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① a) $c_0 f(0) + c_1 f(h) + c_2 f(2h)$

$$= c_0 f(0)$$
$$+ c_1\left(f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \frac{h^3}{3!} f'''(0) + \cdots\right)$$
$$+ c_2\left(f(0) + 2hf'(0) + \frac{(2h)^2}{2} f''(0) + \frac{(2h)^3}{3!} f'''(0) + \cdots\right).$$

So, we need $c_0 + c_1 + c_2 = 0$.

$$hc_1 + 2hc_2 = 1.$$
$$\frac{h^2}{2}c_1 + \frac{4h^2}{2}c_2 = 0.$$

$\therefore$ $c_0 + c_1 + c_2 = 0$

$$c_1 + 2c_2 = 1/h.$$
$$\frac{c_1}{2} + 2c_2 = 0.$$

Thus, $\frac{c_1}{2} = \frac{1}{h}$ $\therefore c_1 = \frac{2}{h}$.

$$\therefore 2c_2 = \frac{1}{h} - \frac{2}{h} = -\frac{1}{h}$$
$$c_2 = -\frac{1}{2h}.$$

$$\therefore c_0 = -c_1 - c_2 = -\frac{2}{h} + \frac{1}{2h} = -\frac{4}{2h} + \frac{1}{2h} = \frac{-3}{2h}.$$

So, $c_0 = -\frac{3}{2h}$

$c_1 = \frac{2}{h}$

$c_2 = -\frac{1}{2h}$.

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b). We notice that with the values of c_0, c_1, c_2 prescribed in (a), we see

$c_0 f(0) + c_1 f(h) + c_2 f(2h)$

$$= f'(0) + \frac{2}{h} \cdot \frac{h^3}{3!} f'''(0) - \frac{1}{2h} \frac{(2h)^3}{3!} f'''(0) + O(h^3)$$

$$= f'(0) + \frac{h^2}{3} f'''(0) - \frac{4h^2}{6} f'''(0) + O(h^3)$$

$$= f'(0) + \left(\frac{h^2}{3} - \frac{2h^2}{3}\right) f'''(0) + O(h^3)$$

$$= f'(0) \underbrace{- \frac{h^2}{3} f'''(0)}_{\text{this is leading term of local truncation error.}} + O(h^3)$$

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② a) Note: $f(0) = -\cos(0) = -1 < 0$

$f(1) = \sqrt{\pi} - \cos(\pi) = \sqrt{\pi} + 1 > 0.$

So, by intermediate value theorem, there is $x \in (0,1)$ s.t. $f(x) = 0$.

b). Let x^* be the root of f.

After k iterations, we have: $|x_k - x^*| \le 2^{-(k+1)}$

So, we want k such that

$$2^{-(k+1)} \le 10^{-5}$$

$$\therefore -k-1 \le \log_2(10^{-5})$$

$$\therefore k \ge -\log_2(10^{-5}) - 1$$

$$= 5\log_2(10) - 1$$

$3 = \log_2(8) \le \log_2(10) \le 4$

So, we want $k \ge 5 \cdot 4 - 1 = 19.$

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(3) a) Given $I - I_T(h) = -\frac{h^2}{12}\left(f'(b) - f'(a)\right) + O(h^4)$

$$I - I_m(h) = \frac{h^2}{24}\left(f'(b) - f'(a)\right) + O(h^4)$$

we note

$$\left(I - I_T(h)\right) + 2\left(I - I_m(h)\right)$$

$$= -\frac{h^2}{12} + 2\frac{h^2}{24} + O(h^4) = O(h^4).$$

b). From the above, we have:

$$3I - I_T(h) - 2I_m(h) = O(h^4).$$

So, $I - \frac{1}{3}I_T(h) - \frac{2}{3}I_m(h) = O(h^4)$.

ie. $I = \frac{1}{3}I_T(h) + \frac{2}{3}I_m(h) + O(h^4)$

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(4). Let $q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$.

$$q(0) = f(0) \Rightarrow a_0 = f(0)$$

$$q'(0) = f'(0) \Rightarrow a_1 = f'(0).$$

$$q''(0) = f''(0) \Rightarrow 2a_2 = f''(0) \Rightarrow a_2 = \frac{f''(0)}{2}.$$

$$q'''(0) = f'''(0) \Rightarrow 6a_3 = f'''(0) \Rightarrow a_3 = \frac{f'''(0)}{6}.$$

$\Rightarrow$

$$q(x_1) = a_0 + a_1 x_1 + a_2 x^2 + a_3 x^3 + a_4 x^4 = f(x_1).$$

Since a_0, a_1, a_2, a_3 are known, we can solve for a_4.

