for your answer.

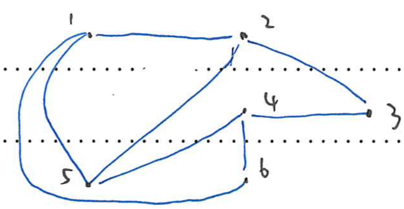
(c) State whether or not G is bipartite giving a reason for your answer.

(a) 12, 15, 16, 23, 25, 34, 45, 46.

G: 

G': 

(b) ~~if a graph is planar, simple, and connected, there must be $e \le 3v-6$ for $v \geq 3$. For G, $e = 8$, $v = 6$, $3v-6 = 18-6 = 12 \geq 8$. So G is planar. In fact~~



It's planar, check the diagram above.

(c) Notice that 12, 15, 25 are adjacent ~~connected~~, so that they have to be in different part of the partition. There's no way G can be partition into 2.

5

More simply G contains a triangle, so G is not bipartite.

Turn over

5. Consider the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) If $a + b = c + d = 1$ show that 1 is an eigenvalue of M.

(b) Find the eigenvectors for M when $a = 2$, $b = -1$, $c = 3$ and $d = -2$.

(a) $b = 1-a, \quad c = 1-d.$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$

$\lambda^2 - (a+d)\lambda + ad - (1-a)(1-d) = 0.$

$\lambda^2 - (a+d)\lambda + \cancel{ad} - 1 - \cancel{ad} + a + d = 0$

1 is a solution to the above hence an eigenvalue of M.

(b) $\lambda^2 - 0 \cdot \lambda + (-4) - (-3) = 0$

$\lambda^2 = 1$

$\lambda = \pm 1.$

- when $\lambda = 1$, $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- when $\lambda = -1$, $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

5

6. Consider the permutation $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}$.

   (a) Find the order of p justifying your answer.

   (b) Find p^2.

   (c) The permutation group G is generated by p. Find the element of G that is of order 2 giving your answer in cycle notation.

(a) $(1\,3\,5\,2\,4\,6)$

order of p is 6 as $p^6 = (1)$.

(b) $p^2 = (1\,3\,5\,2\,4\,6)(1\,3\,5\,2\,4\,6)$

$= (1\,5\,4)(3\,2\,6)$

(c) $p^6 = e$, $(p^3)^2 = e$, so element of order 2 is p^3.

$p^3 = (1\,3\,5\,2\,4\,6)(1\,3\,5\,2\,4\,6)(1\,3\,5\,2\,4\,6)$

$= (1\,2)(3\,4)(5\,6)$

7. A linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ is represented by the matrix $M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 7 & 5 \\ -3 & 1 & 4 \\ 1 & 5 & 4 \end{pmatrix}$.

(a) Find the rank of M.

(b) Find a basis for the range of T.

(c) Find the kernel of T.

$T(\vec{v}) = \begin{pmatrix} & & \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$= x\begin{pmatrix} \\ \end{pmatrix} + y\begin{pmatrix} \\ \end{pmatrix} + z\begin{pmatrix} \\ \end{pmatrix}$

☆ 行数大于列数时加0!!!

(a) $\text{rref}(M) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\text{rank}(M) = 2.$

(b) M 将一个 3×1 矩阵转化为一个 4×1 矩阵.

线性变换的结果一定是 $\begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}$ 与 $\begin{pmatrix} 2 \\ 7 \\ 1 \\ 5 \end{pmatrix}$ 的线性组合.

basis for the range of T 即为 $\begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 1 \\ 5 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$x - z = 0.$

$y + z = 0.$

let $z = \lambda.$

$x = \lambda,$

$y = -\lambda.$

kernel is $\begin{pmatrix} \lambda \\ -\lambda \\ \lambda \end{pmatrix}$.

0

8. Determine whether each of the following infinite series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{3n}{2n^2+5}$

$\frac{n}{2n^2+5} < \frac{n}{2n^2} = \frac{1}{2n}$ diverge

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}$

$\frac{\frac{3}{2n+2+\frac{5}{n+1}}}{\frac{3}{2n+\frac{5}{n}}}$ $\frac{2n+\frac{5}{n}}{2n+2+\frac{5}{n+1}} = \frac{2n^2(n+1)+5(n+1)}{\cancel{\quad}n+5n}{(2n+2)n(n+1)}$

(a) $\sum_{n=1}^{\infty} \frac{3}{2n+5/n}$ converges.

~~as $n \to \infty$, the denominator $\to \infty$~~

Ratio Test: as $n \to \infty$, Ratio $\to 1$.

Limit comparison.

$\sum_{n=1}^{\infty} \frac{3n}{2n^2+5}$ $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\frac{\frac{3n}{2n^2+5}}{\frac{1}{n}} = \frac{3n^2}{2n^2+5} = \frac{3}{2}$ as $n \to \infty$. $0 < \frac{3}{2} < \infty$.

they behave the same.

therefore $\sum_{n=1}^{\infty} \frac{3n}{2n^2+5}$ diverges.

