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① False.

We note for $n \geq 1$, $tr(I_n) = n$, where I_n is the $n \times n$ identity. However, we note that given any A, B, we have $tr(AB) = tr(BA)$. Thus, since $tr(M+N) = tr(M) + tr(N)$, and $tr(M-N) = tr(M) - tr(N)$, we see:

$$tr(AB-BA) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0.$$

Since $\mathbb{C}$ is characteristic zero, $n \neq 0$. Thus, no such A, B exist such that $AB - BA = I_n$.

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(2) Suppose λ is an eigenvalue of B. ~~As B, B^5 are similar~~
Then λ^5 is an eigenvalue of B^5. As B and B^5 are similar, they have the same eigenvalues, so λ^5 is also an eigenvalue of B. Thus if λ is an eigenvalue of B λ^5 is also an eigenvalue, as are $(\lambda^5)^5, \ldots, \lambda^{5^n}$.
We now note that since B is 2x2, it must have at most two ~~eigenval~~ distinct eigenvalues. This means that either $\lambda^5 = \lambda$, or if $\lambda^5 \neq \lambda$, we must have $\lambda^{5^2} = \lambda^{25} = \lambda$ OR $\lambda^{25} = \lambda^5$. In the first case, we can divide by λ, as $\lambda \neq 0$ (as B is given to be invertible), yielding $\lambda^4 = 1$. In the second case, we divide by λ again and yield $\lambda^{24} = 1$.
If it is the case that $\lambda^{25} = \lambda^5$, then we note λ and λ^5 are our distinct eigenvalues. In this case, $\det(B) = \lambda \cdot \lambda^5 = \lambda^6$. We note λ^5 and λ^{25} would then be eigenvalues of B^5, so $\det(B^5) = \lambda^{30}$. However, as B and B^5 are similar, they have the same determinant. So, $\lambda^6 = \lambda^{30}$, and by dividing by λ^6, we see $\lambda^{24} = 1$. So, in all these cases, we have $\lambda^{24} = 1$, meaning λ is a root of unity ~~of order dividing 24~~ with order dividing 24.

(3)

We note: $P(x) = \sum_{\sigma \in S_n} \overset{\text{sgn}(\sigma)}{\downarrow} \prod_{i=1}^{n} (a_{i,\sigma(i)} + x)$, where S_n is the nth symmetric group

and $\text{sgn}(\sigma)$ is the sign of σ

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) (a_{1\,\sigma(1)} + x)(a_{2\,\sigma(2)} + x) \cdots (a_{n\,\sigma(n)} + x)$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left[\prod_{i=1}^{n} a_{i\,\sigma(i)} + x \left(\sum_{i=1}^{n} \prod_{j \neq i} a_{i\,\sigma(i)} \right) + \cdots \right].$$

From this, we note that the term on x^k is

$$\sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{i_1 < i_2 < \cdots < i_k} \prod_{j \neq i_1, i_2, \ldots, i_k} a_{i\,\sigma(i)}).$$

$$\boxed{\sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{i_1 < i_2 < \cdots < i_k} \prod_{\substack{j \neq i_j \\ \forall j = 1, 2, \ldots, k}} a_{i,\sigma(i)}}$$

⑤

For $n=2$, we have $\begin{pmatrix}1&1\\1&-1\end{pmatrix}$. Clearly, $A\begin{bmatrix}1\\-1\end{bmatrix} = -\begin{bmatrix}1\\-1\end{bmatrix}$ and $A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

For $n\geq 2$, we have:
$$T = \begin{array}{cl} \begin{bmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & \ddots & 1 & 1 \\ & & & \cdots & 1 & -1 \\ & & 1 & -1 & & \\ 1 & -1 & & & & \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \frac{n}{2}\text{th row for } n \text{ } \cancel{\text{odd}} \text{ even} \\ \\ \\ \\ \end{array} \end{array}$$

and
$$T = \begin{array}{cl} \begin{bmatrix} 1 & 1 & & & \\ & & 1 & 1 & \\ & & & \ddots & 1 \\ & & 1 & -1 & \\ 1 & -1 & & & \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\text{st row for } n \text{ odd.} \\ \\ \\ \end{array} \end{array}$$

We notice that A fixes all the columns with just 1's and 0's and negates all the columns that have a -1. We also see that T is invertible, as it is readily checked that the columns are linearly independent.

Thus, under the basis as columns of T, ~~A is diag~~ the transformation T is diagonal, meaning TAT^{-1}, is diagonal.

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(6) Suppose λ is an eig. value such that $|\lambda| \geq 1$.

