Quantathon

Team 5

February 19, 2022

1 Problem 1

Observe that

$$
\begin{aligned}
&P(2 \text{ heads on the first 2 tosses} \mid \text{a head on the first toss}) \\
=&\frac{\int_0^1 p^2\, dp}{\int_0^1 p\, dp} \\
=&\frac{2}{3} > \frac{1}{2} = P(\text{a head on the first toss})
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&P(H_n \text{ heads on the first } n \text{ tosses with a head on the} (n+1)^{th} \text{toss} \mid H_n \text{ heads on the first } n \text{ tosses}) \\
=&\frac{\int_0^1 p^{H_n+1}(1-p)^{T_n}\binom{n}{H_n}\, dp}{\int_0^1 p^{H_n}(1-p)^{T_n}\binom{n}{H_n}\, dp} && (H_n + T_n = n) \\
=&\frac{\int_0^1 p^{H_n+1}(1-p)^{T_n}\, dp}{\int_0^1 p^{H_n}(1-p)^{T_n}\, dp} && (H_n + T_n = n) \\
=&\frac{H_n+1}{n+2}
\end{aligned}
$$

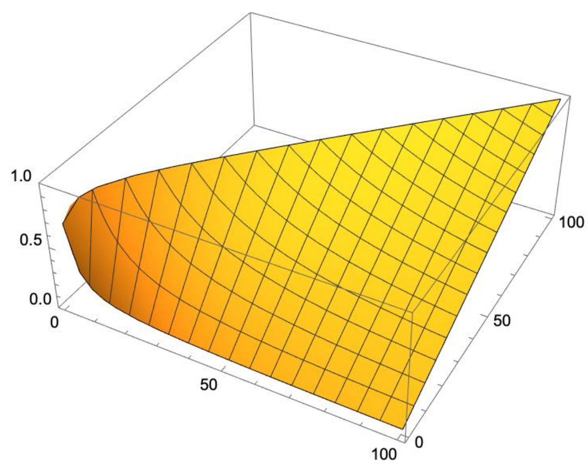

Figure 1: n and H_n are the axes on the xy-plane, with probability on the z-axis

2 Problem 2

We denote head with H and tail with T.
Since the probability of a H on the first toss is both $\frac{1}{2}$ for both Nickel and Dime, without loss of generality, we choose to toss Nickel on the first toss.

- If the first toss is H, then the second toss being a head with Nickel is $\frac{2}{3}$ which is greater than switching to Dime with probability $\frac{1}{2}$ on the second toss with a head.
  - If the first 2 tosses are H ($\frac{1}{2} \times \frac{2}{3}$), then whatever we do on the third toss, the fourth toss with a H is of probability $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$ if the fourth toss is a Nickel.
    - Suppose we continue with Nickel on the third toss, if we end up with H on the third toss, then the probability of H on the fourth toss is $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$; if we end up with T on the third toss, then the probability of H on the fourth toss with Nickel is $\frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} \times \frac{3}{5} = \frac{1}{20}$. Then tossing Nickel on the third toss will give us probability $\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$.
    - Suppose we switch to Dime on the third toss, despite the result of the third toss, we use Nickel on the fourth toss, then the probability of H is $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$.
  - If the second toss is T for Nickel ($\frac{1}{2} \times \frac{1}{3}$), we continue with Nickel on the third toss.
    - If we continue with Nickel on the third toss
      - If the third toss is H, then we still use Nickel on the fourth toss, and the probability of H is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{3}{5} = \frac{1}{20}$.
      - If the third toss is T, then we switch to Dime on the fourth toss, and the probability of H is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{24}$.

      Thus, if the first toss is H and second toss is T and we continue with Nickel, the probability of H on the fourth toss is $\frac{1}{20} + \frac{1}{24} = \frac{11}{120}$.
    - If we switch to Dime on the third toss
      - If the third toss is H, then we use Dime on the fourth toss, and the probability of H on the fourth toss is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{18}$
      - If the third toss is T, then we use Nickel on the fourth toss, and the probability of H on the fourth toss is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{24}$

      Thus, if the first toss is H and second toss is T and we continue with Nickel, the probability of H on the fourth toss is $\frac{1}{18} + \frac{1}{24} = \frac{7}{72} > \frac{11}{120}$.

    Thus, if the second toss is T for Nickel, then we switch to Dime on the third toss, and fourth toss being H has probability $\frac{7}{72}$.

