Parallel MR Image Reconstruction Using Augmented Lagrangian Methods

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1. P1 implementation
2. P2 implementation
3. Reference P2

1. Introduction

1. P1

$$\mathbf{P1}: \min_{\mathbf{u}_1,\mathbf{x}} J_1(\mathbf{x},\mathbf{u}_1) \text{ subject to } \mathbf{u}_1 = \mathbf{Rx}$$

where

$$J_1(\mathbf{x},\mathbf{u}_1) \triangleq \frac{1}{2}\|\mathbf{d}-\mathbf{FSx}\|^2 + \sum_{q=1}^{Q}\lambda_q \sum_{n=1}^{N_q}\Phi_{qn}\left(\sum_{p=1}^{P_q}|[\mathbf{u}_{1pq}]_n|^{m_q}\right)$$

AL-P1: AL Algorithm for solving problem P1

1. Select $\mathbf{x}^{(0)}$ and $\mu > 0$
2. Precompute $\mathbf{S}^{\mathrm{H}}\mathbf{F}^{\mathrm{H}}\mathbf{d}$; set $\boldsymbol{\eta}_1^{(0)} = \mathbf{0}$ and $j = 0$

Repeat:

3. Obtain an update $\mathbf{u}_1^{(j+1)}$ using an appropriate technique as described in **Sections IV-A2 to IV-A6**
4. Obtain an update $\mathbf{x}^{(j+1)}$ by running few CG iterations on **(17)**
5. $\boldsymbol{\eta}_1^{(j+1)} = \boldsymbol{\eta}_1^{(j)} - (\mathbf{u}_1^{(j+1)} - \mathbf{Rx}^{(j+1)})$
6. Set $j = j+1$

Until stop-criterion is met

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where $\boldsymbol{\eta}_1 = -(1/\mu)\boldsymbol{\gamma}_1$

$$\mathcal{L}_1(\mathbf{u},\boldsymbol{\eta}_1,\mu) = J_1(\mathbf{x},\mathbf{u}_1) + \frac{\mu}{2}\|\mathbf{Cu}-\boldsymbol{\eta}_1\|^2 \qquad (14)$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \mathcal{L}_1(\mathbf{u}_1,\mathbf{x}^{(j)},\boldsymbol{\eta}_1^{(j)},\mu) \qquad (15)$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \mathcal{L}_1(\mathbf{u}_1^{(j+1)},\mathbf{x},\boldsymbol{\eta}_1^{(j)},\mu). \qquad (16)$$

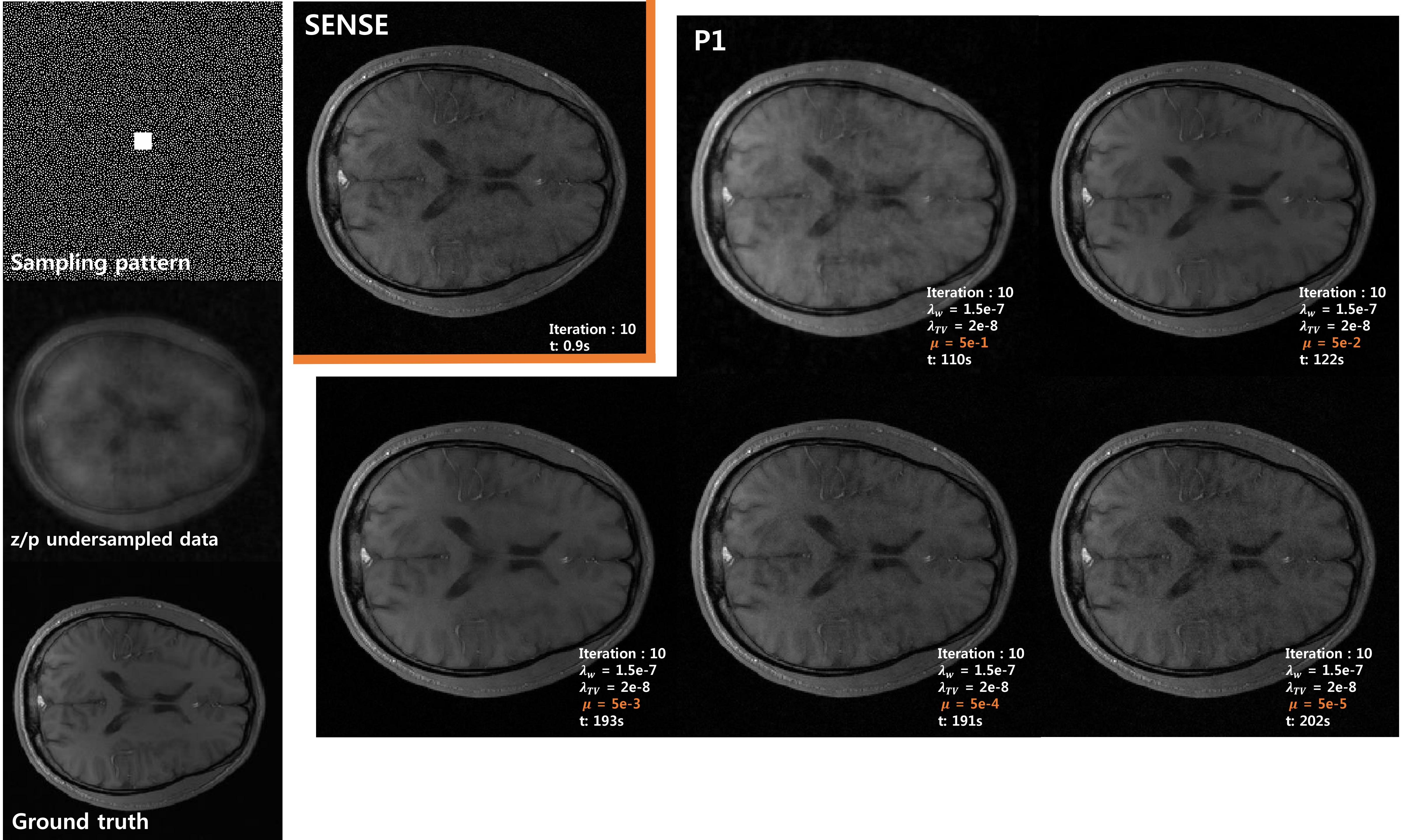
$$\begin{aligned}\mathbf{x}^{(j+1)} &= \arg\min_{\mathbf{x}} \mathcal{L}_1(\mathbf{u}_1^{(j+1)},\mathbf{x},\boldsymbol{\eta}_1^{(j)},\mu).\\ &= \arg\min_{\mathbf{x}}\left\{\frac{1}{2}\|\mathbf{d}-\mathbf{FSx}\|_2^2 + \frac{\mu}{2}\left\|\mathbf{u}_1^{(j+1)}-\mathbf{Rx}-\boldsymbol{\eta}_1^{(j)}\right\|_2^2\right\}\\ &= \mathbf{G}_\mu^{-1}[\mathbf{S}^{\mathrm{H}}\mathbf{F}^{\mathrm{H}}\mathbf{d} + \mu\mathbf{R}^{\mathrm{H}}(\mathbf{u}_1^{(j+1)}-\boldsymbol{\eta}_1^{(j)})] \qquad (17)\end{aligned}$$