Thus, we have the exisence of a polynomial satisfying the desired condition.

To see uniqueness, suppose q_1, q_2 both $\underbrace{\text{satisfying}}_{\text{deg 4 polynomial}}$ the above.

Let $q = q_1 - q_2$. Then:

$$q(0) = 0$$
$$q'(0) = 0$$
$$q''(0) = 0$$
$$q'''(0) = 0$$
and $q(x_1) = 0$.

This tells us that 0 is a root of multp. 4 and x_1 a root of mult. 1. So, $x^4(x - x_1)$ divides q, a degree 4 polynomial. Thus, $q \equiv 0$, so $q_1 = q_2$.

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a) We note that A is symmetric, meaning it has real eigenvalues, and that $\|A\|_{\ell^2}$ is the biggest magnitude of the eigenvalues (i.e. $\|A\|_{\ell^2} = \sup_{\lambda \text{ eigenvalue}} |\lambda|$).

Since we have $\|A(u-v)\| \le \|A\|_{\ell^2} \|u-v\|$, and this is achieved when $u-v$ is an eigenvector of the biggest eigenvalue, the Lipshitz constant is $\|A\|_{\ell^2}$. So,

Lipshitz constant is $\|A\|_{\ell^2} = \sup_{1,\dots,N} \left|\frac{4}{h^2}\sin^2\left(\frac{k\pi}{N+1}\right)\right|$

$$= \frac{4}{h^2}\sup_{1,\dots,N}\left|\sin^2\left(\frac{k\pi}{N+1}\right)\right|.$$

$$= \frac{4}{h^2}\left|\sin^2\left(\frac{\left\lfloor\frac{N+1}{2}\right\rfloor \pi}{N+1}\right)\right| \le \frac{4}{h^2}.$$

b). Euler's method is given by: $y_{n+1} = y_n + \Delta t A y_n$

Let $\tilde{y}_n$ be the true solution at time n. Then we have, by Taylor Exp. $\tilde{y}_{n+1} = \tilde{y}_n + \Delta t \tilde{y}'_n + \frac{(\Delta t)^2}{2} y''_{n+1} + O(\Delta t)^3$.

And as $\tilde{y}'_n = A\tilde{y}_n$, we have

$$\tilde{y}_{n+1} = \tilde{y}_n + \Delta t A \tilde{y}_n + \frac{(\Delta t)^2}{2} A^2 \tilde{y}_n + O(\Delta t)^3.$$

To get Local truncation error we note:

$$\tilde{y}_{n+1} = \tilde{y}_n + \Delta t A \tilde{y}_n + \tau_{n+1}$$

$$\therefore\ -\tau_{n+1} = \frac{(\Delta t)^2}{2} A^2 y_n + O((\Delta t)^3).$$

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So, we see that the local truncation error is order 2.

Now, we find global error. Let $e_n = \tilde{y}_n - y_n$

We have $\tilde{y}_{n+1} = \tilde{y}_n + \Delta t A \tilde{y}_n + \tau_n$

$$y_{n+1} = y_n + \Delta t A y_n$$

$$\therefore \quad e_{n+1} = e_n + \Delta t A e_{n} + \tau_n.$$

$$\therefore \quad \|e_{n+1}\| \leq \|e_n\| + \|\Delta t A e_n\| + \|\tau_n\|$$

$$= (1 + \Delta t \|A\|_{\ell_2}) \|e_n\| + \tau_n.$$

$$= \quad \vdots$$

$$= (1+\Delta t \|A\|_{\ell_2})^{n+1} \|e_0\| + \sum_{i=1}^{n} (1+\Delta t \|A\|_{\ell_2})^{n-i} \tau_i$$