Then there is some $v \in \mathbb{C}^m$ such that $Tv = \lambda v$
WLOG, $\|v\| = 1$.
In particular, we see that $\|T^n v\| = \lambda^n$,

and as $|\lambda| \geq 1$, $|\lambda^n| \not\to 0$.

Thus, it follows that the operator norm (over $\mathbb{C}$), $\sup_{\|u\|=1} \|Tu\|$ does not go to 0, so

$\lim_{n \to \infty} T^n \neq 0$ (as all norms are equivalent in finite dim. vector space).

Now, suppose $|\lambda| < 0$, for all eigenvalues $\lambda_1, \dots, \lambda_m$.

In this case, the Jordan canonical form for T would be

$$T = PJP^{-1}, \quad \text{where } T = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}, \quad J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_k \end{bmatrix}.$$

To show $T^n \to 0$, it is enough to show $J^n \to 0$, since conjugating by an invertible matrix is a continuous operation.
Moreover, to show $J^n \to 0$, it's enough to show $J_k^n \to 0$,
as $J^n = \begin{bmatrix} J_1^n & & \\ & \ddots & \\ & & J_r^n \end{bmatrix}$.

Suppose J_k is size $s \times s$.

$$\cancel{J_k^n = \begin{bmatrix} \lambda_k^n & \binom{n}{1}\lambda_k & & \\ & \ddots & \ddots & \\ & & & \lambda_k^n \end{bmatrix}}$$

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Note:

$$J_k = \begin{bmatrix} \lambda_k^n & \binom{n}{1}\lambda_k^{n-1} & \cdots & \binom{n}{s}\lambda_k^{n-s} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{n}{1}\lambda_k^{n-1} \\ & & & \lambda_k^n \end{bmatrix},$$

That is, the nonzero entries are of the form $\binom{n}{a}\lambda_k^{n-a}$, for $0 \le a \le s$.

Note: $\binom{n}{a}\lambda_k^{n-a} = \frac{n!}{a!(n-a)!}\lambda_k^{n-a} = \frac{n(n-1)\cdots(n-a+1)}{a!}\lambda_k^{n-a}$

Note that $n(n-1)\cdots(n-a+1)$ is a degree a polynomial and $\lambda_k^{n-a} = \frac{\lambda_k^n}{\lambda_k^a}$ is an exponential, so, by repeatedly applying L'hopital's rule, we see $\lim_{n\to\infty}\binom{n}{a}\lambda_k^{n-a} = 0$, since exponential beats polynomial and $|\lambda| < 1$. It thus follows that $\lim_{n\to\infty} J_k^n = 0$ for all k, so $\lim_{n\to\infty} J^n = 0$, so indeed, $\lim_{n\to\infty} J^n = 0$ as well.

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(7) a) Suppose $f(x) < 18$ for all x. As $[0,2]$ is compact, we note that f attains its supremum (as f is continuous), meaning $\sup_{x \in I} f(x) = M < 18$.

From this, since $f(x) \leq M$, we note that

$$\int_0^2 f(x)\,dx \leq \int_0^2 M\,dx = 2M < 36.$$

But this contradicts the fact that $\int_0^2 f(x)\,dx = 36$.
~~we are trying to see that it is not the case that~~

$\boxed{(*)}$ continued on next pg.

b) Seeking a contrad, suppose $g(x,y) < 9$ for all (x,y).

Then as before, since g is continuous, f attains its supremum ~~meaning $\sup_{(x,y)\in I^2} g(x,y) = M$~~ as $[0,2]^2$ is compact.

Thus, $\sup_{(x,y)\in[0,2]^2} g(x,y) = M < 9$.

As before, we see $g(x,y) \leq M < 9$ for all x, y,

so $\iint_{I \times I} g(x,y)\,dx\,dy \leq \iint_{I \times I} M\,dx\,dy = 4M < 36$,

a contradiction to $\iint_{I^2} g(x,y)\,dx\,dy = 36$.

continued on next page.

7(a) continued.

We also can similarly show that it is not the case that $f(x) > 18$ for all x.

Thus, ~~the case x~~ is $x \in I$ such that $f(x) \geq 18$, and $y \in I$ such that $f(y) \leq 18$.

By intermediate value thm (as f is cts), $\exists z$ between x and y such that $f(z) = 18$.

7(b) continued.

So, we see it is not the case that $g(x,y) < 9$ for all (x,y). Similarly, we can show it is not the case that $g(x,y) > 9$ for all x,y.