  Therefore, if the first toss is H, then the probability of H on the fourth toss according to our strategy is $\frac{1}{4} + \frac{7}{72} = \frac{25}{72}$
- If the first toss is T, then we use Dime for the second toss.
  - If the second toss is H ($\frac{1}{2} \times \frac{1}{2}$), we continue with Dime for the third toss
    - If the third toss is H, we use Dime on the fourth toss, and the probability of H on the fourth toss is $\frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{8}$
    - If the third toss is T, we use Dime on the fourth toss, and the probability of H on the fourth toss is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$

    Thus, the probability of H on the fourth toss is $\frac{1}{8} + \frac{1}{24} = \frac{1}{6}$
  - If the second toss is T ($\frac{1}{2} \times \frac{1}{2}$), then tossing Nickel or Dime will behave the same afterwards. Without loss of generality, we use Nickel on the third toss.
    - If the third toss is H, we use Nickel on the fourth toss, and the probability of H on the fourth toss is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$
    - If the third toss is T, we use Dime on the fourth toss, and the probability of H on the fourth toss is $\frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{18}$

    Thus, the probability of H on the fourth toss is $\frac{1}{24} + \frac{1}{18} = \frac{7}{72}$

  Therefore, if the first toss is T, then the probability of H on the fourth toss according to our strategy is $\frac{1}{6} + \frac{7}{72} = \frac{19}{72}$.

In conclusion, based on our strategy and calculation, the highest probability of H on the fourth toss is $\frac{25}{72} + \frac{19}{72} = \frac{11}{18}$

3 Problem 3

Let H_1, N_1, H_2, and N_2 be the number of heads and total tosses of Nickel and Dime respectively. We use Nickel for the first toss.
After some $n = N_1 + N_2$ tosses, then for the $n+1^{th}$ toss, by **Problem 1**, $\frac{H_1+1}{N_1+2}$ is the probability of tossing Nickel on the next toss and get a head, and $\frac{H_2+1}{N_2+2}$ is the probability of tossing Dime on the next toss and getting a head. There are 3 exhaustive cases:

- If $\frac{H_1+1}{N_1+2} \geq \frac{H_2+1}{N_2+2}$, then we choose Nickel for the next toss
- If $\frac{H_1+1}{N_1+2} < \frac{H_2+1}{N_2+2}$, then we choose Dime for the next toss.

Here are 2 python programs that simulate the above process for $n = N_1 + N_2$.

- this program calculates the lower bound of π_n

```
def ratio(h, n):
    return (h + 1.0)/(n + 2.0)
def prob(h1, n1, h2, n2, limit):
    t1 = ratio(h1, n1)
    t2 = ratio(h2, n2)
    if t1 >= t2:
        if n1 + n2 == limit: return t1
        return t1 * prob(h1 + 1, n1 + 1, h2, n2, i) + (1 - t1) * prob(h1, n1 + 1, h2, n2, i)
    else:
        if n1 + n2 == limit: return t2
        return t2 * prob(h1, n1, h2 + 1, n2 + 1, i) + (1 - t2) * prob(h1, n1, h2, n2 + 1, i)
for i in range(1, 100000, 1):
    print((prob(0, 1, 0, 0, i) + prob(1, 1, 0, 0, i)) / 2)
```

(n, π_n): (1, 0.5833333333333333), (2, 0.5833333333333334), (3, 0.6055555555555556), (4, 0.6083333333333334), (5, 0.6148809523809524), (6, 0.6194444444444445), (7, 0.6231746031746033), (8, 0.6242460317460317), (9, 0.6278529341029342), (10, 0.6294522607022608), (11, 0.6303161983519127), (12, 0.6319701000058142), (13, 0.6333577665720522), (14, 0.6340199119663406), (15, 0.6350134572736112), (16, 0.6356582051364965), (17, 0.6363328550213961), (18, 0.6371978587335914), (19, 0.6376203957961493), (20, 0.6379010465251049), (21, 0.638488874487368), (22, 0.6391022497839296), (23, 0.6392816827387082), (24, 0.6396340344909657), (25, 0.6400129781795174)...
The time complexity of the above program is in $O(2^n)$. We observed through the above values that the π_n monotonically increases as n increases and the rate of change is approaching 0.

- Since it is impossible to obtain an exact value of π_n for large n through the above method with the time and resources available, we use **Monte Carlo simulation** method to make an estimation. We take $n = 1000$ to approximate $\lim_{n\to\infty} \pi_n$. We repeat the simulation for 5 million times.

```
import random
def simulate(p, limit):
    head = [0, 0]
    n = [0,0]
    thiscoin = 0
    for i in range(limit):
        result = random.random()
        if (result < p[thiscoin]):
            head[thiscoin]+=1
        n[thiscoin] +=1
        if (head[0] + 1)/ (n[0]+2) > (head[1]+1)/(n[1]+2):
            thiscoin = 0
        else:
            thiscoin = 1
    return result < p[thiscoin]
```

```
def main():
    count = 0
    for i in range(10000):
        p = [random.random(), random.random()]
        if simulate(p, 10000):
            count += 1
    return count
print(main())
```

By running the program, out of 5 million simulations, there are 3242602 times that the last toss is a head. Thus, we conclude that $\lim_{n\to\infty} \pi_n \approx 0.649$