$$\leq (1+\Delta t \|A\|_{\ell_2})^{N} \|e_0\|$$

$$+ \max\|\tau_i\| \sum_{i=1}^{N} (1+\Delta t \|A\|_{\ell_2}).$$

$$\leq \left(e^{\Delta t \|A\|_{\ell_2}}\right)^{N} \|e_0$$

$$+ C \left| \frac{1-(1+\Delta t \|A\|_{\ell_2})^{N}}{1-(1+\Delta t \|A\|_{\ell_2})} \right|$$

$$\leq e^{N\Delta t \|A\|_{\ell_2}} \|e_0\| + C(\Delta t)^2 \frac{1+e^{\Delta t \|A\|_{\ell_2} \cdot N}}{\Delta t \|A\|_{\ell_2}}$$

$$= e^{(T-T_0)\|A\|_{\ell_2}} \|e_0\| + C\left(\frac{1+e^{(T-T_0)\|A\|_{\ell_2}}}{\|A\|_{\ell_2}}\right)\Delta t$$

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So we have

$$|e_n| \leq e^{(T-T_0)\|A\|_{\ell_2}} \|e_0\| + \frac{(\Delta t)}{2} \|A\|^2 \|\tilde{y}\|_{\infty} \cdot \left(\frac{1+e^{(T-t_0)\|A\|_{\ell_2}}}{\|A\|} \right)$$

c) $h = 0.1$. So, $\|A\| = \frac{4}{(0.1^2)} = 400$

So, we want $\frac{(\Delta t)}{2} (400)^{\cancel{2}} \left(\frac{1+e^{(T-t_0)\cdot 400}}{\cancel{400}} \right) \leq 10^{-3}$

$$\therefore \Delta t \leq \frac{2 \cdot 10^{-3}}{400^{\cancel{2}} \cdot \left(\frac{e^{400\cdot(T-t_0)}}{\cancel{400}} \right)}$$

$$= \frac{1}{400 \cdot 10^3 \cdot 10^{(0.43 \cdot 400)(T-t_0)}}.$$

$$= \frac{1}{400 \cdot 10^3 \cdot 10^{172}}.$$

$$= \frac{1}{4} \cdot 10^{-177}.$$

d). No. The size of t as above is very small; There would be issues with roundoff error when computing it with floating point arithmetics.

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(6) Given $\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = c\frac{\partial^2 u}{\partial x^2}$, we have

by differentiation by x

$u_{tx} + u_x u_{xx} = c u_{xxx}$. Let $w = u_x$.

We thus have: $w_t + w w_t = c w_{xx}$.

i.e. $w_t + f(w)_x = c w_{xx}$, for $f(w) = \frac{w^2}{2}$.

To solve for w, we consider the following scheme:

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} + \frac{f(w_{i+1}^n) - f(w_{i-1})}{2\Delta x} = (c + D\Delta x)\frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\Delta x^2}$$

where D is to be determined later.

~~Now we~~

Now, we note that $f(w_{i+1}^n) - f(w_{i-1}) = f'(\xi_i^n)(w_{i+1}^n - w_{i-1}^n)$.

Let $\mu = \frac{\Delta t}{\Delta x^2}$

So, we have: $w_i^{n+1} = \left(-\frac{f'(\xi_i^n)\mu\Delta x}{2} + c\mu + D\Delta x \cdot \mu\right) w_{i+1}^n$

$+ \left(1 - 2\mu(c + D\Delta x)\right) w_i^n$

$+ \left(\frac{f'(\xi_i^n)\mu\Delta x}{2} + c\mu + D\mu\Delta x\right) w_{i-1}^n$

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By choosing μ so that $1 > (1 - 2\mu(C + D\Delta x)) > 1/2$

and D such that $D > \sup\|f'(w)\|$ and Δx

such that everything is positive, we thus see that

we have: $\|w^{n+1}\|_\infty \le \cancel{\left(\frac{f'(\xi_i^n)\mu\Delta x}{2} + C w_i^n \mu + D\Delta x \mu\right)}$

we can pick a D because we are on periodic domain

$$\|w^n\|_\infty \left(\frac{-f'(\xi_i^n)\mu\Delta x}{2} + C\mu + D\Delta x\mu\right)$$

$$+ \|w^n\|_\infty \left(1 - 2\mu(C + D\Delta x)\right)$$

$$+ \|w^n\|_\infty \left(\frac{f'(\xi_i^n)\mu\Delta x}{2} + C\mu + D\Delta x\mu\right)$$

$$= \|w^n\|_\infty$$

Thus, we see our scheme is contracting.