Thus, $\exists (x,y)$ and (x',y') such that $g(x',y') \geq 9$
$g(x,y) < 9$. ~~Let ... (continuous path) for~~

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a cont. path such that $\gamma(0) = (x,y)$, $\gamma(1) = (x',y')$.

We note $g(\gamma(t))$ is a cont. function from $[0,1]$ to $\mathbb{R}$.

Since $g(\gamma(1)) \geq 9$ and $g(\gamma(0)) \leq 9$, we see that $\exists t \in [0,1]$ s.t. $g(\gamma(t)) = 9$ (by int. value thm).

So, at $\gamma(t)$, we have $g(\gamma(t)) = 9$ ✓.

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(8) Since $\int_a^b x^n f(x)dx = 0$ for all n, it follows immediately that $\int_a^b p(x) f(x)\,dx = 0$, for any polynomial p.

Seeking a contradiction, suppose f is not identically zero. As f is continuous, ~~can fin~~ by Stone Weierstrauss theorem, ~~we can find a polynomial p such that given any ε~~ for any $\varepsilon > 0$, we can find p such that $\|p-f\|_\infty < \varepsilon$.

So, given any $\varepsilon > 0$, take p s.t. $\|f-p\|_\infty < \varepsilon / |b-a| \cdot \|f\|_\infty$.

We then note that: [here, $f^2(x) = f(x)^2$]

$$\left|\int_a^b f^2(x)\,dx\right| = \left|\int_a^b f^2(x)\,dx - \int_a^b p(x)f(x)\,dx\right|$$

$$= \left|\int_a^b (f(x)-p(x))\,f(x)\,dx\right|$$

$$\leq \int_a^b \|(f(x)-p(x)) \cdot f(x)\|_\infty\,dx$$

$$\leq \int_a^b \|f-p\|_\infty \cdot \|f\|_\infty\,dx$$

$$= \frac{\varepsilon}{|b-a|\,\|f\|_\infty} \cdot |b-a| \cdot \|f\|_\infty = \varepsilon.$$

So, $\left|\int_a^b f(x)^2 dx\right| \leq \varepsilon$ for any $\varepsilon > 0$. As ε is arbitrary this means $\int_a^b f(x)^2 dx = 0$. Since $f(x)^2$ is non-negative, $f(x)^2 \equiv 0$, which contradicts assumption that $f \not\equiv 0$.

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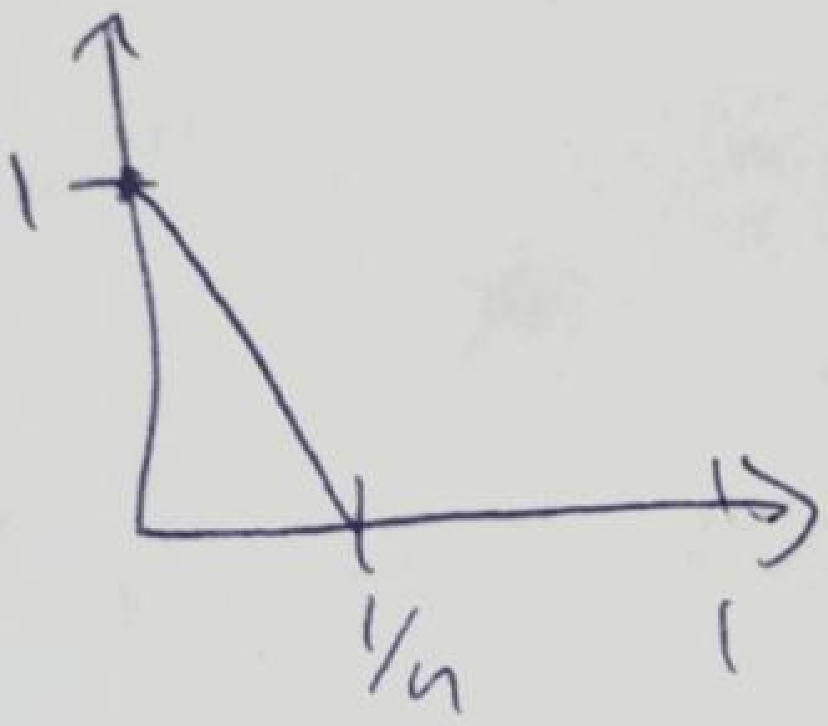
(9) d) we show X is not complete.

To do this, consider $f_n = \begin{cases} 1 & x=0 \\ 0 & x \geq 1/n \\ -nx+1 & x\in[0,\frac{1}{n}] \end{cases}$.