(b) Ratio Test: $\left| \frac{\frac{(2n+2)!}{5^{n+1}[(n+1)!]^2}}{\frac{(2n)!}{5^n (n!)^2}} \right| = \left| \frac{(2n+2)!\, 5^n (n!)^2}{(2n)!\, 5^{n+1} [(n+1)!]^2} \right| = \left| \frac{(2n+2)(2n+1)}{5 \cdot (n+1)^2} \right|$

$= \left| \frac{2(2n+1)}{5(n+1)} \right|$

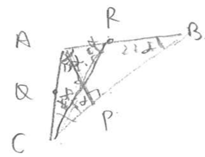
$= \frac{4}{5}$ as $n \to \infty$.

since the ratio < 1, the series converges.

$2\frac{1}{2}$

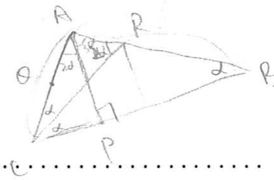
$2\frac{1}{2}$

9. In $\triangle ABC$, the angles are $A = \frac{5\pi}{8}$, $B = \frac{\pi}{8}$ and $C = \frac{\pi}{4}$. Point P is the foot of the perpendicular from A to side $[BC]$, point Q is the midpoint of side $[AC]$, and point R is the intersection of the angle bisector of angle C with side $[AB]$.

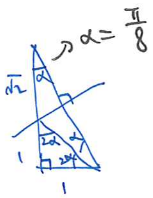


(a) Show $\dfrac{AR}{BR} = \tan\frac{\pi}{8}$.

(b) Show $\dfrac{BP}{CP} = \tan\frac{3\pi}{8}$.



(c) Hence show that the lines (AP), (BQ) and (CR) are concurrent.



(a) $\tan\frac{\pi}{8} = \frac{1}{1+\sqrt{2}}$. Let $AP = a$.

$\angle APC = 90°$, $\angle C = 45°$, $CP = AP = a$, $AC = \sqrt{2}a$.

$\because \tan\frac{\pi}{8} = \frac{1}{1+\sqrt{2}}$, $\therefore PB = (1+\sqrt{2})a$.

According to the angle bisector theorem, $\frac{AR}{BR} = \frac{AC}{BC} = \frac{\sqrt{2}a}{a+(1+\sqrt{2})a}$
$= \frac{\sqrt{2}}{2+\sqrt{2}}$
$= \frac{1}{1+\sqrt{2}}$

Hence, $\frac{AR}{BR} = \tan\frac{\pi}{8}$.

(b) $BP = (1+\sqrt{2})a$, $CP = a$. $\frac{BP}{CP} = \frac{1+\sqrt{2}}{1} = \cot\frac{\pi}{8} = \tan\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \tan\frac{3}{8}\pi$

(c) Ceva's Theorem:

$\frac{AQ}{QC} \cdot \frac{CP}{PB} \cdot \frac{BR}{RA} = 1 \cdot \tan\frac{\pi}{8} \cdot \tan\frac{3\pi}{8} = \cot\left(\frac{\pi}{2} - \frac{\pi}{8}\right) \cdot \tan\frac{3\pi}{8} = 1.$

Therefore, they're concurrent.

/5

10. The random variable $X \sim \mathrm{NB}(r, p)$ has probability generating function $G_X(t) = \dfrac{p^r t^r}{(1-qt)^r}$.

(a) Use this probability generating function to find $\mathrm{E}(X)$.

Consider another independent random variable $Y \sim \mathrm{NB}(s, p)$ and let $W = X + Y$.

(b) i. Find the probability generating function for W.

ii. Hence identify the distribution that W follows and state its parameters.

iii. Given that $r = 2$ and $s = 3$, calculate $P(X = 3 \mid W = 7)$.

(a) $G_X'(t) = \dfrac{rp^r t^{r-1}(1-qt)^r - r(1-qt)^{r-1}(-q)\cdot p^r t^r}{(1-qt)^{2r}}$

$G_X'(1) = \dfrac{rp^r(1-q)^r + r(1-q)^{r-1}q\cdot p^r}{(1-q)^{2r}}$

$= \dfrac{rp^r\cdot p^r + r\cdot p^{r-1}\cdot p^r\cdot q}{p^{2r}}$

$= \dfrac{rp^{2r} + rq\cdot p^{2r-1}}{p^{2r}} = \dfrac{rp + rq}{p} = \dfrac{r}{p}$

(b) i. $G_Y(t) = \dfrac{p^s t^s}{(1-qt)^s}$

$G_W(t) = G_X(t)\,G_W(t) = \dfrac{(pt)^{r+s}}{(1-qt)^{r+s}}$

ii. $W \sim NB(r+s, p)$

iii. $X \sim NB(2,p)$, $Y \sim NB(3,p)$, $W \sim NB(5,p)$

$P(X=3) = \binom{2}{1} p^2 q^1$, $P(W=7) = \binom{6}{4} p^5 q^2$

$P(Y=4) = \binom{3}{2} p^3 q^1$

$P(X=3 \mid W=7) = \dfrac{P(X=3)\cdot P(Y=4)}{P(W=7)}$

$= \dfrac{2\cdot \cancel{\frac{3\times 2}{2\times 1}}}{\cancel{\frac{6\times 5}{2\times 1}}}$

$= \dfrac{2}{5}$

4/4