From this, we see $\|w^1\| \le \|w^0\|$, so if u^0 has image in $[-m, m]$, so does u^1, which tells us that our

$$\sup\|f'(w^1)\| \le \sup\|f'(w^0)\|.$$

Inductively, we thus see $\|w^n\| \le \|w^0\|$, showing stability.

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To see that this converges, we note that if v is the true solution, then we have

$$\frac{v_i^{n+1}-v_i^n + \theta(\Delta t)}{\Delta t} + \frac{f(v_{i+1}^n)-f(v_{i-1}^n)}{2\Delta x} + \theta((\Delta x)^2)$$

$$= (c+D\Delta x)\, \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2} + \theta(\Delta x)$$

$$\left(\text{as } D\Delta x\, \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2} = D\Delta x \cdot (v_{xx} + \theta(\Delta x^2)) = \theta(\Delta x)\right)$$

As we also have

$$\frac{w_i^{n+1}-w_i^n}{\Delta t} + \frac{f(w_{i+1}^n)-f(w_{i-1}^n)}{2\Delta x} = (c+D\Delta x)\,\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2}.$$

subtracting the two equations yields the following, where $e_i^n = v_i^n - w_i^n$:

$$\frac{e_i^{n+1}-e_i^n}{\Delta t} + \frac{f(w_{i+1}^n)-f(v_{i+1}^n)-f(w_{i-1}^n)+f(v_{i-1}^n)}{2\Delta x}$$

$$= (c+D\Delta x)\, \frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{\Delta x^2} + \theta(\Delta x) + \theta(\Delta t)$$

By mean value theorem, we have:

$$\frac{e_i^{n+1}-e_i^n}{\Delta t} + \frac{f'(\xi_{i+1}^n)\, e_{i+1}^n - f'(\xi_{i-1}^n)\, e_{i-1}^n}{2\Delta x}$$

where ξ_i^n between v_i^n and w_i^n

$$= (c+D\Delta x)\, \frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{\Delta x^2} + \theta(\Delta x) + \theta(\Delta t)$$

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From here, we note that

$$e_i^{n+1} = \left(-f'(\xi_{i+1}^n)\mu\Delta x + (\mathcal{L}+\Pi\Delta x)\mu\right) e_{i+1}^n$$

$$+ \left(1 - 2\mu(\mathcal{L}+\Pi\Delta x)\right) e_i^n$$

$$+ \left(\frac{f'(\xi_i^n)\mu\Delta x}{2} \mathcal{L}\mu + \Pi\mu\Delta x\right) e_{i-1}^n + O(\Delta x\Delta t) + O(\Delta t^2).$$

As all terms are positive, we see that, as before,

$$\|e_i^{n+1}\|_\infty \le \|e_i^n\|_\infty + O(\Delta x\Delta t) + O(\Delta t^2).$$

As we have $\lambda = \frac{\Delta t}{\Delta x^2}$, we see $\Delta t = O(\Delta x^2)$.

Thus, $\|e_i^{n+1}\|_\infty \le \|e_i^n\|_\infty + C\Delta x^3$

~~So, we have~~ $\le \dots$

$$\le \|e_i^0\|_\infty + (n+1)C\Delta x^3$$

$$\le \|e^0\|_\infty + NC\Delta x^3$$

Assuming $e_0 = 0$, we have that this is

$$\le NC\Delta x^3$$

$$= NC\frac{\Delta x\Delta t}{\lambda}.$$

$$= \frac{C}{\lambda}\Delta x(T-T_0)$$

$$= O(\Delta x).$$

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We note that as $\epsilon \downarrow 0$, this scheme remains convergent, as we introduced a vanishing viscosity term $D\Delta x \left(\frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\Delta x^2} \right)$, which takes care of the f' terms, and thus, the μ and Δx need not depend on ϵ.