where

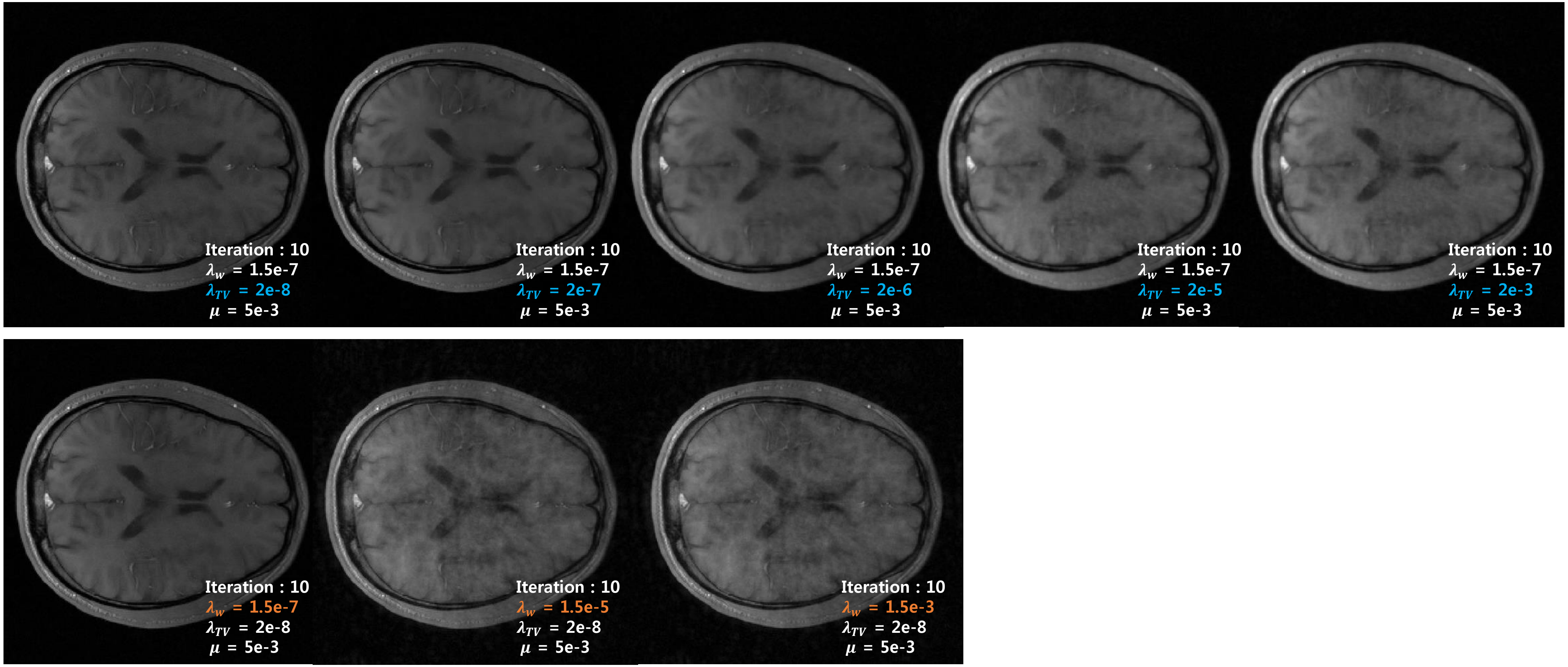
$$\mathbf{G}_\mu = \mathbf{S}^{\mathrm{H}}\mathbf{F}^{\mathrm{H}}\mathbf{FS} + \mu\mathbf{R}^{\mathrm{H}}\mathbf{R}. \quad (18)$$

$$\begin{aligned}\mathbf{u}_1^{(j+1)} &= \arg\min_{\mathbf{u}_1} \mathcal{L}_1(\mathbf{u}_1,\mathbf{x}^{(j)},\boldsymbol{\eta}_1^{(j)},\mu)\\ &= \arg\min_{\mathbf{u}_1}\left\{\begin{array}{l}\sum_{q=1}^{Q}\lambda_q\sum_{n=1}^{N_q}\Phi_{qn}\left(\sum_{p=1}^{P_q}|[\mathbf{u}_{1pq}]_n|^{m_q}\right)\\ +\frac{\mu}{2}\left\|\mathbf{u}_1-\mathbf{Rx}^{(j)}-\boldsymbol{\eta}_1^{(j)}\right\|_2^2\end{array}\right\}. \qquad (19)\end{aligned}$$

$$v_k^{(j+1)} = \mathrm{shrink}\left\{\varrho_k^{(j)} + \beta_k^{(j)}, \frac{\lambda}{\mu}\right\} \; \forall k,$$



