International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – CALCULUS

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1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

The function f is defined on the domain $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ by $f(x) = \ln(1+\sin x)$.

(a) Show that $f''(x) = -\dfrac{1}{(1+\sin x)}$. *[4 marks]*

(b) (i) Find the Maclaurin series for $f(x)$ up to and including the term in x^4.

(ii) Explain briefly why your result shows that f is neither an even function nor an odd function. *[7 marks]*

(c) Determine the value of $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin x)-x}{x^2}$. *[3 marks]*

2. *[Maximum mark: 8]*

Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{x^2 - y^2}\,,\ x > 0,\ x^2 > y^2.$$

(a) Show that this is a homogeneous differential equation. *[1 mark]*

(b) Find the general solution, giving your answer in the form $y = f(x)$. *[7 marks]*

$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$

$v = \sin \ln(x) + c$

$y = x\sin(\ln(x) + c)$

$y = vx$

$(1-v^2)^{\frac{1}{2}} \to \frac{1}{2}(1-v^2)^{-\frac{1}{2}}(-2v)$

$\frac{dy}{dx} = \frac{dv}{dx}x + \cancel{v} = \cancel{v} + \sqrt{1-v^2}$

$(1-v^2)^{-\frac{1}{2}}$

$\int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx$

$\arcsin v = \ln x + c$

p 240. Ex10E2 2,8,8

3. *[Maximum mark: 15]*

Consider the differential equation

$$\frac{dy}{dx} = 2e^x + y\tan x, \text{ given that } y = 1 \text{ when } x = 0.$$

The domain of the function y is $\left[0, \frac{\pi}{2}\right[$.

(a) By finding the values of successive derivatives when $x = 0$, find the Maclaurin series for y as far as the term in x^3. *[6 marks]*

(b) (i) Differentiate the function $e^x(\sin x + \cos x)$ and hence show that

$$\int e^x \cos x \, dx = \frac{1}{2}e^x(\sin x + \cos x) + c.$$

(ii) Find an integrating factor for the differential equation and hence find the solution in the form $y = f(x)$. *[9 marks]*

4. *[Maximum mark: 10]*

Let $f(x) = 2x + |x|$, $x \in \mathbb{R}$.

(a) Prove that f is continuous but not differentiable at the point $(0, 0)$. *[7 marks]*

(b) Determine the value of $\int_{-a}^{a} f(x)\,dx$ where $a > 0$. *[3 marks]*

5. *[Maximum mark: 13]*

Consider the infinite series $\sum_{n=1}^{\infty} \frac{(n-1)x^n}{n^2 \times 2^n}$.

(a) Find the radius of convergence. *[4 marks]*

(b) Find the interval of convergence. *[9 marks]*

1. (a) $f'(x) = \frac{1}{1+\sin x} \cdot \cos x$. $f'(0) = 1$ ✓

$f''(x) = \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \frac{-1-\sin x}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$. $f''(0) = -1$

(b) i. $f'''(x) = \frac{\cos x}{(1+\sin x)^2}$. $f'''(0) = 1$.

$f''''(x) = \frac{-\sin x(1+\sin x)^2 - \cos x \cdot 2(1+\sin x)\cdot \cos x}{(1+\sin x)^4}$. $f''''(0) = -2$.

$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4$. ✓

60/60

Excellent!!

ii. even function only have 0, x^2, x^4, ... terms in the Maclaurin Series.

odd function only have x, x^3, x^5, ... terms in the Maclaurin series.

$P_4(x)$ have both types, so it's neither an even function nor an odd function.

$\boxed{7^*}$

(c) $\lim\limits_{x\to 0} \ln(1+\sin x) - x = \lim\limits_{x\to 0} x^2 = 0$. Apply the L'Hôpital's rule: ✓

$L = \frac{\lim\limits_{x\to 0} \frac{\cos x}{1+\sin x} - 1}{\lim\limits_{x\to 0} 2x}$. Apply again, $L = \frac{\lim\limits_{x\to 0} \frac{-1}{1+\sin x}}{2} = -\frac{1}{2}$.

Alternative Method:

$L = \lim\limits_{x\to 0} \frac{P_4(x) - x}{x^2} = \lim\limits_{x\to 0} \frac{-\frac{1}{2}x^2 + O(x^3)}{x^2} = \lim\limits_{x\to 0} -\frac{1}{2} + O(x) = -\frac{1}{2}$. ✓ /14

2. (a) $\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2-y^2}{x^2}} = \frac{y}{x} + \sqrt{1-\left(\frac{y}{x}\right)^2}$ ✓

(b) Let $y = vx$. $\frac{dy}{dx} = \frac{dv}{dx}x + v$.