So, this gives us a scheme to find the solution for $w = u_x$. We can then integrate with respect to x in order to approximate u.

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7 a) To see well posedness, we look at the symbol for this equation.

$$as^2 + 2bs(i\omega) + c(i\omega)^2 = 0$$

$$\Rightarrow as^2 + (2b\omega i)s - c\omega^2 = 0.$$

We want the real part of the root for s to be bounded above.

Here, $$s = \frac{-2b\omega i \pm \sqrt{-4b^2\omega^2 + 4ac\omega^2}}{2a}$$

$$= \left(-b \pm \frac{\sqrt{4b^2\omega^2 - 4ac\omega^2}}{2a}\right)i$$

In order for this to have real part bounded above, we need s to be imaginary, as otherwise, the $\sqrt{4b^2\omega^2 - 4ac\omega^2}$ term will blow up

So, we need $4b^2\omega^2 - 4ac\omega^2 \geq 0$.

$$\therefore 4\omega^2(b^2 - ac) \geq 0.$$

So, we need ~~$b^2 - 4ac \geq 0$.~~

$$b^2 - ac \geq 0.$$

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b)/c) Let $w_1 = u_t$
$w_2 = u_x$.

We notice that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_t + \underbrace{\begin{bmatrix} \frac{2b}{a} & \frac{c}{a} \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_x = 0,$$ as we have

$a u_{tt} + 2b u_{xt} + c u_{xx} = 0$.

We note that

the char. poly of A is $\lambda^2 - \frac{2b}{a}\lambda + \frac{c}{a}$, which has, as its roots the same as $a\lambda^2 - 2b + c$. Since $b^2 - ac \geq 0$, we note that the roots of the polynomial are real,

Moreover, we can show that the matrix A is in fact diagonalizable. This gives us a way to solve the decoupled system, from which we can us the scheme $\frac{w_i^{n+1} - w_i^n}{\Delta t} + \lambda \frac{w_{i+1}^n - w_i^n}{\Delta x} = 0$

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Or $\dfrac{w_i^{n+1} - w_i^n}{\Delta t} + \lambda \dfrac{w_i^n - w_{i-1}^n}{\Delta x} = 0$ to solve

the decoupled system, depending on sign of λ.

E.g. If $\lambda \geq 0$, we use $\dfrac{w_i^{n+1} - w_i^n}{\Delta t} + \lambda \dfrac{w_i^n - w_{i-1}^n}{\Delta x} = 0$

because here, stability is seen in that

$$w_i^{n+1} = w_i^n + \lambda \left(\frac{\Delta t}{\Delta x}\right)\left(w_{i-1}^n - w_{i+1}^n\right)$$

So by von Neumann analysis, we have

$$g = 1 - \lambda \, (\Delta t/\Delta x) \left(1 - e^{-i\theta}\right).$$

So, we can choose $\left(\frac{\Delta t}{\Delta x}\right)$

So that $\|g\| \leq 1$ (i.e. we want $\left|\lambda \frac{\Delta t}{\Delta x}\right| < \frac{1}{2}$).

So, by solving the decoupled system, we can solve for $w_1 = u_t$ and $w_2 = u_x$, and by using initial conditions, we can obtain u.

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(8) a) we want: ~~$\|h\|$~~ $\|h\|_{L^2(\Gamma_3)} < \infty$

$\|f\|_{L^2(\Omega)} < \infty$

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(8) a) ~~We assume that h is sufficiently smooth so that we can extend h to $\tilde{h}$, where $\tilde{h}|_{T_3} = h$, and $\tilde{h}|_{T_1 \cup T_2} \equiv 0$.~~

Let $V = \{ v \in H^1(T) : v|_{T_1 \cup T_2} \equiv 0 \}$

Given $v \in V$, note: $u \in V$.

