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MATHEMATICS
HIGHER LEVEL
PAPER 3 – DISCRETE MATHEMATICS

反馈
2 (i) 2 edges. not 2 paths. [−1]

Excellent!
分数
59 / 60

Jerry Jiang

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 9]*

(a) Use the Euclidean algorithm to find the greatest common divisor of 259 and 581. *[4 marks]*

581=259*2+63; 259=63*4+7; 63=7*9+0, hence gcd(259,581)=7.

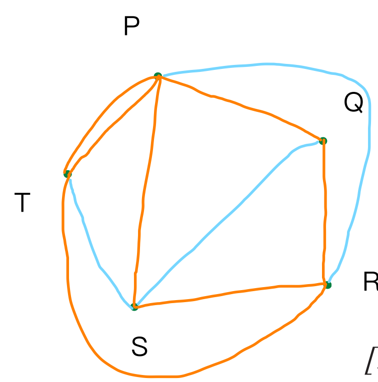
(b) Hence, or otherwise, find the general solution to the diophantine equation $259x+581y=7$. *[5 marks]*

By inspection, one solution is x0=9, y0=−4. Therefore, the general solution is x0=9+83t, y0=−4−37t.

2. *[Maximum mark: 13]*

The graph G has vertices $P,\ Q,\ R,\ S,\ T$ and the following table shows the number of edges joining each pair of vertices.

	P	**Q**	**R**	**S**	**T**
P	0	1	0	1	2
Q	1	0	1	0	0
R	0	1	0	1	1
S	1	0	1	0	0
T	2	0	1	0	0



(a) Draw the graph G as a planar graph. *[2 marks]*

(b) Giving a reason, state whether or not G is

(i) simple;

It is not simple as there are two paths from T to P.

(ii) connected;

It is connected as there’s always a path from one vertex to another.

(iii) bipartite. *[4 marks]*

It is bipartite with partition {P,R} and {S,Q,T}.

(c) Explain what feature of G enables you to state that it has an Eulerian trail and write down a trail. *[2 marks]*

The trail TPTRQPSR contains every edge of the graph, making it an Eulerian trail.

(This question continues on the following page)

(Question 2 continued)

(d) Explain what feature of G enables you to state it does not have an Eulerian circuit. *[1 mark]*

There are two vertices with odd degrees, so there's no way to begin and end at one vertex.

(e) Find the maximum number of edges that can be added to the graph G (not including any loops or further multiple edges) whilst still keeping it planar. *[4 marks]*

K5 is a well known graph that's not planar. In the graph above in light blue, we know we can add TS, SQ, and RP while keeping it planar. The only edge left compared to K5 is TQ, so this is as far as we can go. The maximum number is 3.

3. *[Maximum mark: 12]*

(a) One version of Fermat's little theorem states that, under certain conditions, $a^{p-1} \equiv 1 \pmod{p}$.

http://mathonline.wikidot.com/calculating-f-for-large-positive-integers

(i) Show that this result is not true when $a = 2$, $p = 9$ and state which of the conditions is not satisfied.

2^(9-1)=2^8=256≡4 mod 9 so the result is false. P should be prime but 9 isn't.

(ii) Find the smallest positive value of k satisfying the congruence $2^{45} \equiv k \pmod{9}$. *[6 marks]*

2 and 9 are prime so apply Euler's theorem. Phi(9)=Phi(3^2)=3^(2-1)*(3-1)=6. 2^6≡1 mod 9. 2^(6*7+3)≡2^3≡8 mod 9. k=8.

(b) Find all the integers between 100 and 200 satisfying the simultaneous congruences $3x \equiv 4 \pmod{5}$ and $5x \equiv 6 \pmod{7}$. *[6 marks]*

This is equivalent to x≡3 mod 5 and x≡4 mod 7. lcd(5,7)=35. By inspection, one x can be 18, so the solution should be x≡18 mod 35. Numbers that are between 100 and 200 are 123, 158, and 193.

4. *[Maximum mark: 12]*

The weights of the edges of a graph G with vertices A, B, C, D and E are given in the following table.

	A	**B**	**C**	**D**	**E**
A	–	11	18	12	9
B	11	–	17	13	14
C	18	17	–	16	10
D	12	13	16	–	15
E	9	14	10	15	–

(a) Starting at A, use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G. I will only write the content here instead of the format *[4 marks]*

AE(9)+EC(10)+CD(16)+DB(13)+BA(11)=59

(b) (i) Use Kruskal's algorithm to find and draw a minimum spanning tree for the subgraph obtained by removing the vertex A from G.

EC(10)+BD(13)+BE(14)=37

(ii) Hence use the deleted vertex algorithm to find a lower bound for the travelling salesman problem for G. *[8 marks]*

AB(11)+AE(9)+37=57

5. *[Maximum mark: 14]*

(a) The sequence $\{u_n\}$, $n \in \mathbb{Z}^+$, satisfies the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$. Given that $u_1 = u_2 = 3$, obtain an expression for u_n in terms of n. *[6 marks]*

r^2−5r+6=0, r=2or3. un=a3^n+b2^n. Solving the simultaneous equation and get: un=−3^n+3*2^n.

(b) The sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, satisfies the recurrence relation $v_{n+2} = 4v_{n+1} - 4v_n$. Given that $v_1 = 2$ and $v_2 = 12$, use the principle of strong mathematical induction to show that $v_n = 2^n(2n-1)$. *[8 marks]*

For n=1, we have v1=2(2−1)=2, satisfied.

For n=2, we have v2=4(4−1)=12, satisfied.

Now suppose we have vn=2^n*(2n−1) and v(n−1)=2^(n−1)*(2n−2−1). From the recurrence relation, we have v(n+1)=4*2^n*(2n−1)−4*2^(n−1)*(2n−2−1)=4*2^(n−1)*(4n−2−2n+3)=

2^(n+1)*(2n+1)=2^(n+1)*[2(n+1)−1]

So we can conclude that the given general equation works for all positive integers for the recurrence relation.