Pictorially, f_n looks like:



We first claim f is cauchy. To see this, take $\varepsilon > 0$, and take $N>0$ s.t. $\frac{1}{N} < \varepsilon$. Then for $m>n>N$, we note that f_m and f_n are both zero from $[\frac{1}{N}, 1]$. Thus, it follows that $f_n - f_m$ is zero on $[\frac{1}{N}, 1]$, so $dist(f_n, f_m) \leq \frac{1}{N} < \varepsilon$.

Now, we note that $f_n(0)=1$ for all n. Thus, we see that f_n pointwise converges to $f(x) = \begin{cases} 0 & x>0 \\ 1 & x=0 \end{cases}$

We note in particular that under this metric, $dist(f, f_n) < \varepsilon$ for $n>N$ (for N as above), since both f and f_n are zero from $[\frac{1}{N}, 1]$. It thus follows that $\lim_{n\to\infty} dist(f, f_n) = 0$, i.e. f is the limit of f_n under this metric. However, $f \notin X$, as f not continuous.

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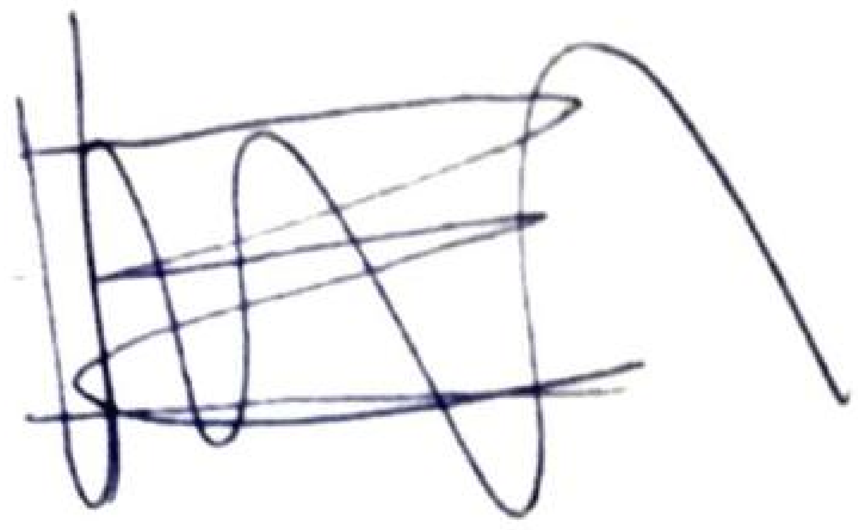

e) To see X is not compact, we first note from d) that X is not ~~continuous~~ complete. Since compact metric spaces are complete and totally bounded, it follows that X is not compact, as it is not complete.

(11) First, we note that $\sum_{n=1}^{\infty} a_n$ converges, as by alt. series test, $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$ converges. Let $S = \{s_1, s_2, \ldots\}$ be the odd terms of a_n and $T = \{t_1, t_2, \ldots\}$ be the even terms of a_n.

We note that $\sum_{i=1}^{\infty} s_i = \infty$, as this is simply $\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which converges as $1/2 < 1$ (p-test).

Similarly, we can perform a comparison test and see that $\sum_{n=1}^{\infty} t_n$ converges.

Now, take any $\beta \in \mathbb{R}$.

First, we start summing terms in S until we get $\sum_{n=1}^{N} s_i > \beta$. (this will happen as $\sum_{i=1}^{\infty} s_i$'s diverge).

Next, we start adding terms in T_n until we get $\sum_{n=1}^{N} s_i + \sum_{n=1}^{m} t_i < \beta$ (this too will happen as $\sum_{n=1}^{\infty} t_n = -\infty$. We then repeat this process indefinitely, to get our bijection $\sigma: \mathbb{N} \to \mathbb{N}$. We note that this process converges to β, since after each time we switch from adding terms in T and S, we have something $\left|\beta - \left(\sum_{n=1}^{N} S_n + \sum_{m=1}^{M} T_m + \sum_{n=N}^{N_2} S_n + \sum_{m=N}^{m_2} T_m \ldots\right)\right| < t_i$, the final term we add before switching to terms in S.

The sum is said ~~[illegible]~~ for switching from adding terms in S to adding terms ~~[illegible]~~ in T. Moreover, we note that ~~[illegible]~~ before switching between the sets to sum, we are only getting closer to β. Thus, as $\lim_{n\to\infty} s_n = 0$ and $\lim_{m\to\infty} t_m = 0$, we see that this sum converges to β. We note that this is indeed a bijection on $\mathbb{N}$, since we continue this process indefinitely, so all terms in T and S will eventually be used.