Sampling pattern
z/p undersampled data
Ground truth
SENSE
Iteration : 10
t: 0.9s
P1
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-1
t: 110s
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-2
t: 122s
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-3
t: 193s
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-4
t: 191s
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-5
t: 202s


Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-3
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-7
μ = 5e-3
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-6
μ = 5e-3
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-5
μ = 5e-3
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-3
μ = 5e-3
Iteration : 10
λ_w = 1.5e-7
λ_{TV} = 2e-8
μ = 5e-3
Iteration : 10
λ_w = 1.5e-5
λ_{TV} = 2e-8
μ = 5e-3
Iteration : 10
λ_w = 1.5e-3
λ_{TV} = 2e-8
μ = 5e-3

2. P2

$$\mathbf{P2}: \min_{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{x}} J_2(\mathbf{u}_0, \mathbf{u}_1) \text{ subject to}$$
$$\mathbf{u}_0 = \mathbf{S}\mathbf{x}, \mathbf{u}_1 = \mathbf{R}\mathbf{u}_2 \text{ and } \mathbf{u}_2 = \mathbf{x}$$

where $\mathbf{u}_0 \in \mathbb{C}^{NL}$, $\mathbf{u}_1 \in \mathbb{C}^{R}$, $\mathbf{u}_2 \in \mathbb{C}^{N}$, and

$$J_2(\mathbf{u}_0, \mathbf{u}_1) \triangleq \frac{1}{2}\|\mathbf{d} - \mathbf{F}\mathbf{u}_0\|^2 + \sum_{q=1}^{Q} \lambda_q \sum_{n=1}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right)$$

AL-P2: AL Algorithm for solving problem P2

1. Select $\mathbf{x}^{(0)}$, $\mathbf{u}_2^{(0)} = \mathbf{x}^{(0)}$, $\nu_{1,2} > 0$, and $\mu > 0$
2. Precompute $\mathbf{F}^{\mathrm{H}}\mathbf{d}$; set $\boldsymbol{\eta}_{20,21,22}^{(0)} = \mathbf{0}$ and $j = 0$

Repeat:

3. Compute $\mathbf{u}_0^{(j+1)}$ from **(30)** using FFTs on **(37)**
4. Compute $\mathbf{u}_1^{(j+1)}$ using an appropriate technique as described in **Section IV-A2 to IV-A6** for problem **(27)**
5. Compute $\mathbf{u}_2^{(j+1)}$ using **(31)**
6. Compute $\mathbf{x}^{(j+1)}$ using **(32)**
7. $\boldsymbol{\eta}_{20}^{(j+1)} = \boldsymbol{\eta}_{20}^{(j)} - (\mathbf{u}_0^{(j+1)} - \mathbf{S}\mathbf{x}^{(j+1)})$
8. $\boldsymbol{\eta}_{21}^{(j+1)} = \boldsymbol{\eta}_{21}^{(j)} - (\mathbf{u}_1^{(j+1)} - \mathbf{R}\mathbf{u}_2^{(j+1)})$
9. $\boldsymbol{\eta}_{22}^{(j+1)} = \boldsymbol{\eta}_{22}^{(j)} - (\mathbf{u}_2^{(j+1)} - \mathbf{x}^{(j+1)})$
10. Set $j = j + 1$

Until stop-criterion is met

$$\mathcal{L}_2(\mathbf{u}, \boldsymbol{\eta}_2, \mu) = J_2(\mathbf{u}_0, \mathbf{u}_1) + \frac{\mu}{2}\|\mathbf{B}\mathbf{u} - \boldsymbol{\eta}_2\|_{\boldsymbol{\Lambda}^2}^2$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{x} \end{bmatrix}, \; f(\mathbf{u}) = J_2(\mathbf{u}_0, \mathbf{u}_1), \; \mathbf{C} = \boldsymbol{\Lambda}\mathbf{B}, \; \mathrm{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where