$$\int_T -v \Delta u + uv = \int_T fv$$

Have: $$\int_T -v\Delta u = \int_T \nabla u \cdot \nabla v - \int_{\partial T} v \nabla u \cdot n$$ when n is outward pointing normal vector

$$= \int_T \nabla u \cdot \nabla v - \int_{T_1 \cup T_2} \overset{0}{\overset{\shortparallel}{v}} \nabla u \cdot n - \int_{T_3} v \nabla u \cdot n$$

$$= \int_T \nabla u \cdot \nabla v - \int_{T_3} vh$$

So we have $$\int_T \nabla u \cdot \nabla v - \int_{T_3} vh + \int_T uv = \int_T fv$$

$$\therefore \int_T \nabla u \cdot \nabla v + \int_T uv = \int_T fv + \int_{T_3} hv.$$

our variational problem is: $a(u,v) = L(v)$

for $a(u,v) = \int_T \nabla u \cdot \nabla v + \int_T uv$. and $L(v) = \int_T fv + \int_{T_3} hv$.

We now show that a is coercive and bounded, and L is bounded.

To see that a is coercive, note that

$$\int_T \nabla u \cdot \nabla u + \int_T u^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = \|u\|_{H^1(T)}$$

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To see that a is bounded, we note that

$$a(u,v) = \int_T \nabla u \cdot \nabla v + \int_T u \cdot v \overset{\text{(Cauchy Schwartz)}}{\leq} \|\nabla u\|_{L^2(T)} \|\nabla v\|_{L^2(T)} + \|u\|_{L^2(T)} \|v\|_{L^2(T)}$$

$$\leq 2\|u\|_{H^1(T)} \|v\|_{H^1(T)}$$

To see that L is bounded, we note that

$$|L(v)| = \left| \int_T f v + \int_{T_3} h v \right| \leq \int_T |fv| + \int_{T_3} |hv|$$

$$\leq \|f\|_{L^2(T)} \|v\|_{L^2(T)} + \|h\|_{L^2(T_3)} \|v\|_{L^2(T_3)}.$$

$$\underset{\text{on } v}{\text{(trace thm)}} \leq \|f\|_{L^2(T)} \|v\|_{L^2(T)} + C \|h\|_{L^2(T_3)} \|v\|_{H^1(T_3)}$$

$$\leq \max\left(\|f\|_{L^2(T)}, C\|h\|_{L^2(T_3)}\right) \|v\|_{H^1(T)}$$

So, as a is coercive and bounded, and L is bounded, we have a unique solution given by Lax Milgram.

b). Let T be a triangulation with no triangle too thin, and h = max diameter of a triangle. Let V_h be the set of piecewise linear functions in V such that for $f \in V_h$, f restricted to any triangle is linear. Let $N_1, \dots, N_n$ be the nodes for this triangulation, not including nodes on $T_1 \cup T_2$. Let $\phi_1, \dots, \phi_N$ be the set of corresponding basis functions, where $\phi_i(N_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$.

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We thus have the following problem:
To find $u_h \in V_h$ such that $a(u_h, v) = L(v)$ for all $v \in V_h$.
In particular, we want $a(u_h, \phi_j) = L(\phi_j)$ for all basis functions.

For $u_h = c_1\phi_1 + \dots + c_N\phi_N$, we have $\sum_{i=1}^{N} c_i\, a(\phi_i, \phi_j) = L(\phi_j)$.

~~This thus gives us~~

Solving for u_h is akin to solving the linear system $Ax = b$, where $A_{ji} = a(\phi_i, \phi_j)$ and $b_j = L(\phi_j)$.
We note that there is a unique solution, since A is symmetric, and $\langle Ax, x\rangle > 0$ for $x > 0$, given by coercivity of a. Thus, as A symm and pos. definite A is invertible. Thus, for $x = \begin{bmatrix} c_1 \\ c_N \end{bmatrix}$, we have our $u_h = c_1\phi_1 + \dots c_N\phi_N$.
We note here that A is sparse, as $a(\phi_j, \phi_i) = 0$ unless they share a triangle.
We have $\|u - u_h\|_a \le \|u - v\|_a$ for all $v \in V_h$, where $\|\cdot\|_a$ is the norm induced by a.

Moreover, we have $\|u - \pi_V u\|_a \le C h^2$, where C is dependent on u, and not h (here, $\pi_V u$ is the projection onto V_h by a).

Thus, we have $\|u - u_h\|_a \le C h^2$ ✓.