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(12) a) Suppose f is convex. We show: f is convex if and only if for $a<b<c$, we have:

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(b)}{b-a}.$$

First, we note that we can write $b = ta+(1-t)c$

Thus, $$\frac{f(b)-f(a)}{b-a} = \frac{f(ta+(1-t)c)-f(a)}{(ta+(1-t)c)-a} = \frac{f(ta+(1-t)c)-f(a)}{(1-t)(c-a)}$$

So, $$\frac{f(b)-f(a)}{b-a} = \frac{f(ta+(1-t)c)-f(a)}{(1-t)(c-a)} \leq \frac{tf(a)+(1-t)f(c)-f(a)}{(1-t)(c-a)}$$

$$= \frac{(1-t)(f(c)-f(a))}{(1-t)(c-a)}$$

$$= \frac{f(c)-f(a)}{c-a}$$

Similarly: $$\frac{f(c)-f(b)}{c-b} =$$

$$\frac{f(c)-f(ta+(1-t)c)}{c-ta-(1-t)c}$$

$$\geq \frac{f(c)-tf(a)-(1-t)f(c)}{t(c-a)}$$

$$= \frac{t(f(c)-f(a))}{t(c-a)} = \frac{f(c)-f(a)}{c-a}.$$

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12 cont: So, we have $\frac{f(b)-f(a)}{b-a} \le \frac{f(c)-f(a)}{c-a} \le \frac{f(c)-f(b)}{c-b}$ ✓

Now, suppose for $a<b<c$ we have $\frac{f(b)-f(a)}{b-a} \le \frac{f(c)-f(a)}{c-a} \le \frac{f(c)-f(b)}{c-b}$.

Now, take $a<c$.

we wish to show $f(ta+(1-t)c) \le tf(a)+(1-t)f(c)$.

Let $b = ta+(1-t)c$.

note: $f(b)-f(a) = tf(a)+(1-t)f(c)-f(a)$

$= (1-t)(f(c)-f(a))$.

So, $\frac{f(b)-f(a)}{b-a} = \frac{(1-t)(f(c)-f(a))}{(1-t)(c-a)} \le \frac{f(c)-f(a)}{c-a}$

So, $f(b)-f(a) \le \underbrace{(ta+(1-t)c-a)}_{=(b-a)} \frac{f(c)-f(a)}{c-a}$

$= (1-t)(c-a) \frac{(f(c)-f(a))}{c-a}$

$= (1-t)(f(c)-f(a))$

$\therefore f(b) \le (1-t)f(c)+tf(a)$.

$\therefore f(ta+(1-t)c) \le tf(a)+(1-t)f(b)$. as desired

We can similarly show this for $c>a$.

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(12) a) ct.

With this characterization, note that if f is convex,

given $x < y$, we have:

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(z)-f(x)}{z-x} \quad \text{for any } z \text{ between } x \text{ and } y.$$

Thus, taking $z \to x$, we see that

$$\frac{f(y)-f(x)}{y-x} \geq \lim_{z \to x} \frac{f(z)-f(x)}{z-x} = f'(x).$$

So, as $y-x>0$, we have:

$$f(y) - f(x) \geq (y-x) f'(x)$$

$$\therefore f(y) \geq f(x) + (y-x) f'(x).$$

We can similarly show the same for $y > x$ (in this case, $y-x<0$, so division flips the inequality).

Now, suppose ~~f(y) f(x)~~ $f(y) \geq f(x) + (y-x) f'(x)$ for all x, y.

~~we note that~~

~~we want to show f(x)~~

We wish to show:

$$f(tx+(1-t)y) \leq t f(x) + (1-t) y.$$

Let $z = tx+(1-t)y$.

we have:

$$f(z) \leq f(y) - (y-z) f'(y)$$

$$= f(y) - (y - tx - (1-t)y) f'(y))$$

$$= f(y) - t(y-x) \left(\frac{f(y)-f(z)}{y-z} \right)$$

$$\leq f(y) - t(y-x) \left(\frac{f(y)-f(x)}{y-x} \right)$$

$$= f(y) - t(f(y) - f(x))$$

$$= t f(x) + (1-t) f(y).$$ as desired.

5) Suppose $f'' \geq 0$.

In this case, we note: for $y > x$

$$\frac{f(y)-f(x)}{y-x} = f'(\xi)$$ for ξ between x and y

As $f'' \geq 0$, f' is increasing, so

$$\frac{f(y)-f(x)}{y-x} = f'(\xi) \geq f'(x)$$

$\therefore$ $f(y) \geq f(x) + (y-x) f'(x)$ as desired