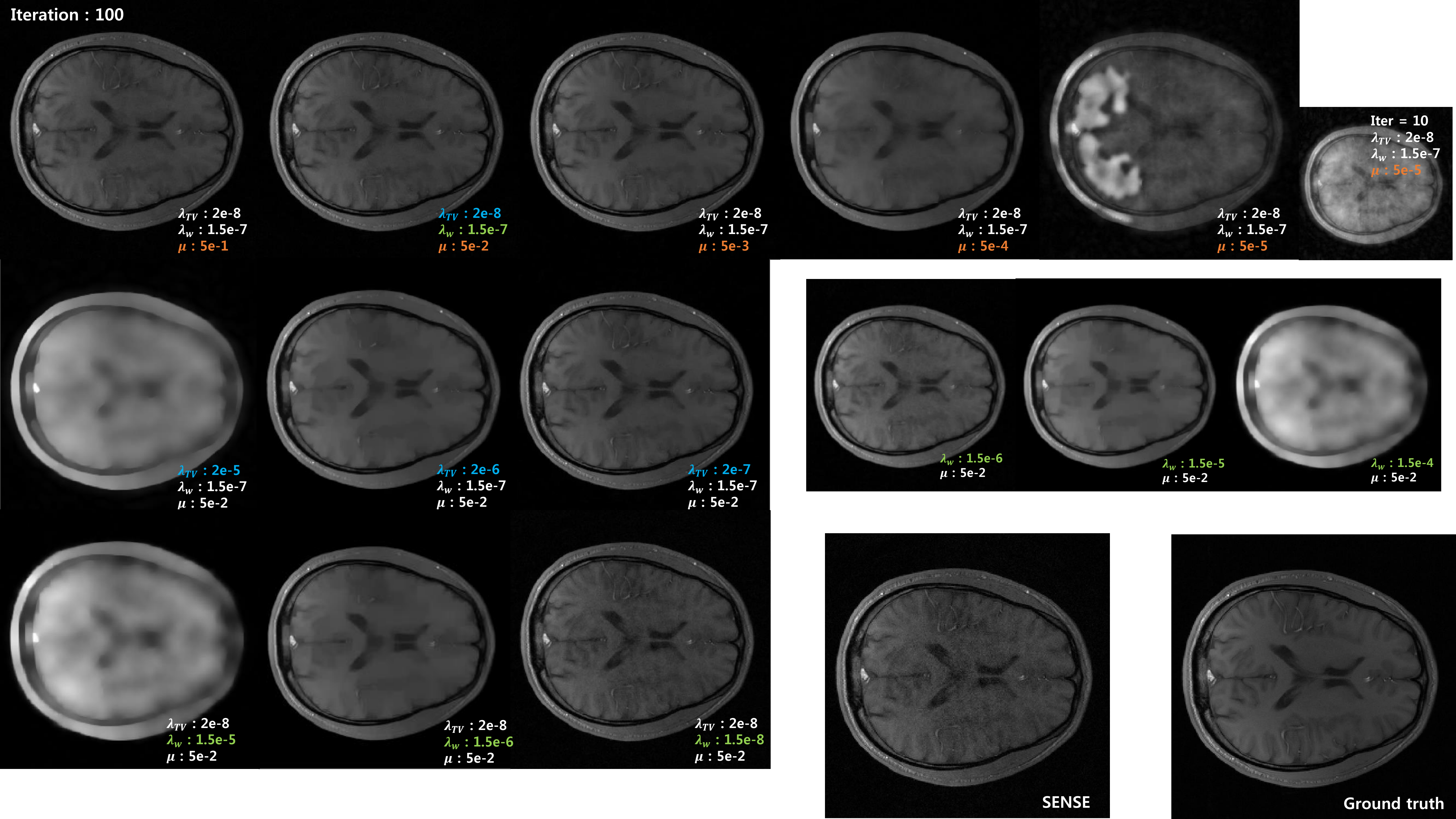
$$\boldsymbol{\Lambda} = \begin{bmatrix} \mathbf{I}_{NL} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{\nu_1}\mathbf{I}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sqrt{\nu_2}\mathbf{I}_N \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{NL} & \mathbf{0} & \mathbf{0} & -\mathbf{S} \\ \mathbf{0} & \mathbf{I}_R & -\mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_N & -\mathbf{I}_N \end{bmatrix}.$$

$$\mathbf{u}_0^{(j+1)} = \arg\min_{\mathbf{u}_0} \left\{ \begin{array}{l} \frac{1}{2}\|\mathbf{d} - \mathbf{F}\mathbf{u}_0\|_2^2 \\ + \frac{\mu}{2}\|\mathbf{u}_0 - \mathbf{S}\mathbf{x}^{(j)} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \end{array} \right\} \tag{26}$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \left\{ \begin{array}{l} \sum_{q=1}^{Q} \lambda_q \sum_{n=q}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right) \\ + \frac{\mu\nu_1}{2}\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2^{(j)} - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \end{array} \right\}$$

$$\mathbf{u}_2^{(j+1)} = \arg\min_{\mathbf{u}_2} \left\{ \begin{array}{l} \frac{\mu\nu_1}{2}\|\mathbf{u}_1^{(j+1)} - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \\ + \frac{\mu\nu_2}{2}\|\mathbf{u}_2 - \mathbf{x}^{(j)} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\} \tag{28}$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \left\{ \begin{array}{l} \frac{\mu}{2}\|\mathbf{u}_0^{(j+1)} - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \\ + \frac{\mu\nu_2}{2}\|\mathbf{u}_2^{(j+1)} - \mathbf{x} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\}. \tag{29}$$


Iteration : 100
λ_{TV} : 2e-8
λ_w : 1.5e-7
μ : 5e-1
λ_{TV} : 2e-8
λ_w : 1.5e-7
μ : 5e-2
λ_{TV} : 2e-8
λ_w : 1.5e-7
μ : 5e-3
λ_{TV} : 2e-8
λ_w : 1.5e-7
μ : 5e-4
λ_{TV} : 2e-8
λ_w : 1.5e-7
μ : 5e-5
Iter = 10
λ_{TV} : 2e-8
λ_w : 1.5e-7
μ : 5e-5
λ_{TV} : 2e-5
λ_w : 1.5e-7
μ : 5e-2
λ_{TV} : 2e-6
λ_w : 1.5e-7
μ : 5e-2
λ_{TV} : 2e-7
λ_w : 1.5e-7
μ : 5e-2
λ_w : 1.5e-6
μ : 5e-2
λ_w : 1.5e-5
μ : 5e-2
λ_w : 1.5e-4
μ : 5e-2
λ_{TV} : 2e-8
λ_w : 1.5e-5
μ : 5e-2
λ_{TV} : 2e-8
λ_w : 1.5e-6
μ : 5e-2
λ_{TV} : 2e-8
λ_w : 1.5e-8
μ : 5e-2
SENSE
Ground truth