So $\frac{dv}{dx}x + v = v + \sqrt{1-v^2}$.

$\int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx$.

$\arcsin v = \ln x + c$

$v = \sin(\ln x + c)$ ✓

Hence, $y = x\sin(\ln x + c)$. /8

3. (a) $\frac{d^2y}{dx^2} = 2e^x + \frac{dy}{dx}\tan x + y\sec^2 x = 2e^x + 2e^x\tan x + y(\tan^2 x + \sec^2 x)$, $\left.\frac{d^2y}{dx^2}\right|_{x=0,y=1} = 3.$

$\frac{d^3y}{dx^3} = 2e^x + 2e^x\cdot\sec^2 x + 2e^x\cdot\tan x + \frac{dy}{dx}(\tan^2 x + \sec^2 x) + y(2\sec^2 x\tan x + 2\sec^2 x\tan x).$

$\left.\frac{d^3y}{dx^3}\right|_{x=0,y=1} = 6.$

$P_3(x) = 1 + 2x + \frac{3}{2}x^2 + x^3.$

(b) i. $(e^x(\sin x + \cos x))' = e^x\cdot(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x\cos x.$

Hence $\int 2e^x\cos x\,dx = e^x(\sin x + \cos x) + C,$

$\int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + C.$

ii. $\frac{dy}{dx} - \tan x y = 2e^x.$ $\int -\tan x\,dx = \int \frac{-\sin x}{\cos x}dx = \ln\cos x + C.$

integrating factor $= e^{\ln\cos x} = \cos x.$

$\cos x\frac{dy}{dx} - \sin x\cdot y = 2e^x\cos x.$

$(\cos x\cdot y)' = 2e^x\cos x$

$\cos x\cdot y = \int 2e^x\cos x = e^x(\sin x + \cos x) + C$

$y = \tan x\cdot e^x + e^x + \frac{C}{\cos x}.$

$x=0,\ y=1,\ C=0$

Therefore, $y = \tan x\cdot e^x + e^x.$

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4. (a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} 2x + \lim_{x\to 0}|x| = 0.$

$f(0) = 0 + 0 = 0 = \lim_{x\to 0} f(x).$ <u>continuous</u>.

$f'(x) = \lim_{h\to 0}\frac{2(x+h) + |x+h| - 2x - |x|}{h}.$

$f'(0) = \lim_{h\to 0}\frac{2h + |h|}{h}.$ ① $\lim_{h\to 0^+}\frac{2h+|h|}{h} = 3$, ② $\lim_{h\to 0^-}\frac{2h+|h|}{h} = 1.$ <u>not differentiable</u>

(b). $\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_{-a}^0 f(x)dx$

$= \int_0^a 2x + x\,dx + \int_{-a}^0 2x - x\,dx$

$= \left[\frac{3}{2}x^2\right]_0^a + \left[\frac{1}{2}x^2\right]_{-a}^0$

$= \frac{3}{2}a^2 + \left[0 - \frac{1}{2}(-a)^2\right]$

$= a^2.$

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5. (a). Let $U_n = \frac{(n-1)x^n}{n^2 \times 2^n}$.

Ratio Test: $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{\frac{n\, x^{n+1}}{(n+1)^2 \times 2^{n+1}}}{\frac{(n-1)\, x^n}{n^2 \times 2^n}}\right| = \left|\frac{x \cdot n^3}{2 \cdot (n+1)^2 (n-1)}\right| = \left|\frac{x}{2}\right|$ as $x \to \infty$.

$\left|\frac{x}{2}\right| < 1$. $-2 < x < 2$. $\underline{R = 2}$

(b) ① when $x = 2$. $U_n = \frac{n-1}{n^2} = \frac{1}{n} - \frac{1}{n^2}$ = divergent series - convergent series = divergent series.

② when $x = -2$, $U_n = (-1)^n \frac{(n-1)}{n^2}$. $\lim_{n\to\infty} U_n = 0$.

$|U_n| - |U_{n+1}| = \frac{n-1}{n^2} - \frac{n}{(n+1)^2} = \frac{n^2-n-1}{n^2(n+1)^2} > 0$ when $n \geq 2$.

According to the alternating series test, the series converges.

Therefore, the interval of convergence is $[-2, 2[$.

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