3. Reference P2

$$\mathcal{L}_2(\mathbf{u},\boldsymbol{\eta}_2,\mu) = J_2(\mathbf{u}_0,\mathbf{u}_1) + \frac{\mu}{2}\|\mathbf{Bu} - \boldsymbol{\eta}_2\|^2_{\boldsymbol{\Lambda}^2}$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{x} \end{bmatrix},\ f(\mathbf{u}) = J_2(\mathbf{u}_0,\mathbf{u}_1),\ \mathbf{C} = \boldsymbol{\Lambda}\mathbf{B},\ \mathrm{b} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \mathbf{I}_{NL} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{\nu_1}\mathbf{I}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sqrt{\nu_2}\mathbf{I}_N \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{NL} & \mathbf{0} & \mathbf{0} & -\mathbf{S} \\ \mathbf{0} & \mathbf{I}_R & -\mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_N & -\mathbf{I}_N \end{bmatrix}.$$

$$\mathbf{u}_0^{(j+1)} = \arg\min_{\mathbf{u}_0} \left\{ \begin{array}{l} \frac{1}{2}\|\mathbf{d} - \mathbf{Fu}_0\|_2^2 \\ +\frac{\mu}{2}\|\mathbf{u}_0 - \mathbf{Sx}^{(j)} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \end{array} \right\} \tag{26}$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \left\{ \begin{array}{l} \sum_{q=1}^{Q} \lambda_q \sum_{n=q}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right) \\ +\frac{\mu\nu_1}{2}\|\mathbf{u}_1 - \mathbf{Ru}_2^{(j)} - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \end{array} \right\}$$

$$\mathbf{u}_2^{(j+1)} = \arg\min_{\mathbf{u}_2} \left\{ \begin{array}{l} \frac{\mu\nu_1}{2}\|\mathbf{u}_1^{(j+1)} - \mathbf{Ru}_2 - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2 - \mathbf{x}^{(j)} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\} \tag{28}$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \left\{ \begin{array}{l} \frac{\mu}{2}\|\mathbf{u}_0^{(j+1)} - \mathbf{Sx} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2^{(j+1)} - \mathbf{x} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\}. \tag{29}$$

$$\mathbf{u}_0^{(j+1)} = \mathbf{H}_\mu^{-1}\left[\mathbf{F}^{\mathrm{H}}\mathbf{d} + \mu(\mathbf{Sx}^{(j)} + \boldsymbol{\eta}_{20}^{(j)})\right] \tag{30}$$

$$\mathbf{u}_2^{(j+1)} = \mathbf{H}_{\nu_1\nu_2}^{-1}\left[\mathbf{R}^{\mathrm{H}}(\mathbf{u}_1^{(j+1)} - \boldsymbol{\eta}_{21}^{(j)}) + \frac{\nu_2}{\nu_1}(\mathbf{x}^{(j)} + \boldsymbol{\eta}_{22}^{(j)})\right] \tag{31}$$

$$\mathbf{x}^{(j+1)} = \mathbf{H}_{\nu_2}^{-1}\left[\mathbf{S}^{\mathrm{H}}(\mathbf{u}_0^{(j+1)} - \boldsymbol{\eta}_{20}^{(j)}) + \nu_2(\mathbf{u}_2^{(j+1)} - \boldsymbol{\eta}_{22}^{(j)})\right] \tag{32}$$

where

$$\mathbf{H}_\mu = \mathbf{F}^{\mathrm{H}}\mathbf{F} + \mu\mathbf{I}_{NL} \tag{33}$$

$$\mathbf{H}_{\nu_1\nu_2} = \mathbf{R}^{\mathrm{H}}\mathbf{R} + \frac{\nu_2}{\nu_1}\mathbf{I}_N \tag{34}$$

$$\mathbf{H}_{\nu_2} = \mathbf{S}^{\mathrm{H}}\mathbf{S} + \nu_2\mathbf{I}_N. \tag{35}$$

3. Reference P2

$$\mathbf{u}_0^{(j+1)} = \arg\min_{\mathbf{u}_0} \left\{ \begin{array}{l} \frac{1}{2}\|\mathbf{d} - \mathbf{F}\mathbf{u}_0\|_2^2 \\ +\frac{\mu}{2}\|\mathbf{u}_0 - \mathbf{S}\mathbf{x}^{(j)} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \end{array} \right\} \quad (26)$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \left\{ \begin{array}{l} \sum_{q=1}^{Q} \lambda_q \sum_{n=q}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right) \\ +\frac{\mu\nu_1}{2}\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2^{(j)} - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \end{array} \right\}$$

$$\mathbf{u}_2^{(j+1)} = \arg\min_{\mathbf{u}_2} \left\{ \begin{array}{l} \frac{\mu\nu_1}{2}\|\mathbf{u}_1^{(j+1)} - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2 - \mathbf{x}^{(j)} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\} \quad (28)$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \left\{ \begin{array}{l} \frac{\mu}{2}\|\mathbf{u}_0^{(j+1)} - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2^{(j+1)} - \mathbf{x} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\}. \quad (29)$$

$$\mathbf{u}_0^{(j+1)} = \mathbf{H}_\mu^{-1}\left[\mathbf{F}^{\mathrm{H}}\mathbf{d} + \mu(\mathbf{S}\mathbf{x}^{(j)} + \boldsymbol{\eta}_{20}^{(j)})\right] \quad (30)$$

$$\mathbf{u}_2^{(j+1)} = \mathbf{H}_{\nu_1\nu_2}^{-1}\left[\mathbf{R}^{\mathrm{H}}(\mathbf{u}_1^{(j+1)} - \boldsymbol{\eta}_{21}^{(j)}) + \frac{\nu_2}{\nu_1}(\mathbf{x}^{(j)} + \boldsymbol{\eta}_{22}^{(j)})\right] \quad (31)$$

$$\mathbf{x}^{(j+1)} = \mathbf{H}_{\nu_2}^{-1}\left[\mathbf{S}^{\mathrm{H}}(\mathbf{u}_0^{(j+1)} - \boldsymbol{\eta}_{20}^{(j)}) + \nu_2(\mathbf{u}_2^{(j+1)} - \boldsymbol{\eta}_{22}^{(j)})\right] \quad (32)$$

where

$$\mathbf{H}_\mu = \mathbf{F}^{\mathrm{H}}\mathbf{F} + \mu\mathbf{I}_{NL} \quad (33)$$

$$\mathbf{H}_{\nu_1\nu_2} = \mathbf{R}^{\mathrm{H}}\mathbf{R} + \frac{\nu_2}{\nu_1}\mathbf{I}_N \quad (34)$$

$$\mathbf{H}_{\nu_2} = \mathbf{S}^{\mathrm{H}}\mathbf{S} + \nu_2\mathbf{I}_N. \quad (35)$$

$$(F^H F + \mu I)u_0 = [F^H d + \mu(Sx + \eta_{20})]$$

$$\text{->}\ (F_u^H F_u + \mu I)u_0 = [F_u^H d + \mu(Sx + \eta_{20})]$$

$$(F^H M^H M F + \mu I)u_0 = [F^H M^H d + \mu(Sx + \eta_{20})]$$

$$= b$$

$$F * (F^H M^H M F + \mu I)u_0 = F * b$$

$$(MF + \mu F)u_0 = Fb$$

$$(M + \mu I)Fu_0 = Fb$$

$$Fu_0 = (M + \mu I)^{-1} Fb$$

$$\boldsymbol{u_0} = \boldsymbol{F^H}[(\boldsymbol{M} + \boldsymbol{\mu I})^{-1}\boldsymbol{Fb}]$$

$$M = \begin{bmatrix} 1 & & & & & \\ & 0 & & \cdots & 0 & \\ & & 1 & & & \\ & \vdots & & \ddots & \vdots & \\ & 0 & & \cdots & 0 & \\ & & & & & 1 \end{bmatrix}$$

3. Reference P2

$$\mathbf{u}_0^{(j+1)} = \arg\min_{\mathbf{u}_0} \left\{ \begin{array}{l} \frac{1}{2}\|\mathbf{d} - \mathbf{F}\mathbf{u}_0\|_2^2 \\ +\frac{\mu}{2}\|\mathbf{u}_0 - \mathbf{S}\mathbf{x}^{(j)} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \end{array} \right\} \tag{26}$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \left\{ \begin{array}{l} \sum_{q=1}^{Q} \lambda_q \sum_{n=q}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right) \\ +\frac{\mu\nu_1}{2}\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2^{(j)} - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \end{array} \right\}$$

$$\mathbf{u}_2^{(j+1)} = \arg\min_{\mathbf{u}_2} \left\{ \begin{array}{l} \frac{\mu\nu_1}{2}\|\mathbf{u}_1^{(j+1)} - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2 - \mathbf{x}^{(j)} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\} \tag{28}$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \left\{ \begin{array}{l} \frac{\mu}{2}\|\mathbf{u}_0^{(j+1)} - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2^{(j+1)} - \mathbf{x} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\}. \tag{29}$$

$$\mathbf{u}_0^{(j+1)} = \mathbf{H}_\mu^{-1}\left[\mathbf{F}^{\mathrm{H}}\mathbf{d} + \mu(\mathbf{S}\mathbf{x}^{(j)} + \boldsymbol{\eta}_{20}^{(j)})\right] \tag{30}$$

$$\mathbf{u}_2^{(j+1)} = \mathbf{H}_{\nu_1\nu_2}^{-1}\left[\mathbf{R}^{\mathrm{H}}(\mathbf{u}_1^{(j+1)} - \boldsymbol{\eta}_{21}^{(j)}) + \frac{\nu_2}{\nu_1}(\mathbf{x}^{(j)} + \boldsymbol{\eta}_{22}^{(j)})\right] \tag{31}$$

$$\mathbf{x}^{(j+1)} = \mathbf{H}_{\nu_2}^{-1}\left[\mathbf{S}^{\mathrm{H}}(\mathbf{u}_0^{(j+1)} - \boldsymbol{\eta}_{20}^{(j)}) + \nu_2(\mathbf{u}_2^{(j+1)} - \boldsymbol{\eta}_{22}^{(j)})\right] \tag{32}$$

where

$$\mathbf{H}_\mu = \mathbf{F}^{\mathrm{H}}\mathbf{F} + \mu\mathbf{I}_{NL} \tag{33}$$

$$\mathbf{H}_{\nu_1\nu_2} = \mathbf{R}^{\mathrm{H}}\mathbf{R} + \frac{\nu_2}{\nu_1}\mathbf{I}_N \tag{34}$$

$$\mathbf{H}_{\nu_2} = \mathbf{S}^{\mathrm{H}}\mathbf{S} + \nu_2\mathbf{I}_N. \tag{35}$$

$$(F^H F + \mu I)u_0 = [F^H d + \mu(Sx + \eta_{20})]$$

-> $(F_u^H F_u + \mu I)u_0 = [F_u^H d + \mu(Sx + \eta_{20})]$

$$(F^H M^H M F + \mu I)u_0 = [F^H M^H d + \mu(Sx + \eta_{20})]$$

$$= b$$

$$F * (F^H M^H M F + \mu I)u_0 = F * b$$

LTI

$$F(F^H M^H M F + \mu I)\ X\ F(u_0) = F(b)$$

$$F(u_0) = \frac{1}{F(F^H M^H M F + \mu I)} F(b)$$

$$\boldsymbol{u_0 = F^H\left[\frac{1}{F(F^H M^H M F + \mu I)} F(b)\right]}$$

3. Reference P2

$$\mathbf{u}_0^{(j+1)} = \arg\min_{\mathbf{u}_0} \left\{ \begin{array}{l} \frac{1}{2}\|\mathbf{d} - \mathbf{F}\mathbf{u}_0\|_2^2 \\ + \frac{\mu}{2}\|\mathbf{u}_0 - \mathbf{S}\mathbf{x}^{(j)} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \end{array} \right\} \quad (26)$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \left\{ \begin{array}{l} \sum_{q=1}^{Q} \lambda_q \sum_{n=q}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right) \\ + \frac{\mu\nu_1}{2}\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2^{(j)} - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \end{array} \right\}$$

$$\mathbf{u}_2^{(j+1)} = \arg\min_{\mathbf{u}_2} \left\{ \begin{array}{l} \frac{\mu\nu_1}{2}\|\mathbf{u}_1^{(j+1)} - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \\ + \frac{\mu\nu_2}{2}\|\mathbf{u}_2 - \mathbf{x}^{(j)} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\} \quad (28)$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \left\{ \begin{array}{l} \frac{\mu}{2}\|\mathbf{u}_0^{(j+1)} - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \\ + \frac{\mu\nu_2}{2}\|\mathbf{u}_2^{(j+1)} - \mathbf{x} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\}. \quad (29)$$

$$\mathbf{u}_0^{(j+1)} = \mathbf{H}_\mu^{-1}\left[\mathbf{F}^{\mathrm{H}}\mathbf{d} + \mu(\mathbf{S}\mathbf{x}^{(j)} + \boldsymbol{\eta}_{20}^{(j)})\right] \quad (30)$$

$$\mathbf{u}_2^{(j+1)} = \mathbf{H}_{\nu_1\nu_2}^{-1}\left[\mathbf{R}^{\mathrm{H}}(\mathbf{u}_1^{(j+1)} - \boldsymbol{\eta}_{21}^{(j)}) + \frac{\nu_2}{\nu_1}(\mathbf{x}^{(j)} + \boldsymbol{\eta}_{22}^{(j)})\right] \quad (31)$$

$$\mathbf{x}^{(j+1)} = \mathbf{H}_{\nu_2}^{-1}\left[\mathbf{S}^{\mathrm{H}}(\mathbf{u}_0^{(j+1)} - \boldsymbol{\eta}_{20}^{(j)}) + \nu_2(\mathbf{u}_2^{(j+1)} - \boldsymbol{\eta}_{22}^{(j)})\right] \quad (32)$$

where

$$\mathbf{H}_\mu = \mathbf{F}^{\mathrm{H}}\mathbf{F} + \mu\mathbf{I}_{NL} \quad (33)$$

$$\mathbf{H}_{\nu_1\nu_2} = \mathbf{R}^{\mathrm{H}}\mathbf{R} + \frac{\nu_2}{\nu_1}\mathbf{I}_N \quad (34)$$

$$\mathbf{H}_{\nu_2} = \mathbf{S}^{\mathrm{H}}\mathbf{S} + \nu_2\mathbf{I}_N. \quad (35)$$

CG OK

$$(R^H R + \rho I)u_2 = [R^H(u_1 - \eta_{21}) + \rho(x + \eta_{22})]$$

$$->(R^H R + \rho I)u_2 = b$$

$$F * (R^H R + \rho I)u_2 = F * b$$

$\Downarrow$ **LTI**

RhR : Toeplitz matrix
(Translation invariant system)

$$F(R^H R + \rho I)\ X\ F(u_2) = F(b)$$

$$\boldsymbol{u_2} = \boldsymbol{F^H}[\frac{\mathbf{1}}{\boldsymbol{R^H R + \rho}}\boldsymbol{Fb}]$$

3. Reference P2

$$\mathbf{u}_0^{(j+1)} = \arg\min_{\mathbf{u}_0} \left\{ \begin{array}{l} \frac{1}{2}\|\mathbf{d} - \mathbf{F}\mathbf{u}_0\|_2^2 \\ +\frac{\mu}{2}\|\mathbf{u}_0 - \mathbf{S}\mathbf{x}^{(j)} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \end{array} \right\} \quad (26)$$

$$\mathbf{u}_1^{(j+1)} = \arg\min_{\mathbf{u}_1} \left\{ \begin{array}{l} \sum_{q=1}^{Q} \lambda_q \sum_{n=q}^{N_q} \Phi_{qn} \left(\sum_{p=1}^{P_q} |[\mathbf{u}_{1pq}]_n|^{m_q} \right) \\ +\frac{\mu\nu_1}{2}\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2^{(j)} - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \end{array} \right\}$$

$$\mathbf{u}_2^{(j+1)} = \arg\min_{\mathbf{u}_2} \left\{ \begin{array}{l} \frac{\mu\nu_1}{2}\|\mathbf{u}_1^{(j+1)} - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_{21}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2 - \mathbf{x}^{(j)} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\} \quad (28)$$

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \left\{ \begin{array}{l} \frac{\mu}{2}\|\mathbf{u}_0^{(j+1)} - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_{20}^{(j)}\|_2^2 \\ +\frac{\mu\nu_2}{2}\|\mathbf{u}_2^{(j+1)} - \mathbf{x} - \boldsymbol{\eta}_{22}^{(j)}\|_2^2 \end{array} \right\}. \quad (29)$$

$$\mathbf{u}_0^{(j+1)} = \mathbf{H}_\mu^{-1} \left[\mathbf{F}^{\mathrm{H}}\mathbf{d} + \mu(\mathbf{S}\mathbf{x}^{(j)} + \boldsymbol{\eta}_{20}^{(j)})\right] \quad (30)$$

$$\mathbf{u}_2^{(j+1)} = \mathbf{H}_{\nu_1\nu_2}^{-1} \left[\mathbf{R}^{\mathrm{H}}(\mathbf{u}_1^{(j+1)} - \boldsymbol{\eta}_{21}^{(j)}) + \frac{\nu_2}{\nu_1}(\mathbf{x}^{(j)} + \boldsymbol{\eta}_{22}^{(j)})\right] \quad (31)$$

$$\mathbf{x}^{(j+1)} = \mathbf{H}_{\nu_2}^{-1} \left[\mathbf{S}^{\mathrm{H}}(\mathbf{u}_0^{(j+1)} - \boldsymbol{\eta}_{20}^{(j)}) + \nu_2(\mathbf{u}_2^{(j+1)} - \boldsymbol{\eta}_{22}^{(j)})\right] \quad (32)$$

where

$$\mathbf{H}_\mu = \mathbf{F}^{\mathrm{H}}\mathbf{F} + \mu\mathbf{I}_{NL} \quad (33)$$

$$\mathbf{H}_{\nu_1\nu_2} = \mathbf{R}^{\mathrm{H}}\mathbf{R} + \frac{\nu_2}{\nu_1}\mathbf{I}_N \quad (34)$$

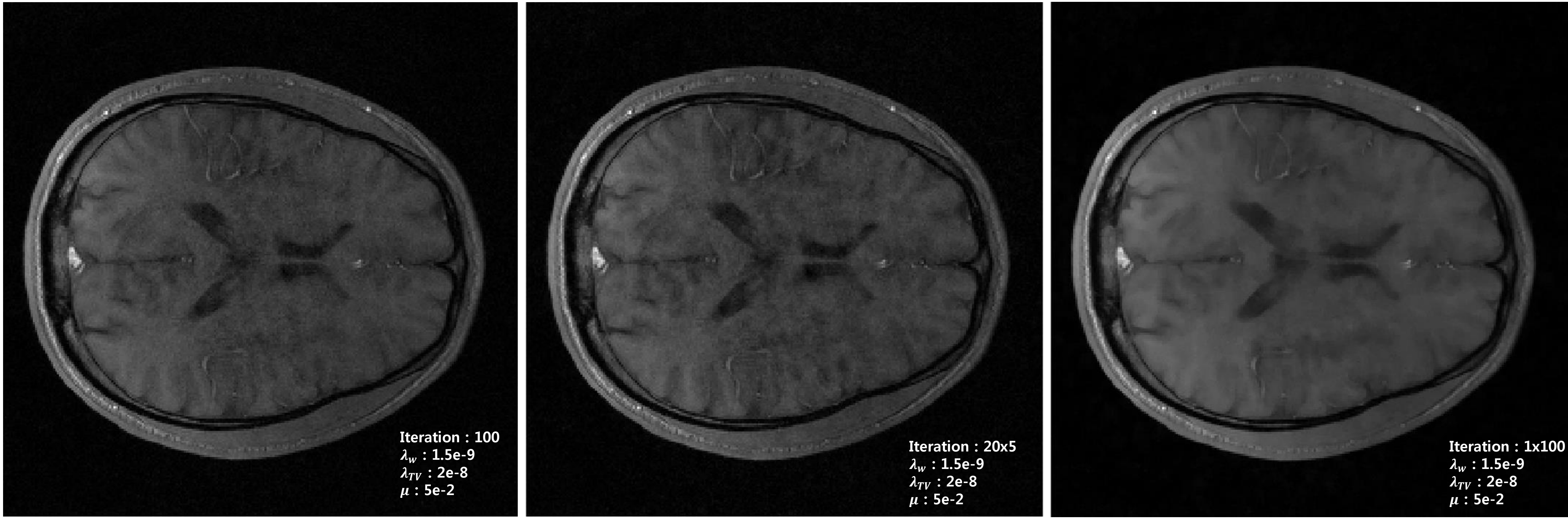
$$\mathbf{H}_{\nu_2} = \mathbf{S}^{\mathrm{H}}\mathbf{S} + \nu_2\mathbf{I}_N. \quad (35)$$

CG OK

$$(S^H S + \nu_2 I)x = [S^H(u_0 - \eta_{20}) + \nu_2(u_2 - \eta_{22})]$$

$$\rightarrow \quad (S^H S + \nu_2 I)x = b$$

$$\boldsymbol{x} = \left[\frac{\mathbf{1}}{\boldsymbol{S^H S + \nu_2}}\boldsymbol{b}\right]$$


Iteration : 100
λ_w : 1.5e-9
λ_{TV} : 2e-8
μ : 5e-2
Iteration : 20x5
λ_w : 1.5e-9
λ_{TV} : 2e-8
μ : 5e-2
Iteration : 1x100
λ_w : 1.5e-9
λ_{TV} : 2e-8
μ : 5e-2